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Research Article

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A note on Kazdan–Warner equation on networks

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Abstract: We investigate the Kazdan–Warner equation on a network. In this case, the differential equation is defined on each edge, while appropriate transition conditions of Kirchhoff type are prescribed at the vertices. We show that the whole Kazdan–Warner theory, both for the noncritical and the critical case, extends to the present setting.

Keywords: Kazdan–Warner equation, network, Kirchhoff condition

MSC 2010: 35A15, 35J60, 35R02

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1 Introduction

The Kazdan–Warner equation

$$\Delta u = c - he^{u}, \tag{1.1}$$

where c is a constant and h a given function, was introduced in [15] in connection with the problem of prescribing the Gaussian curvature of a compact manifold M. The resolvability of (1.1) depends on the sign of c. Let $\bar{h}$ denote the average of h on M. In [15], it is shown that,

(i) if $c = 0$ and $h \not\equiv 0$, then (1.1) is solvable if and only if h changes sign and $\bar{h} < 0$;
(ii) if $c > 0$, then (1.1) is solvable if and only if the set $\{h > 0\}$ is not empty;
(iii) if $c < 0$, if (1.1) is solvable, then $\bar{h} < 0$. For $\bar{h} < 0$, there exists a constant $-\infty \le c(h) < 0$ such that (1.1) is solvable for any $c \in (c(h), 0)$ and not solvable for any $c < c(h)$. Moreover, $c(h) = -\infty$ if and only if $h \le 0$ in M.

If $c < 0$, $c = c(h)$ is not included in the previous cases and deserves particular attention. It has been shown in [8] that, if $c(h) > -\infty$, then (1.1) can be also solved for $c = c(h)$.

The previous theory has been recently extended in [13, 14] to the case of a combinatorial graph. Here the Laplacian is replaced by a finite difference operator, the so-called graph Laplacian, and most of the effort is to reproduce in a finite-dimensional setting some crucial properties as the maximum principle and the Moser–Trudinger inequality.

An intermediate situation between a compact manifold and a combinatorial graph is given by a network (or metric graph) Γ, which is given by a finite collection of vertices connected by continuous non-self-intersecting edges (see Section 2 for the precise definition). On a network, differential equation (1.1) is defined on each edge, while appropriate transition conditions of Kirchhoff type are prescribed at the vertices. In this

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paper, we obtain the same conclusions of the manifold and the graph cases, showing that the Kazdan–Warner theory remains unchanged for different classes of manifolds, also non-regular such as in the case of networks. To prove these results, we shall adapt the method by Kazdan and Warner [15, Theorem 5.3] (see also [14, Theorem 2]) and, for the critical case, some techniques of [8, 13] with some specific arguments for networks.

The study of the theory of differential equations on networks is motivated by two features: the networks provide simplified mathematical models for physics, chemistry and engineering where one can investigate phenomena in a "quasi-one-dimensional" environment. Moreover, being encompassed in the larger family of non-regular manifold, they are a first step toward the approach to more general manifolds. The recent study of differential equations on networks steamed mainly in the framework of hyperbolic systems and spectral theory (we refer the reader to the monographs [3, 5, 18] and references therein). A general theory for linear and semilinear differential equations has been developed mainly employing the variational structure of the problem, and in this framework, the natural transition conditions at the vertices are the Kirchhoff conditions (for instance, see [17–19]; see also [10, 11] for a stochastic interpretation). The theory has been also extended to some nonlinear problem such as traffic flows [12], optimal control problems through the corresponding Hamilton–Jacobi equations [4, 16] and mean field games [1, 7, 16], nonlinear Schrödinger equations [2, 9]. For nonlinear problems, different transition conditions at the vertices can arise.

The paper is organized as follows. In Section 2, we introduce some notation and preliminary results. In Sections 3, 4 and 5, we study respectively the cases $c = 0$, $c > 0$ and $c < 0$. In Section 5, we also discuss the critical case $c = c(h)$.

2 Notation, definitions and preliminary results

A network $\Gamma = (V, E)$ is a finite collection of points $V := \{\mathrm{v}_i\}_{i\in I}$ in $\mathbb{R}^n$ connected by continuous edges $E := \{e_j\}_{j\in J}$, where each edge does not intersect itself and any two edges can only have intersection at a vertex. For $i \in I$, we set

$$\mathrm{Inc}_i := \{j \in J : \ e_j \text{ is incident to } \mathrm{v}_i\}.$$

A coordinate $\pi_j : [0, l_j] \to \mathbb{R}^n$, with $l_j > 0$, is chosen to parametrize e_j, i.e. $e_j := \pi_j((0, l_j))$. We assume that Γ is compact and connected, and we denote by $|\Gamma|$ the sum of the lengths of the edges e_j, $j \in J$.

For a function $u : \Gamma \to \mathbb{R}$, we denote by $u_j : [0, l_j] \to \mathbb{R}$ the restriction of u to e_j, i.e. $u(x) = u_j(y)$ for $x \in e_j$, $y = \pi_j^{-1}(x) \in (0, l_j)$. Given $\mathrm{v}_i \in V$, we denote by $\partial_j u(\mathrm{v}_i)$ the oriented derivative at v_i along the arc e_j defined by

$$\partial_j u(\mathrm{v}_i) = \lim_{x\in e_j,\, x\to \mathrm{v}_i} \frac{u_j(\pi_j^{-1}(x)) - u_j(\pi_j^{-1}(\mathrm{v}_i))}{|\pi_j^{-1}(x) - \pi_j^{-1}(\mathrm{v}_i)|}$$

if the limit exists, where π_j is the parametrization of arc e_j. For a function $\phi : \Gamma \to \mathbb{R}$ and $A \subset \Gamma$, we set

$$\int_A \phi(x)\, dx := \sum_j \int_{(0,l_j)\cap\pi_j^{-1}(A)} \phi(r)\, dr.$$

A function u is said to be continuous on Γ if it is continuous with respect to the subspace topology of Γ, i.e. $u_j \in C([0, l_j])$ for any $j \in J$ and $u_j(\pi_j^{-1}(\mathrm{v}_i)) = u_k(\pi_k^{-1}(\mathrm{v}_i))$ for any $i \in I$, $j, k \in \mathrm{Inc}_i$.

We introduce some functional spaces for functions defined on the network. The space $L^p(\Gamma)$, $p \geq 1$, consists of the functions that are measurable and p-integrable on each edge e_j, $j \in J$. We set

$$\|f\|_{L^p} := \Big(\sum_{j\in J} \|f_j\|^p_{L^p(e_j)}\Big)^{\frac{1}{p}}.$$

The space $L^\infty(\Gamma)$ consists of the functions that are measurable and bounded on each edge e_j, $j \in J$. We set

$$\|f\|_{L^\infty} := \sup_{j\in J} \|f_j\|_{L^\infty(e_j)}.$$

The Sobolev space $W^{k,p}(\Gamma)$, $k \in \mathbb{N}$ and $p \geq 1$, consists of all continuous functions on Γ that belong to $W^{k,p}(e_j)$ for each $j \in J$. We set

$$\|f\|_{W^{k,p}} := \left(\sum_{l=0}^{k} \|\partial^l f\|_{L^p}^p\right)^{\frac{1}{p}}.$$

As usual, we set $H^k(\Gamma) := W^{k,2}(\Gamma)$, $k \in \mathbb{N}$. The space $C^k(\Gamma)$ for $k \in \mathbb{N}$ consists of all continuous functions on Γ that belong to $C^k(e_j)$ for $j \in J$. The space $C^k(\Gamma)$ is a Banach space with the norm

$$\|f\|_{C^k} = \max_{\beta \leq k} \|\partial^\beta f\|_{L^\infty}.$$

The following lemma gives a Poincaré inequality for compact networks.

Lemma 2.1. *For every function $f \in H^1(\Gamma)$ with $\int_\Gamma f(x)\,dx = 0$, there hold*
(i) $|f(x)| \leq \sqrt{|\Gamma|}\|\partial f\|_{L^2}$,
(ii) $\int_\Gamma f^2(x)\,dx \leq |\Gamma|^2 \int_\Gamma |\partial f(x)|^2\,dx$.

Proof. By definition of H^1, the function f is continuous on Γ; hence there exists a point $x_0 \in \Gamma$ such that $f(x_0) = 0$. Since Γ is connected, for any point $x \in \Gamma$, there exists a path $\gamma\colon (0,r) \to \Gamma$ on the network such that $\gamma(0) = x_0$, $\gamma(r) = x$, $|\gamma'(s)| = 1$ and $r \leq |\Gamma|$. Hence we have

$$|f(x)| = \left|f(x_0) + \int_0^r (f \circ \gamma)'(s)\,ds\right| \leq \int_0^r |\partial f(\gamma(s))|\,ds \leq \sqrt{r}\|\partial f\|_{L^2(\gamma)} \leq \sqrt{|\Gamma|}\|\partial f\|_{L^2}.$$

We deduce that

$$\int_\Gamma f^2(x)\,dx \leq \int_\Gamma |\Gamma|\|\partial f\|_{L^2}^2\,dx = |\Gamma|^2\|\partial f\|_{L^2}^2. \qquad \square$$

We also give an analogue of the Trudinger–Moser inequality for compact networks.

Lemma 2.2. *For any $\beta, \delta \in \mathbb{R}$ with $\delta > 0$, there exists a constant C (depending only on β, δ and the network) such that, for all functions $f \in H^1(\Gamma)$ with $\int_\Gamma |\partial f|^2 \leq \delta$ and $\int_\Gamma f = 0$, there holds*

$$\int_\Gamma e^{\beta f^2(x)}\,dx \leq C.$$

Proof. We adapt the argument of [14, Lemma 7]. The case $\beta \leq 0$ is obvious because Γ has a bounded total length. Fix $\beta > 0$, and consider a function f as in the statement. By Lemma (2.1) (i) and the assumption $\|\partial f\|_{L^2}^2 \leq \delta$, we have

$$\int_\Gamma e^{\beta f^2(x)}\,dx \leq \int_\Gamma e^{\beta|\Gamma|\|\partial f\|_{L^2}^2}\,dx \leq e^{\beta|\Gamma|\delta}|\Gamma|. \qquad \square$$

We consider the Kazdan–Warner equation on the network Γ,

$$\begin{cases} \partial^2 u = c - he^u, & x \in e_j,\ j \in J, \\ u_j(\mathrm{v}_i) = u_k(\mathrm{v}_i), & j,k \in \mathrm{Inc}_i,\ \mathrm{v}_i \in V, \\ \sum_{j \in \mathrm{Inc}_i} \partial_j u(\mathrm{v}_i) = 0, & \mathrm{v}_i \in V, \end{cases} \tag{2.1}$$

where c is a given constant and h is a continuous function on Γ. Note that the Kazdan–Warner equation is defined on each edge, while, at the vertices, we impose the continuity of u and the Kirchhoff condition, a classical condition for differential equations defined on networks (see [18, 20]).

Definition 2.1. We introduce the notion of solution to problem (2.1).
(a) A strong solution to problem (2.1) is a function $u \in C^2(\Gamma)$ which satisfies (2.1) in a pointwise manner.
(b) A weak solution to problem (2.1) is a function $u \in H^1(\Gamma)$ such that

$$\int_\Gamma \partial u \partial \phi\,dx = -c\int_\Gamma \phi\,dx + \int_\Gamma he^u \phi\,dx \quad \text{for all } \phi \in H^1(\Gamma). \tag{2.2}$$

Remark 2.1. One can easily check that, if $u \in C^2(\Gamma)$ is a weak solution of (2.1), then it is also a strong solution. Moreover, any weak solution of (2.1) is also a strong solution. Actually, a weak solution u fulfils $\partial^2 u = c - he^u$ in distributional sense inside each edge e_j. The right-hand side of this equality is continuous; hence, by standard theory, $u \in C^2(e_j)$ for every $j \in J$. Being a weak solution, u also belongs to $H^1(\Gamma)$; hence, u belongs to $C^2(\Gamma)$ and is a classical solution of the differential equation in (2.1) inside each edge. By these last properties, integrating by parts (2.2), we obtain that u also fulfils the Kirchhoff condition in (2.1). In conclusion, u is a strong solution to (2.1).

In the next three sections, we discuss the solvability of (2.1) in the cases $c = 0$, $c > 0$ and $c < 0$.

3 The Kazdan–Warner equation with case $c = 0$

Theorem 3.1. *Assume $c = 0$ and $h \not\equiv 0$. Then problem* (2.1) *has a solution u if and only if h changes sign and $\int_\Gamma h < 0$.*

Proof. Assume that u is a solution to problem (2.1) with $c = 0$. We note that the hypothesis $h \not\equiv 0$ prevents u to be constant. Letting $\phi \equiv 1$ in (2.2), we get $\int_\Gamma he^u\,dx = 0$, which implies that h must change sign. Integrating $e^{-u}\partial^2 u = -h$ by parts on Γ, we get

$$\int_\Gamma (\partial u)^2 e^{-u}\,dx + \sum_{i \in I}\sum_{j \in \mathrm{Inc}_i} e^{-u(\mathrm{v}_i)}\partial_j u(\mathrm{v}_i) = -\int_\Gamma h\,dx.$$

Taking advantage of the Kirchhoff condition and of the continuity of u at each vertex, we obtain

$$\int_\Gamma (\partial u)^2 e^{-u}\,dx = -\int_\Gamma h\,dx.$$

Since u cannot be constant, we deduce $\int_\Gamma h\,dx < 0$.

Conversely, we prove that, for any h which changes sign and satisfies $\int_\Gamma h < 0$, there exists a solution to (2.1). We define the set

$$B := \Big\{v \in H^1(\Gamma) \;\Big|\; \int_\Gamma he^v\,dx = 0,\ \int_\Gamma v\,dx = 0\Big\}.$$

We claim that B is not empty. Since h changes sign, there exists a point $x_0 \in \Gamma$ such that $h(x_0) > 0$. By the continuity of h, without any loss of generality, we can assume $x_0 \in e_{\bar{j}}$ for some $\bar{j} \in J$; namely, there exist $\bar{j} \in J$ and $y_0 \in (0, l_{\bar{j}})$ such that $h_{\bar{j}}(y_0) > 0$. Moreover, still by the continuity of h, there exists $\varepsilon > 0$ such that $(y_0 - \varepsilon, y_0 + \varepsilon) \subset (0, l_{\bar{j}})$ and $h_{\bar{j}}(y) > \frac{h_{\bar{j}}(y_0)}{2}$ for all $y \in (y_0 - \varepsilon, y_0 + \varepsilon)$. Consider a function $w \in C^2(\Gamma)$ such that $w_{\bar{j}}(y) = 1$ if $y \in (y_0 - \frac{\varepsilon}{2}, y_0 + \frac{\varepsilon}{2})$, $w_{\bar{j}}(y) = 0$ if $y \notin (y_0 - \varepsilon, y_0 + \varepsilon)$ and $w_j \equiv 0$ if $j \in J \setminus \{\bar{j}\}$. For $\ell > 0$, the function $w_\ell(\cdot) := \ell w(\cdot)$ fulfils

$$\int_\Gamma he^{w_\ell}\,dx = \int_{e_{\bar{j}}} he^{w_\ell}\,dx + \sum_{j \in J\setminus\{\bar{j}\}} \int_{e_j} he^{w_\ell}\,dx \ge \int_{y_0-\frac{\varepsilon}{2}}^{y_0+\frac{\varepsilon}{2}} h(y)e^{w_\ell(y)}\,dy - \int_\Gamma |h|\,dx \ge \frac{\varepsilon h_{\bar{j}}(y_0)e^\ell}{2} - \int_\Gamma |h|\,dx > 0 \tag{3.1}$$

provided that ℓ is sufficiently large. On the other hand, for $\ell = 0$, we have $w_0(x) \equiv 0$ and, by assumptions,

$$\int_\Gamma he^{w_0(x)}\,dx = \int_\Gamma h\,dx < 0.$$

Therefore, there exists $\ell_0 > 0$ such that $\int_\Gamma he^{w_{\ell_0}} = 0$. Hence the function $\hat{w}(\cdot) := w_{\ell_0}(\cdot) - \int_\Gamma w_{\ell_0}/|\Gamma|$ belongs to B, and the claim is proved.

Consider the functional

$$\mathcal{J}(v) := \frac{1}{2}\int_\Gamma |\partial v|^2\,dx \quad \text{for all } v \in B.$$

Let $\{v_n\}_{n\in\mathbb{N}}$ be a minimizing sequence for $\mathcal{J}$ over B, i.e. $\lim_{n\to+\infty}\mathcal{J}(v_n)=\inf_B\mathcal{J}$. By Lemma 2.1 (ii), we have that the functions v_n are uniformly bounded in $H^1(\Gamma)$. We deduce that, possibly passing to a subsequence, there exists $\bar{u}\in H^1(\Gamma)$ such that, as $n\to+\infty$, $v_n\rightharpoonup\bar{u}$ weakly in $H^1(\Gamma)$ and $v_n\to\bar{u}$ uniformly on Γ. In particular, we get that $\bar{u}$ belongs to B, and it is a minimizer of $\mathcal{J}$ on B.

We claim that $\bar{u}$ is a strong solution to problem (2.1). Actually, by standard Lagrangian multiplier theory, there exist $\lambda,\mu\in\mathbb{R}$ such that

$$0=\frac{d}{dt}\Big(\mathcal{J}(\bar{u}+t\phi)-\lambda\int_\Gamma he^{\bar{u}+t\phi}\,dx-\mu\int_\Gamma(\bar{u}+t\phi)\,dx\Big)\Big|_{t=0}=\int_\Gamma\partial\bar{u}\partial\phi\,dx-\lambda\int_\Gamma he^{\bar{u}}\phi\,dx-\mu\int_\Gamma\phi\,dx$$

for every $\phi\in H^1(\Gamma)$. Choosing $\phi\equiv1$, since $\bar{u}\in B$, we get $\mu=0$. Arguing as in Remark 2.1, inside each edge e_j, there holds $\partial^2\bar{u}+\lambda he^{\bar{u}}=0$ in distributional sense. By the continuity of $\bar{u}$, we infer that $\bar{u}\in C^2(e_j)$ and, since $\bar{u}\in H^1(\Gamma)$, also that $\bar{u}\in C^2(\Gamma)$. Moreover, $\bar{u}$ is a strong solution to

$$\begin{cases}\partial^2\bar{u}=-\lambda he^{\bar{u}}, & x\in e_j,\ j\in J,\\ \sum_{j\in\mathrm{Inc}_i}\partial_j\bar{u}(\mathrm{v}_i)=0, & \mathrm{v}_i\in V,\\ \bar{u}_j(\mathrm{v}_i)=\bar{u}_k(\mathrm{v}_i), & j,k\in\mathrm{Inc}_i,\ \mathrm{v}_i\in V.\end{cases}$$

We claim that $\lambda>0$. The function $\bar{u}$ also solves $e^{-\bar{u}}\partial^2\bar{u}=-\lambda h$; integrating this relation, by Kirchhoff and continuity conditions, we get

$$\int_\Gamma(\partial\bar{u})^2e^{-\bar{u}}\,dx=-\lambda\int_\Gamma h\,dx.$$

Let us first prove that the left-hand side of this equality is positive. We proceed by contradiction assuming $\int_\Gamma(\partial\bar{u})^2e^{-\bar{u}}=0$. Hence $\partial\bar{u}\equiv0$, and in particular, $\bar{u}$ is constant. Since $\bar{u}\in B$, we get $e^{\bar{u}}\int_\Gamma h=0$ contradicting the assumption $\int_\Gamma h<0$. Therefore, the left-hand side in the last equality is positive; again by virtue of $\int_\Gamma h<0$, the constant λ must be positive. Finally, the function $u(\cdot):=\bar{u}(\cdot)+\log(\lambda)$ is a strong solution to (2.1). □

4 The Kazdan–Warner equation with case $c>0$

Theorem 4.1. *Assume $c>0$. Then problem* (2.1) *has a solution u if and only if h is positive somewhere.*

Proof. Assume that u is a solution of (2.1); choosing $\phi\equiv1$ as test function in (2.2), we get $\int_\Gamma he^u=c|\Gamma|>0$. Hence $\{x\in\Gamma\mid h(x)>0\}\neq\emptyset$.

Conversely, for any $h\in C^0(\Gamma)$ with $\{h>0\}\neq\emptyset$, we prove that problem (2.1) admits at least one solution. To this end, it is expedient to introduce the set

$$B:=\Big\{v\in H^1(\Gamma)\ \Big|\ \int_\Gamma he^v\,dx=c|\Gamma|\Big\}.$$

We claim that B is not empty. For $\ell\geq0$, we introduce the function w_ℓ as in the proof of Theorem 3.1, while for $\ell\leq0$, we set $\bar{w}_\ell\equiv\ell$. Since $w_0\equiv\bar{w}_0$, the function

$$g(\ell):=\begin{cases}\int_\Gamma he^{w_\ell}\,dx & \text{if }\ell\geq0,\\ \int_\Gamma he^{\bar{w}_\ell}\,dx & \text{if }\ell<0\end{cases}$$

is well defined and continuous, it fulfils $\lim_{\ell\to+\infty}g(\ell)=+\infty$ (by virtue of estimate (3.1)) and

$$\lim_{\ell\to-\infty}g(\ell)=\lim_{\ell\to-\infty}e^\ell\int_\Gamma h=0.$$

Hence there exists $\bar{\ell}\in\mathbb{R}$ such that $g(\bar{\ell})=c|\Gamma|$, namely, $B\neq\emptyset$.

We consider the functional

$$\mathcal{J}(u) := \frac{1}{2}\int_\Gamma |\partial u|^2\,dx + c\int_\Gamma u\,dx \quad \text{for all } u \in B.$$

As a first step, let us prove that $\mathcal{J}$ is bounded from below in B. To this end, for any $u \in B$, we set $\bar{u} := \int_\Gamma u/|\Gamma|$ and $v := u - \bar{u}$. Note $\int_\Gamma v = 0$ and $\partial v \equiv \partial u$. Since $u \in B$, it holds $\int_\Gamma he^v\,dx = c|\Gamma|e^{-\bar{u}}$, which implies

$$\bar{u} = \log(c|\Gamma|) - \log\Big(\int_\Gamma he^v\,dx\Big);$$

replacing this equality in the definition of $\mathcal{J}$, we get

$$\mathcal{J}(u) = \frac{1}{2}\|\partial u\|_{L^2}^2 + c|\Gamma|\log(c|\Gamma|) - c|\Gamma|\log\Big(\int_\Gamma he^v\,dx\Big). \tag{4.1}$$

Let us now estimate $\int_\Gamma he^v$; if v is constant, then, by $\int_\Gamma v = 0$, it must be $v \equiv 0$ and, in particular $\int_\Gamma he^v = \int_\Gamma h$. For v non-constant, it is expedient to introduce the function $\tilde{v} := v/\|\partial v\|_{L^2}$ which verifies $\tilde{v} \in H^1(\Gamma)$, $\int_\Gamma \tilde{v} = 0$ and $\|\partial\tilde{v}\|_{L^2} = 1$. Lemma 2.1 (ii) and Lemma 2.2 guarantee that, for any $\beta \in \mathbb{R}$, there exists a constant K_β (depending only on β) such that

$$\|\tilde{v}\|_{L^2} \le |\Gamma|, \qquad \int_\Gamma e^{\beta\tilde{v}^2(x)}\,dx \le K_\beta.$$

For every ε positive, for $\beta_\varepsilon := \frac{1}{4\varepsilon}$, there holds

$$\int_\Gamma he^v\,dx \le \|h\|_{L^\infty}\int_\Gamma e^{\varepsilon\|\partial v\|_{L^2}^2 + \frac{v^2}{4\varepsilon\|\partial v\|_{L^2}^2}}\,dx \le \|h\|_{L^\infty}e^{\varepsilon\|\partial v\|_{L^2}^2}K_{\beta_\varepsilon}.$$

Replacing this estimate in (4.1), we obtain

$$\mathcal{J}(u) \ge \frac{1}{2}\|\partial u\|_{L^2}^2 + c|\Gamma|[\log(c|\Gamma|) - \varepsilon\|\partial u\|_{L^2}^2 - \log(\|h\|_\infty K_{\beta_\varepsilon})]$$

and, in particular, for $\varepsilon_0 := \frac{1}{4c|\Gamma|}$,

$$\mathcal{J}(u) \ge \frac{1}{4}\|\partial u\|_{L^2}^2 + c|\Gamma|[\log(c|\Gamma|) - \log(\|h\|_{L^\infty}K_{\beta_{\varepsilon_0}})]. \tag{4.2}$$

Hence the proof that $\mathcal{J}$ is bounded in B from below is accomplished.

Let $\{u_n\}_{n\in\mathbb{N}}$ be a minimizing sequence for $\mathcal{J}$ in B; set $\bar{u}_n := \int_\Gamma u_n/|\Gamma|$ and $v_n := u_n - \bar{u}_n$; hence $\partial u_n \equiv \partial v_n$, and by estimate (4.2), ∂v_n is bounded in $L^2(\Gamma)$, uniformly in n. By Lemma 2.1 (ii), also v_n is uniformly bounded in $L^2(\Gamma)$, and therefore the functions v_n are uniformly bounded in $H^1(\Gamma)$. Moreover, by the definition of $\mathcal{J}$, we get that $\int_\Gamma u_n$ are uniformly bounded, and consequently also $\bar{u}_n$ are uniformly bounded. Being $u_n = v_n + \bar{u}_n$, also the functions u_n are uniformly bounded in $H^1(\Gamma)$. Possibly passing to a subsequence, there exists $u \in H^1(\Gamma)$ such that, as $n \to +\infty$, $u_n \rightharpoonup u$ in the weak topology of $H^1(\Gamma)$, $u_n \to u$ uniformly, $u \in B$ and $\mathcal{J}(u) = \min_B \mathcal{J}$.

We claim that u is a solution to (2.1). By standard Lagrangian theory, there exists $\lambda \in \mathbb{R}$ such that, for every $\phi \in H^1(\Gamma)$,

$$\begin{aligned} 0 &= \frac{d}{dt}\Big(\int_\Gamma \frac{\partial(u+t\phi)^2}{2}\,dx + c\int_\Gamma (u+t\phi)\,dx - \lambda\Big(c|\Gamma| - \int_\Gamma he^{u+t\phi}\,dx\Big)\Big)\Big|_{t=0} \\ &= \int_\Gamma \partial u\partial\phi\,dx + c\int_\Gamma \phi\,dx - \lambda\int_\Gamma he^u\phi\,dx. \end{aligned} \tag{4.3}$$

Choosing $\phi \equiv 1$, we get $c|\Gamma| = \lambda\int_\Gamma he^u$; since $u \in B$, we get $\lambda = 1$. In conclusion, relation (4.3) with $\lambda = 1$ is equivalent to the definition of weak solution to (2.1). □

5 The Kazdan–Warner equation with case $c < 0$

Theorem 5.1. *Assume $c < 0$.*

(i) *If* (2.1) *has a solution, then $\int_\Gamma h < 0$.*

(ii) *If $\int_\Gamma h < 0$, then there exists a constant $c(h) \in [-\infty, 0)$ such that* (2.1) *has a solution for any $c(h) < c < 0$ and no solution for $c < c(h)$.*

(iii) *For $\int_\Gamma h < 0$, let $c(h)$ be defined as in* (ii). *Then $c(h) = -\infty$ if and only if $h \le 0$ in Γ.*

We introduce the definition of upper and lower solution to (2.1).

Definition 5.1. A function $u \in C^2(\Gamma)$ is said to be a lower (respectively, an upper) solution of (2.1) if

$$\begin{cases} \partial^2 u - c + he^u \ge 0, & x \in e_j,\ j \in J, \\ \sum\limits_{j \in \mathrm{Inc}_i} \partial_j u(\mathrm{v}_i) \ge 0, & \mathrm{v}_i \in V, \end{cases} \quad \left(\text{resp.,} \quad \begin{cases} \partial^2 u - c + he^u \le 0, & x \in e_j,\ j \in J, \\ \sum\limits_{j \in \mathrm{Inc}_i} \partial_j u(\mathrm{v}_i) \le 0, & \mathrm{v}_i \in V. \end{cases} \right)$$

In order to prove Theorem 5.1, we need some preliminary results.

Lemma 5.1. *If there exist a lower solution u_- and an upper solution u_+ of* (2.1) *such that $u_- \le u_+$, then there exists a solution u of* (2.1) *such that $u_- \le u \le u_+$.*

Proof. Set $k_1(x) = \max\{1, -h(x)\}$ and $k(x) = k_1(x)e^{u_+(x)}$, and consider the sequence of functions $\{u_n\}_{n\in\mathbb{N}}$ defined inductively as $u_0 = u_+$ and $u_{n+1} \in C^2(\Gamma)$ the solution of

$$\begin{cases} \mathcal{L}u_{n+1} = f(x, u_n) - ku_n, & x \in e_j,\ j \in J, \\ \sum\limits_{j \in \mathrm{Inc}_i} \partial_j u_{n+1}(\mathrm{v}_i) = 0, & \mathrm{v}_i \in V, \end{cases} \tag{5.1}$$

where $\mathcal{L}u = \partial^2 u - ku$ and $f(x, u) = c - h(x)e^u$. We first observe that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is well defined. Indeed, since $k(x) \ge e^{-\|u_+\|_{L^\infty}} =: \lambda > 0$, the differential equation in (5.1) can be written as

$$-\partial^2 u_{n+1} + H(x, u_{n+1}) + \lambda u_{n+1} = 0$$

with $H(x, r) := (k(x) - \lambda)r + f(x, r) - k(x)u_n(x)$; hence the result in [7, Proposition 10] ensures the existence of a solution to (5.1). Moreover, by the linearity of (5.1), the maximum principle in [7, Proposition 12] guarantees the uniqueness of this solution. We claim that

$$u_- \le u_{n+1} \le u_n \le u_+ \quad \text{for any } n \in \mathbb{N}. \tag{5.2}$$

Since

$$\begin{cases} \mathcal{L}(u_1 - u_0) = f(x, u_0) - ku_0 - \partial^2 u_0 + ku_0 \ge 0, & x \in e_j,\ j \in J, \\ \sum\limits_{j \in \mathrm{Inc}_i} \partial_j (u_1 - u_0)(\mathrm{v}_i) \ge 0, & \mathrm{v}_i \in V, \end{cases}$$

the inequality $u_1 \le u_0 = u_+$ on Γ follows immediately by the maximum principle (see [7, Proposition 12]). Assuming inductively that $u_n \le u_{n-1}$, we have

$$\begin{aligned} \mathcal{L}(u_{n+1} - u_n) &= k(x)(u_{n-1} - u_n) + h(x)(e^{u_{n-1}} - e^{u_n}) \\ &\ge k_1(x)e^{u_+(x)}(u_{n-1} - u_n) - k_1(x)(e^{u_{n-1}} - e^{u_n}) \\ &\ge k_1(x)(e^{u_+(x)} - e^{\xi(x)})(u_{n-1} - u_n) \quad \text{for } x \in e_j,\ j \in J, \end{aligned}$$

where $\xi(x) \in [u_n(x), u_{n-1}(x)]$. By induction, we have $u_+ \ge u_{n-1}$, and recalling the condition at the vertices, we get

$$\begin{cases} \mathcal{L}(u_{n+1} - u_n) \ge 0, & x \in e_j,\ j \in J, \\ \sum\limits_{j \in \mathrm{Inc}_i} \partial_j (u_{n+1} - u_n)(\mathrm{v}_i) \ge 0, & \mathrm{v}_i \in V. \end{cases}$$

We conclude again by the maximum principle that $u_{n+1} \le u_n$ in Γ. We finally observe that, arguing as before, we have

$$\begin{cases} \mathcal{L}(u_- - u_{n+1}) \ge k(x)(u_n - u_-) + h(x)(e^{u_n} - e^{u_-}) \ge 0, & x \in e_j,\ j \in J, \\ \sum\limits_{j \in \mathrm{Inc}_i} \partial_j(u_- - u_{n+1})(\mathrm{v}_i) \ge 0, & \mathrm{v}_i \in V, \end{cases}$$

and therefore $u_- \le u_{n+1}$ on Γ for all n. Hence the claim (5.2) is proved.

By [7, Proposition 10], there exists a positive constant C (independent of n) such that $\|u_n\|_{H^1} \le C$ and, in particular, $\|u_n\|_{L^\infty} \le C$ for every $n \in \mathbb{N}$. Replacing the bounds (5.2) in the differential equation of (5.1), we obtain that $\|\partial^2 u_n\|_{L^\infty}$ are uniformly bounded; this property and again the bounds (5.2) ensure $\|u_n\|_{H^2} \le C$. The Ascoli–Arzela theorem yields that, up to passing to a subsequence, $\{u_n\}$ converges in $H^1(\Gamma)$ to a function $u \in H^1(\Gamma)$ which is a weak solution to (2.1) with $u_- \le u \le u_+$. Finally, by Remark 2.1, u is a classical solution to (2.1). □

In the next lemma, we show that (2.1) admits a lower solution u_- for any $c < 0$.

Lemma 5.2. *If $c < 0$, then there exists a lower solution u_- of* (2.1).

Proof. Set $u_- \equiv -A$ for some constant $A > 0$. Then the function u_- fulfils the Kirchhoff condition in (2.1) and also

$$\partial^2 u_-(x) - c + h(x)e^{u_-(x)} = -c + h(x)e^{-A} \ge 0, \quad x \in e_j,\ j \in J,$$

for A sufficiently large. Hence u_- is a lower solution to (2.1). □

Proof of Theorem 5.1. Assume that there exists a solution u of (2.1). Then, integrating (2.1) on Γ, we get

$$-\int_\Gamma h(x)\,dx = \int_\Gamma (\partial u(x))^2 e^{-u(x)}\,dx - c\int_\Gamma e^{-u(x)}\,dx > 0$$

and therefore (i).

We now assume that $\int_\Gamma h(x)\,dx < 0$. Recall that, by Lemma 5.1 and Lemma 5.2, problem (2.1) has a solution if and only if there exists an upper solution u_+. Moreover, it is easy to see that, if u_+ is an upper solution for a given $\bar{c} < 0$, then it is also an upper solution for any c such that $\bar{c} \le c < 0$. Hence it follows that there exists a constant $c(h)$ with $-\infty \le c(h) \le 0$ such that (2.1) admits a solution for $c > c(h)$ and no solution for $c < c(h)$.

We show that $c(h) < 0$. Let $m \in C^2(\Gamma)$ be a solution of

$$\begin{cases} \partial^2 m(x) = \int_\Gamma h(x)\,dx - h(x), & x \in e_j,\ j \in J, \\ \sum\limits_{j \in \mathrm{Inc}_i} \partial_j m(\mathrm{v}_i) = 0, & \mathrm{v}_i \in V \end{cases} \tag{5.3}$$

(existence of a weak solution to (5.3) is proved in [7, Proposition 13], while the regularity follows by Remark 2.1), and let a be a positive constant such that

$$\max_{x\in\Gamma} |e^{am(x)} - 1| \le \frac{-\int_\Gamma h(x)\,dx}{2\|h(x)\|_{L^\infty}}.$$

We define $b = \ln(a)$, $c = \frac{1}{2}a\int_\Gamma h(x)\,dx$ and $u_+(x) = am(x) + b$. Then $c < 0$ and

$$\partial^2 u_+(x) - c + h(x)e^{u_+(x)} = ah(x)(e^{am(x)} - 1) + \frac{a\int_\Gamma h(x)\,dx}{2} \le a\|h(x)\|_{L^\infty}|e^{am(x)} - 1| + \frac{a\int_\Gamma h(x)\,dx}{2} \le 0.$$

Moreover, by (5.3), u_+ is continuous and verifies the Kirchhoff condition because m enjoys the same properties. Hence u_+ is an upper solution, and therefore we conclude that

$$c(h) \le \frac{a}{2}\int_\Gamma h(x)\,dx < 0.$$

We finally prove (iii). Note that $\int h < 0$ ensures $h \not\equiv 0$.

We first show that, if $h \le 0$ in Γ, then (2.1) is solvable for any $c < 0$, and therefore $c(h) = -\infty$. Fixed $c < 0$, let m be a solution of (5.3), and choose two constants a, b such that $a \int_\Gamma h(x)\,dx < c$ and $e^{am(x)+b} - a > 0$ for $x \in \Gamma$. We show that the function $u_+(x) = am(x) + b$ is an upper solution of (2.1). Indeed, there holds

$$\partial^2 u_+(x) - c + h(x)e^{u_+(x)} = a \int_\Gamma h(x)\,dx - ah(x) - c + h(x)e^{am(x)+b} \le h(x)(e^{am(x)+b} - a) \le 0,$$

while the continuity and the Kirchhoff conditions for u_+ come again from those of m. Hence u_+ is an upper solution to (2.1), and therefore, for any $c < 0$, there exists a solution to (2.1).

Conversely, let us prove that $c(h) = -\infty$ implies $h \le 0$ in Γ. To this end, as in [13, Theorem 2.3], we argue by contradiction assuming that $\{h > 0\}$ is not empty. For any $c < 0$, let u be a solution to (2.1) (whose existence is ensured by $c(h) = -\infty$), and let $\phi_c \in C^2(\Gamma)$ be a solution to the problem

$$\begin{cases} \partial^2\phi_c + c\phi_c = h, & x \in e_j,\ j \in J, \\ \sum_{j\in\mathrm{Inc}_i} \partial_j\phi_c(\mathrm{v}_i) = 0, & \mathrm{v}_i \in V \end{cases} \tag{5.4}$$

(whose existence is ensured by [7, Proposition 10]). We claim

$$\phi_c(x) \ge e^{-u(x)} > 0 \quad \text{for all } x \in \Gamma.$$

In order to prove this relation, by the maximum principle [7, Proposition 12], it suffices to prove that e^{-u} is a lower solution to (5.4). Actually, there holds

$$\partial^2(e^{-u}) + ce^{-u} = e^{-u}[-\partial^2 u + |\partial u|^2 + c] = e^{-u}[he^u + |\partial u|^2] \ge h;$$

moreover, e^{-u} is continuous and satisfies the Kirchhoff condition because u does it. Hence our claim is proved.

Let us assume for the moment that there holds

$$\lim_{c\to-\infty} c\phi_c(x) = h \quad \text{for all } x \in \Gamma; \tag{5.5}$$

then this property would contradict $\phi_c \ge 0$ in $\{h > 0\}$, accomplishing the proof. Therefore, it remains to prove (5.5); to this end, we observe that the operator $A := -\partial^2$, coupled with Kirchhoff condition, is a maximal monotone operator (for the precise definition and main properties, we refer the reader to the monograph [6, p. 101]). Applying [6, Proposition VII.2 (c)] with λ equal to $\frac{1}{c}$, we obtain $\lim_{c\to-\infty} u_c = h$, where u_c solves $(I + \frac{A}{c})(u_c) = h$. By linearity, we get $u_c = c\phi_c$, accomplishing the proof of (5.5). □

5.1 The critical case $c = c(h)$

Proposition 5.1. *For $\int_\Gamma h < 0$ and $c(h) > -\infty$, problem (2.1) with $c = c(h)$ admits a solution.*

Proof. Note that Theorem 5.1 (iii) ensures that h changes sign (and obviously, $h \not\equiv 0$). Given a decreasing sequence $\{c_k\}_{k\in\mathbb{N}}$ with $c(h) < c_k < 0$ converging to $c(h)$ as $k \to +\infty$, we consider

$$\begin{cases} \partial^2 u = c_k - he^u, & x \in e_j,\ j \in J, \\ \sum_{j\in\mathrm{Inc}_i} \partial_j u(\mathrm{v}_i) = 0, & \mathrm{v}_i \in V. \end{cases} \tag{5.6}$$

The idea is to show that a sequence of solutions u_k of (5.6), appropriately chosen, converges for $k \to \infty$ to a solution of (2.1) with $c = c(h)$.

Lemma 5.3. *For each $k \in \mathbb{N}$, there exist a lower solution $\phi_k \equiv -A \in \mathbb{R}$ and an upper solution ψ_k to (5.6) with $\psi_k > \phi_k$.*

Proof. To show the existence of a lower solution, it suffices to argue as in Lemma 5.2 choosing A sufficiently large so that

$$-c_k + h(x)e^{-A} \ge -c_k - \|h\|_{L^\infty} e^{-A} =: \delta > 0. \tag{5.7}$$

For the upper solution, we choose ψ_k as a solution to (2.1) with c replaced by any $\tilde{c}_k \in (c(h), c_k)$ (whose existence is established in Theorem 5.1).

Finally, it remains to prove the inequality $\psi_k > -A$ (see also [21, Claim 3] for a similar argument). Denoting by $\tilde{x}$ a minimum point of ψ_k on Γ, we claim that $\psi_k(\tilde{x}) > -A$.

Assume first that $\tilde{x} \in e_j$ for some $j \in J$. The first equation in (2.1) yields

$$h(\tilde{x})e^{\psi_k(\tilde{x})} = \tilde{c}_k - \partial^2\psi_k(\tilde{x}) \le \tilde{c}_k < 0$$

and, in particular, $h(\tilde{x}) < 0$. On the other hand, the function $\phi_k \equiv -A$ satisfies $h(\tilde{x})e^{-A} > \tilde{c}_k$. The last three relations give $e^{\psi_k(\tilde{x})} - e^{-A} > 0$, which is equivalent to $\psi_k(\tilde{x}) > -A$.

Assume now $\tilde{x} = \mathrm{v}_i$ for some $i \in I$ and, for later contradiction, $\psi_k(\mathrm{v}_i) \le -A$. Observe that, for any $j \in \mathrm{Inc}_i$, the restriction of ψ_k to e_j attains its minimum at v_i, and consequently, $\partial_j\psi_k(\mathrm{v}_i) \ge 0$. Taking into account the Kirchhoff condition in (2.1), we deduce $\partial_j\psi_k(\mathrm{v}_i) = 0$ for all $j \in \mathrm{Inc}_i$. On the other hand, by (5.7) and the continuity of h, there exists $\eta > 0$ such that

$$\tilde{c}_k + \|h\|_{L^\infty}e^{-A+\eta} < -\frac{\delta}{2}. \tag{5.8}$$

Moreover, by the continuity of ψ_k and $\psi_k(\mathrm{v}_i) \le -A$, (5.8) ensures

$$\partial_j^2\psi_k(x) = \tilde{c}_k - h(x)e^{\psi_k(x)} \le \tilde{c}_k + \|h\|_{L^\infty}e^{-A+\eta} < -\frac{\delta}{2} < 0$$

for any $x \in e_j$ sufficiently close to v_i. In conclusion, near v_i, the function $\partial_j\psi_k$ is strictly decreasing with $\partial_j\psi_k(\mathrm{v}_i) = 0$, and therefore ψ_k is strictly decreasing. This fact contradicts that ψ_k attains its minimum at v_i. □

Lemma 5.4. *Fix $k \in \mathbb{N}$. The minimum of the problem*

$$\inf\{\mathcal{J}_k(u) : u \in H^1(\Gamma),\ -A \le u(x) \le \psi_k(x) \text{ for all } x \in \Gamma\}, \tag{5.9}$$

where

$$\mathcal{J}_k(u) := \frac{1}{2}\int_\Gamma |\partial u|^2\,dx + c_k\int_\Gamma u\,dx - \int_\Gamma he^u\,dx,$$

is attained by some function $\bar{u}$ with

$$-A < \bar{u} < \psi_k. \tag{5.10}$$

Moreover, $\bar{u}$ is a solution of (5.6).

Proof. Let $\{v_n\}_n$ be a minimizing sequence for $\mathcal{J}_k$. Then there holds

$$\mathcal{J}_k(v_n) + o(1) \le \mathcal{J}_k(-A) = c_k(-A)|\Gamma| - e^{-A}\int_\Gamma h \le C$$

for some constant C (independent of k). Moreover, we have

$$\begin{aligned} C + o(1) \ge \mathcal{J}_k(v_n) &= \frac{1}{2}\int_\Gamma |\partial v_n|^2\,dx + c_k\int_\Gamma v_n\,dx - \int_\Gamma he^{v_n}\,dx \\ &\ge \frac{1}{2}\int_\Gamma |\partial v_n|^2\,dx + c_k\int_\Gamma \psi_k\,dx - \|h\|_{L^\infty}\int_\Gamma e^{\psi_k}\,dx, \end{aligned} \tag{5.11}$$

where the inequality is due to the constraint $-A \le v_n \le \psi_k$. We deduce that $\|\partial v_n\|_{L^2}$ are uniformly bounded; on the other hand, also $\|v_n\|_{L^\infty}$ are uniformly bounded. Therefore, the sequence $\{v_n\}_n$ is uniformly bounded in $H^1(\Gamma)$. We infer that, possibly passing to a subsequence, there exists $\bar{u} \in H^1(\Gamma)$ with $-A \le \bar{u} \le \psi_k$ such that $v_n \to \bar{u}$ uniformly and $v_n \rightharpoonup \bar{u}$ weakly in H^1. By the lower semicontinuity of $\mathcal{J}_k$, we get $\mathcal{J}_k(\bar{u}) \le \liminf_n \mathcal{J}_k(v_n)$; hence $\bar{u}$ is a minimum point for (5.9). Let us assume for the moment that inequalities (5.10) hold true; then, by standard Lagrange multipliers method, we have $\frac{d}{dt}\mathcal{J}_k(\bar{u} + t\phi)|_{t=0} = 0$ for any $\phi \in H^1(\Gamma)$, from which we get (2.2). Arguing as in Remark 2.1, we get that $\bar{u}$ is a strong solution to (5.6). Therefore, it remains to prove the inequalities in (5.10); since the proofs are similar, we shall only provide the one for the first relation. As a first step, we claim that it cannot exist an interval $[x_1, x_2] \subset e_j$ (for some $j \in J$) such that $\bar{u} = -A$ on $[x_1, x_2]$. Actually, for later contradiction, assume that there exists such an interval $[x_1, x_2]$. Without any loss of generality, we can assume $x_1 = 0$. We introduce the function

$$\hat{u}(x) := \begin{cases} -A + \alpha \min\{x, x_2 - x\} & \text{if } x \in (0, x_2), \\ \bar{u}(x) & \text{otherwise}, \end{cases}$$

where $\alpha > 0$ is a constant so small that $\bar{u}$ is admissible for (5.9) and to be suitably chosen later on. We have

$$\mathcal{J}_k(\bar{u}) - \mathcal{J}_k(\hat{u}) \geq -\frac{\alpha^2 x_2}{2} - \frac{\alpha c_k x_2^2}{4} - \|h\|_{L^\infty} \int_0^{x_2} (e^{\hat{u}} - e^{\bar{u}})\, dx.$$

By the Lagrange theorem, for every $x \in (0, \frac{x_2}{2})$, there exists $\xi \in (0, x)$ such that

$$e^{\hat{u}(x)} - e^{\bar{u}(x)} = \alpha e^{-A+\alpha\xi} x \leq \alpha e^{-A+\alpha\frac{x_2}{2}} \frac{x_2}{2};$$

by symmetry, there holds also $e^{\hat{u}(x)} - e^{\bar{u}(x)} \leq \alpha e^{-A+\alpha\frac{x_2}{2}} \frac{x_2}{2}$ for every $x \in (\frac{x_2}{2}, x_2)$. Replacing these two inequalities in the previous one, we get

$$\mathcal{J}_k(\bar{u}) - \mathcal{J}_k(\hat{u}) \geq -\alpha^2 \frac{x_2}{2} + \alpha[-c_k - 2\|h\|_{L^\infty} e^{-A+\alpha\frac{x_2}{2}}] \frac{x_2^2}{4}.$$

For A sufficiently large and α sufficiently small, we get a contradiction. Finally, we claim that, if $\bar{u} > -A$ on some interval $(x_1, x_2) \subset e_j$ (for some $j \in J$), then $\bar{u} > -A$ on $\bar{e}_j$. Indeed, let us assume $-A < \bar{u} < \psi_k$ on some $(\bar{x}_1, x_2)$ and, by later contradiction, $\bar{u}(x_2) = -A$. Then $\phi := \bar{u} + A$ attains a minimum at x_2. Moreover, as before, by the Lagrange multipliers method, $\bar{u}$ solves (5.6) on $(\bar{x}_1, x_2)$. Then, for $x \to x_2^-$, by (5.7), we have

$$\partial^2 \phi = c_k - h e^{\bar{u}} = c_k - h e^{-A} + o(1) < 0,$$

which provides a contradiction. □

We can now conclude the proof of Proposition 5.1. Denote by u_k, $k \in \mathbb{N}$, a solution of (5.6) given by Lemma 5.4. Assume for the moment that the sequence $\{u_k\}_k$ is bounded in $H^1(\Gamma)$. Hence there exists $u \in H^1(\Gamma)$ such that, as $k \to +\infty$, up to a subsequence, $u_k \rightharpoonup u$ in the weak topology of $H^1(\Gamma)$ and $u_k \to u$ uniformly. Passing to the limit in the weak formulation of (5.6), we get that u is a weak, and therefore also a strong, solution to (2.1) with $c = c(h)$.

It remains to prove that $\{u_k\}_k$ is bounded in H^1. To this end, fix $0 < \delta < \max_\Gamma h$, an interval D inside some edge e_j such that $D \subset \{h(x) \geq \delta\}$ and a point $\bar{x} \in D$; by the same arguments as in [8, p. 743] (note that we can use [8, Lemma 2.1] because any solution of the equation in D is also a solution in a 2-dimensional domain), we get that the u_k's are uniformly bounded in D. Therefore, the functions $w_k(x) := u_k(x) - u_k(\bar{x})$ satisfy $w_k(\bar{x}) = 0$, and there exists $C_1 > 0$ such that $|u_k(\bar{x})| \leq C_1$ for any k. Arguing as in Lemma 2.1 (i), we get $\|w_k\|_{L^\infty} \leq |\Gamma|^{\frac{1}{2}} \|\partial w_k\|_{L^2} = |\Gamma|^{\frac{1}{2}} \|\partial u_k\|_{L^2}$, and we deduce

$$\|u_k\|_{L^\infty} \leq |u_k(\bar{x})| + \|w_k\|_{L^\infty} \leq C_1 + |\Gamma|^{\frac{1}{2}} \|\partial u_k\|_{L^2}. \tag{5.12}$$

On the other hand, choosing $\phi \equiv 1$ as test function in the weak formulation of (5.6), we get

$$\int_\Gamma h e^{u_k}\, dx = c_k |\Gamma|. \tag{5.13}$$

Since c_k are negative, relations (5.11) with $v_n = u_k$ and (5.13) entail

$$\begin{aligned} C \geq \frac{1}{2} \int_\Gamma |\partial u_k|^2\, dx + c_k \int_\Gamma u_k\, dx - \int_\Gamma h e^{u_k}\, dx &\geq \frac{\|\partial u_k\|_{L^2}^2}{2} + c_k \int_\Gamma |u_k|\, dx - c_k |\Gamma| \\ &\geq \frac{\|\partial u_k\|_{L^2}^2}{2} + c_k C_1 |\Gamma| + c_k |\Gamma|^{\frac{3}{2}} \|\partial u_k\|_{L^2} - c_k |\Gamma|, \end{aligned}$$

where the last inequality is due to (5.12). Hence ∂u_k are uniformly bounded in L^2; by (5.12), the u_k's are uniformly bounded in L^∞ and consequently also in H^1. □

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