Forum Math. 7 (1995), 247–277

Forum Mathematicum
© de Gruyter 1995

The hierarchy of uniserial modules over a valuation domain

Silvana Bazzoni[1], Laszlo Fuchs[2] and Luigi Salce[1]

(Communicated by Rüdiger Göbel)

Abstract. The class of uniserial modules (i.e. modules whose submodules form a chain under inclusion) is studied over a valuation domain R. The isomorphy classes of torsion uniserial R-modules form a monoid Unis R under the operation Tor. In this paper, certain submonoids of Unis R are investigated, which consist of nonfinitely annihilated uniserials; these include all the nonstandard uniserial modules. Some of the submonoids turn out to be Clifford semigroups (i.e. unions of groups). Several results give information about the structure of monoids and about their group constituents.

The non-finitely annihilated uniserials are classified into six classes; this classification is slightly different from the one for non-standard uniserials due to Bazzoni-Salce.

1991 Mathematics Subject Classification: 13C05; 13A18, 13C12, 13F30.

Introduction

Throughout, R will denote a valuation domain, i.e. a commutative domain with 1 in which the ideals form a chain under inclusion. Q $(\neq R)$ will stand for its field of quotients (mostly viewed as an R-module), and P for its maximal ideal.

In recent publications on modules over valuation domains, a most interesting class of modules has played an increasing role: the non-standard uniserials. The existence of this class has been in doubt for a while, but since their existence over suitable valuation domains has been established, their properties are thoroughly studied, and we are getting a better grasp on them. Their peculiar behavior is full of surprises.

An R-module U is said to be *uniserial* if its submodules form a chain under inclusion. U is *standard* if $U \cong J/I$ for some submodules $I \leq J$ of Q. Over suitable

[1] Lavoro eseguito con il contributo del M.U.R.S.T.
[2] Partially supported by NSF Grant.

valuation domains, there do exist non-standard uniserial modules; these are necessarily uncountably generated and torsion. For their existence see Shelah [S], Fuchs-Salce [FS], Franzen-Göbel [FG], Bazzoni-Salce [BS1], and more recently, Eklof [E], Osofsky [O1], [O2], and Eklof-Shelah [ES]. Recent advances in the structural study of non-standard uniserials have come about as a result of three papers [BS1]–[BS3] by Bazzoni-Salce where various features, in particular, uniserial extensions and mutual epimorphisms have been studied extensively.

In the present paper, we initiate a totally different approach. It is motivated by the observation that several aspects of non-standard uniserials over a valuation domain can be better understood by focusing on the collection of the isomorphy classes of all torsion uniserials rather than on the individual uniserial modules themselves, and by studying the interrelations between these classes.

A natural vehicle for dealing with these problems is to equip the set of isomorphy classes of uniserials over R with an algebraic structure. This can be done by exploiting the idea that the isomorphy classes of torsion uniserials over R form a commutative monoid Unis R under the operation "Tor_1^R"; cf. Fuchs [F]. However, this monoid turns out to be too large to be manageable. The standard uniserials are easily classified [FS, p. 142], so it it natural to concentrate on the non-standard ones. Since the isomorphy classes of the non-standard uniserials do not form a subsemigroup in Unis R, we have selected a class in between, and focus our attention on families of subsemigroups of Unis R which collectively contain the isomorphy classes of all the non-standard ones and which allow a more satisfactory and aesthetically more pleasing formulation of the results. These uniserials are singled out by the condition of not being annihilated by the annihilator of any of their elements (we shall call them non-finitely annihilated).

The majority of our results deal with the collection of isomorphy classes of non-finitely annihilated uniserials U in Unis R. Our goal of investigating this subset of Unis R is accomplished by breaking it into disjoint pieces: with the aid of the newly introduced concept of 'level' or the prime ideal $U_{\#}$ associated with U. The isomorphy classes of uniserials at a fixed level K form a Clifford semigroup Unis^K (i.e. a union of groups), and so do those with the same prime $U_{\#}$. Moreover, the isomorphy classes of uniserials whose elements have principal ideal annihilators form a subsemigroup Unis_R in Unis R which is likewise a Clifford semigroup. The abundance of Clifford semigroups in Unis R is a pleasant surprise. Their structures will be described in more details.

Another point of view furnishing more information about Unis R stems from a categorical approach: certain subsemigroups of Unis R can be regarded as skeletons of full subcategories of the module category Mod-R. Category equivalences will be established between certain subgroups and their cosets in Unis R at the same level.

The way our Clifford semigroups are built up from their group constituents depends solely on the value group of the valuation domain R. But the structures of these groups reflect completeness properties of R (in particular, the sizes of these subgroups depend primarily on the so-called Gamma invariants, introduced in [ES], measuring the deviation of certain quotients of R from almost maximality); many of these groups are trivial if there are non non-standard uniserials in the Clifford semigroup considered. Our main objective is to analyze the interrelation between the group constituents. Subsequent theorems will show that it suffices to find the structure of some of them.

Our results are convincing evidence that there is more order in the hierarchy of uniserials over a valuation domain than previously anticipated.

1. Preliminaries

For the following definitions see [FS] and [BS1].

Let U be a torsion uniserial R-module, and $0 \neq u \in U$. Setting $I = \operatorname{Ann} u = \{r \in R \mid ru = 0\}$ (called the *annihilator* ideal of u) and $J = H(u) = \{r^{-1} \mid u \in rU\}$ (called the *height* ideal of u), we say that U is *of type* $[J/I]$ (the isomorphism class of the standard uniserial module J/I), and write $t(U) = [J/I]$. The type $t(U)$ does not depend on the choice of $0 \neq u \in U$; indeed, if $0 \neq v \in U$ is such that $v = su$ $(s \in R)$, then $\operatorname{Ann} v = s^{-1}I$, $H(v) = s^{-1}J$, and $J/I \cong s^{-1}J/s^{-1}I$. Note that $su \neq 0$ exactly if $s \notin I$.

By definition of type, technically $t(U) = [J/I]$ implies $I < R$ and $J \geq R$. However, we will not always impose the condition $I < R$ on the representative J/I of the type of a torsion uniserial module U. Actually, we will often use the symbol $[J/R]$ rather than $[aJ/aR]$ to represent the type of a uniserial module whose elements have principal ideal annihilators.

The definition of „type“ for all uniserials is motivated by the fact that all uniserials of type $[J/I]$ can be obtained in a unified fashion as direct limits of the same collection of modules, using only different connecting maps. In fact, let $R < J_1 < \ldots < J_\nu < \ldots$ be a continuous well-ordered ascending chain of fractional ideals whose union is J. Then J/I is just the direct limit of the uniserial modules J_ν/I where the connecting maps $\eta_\nu : J_\nu/I \to J_{\nu+1}/I$ are the canonical (i.e. induced by the embedding of J_ν in $J_{\nu+1}$). Likewise, any non-standard uniserial of the same type is the direct limit of the same set of modules, only the maps $J_\nu/I \to J_{\nu+1}/I$ are modified: the canonical η_ν is preceded by an automorphism α_ν of J_ν/I; cf. [FS, p. 149].

With a uniserial module U (of type $[J/I]$ one associates the following ideals (see [FS]):

$$U^{\#} = \{r \in R \mid rU < U\}, \quad U_{\#} = \bigcup_{0 \neq u \in U} \operatorname{Ann} u.$$

The ideals $U^{\#} = J^{\#}$ and $U_{\#} = I^{\#}$ are always prime. It is easily seen that for an ideal I of R, $I^{\#}$ is the union of all proper ideals of R which are isomorphic to I; thus $I^{\#}$ is the maximal ideal P if I is principal, otherwise $I^{\#} = \bigcup_{r \in R \setminus I} r^{-1}I = (R : I)I$. An ideal I of R is called *archimedean* if $I^{\#} = P$.

Lemma 1.1. *For an ideal I of R, we have*

(i) $R_{I^{\#}}I = I$ *(i.e. I is an ideal over the localization of R at $I^{\#}$);*

(ii) $I : I = R_{I^{\#}}$;

(iii) *$R:(R:I)=I$ unless P is not principal and $I=Pr$ for some $r\in R$, in which case $R:(R:I)=Rr$;*

(iv) $(R:I)^{\#}=I^{\#}$;

(v) *$I^{\#}I=I$ if and only if I is not principal as an R_{I^*}-ideal;*

(vi) *$R_{I^*}: I=R:I$ provided that I is not principal as an R_{I^*}-ideal;*

(vii) *$I:I^{\#}=I$ provided that I is not isomorphic to $I^{\#}$.*

Proof. (i) is evident, since $sI=I$ for $s\in R\backslash I^{\#}$.

(ii) Because of (i) it suffices to verify the inclusion $I:I\leq R_{I^*}$. If $r\in R$ satisfies $r^{-1}I\leq I$, then $I\leq rI$ which amounts to $r\in R\backslash I^{\#}$, i.e. $r^{-1}\in R_{I^*}$.

(iii) The inclusion $R:(R:I)\geq I$ is obvious; it implies the equality $R:(R:(R:I))=R:I$. If $I<J$ are ideals and if there is an $r\in R$ such that $I\leq Rr<J$, then $r^{-1}\in R:I$, but $\notin R:J$. Hence in this case $R:I>R:J$. Such an r fails to exist only in the indicated case. Hence the desired equality follows.

(iv) If $r\notin I^{\#}$, then $rI=I$, so from $(R:I)\ r^{-1}I=(R:I)\ r^{-1}rI\leq R$ we obtain $(R:I)\ r^{-1}\leq R:I$ and $r\notin(R:I)^{\#}$. This, along with (iii) implies the converse.

(v) See [FS, I.4.8].

(vi) requires proof in one direction only. So assume $a\in R_{I^*}: I$, i.e. $aI\leq R_{I^*}$. The inclusion $aI\leq R$ is obvious whenever $a\in R$ or $a^{-1}\in R\backslash I$. In the remaining case $a^{-1}\in I$, $R_{I^*}a^{-1}\leq R_{I^*}I=I$, thus $I=R_{I^*}a^{-1}$ is a principal R_{I^*}-ideal – this case has been excluded. Hence $a\in R:I$.

(vii) The inclusion $\geq$ being trivial, assume that $rI^{\#}\leq I$ holds for some $r\in R$. Then $I^{\#}\leq r^{-1}I$. Since $I^{\#}$ is the union of proper ideals $\cong I$, this can only happen if $R\leq r^{-1}I$ or $I^{\#}=r^{-1}I$. Now $r\in I$ in the first case, and $I\cong I^{\#}$ in the second case. □

Lemma 1.2. *For all (fractional) ideals I and L, $(IL)^{\#}=I^{\#}\cap L^{\#}$ holds true.*

Proof. If L is principal, then the claim is immediate. Assume L is not principal. The inclusion $(IL)^{\#}\leq I^{\#}\cap L^{\#}$ being obvious, suppose $I^{\#}\leq L^{\#}$. Let $r\in R$ satisfy $rIL=IL$. Multiplying it by $R:L$ we get $rIL^{\#}=IL^{\#}$. If $IL^{\#}=$ I, then $rI=I$, so $r\notin I^{\#}$. Otherwise $IL^{\#}<I$, $I^{\#}=L^{\#}$, and so by (1.1)(v) I is principal over R_{I^*}. In this case $IL\cong L$, and the assertion follows. □

The *annihilator*

$$\operatorname{Ann} U=\{r\in R\,|\,rU=0\}=I:J=\{r\in R\,|\,rJ\leq I\}$$

of a uniserial module U is an ideal A of R. We distinguish between the cases when there is or there is not an element $u\in U$ with $\operatorname{Ann} u=A$. In the first alternative, we shall call *U finitely annihilated.* Note that this is the case if and only if $R_{I^*}\otimes_R U$ is finitely generated (i.e. principal) as an R_{I^*}-module. The symbol "n.f.a." will be used as an abbreviation for non-finitely annihilated. (In the faithful case, this distinction has been introduced by Shores-Lewis [SL].) All torsion-free uniserials are finitely annihilated (with annihilator 0). A non-standard uniserial is necessarily torsion and n.f.a. The property of being n.f.a. is common to all uniserials of the same type.

Observe that a proper submodule of a n.f.a. uniserial has a larger annihilator. However, it is not uncommon for proper quotients to have the same annihilator (cf. (1.6)) – this simple fact will turn out most relevant in our study.

Lemma 1.3. *If U is a n.f.a. uniserial module of type $[J/I]$ with annihilator A, then*

(i) $J^{\#} \leq I^{\#}$;

(ii) $A^{\#} = J^{\#}$;

(iii) *J is not a cyclic R_{I^*}-module.*

Proof. (i) (see [FS, VII]) Suppose $r \in J^{\#} \setminus I^{\#}$, i.e. $r \in R$ satisfies $rU < U$ and does not annihilate any nonzero $u \in U$. Let $a \in U \setminus rU$. Since U is n.f.a., there an $s \in R$ such that $sa = 0$ but $sU \neq 0$. Hence $srU \leq sRa = 0$, thus by the choice of r, $sU = 0$, a contradiction.

(ii) If $r \notin J^{\#}$, then from $(I:J)r^{-1}J = (I:J)J \leq I$ we obtain $r^{-1}(I:J) \leq I:J$, and so $r \notin (I:J)^{\#}$. On the other hand, if $r \in J^{\#}$, then $rJ < J$ implies $I:J = \operatorname{Ann} J/I < \operatorname{Ann} rJ/I = r^{-1} \operatorname{Ann} J/I = r^{-1}(I:J)$, and therefore $r \in (I:J)^{\#}$.

(iii) If $J = aR_{I^*}$, then $\operatorname{Ann}(a + I) = \operatorname{Ann} U$ would be a contradiction. □

The following is an easy and most useful necessary and sufficient condition for a uniserial to be n.f.a.

Lemma 1.4. *A uniserial module of type $[J/I]$ is n.f.a. if and only if $I^{\#}J = J$.*

Proof. First suppose J/I is n.f.a. If $I^{\#} > J^{\#}$, then for $s \in I^{\#} \setminus J^{\#}$ we have $sJ = J$, and thus $I^{\#}J = J$. If $I^{\#} = J^{\#}$, then again $J^{\#}J = J$; in fact, otherwise $J = aR_{J^*}$ for some $a \in Q$ (see (1.1)(v)), in which case $I^{\#} = J^{\#}$, a contradiction to (1.3) (iii).

Conversely, let $I^{\#}J = J$, and by way of contradiction assume J/I is finitely annihilated. Thus [FS, VII.2.2] implies $I^{\#} \leq J^{\#}$. If $I^{\#} < J^{\#}$ and $r \in J^{\#} \setminus I^{\#}$, then $I^{\#}J \leq rJ < J$, a contradiction. If $I^{\#} = J^{\#}$, then choose $x + I \in J/I$ with annihilator $I:J$. There are $y \in J$ and $r \in J^{\#}$ such that $x = ry$. For such a y we have Ann $(y + I) = r\operatorname{Ann}(x + I) = r(I:J)$ which is by (1.3)(ii) strictly smaller than $I:J$. This is impossible. □

The two *threshold* submodules of a uniserial U were introduced in [BS1]; they are defined as follows:

$$U_\circ = \bigcup_{r \notin J^*} U[r] \quad \text{and} \quad U^\circ = \bigcap_{r \in J^*} U[r],$$

where $U[r] = \{u \in U \mid ru = 0\}$. Manifestly, $U_\circ$ is nothing else than the kernel of the localization map $U \to R_{J^*} \otimes_R U$ at the prime $J^{\#}$, while U° is equal to $U[U^{\#}] = \{u \in U \mid ru = 0 \text{ for all } r \in U^{\#}\}$. We always have $U_\circ \leq U^\circ$. The upper threshold submodule U° will play a relevant role in the sequel. (In [BS1], they were denoted U^c, U_c.)

For the proof of the following lemma we refer to [BS1, (2.2)].

Lemma 1.5. *For a n.f.a. uniserial module U of type $[J/I]$ we have:*

(i) $U^\circ = U$ *if and only if* $J^\# = \mathrm{Ann}\, U$;

(ii) $U_\circ = 0$ *exactly if* $J^\# = I^\#$;

(iii) $J^\# < I^\#$ *implies* $U_\circ = U^\circ$. □

The relevance of threshold submodules is apparent from

Lemma 1.6. *For a submodule V of a n.f.a. uniserial module U of type $[J/I]$, the following hold:*

(i) $V > U^\circ$ *implies* $\mathrm{Ann}\, U/V > \mathrm{Ann}\, U$;

(ii) $V < U_\circ$ *implies* $\mathrm{Ann}\, U/V = \mathrm{Ann}\, U$;

(iii) $\mathrm{Ann}\, U/U^\circ = \mathrm{Ann}\, U$ *if and only if* $J^\# U = U$.

Proof. (i) $V > U^\circ$ means that $U[r] \leq V$ for some $r \in J^\#$. If $u \in U \backslash rU$, then $\mathrm{Ann}\, u$ annihilates $rU \cong U/U[r]$ and thus U/V, but it cannot annihilate U.

(ii) If $V < U_\circ$, then $V \leq U[r]$ for some $r \in R \backslash J^\#$. The existence of an epimorphism $U/V \to U/U[r] \cong rU = U$ shows that $\mathrm{Ann}\, U/V$ cannot be larger than $\mathrm{Ann}\, U$.

(iii) [BS2, (1.1)] Suppose that $J^\# U = U$, and let $rU \leq U^\circ$ for some $r \in R$. By definition, every element of U° is annihilated by $J^\#$, thus $rUJ^\# = 0$. Hence $rU = 0$ and $\mathrm{Ann}\, U/U^\circ = \mathrm{Ann}\, U$. On the other hand, if $J^\# U < U$, then choose a $u \in U \backslash J^\# U$. To show that $\mathrm{Ann}\, U/U^\circ > \mathrm{Ann}\, U$ it suffices to verify that if $su \in U^\circ$ for some $s \in R$, then $sU \leq U^\circ$, because then $\mathrm{Ann}\, U/U^\circ = \mathrm{Ann}\,(u + U^\circ) \geq \mathrm{Ann}\, u > \mathrm{Ann}\, U$. If $v \in Ru$, then trivially $sv \in U^\circ$. If $v \in U \backslash Ru$, then $u = rv$ for some $r \in R$ which is $\notin J^\#$. Since $J^\#$ is a prime, for each $t \in J^\#$ there is a $t' \in J^\#$ such that $t = rt'$. Now $su \in U^\circ$ implies $tsv = rt'sv = t'su = 0$, U° being annihilated by t'. Hence sv is annihilated by $J^\#$, and so $sv \in U^\circ$. □

Let us record here a simple result on the elements of U°.

Lemma 1.7. *If U is a n.f.a. uniserial module, then*

(i) $H(u)$ *is the same for each non-zero* u *in* U°;

(ii) *if* $U^\circ = 0$, *then for each* $u \neq 0$ *there exists a* $u' \neq 0$ *such that* $H(u) < H(u')$.

Proof. (i) Let $I = \mathrm{Ann}\ u$, $J = H(u)$ and $ru = u'$ $(r \in R \backslash I)$. Since $u \in U^\circ$ means that u is annihilated by every element of $J^\#$, it is clear that $r \notin J^\#$. Therefore, $H(u') = H(ru) = r^{-1}H(u) = r^{-1}J = J = H(u)$.

(ii) Assume that $I = \operatorname{Ann} u$ and $J = H(u)$. Now $u \neq 0$ implies $I < J^{\#} = I^{\#}$ (see (1.5)(ii)); choose $r \in J^{\#} \setminus I$ and let $u' = ru \neq 0$. Then $H(u') = H(ru) = r^{-1}H(u) = r^{-1}J > J = H(u)$. □

A uniserial module U is said to be *strongly n.f.a.* if all non-zero quotients of U have the same annihilator as U; it is *barely n.f.a.* if every proper quotient of U has a larger annihilator. U is called *equiannihilated* if $\operatorname{Ann} U/U^{\circ} = \operatorname{Ann} U$. For the proof of the following we refer to [BS1, (2.3)], the definition of barely n.f.a., and [BS2, (1.1)], respectively.

Lemma 1.8. *Let U be a n.f.a. uniserial module of type $[J/I]$. Then*

(i) *U is strongly n.f.a. if and only if $J^{\#} = \operatorname{Ann} U$;*

(ii) *U is barely n.f.a. if and only if $U^{\circ} = 0$.* □

In her thesis [So] Soileau has demonstrated that if U, V are uniserial modules, then so are $\operatorname{Tor}_1^R(U, V)$ and $U \otimes_R V$. Furthermore, for standard uniserials, $U = K/L$ and $V = J/I$, one has

$$\operatorname{Tor}_1^R(K/L, J/I) \cong (IK \cap JL)/IL, \quad K/L \otimes J/I \cong KJ/(LJ + KI).$$

For the proofs of this, see [FS, p. 67]. As Tor commutes with direct limits, it is easy to see that for all uniserial modules U and V we have:

Lemma 1.9. *$t(U) = [K/L]$ and $t(V) = [J/I]$ imply*

(i) $t(\operatorname{Tor}(U, V)) = [(IK \cap JL)/IL]$ and

(ii) $t(U \otimes V) = [KJ/(LJ + KI)]$. □

2. The level of a uniserial module

A most useful concept which we introduce here is the level of a uniserial module U. This is a fractional ideal associated with U. We shall see that uniserial modules of the same level share several relevant properties.

The *level* of a uniserial module U of type $[J/I]$ (with $I < R$) is defined as

$$\operatorname{Lev} U = K = \bigcup_{r \in R \setminus I} r^{-1} J.$$

We set $\operatorname{Lev} U = J$ if U is of type $[J/R]$. $\operatorname{Lev} U$ is well defined, as it is independent of the way the type of U is represented. Obviously, $J \leq K$ always, and $K = J$ if $J = Q$. (We will see in (2.6) infra that $K \cong J$ holds in most cases.) For example, let $r \in P$. Then $\operatorname{Lev}(J/rR) = r^{-1}PJ$, so $\operatorname{Lev}(J/rR) = r^{-1}J$ if J is not principal and $r^{-1}a^{-1}P$ if $J = a^{-1}P$ is a principal ideal. Thus always $\operatorname{Lev}(J/R) = PJ$.

The following result points out a property that distinguishes $\operatorname{Lev} U$ from the other fractional ideals.

Lemma 2.1. *Let K be the level of the n.f.a. uniserial module U. Then for a fractional ideal $L \geq R$ we have*

$\operatorname{Tor}(L/R, U) \cong U$ *if and only if* $L \geq K$.

Proof. Let U be of type $[J/I]$. The exact sequence $0 \to R \to L \to L/R \to 0$ induces the exact sequence

$$0 \to \operatorname{Tor}(L/R, U) \to R \otimes U \cong U \xrightarrow{\alpha} L \otimes U \to L/R \otimes U \to 0.$$

First let $L \geq K$. To show that α is the zero map, choose $1 \otimes u$ ($u \in U$) where w.l.o.g. $\operatorname{Ann} u > I$ can be assumed. Since $\operatorname{Ann} u > \operatorname{Ann} U$, there is an $s \in R \backslash I$ such $su = 0$ but $sU \neq 0$. Then $s^{-1} \in s^{-1}J \leq K \leq L$ implies that $1 \otimes u \in R \otimes U$ maps upon $1 \otimes u = s^{-1} \otimes su = 0$.

Next let $L < K$. Then $LI < J$, for otherwise $J \leq LI$ implies that for any $r \notin I$, $J \leq LI \leq Lr$. Hence $r^{-1}J \leq L$ and $K \leq L$ would follow. $\operatorname{Tor}(L/R, U)$ is of type $[IL/I]$ by (1.9), and from $LI < J$ we infer that it has a larger annihilator than U. □

The uniserial module U is said to be *at the same level as* or *of higher (lower) level than the uniserial module* V according as $\operatorname{Lev} U = \operatorname{Lev} V$ or $\operatorname{Lev} U > \operatorname{Lev} V$ ($\operatorname{Lev} U < \operatorname{Lev} V$). The level cannot increase by passing to a submodule or a quotient. If U is a n.f.a. uniserial, then $\operatorname{Lev} V < \operatorname{Lev} U$ for all proper submodules V of U, but for proper quotients equality may hold.

Example 2.2. Let I, J be submodules of Q such that $I < R < J$ and $I^{\#}J = J$. Then J/I and $J/I^{\#}I$ are both n.f.a. with the same annihilators and the same levels. They are different whenever I is a principal $R_{I^{\#}}$-ideal. In fact, by (1.4), J/I is n.f.a. Now (1.2) implies that $(I^{\#}I)^{\#} = I^{\#}$, thus $J/I^{\#}I$ is likewise n.f.a. Evidently, $\operatorname{Ann} J/I \geq \operatorname{Ann} J/I^{\#}I$, while the converse inclusion follows from the fact that $rJ \leq I$ implies $rJ \leq I^{\#}I$. (2.3)(ii) will show that they have the same level.

From the definition it is clear that uniserial modules which are epimorphic images of each other are of the same level.

In the balance of this section, we shall deal exclusively with n.f.a. uniserial modules. (Levels of finitely annihilated uniserials – which are not needed here – behave differently.) A few useful elementary properties of the level of a n.f.a. uniserial are collected in the next two technical lemmas. We shall use the notation

$$I^{-1} = \langle r^{-1} \mid r \in R \backslash I \rangle.$$

Lemma 2.3. *Let U be a n.f.a. uniserial module of type $[J/I]$ with $I < R < J$ and* $\operatorname{Ann} U = A$. *Then the following hold for* $K = \operatorname{Lev} U$:

(i) $K = \bigcup_{0 \neq u \in U} H(u)$;

(ii) $K = (R : I) J = I^{-1}J$;

(iii) $KI = J$;

(iv) $K^{\#} = J^{\#}$;

(v) $I^{\#}K = K$.

Proof. (i) follows at once from the definition by observing that if $H(u) = J$, then $H(ru) = r^{-1}J$ for $ru \neq 0$, i.e. for $r \notin I$.

(ii) If I is not principal, then $R:I = I^{-1}$, and so $\bigcup_{r\in R\setminus I} r^{-1}J = (R:I)J$. If I is a principal ideal, say $I = Ra$ $(a \in R)$, then $K = \bigcup_{p\in P} a^{-1}pJ = a^{-1}J = (R:Ra)J = (R:I)J$; here we have used $J \geq PJ \geq I^{\#}J = J$ (see (1.4)).

(iii) Evidently, $KI = \bigcup_{r\in R\setminus I} r^{-1}JI = \bigcup_{r\in R\setminus I} r^{-1}IJ$ is equal to $I^{\#}J$. Again, (1.4) implies that this product is J.

(iv) If $t \notin K^{\#}$, then $tK = K$, thus by (iii) we obtain $tJ = tKI = KI = J$. Hence $K^{\#} \geq J^{\#}$. Analogously, $K^{\#} \leq J^{\#}$ follows from (ii).

(v) If $I^{\#} = J^{\#}$, then by (1.4), we have $J^{\#}J = J$ which implies $I^{\#}K = J^{\#}K = J^{\#}(R:I)J = (R:I)J = K$. If $I^{\#} > J^{\#}$, then (iv) implies $I^{\#}K = K$. □

Remark. If U is as in (2.3), its level remains K, even if U is viewed as an R_{I^*}-module. All the statements of (2.3) hold true whenever $R:I$ is replaced by $R_{I^*}:I$. This is essentially due to the inclusion relation $J^{\#} \leq I^{\#}$.

Lemma 2.4. *Under the hypotheses of* (2.3), *we also have*

(vi) $K > R_{I^*}$, *and*

(vii) Ann $(K/R_{I^*}) = \text{Ann } U = \text{Ann }(K/I^{\#}) = \text{Ann }(K/R)$.

Proof. (vi) Since $K^{\#} = J^{\#} \leq I^{\#}$, K is an R_{I^*}-module. $1 \in K$ implies $K \geq R_{I^*}$. Proper inclusion will follow from (vii).

(vii) If $I = Ra$ is a principal ideal, then $K = a^{-1}J$ (see the proof of (2.3)(ii)). Thus $K/R \cong J/I$, which implies $\text{Ann } U = \text{Ann } K/R = \text{Ann } K/P$. Suppose now that I is non-principal. From $I^{\#} < R \leq R_{I^*}$ we conclude that $\text{Ann } K/I^{\#} \leq \text{Ann } K/R \leq \text{Ann } K/R_{I^*}$. Now if $a \in \text{Ann } U$, i.e. $aJ \leq I$, then $aK = a(R:I)J \leq (R:I)I = I^{\#}$ implies the inclusion $\text{Ann } U \leq \text{Ann } K/I^{\#}$. On the other hand, if $a \in R$ satisfies $aK \leq R_{I^*}$, then $aJ = aKI \leq IR_{I^*} = I$, i.e. $a \in \text{Ann } U$. Consequently, the sets of annihilators of K/R, K/R_{I^*}, U, and $K/I^{\#}$ must all be equal. In order to show that K/R_{I^*} and $K/I^{\#}$ are n.f.a., by (1.4) it is enough to note that $I^{\#}K = K$ because of (2.3)(v). □

Observe that, using the notations above, $I:J = R:K$ follows from (vii).

A natural companion to the preceeding lemma is the following:

Lemma 2.5. *Let I be a proper ideal of R and K a submodule of Q containing R_{I^*} such that $I^{\#}K = K$. Then for the fractional ideal $J = KI$, the uniserial module J/I is n.f.a. and* $\text{Lev}(J/I) = K$.

Proof. Evidently, $I^{\#}K = K$ implies $I^{\#}J = I^{\#}KI = KI = J$, i.e. J/I is n.f.a. From (2.3)(ii) we infer $\text{Lev}(J/I) = \text{Lev}(KI/I) = (R:I)KI = K$, since $(R:I)I$ equals R or $I^{\#}$, according as I is principal or not, and $I^{\#}K = K$. □

If P is not a principal ideal, then choosing $I = P$, we see that every non-principal fractional ideal $K > R$ is the level of a n.f.a. uniserial.

Some additional information on n.f.a. uniserials is as follows.

Lemma 2.6. *Let J/I be a n.f.a. uniserial module and L a fractional ideal.*

(i) *If $I^* \leq L^*$, then $J/I \otimes L \cong JL/IL$ is likewise n.f.a., and has the same annihilator and the same level.*

(ii) *If* $\operatorname{Ann} JL/IL = \operatorname{Ann} J/I$, *then JL/IL is again n.f.a.*

Proof. (i) Since $J = KI$ by (2.3)(iii) where $K = \operatorname{Lev} J/I$, (2.4) and (2.5) imply that $JL/IL = KIL/IL$ must also be n.f.a. with the same annihilator and the same level K as J/I.

(ii) If there existed an $xy + IL$ with $x \in J$, $y \in L$ whose annihilator was equal to $\operatorname{Ann} J/I$, then $\operatorname{Ann}(x + I) \leq \operatorname{Ann}(xy + IL) = \operatorname{Ann}(Jl/IL) = \operatorname{Ann} J/I$ would contradict the hypothesis of J/I being n.f.a. □

We can now easily describe the levels of n.f.a. uniserial modules.

Theorem 2.7. *Let U be a n.f.a. uniserial module of type $[J/I]$ with $I < R < J$, and set $K = \operatorname{Lev} U$. Then $K \cong J$ except when $U^\circ = 0$ and I is not principal as an R_{I^*}-ideal.*

More precisely, $K = a^{-1}J$ for some $a \notin I$ if $U^\circ \neq 0$; $K = a^{-1}J$ for some $a \in I$ if $U^\circ = 0$ and $I \cong R_{I^*}$. In the exceptional case, $K \cong J^*$ or $K \cong J : I$ according as $J \cong I$ or not.

Proof. First assume $U^\circ \neq 0$. By (1.7)(i) $K = H(u)$ for each nonzero $u \in U^\circ$. From (2.3)(i) we infer that $K = a^{-1}J$ for some $a \notin I$, and thus $K \cong J$.

Next let $U^\circ = 0$ and $I = aR_{I^*}$ for some $a \in I$. Clearly, $r \notin I$ is equivalent to $r = at^{-1}$ for some $t \in I^*$. Therefore, we have $K = \bigcup_{r \in R \setminus I} r^{-1}J = a^{-1}(\bigcup_{t \in I^*} tJ)$ $= a^{-1}I^*J = a^{-1}J$ (see (1.4)).

It remains to consider the case in which $U^\circ = 0$ and $I \ncong R_{I^*}$. Suppose by way of contradiction that $K \cong J$, i.e. $K = a^{-1}J$ for some $a \in Q$. In view of (1.7)(ii), a must then belong to I. Note that by (2.3)(iii) $a^{-1}JI = KI = J$. Since I is not principal as an R_{I^*}-module, $a^{-1}I > R_{I^*}$. If $a^{-1}x \notin R_{I^*}$ for an $x \in I$, then $ax^{-1} \in I^* = J^*$, thus $a^{-1}xJ > J$ (recall that $U_\circ = 0$ amounts to $I^* = J^*$, see (1.5)(ii)). This implies $a^{-1}JI > J$, a contradiction.

In case $I = aJ$ for some $a \in I$, by (2.3)(ii) we have $K = JI^{-1} = a^{-1}II^{-1} = a^{-1}I^*$ $= a^{-1}J^*$. Conversely, if $K = a^{-1}J^*$ $(a \in I)$, then $J = KI = a^{-1}J^*I = a^{-1}I$ in view of (1.2)(v), since $I \ncong R_{I^*}$.

Let $I \ncong J$. The inclusion $K \leq J : I = \bigcap_{r \in I} r^{-1}J$ is obvious. Proper inclusion would imply the existence of an $a \in Q \setminus K$ such that $J = KI \leq aI \leq J$, contrary to $I \ncong J$. Hence $K = J : I$. □

The last result provides a full description of the levels. It shows that the only case in which the level of a n.f.a. uniserial module U of type $[J/I]$ fails to be isomorphic to J is when $U \in \mathcal{V}_6$. (For the definition of the class $\mathcal{V}_6$ see Section 8).

Next a necessary and sufficient condition is given in order that two uniserials be of the same level.

Proposition 2.8. *Let U_i ($i = 1, 2$) be n.f.a. uniserial modules, and let $t(U_i) = [J_i/I_i]$ where $I_1^\# \le I_2^\#$. Then the following are equivalent:*

1) $\operatorname{Lev} U_1 = \operatorname{Lev} U_2$;

2) $J_1 I_2 = J_2 I_1$, $J_1^\# = J_2^\#$, *and if* $J_2^\# = I_1^\# < I_2^\#$, *then* $J_2 J_2^\# = J_2$.

Proof. 1) ⇒ 2) Write $K_i = \operatorname{Lev} U_i$ ($i = 1, 2$). By (2.3)(iii), we have $J_i = K_i I_i$, so from $K_1 = K_2$ we deduce that $J_1 I_2 = J_2 I_1$. Furthermore, $J_1^\# = K_1^\# = K_2^\# = J_2^\#$ (see (2.3)(iv)). Let now $J_2^\# = I_1^\# < I_2^\#$; then, by (2.3)(v), $K_2 J_2^\# = K_2$. But by (1.5)(ii)–(iii) $J_2^\# < I_2^\#$ implies $U_2^\circ > 0$ in which case $K_2 \cong J_2$ holds owing to (2.6); hence also $J_2 J_2^\# = J_2$, and we are done.

2) ⇒ 1) From the equality $J_1 I_2 = J_2 I_1$ we obtain

$$(*) \qquad J_1 I_2 (R : I_1)(R : I_2) = J_2 I_1 (R : I_1)(R : I_2).$$

The left side of $(*)$ becomes either K_1 or $K_1 I_2^\#$ (according as I_2 is principal or not) where, by our assumption $I_1^\# \le I_2^\#$ and by (2.3)(v), $K_1 I_2^\#$ is equal to K_1. The right side of $(*)$ becomes either K_2 or $K_2 I_1^\#$ (according as I_1 is principal or not). The last module equals K_2 whenever $I_2^\# = I_1^\#$ or $J_2^\# < I_1^\#$. Hence only the case remains to be considered in which $J_2^\# = I_1^\# < I_2^\#$. But by virtue of (2.6) $J_2^\# < I_2^\#$ (which is equivalent to $U_2^\circ > 0$) implies that $K_2 \cong J_2$, thus $K_2 K_2^\# = K_2$ by hypothesis. From $K_2^\# = J_2^\# \le I_1^\#$ we conclude that the right side $K_2 I_1^\#$ is equal to K_2. □

Concerning the level of the torsion product of two uniserials, we prove the following result and derive a most relevant corollary.

Proposition 2.9. *Let U and V be n.f.a. uniserial modules such that* $\operatorname{Lev} U \le \operatorname{Lev} V$. *If $U_\# \le V_\#$, then $T =$* $\operatorname{Tor}(U, V)$ *is n.f.a. and satisfies*

$$\operatorname{Ann} T = \operatorname{Ann} U, \quad T_\# = U_\# \quad \text{and} \quad \operatorname{Lev} T = \operatorname{Lev} U.$$

Proof. If we write $\operatorname{Lev} U = K$ and $\operatorname{Lev} V = K'$, then, by (2.3), we have $t(U) = [KI/I]$, $t(V) = [K'L/L]$ for some ideals I, L in R. In view of (1.9), $\operatorname{Tor}(U, V)$ is of type $[KIL/IL]$, where $(IL)^\# = I^\#$ because of (1.2). Consequently, $T_\# = U_\#$. That T is n.f.a. follows from (1.4). Furthermore, by (2.6)(i) $I^\# \le L^\#$ implies Ann $(KIL/IL) = \operatorname{Ann} KI/I$, and thus $\operatorname{Lev} T = K$. □

Corollary 2.10. *The torsion product of two n.f.a. uniserials of the same level is always a n.f.a. uniserial of the same level.* □

However, it can very well happen that the torsion product of two n.f.a. uniserials of different levels is finitely annihilated. For instance, let the maximal ideal P be non-principal and L ($< P$) a nonzero idempotent prime. Set $U = a^{-1}P/R$,

$V = a^{-1}L/aR_L$ for some $a \in L$; then both U and V are n.f.a. In this case, $\mathrm{Tor}(U, V)$ is of type $[(PR_L \cap a^{-1}L)/aR_L] = [R_L/aR_L]$, thus it is annihilated by $\mathrm{Ann}(1 + R_L)$.

Let us point out that if U, V are n.f.a. uniserials and $T = \mathrm{Tor}(U, V)$ satisfies $\mathrm{Ann}\, T = \max(\mathrm{Ann}\, U, \mathrm{Ann}\, V)$, then T is again n.f.a. In fact, if $t(U) = [J/I]$ and $t(V) = [K/L]$, then $t(T) = [(IK \cap JL)/IL]$. Hence $\mathrm{Ann}\, T = IL : (IK \cap JL) = (IL : IK) \cup (IL : JL)$, and therefore either $\mathrm{Ann}\,(IK/IL) = \mathrm{Ann}\, K/L$ or $\mathrm{Ann}\,(JL/IL) = \mathrm{Ann}\, J/I$. The claim now follows from (2.6)(ii).

3. The semigroup Unis R

It is immediately seen that the isomorphy classes of torsion uniserial R-modules form a commutative semigroup under the operation "Tor_1^R"; in fact, the commutativity of Tor is obvious from R being commutative, while associativity is a consequence of the associative behavior of Tor over semi-hereditary rings (cf. Cartan-Eilenberg [CE]). We shall denote this semigroup by **Unis R**. Since $\mathrm{Tor}_1^R(Q/T, T) \cong T$ for all torsion modules T, the isomorphy class of Q/R is the neutral element of Unis R, hence Unis R is a monoid.

Though Unis R does not seem to carry much interessing structure, one can single out in Unis R tractable pieces of interest.

The example after (2.10) shows that the isomorphy classes of n.f.a. uniserials do not form a subsemigroup in Unis R; therefore we focus on certain subsets of n.f.a. uniserials. The isomorphy classes represented by the n.f.a. uniserials U of type $[K/R]$ (with K running over the non-principal submodules of Q containing R) form a submonoid Unis_R of Unis R; this is a simple consequence of (1.9). Unis_R consists of those n.f.a. uniserials whose elements have principal ideal annihilators. The n.f.a. uniserials of type $[K/R]$ for a fixed K form a subsemigroup in Unis_R; we will show that this is actually an abelian group (which we shall denote by $\mathrm{Gp}[K/R]$; see (3.1)).

Recall that by a *Clifford semigroup* C is meant a semigroup which is the union of pairwise disjoint groups G_σ ($\sigma \in S$) indexed by a meet-semilattice S such that

(i) for every pair $\sigma, \tau \in S$ with $\sigma \geq \tau$ there is a 'bounding' homomorphism $f_{\sigma,\tau}$: $G_\sigma \to G_\tau$ where $f_{\sigma,\sigma}$ is the identity map and $f_{\sigma,\tau} f_{\varrho,\sigma} = f_{\varrho,\tau}$ holds for all $\varrho \geq \sigma \geq \tau$;

(ii) multiplication is defined via $cd = f_{\varrho,\tau} c f_{\sigma,\tau} d$ for $c \in G_\varrho$, $d \in G_\sigma$ where τ is the meet of ϱ and σ in S and the last product is computed in G_τ. (See [CP, p. 128].)

Theorem 3.1. *Under the operation* Tor_1^R, *the monoid* Unis_R *is a commutative Clifford semigroup with totally ordered index set. The isomorphy classes of uniserials in* Unis_R *of a fixed type* $[K/R]$ *form an abelian group* **Gp[K/R]** *whose neutral element is the isomorphy class of the standard uniserial* K/R. *The bonding homomorphisms are all trivial.*

Proof. That the uniserials of the fixed type $[K/R]$ in Unis_R form an abelian group under "Tor" can be proved in the same way as it was done for divisible uniserials in Fuchs-Shelah [FSh]. In fact, it is straightforward to show that if U is the direct limit of the modules J_ν/R using the maps $\eta_\nu\alpha_\nu$: $J_\nu/R \to J_{\nu+1}/R$ where α_ν is an automorphism of J_ν/R (see Section 1 above), then the inverse of the isomorphy class

$[U]$ of U in Unis_R can be represented by the direct limit of the same set of modules replacing the α_v by their inverses.

Let $[U], [V] \in \text{Unis}_R$ be of types $[K/R]$ and $[K'/R]$, respectively, where $K > K'$. Then there exists an $r \in R$ such that $rV = 0$, but $rU \neq 0$. Choose $u \in U$ with $\text{Ann}\, u = Rr$. The exact sequence $0 \to Rr \to R \to R/Rr \to 0$ induces the exact sequence.

$$0 \to \text{Tor}_1^R(Ru, V) \cong \text{Tor}_1^R(R/Rr, V) \to Rr \otimes V \cong V \to R \otimes V.$$

The last map is trivial, since it sends $r \otimes v$ to $1 \otimes rv = 0$ $(r \in R,\, v \in V)$. We infer that $\text{Tor}(Ru, V) \cong V$. As the canonical map $\text{Tor}(Ru, V) \to \text{Tor}(U, V)$ is an isomorphism, we obtain $\text{Tor}(U, V) \cong V$. This completes the proof. □

The fundamental subsets are the subsemigroups Unis_L^K. For a submodule K of Q containing R and for $L \in \text{Spec}\, R$, $\mathbf{Unis}_L^K$ will denote the subset of Unis R consisting of the isomorphy classes of those uniserials U which satisfy:

(i) U is n.f.a.;

(ii) $\text{Lev}\, U = K$; and

(iii) $U_\# = L$.

In view of (1.4), Unis_L^K is empty unless $LK = K$; thus we shall always assume that $L \geq K^\#$ and consider $\text{Unis}_{K^\#}^K$ only if $K^\# K = K$. By (2.5), we always have $K > R_L$. The uniserial modules U in Unis_L^K are of type $[KI/I]$ where the ideal I satisfies $I^\# = L$, and therefore all the uniserials in Unis_L^K are in a natural way R_L-modules.

In general, the semigroup Unis_L^K contains several groups. The existence (but not the structure) of these groups is determined by the value group of R. One of these groups is $\mathbf{Gp[K/R_L]}$ whose elements are the n.f.a. uniserials of type $[K/R_L]$. That $\text{Gp}[K/R_L]$ is in fact a group follows at once from the application of (3.1) to R_L-modules.

In addition, we define $\mathbf{Unis[KI/I]}$ (with $I^\# = L$) to be the subset of Unis_L^K which consists of the isomorphy classes of modules of type $[KI/I]$. In case $I \cong R_L$, $\text{Unis}[KI/I]$ becomes $\text{Gp}[K/R_L]$.

From these definitions and from (2.10) the following result is readily derived.

Proposition 3.2. *Let L be a prime ideal of R and K a submodule of Q such that $K > R_L$ and $LK = K$.*

(i) *Unis_L^K is a monoid with $[K/R_L]$ as neutral element.*

(ii) *The isomorphy classes of uniserials of type $[K/R_L]$ form a subgroup* $\mathbf{Gp[K/R_L]}$ *of* Unis_L^K.

(iii) *Unis_L^K is the disjoint union of $\text{Gp}[K/R_L]$ and of subsets $\text{Unis}[KI/I]$ with I ranging over the isomorphy classes of ideals of R such that $I^\# = L$, $K > R_{I^\bullet}$ and $I \not\cong R_L$:*

$$\text{Unis}_L^K = \text{Gp}[K/R_L] \cup \bigcup_{I^\# = L} \text{Unis}[KI/I]. \quad □$$

We illustrate this monoid in a particular situation.

Example 3.3. Unis_L^Q is the monoid consisting of the isomorphy classes of divisible torsion uniserials where the annihilators I of elements satisfy $I^\# = L$. In particular, the monoid Unis_P^Q consists of the isomorphy classes of divisible uniserials with archimedean annihilator ideals; it contains $\mathrm{Gp}\,[Q/R]$ as a subgroup.

4. Category equivalance between Gp [K/R$_{I^*}$] and Unis [KI/I]

The monoid Unis_L^K (introduced in the preceding section) proves to be a very nicely structured object: it can be thought of not only as a monoid but also as the skeleton of a full subcategory of Mod-R. More importantly, in view of (3.2), it breaks down into disjoint pieces which turn out to be – as we shall see – equivalent as categories.

Adopting this point of view, in the decomposition (3.2)(iii) of Unis_L^K both $\mathrm{Gp}\,[K/R_L]$ and the $\mathrm{Unis}\,[KI/I]$ will be considered as skeletons of full subcategories of Mod-R, or, equivalently, of Mod-R_L. The primary goal of this section is to establish a category equivalence between these skeleton categories. Furthermore, we want to show that $\mathrm{Gp}\,[K/R_L]$ acts transitively and faithfully on each $\mathrm{Unis}\,[KI/I]$.

To start with, we prove three lemmas encompassing results which play indispensible roles in our functorial approach.

Lemma 4.1. *Let I be an ideal of R, KI/I a n.f.a. uniserial module of level K, and U a uniserial module of type $[K/R_{I^*}]$. There is a natural isomorphism*

$$\mathrm{Tor}_1^R(KI/I, U) \cong I \otimes_R U;$$

these uniserial modules are of type $[KI/I]$.

Proof. The exact sequence $0 \to I \to J \to J/I \to 0$ induces the exact sequence

$$0 \to \mathrm{Tor}(J/I, U) \to I \otimes U \to J \otimes U \to J/I \otimes U \to 0.$$

If $J = KI$, then the map between the first two tensor products is the zero map, since by (2.4)(vi) $\mathrm{Ann}(K/R_{I^*}) = I : J$. Hence the stated isomorphism follows. The claim on the type is a consequence of the second formula in (1.9), we just have to observe that $IK \cap R_{I^*}J = J$ and $R_{I^*}I = I$. $\square$

Lemma 4.2. *Let V be a n.f.a. uniserial module of type $[KI/I]$, where I is not principal as an R_{I^*}-ideal. There is a natural isomorphism*

$$I^{-1} \otimes V \cong \mathrm{Tor}_1^R(KI^{-1}/I^{-1}, V).$$

These uniserial modules are of type $[K/I^\#]$.

Proof. If I is not principal as an R_{I^*}-ideal, then $I^{-1} = R_{I^*} : I$ (see (1.1)(vi)). The claim on the type follows from (1.9) and properties of K listed in (2.3). The exact sequence

$$0 \to I^{-1} \to KI^{-1} \to KI^{-1}/I^{-1} \to 0$$

induces the long exact sequence

$$0 \to \mathrm{Tor}^1(V, KI^{-1}/I^{-1}) \to V \otimes I^{-1} \to V \otimes KI^{-1} \to V \otimes (KI^{-1}/I^{-1}) \to 0.$$

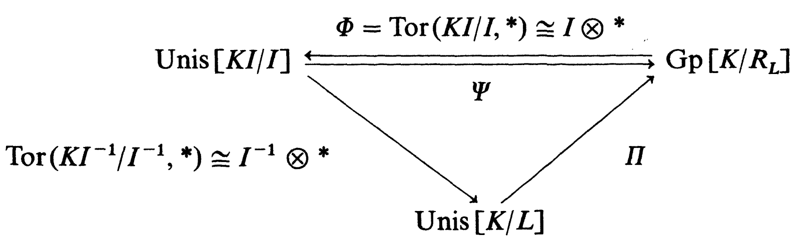
To see that the map between the first two tensor products is the zero map, let $v \otimes a \in V \otimes I^{-1}$ with $v \in V$ and $a \in I^{-1}$. Choose $r \in KI/R$ such that $\mathrm{Ann}\, v \geq r^{-1}I$. Clearly, $ra \in KIa \leq KR = K = KI^{\#} = KI^{-1}I$, thus there exist $x \in KI^{-1}$ and $i \in I$ with $ra = xi$. Therefore, in the tensor product $V \otimes KI^{-1}$, $v \otimes a$ becomes equal to $v \otimes r^{-1}xi = r^{-1}iv \otimes x = 0$. □

The final preparatory lemma is very simple.

Lemma 4.3. *Let I be a proper ideal of R, $I^{\#} = L$, and K a submodule of Q containing R_L such that $LK = K$. For a uniserial module W of type $[K/L]$ we have: $W[L] \leq W^{\circ}$ and $W/W[L]$ is of type $[K/R_L]$.*

Proof. As noted in (1.3)(i), $K^{\#} \leq L$, so obviously $W[L] \leq W^{\circ} = W[K^{\#}]$. Since $W[L] \cong R_L/L$, the claim follows. □

The preceding results can be interpreted as statements concerning three functors acting between three skeleton categories: $\mathrm{Unis}[KI/I]$, $\mathrm{Gp}[K/R_L]$, and $\mathrm{Unis}[K/L]$, where $I^{\#} = L$. In fact, (4.1) shows that $\Phi = \mathrm{Tor}_1(KI/I, *) \cong I \otimes *$ is a functor assigning to $[U] \in \mathrm{Gp}[K/R_L]$ the object $[\mathrm{Tor}_1(KI/I, U)] \in \mathrm{Unis}[KI/I]$, while (4.2) asserts that the functor $\mathrm{Tor}_1(KI^{-1}/I^{-1}, *) \cong I^{-1} \otimes *$ associates $[I^{-1} \otimes V] \in \mathrm{Unis}[K/L]$ with $[V] \in \mathrm{Unis}[KI/I]$. Furthermore, the correspondence $[W] \mapsto [W/W[L]]$ induces a functor Π: $\mathrm{Unis}[K/L] \to \mathrm{Gp}[K/R_L]$. This leads us to the following commutative diagram of functors between full skeleton subcategories of Mod-R_L:



where Ψ denotes the composition map $\Pi \circ [I^{-1} \otimes *]$.

We can now state and prove the main result of this section.

Theorem 4.4. *Suppose that $I < R$, $I^{\#} = L$, $K > R_L \not\geq I$ are such that $LK = K$. Then Φ is a category equivalence between skeletons of full subcategories* $\mathrm{Unis}[KI/I]$ *and* $\mathrm{Gp}[K/R_L]$ *of* Mod-R *(or, equivalently, of* Mod-R_L*). Ψ is the inverse of Φ.*

Proof. What we have to prove is that, given $[U] \in \mathrm{Gp}[K/R_L]$, we have $(\Psi \circ \Phi)$ $[U] = [U]$, i.e.

(1) $$(U \otimes I \otimes I^{-1})/(U \otimes I \otimes I^{-1})[L] \cong U,$$

and that, given $[V] \in \mathrm{Unis}[KI/I]$, $[V] = (\Phi \circ \Psi)[V]$ holds, i.e.

(2) $$((V \otimes I^{-1})/(V \otimes I^{-1})[L]) \otimes I \cong V.$$

To verify (1), note that, since $R_L \ncong I$, the left hand side is isomorphic to $(U \otimes L)/(U \otimes L)[L]$. From the exact sequence $0 \to L \to R \to R/L \to 0$ we derive the exact sequence

$$0 \to \mathrm{Tor}(U, R/L) \to U \otimes L \to U \to U/LU \to 0.$$

Here the image of $\mathrm{Tor}(U, R/L)$ in $U \otimes L$ is exactly $(U \otimes L)[L]$. Furthermore, $U/LU = 0$, since $KL = K$. Hence the desired isomorphism (1) follows.

The proof of (2) is more delicate. Manifestly, the left hand side is isomorphic to $(V \otimes I^{-1} \otimes I)/((V \otimes I^{-1})[L] \otimes I)$, since I is flat as an R-module. The canonical homomorphism

$$\psi : V \otimes I^{-1} \otimes I \to V$$

sends $v \otimes a^{-1} \otimes i \in V \otimes I^{-1} \otimes I$ to $va^{-1}i \in V$ $(v \in V, a \in R \setminus I, i \in I)$. Clearly, ψ is surjective, since $\mathrm{Im}\,\psi = VL = V$ (recall that $KIL = KI$). Thus it suffices to verify that $\mathrm{Ker}\,\psi = (V \otimes I^{-1})[L]) \otimes I$. A typical element of the module $(V \otimes I^{-1})[L]$ is of the form $r(1 + I) \otimes 1$ $(r \in R)$, hence a typical element of $(V \otimes I^{-1})[L]) \otimes I$ has the form $(r(1 + I) \otimes 1) \otimes i$ $(i \in I)$. Obviously, $ir(1 + I) = 0$, whence we conclude $(V \otimes I^{-1})[L]) \otimes I \leq \mathrm{Ker}\,\psi$. Conversely if $va^{-1}i = 0$ in V $(a^{-1} \in I^{-1}, i \in I)$, then $a^{-1}i \in \mathrm{Ann}\,v$. But $\mathrm{Ann}\,v = j^{-1}I$ for some $j \in J$, hence $ja^{-1}i \in I$. Since $j \notin R$ can be assumed, we obtain

$$v \otimes a^{-1} \otimes i = (1 + I) \otimes 1 \otimes ja^{-1}i \in (V \otimes I^{-1})[L]) \otimes I,$$

completing the proof. □

Note that the category equivalence stated in (4.4) holds true even if $R_L \cong I$. Indeed, in this case $\mathrm{Unis}[KI/I]$ coincides with $\mathrm{Gp}[K/R_L]$, and Φ and Ψ are the identity functors (Ψ is now just the functor obtained by tensoring with $I^{-1} \cong R_L$ which maps $\mathrm{Gp}[K/R_L]$ onto itself).

It is a basic observation that the group $\mathrm{Gp}[K/R_L]$ operates, via the functor Tor, on the skeleton category $\mathrm{Unis}[KI/I]$, if I is an ideal such that $I^* = L$, i.e. if $\mathrm{Unis}[KI/I] \leq \mathrm{Unis}_L^K$. In fact, if V is n.f.a. of type $[KI/I]$ and U is of type $[K/R_L]$, then $\mathrm{Tor}(V, U)$ is also of type $[KI/I]$. Using the lemmas at the beginning of this section, we can easily verify the following theorem.

Theorem 4.5. *If L is a prime ideal of R, and K is a submodule of Q such that $K > R_L$ and $LK = K$, then the group* $\mathrm{Gp}[K/R_L]$ *operates via* Tor *transitively and faithfully on* $\mathrm{Unis}[KI/I]$, *for each ideal I of R such that $I^* = L$.*

Proof. Note that $LK = K$ implies that $K^{\#} \leq L$. We must show that, given $[V]$ and $[V']$ in Unis$[KI/I]$, there exists an $[U_0] \in \mathrm{Gp}[K/R_L]$ such that $\mathrm{Tor}(V, U_0) \cong V'$. Let $[V] = \Phi[U]$ and $[V'] = \Phi[U']$. Then, by (4.1), $V \cong \mathrm{Tor}(KI/I, U)$ and $V' \cong \mathrm{Tor}(KI/I, U')$. Let U^{-1} denote the uniserial module such that $\mathrm{Tor}(U, U^{-1}) \cong K/R_L$ (thus $[U^{-1}]$ is the inverse of $[U]$ in $\mathrm{Gp}[K/R_L]$), and let $U_0 = \mathrm{Tor}(U', U^{-1})$. Then

$$\begin{aligned}&\mathrm{Tor}(U, U_0)\\ &= \mathrm{Tor}(U, \mathrm{Tor}(U', U^{-1})) \cong \mathrm{Tor}(U', \mathrm{Tor}(U, U^{-1}) \cong \mathrm{Tor}(U', K/R_L) \cong U'.\end{aligned}$$

Therefore we have:

$$\begin{aligned}V' \cong \mathrm{Tor}(KI/I, U') &\cong \mathrm{Tor}(KI/I, \mathrm{Tor}(U, U_0)) \cong \mathrm{Tor}(\mathrm{Tor}(KI/I, U), U_0)\\ &\cong \mathrm{Tor}(V, U_0).\end{aligned}$$

To show that $\mathrm{Gp}[K/R_L]$ acts faithfully, we must convince ourselves that $U \ncong U'$ in $\mathrm{Gp}[K/R_L]$ implies $U \otimes I \ncong U' \otimes I$. But this is clear, since (4.4) guarantees that Φ is injective. □

5. Unis_L^K as union of two groups

In the endeavor of finding adequate tools elucidating the monoid structure of Unis_L^K (for prime ideals L with $LK = K$), one is naturally led to the consideration of the set of isomorphy classes of fractional ideals I such that $I^{\#} = L$. Here again, we try to equip this set with an algebraic operation.

For a prime ideal L of R, let $\mathbf{G}_L$ denote the set of isomorphy classes [I] of submodules I of Q such that $I^{\#} = L$ and $I \ncong R_L$. It should not bother us that the exclusion of $[R_L]$ from $\mathbf{G}_L$ may result in $\mathbf{G}_L$ being empty; this exclusion is unavoidable if we want to ensure that in all other cases $\mathbf{G}_L$ becomes an abelian group (which has an important role in the description of Unis_L^K).

For example, we have $\mathbf{G}_0 = \{[0]\}$. $\mathbf{G}_P$ consists of the isomorphy classes of all nonprincipal archimedean ideals.

Lemma 5.1. *Given $0 \neq L \in \mathrm{Spec}\, R$, $\mathbf{G}_L$ is empty if and only if $L^2 < L$.*

Proof. $\mathbf{G}_L = \emptyset$ if and only if each ideal I with $I^{\#} = L$ is a principal R_L-ideal. It is readily seen that this is equivalent to L being a principal R_L-ideal, so the claim follows. □

Proposition 5.2. *If $0 \neq L \in \mathrm{Spec}\, R$ and $\mathbf{G}_L \neq \emptyset$, then $\mathbf{G}_L$ is an abelian group under the multiplication $[I]\,[J] = [IJ]$. Its neutral element is $[L]$.*

Proof. Multiplication is obviously well defined. Let I and J be submodules of Q with $I^{\#} = L = J^{\#}$. From (1.2) we obtain that $(IJ)^{\#} = L$, so $\mathbf{G}_L$ is a commutative semigroup under multiplication. If I is not principal as an R_L-ideal, then $IL = I$ and $[L]$ is the neutral element in $\mathbf{G_L}$. By (1.2) and (1.1)(iv) we have the equalities $II^{-1} = L$ and $(I^{-1})^{\#} = I^{\#} = L$. □

Not unexpectedly, the structure of $\mathbf{G_L}$ depends only on the value group of R_L. This dependence is illustrated by the next proposition which improves on Theorem 1 in [B].

Proposition 5.3. *Let L be a prime ideal of R. $\mathbf{G_L}$ is either empty or isomorphic to the group $\hat{\Gamma}/\Gamma$, where Γ is the value group of R_L and $\hat{\Gamma}$ denotes its completion (in the order topology).*

Proof. It is well known (see e.g. [FS, I.3.2]) that there exists a natural bijection Θ between the set of ideals of R_L and the set of filters of the positive cone Γ^+ of Γ. Under this correspondence, the archimedean ideals of R_L correspond to the Cauchy filters (i.e. those filters which represent elements of the completion $\hat{\Gamma}$). We claim that

$$[I] \to \Theta(I) + \Gamma$$

is a desired isomorphism from $\mathbf{G_L}$ to $\hat{\Gamma}/\Gamma$. In fact, since $I \cong I'$ if and only if $I = sI'$ for some $s \in R_L$, $[I] = [I']$ is equivalent to $\Theta(I) = \gamma + \Theta(I')$ with $\gamma (\in \Gamma)$ the value of s. Hence the correspondence is well defined and injective; it is clearly surjective, since so is Θ. Finally, it is a group homomorphism, because the product II' is sent by Θ into the filter $\Theta(I) + \Theta(I')$ of Γ. □

As a very special case, let us mention that if L is a prime ideal of R with an immediate predecessor $L' < L$ in Spec R, and if $\mathbf{G_L} \neq \emptyset$, then $\mathbf{G_L}$ is isomorphic to a proper quotient of the additive group $\mathbb{R}$ of the real numbers, hence it is divisible. In fact, it is easy to see that every element of $\mathbf{G_L}$ can be represented by an archimedean ideal of R_L/L'. Now R_L/L' is a valuation domain with archimedean value group. therefore this value group is isomorphic to a subgroup S of the additive group of the real numbers. Moreover, by the hypothesis $\mathbf{G_L} \neq \emptyset$, S is not discrete, hence its completion coincides with $\mathbb{R}$. Consequently, by (5.3), $\mathbf{G_L}$ is isomorphic to $\mathbb{R}/S$.

If $\mathbf{G_L} = \emptyset$, then $\text{Unis}_L^K = \text{Gp}[K/R_L]$. If $\mathbf{G_L} = \{[0]\}$, then Unis_L^K is the disjoint union of $\text{Gp}[K/R_L]$ and $\text{Unis}[K/L]$. For various examples of non-zero groups $\mathbf{G_P}$ we refer to Bazzoni [B].

Next we focus our attention on the monoid Unis_L^K, in particular, on the complement H_L of $\text{Gp}[K/R_L]$ in Unis_L^K. Assume H_L is not empty, i.e. $\mathbf{G_L} \neq \emptyset$. We know that H_L is the union of its subsets $\text{Unis}[KI/I]$ for all $[I] \in \mathbf{G_L}$. One of these is $\text{Unis}[K/L]$ which is evidently a subgroup; to emphasize this fact, let us denote it by $\mathbf{Gp[K/L]}$. One can find another group in H_L: the isomorphy classes of standard uniserials KI/I form a subgroup $\mathbf{St_L^K}$ with identity $[K/L]$ such that $\text{St}_L^K \cong \mathbf{G_L}$. It is clear that the correspondence $[I] \mapsto [KI/I]$ is a bijection between $\mathbf{G_L}$ and St_L^K. It is an isomorphism, since (1.9) implies $\text{Tor}(KI/I, KI'/I') \cong KII'/II'$.

About the algebraic structure of H_L an explicit information can be given by the following stunning result.

Theorem 5.4. *Let L be a prime ideal of R, $L^2 = L$, and K a submodule of Q such that $K > R_L$ and $KL = K$. The disjoint union H_L of the classes* $\mathrm{Unis}[KI/I]$, *with* $[I]$ *ranging over* $\mathbf{G_L}$, *is an abelian group. Moreover, it is the direct product of two subgroups,*

$$\mathrm{H}_L = \mathrm{Gp}[K/L] \times \mathrm{St}_L^K$$

where $\mathrm{Gp}[K/L] \cong \mathrm{Gp}[K/R_L]$ *and* $\mathrm{St}_L^K \cong \mathbf{G_L}$.

Proof. By (5.1), H_L is not empty. The subgroups $\mathrm{Gp}[K/L]$, St_L^K of H_L intersect obviously only in $[K/L]$, thus $\mathrm{Gp}[K/L] \times \mathrm{St}_L^K$ is a subgroup of H_L. In order to justify our claim, we have to prove that every uniserial V of type $[KI/I]$ (with $[I] \in \mathbf{G_L}$) is contained in this direct product. From (4.2) we conclude that $[\mathrm{Tor}(KI^{-1}/I^{-1}, V)] \in \mathrm{Gp}[K/L]$, and owing to (4.1), $\mathrm{Tor}(KI/I, \mathrm{Tor}(KI^{-1}/I^{-1}, V))$ $= \mathrm{Tor}(KI^{\#}/I^{\#}, V) \cong I^{\#} \otimes V \cong V$.

To verify the second statement, consider the map $[U] \mapsto [L \otimes_R U]$ from $\mathrm{Gp}[K/R_L]$ into $\mathrm{Gp}[K/L]$. The diagram in Section 4 shows that this is just the composite of Φ with $I^{-1} \otimes *$ whose inverse is Π, so it is a bijection. In order to prove that it preserves multiplication, we have to convince ourselves that $\mathrm{Tor}(L \otimes U, L \otimes U')$ is isomorphic to $L \otimes \mathrm{Tor}(U, U')$, for uniserial modules U, U' of type $[K/R_L]$. But this is straightforward to verify by using the associativity of the functor Tor and the isomorphism $L \otimes U \cong \mathrm{Tor}(K/L, U)$ for uniserial modules U of type $[K/R_L]$, which was proved in (4.1). The second isomorphism has been established above. □

Consequently, the semigroup Unis_L^K is either a group or the disjoint union of two groups. In order to be more specific about the way these two groups interact within the monoid Unis_L^K, let us introduce the monoid St_L^{K*} as the union of the group St_L^K with $[K/R_L]$ (acting as neutral element) attached. Combining (3.2) and (5.4) we have at once:

Theorem 5.5. *Suppose that L is a prime ideal of R, $L^2 = L$, and K is a submodule of Q such that $K > R_L$ and $KL = K$. Then the monoid* Unis_L^K *is the direct product of the group* $\mathrm{Gp}[K/R_L]$ *and the monoid* St_L^{K*}:

$$\mathrm{Unis}_L^K = \mathrm{Gp}[K/R_L] \times \mathrm{St}_L^{K*}. \quad \square$$

We close this section with a rather trivial but pertinent consequence of (5.4): $\mathrm{Unis}[KI/I]$ is precisely the orbit of the isomorphy class of the standard uniserial module KI/I under the group $\mathrm{Gp}[K/R_L]$ in Unis_L^K; furthermore, $\mathrm{Gp}[K/R_L]$ acts faithfully on the set $\mathrm{Unis}[KI/I]$.

6. Localizing Unis_P^K

Localization at a prime ideal L is the functor $R_L \otimes_R *$ from Mod-R to Mod-R_L; we know that it carries uniserial modules into uniserials. Bearing in mind that the objects in Mod-R_L can also be viewed as objects in Mod-R, it is clear that this functor induces a map of Unis R into itself. In particular, as the isomorphy class of the tensor product of R_L with a uniserial module U (where $[U] \in \mathrm{Unis}_P^K$) belongs to Unis_L^K, this map carries Unis_P^K into Unis_L^K. Localization proves to be a decisive tool in relating Unis_P^K (in case $P > L \geq K^*$) to Unis_L^K. More importantly, localization at L gives rise to a morphism from Gp $[K/R]$ into Gp $[K/R_L]$, which surprisingly, turns out to be an isomorphism. (Caution should be exercised in case $KK^* < K$ in which Gp $[K/R_{K^*}]$ fails to exist.) Consequently, the groups Gp $[K/R_L]$ in Unis R are just replicas of Gp $[K/R]$ under natural isomorphisms. This together with the fact (shown by (5.5)) that the existence of n.f.a. uniserials of level K is already determined by uniserials of type $[K/R_L]$ is a strong indication that the structure Unis R is not as complex after all as one expects and, definitely, it is most worthy of further study.

It is convenient to start our investigation into the localization process with a simple lemma which describes more explicitly how localizatzion acts on certain uniserials.

Lemma 6.1. *If L is a prime ideal of R and U is a uniserial R-module with $U^* \leq L$, then the localization map $U \to R_L \otimes U$ gives rise to the exact sequence*

$$0 \to \mathrm{Tor}_1^R(R_L/R, U) \to U \to R_L \otimes U \to 0.$$

Proof. The exact sequence $0 \to R \to R_L \to R_L/R \to 0$ induces the exact sequence

$$0 \to \mathrm{Tor}(R_L/R, U) \to U \to R_L \otimes U \to (R_L/R) \otimes U \to 0.$$

The last tensor product has to vanish, since the annihilators of the elements in R_L/R are in $R \setminus L$, while $sU = U$ for each $s \in R \setminus L$. Evidently, the kernel of the localization map is the submodule $\bigcup_{r \notin L} U[r]$ of U. □

A closer look at the localization process leads us to the following conclusion.

Lemma 6.2. *Assume L is a prime ideal of R, and K is a submodule of Q such that $K > R_L$ and $KL = K$. Localizations of the elements of the monoid* Unis_P^K *(of the group* Gp $[K/R]$*) at L commute with the semigroup operation, i.e., for* $[U], [U'] \in \mathrm{Unis}_P^K$ *we have*

$$R_L \otimes \mathrm{Tor}_1^R(U, U') \cong \mathrm{Tor}_1^R(R_L \otimes U, R_L \otimes U').$$

Proof. In order to apply (6.1), we have to ascertain that $U^* \leq L$ (and the same for U'); but this is clear from $U^* = K^*$. (6.1) yields an exact sequence

$$0 \to \mathrm{Tor}(R_L/R, U) \to U \to R_L \otimes U \to 0.$$

Tensor it with U' to obtain the exact sequence

$$0 \to \mathrm{Tor}(\mathrm{Tor}(R_L/R, U), U') \to \mathrm{Tor}(U, U') \to \mathrm{Tor}(R_L \otimes U, U') \\ \to \mathrm{Tor}(R_L/R, U) \otimes U' = 0 .$$

The comparison of this sequence with the localization exact sequence of $\mathrm{Tor}(U, U')$:

$$0 \to \mathrm{Tor}(R_L/R, \mathrm{Tor}(U, U')) \to \mathrm{Tor}(U, U') \to R_L \otimes \mathrm{Tor}(U, U') \to 0$$

yields a natural isomorphism

$$R_L \otimes \mathrm{Tor}(U, U') \cong \mathrm{Tor}(R_L \otimes U, U') .$$

Tensoring both sides with R_L, from this isomorphism along with $R_L \otimes R_L = R_L$ we derive the desired conclusion. □

What we have just proved amounts to saying that the localization functor $U \to R_L \otimes U$ at a prime ideal L induces a homomorphism from the monoid Unis_P^K into the monoid Unis_L^K. Its restriction to $\mathrm{Gp}[K/R]$ behaves in a straightforward manner:

Theorem 6.3. *Let L be a prime ideal of R, $L^2 = L$, and K a submodule of Q such that $R_L < K$ and $LK = K$. Then the correspondence*

$$\gamma_L \colon \mathrm{Gp}[K/R] \to \mathrm{Gp}[K/R_L]$$

induced by the localization map at L is a group isomorphism.

Proof. Note that $LK = K$ implies that $R_L K = K$, so for a uniserial U of type $[K/R]$ we have $t(R_L \otimes U) = [K/R_L]$. Thus γ_L is a map between the indicated groups. That it is a group homomorphism is an immediate consequence of (6.2).

The proof of (6.1) shows that, for a given $[U] \in \mathrm{Gp}[K/R]$, we have $R_L \otimes U \cong U/U[L]$. By [BS2, Thm 3.7], all elongations of non-standard uniserials $U/U[L]$ by the same $U[L] \cong R_L/R$ are isomorphic (the same for standard uniserials is trivial). Consequently, γ_L is monic. In order to verify that it is epic as well, we must find, for each $[V] \in \mathrm{Gp}[K/R_L]$, an elongation $U \to V$ by R_L/R; then U is necessarily of type $[K/R]$ and has the same annihilator as V. By [BS2], such an elongation exists if and only if either $R : R_L > K^*$, or $R : R_L = K^*$ and $KK^* = K$. In the present case, $R : R_L = L$, so that the desired U does exist. Note that $U/U[L] \cong V$, hence $[V] = [R_L \otimes U] = \gamma_L[U]$. □

We proceed to study the influence of the localization on the monoid Unis_P^K. Needless to say, it suffices to investigate how it effects the various categories $\mathrm{Unis}[KI/I]$, where $I^* = P > K^*$ and $K > R_{I^*}$.

Proposition 6.4. *Let $K > R_{I^*}$ and $I^\# = P > K^\#$, so that* $\mathrm{Unis}[KI/I] \leq \mathrm{Unis}_P^K$. *For a prime ideal L satisfying $P > L \geq K^\#$ and $KL = K$, the following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{Unis}\,[KI/I] & \xrightarrow{\quad \Psi \quad} & \mathrm{Gp}\,[K/R] \\ & {\scriptstyle * \otimes R_L} \searrow \quad \swarrow {\scriptstyle \gamma_L} & \\ & \mathrm{Gp}\,[K/R_L] & \end{array}$$

Proof. Since $I^\# = P$, we can assume, without loss of generality, that $I > L$. Recall that, if $[U] \in \mathrm{Unis}[KI/I]$, then $\Psi[U] = [U \otimes I^{-1}/(U \otimes I^{-1})[P]]$. Thus $\gamma_L \Psi$ sends $[U]$ into the isomorphy class of

$$(U \otimes I^{-1}/(U \otimes I^{-1})[P]) \otimes R_L \cong U \otimes I^{-1} \otimes R_L/((U \otimes I^{-1})[P] \otimes R_L).$$

The denominator on the right hand side vanishes, since $(U \otimes I^{-1})[P] \cong R/P$. The conclusion now follows from the fact that $I > L$ implies $I^{-1} R_L = R_L$, thus $U \otimes I^{-1} \otimes R_L \cong U \otimes R_L$. □

It is clear that what we have proved gives also full information about how the localization at L effects $\mathrm{Unis}_{L'}^K$ where $P > L' > L$; in fact, it is enough to apply the preceding results, starting with the ring $R_{L'}$ rather than R. We can now conclude that localization at the prime $L'(<L)$ carries Unis_L^K onto the group $\mathrm{Gp}\,[K/R_{L'}]$. Thus

Proposition 6.5. *If $L > L'$ are prime ideals, $R < K \leq Q$ and $KL = KL' = K$, then the localization map $- \otimes R_{L'}$ induces a homomorphism of the monoid* $\mathrm{Unis}_L^K = \mathrm{Gp}\,[K/R_L] \times \mathrm{St}_L^{K*}$ *onto the group* $\mathrm{Gp}\,[K/R_{L'}]$. *This map, restricted to* $\mathrm{Gp}\,[K/R_L]$, *is an isomorphism, and it acts trivially on* St_L^{K*}. □

7. Clifford semigroups in Unis R

Having investigated the structures of the monoid Unis_L^K and their behavior under the localization process, we would like to find out how the individual semigroups Unis_L^K fit in larger subsemigroups of Unis R. To this end, we introduce two kinds of subsemigroups, Unis^K and Unis_L, by fixing the level K and by fixing the prime L associated with the annihilators of elements, respectively. Amazingly, both of them will prove to be Clifford semigroups with totally ordered sets of subgroups.

Unis^K will denote the set of isomorphy classes of n.f.a. uniserial modules U in Unis R of a fixed level K. (2.10) guarantees that this is a subsemigroup of Unis R:

Proposition 7.1. *Let K be a non-principal submodule of Q, $K > R$.*

(i) Unis^K *is a subsemigroup of* Unis R, *which is a monoid with* $[K/R]$ *as neutral element.*

(ii) Unis^K *is the disjoint union of the subsemigroups* Unis_L^K *where* L *ranges over the set of the prime ideals such that* $LK = K$:

$$\mathrm{Unis}^K = \bigcup_{LK=K} \mathrm{Unis}_L^K. \quad \square$$

We want to examine Unis^K more carefully. Manifestly, the isomorphy classes of the standard uniserials in Unis^K form a submonoid which we shall denote by $\mathbf{St}^{K*}$. We are now prepared to prove one of our structure theorems.

Theorem 7.2. *For a non-principal submodule of* Q, $K > R$, Unis^K *is a commutative Clifford semigroup over a totally ordered index set, where the bonding homomorphisms are either isomorphisms or projections onto a direct factor followed by an isomorphism.*

Moreover, Unis^K *is the direct product of a group and a Clifford semigroup:*

$$\mathrm{Unis}^K = \mathrm{Gp}[K/R] \times \mathrm{St}^{K*}.$$

Proof. Given K, define the index set as $S = \{\varrho_L, \sigma_L \mid L \in \mathrm{Spec}\, R, LK = K\}$, where a total order is defined by declaring $\varrho_L > \sigma_L$ and $\sigma_L > \varrho_{L'}$ whenever $L > L'$. Set $G_{\varrho_L} = \mathrm{Gp}[K/R_L]$ and $G_{\sigma_L} = \mathrm{Gp}[K/R_L] \times \mathrm{St}_L^K$ with bonding homomorphisms $G_{\varrho_L} \to G_{\sigma_L}$, $G_{\sigma_L} \to G_{\varrho_{L'}}$ acting as follows: $[U] \mapsto [L \otimes_R U]$ and $[U] \mapsto [R_{L'} \otimes_R U]$, respectively. Thus the first homomorphism is the embedding of $\mathrm{Gp}[K/R_L]$ in the first factor of G_{σ_L}, while the second maps $\mathrm{Gp}[K/R_L]$ isomorphically onto $\mathrm{Gp}[K/R_{L'}]$ and acts trivially on St_L^K; hence it is the projection onto the first factor followed by an isomorphism. An appeal to (4.1) and (6.5) shows that we obtain a Clifford semigroup indeed.

The subsemigroup St^{K*} is itself a Clifford semigroup, which is the union of the Clifford semigroups St_L^{K*} ($L \in \mathrm{Spec}\, R$, $LK = K$) with trivial bonding homomorphisms. The claim on the direct decomposition is an immediate consequence of the first one, by taking into account the action of $[K/R]$ via Tor on the classes of standard uniserials in Unis^K. $\square$

Corollary 7.3. *There exists a non-standard uniserial R-module of level K if and only if* $\mathrm{Gp}[K/R] \neq 0$. *R admits a non-standard uniserial module exactly if the Clifford semigroup* Unis_R *contains non-trivial groups.*

Proof. The first assertion is evident from (7.2), while the second follows from (3.1). $\square$

Next we group together the semigroups Unis_L^K in a different way: keeping L fixed. Accordingly, for a fixed prime ideal L of R, we consider all uniserials of type $[KI/I]$ with K ranging over the set of fractional ideals $> R_L$ such that $LK = K$ and I is an ideal of R with $I^\# = L$. All these uniserials are n.f.a. Their isomorphy classes form a subsemigroup $\mathbf{Unis}_L$ of Unis R. In fact, if the uniserials U, V are of types $[KI/I]$

and $[K'I'/I']$, respectively, where $I^* = L = I'^*$, then $\operatorname{Tor}(U, V)$ is of type $[(K'I'I \cap KII')/II']$ where $(II')^* = L$. Evidently, we have

Proposition 7.4. *For a prime ideal L,*

(i) Unis_L *is a subsemigroup of* Unis *R, and* $[Q/R_L]$ *is its neutral element.*

(ii) Unis_L *is the disjoint union of subsemigroups* Unis_L^K *for submodules of Q such that* $K > R_L$ *and* $LK = K$:

$$\operatorname{Unis}_L = \bigcup_{KL=K} \operatorname{Unis}_L^K. \quad \square$$

Visibly, Unis_L^K is the intersection of Unis^K and Unis_L.

The next theorem is an analogue of (7.2). Before stating it, note that the isomorphy classes of uniserials of types $[K/R_L]$ with $LK = K$ form a (Clifford) semigroup $\operatorname{Unis}_{R_L}$ which is the union of the groups $\operatorname{Gp}[K/R_L]$ with $LK = K$. $\mathbf{G}_{L^*}$ will denote the group $\mathbf{G}_L$ with an additional neutral element adjoined. $\mathbf{G}_{L^*}$ is isomorphic to St_L^{K*} for each $K > R_L$ with $KL = K$.

Theorem 7.5. *For a non-zero prime ideal L of R, the monoid* Unis_L *is a commutative Clifford semigroup where the bonding homomorphisms are either isomorphisms or projections onto a direct factor followed by an isomorphism.*
In addition, we have:

$$\operatorname{Unis}_L \cong \operatorname{Unis}_{R_L} \times \mathbf{G}_{L^*}.$$

Proof. For a given L, define the index set $T = \{\varrho^K, \sigma^K \mid K > R_L, LK = K\}$, and set $\varrho^K > \sigma^K$ for all K, and $\varrho^K > \varrho^{K'}$ and $\sigma^K > \sigma^{K'}$ whenever $K > K'$. Let $G_{\varrho^K} = \operatorname{Gp}[K/R_L]$ and $G_{\sigma^K} = \operatorname{Gp}[K/R_L] \times \operatorname{St}_L^K$ with bonding homomorphisms $G_{\sigma^K} \to G_{\sigma^K}$, $G_{\sigma^K} \to G_{\sigma^{K'}}$ acting as follows: $[U] \mapsto [L \otimes_R U]$ and $[U] \mapsto [\operatorname{Tor}(K'/R_L, U)]$, respectively, while $G_{\varrho^K} \to G_{\varrho^{K'}}$ is trivial. (Thus the first homomorphism maps isomorphically onto the first factor, while the second one acts trivially on the first factor and maps St_L^K isomorphically onto $\operatorname{St}_L^{K'}$; recall that both are $\cong \mathbf{G}_L$.) A glance at (5.5) shows that this construction leads to Unis_L. The direct decomposition of Unis_L is straightforward. $\square$

Let us point out that (7.2) and (7.5) show that the set of isomorphy classes of n.f.a. uniserials is the disjoint union of numerous groups. The structure of these groups is determined by the groups $\operatorname{Gp}[K/R]$ (for all non-principal submodules K of Q) and by the groups $\mathbf{G}_L$ (for all primes L).

8. The six classes of n.f.a. uniserials

Bazzoni and Salce [BS2] distinguish six different classes $\mathscr{U}_i$ $(i = 1, \ldots, 6)$ of non-standard uniserials, and show that in the constructible universe L none of these classes is empty. Eklof [E] derives that the same holds in ZFC. Osofsky [O1] constructs valuation domains in ZFC for which none of these six classes is empty. These classes are as follows (A stands for Ann U):

$\mathcal{U}_1$	$0 = A = U^* < U_\#$	$0 < U_\circ = U^\circ = U$	divisible, strongly non-standard
$\mathcal{U}_2$	$0 < A = U^* < U_\#$	$0 < U_\circ = U^\circ = U$	bounded, strongly non-standard
$\mathcal{U}_3$	$0 < A < U^* < U_\#$	$0 < U_\circ = U^\circ < U$	Ann $U/U^\circ >$ Ann U
$\mathcal{U}_4$	$0 < A < U^* < U_\#$	$0 < U_\circ = U^\circ < U$	Ann $U/U^\circ =$ Ann U (equiannihilated)
$\mathcal{U}_5$	$0 < A < U^* = U_\#$	$0 = U_\circ < U^\circ < U$	U/U° non-standard, equiannihilated
$\mathcal{U}_6$	$0 < A < U^* = U_\#$	$0 = U_\circ = U^\circ < U$	barely non-standard, equiannihilated

The classes $\mathcal{U}_5$ and $\mathcal{U}_6$ are divided into subclasses

$$\mathcal{U}_{54} = \{U \in \mathcal{U}_5 \mid U^* < P, U[P] < U^\circ\},$$
$$\mathcal{U}_{56} = \{U \in \mathcal{U}_5 \mid U^* = P, U[P] = U^\circ\},$$

and

$$\mathcal{U}_{64} = \{U \in \mathcal{U}_6 \mid U^* < P, I \cong R_{I^*}\},$$
$$\mathcal{U}_{65} = \{U \in \mathcal{U}_6 \mid I \cong R\},$$
$$\mathcal{U}_{66} = \{U \in \mathcal{U}_6 \mid I \ncong R_{I^*}, I \ncong I^* = J^*\},$$

respectively. If $U \in \mathcal{U}_{54}$, then $U/K \in \mathcal{U}_4$ for $0 < K < U^\circ$ and $U/U^\circ \in \mathcal{U}_6$. If $U \in \mathcal{U}_{56}$, then $U/U^\circ = U/U[P] \in \mathcal{U}_6$. The modules $U \in \mathcal{U}_{64}$ are quotients of modules in $\mathcal{U}_4$, the modules $U \in \mathcal{U}_{65}$ are quotients of modules in $\mathcal{U}_5$ but not in $\mathcal{U}_4$, while those in $\mathcal{U}_{66}$ are not proper quotients of uniserial modules.

It seems inevitable to change this classification into a more systematic one which embraces the n.f.a. uniserials as well and relies on properties which turned out to have the greatest impact.

Given a n.f.a. uniserial module U, we divide the submodules V of U into two subsets (upper and lower) as follows:

$$\mathcal{U} = \{V \leq U \mid \mathrm{Ann}(U/V) > \mathrm{Ann}\, U\}$$

and

$$\mathcal{L} = \{V \leq U \mid \mathrm{Ann}(U/V) = \mathrm{Ann}\, U\}.$$

Evidently, $U \in \mathcal{U}$ and $0 \in \mathcal{L}$, so neither of these is empty. (This is actually a Dedekind cut in the totally ordered set of submodules.) We need the following simple fact:

Lemma 8.1. *The upper threshold submodule U° is either the maximum of $\mathcal{L}$ or the minimum of $\mathcal{U}$. If $\mathcal{L}$ ($\mathcal{U}$) contains U°, then $\mathcal{U}$ ($\mathcal{L}$) has no minimum (maximum).*

Proof. If $U_\circ = U^\circ$, then the claim follows from (1.6). So assume $U_\circ < U^\circ$ and let $u \in U^\circ \setminus U_\circ$. Set $H(u) = J$ and Ann $u = I$. By definition, $u \in U^\circ$ means Ann $u \geq J^*$, and $u \notin U_\circ$ means Ann $u \leq J^*$, thus Ann $u = J^*$. Hence $I \cong J^*$. By (1.3), $J^* J = I^* J = J$ whence we infer from (1.6)(iii) that $U^\circ \in \mathcal{L}$. □

By an *elongation* of a uniserial module U is meant an epimorphism ϕ: $W \to U$ where W is a uniserial module; the elongation ϕ: $W \to U$ is *proper* if ϕ is not an isomorphism. We shall call U *elongable* if it admits a proper elongation ϕ: $W \to U$ such that Ann $W =$ Ann U. Elongations of non-standard uniserials have been studied by Bazzoni-Salce [BS2].

Lemma 8.2. *If ϕ: $W \to U$ is an elongation of the n.f.a. uniserial U, then W is equiannihilated if and only if U is.*

Proof. From (8.1) it is evident that the complete inverse image of U° under ϕ has to be W°. The claim now follows from the isomorphism $W/W^\circ \cong U/U^\circ$. □

The next result gives detailed information about the elongations of a n.f.a. uniserial module. (Cp. [BS2].)

Lemma 8.3. *Let U be a n.f.a. uniserial module of type $[J/I]$.*

(i) *If U is equiannihilated, then it has a n.f.a. elongation ϕ: $W \to U$ where W is of type $[J/J(I:J)]$, has the same annihilator as U and is no longer elongable.*

(ii) *If U is not equiannihilated, then it has proper n.f.a. elongations of types $[J/L]$ with $L > J(I:J)$, which have the same annihilator as U. In case U is standard, there is a finitely annihilated elongation of U of type $[J/J(I:J)]$, which has the same annihilator as U.*

Proof. We first show that the smallest ideal I' of R for which $\operatorname{Ann} J/I' = \operatorname{Ann} J/I$ is $J(I:J)$. Evidently, $I' < J(I:J)$ implies $I':J < I:J$. On the other hand, $J(I:J) \leq I$ implies $I:J \geq I':J = J(I:J):J \geq I:J$.

By (1.2) and (1.3)(ii) we have $(J(I:J))^\# = J^\# \cap (I:J)^\# = J^\#$, so (1.4) shows that $J/J(I:J)$ is n.f.a. exactly if $J^\# J = J$, i.e. if J/I is equiannihilated. This proves (i).

To verify (ii), start with an element $a \in I^\# \setminus J^\#$ (which exists by (1.6)(ii) and (1.4)) and form the exact sequence $0 \to Ra \to R \to R/Ra \to 0$; it induces the exact sequence

$$0 \to \operatorname{Tor}(R/Ra, U) \to Ra \otimes U \to R \otimes U \cong U \to R/Ra \otimes U \to 0.$$

Since $aU = U$, (1.9) tells us that the last tensor product vanishes. Thus this sequence becomes $0 \to I/aI \to Ra \otimes U \xrightarrow{\phi} U \to 0$. As $I/aI \neq 0$, ϕ: $Ra \otimes U \to U$ is a desired proper elongation of U. If U is standard, i.e. $\cong J/I$, then ϕ: $J/J(I:J) \to U$ is a proper elongation with the only change that $J/J(I:J)$ becomes finitely annihilated: $(J(I:J))^\# = J^\#$ and $JJ^\# < J$. □

Remark. It is worth while pointing out that the ideal $J(I:J)$ in (8.3) coincides with the ideal $IJ^\#$; this fact follows from (2.3). In fact, if K denotes the level of J/I, then $KI = J$, $I:J = R:K$, and $K^\# = J^\#$ imply $J(I:J) = KI(I:J) = KI(R:K) = K^\# I$ (as K is not principal).

The following corollary to the preceding lemma will be useful in subsequent considerations.

Proposition 8.4. *A n.f.a. uniserial of type* $[J/I]$ *is not elongable if and only if* $I^* = J^*$ *and* $II^* = I$.

Proof. By (8.3) a n.f.a. uniserial of type $[J/I]$ is not elongable if and only if it is equiannihilated and $J(I:J) = I$. By (1.2) and (1.3)(ii) we have $I^* = J^*$, and the claim follows from the preceding Remark. □

We are now ready to specify the division of the class of n.f.a. uniserials into six mutually disjoint classes. Intuitively, the following properties are likely to have a great deal of influence on the behavior of uniserials and are therefore chosen as the basis of division: 1. the location of the threshold submodule U°; 2. being equiannihilated or not; 3. elongable or not. This leads to twelve classes of which six are empty for the following reasons. Two classes are empty because $U^\circ = U$ implies that U cannot be equiannihilated. Two classes are empty, since U must be equiannihilated in case $U^\circ = 0$. Finally, two additional classes are empty, since by (8.3)(ii) non-equiannihilated uniserials are elongable. (9.4) will show that the remaining six classes are in general not empty. The non-empty ones are listed in the following table.

	location of U°	equiannihilation	elongable?	remarks
$\mathscr{V}_1$	$0 < U^\circ = U$	$\operatorname{Ann} U/U^\circ > \operatorname{Ann} U$	yes	strongly n.f.a.
$\mathscr{V}_2$	$0 < U^\circ < U$	$\operatorname{Ann} U/U^\circ > \operatorname{Ann} U$	yes	
$\mathscr{V}_3$	$0 < U^\circ < U$	$\operatorname{Ann} U/U^\circ = \operatorname{Ann} U$	yes	
$\mathscr{V}_4$	$0 < U^\circ < U$	$\operatorname{Ann} U/U^\circ = \operatorname{Ann} U$	no	
$\mathscr{V}_5$	$0 = U^\circ < U$	$\operatorname{Ann} U/U^\circ = \operatorname{Ann} U$	yes	barely n.f.a.
$\mathscr{V}_6$	$0 = U^\circ < U$	$\operatorname{Ann} U/U^\circ = \operatorname{Ann} U$	no	barely n.f.a.

For a n.f.a. uniserial U of type $[J/I]$, the properties listed in the third and fourth columns translate into relations between I and J; cf. (1.6)(iii), and (8.4), (1.5)(ii).

	equiannihilation	elongable	J^* related to I^*	remarks
$\mathscr{V}_1$	$J^*J < J$	$J^*I < I$	$J^* < I^*$	$\operatorname{Ann} U = J^*$
$\mathscr{V}_2$	$J^*J < J$	$J^*I < I$	$J^* < I^*$	
$\mathscr{V}_3$	$J^*J = J$	$J^*I < I$	$J^* < I^*$	
$\mathscr{V}_4$	$J^*J = J$	$J^*I = I$	$J^* = I^*$	$I \cong I^*$, $0 = U_\circ < U^\circ$
$\mathscr{V}_5$	$J^*J = J$	$J^*I < I$	$J^* = I^*$	$I \cong R_I$.
$\mathscr{V}_6$	$J^*J = J$	$J^*I = I$	$J^* = I^*$	$I^* \ncong I \ncong R_I$.

The proof of (8.1) shows that if $U_\circ < U^\circ$ then $I \cong J^*$, so $J^* = I^*$, which implies, by (1.5)(ii), that $U_\circ = 0$. Hence $U \in \mathscr{V}_4$. Conversely, if $U \in \mathscr{V}_4$ then (8.4) shows that $I^* = J^*$ and $II^* = I$, so again $U_\circ = 0$ and $I \cong J^*$ (since $U^\circ \neq 0$).

In case $J^* = I^*$, the existence of proper elongations means $I^* I < I$ which is, in view of (1.1)(v), equivalent to $I \cong R_{I^*}$.

Theorem 8.5. *We have the following inclusions:*

$$\mathscr{U}_1, \mathscr{U}_2 \subset \mathscr{V}_1; \mathscr{U}_3 \subset \mathscr{V}_2; \mathscr{U}_4 \subset \mathscr{V}_3; \mathscr{U}_5 \subset \mathscr{V}_4; \mathscr{U}_{64}, \mathscr{U}_{65} \subset \mathscr{V}_5; \mathscr{U}_{66} \subset \mathscr{V}_6.$$

Proof. Compare the tables above. □

9. The six classes $\mathscr{V}_i$ and Unis_K^L

Since we have gathered a fairly large amount of information about various subsemigroups in Unis R, we can tackle the problem of relating the classification of the n.f.a. uniserial modules into the six classes $\mathscr{V}_i$ ($1 \le i \le 6$) mentioned in Section 8 with the structure of the monoid Unis_L^K.

We proceed as follows. For a fixed level K (where as usual K is a submodule of Q properly containing R), we consider all the submonoids Unis_L^K of Unis^K with L ranging over the set of prime ideals of R containing K^*. We then verify that for a fixed $L > K^*$, all the n.f.a. uniserials whose isomorphy classes belong to Unis_L^K are members of the same class $\mathscr{V}_i$ for some $i \le 3$. Significantly, the index i depends only on K. Then we concentrate on the monoid $\mathrm{Unis}_{K^*}^K$ [in case $KK^* = K$] which behaves in a totally different way. Whenever $\mathrm{Gp}[K/R_{K^*}] \neq \emptyset$, $\mathrm{Unis}_{K^*}^K$ contains uniserials from all the classes $\mathscr{V}_4$, $\mathscr{V}_5$ and possibly from $\mathscr{V}_6$ (in case $G_{K^*} \neq 0$).

We start with the characterization of the levels of n.f.a. uniserial modules in the various classes $\mathscr{V}_i$.

Theorem 9.1. *Let U a n.f.a. uniserial module of type $[J/I]$ with $I < R < J$, and set $K = \mathrm{Lev}\, U$.*

(i) *If $U \in \mathscr{V}_i$ ($i \le 4$), then $K = a^{-1}J$ for some $a \notin I$.*

(ii) *If $U \in \mathscr{V}_5$, then $K = a^{-1}J$ for some $a \in I$.*

(iii) *If $U \in \mathscr{V}_6$, then $K \ncong J$. More precisely, $K = a^{-1}J^*$ for some $a \in I$ or $K = J : I$ according as $I \cong J$ or $I \ncong J$.*

Proof. This follows immediately from (2.6) once we observe that (i) $U \in \mathscr{V}_i$ for $i \le 4$ if and only if $U^\circ \neq 0$; (ii) $U \in \mathscr{V}_5$ if and only if $U^\circ = 0$ and I is principal as an R_{I^*}-module; (iii) $U \in \mathscr{V}_6$ exactly if $U^\circ = 0$ and I is not a principal R_{I^*}-module. □

Let us turn our attention to the monoids Unis_L^K for $L > K^*$. We are in a position to obtain a complete picture of the class membership of the isomorphy classes of uniserials in these semigroups.

Theorem 9.2. *Let K be a module with $R < K \leq Q$, and $L > K^*$ a prime ideal. All the uniserials in* Unis_L^K *belong to the same class* $\mathscr{V}_i$ $(i \leq 3)$.

More specifically,

(i) $\mathrm{Unis}_L^K \subset \mathscr{V}_1$ *if and only if* $R:K = K^*$;

(ii) $\mathrm{Unis}_L^K \subset \mathscr{V}_2$ *if and only if* $R:K < K^*$ *and* $KK^* < K$;

(iii) $\mathrm{Unis}_L^K \subset \mathscr{V}_3$ *if and only if* $R:K < K^*$ *and* $KK^* = K$.

Proof. $L > K^*$ implies that Unis_L^K exists. If $U \in \mathrm{Unis}_L^K$ is of type $[J/I]$, then by (2.3)(iv) $I^* = L$ implies $I^* > J^*$, which is equivalent to $U \in \mathscr{V}_i$ for some $i \leq 3$ (see last table in Section 8).

We always have $A = I:J = R:K$ (cf. (2.4)(vii)) and $R:K \leq K^*$ (since $R < K$ and (1.1)(iv)). Because of (1.8)(i), U is strongly n.f.a. if and only if $A = J^* = K^*$. Hence $U \in \mathscr{V}_1$ if and only if $R:K = K^*$.

In the remaining cases $R:K < K^*$. By (1.6)(iii) U is equiannihilated if and only if $JJ^* = J$. From (2.3)(ii)–(iii) we can easily verify that this is equivalent to $KK^* = K$. Hence (ii) and (iii) are obvious. □

It remains to examine the monoid $\mathrm{Unis}_{K^*}^K$ which exists exactly if $KK^* = K$.

Theorem 9.3. *Let K be a submodule of Q such that $R < K < Q$ and $KK^* = K$. Then*

(i) $\mathrm{Gp}[K/K^*] \subset \mathscr{V}_4$;

(ii) $\mathrm{Gp}[K/R_{K^*}] \subset \mathscr{V}_5$;

(iii) $\mathrm{Unis}[KI/I] \subset \mathscr{V}_6$ *for all ideals I such that* $K^* = I^* \gneqq I \gneqq R_{I^*}$.

Proof. Let $U \in \mathrm{Unis}_{K^*}^K$ be a uniserial module of type $[J/I]$. $U_\# = K^* = U^*$ implies that U belongs to $\mathscr{V}_i$ for some $i = 4, 5, 6$.

The last table in Section 8 shows that $U \in \mathscr{V}_4$ implies $I \cong I^*$. In the present situation, this means $t(U) = [K/K^*]$, i.e. $[U] \in \mathrm{Gp}[K/K^*]$. If $U \in \mathscr{V}_5$, then $I \cong R_{I^*}$. Hence $[U] \in \mathrm{Gp}[K/R_{K^*}]$ and the claim in (ii) is evident. Finally, $U \in \mathscr{V}_6$ must represent the remaining case in which $[U] \in \mathrm{Unis}[KI/I]$ for an ideal I such that $K^* = I^* \gneqq I \gneqq R_{I^*}$. □

Evidently, a uniserial in $\mathrm{Gp}[K/K^*]$ maps homomorphically onto a uniserial in $\mathrm{Gp}[K/R_{K^*}]$. Hence modules in $\mathscr{V}_5$ are quotients of modules in $\mathscr{V}_4$, and modules in $\mathscr{V}_4$ have elongations in $\mathscr{V}_5$.

Finally, we examine when the classes $\mathscr{V}_i$ are not empty. For uniserials in the first four classes, we can restrict our consideration to the existence of uniserials with principal annihilators (since $U^\circ \neq 0$).

$\mathscr{V}_1$ is never empty: it contains all divisible uniserials (their types are $[Q/R]$).

$\mathscr{V}_2$ is not empty if and only if there exists a nonzero prime $L < P$; the type is necessarily $[s^{-1}R_L/R]$ with $s \in L$.

$\mathscr{V}_3$ is not empty exactly if there exists a nonzero idempotent prime $L < P$. A possible type is $[s^{-1}L/R]$ with $s \in L$.

$\mathscr{V}_4$ and $\mathscr{V}_5$ are not empty if and only if there exists an idempotent nonzero prime L. In these cases, possible types are $[s^{-1}L/L]$ and $[s^{-1}L/R_L]$, respectively.

$\mathscr{V}_6$ is not empty exactly if there exists an idempotent nonzero prime L for which $\mathbf{G_L}$ is a non-trivial group (equivalently, the quotient group of the value group of R modulo the convex subgroup associated with the prime L is incomplete).

To sum up:

Proposition 9.4. *For a valuation domain R, each of the classes $\mathscr{V}_i$ $(i \leq 5)$ is non-empty if and only if there exists a nonzero idempotent prime ideal L different from P. $\mathscr{V}_6$ is non-empty exactly if the value group of R_L is not complete for some nonzero idempotent prime ideal L.* □

Examples. 1) If R is a discrete valuation domain, then $\mathscr{V}_1$ is the only not empty class.

2) R archimedean not discrete implies that $\mathscr{V}_1$, $\mathscr{V}_4$ and $\mathscr{V}_5$ are not empty, while $\mathscr{V}_2$, $\mathscr{V}_3$ are empty; $\mathscr{V}_6$ is not empty precisely if the value group is isomorphic to a proper subgroup of the reals.

3) R discrete of rank $n > 1$ implies that $\mathscr{V}_1$, $\mathscr{V}_2$ are not empty, while $\mathscr{V}_3$, $\mathscr{V}_4$, $\mathscr{V}_5$, and $\mathscr{V}_6$ are empty.

References

[B] Bazzoni, S: Archimedean ideals of valuation domains. Comm. Algebra **18** (1990), 3613–3618

[BS1] Bazzoni, S., Salce, L.: On non-standard uniserial modules over valuation domains and their quotients. J. Algebra **128** (1990), 292–305

[BS2] Bazzoni, S., Salce, L.: Elongations of uniserial modules over valuation domains. J. Algebra **146** (1992), 304–317

[BS3] Bazzoni, S., Salce, L.: Equimorphism classes of uniserial modules over valuation domains. Arch. Math. **58** (1992), 14–23

[CE] Cartan, H., Eilenberg, S.: Homological Algebra. Princeton University Press 1956

[CP] Clifford, A.H., Preston, G.B.: The algebraic theory of semigroups. Math. Surveys, Nr. 7, Amer. Math. Soc., Providence 1967

[E] Eklof, P.: A transfer theorem for non-standard uniserials. Proc. Amer. Math. Soc. **114** (1992), 593–600

[ES] Eklof, P., Shelah, S.: On a conjecture regarding non-standard uniserial modules. Trans. Amer. Math. Soc. **340** (1993), 337–351.

[FG] Franzen, B., Göbel, R.: Nonstandard uniserial modules over valuation domains. Results in Math. **12** (1987), 86–94

[F] Fuchs, L.: On a semigroup with operation "Tor". Proceedings of the 1986 Louisiana State University Semigroup Conference, 44–45

[FS] Fuchs, L., Salce, L.: Modules over Valuation Domains. Lecture Notes in Pure Appl. Math. 97. Marcel Dekker 1985

[FSh] Fuchs, L., Shelah, S.: Divisible uniserial torsion modules over valuation domains. Preprint, 1987

[O1] Osofsky, B.: A construction of non-standard uniserial modules over valuation domains. Bull. Amer. Math. Soc. **25** (1991), 89–97

[O2] Osofsky, B.: Constructing non-standard uniserial modules over valuation domains. Proceedings of the Azumaya Conference (Bloomington), May 1990. Contemporary Math. **124** (1992), 151–164

[S] Shelah, S.: Non-standard uniserial modules over a uniserial domain exist. Around Classification Theory of Models, Lecture Notes in Math. 1182. Springer, 1986, 135–150

[SL] Shores, T.S., Lewis, W.J.: Serial modules and endomorphism rings. Duke Math. J. **41** (1974), 889–909

[So] Soileau, P.L.: The tensor and torsion products of modules over valuation domains. Ph. D. Thesis, Tulane University, 1984

Received April 27, 1993

Silvana Bazzoni and Luigi Salce, Dipartimento di Matematica Pura e Applicata, Università di Padova, 35131 Padova, Via Belzoni 7, Italy

Laszlo Fuchs, Department of Mathematics, Tulane University, New Orleans, Louisiana 70118, USA