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The graph p-Laplacian eigenvalue problem

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Abstract

In this thesis we discuss the graph p-Laplacian eigenvalue problem. In particular, after reviewing the state of the art, we present new results on the nodal domain count of the p-Laplacian eigenpairs, on the graph ∞-Laplacian eigenproblem, and on the computation of the p-Laplacian eigenpairs.

Concerning the nodal domain count, we prove that the number of nodal domains induced by a p-Laplacian eigenfunction can be bounded, both from above and below, in terms of the position of the corresponding eigenvalue in the variational spectrum. Moreover, we prove that on trees the variational spectrum exhausts the entire spectrum, and the number of nodal domains induced by an everywhere nonzero eigenfunction is equal to the variational index of the corresponding eigenvalue. These results allow us to derive, from the p-Laplacian spectrum, topological information about the graph. Indeed, when p is equal to 1 and ∞, the p-Laplacian eigenvalues approximate the Cheeger constants and the packing radii of the graph, respectively.

The study of the ∞-Laplacian eigenproblem is another major contribution of this thesis. In particular, we compare different formulations of this degenerate eigenproblem. In the first case we study the ∞-eigenpairs as solutions of the limiting eigenvalue equation, in the second case we define the ∞-eigenpairs as generalized critical points of the ∞-Rayleigh quotient $\mathcal{R}_\infty(f) = \|\nabla f\|_\infty / \|f\|_\infty$.

Then, we relate the ∞-variational eigenvalues to the packing radii of the graph. Here, among other things, we prove that the first and the second ∞-variational eigenvalues are exactly equal to the first and the second packing radii of the graph.

Finally, we present a novel approach to compute the p-Laplacian eigenpairs both in the smooth case $2 < p < \infty$, and in the degenerate case $p = \infty$. To this end, we observe that the p-Laplacian eigenvalue problem, both for $2 < p < \infty$ and $p = \infty$, can be reformulated as a constrained linear weighted-Laplacian eigenvalue problem. Based on this remark, we introduce a family of energy functions whose domain is the space of positive measures on the edges and on the nodes of the graph. Then, we first prove that the unique saddle point of the first energy function corresponds to the unique first eigenpair of the p-Laplacian. Second, we prove that smooth saddle points of the k-th energy function correpond to p-Laplacian eigenpairs (f, λ), such that the Morse index of the p-Rayleigh quotient $\mathcal{R}_p(f) = \|\nabla f\|_p / \|f\|_p$ in f is equal to k. Based on such results, we introduce gradient flows suited to compute saddle points of the proposed energy functions and we discuss the results of their numerical integration. Practically, the integration of the gradient flows, at each step, requires only the computation of a linear eigenpair. Hence we are able to use all the theoretical and numerical advantages of the linear setting to overcome the difficulties of solving a nonlinear equation.

However, the theoretical study of the gradient flows remains an open problem, which deserves a future in-depth study.

Contents

1 Introduction

In this thesis we review the graph p-Laplacian eigenvalue problem and present our contributions to the theoretical and numerical investigation of this topic. The p-Laplacian operator arises as a natural generalization of the Laplace-Beltrami operator when one considers variational problems involving the p-norm of the gradient of an objective function $\|\nabla f\|_p$. The linear Laplacian operator corresponds to the case $p = 2$, while a nonlinear p-Laplacian operator is obtained when $p \neq 2$. The numerous applications of these kind of problems make the p-Laplacian one of the most studied nonlinear operators both in the continuous and in the discrete setting. In particular, in many applications, it has been observed numerically or theoretically that using a suitable p-norm in place of the 2-norm it is possible to achieve better results and avoid pathological situations. We mention for example the case $p \in [1, 2]$, that is of particular interest in signal processing and variational filtering strategies where given noisy data we want to denoise them by minimizing the distance from the observed signal plus a regularization term based on the p-Laplacian operator [73, 81]. For example, given a partial sample of the data, if we want to recover the entire signal promoting its smoothness as typical of semi-supervised learning, the case $p \in [2, \infty]$ becomes fundamental [36, 82]. The same case has significant applications in optimal transport problems [40, 42] and image reconstruction [37, 38]. Moreover, p-Laplacian eigenpairs find application in spectral representation and approximation theory [18] as well as unsupervised learning and graph partitioning [13, 20, 50].

Since the p-Laplacian operator is a generalization of the Laplace-Beltrami operator, its study often starts from the extension to the nonlinear case of classical results about the Laplace operator. We cite for example the extension of the maximum principle that allows one to prove the uniqueness of the solution to the p-Poisson problem [69]. Within this framework, it is natural to investigate the notion of spectrum and its applications in the non-linear case. Indeed, it is well known that the spectrum of the Laplacian is a fundamental tool in the study of solutions, exact or approximated, of many physical problems, such as the Poisson equation, the heat equation or the wave equation, just to mention some. On the other hand, it is also worth to recall the relevance of the Laplace-Beltrami operator in relation to the study of the geometry of the domain. Think for example to the Cheeger inequality [23], which bounds the smallest eigenvalue of the Laplacian by the Cheeger constant, i.e., the minimal ratio between the area of the boundary and the volume of a subdomain, among all the possible subdomains. Think also of the famous *Can one hear the shape of a drum?* problem [60], which poses he question of whether or not the spectrum of the Laplace-Beltrami operator fully characterizes the domain. This conjecture was then proved to be false [48], but in the meantime many different geometrical properties of the domain had been related to properties of the Laplacian spectrum [79].

Generalizing the notion of eigenpair to the p-Laplacian is not difficult. Recall that the Laplacian eigenpairs correspond to the critical values of the 2-Rayleigh quotient $\mathcal{R}_2(f) = \|\nabla f\|_2/\|f\|_2$, and that the p-Laplacian operator arises naturally when we consider the variation of the p-norm of the gradient. It is then natural to define the p-Laplacian eigenpairs as the critical points of a Rayleigh quotient of the form:

$$\mathcal{R}_{p,q} = \|\nabla f\|_p/\|f\|_q \,.$$

Here the norm in the denominator could, in principle, be chosen arbitrarily, however the most studied case is surely $q = p$, which we denote as $\mathcal{R}_p(f)$. The critical point condition $\partial_f \mathcal{R}_p(f) = 0$ leads to the equation:

$$\Delta_p f = \lambda |f|^{p-2} f \,, \tag{1.1}$$

where Δ_p denotes the p-Laplacian operator. The choice $p = q$ is particularly interesting because on the one hand, the Laplacian eigenpairs are recovered as a particular case and, on the other hand, the Poincaré inequality guarantees to have a "positive definite" p-Laplacian operator for any p. However, we mention that also the case $q = 2$ and $p \neq 2$, which allows us to work in the Hilbert space L^2 has received particular attention [14, 17], as well as all of the cases $p \geq q$, in which the Sobolev inequalities combined with the Poincaré inequalities allow to have a positive definite p-Laplacian operator as well [45, 56, 74]. Singular cases may arise for other choices of q, see for example [10].

Besides the intuitive definition, extending classical results from the linear case to the nonlinear one is not an easy task and many problems are still open. First of all, there is the problem of quantifying the number or eigenpairs. In the linear case, the Laplacian operator is self adjoint and we know that the cardinality of its spectrum is countable in the continuous setting, and equal to the dimension of the space in the discrete one. For the p-Laplacian, instead, such information is unknown and the countability or finiteness of its spectrum are open problems. It is, however, quite easy to show that in the discrete setting the number of eigenvalues of the p-Laplacian can exceed the dimension of the space [2, 90], see also Chapter 1. Another major problem concern the properties of the eigenfunctions. In fact, in the linear case we know that it is always possible to extract a base of orthogonal eigenfunctions and that, the multiplicity of an eigenvalue equals the dimension of the corresponding eigenspace. This is fundamental information when we are interested in decomposing or approximating a signal with some or all of its frequencies. Unfortunately, these properties are again lost when we consider the nonlinear p-Laplacian. Indeed, the fact that the cardinality of the spectrum is unknown rules out the possibility of defining an algebraic multiplicity, while, the loss of linearity implies the existence of non orthogonal eigenfunctions and multiple, but not necessarily infinite, eigenfunctions with the same eigenvalue [2], implying that also a notion of geometrical multiplicity is not well defined.

These problems are partially overcome by the definition of the variational eigenvalues, i.e., a family of eigenvalues whose cardinality, depending on the setting, equals the dimension of the space (discrete setting) or is countable (continuous setting). There are several ways to define such eigenvalues, the most classical one is based on the Lusternik-Schnierelman theory and the notion of Krasnoselskii genus [46, 47, 85]. In this case, the strategy is to define a generalized notion of dimension, the genus, for symmetric subsets and then, similarly to the linear case, to prove that

$$\lambda_k(\Delta_p) = \min_{genus(A)\geq k} \max_{f\in A} \mathcal{R}_p(f)$$

is a critical value of the p-Rayleigh quotient $\mathcal{R}_p$, for any k. A relevant advantage of these eigenvalues is the possibility to define a notion of algebraic multiplicity, i.e., the number of times that an eigenvalue is repeated in the variational sequence. If a variational eigenvalue, λ, has multiplicity k, then there exists a subset of genus greater than k of eigenfunctions associated to λ, i.e. a kind of eigenspace. Moreover, the existence of a subset with genus greater than k, implies the existence of at least k linearly independent eigenfunctions associated to λ [85].

Surprisingly, the introduction of the variational eigenvalues allows one, not only to recover but also to improve the relationships between some geometrical properties of the domain and the spectrum of the Laplacian. This is the case for example for the Cheeger inequality, which becomes an equality when p goes to one, i.e. the limit of the first non zero eigenvalue of the p-Laplacian operator when p goes to 1 is equal to the Cheeger constant of the domain. This result holds both in the continuous setting [62, 76] and in the graph one [13, 20]. Similarly, if we consider the limit, for p that goes to one, of higher variational eigenvalues of the graph p-Laplacian, it is possible to show that

$$\lim_{p\to 1} \lambda_k(\Delta_p) \leq h_k(\mathcal{G}),$$

where $h_k(\mathcal{G})$ is the higher order Cheeger constant of index k, [86].

Another example comes from the study of the ∞-limit of the variational p-Laplacian eigenvalues. In this case it is possible to prove that the first and the second eigenvalue of the p-Laplacian converge respectively toward the reciprocal of the radius of the maximal ball which can be inscribed in domain and to the reciprocal of the maximal radius which allows us to inscribe two disjoint balls in the domain [57, 58], see also Chapter 5. Moreover, again considering the limit of higher variational eigenvalues, it is possible to show that

$$\lim_{p\to\infty} \lambda_k(\Delta_p) \leq 1/R_k,$$

where R_k is the packing radius of order k [49], i.e., the maximal radius that allows to inscribe k disjoint balls in the domain, see [57, 58] for the details.

In the study of the above relations of the p-Laplacian eigenvalues with the higher order Cheeger constants and the radii of the domain it is worth to mention the relevance of the study of the nodal domains. A nodal domain induced by an eigenfunction, f, is one of the maximal subdomains where f is strictly positive or negative. The relevance of the nodal domains is twofold. First of all, observe that if we have an eigenpair (f, λ) on a domain Ω with homogeneous Dirichlet boundary conditions, and a nodal domain A induced by f, then it is easily observed that $(f|_A, \lambda)$ is an eigenpair on the domain A still with homogeneous Dirichlet boundary conditions. This means that the nodal domains induced by some eigenfunction f are subdomains of Ω that share the same eigenvalue λ induced by f on Ω. Using this property, if we denote by $\mathcal{N}(f)$ the number of nodal domains induced by a function f it is possible to provide a lower bound for the limit of the Δ_p-eigenvalues:

$$h_{\mathcal{N}(f_1)}(\mathcal{G}) \leq \lim_{p\to 1} \lambda_k(\Delta_p) \leq h_k(\mathcal{G})\,, \qquad 1/R_{\mathcal{N}(f_\infty)} \leq \lim_{p\to\infty} \lambda_k(\Delta_p) \leq 1/R_k, \quad (1.2)$$

where f_1 and f_∞ are proper limits of any eigenfunction of $\lambda_k(\Delta_p)$. On the other hand, the number of nodal domains induced by an eigenfunction is somehow capable to reproduce the frequency (index) of the corresponding eigenvalue.

This has been originally observed for the Laplacian operator in Sturm's oscillation theorem that states that the zeros of the k-th mode of vibration of an oscillating string induce k nodal domains. Later, Courant extended this result to higher dimensions, proving that the k-th eigenfunction of an oscillating membrane admits no more than k subdomains [26]. The count of the nodal domains of the Laplacian operator in the discrete graph setting has then lead to observe that trees behave like strings, i.e. an eigenvector f_k for the k-th eigenvalue, if everywhere non-zero, induces exactly k nodal domains [5, 7]. In addition, in the general case of eigenvalues with any multiplicity and eigenvectors with possibly some zero entry on general graphs, it was proved in [30, 34, 87] that the following inequality holds for the number of nodal domains induced by any eigenfunction of the k-th eigenvalue of the Laplacian operator:

$$k + r - 1 - \beta - z \leq \mathcal{N}(f_k) \leq k + r - 1\,,$$

where β is the total number of independent loops of the graph, z is the number of zeros of f_k and r is the multiplicity of λ_k.

The main contribution of Chapter 3 of this thesis is to show that the above result holds almost unchanged for the nodal domains of the eigenfunctions of the p-Laplacian operator, for any $p > 1$. An upper bound was provided in [86], where it is shown that, for any eigenfunction f_k of the k-th variational eigenvalues of the p-Laplacian, the number of nodal domains is bounded above as

$$\mathcal{N}(f_k) \leq k + r - 1,$$

where r is the variational multiplicity of the corresponding eigenvalue. In this thesis, instead, we consider the case of trees, proving analogous results to the linear case, and propose a lower bound for general graphs that states that if f is a p-Laplacian eigenfunction with eigenvalue λ such that $\lambda > \lambda_k(\Delta_p)$, then

$$\mathcal{N}(f) \geq k - \beta - z + 1\,,$$

where β is the total number of independent loops of the graph and z is the number of zeros of f. These bounds, combined with the inequalities (1.2), lead to interesting inequalties between the p-Laplacian eigenvalues and the geometrical quantities R_k and h_k.

Considering the above discussion and in view of (1.2), it is natural to futher focus the attention on the investigation of the 1-Laplacian and the ∞-Laplacian eigenvalue problems. However the definition of eigenpairs as critical values/points does not transfer directly to the 1- and ∞-case, due to the lack of differentiability of the corresponding Rayleigh quotients. The approaches used to overcome this difficulty have been various. In [57, 58], the authors study the ∞-eigenpairs defined as the solutions to an ∞-limit eigenvalue equation. In [50], studying the 1-Laplacian eigenvalue problem, the authors propose the definition of a 1-eigenfunction as a Clarke critical point of $\mathcal{R}_1$, i.e., an f such that

$$0 \in \partial^{Cl}\mathcal{R}_1(f), \tag{1.3}$$

where $\partial^{Cl}\mathcal{R}_1(f)$ is the Clarke subgradient of the locally Lipschitz fucntion $\mathcal{R}_1$ [25]. Finally, investigating the graph 1-eigenvalue problem, the author of [20] proposes the definition of a 1-eigenfunction, f, as a function that satisfies:

$$0 \in \partial\|\nabla f\|_1 \cap \bigcup_{\lambda\geq 0} \lambda\partial\|f\|_1, \tag{1.4}$$

where $\partial\|\nabla f\|_1$ and $\partial\|f\|_1$ are the subgradients of the two convex functions $f \mapsto \|\nabla f\|_1$ and $f \mapsto \|f\|_1$ [35, 80]. It is worth noting that the formulation (1.4) is a generalization of the one in (1.3). In fact, it follows directly from the properties of the Clarke subdifferential that any solution to (1.3) solves (1.4). However, examples exist of functions that solve (1.4) but not (1.3), see [90] for an example. This reformulation has then been generalized to general functions on convex polytopes in [22]. Interestingly, the formulation (1.4) admits a useful geometrical interpretation. Indeed, if f solves (1.4) with $\|f\|_1 = 1$, then $\partial\|\nabla f\|_1$, which generalizes the definition of $\Delta_p f$ to the case $p = 1$, intersects the normal outward cone to the piecewise-linear manifold $S_1 = \{f \mid \|f\|_1 = 1\}$, i.e. $\bigcup_{\lambda\geq 0} \lambda\partial\|f\|_1$. Observe that, when $p > 1$, the normal cone to $S_p = \{f \mid \|f\|_p = 1\}$ reduces to $\bigcup_{\lambda\geq 0} \lambda|f|^{p-2}f$, that immediately shows that equation (1.4) actually generalizes (1.1).

Based on the approaches above, in Chapter 5 we consider ∞-Laplacian eigenpairs on graphs and show, among other things, that the inequalities in (1.2) hold true when $\lim_{p\to\infty} \lambda_k(\Delta_p)$ is replaced by the properly defined variational ∞-eigenpairs. Analogous results were previously shown for the graph 1-Laplacian [20, 50] and for the first ∞-eignvalue, $\lambda_1(\Delta_\infty)$, in both the graph and the continuous setting [16, 17].

Given the remarkable theoretical spectral properties of the p-Laplacian, a critical issue that arises is the problem of how to compute p-Laplacian eigenpairs and where to locate a computed eigenvalue with respect to the variational spectrum. Both are still partially open problems that have been faced by different authors in recent years [14, 50, 88]. The majority of the existing methods, such as the inverse nonlinear power method [50] or gradient flow methods for the functional $\mathcal{R}_p$ [14], are mainly suited to compute extremal eigenvalues, i.e. the maximal or the minimal eigenvalue. From a theoretical point of view, these methods have the advantage that it is always possible to prove the convergence, but the drawback that the convergence toward an extremal eigenpair depends on the initial point and thus can be guaranteed only a-posteriori. Other methods, like the local minmax method [88], are suited to compute families of nonlinear eigenpairs, $\{(f_i, \lambda_i)\}_{i=1}^k$, with increasing local minmax index, i.e. number of decreasing directions of the functional $\mathcal{R}_p$ in f_i. However, to compute the k-th eigenvalue, these methods need to know the first $k-1$ eigenpairs and are suited to compute an eigenpair, (f, λ), if and only if λ is a local maximum of $\mathcal{R}_p$ in the space spanned by f and the previously computed $f_1, \dots, f_{k-1}$. Note that the latter is a nontrivial condition in the case of the p-Laplacian and, in principle, could never be satisfied. The theory about how to compute nonlinear eigenpairs is thus still far from being complete. We face this and other problems in the next chapters. In chapter 4 we contribute to this challenging line of work with a novel approach based on a suitable linearization of the p-Laplacian eigenproblem inspired by recent gradient flow formulations of the Monge-Kantorovich optimal transport equation [41, 42, 43].

We divide the reminder of this thesis as follows:

- Chapter 2. In this chapter we fix the notation and review the state of the art, settling in a rigorous mathematical framework the results discussed in this introduction. At the end of this chapter we present also a brief but precise description of our contributions and main results, which are contained in the next three chapters.

- Chapter 3 is devoted to a discussion on the nodal domains of the p-Laplacian eigenfunctions. Here we face the case of trees and provide novel lower bounds for the number of nodal domains induced by the p-eigenfunctions in terms of the position of the eigenvalue in the variational spectrum.

- Chapter 4 is devoted to a reformulation of the p-Laplacian eigenvalue problem in terms of a constrained linear weighted Laplacian eigenvalue problem. On the one hand, this allows us to characterize a family of p-eigenparis in terms of the critical points of a family of energy functions, and on the other it allows to introduce new numerical methods for the computation of the p-Laplacian eigenpairs.

- Chapter 5 is devoted to the study of the graph ∞-eigenvalue problem. Here we compare different formulations of the ∞-eigenvalue problem, in terms of subgradients and in terms of ∞-limit eigenvalue equation. We recover results known in the continuous setting that relate the ∞-variational eigenvalues to the readii of the graph and finally we extend to the ∞-Laplacian case results proved in Chapter 4 for the case $p \in (2, \infty)$.

We point out that every chapter can be of independent interest, even though the various results are certainly interconnected. For this reason, we chose to write the chapters as self-consistent as possible. In the introduction of every chapter, we recall the basic notation and results (from the literature and the other chapters) needed for the reading and comprehension of the chapter itself.

2 Notation and Preliminaries

We devote this chapter to establish the notation and summarize the state of the art in the study of the spectral properties the graph p-Laplacian. We conclude with a short but complete summary of our main contributions.

2.1 The Graph Setting

An undirected graph, $\mathcal{G}$, can be denoted by a triple $\mathcal{G} := (V, E, \omega)$, where V is the discrete set of nodes (or vertices) of the graph, $E \subset V \times V$ denotes the set of edges and is such that if $(u, v) \in E$ then also $(v, u) \in E$, and finally $\omega : E \to \mathbb{R}$ is a function on the edges such that $\omega(u, v) = \omega(v, u)$. The value of $\omega(u, v)$ can be thought of as representing the reciprocal of the edge length. To simplify the notation, in the following we will often write $\omega_{uv} := \omega(u, v)$ Using these definitions, we can introduce a distance between two nodes u and v of the graph defined as the length of the shortest path joining them:

$$d(u, v) = \min_{\Gamma \in \mathrm{path}_{u,v}} \mathrm{length}(\Gamma)\,,$$

where $\mathrm{path}_{u,v}$ denotes the set of paths joining u to v:

$$\mathrm{path}_{u,v} = \big\{\Gamma = \{u_i\}_{i=1}^n | u_1 = u,\ u_n = v,\ (u_i, u_{i+1}) \in E,\ n \text{ arbitrary}\big\}$$

and the length of a path $\Gamma = \{u_i\}_{i=1}^n$ is defined as

$$\sum_{i=1}^{n-1} \frac{1}{\omega(u_i, u_{i+1})}\,.$$

Denote by $\mathcal{H}(V)$ and $\mathcal{H}(E)$ the Hilbert spaces of the functions on the nodes and on the edges of the graph, respectively, endowed with the scalar products:

$$\langle f, g\rangle_{\mathcal{H}(V)} = \sum_{u \in V} f(u)g(u) \qquad \langle F, G\rangle_{\mathcal{H}(E)} = \frac{1}{2}\sum_{uv \in E} F(u, v)G(u, v)\,.$$

We can then introduce the graph equivalent of the differential operators used in the continuous setting. Let us start with the gradient of a function in $\mathcal{H}(V)$

defined as the function that reproduces the slope of f on the edges:

$$\begin{aligned}\nabla :\ \mathcal{H}(V) &\longrightarrow \mathcal{H}(E)\\ f\ &\longrightarrow \nabla f(u,v) = \omega_{uv}(f(v) - f(u))\,,\end{aligned}$$

with u and v being vertices of the edge (u, v) and with the obvious property that $\nabla f(u, v) = -\nabla f(v, u)$. Next we introduce the divergence operator. Not considering a boundary on graphs is usually understood to be analogous to having homogeneous Neumann boundary conditions. Thus, to preserve the classical divergence theorem in the continuous setting, i.e.

$$-\langle f, \operatorname{div} G\,\rangle_{\mathcal{H}(V)} = \langle \nabla f, G\rangle_{\mathcal{H}(E)}\,,$$

we may define the divergence as the half of minus the adjoint of the gradient, that in matrix form reads $\operatorname{div} = -\frac{1}{2}\nabla^T$, i.e.

$$\begin{aligned}\operatorname{div} :\ \mathcal{H}(E) &\longrightarrow \mathcal{H}(V)\\ G\ &\longrightarrow \operatorname{div} G(u) = \frac{1}{2}\sum_{v\sim u}\omega_{uv}\big(G(u,v) - G(v,u)\big)\,,\end{aligned}$$

where $\{v\,|\,v \sim u\}$ are the nodes connected to the node u by an edge, i.e. such that $(u, v) \in E$. Given the definitions of gradient and divergence we can introduce the graph Laplacian operator $(p = 2)$ and the more general p-Laplacian operator $\big(p \in (1,\infty)\big)$, whose definitions are similar to the one used in the continuous setting:

$$\Delta_p f(u) = -\operatorname{div}\big(|\nabla f|^{p-2} \odot \nabla f\big)(u) = \sum_{v\sim u}\omega_{uv}|\nabla f(v,u)|^{p-2}\nabla f(v,u)\,,$$

where $|\nabla f|^{p-2}$ has to be understood entrywise and $\odot$ denotes the entrywise product (we will omit this symbol in the following). When $p = 2$, it is possible to check that such definition matches the classical definition of Δ_2 in terms of the adjacency matrix. Consider A, the weighted adjacency matrix of the graph, i.e. $A_{uv} = \omega_{uv}\mathbb{1}_E\big((u,v)\big)$, where $\mathbb{1}$ denotes the indicator function, then a direct computation shows that

$$\Delta_2 = \operatorname{diag}(A * \mathbf{1}) - A\,,$$

where $\mathbf{1}$ denotes the vector entrywise equal to one.

Boundary case. Finally, observe that sometimes we are interested in studying problems on graphs with some boundary conditions, in this case we define the boundary of the graph, $B \subset V$, as a subset of the nodes. Then, if we impose homogeneous boundary conditions, we set

$$\mathcal{H}_0(V) := \{f : V \setminus B \to \mathbb{R}\} \cong \{f : V \to \mathbb{R} \mid f(u) = 0\ \forall u \in B\}\,,$$

and we consider the gradient operator

$$\begin{aligned} \nabla : \mathcal{H}_0(V) &\longrightarrow \mathcal{H}(E) \\ f \mapsto \nabla f(u,v) &= \begin{cases} \omega_{uv}\big(f(v) - f(u)\big) & \text{if } u,v \in V \setminus B \\ \omega_{uv} f(v) & \text{if } u \in B,\ v \in V \setminus B \\ -\omega_{uv} f(u) & \text{if } u \in V \setminus B,\ v \in B \end{cases} . \end{aligned}$$

As before, we introduce the divergence operator, div : $\mathcal{H}(E) \to \mathcal{H}_0(V)$, in such a way to preserve the divergence theorem, i.e., $-\mathrm{div} = \frac{1}{2}\nabla^T$.

2.2 p-Laplacian eigenvalue problem

Now we can introduce the p-Laplacian eigenvalue problem. Mimicking the continuous setting, we consider the Rayleigh quotient:

$$\mathcal{R}_p(f) = \frac{\|\nabla f\|_p}{\|f\|_p} = \frac{\Big(\frac{1}{2}\sum_{(u,v)\in E} |\nabla f(u,v)|^p\Big)^{\frac{1}{p}}}{\Big(\sum_{u\in V} |f(u)|^p\Big)^{\frac{1}{p}}} ,$$

whose critical point equation for $p \in (1,\infty)$, reads:

$$\Delta_p f(u) = \big(\mathcal{R}_p(f)\big)^p |f(u)|^{p-2} f(u) \quad \forall u \in V ,$$

up to rescaling. We thus define (f,λ) to be a p-Laplacian eigenapair iff

$$\Delta_p f(u) = \lambda |f(u)|^{p-2} f(u) \quad \forall u \in V . \tag{2.1}$$

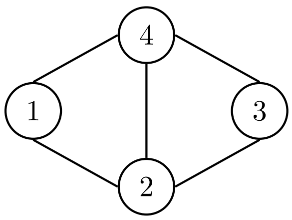
Multiplying the above equation by $f(u)$ and then summing over the vertices shows that if λ is an eigenvalue corresponding to the eigenfunction f, necessarily $\lambda = \big(\mathcal{R}_p(f)\big)^p$.

Remark 2.2.1. *In more generality, we mention that the Hilbert spaces $\mathcal{H}(E)$ and $\mathcal{H}(V)$ can be provided each one of a measure, $\mu : E \to \mathbb{R}$ and $\nu : V \to \mathbb{R}$, that produce the norms*

$$\|f\|^p_{\nu,p} = \sum_{u\in V} \nu_u |f(u)|^p \qquad \|G\|^p_{\mu,p} = \frac{1}{2}\sum_{uv\in E} \mu_{uv} |G(u,v)|^p .$$

In this case, differentianting the Rayleigh quotient $\mathcal{R}_{p,\mu,\nu}(f) = \|\nabla f\|_{p,\mu}/\|f\|_{p,\nu}$ we derive the (weighted) p-Laplacian eigenvalue equation

$$\Delta_{p,\mu} f(u) = -\mathrm{div}\big(\mu|\nabla f|^{p-2}\nabla f\big)(u) = \lambda \nu_u |f(u)|^{p-2} f(u) \quad \forall u \in V .$$



1. $f_1 = (1,1,1,1), \quad \lambda_1 = 0$
2. $f_2 = (1,0,-1,0), \quad \lambda_2 = 2$
3. $f_3 = (0,1,0,-1), \quad \lambda_3 = 2 + 2^{p-1}$
4. $f_4 = (1,0,1,-2^{\frac{1}{p-1}}),\ (1,-2^{\frac{1}{p-1}},1,0)$
   $\lambda_4 = 1 + \big(1 + 2^{\frac{1}{p-1}}\big)^{p-1}$
5. $f_5 = (1,-1,1,-1), \quad \lambda_5 = 2^p$

Figure 2.1: Left: Example graph in which the corresponding p-Laplacian Δ_p with $\omega_{uv} = 1\ \forall (u,v) \in E$, has more eigenvalues then the dimesion of the space. Right: Set of five eigenvalues and corresponding eigenfunctions.

Next, we would like to highlight, that differently from the linear case $p = 2$ where Δ_2 is a symmetric positve semidefinite matrix, for a generic p the number of p-Laplacian eigenpairs can be greater than the dimension of the space, i.e $|V|$, the eigenpairs are not in general orthogonal, and there is no clear notion of eigenspace or multiplicity, see Figure 2.1. We will come back to this topic in Chapter 3. Morover, we refer to [2, 90] for other examples and discussions on the problem. In particular, we mention that in [2] the author computes all the eigenvalues of Δ_p on complete graphs, showing that their number is equal to $\lfloor N/2 \rfloor (N - \lfloor N/2 \rfloor) + 1$ where N is the cardinality of the nodes. The finiteness of the p-Laplacian spectrum on general graphs as well as the existence of upper bounds for the cardinality of the spectrum remain open problems.

Despite all the difficulties highlighted so far, using some classical results from calculus of variations it is always possible to characterize a set of "variational" eigenvalues whose number, counted with their multiplicity, equal in number the dimension of the space, $|V|$. To define such eigenpairs, we observe that, because of the homogeneity of the Rayleigh quotient $\mathcal{R}_p$, we can restrict the study of its critical points to the p-sphere, $S_p := \{f \in \mathcal{H}(V) \,|\, \|f\|_p = 1\}$. Having $\mathcal{R}_p^c := \{f \mid \mathcal{R}_p(f) < c\}$, we consider the following Deformation Lemma and its direct consequence given in Theorem 2.2.3 (they are particular cases of more general and classic results, see e.g. [46, 47, 75, 85]).

Lemma 2.2.2 (Deformation Lemma)**.** *Assume c to be a regular value of $\mathcal{R}_p$, then there exist $\epsilon > 0$ and a continuous family of deformations $\phi \in C([0,1] \times S_p, S_p)$ such that $\phi(t,f) = -\phi(t,-f)\ \forall (t,f)$, $\phi(1, \mathcal{R}_p^{c+\epsilon}) \subset \mathcal{R}_p^{c-\epsilon}$, and $\phi(0,f) = f$.*

Proof. We give here a sketch of the proof that is quite intuitive. Consider a neighborhood B of $\{f \mid \mathcal{R}_p(f) = c\}$ without critical points, a cutoff function $\xi(f)$ that is zero outside B, and the projection of the gradient of $\mathcal{R}_p(f)$ on the tangent space of S_p, denoted, with a small abuse of notation, by $\frac{\partial}{\partial f}\mathcal{R}_p(\cdot)$. Finally, define

$\phi(t, f)$ as the solution to the gradient flow

$$\begin{cases} \frac{\partial}{\partial t}\phi(t, f) = -\xi(\phi(t, f))\frac{\partial}{\partial f}\mathcal{R}_p(\phi(t, f)) \\ \phi(0, f) = f \end{cases}$$

which is a continuous mapping from $[0, 1] \times S_p$ to S_p. □

Theorem 2.2.3. *Assume $\mathcal{F}$ to be a family of subsets of S_p such that for any regular value $c \in \mathbb{R}$ of $\mathcal{R}_p$, there exist $\epsilon > 0$ and a continuous deformation of the domain $\phi : [0, 1] \times S_p \to S_p$ s.t.*

$$\begin{cases} \phi : (0, \cdot) = id_{S_p}(\cdot) \\ \phi(1, \mathcal{R}_p^{c+\epsilon}) \subset \mathcal{R}_p^{c-\epsilon} \\ \phi(t, A) \in \mathcal{F}\,,\ \forall A \in \mathcal{F}\,,\ \forall t \in [0, 1] \end{cases}$$

Then

$$\Lambda := \inf_{A\in\mathcal{F}} \sup_{f\in A} \mathcal{R}_p(f)\,,$$

is a critical value of $\mathcal{R}_p$, i.e. the p-th root of an eigenvalue of Δ_p.

Proof. The proof is a direct consequence of the Deformation Lemma 2.2.2. □

Based on the above theorem, we can introduce the variational eigenapairs of the p-Laplacian. Theorem 2.2.3 states that we have to find families, $\mathcal{F}_k$, of subsets stable with respect to deformations, i.e, if $A \in \mathcal{F}_k$ and ϕ is a deformation, also $\phi(A) \in \mathcal{F}_k$. To understand how this works, recall the Fisher-Courant min max characterization of the eigenvalues of a symmetric matrix (for example the graph Laplacian) i.e.

$$\lambda_k(\Delta_2) = \min_{\dim(A)\geq k} \max_{f\in A\setminus\{0\}} \frac{\langle \Delta_2 f, f\rangle}{\langle f, f\rangle} = \min_{\dim(A)\geq k} \max_{f\in A\setminus\{0\}} \big(\mathcal{R}_2(f)\big)^2\,.$$

A possible strategy (not the unique one) to generalize this min max theorem to the nonlinear case, using Theorem 2.2.3, is based on the idea of considering a generalized notion of dimension, the Krasnoselskii genus, that is related to the Lyusternik–Schnirelmann category of a space [46, 47, 85]. First of all, observe that, as we are interested in studying critical points of $\mathcal{R}_p$, which is an even functional, it would be enough to generalize the notion of dimension to the symmetric subsets. Thus we introduce the family $\mathcal{A}$ of subsets of $\mathbb{R}^n$ that are symmetric and closed, i.e.:

$$\mathcal{A} = \{A \subseteq \mathbb{R}^n |\ A \ \text{closed}\,,\ \ A = -A\}\,.$$

Then, we observe that in the case A is a linear subspace of dimension k, $A \setminus \{0\}$ can be retracted with continuity on a sphere of dimension $k - 1$, S^{k-1}. This notion can be generalized by defining, for any $A \in \mathcal{A}$, the Krasnoselskii genus of A:

$$\gamma(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ \inf\{k \in \mathbb{N} \mid & \exists\, \psi \in C(A, S^{k-1}) \ \ s.t. \ \psi(x) = -\psi(-x)\} \\ +\infty & \text{if } \nexists\, k \text{ as above} \end{cases}$$

Note that, if $\gamma(A) \geq k$ and $\phi \in C(\mathbb{R}^n, \mathbb{R}^n)$, then $\gamma(\phi(A)) \geq \gamma(A)$. Hence, the families $\mathcal{F}_k(S_p) = \{A \subseteq \mathcal{A} \cap S_p \,|\, \gamma(A) \geq k\}$ satisfy the hypotheses of Theorem 2.2.3. Thus, we can define the Krasnoselskii variational eigenvalues of Δ_p as

$$\lambda_k = \inf_{A \in \mathcal{F}_k} \sup_{f \in A} \big(\mathcal{R}_p(f)\big)^p . \tag{2.2}$$

Since we are working in a finite dimensional space, it is also possible to prove that the above inf sup is actually a min max (see also Chapter 4). The advantage of defining these eigenvalues is twofold. On the one hand it allows us to select a number of eigenvalues that equals the dimension of the space. On the other hand, thanks to the properties of the Krasnoselskii genus, we can recover some multiplicity results. In particular, we recall the following results from [85]:

Lemma 2.2.4. *(See Lemma 5.6 Chapter II [85]). Suppose for some k, m there holds*

$$-\infty < \lambda = \lambda_k = \cdots = \lambda_{k+m-1} < \infty .$$

Then, $\gamma(K_\lambda) \geq m$, where $K_\lambda = \{f \,|\, \mathcal{R}_p(f) = \lambda ,\ \partial_f \mathcal{R}_p(f) = 0\}$ is the set of critical points associated to λ.

Proposition 2.2.5. *(See Proposiiton 5.3 Chapter II [85]). Suppose $A \subset V$ is a compact symmetric subset of a Hilbert space V and suppose $\gamma(A) = m < \infty$. Then A contains at least m mutually orthogonal vectors $\{v_i\}_{i=1}^m$.*

The last two results show that if a variational eigenvalue, λ, has multiplicity m, then there exists a sort of eigenspace associated to it, whose genus (intended as its generalized dimension) is lower bounded by the multiplicity. In turn, this implies that there exist at least m mutually orthogonal eigenvectors associated to λ.

Boundary case. We conclude this paragraph by recalling that the p-Laplacian eigenvalue problem can also be studied on a graph with a boundary, B, and homogeneous Dirichlet boundary conditions (see section 2.1).

Then, the p-Rayleight quotient of a function in $\mathcal{H}_0(V)$, i.e. which is zero on the boundary B, reads

$$\mathcal{R}_p^p(f) = \frac{2^{-1} \sum_{\substack{(u,v) \in E \\ u,v \in V \setminus B}} \omega_{uv}^p |f(u) - f(v)|^p + \sum_{\substack{(u,v) \in E \\ u \in V \setminus B,\, v \in B}} \omega_{uv}^p |f(u)|^p}{\sum_{u \in V \setminus B} |f(u)|^p}, \quad f \in \mathcal{H}_0(V)$$

and its critical point equation, i.e., the p-Laplacian eigenvalue problem, can be written as:

$$\begin{cases} \Delta_p f(u) = \lambda |f(u)|^{p-2} f(u) & \forall u \in V \setminus B \\ f(u) = 0 & \forall u \in B \end{cases}$$

where given $u \in V \setminus B$,

$$\Delta_p f(u) = \sum_{\substack{v \sim u \\ v \in V \setminus B}} \omega_{uv}^p |f(u) - f(v)|^p + \sum_{\substack{v \sim u \\ v \in B}} \omega_{uv}^p |f(u)|^p \,. \tag{2.3}$$

Clearly, also in this case, considering the symmetric subsets of $\mathcal{H}_0(V) \cap S_p$ whose genus is greater than k, we can introduce the k-th variational eigenvalue. In particular, note that $\lambda_1(\Delta_p) = 0$ in the case we have no boundary conditions, as $\lambda_1(\Delta_p) = \min_f (\mathcal{R}_p(f))^p = 0$ which is obtained on nodewise constants. Differently, if we consider a boundary, it can be proved that $\lambda_1(\Delta_p) \neq 0$ and also the study of the first eigenfunction becomes of interest. In chapter 3, we will present a complete study of the first eigenfunction of a class of generalized p-Laplacian operators that include the homogeneous-Dirichlet p-Laplacian operator (2.3).

2.2.1 Cases $p = 1, \infty$

A particular discussion is necessary for the two extreme cases $p = 1$ and $p = \infty$. Observe that in these cases the Rayleigh quotients $\mathcal{R}_1(f)$ and $\mathcal{R}_\infty(f)$ are still well defined but not differentiable anymore. This opens the problem of how to define the one and the infinity eigenpairs. The answer to this problem is not unique, and different approaches have been proposed in the literature. Here, we discuss an approach that has been initially used for the case $p = 1$ [20, 50], but that has been recently used also in the continuous setting for the infinity case [16, 17]. The idea is to define a generalized notion of critical points for the Rayleigh quotients $\mathcal{R}_1(f)$ and $\mathcal{R}_\infty(f)$. To this aim, we first note that, given a p-Laplacian eigenfunction f, we can assume w.lo.g. $\|f\|_p = 1$, which is equivalent to saying that f is a point of the unit sphere S_p. Then, observe that $|f|^{p-2} f$ is the outward normal to S_p in f. As a consequence, from eq.(2.1), since f is a p-Laplacian eigenfunction, $\partial_f \|\nabla f\|_p$ $(= C\Delta_p(f))$[1] is equal, up to rescaling, to the outward normal to the manifold S_p in the point f. We immediately encounter two difficulties when trying to generalize this idea. The first is the non differentiability of the one or infinity norms of the gradient, and the second is the fact that the outward normal to the spheres S_1 and S_∞ is not everywhere well defined. A solution to both of these problems comes from the notion of subgradients of a convex function [80]. Let $\Psi : \mathbb{R}^n \to \mathbb{R}$ be a convex function, e.g a norm. Its subgradient at a point f_0 is defined as:

$$\partial \Psi(f_0) = \left\{ \xi \mid \Psi(g) - \Psi(f_0) \geq \langle \xi, g - f_0 \rangle \; \forall g \in \mathbb{R}^n \right\}.$$

[1] here $\partial_f \|\nabla f\|_p$ denotes the usual gradient in $\mathbb{R}^n$ of the function $f \mapsto \|\nabla f\|_p$. We prefer the symbol ∂_f to ∇ to avoid confusing the gradient on the graph with the gradient in $\mathbb{R}^n$

This is a generalization of the notion of gradient: if the function Ψ is differentiable at the point f_0, then $\partial\Psi(f_0) = \big(\partial_f \Psi\big)(f_0)$, where $\partial_f \Psi$ denotes the usual gradient in $\mathbb{R}^n$. Morover it is possible to characterize the composition of the subdifferential of a convex function with a linear transformation (see Theorem 23.9 [80]):

Theorem 2.2.6. *Let $\Phi(f) = \Psi(Af)$, where Ψ is a convex function on $\mathbb{R}^m$, $|\Psi(f)| < +\infty \; \forall f \in \mathbb{R}^m$ and A is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, then*

$$\partial\Phi(f) = A^T \partial(\Psi(Af))$$

These results allow us to define a generalized notion of $\partial_f \|\nabla f\|$, meaningful also in the case of the one and the infinity norms and that matches the classical definition for $1 < p < \infty$.

Now, we need to generalize the notion of outward normal to the spheres of p-norm with p equal to one or infinity. As the outward normal does not change, instead of considering the sphere, we consider the corresponding closed ball, i.e.:

$$D_p = \{f \mid \|f\|_p \leq 1\}\,.$$

It is easy to see that D_p is a convex set for any p and we can define the convex external cone in the generic point, f_0 s.t. $\|f_0\|_p = 1$ as

$$C_{Ext}(f_0) = \{\xi \mid \langle \xi, g - f_0\rangle \leq 0 \;\; \forall\, g \in D_p\}\,.$$

Then, we observe that the external cone can be related to the subgradient of $f \to \|f\|_p$ (see Theorem 23.7 [80] and [22] for more general results):

Lemma 2.2.7. *Let $\|\cdot\|$ be a norm and $D := \{f | \|f\| \leq 1\}$. Then, for any $f_0 \in D_p$ with $\|f_0\| = 1$, it holds the following equality:*

$$C_{Ext}(f_0) = \bigcup_{\lambda \geq 0} \lambda \partial \|f_0\|\,.$$

Here $C_{ext}(f_0) = \{\xi | \langle \xi, g - f_0\rangle \leq 0 \; \forall g \in D\}$ is the external cone to D in f_0 and $\partial\|f_0\| = \big\{\xi \mid \|g\| - \|f_0\| \geq \; \langle \xi, g - f_0\rangle \; \forall g \in \mathbb{R}^n\big\}$. is the subrgadient of the norm in f_0

Proof. The inclusion

$$\bigcup_{\lambda \geq 0} \lambda \partial \|f_0\|_p \subset C_{Ext}(f_0)$$

is a direct consequence of the fact that $\|g\|_p - \|f_0\|_p \leq 0 \; \forall g \in D_p$. Indeed from the last inequality, if $\xi \in \lambda \partial \|f_0\|_p$ with $\lambda > 0$, then

$$\langle \xi, g - f_0\rangle \leq \lambda\Big(\|g\| - \|f_0\|\Big) \leq 0\,.$$

On the other hand, assume that $C_{Ext}(f_0) \neq \bigcup_{\lambda \geq 0} \lambda \partial \|f_0\|_p$, then there exists $\xi \in C_{Ext}(f_0)$ such that, for any $\lambda \geq 0$, $\xi \notin \lambda \partial \|f_0\|_p$, where given $\lambda \geq 0$:

$$\lambda \partial \|f_0\|_p := \{\xi | \lambda \Big(\|g\| - \|f_0\| \Big) \geq \langle \xi, g - f_0 \rangle \ \forall g \in \mathbb{R}^n \}$$

This implies that $\xi \neq 0$ and that, for any $\lambda > 0$, there exists some $g_\lambda \in \mathbb{R}^n$ such that

$$\langle \xi, g_\lambda - f_0 \rangle > \lambda \Big(\|g_\lambda\|_p - \|f_0\|_p \Big) \,. \tag{2.4}$$

Note that, since $\xi \in C_{Ext}(f_0)$, for any $g \in \mathbb{R}^n$ it holds the following inequality:

$$\Big\langle \xi, \frac{g}{\|g\|_p} \Big\rangle \leq \langle \xi, f_0 \rangle \,, \tag{2.5}$$

where we write $g/\|g\| = 0$, if $g = 0$. Now if we use the above eq.(2.4) joint with eq.(2.5) and the fact that $\|f_0\|_p = 1$, we obtain the following set of inequalities for any $\lambda > 0$.

$$\begin{aligned} \Big(\|g_\lambda\|_p - 1 \Big) \langle \xi, f_0 \rangle =& \|g_\lambda\|_p \langle \xi, f_0 \rangle - \langle \xi, f_0 \rangle \geq \\ \geq& \|g_\lambda\|_p \Big\langle \xi, \frac{g_\lambda}{\|g_\lambda\|} \Big\rangle - \langle \xi, f_0 \rangle = \langle \xi, g_\lambda - f_0 \rangle > \lambda \Big(\|g_\lambda\|_p - 1 \Big) \,, \end{aligned} \tag{2.6}$$

where as before we say $g_\lambda / \|g_\lambda\| = 0$ if $g_\lambda = 0$. From (2.6), we deduce that necessarily $\langle \xi, f_0 \rangle \leq 0$, otherwise taking $\lambda = \langle \xi, f_0 \rangle$ in (2.6) would lead to a contradiction. However, also $\langle \xi, f_0 \rangle leq 0$ leads to a contradiction. Indeed, since $\xi \in C_{Ext}(f_0) \setminus \{0\}$, we note $\xi / \|\xi\| \in D_p$ and from the definition of external cone we deduce the opposite inequality:

$$\Big\langle \xi, \frac{\xi}{\|\xi\|} - f_0 \Big\rangle \leq 0 \,, \qquad \text{i.e.} \qquad \langle \xi, f_0 \rangle \geq \frac{\|\xi\|_2^2}{\|\xi\|} > 0 \,.$$

□

It follows that saying that f is an eigenfunction of the p-Laplacian is equivalent to asking that there exist $\Lambda > 0$ such that

$$\emptyset \neq \partial \|\nabla f\|_p \cap \Lambda \partial \|f\|_p \,, \tag{2.7}$$

and now this definition makes sense also when $p = 1$ and $p = \infty$.

To complete the discussion about the two nonsmooth cases we need to characterize the sets $\partial \|f\|_1$ and $\partial \|f\|_\infty$. To this end we recall the following result [18].

Lemma 2.2.8. *Given a function f_0 and a norm $\| \cdot \|$,*

$$\partial \|f_0\| = \{\xi \mid \|g\| \geq \langle \xi, g \rangle \ \forall\, g, \ \ \|f_0\| = \langle \xi, f_0 \rangle \}$$

Proof. The inclusion $\{\xi \mid \|g\| \geq \langle \xi, g\rangle \;\forall\, g,\;\; \|f_0\| = \langle \xi, f_0\rangle\} \subset \partial\|f_0\|$ is trivially proved. Indeed if $\xi \in \{\xi \mid \|g\| \geq \langle \xi, g\rangle \;\forall\, g,\;\; \|f_0\| = \langle \xi, f_0\rangle\}$, then

$$\langle \xi, g - f_0\rangle \leq \|g\| - \|f_0\|\,,$$

which is the definition of subgradient. To prove the opposite inclusion, consider $\xi \in \partial\|f_0\|$, then the triangular inequality and the definition of $\partial\|f_0\|$ yield:

$$\langle \xi, h - f_0\rangle = \langle \xi, h\rangle - \langle \xi, f_0\rangle \leq \|h\| - \|f_0\| \leq \|h - f_0\| \quad \forall h$$

Then, as $g := (h - f_0)$ spans the whole $\mathbb{R}^n$, the sup in the above inequality yields

$$\sup_g \Big(\langle \xi, g\rangle - \|g\|\Big) = \sup_h \Big(\langle \xi, h - f_0\rangle - \|h - f_0\|\Big) \leq 0\,,$$

which reads

$$\|g\| \geq \langle \xi, g\rangle \qquad \forall g\,. \tag{2.8}$$

Furthermore, taking $g = 0$ we obtain the opposite inequality

$$\sup_g \Big(\langle \xi, g\rangle - \|g\|\Big) \geq 0\,,$$

which yields $\sup_g \Big(\langle \xi, g\rangle - \|g\|\Big) = 0$. Finally, from the subgradient definition, we have:

$$0 = \Big(\sup_g \langle \xi, g\rangle - \|g\|\Big) \geq \langle \xi, f_0\rangle - \|f_0\| \geq \Big(\sup_g \langle \xi, g\rangle - \|g\|\Big) = 0\,,$$

which reads:

$$\langle \xi, f_0\rangle = \|f_0\|\,. \tag{2.9}$$

The last equality (2.9) joint with the inequality (2.8) proves the inclusion $\partial\|f_0\| \subset \{\xi \mid \|g\| \geq \langle \xi, g\rangle \;\forall\, g,\;\; \|f_0\| = \langle \xi, f_0\rangle\}$ and concludes the proof. □

Observe that from this Lemma it follows that necessarily, if f is an eigenfunction as in eq.(2.7), then $\Lambda = \mathcal{R}_p(f)$ i.e., it is the p-th root of the eigenvalue defined in (2.1). However, observe that, when $p = \infty$, $\mathcal{R}_\infty^\infty$ is not defined, while for $p = 1$ there is no difference between $\mathcal{R}_1(f)$ and the 1-st root. Thus, for the two extreme cases, $p = 1$ and $p = \infty$, we will call, with a small abuse of notation, Λ in eq.(2.7) the eigenvalue corresponding to f.

The subgradients of the one and infinity norms can be calculated from Lemma 2.2.8 and Theorem 2.2.6, yielding the following formulas, where ξ and Ξ denote functions on $\mathcal{H}(V)$ and $\mathcal{H}(E)$, respectively.

$$\begin{aligned} \partial\|f\|_1 &= \Big\{\xi \;\Big|\; \xi(u) = \operatorname{sign}\big(f(u)\big)\Big\} \\ \partial\|\nabla f\|_1 &= \Big\{-\operatorname{div}\Xi \;\Big|\; \Xi(u,v) = \operatorname{sign}\big(\nabla f(u,v)\big)\Big\} \end{aligned} \tag{2.10}$$

where sign(x) is the set valued function, $\mathrm{sign}(x) = \begin{cases} 1 & if\ x > 0 \\ [-1,1] & if\ x = 0 \\ -1 & if\ x < 0 \end{cases}$.

$$\partial\|f\|_\infty = \left\{ \xi \;\middle|\; \begin{array}{l} \|\xi\|_1 = 1,\;\; \xi(u) = 0\;\; if\; |f(u)| < \|f\|_\infty \\ \xi(u)f(u) = \big|\xi(u)f(u)\big| \end{array} \right\}$$

$$\partial\|\nabla f\|_\infty = \left\{ -\mathrm{div}\,\Xi \;\middle|\; \begin{array}{l} \|\Xi\|_1 = 1,\; \Xi(u,v) = 0\;\; if\; |\nabla f(u,v)| < \|\nabla f\|_\infty \\ \Xi(u,v)\nabla f(u,v) = \big|\Xi(u,v)\nabla f(u,v)\big| \end{array} \right\}$$

We conclude this part recalling that, also in the degenerate cases, the min max in eq.(2.2) characterizes eigenvalues as generalized critical values of eq.(2.7), allowing to define the variational eigenvalues also for $p = 1$ and $p = \infty$. This fact follows from Theorems 6.1, 6.4 and Theorems 5.1, 5.8 of [22] (see also Theorem 2.2.3 above) applied to the boundary of fine polytopes such as S_1 and S_∞. For completeness, we recall the deformation Theorem 5.1 from [22]. Here, for simplicity, it is stated only for the 1-norm and ∞-norm spheres, while in [22] it is stated for more general polytopes (see Theorems 6.1, 6.4 [22]). The existence of the one and infinity variational eigenvalues follow then from Theorem 2.2.3 and Lemma 2.2.9.

Lemma 2.2.9. *Let Φ be a Lipschitz convex function on $\mathbb{R}^N$. Define $\tilde{\Phi} := \Phi|_X$ as the restriction of f to the boundary, X of one of the convex polytopes $P = \{f \| \|f\|_1 \leq 1\}$ or $P = \{f \| \|f\|_\infty \leq 1\}$. Let $c \in \mathbb{R}$, be an isolated critical value, $K_c = K \cap \tilde{\Phi}^{-1}(c)$ and $N \subset X$ be a neighborhood of K_c, where K is the set of critical points of Φ on X. Then $\forall \epsilon_0 > 0$, there exist $\epsilon \in (0, \epsilon_0)$ and a deformation $\eta : X \times [0,1] \to X$ satisfying:*

1. *$\eta(x, 0) = x\,,\ \forall x \in X$;*
2. *$\eta(x,t) = x\,,\ \forall x \notin \tilde{\Phi}^{-1}[c-\epsilon_0, c+\epsilon_0], \forall t \in [0,1]$;*
3. *$\eta(\tilde{f}_{c+\epsilon} \setminus N, 1) \subset \tilde{\Phi}_{c-\epsilon}$, where $\tilde{\Phi}_b$ is the level set of $\tilde{\Phi}$ below or equal to b;*
4. *$\eta(\tilde{\Phi}_{c+\epsilon}, 1) \subset \tilde{\Phi}_{c-\epsilon}$ if $K_c = \emptyset$.*

2.3 Nodal Domains

The nodal domains induced by a function f are generally the maximal subdomains where f has constant sign. Here we briefly discuss and recall a number of results about the nodal domains induced by the p-Laplacian eigenfunctions. We will return on this topic in Chapter 3, where we face the problem of bounding the number of nodal domains induced by a p-Laplacian eigenfunction in terms of the position of the eigenvalue with respect to the variational spectrum. We start from the definition of nodal domain:

Definition 2.3.1 (Nodal domains)**.** *Given a graph $\mathcal{G}$ and a function $f : V \to \mathbb{R}$, a subset of the vertices, $A \subseteq V$, is a nodal domain induced by f if the subgraph $\mathcal{G}_A \subset \mathcal{G}$ with vertices in A is a maximal connected subgraph of $\mathcal{G}$ where f is nonzero and has constant sign. We denote by $\mathcal{N}(f)$ the number of nodal domains induced by a function f.*

A classical result relates the number of nodal domains induced by an eigenfunction of the linear Laplacian with the corresponding frequency [5, 6, 87]. The same result can be transfered to the nonlinear p-Laplacian setting. In particular, without entering in the details of the sharpeness of the bounds, that we will discuss in Chapter 3, from [86] and the results of this thesis as published in [31], it can be shown that

Theorem 2.3.2. *Suppose that $\mathcal{G}$ is a connected graph, $1 < p < \infty$ and $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$ are the variational eigenvalues of Δ_p.*

- *If f is an eigenfunction of Δ_p with eigenvalue λ such that $\lambda < \lambda_k$, then*

$$\mathcal{N}(f) \leq k - 1\,;$$

- *if f is an eigenfunction of Δ_p with eigenvalue λ such that $\lambda > \lambda_k$, then*

$$\mathcal{N}(f) \geq k - \beta - z(f) + 1\,,$$

  where β is the number of independent loops of the graph, i.e. $\beta = |E| - |V| + 1$, and $z(f)$ is the number of nodes where f is zero.

Given a nodal domain A, define by $E(A, A^c)$ its boundary given by:

$$E(A, A^c) = \{(u, v) \in E \text{ s.t. } u \in A,\ v \in A^c \text{ or } u \in A^c,\ v \in A\}\,, \tag{2.11}$$

where $A^c = V \setminus A$. Then, given an eigenpair and its nodal domains it is possible to establish the following equality. In the next Lemma, with a small abuse of notation, we write $\|f\|_{p-1}^{p-1} = \sum_u |f(u)|^{p-1}$ and $\|\nabla f\|_{\omega,p-1}^{p-1} = (1/2)\sum_{(u,v)} \omega_{uv} |\nabla f(u, v)|^{p-1}$ also in the case $p \in (1, 2)$ in which the above functionals are not norms.

Lemma 2.3.3. *Let (λ, f) be a p-Laplacian eigenpair, $p > 1$, A a nodal domain induced by f and $E(A, A^c)$ the boundary of A. Then*

$$\lambda = \frac{\|\nabla f|_{E(A,A^c)}\|_{\omega,p-1}^{p-1}}{\|f|_A\|_{p-1}^{p-1}}$$

where $\nabla f|_{\partial A}(u, v) = \mathbb{1}_{E(A,A^c)}(u, v)\nabla f(u, v)$ and $f|_A(u) = \mathbb{1}_A(u)f(u)$.

Proof. The proof easily follows summing over all the nodes u that belong to A the eigenvalue equation of f. □

These results about the nodal domains, jointly with the fact that the nodal domains are related with a form of balanced partition of the graph, lead to the idea of using the eigenvalues of the p-Laplacian to approximate some "optimal" partitions of the graph. This idea, as we will see in the next paragraphs, allows us to derive some geometrical information about the graph from the spectrum of the one and infinity Laplacians.

2.3.1 $p = 1$ and Cheeger constants

We start this subsection by introducing a family of Cheeger constants of a graph. These constants are typically used in data analysis applications and they are useful to provide information about the number and the quality of the clusters of a graph. Let $A \subset V$ be a subset of the nodes and let $E(A, A^c)$ be defined as in eq.(2.11). Consider the quantity

$$c(A) = \frac{\|\omega\big(E(A, A^c)\big)\|_1}{|A|} = \frac{\frac{1}{2}\sum_{(u,v)\in E(A,A^c)} \omega(u,v)}{|A|}\,,$$

and, given an integer k, all the possible families of k nonempty and disjoint subsets of V

$$\mathcal{D}_k(\mathcal{G}) = \{A_1, \dots, A_k \subset V \mid A_i \neq \emptyset,\ A_i \cap A_j = \emptyset\ \ \forall\, i, j\}\,.$$

Define the k-th Cheeger constant, see [28, 65, 66] as

$$h_k(\mathcal{G}) := \min_{\{A_1,\dots A_k\}\in\mathcal{D}_k(\mathcal{G})} \max_{i=1,\dots,k} c(A_i)\,,$$

Observe that having a "small" value of $h_k(\mathcal{G})$ means that there exist k subsets of nodes that are at the same time quite large ($|A_i|$ is sufficiently big for any i) and poorly connected to each other ($\sum_{(u,v)\in E(A_i,A_i^c)} \omega(u,v)$ small for any i). This is exactly what k clusters of nodes should be. The constant $h_k(\mathcal{G})$ can thus be considered as an indicator of how well the graph can be clustered into k subgraphs, with the corresponding family of subsets being the approximate clusters.

Now observe that, given a subset $A \subset V$ and considered its characteristic function χ_A, we have that

$$\mathcal{R}_1(\chi_A) = \frac{\frac{1}{2}\sum_{(u,v)\in E} \omega_{uv}|\chi_A(u) - \chi_A(v)|}{\sum_{v\in V} |\chi_A(u)|} = c(A)\,.$$

Then, consider a maximizing family of nonempty disjoint subsets $\{A_1, \dots, A_k \subset \mathcal{D}_k(\mathcal{G})$ in the definition of $h_k(\mathcal{G})$, i.e.

$$h_k(\mathcal{G}) = \max_{i=1,\dots,k} c(A_i)\,.$$

Named Λ_k^1 the k-th variational eigenvalue of the 1-Laplacian, i.e.,

$$\Lambda_k^1 = \min_{A \in \mathcal{F}_k(S_1)} \max_{f \in A} \mathcal{R}_1(f)\,,$$

a simple argument shows that

$$\Lambda_k^1 \leq \max_{f \in \mathrm{span}\{\chi_{A_i}\,|\, i=1,\dots,k\}} \mathcal{R}_1(f) \leq \max_i \mathcal{R}_1(\chi_{A_i}) = \max_{i=1,\dots,k} c(A_i) = h_k(\mathcal{G})\,.$$

Moreover if f is a 1-Laplacian eigenfunction associated to the eigenvalue Λ, there exist $\xi \in \partial\|f\|_1$ and $\Xi \in \partial\|\nabla f\|_1$ such that

$$-div\Xi(u) = \frac{1}{2}\sum_{v\sim u} \omega_{uv}\Big(\Xi(v,u) - \Xi(u,v)\Big) = \Lambda\xi(u)\,. \tag{2.12}$$

Then, if we consider a nodal domain A induced by f, where we assume w.l.o.g. $f > 0$ over A, and we sum the eigenvalue equation (2.12) over A we get

$$\Lambda = \frac{\sum_{u\in A,\, v\in A^c,\, v\sim u} \omega_{uv}}{|A|} = c(A)\,,$$

where we have used the characterization of the subgradients (2.10), i.e. $\xi(u) = 1\;\forall\, u \in A$ and $\Xi(v,u) = -\Xi(u,v) = 1\;\forall u \in A,\; v \in A^c$ such that $(u,v) \in E$.

The remarks above show that the study of the Cheeger constants is tightly related the the study of the eigenpairs of the 1-Laplacian. What is actually possible to prove is the following theorem [13, 20, 29, 50, 86]

Theorem 2.3.4. *Let (f, Λ_k^1) be the k-th variational eigenpair of the 1-Laplacian, then*

$$\Lambda_2^1 = h_2(\mathcal{G})\,, \qquad h_{\mathcal{N}(f)}(\mathcal{G}) \leq \Lambda_k^1 \leq h_k(\mathcal{G}) \quad \forall k,$$

where $\mathcal{N}(f)$ is the number of nodal domains induced by f.

Moreover using the p-Laplacian eigenpairs, when p goes to 1, it is possibile to prove the following theorem which relates the p-eigenpairs to the Cheeger constants, [29, 86]:

Theorem 2.3.5. *Let $(f_k, \lambda_k^{(p)})$ be the k-th variational eigenpair of the p-Laplacian, $p > 1$, then*

$$\frac{2^{p-1}}{\tau(\mathcal{G})^{p-1}} \frac{\big(h_{\mathcal{N}(f)}(\mathcal{G})\big)^p}{p^p} \leq \lambda_k^{(p)} \leq 2^{p-1} h_k(\mathcal{G})\,,$$

where $\tau(\mathcal{G}) = \max_{u\in V}\sum_{v\sim u}\omega(u,v)$

Combining the last two theorems, we observe that whenever we have a variational eigenfunction whose nodal domain count reflects the corresponding frequency, letting p go to one, the eigenvalue reproduces exactly an higher-order Cheeger constant.

2.3.2 $p = \infty$ and packing radii of the graph

Similar results relate the ∞-eigenpairs to the maximal distance between k nodes. It is worth mentioning that these results are similar to those obtained in the continuous setting by [39, 57, 58], using an approach different from the one that employs the subgradients. We enter in the details of this topic in Chapter 5, where we provide also a comparison between the formulation of ∞-eigenpairs proposed by Lindqvist et al. [39, 57, 58] and the formulation in terms of subgradients.

Let us start by introducing the k-th packing radius of the graph:

$$R_k = \max_{v_1,\ldots,v_k \in V} \min_{i,j=1,\ldots,k} \frac{d(v_i, v_j)}{2},$$

which can also be written as

$$R_k = \max\{r \mid \exists\, v_1, \ldots, v_k \in V \ \ s.t.\ \ d(v_i, v_j) \geq 2r \ \ \forall\, i, j = 1, \ldots, k\}.$$

Observe that $2R_2$ is obviously the diameter of the graph, while, for $k > 2$ we are computing the maximal reciprocal distance among k nodes, following [49] we name this quantity "packing radii of order k" . In terms of information about a set of data represented by the graph, R_k measures a sort of distribution width of the data.

Similarly to what shown before for the 1-eigenpairs, it is not difficult to show that, if (f, Λ) is an ∞-eigenpair, then there exist two nodes $u, v \in V$ such that $f(u) > 0, f(v) < 0$ and

$$\Lambda = \frac{\|\nabla f\|_\infty}{\|f\|_\infty} = \frac{2}{d(u, w)}.$$

Since any ∞-eigenpair can be related to a distance between nodes in different nodal domains, it makes again sense to relate the infinite variational eigenvalues to the packing radii of the graph. The following results, very similar to the one presented for the $p = 1$ case, are part of the contributions of this thesis, we refer to Chapter 5 for the details.

Theorem 2.3.6. *Let $\Lambda_k^{(\infty)}$ be the k-th ∞-variational eigenvalue, then*

$$\Lambda_2^{(\infty)} = \frac{1}{R_2}, \qquad \Lambda_k^{(\infty)} \leq \frac{1}{R_k} \quad \forall k$$

Moreover, in the case of eigenpairs obtained as limit for $p \to \infty$ of p-Laplacian eigenapairs we have the following

Theorem 2.3.7. *Let (f, Λ^∞) be an ∞-eigenpair that is a limit of p-Laplacian eigenpairs as $p \to \infty$, then*

$$\frac{1}{R_{\mathcal{N}(f)}} \leq \Lambda^{(\infty)}$$

Thus, as in the case $p = 1$, whenever we have a variational eigenfunction whose nodal domain count reflects the corresponding frequency, letting p go to infinity, the eigenvalue reproduces exactly a packing radii.

We point out that the above results hold also in the more general setting of a graph with a boundary, $B \subset V$. In this case we only have to change the definition of the packing radius R_k, which becomes:

$$R_k^B := \sup\{r \mid \exists\, v_1, \dots, v_k \ \text{ with } d(v_i, v_j \geq 2r \ d(v_i, B) \geq r \ \forall i, j = 1, \dots, k\}.$$

Moreover, if the boundary is nonempty, from the above Theorems 2.3.6 and 2.3.7, we can also derive the equality

$$\Lambda_1 = \frac{1}{R_1^B} = \frac{1}{max_u d_B(u)} \, .$$

2.4 Computing the p-Laplacian eigenpairs

In the last paragraphs we have shown that the graph p-Laplacian eigenpairs, with particular attention to the limit cases $p = 1$ and $p = \infty$, can be used to deduce or approximate topological properties of the graph itself. Since the computation of these topological invariants is usually difficult and expensive, it is natural to wonder if it is possible to bypass this problem by computing instead the p-Laplacian eigenpairs. Unfortunately this is a complicated task itself, and, to the best of our knowledge, at the moment there is no method able to compute all the p-Laplacian eigenpairs. Moreover there is no method able to locate a given eigenvalue in the variational spectrum. Among the few methods proposed in the literature, we can mention the inverse nonlinear power method [50] and the gradient flows proposed in [14, 15]. Both these methods are aimed at the computation of extremal (minimal or maximal) eigenpairs. The strenght of these approaches is the possibility to prove the convergence toward an eigenpair. However the limit eigenpair depends on the starting point and, from a theoretical point of view, there is no a-priori information about the eigenpair that will be computed. On the other hand, there are methods like the local minmax method proposed in [88], that are able to compute sequences of $|V|$ eigenpairs with an increasing number of local decreasing directions of $\mathcal{R}_p$. However these methods, once the first k-eigenpairs $\{(f_i, \lambda_i)\}_{i=1}^k$ have been computed, are able to compute the $k+1$-th, (f_{k+1}, λ_{k+1}), if and only if

$$\lambda_{k+1} = \operatorname*{local\,max}_{f \in span\{f_i\}_{i=1}^{k+1}} \mathcal{R}_p(f) \, ,$$

where $\operatorname{local\,max}_{f \in span\{f_i\}_{i=1}^{k+1}} \mathcal{R}_p(f)$ is the set of local maxima of the function $\mathcal{R}_p$ on $\operatorname{span}\{f_i\}_{i=1}^{k+1}$. This property, however, is not trivially satisfied by the p-Laplacian eigenpairs and in principle could be satisfied by none of the p-Laplacian eigenpairs. In Chapters 4 and 5 we propose a novel method suited to work both in the $p \in [2, \infty)$ and $p = \infty$ case. For this method we provide theoretical and a-priori guarantee of convergence towards the first p-Laplacian eigenpair. Moreover, given an index k and without using any other nonlinear eigenpair, our method is suited to compute non-extremal p-Laplacian eigepairs, (f, λ), such that the Morse index of $\mathcal{R}_p$ in f is equal to k, where the Morse index denotes the number of negative eigenvalues of the Hessian matrix of $\mathcal{R}_p$ i.e. decreasing directions. In addition, we provide bounds for the Morse index of some properly defined variational eigenfunction in terms of the variational index.

In more details, assuming $p \in [2, \infty]$, in Chapter 4 and 5 we consider a reformulation of the p-Laplacian eigenavalue problem in terms of a weighted linear Laplacian eigenvalue problem

$$\text{if } p \in (2,\infty) \;\Rightarrow \begin{cases} \Delta_{\mu_0} f(u) = -\mathrm{div}\big(\mu_0 \nabla f\big)(u) = \lambda \nu_{0u} f(u) & \forall u \in V \\ \mu_{0uv} = |\nabla f(u,v)|^{p-2} & \forall (u,v) \in E \\ \nu_{0u} = |f(u)|^{p-2} & \forall u \in V \end{cases}$$

$$\text{if } p = \infty \quad \Rightarrow \begin{cases} \Delta_{\mu_0} f(U) = -\mathrm{div}(\mu_0 \nabla f)(u) = \Lambda \nu_{0u} f(u) & \forall u \in V \\ |\nabla f(u,v)| = \|\nabla f(u,v)\|_\infty & \text{if } \mu_{0uv} > 0 \\ |f(u)| = \|f(u)\|_\infty & \text{if } \nu_{0u} > 0 \\ \|\mu_0 \nabla f\|_{1,E} = 1 \\ \|\nu_0 f\|_{1,V} = 1 \end{cases}$$

This reformulation allows us to introduce a class of energy functions in the variables (μ, ν) whose saddle points correspond to eigenpairs of the p-Laplacian. Indeed, for any couple of positive measures $\mu : E \to \mathbb{R}^+$ $\nu : V \to \mathbb{R}^+$, we consider the eigenpairs $\Big(f(\mu,\nu), \lambda(\mu,\nu)\Big)$ of the generalized linear eigenvalue problem

$$\Delta_\mu f(u) = \Big(-\operatorname{div}\,\big(\mathrm{diag}(\mu)\big)\nabla f\Big)(u) = \lambda \nu_u f(u)\,. \tag{2.13}$$

Observe that these are eigenpairs of a linear eigenvalue problem and thus can be enumerated from 1 to $|V|$, the cardinality of the node space, and can be easily computed. Then, for any $1 \le k \le |V|$, we introduce the function

$$\mathcal{E}_k^p(\mu,\nu) = \frac{1}{\lambda_k(\mu,\nu)} + \frac{p-2}{p} \sum_{(u,v)\in E} (\mu_{uv}^{\frac{p}{p-2}}) - \frac{p-2}{p} \sum_{v \in V} (\nu_v^{\frac{p}{p-2}})\,, \tag{2.14}$$

and the sets $\mathcal{M}^+(V) := \{\nu : V \to \mathbb{R}_{\ge 0}\}$, $\mathcal{M}^+(E) := \{\mu : E \to \mathbb{R}_{\ge 0}\}$. With these definitions we prove the following:

Theorem 2.4.1. *Let $p \in (2, \infty]$, then $\mathcal{E}_1^p(\mu, \nu)$ admits a unique saddle point $\big(\max_\nu \min_\mu\big)$ and*

$$(\nu^*, \mu^*) := \underset{\nu \in \mathcal{M}^+(V)}{\arg\max}\; \underset{\mu \in \mathcal{M}^+(E)}{\arg\min}\; \mathcal{E}_1^p(\mu,\nu)$$

is such that $\big(\lambda_1^{\frac{p}{2}}(\mu^*,\nu^*), f_1(\mu^*,\nu^*)\big) = \big(\lambda_1(\Delta_p), f_1(\Delta_p)\big)$ *is the unique, up to scaling,* 1*-st p-Laplacian eigenpair.*

Theorem 2.4.2. *Let,* $p \in (2,\infty)$ *and*

$$(\nu^*,\mu^*) := \underset{\nu\in\mathcal{M}^+(V)}{\arg\max}\ \underset{\mu\in\mathcal{M}^+(E)}{\arg\min}\ \mathcal{E}_k^p(\mu,\nu)$$

be a smooth saddle point of the function $\mathcal{E}_k^p(\mu,\nu)$*, then* $\big(\lambda_k^{\frac{p}{2}}(\mu^*,\nu^*), f_k(\mu^*,\nu^*)\big)$ *is a p-Laplacian eigenpair.*

Moreover, given a p-Laplacian eigenpair, (f,λ), we can consider the index of the eigenvalue thought as an eigenvalue of the corresponding linear eigenvalue problem (2.13) and use it to derive information about the Morse index of the p-Rayleigh quotient in f.

These results finally lead to the construction of numerical algorithms for the computation of p-Laplacian eigenpairs based on gradient flows for the functionals $\mathcal{E}_k^p$. These algorithms, at each step, only require the computation of an eigenpair of a weighted linear Laplacian, thus allowing us to use all the advantages of linearity to numerically solve a nonlinear problem.

2.5 Our contributions

In this section we give a short but precise list of our main contributions joint with an overview of the structure of the thesis. At the end of the section we will add a diagramatic map summarizing our contributions.

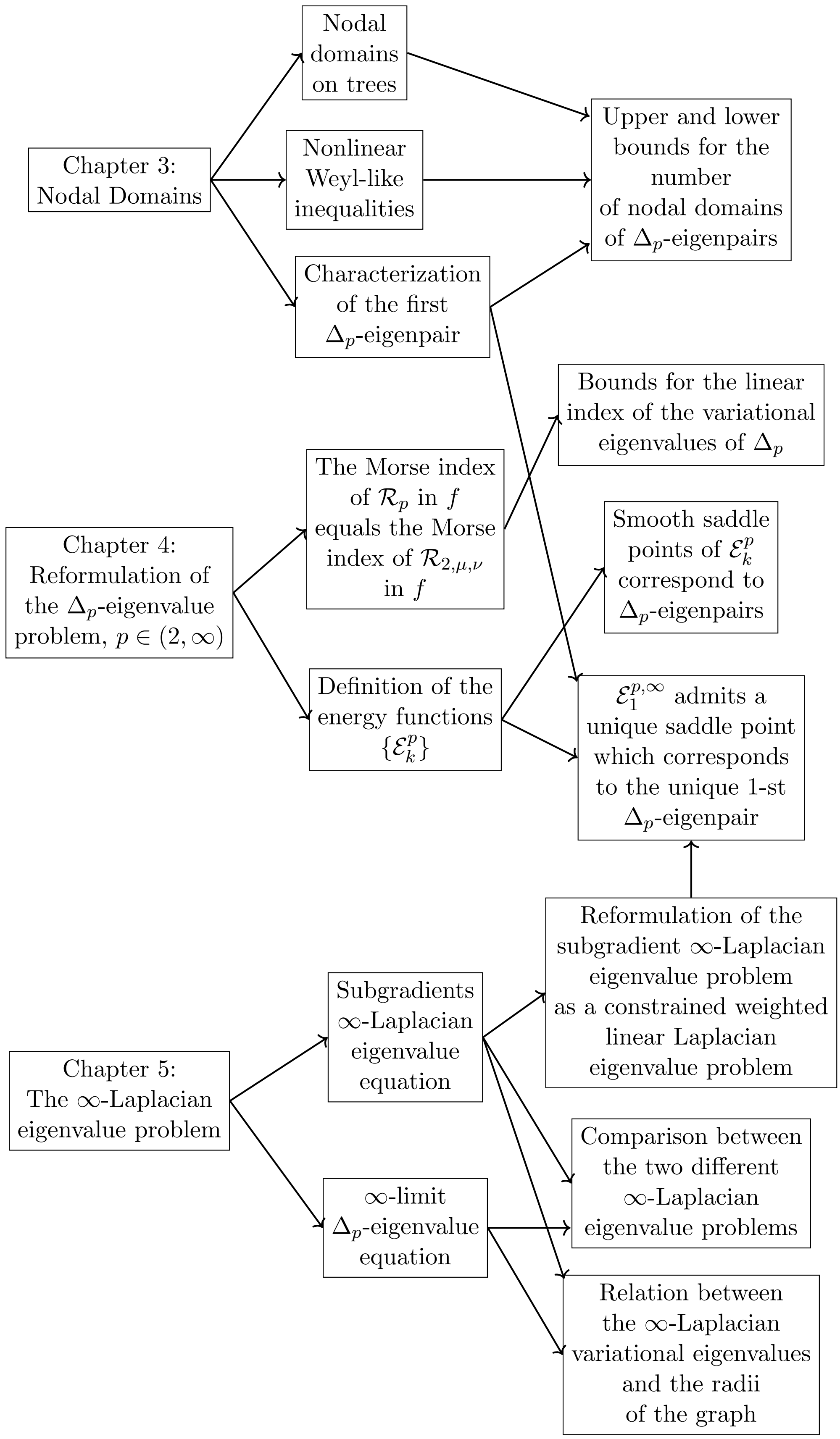
- **Chapter 3: Nodal Domains.**
  In this chapter we study the nodal domains of the eigenfunctions of a generalized class of p-Laplacian operators. In particular, we prove the uniqueness of the first eigenfunction of the generalized p-Laplacian and its characterization as the only eignfunction that induces only one nodal domain. Then, we study how the spectrum of p-Laplacian operators changes after different kinds of perturbation of the graph and we prove novel nonlinear Weyl-like inequalities. After that, we study the spectrum of p-Laplacian operators on trees. Here we prove, first, that the variational spectrum exhausts all the spectrum, and, second, that for any simple variational eigenvalue, λ, the number of nodal domains induced by the corresponding eigenfunction equals the variational index of λ. Finally, using the previous results, we prove old and new bounds for the number of nodal domains induced by an eigenfunction. These bounds depend on the position of the corresponding eigenvalue with respect to the variational spectrum.

- **Chapter 4: A reformulation of the p-Laplacian eigenvalue problem.**

In this chapter we study the reformulation of the p-Laplacian eigenpairs as constrained weighted Laplacian eigenpairs when $p \in (2, \infty)$. We prove that given an eigenpair (f, λ), the Morse index of the p-Rayleigh quotient in f matches the Morse index of the corresponding weighted 2-Rayleigh quotient in f which, in turn, corresponds to the linear index of the eigenvalue λ. Using these comparisons, for any variational eigenvalue of the p-Laplacian, we give bounds for its linear index in terms of the variational index. Then, we introduce the energy functions $\mathcal{E}_k^p$ eq.(2.14) and we prove that $\mathcal{E}_1^p$ admits a unique saddle point corresponding to the unique first eigenfunction of the p-Laplacian. In addition, we prove that smooth saddle points of the higher energy functions $\mathcal{E}_k^p$ for $k > 1$ correspond to higher p-Laplacian eigenpairs. It is worth mentioning that, to prove the latter results, we need to study in some depth the so-called $(p, 2)$-eigenpairs, i.e., the critical values and points of the Rayleigh quotient $\mathcal{R}_{p,2}(f) = \|\nabla f\|_p / \|f\|_2$.

- **Chapter 5: The Infinity eigenvalue problem.**
  In this chapter we study the ∞-Laplacian eigenvalue problem. In the first part we consider the discrete analogue of the approach proposed by Lindqvist and Juutinen [57, 58], i.e. we look at the solutions of the ∞-limit eigenvalue equation. Within this approach, we prove inequalities between the ∞-limit variational eigenvalues and the packing radii of the graph that are finite-dimensional counterparts of the continuous inequalities presented in [57, 58].

  In the second part of the chapter, we consider the ∞-Laplacian eigenvalue problem expressed in terms of subgradients of the infinity norms. Within this approach, we prove, first, inequalities between the variational eigenvalues and the packing radii of the graph and, second, a geometrical characterization of the ∞-eigenpairs. Using the latter characterization, we are able to compare the two formulations of the ∞-eigenvalue problem, in particular we prove that the ∞-limit eigevalue problem proposed in the first part of the chapter is stronger than the subgradient ∞-Laplacian eigenvalue problem. We conclude the chapter proposing a reformulation of the subgradient ∞-Laplacian eigenvalue problem in terms of a constrained weighted linear Laplacian eigenvalue problem. This reformulation allows us to prove that, also in the ∞-case, the function $\mathcal{E}_1^\infty$ (2.14) admits a unique saddle point and that such saddle point corresponds to the first eigenvalue of the ∞-Laplacian.


Chapter 3: Nodal Domains
Nodal domains on trees
Nonlinear Weyl-like inequalities
Characterization of the first Δ_p-eigenpair
Upper and lower bounds for the number of nodal domains of Δ_p-eigenpairs
Bounds for the linear index of the variational eigenvalues of Δ_p
Chapter 4: Reformulation of the Δ_p-eigenvalue problem, $p \in (2, \infty)$
The Morse index of $\mathcal{R}_p$ in f equals the Morse index of $\mathcal{R}_{2,\mu,\nu}$ in f
Smooth saddle points of $\mathcal{E}_k^p$ correspond to Δ_p-eigenpairs
Definition of the energy functions $\{\mathcal{E}_k^p\}$
$\mathcal{E}_1^{p,\infty}$ admits a unique saddle point which corresponds to the unique 1-st Δ_p-eigenpair
Reformulation of the subgradient ∞-Laplacian eigenvalue problem as a constrained weighted linear Laplacian eigenvalue problem
Subgradients ∞-Laplacian eigenvalue equation
Chapter 5: The ∞-Laplacian eigenvalue problem
Comparison between the two different ∞-Laplacian eigenvalue problems
∞-limit Δ_p-eigenvalue equation
Relation between the ∞-Laplacian variational eigenvalues and the radii of the graph

3 Nodal Domains

3.1 Introduction

We have seen in Chapter 2 that the p-Laplacian nodal domains play an important role in the study of the k-th order isoperimetric constant $h_k(\mathcal{G})$ of the graph. Indeed, as we have recalled in the introdution (see also [86]), this fundamental graph invariants can be bounded from above and from below using the variational spectrum λ_k of the p-Laplacian and its nodal domain count via the Cheeger-like inequality

$$\lambda_{\mathcal{N}(f_k)} \leq h_{\mathcal{N}(f_k)} \leq c(p)\lambda_k^{1/p}, \tag{3.1}$$

where $c(p) \to 1$ as $p \to 1$ and f_k is any eigenfunction of λ_k. This result clearly highlights the importance of the nodal domain count in connection to, for example, the quality of p-Laplacian graph embeddings for data clustering, for which there is a wealth of empirical evidence [11, 12, 13, 38]. In Chapter 5, moreover, we will see similar inequalities, involving the number of nodal domains, connecting the p-Laplacian eigenpairs to the packing radii of the graph.

Because of these reasons, the estimation of the number of nodal domains of the Laplacian and p-Laplacian eigenfunctions, both on continuous manifolds and on discrete and metric graphs, has been an active field of research in the past years.

In the discrete graph setting, it was proved in [5, 7] that trees behave like strings and that the k-th eigenvector f_k of the Laplacian (or more generally Schrödinger) operator, if everywhere non-zero, induces exactly k nodal domains. Moreover, again under the assumption that the k-th eigenvector f_k of the graph Laplacian operator is everywhere non-zero, it was proved in [6] that for general graphs the following inequality holds for the number of nodal domains $\mathcal{N}(f_k)$ of f_k, provided the corresponding eigenvalue is simple:

$$k - \beta + l(f_k) \leq \mathcal{N}(f_n) \leq k\,.$$

Here β is the total number of independent loops of the graph and $l(f_k)$ is the number of independent loops where f_k has constant sign. In the general case of eigenvalues with any multiplicity and eigenvectors with possibly some zero entry, it was proved in [30, 34, 87] that the following inequality holds:

$$k + r - 1 - \beta - z \leq \mathcal{N}(f_k) \leq k + r - 1\,,$$

where f_k is an eigenvector of the eigenvalue λ_k, z is the number of zeros of f_k and r is the multiplicity of λ_k.

A nodal domain theorem for the graph p-Laplacian is provided in [86], where it is shown that, for any eigenfunction f_k of the p-Laplacian, the number of nodal domains is bounded above as $\mathcal{N}(f_k) \leq k + r - 1$, where r is the multiplicity of the corresponding variational eigenvalue. Analogous results are proved in [21] for the case $p = 1$. However, no lower bounds for $\mathcal{N}(f_k)$ were known in the general case.

The final aim of this chapter is thus to provide lower bounds on the number of nodal domains of the generic eigenfunction of the p-Laplacian. To this end, we will introduce a class of generalized p-Laplacian operators, already addressed in [78]. Such operators, similar to the generalized linear Laplacian or Schrödinger operator [5, 87], are also largely related to p-Laplacian problems with zero Dirichlet boundary conditions, see the Introduction and [54]. Thus, all our results apply to both the classical p-Laplacian and the generalized p-Schrödinger operators. We prove a classical characterization of the first and the second variational eigenpairs and nonlinear Weyl's like inequalities. These are fundamental instruments to study the nodal domains of the generic eigenfunction. Our general strategy, inspired by the work of [5], consists of defining appropriate rules to remove nodes or edges from the graph without changing an eigenpair. Repeated applications of this procedure allows us to arrive to a structured graph (e.g. a tree or the disjoint union of the nodal domains) for which nodal domain numbers can be fully characterized. This characterization can be brought back to the original graph by reversing the proposed procedure. This strategy allows us to find new lower bounds as well as retrieve known upper bounds for the number of nodal domains of any eigenfunction, as a function of the position of the corresponding eigenvalue in the variational spectrum. In addition, our estimates, with $p = 2$, provide an improvement on the known results for the linear case.

An important side result of our work is that we are able to prove that, if the graph is a tree, the variational eigenvalues are all and only the eigenvalues of the p-Laplacian operator and that the k-th eigenfunction, if everywhere nonzero, admits exactly k nodal domains. This result extends what is already known in the particular cases of the path graph [86] and the star graph [3], and is a generalization to the p-Laplacian of analogous findings known in the linear case [5, 7]. This is of independent interest for its potential applications to nonlinear spectral graph sparsification, expander graphs, and graph clustering [84]. In particular, note that our findings, in combination with (3.1), show that the k-th order isoperimetric constant of a tree coincides with the k-th variational eigenvalue of the 1-Laplacian.

The results presented in this chapter has been collected in the paper: "Nodal domain count for the generalized graph p-Laplacian", published on the journal "Applied and Computational Harmonic Analysis", [31].

3.2 Notation

Let $\mathcal{G} = (V, E)$ be a connected undirected graph, where V and E are the sets of nodes and edges endowed with positive measures $\nu : V \to \mathbb{R}_+$ and $\mu : E \to \mathbb{R}_+$, respectively. Given a function $f : V \to \mathbb{R}$, for any $p > 1$ consider the p-Laplacian operator:

$$(\Delta_p f)(u) := \sum_{v \sim u} \mu_{uv} |f(u) - f(v)|^{p-2} (f(u) - f(v)) \qquad \forall u \in V\,,$$

where $v \sim u$ denotes the presence of an edge between v and u. We highlight that the above definition of the p-Laplacian operator differs from the one given in the introduction (see Remark 2.2.1), here indeed we are neglecting the term ω^p. However, since in this chapter we are not interested in varying p, we observe that this choice corresponds to redefining $\mu := \mu\omega^p$ and thus does not affect the results. On the other hand, this choice allows us to lighten the notation. Observe also that, while the p-Laplacian can be studied also for $p = 1$ and $p = \infty$, throughout this chapter we will not consider this limit cases and we will always implicitly assume that $p \in (1, \infty)$. Throughout the chapter, we will often use the function $\phi_p(x) := |x|^{p-2}x$, so that Δ_p can be compactly written as:

$$(\Delta_p f)(u) := \sum_{v \sim u} \mu_{uv} \phi_p\left(f(u) - f(v)\right) \qquad \forall u \in V\,.$$

In analogy to the linear case, where the generalized Laplacian is defined as the Laplacian plus a diagonal matrix [87], we define the generalized p-Laplacian (or p-Schrödinger) operator as

$$(\mathcal{H}_p f)(u) := (\Delta_p f)(u) + \kappa_u |f(u)|^{p-2} f(u) \qquad \forall u \in V\,,$$

where κ_u is a real coefficient. We say that f is an eigenfunction of $\mathcal{H}_p$ if there exists $\lambda \in \mathbb{R}$ such that

$$(\mathcal{H}_p f)(u) = \lambda\, \nu_u |f(u)|^{p-2} f(u) \qquad \forall u \in V\,. \tag{3.2}$$

Similarly to the linear case, generalized p-Laplacians and their eigenfunctions are directly connected with the solutions of Dirichlet problems on graphs for the p-Laplacian operator. In fact, assuming to have a graph $\mathcal{G} = (V, E)$ with boundary B, if f is a solution to the Dirichlet problem

$$\begin{cases} (\Delta_p f)(u) = \lambda\, \nu_u |f(u)|^{p-2} f(u) & \forall u \in V \setminus B \\ f(u) = 0 & \forall u \in B \end{cases}\,,$$

we deduce that f is automatically also solution to the following eigenvalue equation for a generalized p-Laplacian, where the information about the boundary nodes has been condensed in the nodal weights (see (2.3)):

$$\sum_{v\in V_I}\mu_{uv}|f(u)-f(v)|^{p-2}(f(u)-f(v))+\Big(\sum_{v\in V_B}\mu_{uv}\Big)|f(u)|^{p-2}f(u)$$
$$=\lambda\,\nu_u|f(u)|^{p-2}f(u)\,.$$

In other words, the p-Laplacian Dirichlet problem with zero boundary conditions is equivalent to the eigenvalue problem (3.2) for the generalized p-Laplacian $\mathcal{H}_p$ with $\kappa_u=\sum_{v\in V_B}\mu_{uv}$.

Finally, the following definition introduces the concept of strong nodal domains of $\mathcal{G}$, corresponding to a given function $f:V\to\mathbb{R}$.

Definition 3.2.1 (Nodal domains)**.** *Consider a graph $\mathcal{G}=(V,E)$ and a function $f:V\to\mathbb{R}$. A set of vertices $A\subseteq V$ is a nodal domain induced by f if the subgraph $\mathcal{G}_A$ with vertices in A is a maximal connected subgraph of $\mathcal{G}$ where f is nonzero and has constant sign. For convenience, in the following we will refer interchangeably to both A and $\mathcal{G}_A$ as the nodal domain induced by f.*

Sometimes it is useful to distinguish between maximal subgraphs, where the sign is strictly defined, and those where zero entries are allowed. In particular, when zero entries of f are allowed in the definition above, the maximal subgraphs are called weak nodal domains, whereas the maximal subgraphs with strictly positive or strictly negative sign, as in Definition 3.2.1, are called strong nodal domains. However, as in this work we are not interested in weak nodal domains, throughout we shall simply use the term "nodal domain" to refer to the strong nodal domains, as defined above.

3.3 Variational spectrum and main results

In this section we state our main results and will devote the remainder of the chapter to their proof. We first recall the notion of variational spectrum. A set of N variational eigenvalues of the generalized p-Laplacian on the graph $\mathcal{G}=(V,E)$ can be defined via the Lusternik–Schnirelman theory and the min-max procedure based on the Krasnoselskii genus, which we review below [63].

Definition 3.3.1 (Krasnoselksii genus)**.** *Let X be a Banach space and consider the class $\mathcal{A}$ of closed symmetric subsets of X, $\mathcal{A}=\{A\subseteq X|\ A\ \text{closed}\,,\ A=-A\}$. For any $A\in\mathcal{A}$ consider the space of the Krasnoselskii test maps on A of dimension k:*

$$\Lambda_k(A)=\{\varphi:A\to\mathbb{R}^k\,\textit{continuous and such that}\ \varphi(x)=-\varphi(-x)\}.$$

The Krasnoselskii genus of A is the number $\gamma(A)$ defined as

$$\gamma(A)=\begin{cases}\inf\{k\in\mathbb{N}\,:\,\exists\,\varphi\in\Lambda_k(A)\ \textit{s.t.}\ 0\notin\varphi(A)\} & \\ \infty & \textit{if}\ \nexists\, k\ \textit{as above}\\ 0 & \textit{if}\ A=\emptyset\end{cases}.$$

Our reference Banach space is the space of vertex states $X = \{f : V \to \mathbb{R}\} = \mathbb{R}^N$ and $\mathcal{A}$ denotes the family of all closed symmetric subsets of $\mathbb{R}^N$. Let $\mathcal{S}_p = \{f \in X : \|f\|_p = 1\}$ be the p-unit sphere on X and for $1 \le k \le N$ consider the family of closed symmetric subsets of $\mathcal{S}_p$ of genus greater than k

$$\mathcal{F}_k(\mathcal{S}_p) := \{A \in \mathcal{A} \cap \mathcal{S}_p \,|\, \gamma(A) \ge k\}\,.$$

In order to define the variational eigenvalues of $\mathcal{H}_p$, we consider the Rayleigh quotient functional

$$\mathcal{R}_{\mathcal{H}_p}(f) = \frac{\sum_{uv \in E} \mu_{uv}|f(u) - f(v)|^p + \sum_{u \in V} \kappa_u |f(u)|^p}{\sum_{u \in V} \nu_u |f(u)|^p}\,.$$

As $\mathcal{R}_{\mathcal{H}_p}$ is positively scale invariant, i.e. $\mathcal{R}_{\mathcal{H}_p}(\alpha f) = \mathcal{R}_{\mathcal{H}_p}(f)$ for all $\alpha > 0$, it is not difficult to observe that the eigenvalues and eigenfunctions of the generalized p-Laplacian operator are the critical values and the critical points of $\mathcal{R}_{\mathcal{H}_p}$ on $\mathcal{S}_p$. The Lusternik-Schnirelman theory allows us to define a set of N variational such critical values, via the following principle

$$\lambda_k = \min_{A \in \mathcal{F}_k(\mathcal{S}_p)} \max_{f \in A} \mathcal{R}_{\mathcal{H}_p}(f)\,. \tag{3.3}$$

We emphasize that the Krasnoselskii genus is a homeomorphism-invariant generalization to symmetric sets of the notion of dimension. In particular, if $A \in \mathcal{A}$ is the intersection of any subspace of dimension k with $\mathcal{S}_p$, then $\gamma(A) = k$. Moreover, note that any A such that $\gamma(A) \ge k$ contains at least k mutually orthogonal functions (see e.g. [83]). Therefore, the definition in (3.3) is a generalization of the Courant-Fisher min-max characterization of the eigenvalues of a symmetric matrix, as $\mathcal{F}_k(\mathcal{S}_p)$ contains all subspaces of dimension greater than k. However, while Courant-Fisher applies directly to the case $p = 2$, linear subspaces alone are not sufficient to provide critical points in the general case $p \ne 2$.

3.3.1 Multiplicity and γ-multiplicity

Similarly to the case of symmetric matrices, we note that the variational eigenvalues $\{\lambda_k\}$ are by definition an increasing sequence. This allows us to define a notion of multiplicity for variational eigenvalues:

Definition 3.3.2. *Let λ_k be a variational eigenvalue of $\mathcal{H}_p$. If λ_k appears m times in the sequence of the variational eigenvalues*

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_{k-1} < \lambda_k = \cdots = \lambda_{k+m-1} < \lambda_{k+r} \le \cdots \le \lambda_N\,.$$

we say that λ_k has multiplicity m and we write $\mathrm{mult}_{\mathcal{H}_p}(\lambda_k) = m$ *or simply* $\mathrm{mult}(\lambda_k) = m$ *when no ambiguity may occur.*

The notion of multiplicity defined above applies only to variational eigenvalues. In the case of a generic eigenvalue λ, we can use the Krasnoselskii genus to extend the notion of geometric multiplicity to the nonlinear setting:

Definition 3.3.3. *Let λ be an eigenvalue of $\mathcal{H}_p$. If*

$$\gamma\Big(\{f \in \mathcal{S}_p \,:\, \mathcal{H}_p(f) = \lambda\nu|f|^{p-2}f\}\Big) = m$$

we say that λ has γ-multiplicity m and we write $\gamma\text{-mult}_{\mathcal{H}_p}(\lambda) = m$, or simply $\gamma\text{-mult}(\lambda) = m$ when no ambiguity may occur.

Finally, we define simple eigenvalues

Definition 3.3.4. *We say that λ is a simple eigenvalue of $\mathcal{H}_p$ if λ has a unique eigenfunction $f \in \mathcal{S}_p$.*

Notice that the notions of multiplicity and γ-multiplicity do not coincide and an eigenvalue with γ-multiplicity equal to one is not necessarily simple. Viceversa, if λ is a simple eigenvalue, then necessarily $\gamma\text{-mult}(\lambda) = 1$ and, if λ is variational then also $\text{mult}(\lambda) = 1$. This result is a direct consequence of the next lemma, whose proof follows directly from Lemma 5.6 and Proposition 5.3, Chapter II of [85]:

Lemma 3.3.5. *If λ is a variational eigenvalue, then*

$$\gamma\text{-mult}(\lambda) \geq \text{mult}(\lambda)\,.$$

Note that the inequality above implies, in particular, that, to any variational eigenvalue λ, there correspond at least $\text{mult}(\lambda)$ orthogonal eigenfunctions. Finally, we remark the following direct consequence of Lemma3.3.5

Corollary 3.3.6. *Let $\mathcal{H}_p$ be the generalized p-Laplacian operator on a graph $\mathcal{G}$ with N nodes. Let $\{\lambda_i\}_{i=1}^n$ be the variational eigenvalues of $\mathcal{H}_p$ counted without multiplicity, i.e. $\lambda_i \neq \lambda_j$ $\forall i \neq j$. Then*

$$\sum_{i=1}^{n} \gamma\text{-mult}(\lambda_i) \geq \sum_{i=1}^{n} \text{mult}(\lambda_i) = N\,,$$

with the equality holding if and only if $\gamma\text{-mult}(\lambda_i) = \text{mult}(\lambda_i)$, for all $i = 1, \ldots, n$.

3.3.2 Main results

We present below our main results. Recalling the idea summarized in the introduction, our strategy for counting nodal domains of generalized p-Laplacians is to come up with algorithmic steps to remove vertices and edges from the original graph in such a way that the original eigenpairs can be recovered from the eigenpairs of the new graph. Since the proofs of our main results require relatively long arguments, we state the results here and devote the remainder of the chapter to their proofs. In particular, after discussing in Sections 3.4 and 3.5 a number of preliminary observations and results, which are of independent interest, Section

3.6 will provide proofs for Theorems 3.3.7 and 3.3.8, which deal with the special case of trees and forests, whereas Section 3.7 will present the proofs of Theorems 3.3.9 and 3.3.10, which address the case of general graphs.

Notice that, unlike linear operators, the variational spectrum does not cover the entire spectrum of the generalized p-Laplacian and, in general, establishing whether a certain eigenvalue is variational or not is still an open problem. For example, Amghibech shows in [2] that the p-Laplacian on a complete graph admits more than just the variational eigenvalues. Another simple example for the setting $\mu \equiv 1$, $\nu \equiv 1$ and $\kappa \equiv 0$ is provided by Figure 2.1 in the introduction, while a more refined analysis of non-variational eigenvalues is recently provided by Zhang in [90].

Our first main result shows that the situation is different for the special case of trees and, more in general, forests. In fact, as for the standard linear case, we prove that when $\mathcal{G}$ is a forest, the variational spectrum covers all the eigenvalues of the generalized p-Laplacian. Here and in the following, we use the symbol $\sqcup$ to denote disjoint union.

Theorem 3.3.7. *Let $\mathcal{G} = \sqcup_{i=1}^{k} \mathcal{T}_i$ be a forest, $\mathcal{H}_p$ a generalized p-Laplacian operator on $\mathcal{G}$, $p > 1$, and $\mathcal{H}_p(\mathcal{T}_i)$ the restriction of $\mathcal{H}_p$ to the i-th tree $\mathcal{T}_i$. Then $\mathcal{H}_p$ admits only variational eigenvalues and for any such eigenvalue λ it holds*

$$\mathrm{mult}_{\mathcal{H}_p}(\lambda) = \gamma\text{-}\mathrm{mult}_{\mathcal{H}_p}(\lambda) = \sum_{i=1}^{k} \mathrm{mult}_{\mathcal{H}_p(\mathcal{T}_i)}(\lambda)$$

where $\mathrm{mult}_{\mathcal{H}_p(\mathcal{T}_i)}(\lambda) = 0$ *if λ is not an eigenvalue of $\mathcal{H}_p(\mathcal{T}_i)$.*

In addition, we are able to prove the following theorem about the number of nodal domains induced on a forest, which generalizes well-known results for the case of the linear Laplacian [5, 7, 44].

Theorem 3.3.8. *Let $\mathcal{G} = \sqcup_{i=1}^{m} \mathcal{T}_i$ be a forest and consider the generalized p-Laplacian operator $\mathcal{H}_p$, $p > 1$, on $\mathcal{G}$. If f_k is an everywhere nonzero eigenfunction associated to the eigenvalue $\lambda_k = \cdots = \lambda_{k+m-1}$ of $\mathcal{H}_p$, then f_k changes sign on exactly $k-1$ edges. In other words, f_k induces exactly $k-1+m$ nodal domains.*

Next, we address the case of general graphs. A tight upper bound for the number of nodal domains of the eigenfunctions of the p-Laplacian on graphs is provided in [86]. It is not difficult to observe that the same upper bound carries over unchanged to the generalized p-Laplacian case. This is summarized in the following result.

Theorem 3.3.9. *Suppose that $\mathcal{G}$ is connected and $\lambda_1 < \lambda_2 \le \cdots \le \lambda_N$ are the variational eigenvalues of $\mathcal{H}_p$, $p > 1$. Let λ be an eigenvalue of $\mathcal{H}_p$ such that $\lambda < \lambda_k$. Any eigenfunction associated to λ induces at most $k-1$ nodal domains.*

Finally, the following theorem provides novel lower bounds for the number of nodal domains of $\mathcal{H}_p$ in the case of general graphs. Morever, when tailored to the case $p = 2$, it provides improved estimates of the nodal domain count that are strictly tighter than the currently available results [6, 87]. We will discuss these properties in more details below.

Theorem 3.3.10. *Suppose that $\mathcal{G}$ is a connected graph with $\beta = |E| - |V| + 1$ independent loops, and let $\lambda_1 \le \cdots \le \lambda_N$ be the variational eigenvalues of $\mathcal{H}_p$, $p > 1$. For a function $f : V \to \mathbb{R}$, let $\mathcal{N}(f)$ be the number of nodal domains induced by f, $l(f)$ the number of independent loops in $\mathcal{G}$ where f has constant sign and $\{v_i\}_{i=1}^{z(f)}$ the nodes such that $f(v_i) = 0$, with $z(f)$ being the number of such nodes. Let $\mathcal{G}' = \mathcal{G} \setminus \{v_i\}_{i=1}^{z(f)}$ be the graph obtained by removing from $\mathcal{G}$ all the nodes where f is zero as well as all the edges connected to those nodes. Let $c(f)$ be number of connected components of $\mathcal{G}'$ and $\beta'(f) = |E'| - |V'| + c(f)$ the number of independent loops of the graph $\mathcal{G}'$. Then:*

P1. If f is an eigenfunction of $\mathcal{H}_p$ with eigenvalue λ such that $\lambda > \lambda_k$, then f induces strictly more than $k - \beta + l(f) - z(f)$ nodal domains. Precisely, it holds $\mathcal{N}(f) \ge k - \beta'(f) + l(f) - z(f) + c(f)$.

P2. If f is an eigenfunction of $\mathcal{H}_p$ corresponding to the variational eigenvalue $\lambda_k > \lambda_{k-1}$ with $\mathrm{mult}_{\mathcal{H}_p}(\lambda_k) = m$, then $\mathcal{N}(f) \ge k+m-1-\beta'(f)+l(f)-z(f)$.

Before moving on, we would like to briefly comment on the above results and provide a comparison with respect to lower bounds available for the linear case $p = 2$. First, note that both P1 and P2 in Theorem 3.3.10 apply to variational eigenvalues of $\mathcal{H}_p$. However they are not corollaries of each other in the sense that there are settings where P1 is more informative than P2 and vice-versa. Indeed, if λ_k is a variational eigenvalue of multiplicity equal to one, then $\lambda_k > \lambda_{k-1}$ and from P1 we obtain

$$\mathcal{N}(f) \ge k - \beta'(f) + l(f) - z(f) + (c(f) - 1) \tag{3.4}$$

for any eigenfunction f of λ_k, which is strictly tighter than the lower bound in P2. However, in P2, when λ_k has multiplicity $m > 1$, we have $\lambda_k > \lambda_{k-1}$ and the two lower bounds in P1 and P2 cannot be compared a-priori. Instead, their combination leads to

$$\mathcal{N}(f) \ge \max\Big\{\Big(k-\beta'(f)+l(f)-z(f)+(c(f)-1)\Big), \Big(k-\beta'(f)+l(f)-z(f)+(m-1)\Big)\Big\} \tag{3.5}$$

for any eigenfunction f of λ_k. These observations allow us to draw new lower bounds for the eigenvalues of $\mathcal{H}_2$, which are all variational. In fact, for a simple eigenvalue λ_k of $\mathcal{H}_2$ with an everywhere nonzero eigenfunction f, it was proved in [6] that $\mathcal{N}(f) \ge k - \beta + l(f)$. Point P1 of Theorem 3.3.10 improves this result by allowing eigenfunctions with zero nodes via inequality (3.4). Note that this implies in particular $\mathcal{N}(f) \geq k-\beta+l(f)-z(f)$, as $c > 1$ and $\beta'(f) \leq \beta$. Similarly, when λ_k is a multiple eigenvalue of multiplicity m and f is any corresponding eigenfunction, it was proved in [87] for the linear case that $\mathcal{N}(f) \geq k + m - 1 - \beta - z(f)$. Combining P1 and P2 allows us to improve this bound via the sharper version given in (3.5), which further accounts for the number of independent loops of f, the number of connected components of $\mathcal{G}'$ and its number of independent loops.

3.4 Properties of the p-Laplacian eigenfunctions

In this section we present a brief review of the main results about p-Laplacian eigenpairs and discuss how to extend them to the generalized p-Laplacian case. We start with the characterization of the first and the last variational eigenvalues. Classical results available for the p-Laplacian equation in the continuous case [67, 68], have been extended to the discrete case in [54, 86]. In the following we present analogous results for the generalized p-Laplacian operator on graphs.

3.4.1 The smallest variational eigenvalue

We consider in this section the first (smallest) variational eigenvalue λ_1 of $\mathcal{H}_p$, defined as:

$$\lambda_1 = \min_{f \in \mathcal{S}_p} \mathcal{R}_{\mathcal{H}_p}(f)\,. \tag{3.6}$$

Since obviously $\mathcal{R}_{\mathcal{H}_p}(f) \geq \mathcal{R}_{\mathcal{H}_p}(|f|)$ for all $f \in \mathcal{S}_p$, we can assume that the first eigenfunction f_1 is always greater than or equal to zero. On the other hand, if $f_1(u) = 0$ for some $u \in V$, then from the eigenvalue equation (3.2) we get

$$\mathcal{H}_p(f_1)(u) = -\sum_{v \in V} \Big(\mu_{uv}|f_1(v)|^{p-2} f_1(v)\Big) = 0\,,$$

which shows that f_1 assumes both positive and negative values, contradicting the previous assumption. We deduce that any eigenfunction corresponding to λ_1 must be everywhere strictly positive, i.e., $f_1(u) > 0$ $\forall u$. This observation generalizes a well-known result for the standard p-Laplacian ($\kappa_u = 0$) on a graph with no boundary for which $\lambda_1 = 0$ and any corresponding eigenfunction is positive and has constant values [2]. We formalize the characterization of the first eigenfunction of the generalized p-Laplacian in the following theorem.

Theorem 3.4.1. *Let λ_1 be the first eigenvalue of $\mathcal{H}_p$ on a connected graph $\mathcal{G}$ as in* (3.6)*. Then*

1. *λ_1 is simple and the corresponding eigenfunction f_1 is strictly positive, i.e., $f_1(u) > 0$ $\forall u \in V$;*

2. *if g is an eigenfunction associated to an eigenvalue λ of $\mathcal{H}_p$ and $g(u) > 0$ $\forall u \in V$, then $\lambda = \lambda_1$.*

Proof. We have already observed that any eigenfunction f of λ_1 must be strictly positive so it remains to prove that for any strictly positive eigefunction g associated to an eigenvalue λ, it holds $g = f_1$ and $\lambda = \lambda_1$. From the eigenvalue equation, we have

$$\sum_{v\sim u} \mu_{uv}\phi_p(f_1(u) - f_1(v)) = \big(\lambda_1\nu_u - \kappa_u\big)\, f_1(u)^{p-1} \tag{3.7}$$
$$\sum_{v\sim u} \mu_{uv}\phi_p(g(u) - g(v)) = \big(\lambda\nu_u - \kappa_u\big)\, g(u)^{p-1} \tag{3.8}$$

where ϕ_p is defined in Section 3.2. If we multiply both sides of (3.7) by the function $f_1(u) - g(u)^p f_1(u)^{1-p}$ and both sides of (3.8) by $g(u) - f_1(u)^p g(u)^{1-p}$, we obtain

$$\sum_{v\sim u} \mu_{uv}\phi_p\big(f_1(u) - f_1(v)\big)\Big(f_1(u) - g(u)^p f_1(u)^{1-p}\Big) = \big(\lambda_1\nu_u - \kappa_u\big)\Big(f_1(u)^p - g(u)^p\Big),$$
$$\sum_{v\sim u} \mu_{uv}\phi_p\big(g(u) - g(v)\big)\Big(g(u) - f_1(u)^p g(u)^{1-p}\Big) = \big(\lambda\nu_u - \kappa_u\big)\Big(g(u)^p - f_1(u)^p\Big).$$

Summing the two equations first together and then over all the vertices, we obtain

$$S(f_1, g) + S(g, f_1) = \big(\lambda_1 - \lambda\big)\sum_{u\in V} \nu_u\Big(f_1(u)^p - g(u)^p\Big) \tag{3.9}$$

with

$$S(f, g) = \sum_{uv\in E} \mu_{uv}\left(|g(u) - g(v)|^p - \phi_p(f(u) - f(v))\Big(\frac{g(u)^p}{f(u)^{p-1}} - \frac{g(v)^p}{f(v)^{p-1}}\Big)\right).$$

If we apply Lemma B.0.1 to the above sums first with $\alpha = f_1(u)/f_1(v) > 0$ and then with $\alpha = g(u)/g(v) > 0$, we deduce that both $S(f_1, g)$ and $S(g, f_1)$ are non-negative. Thus, if $\lambda = \lambda_1$, in which case $S(f_1, g) = S(g, f_1) = 0$, again using Lemma B.0.1, we obtain

$$\frac{g(u)}{g(v)} = \frac{f_1(u)}{f_1(v)}\,,$$

which shows that, since the graph is connected, g is proportional to f_1, implying λ_1 simple. This allows us to conclude that f_1 and g are the same eigenfunction. Assume now that there exists an eigenvalue $\lambda > \lambda_1$ with the associated eigenfunction g being strictly positive. For any $\varepsilon > 0$, the function εg is also a strictly positive eigenfunction associated with λ. Thus we can find a $\varepsilon > 0$ such that $f_1(u) > \varepsilon g(u)$ for all $u \in V$. This yields an absurd in (3.9) as the left hand side term is strictly positive and the right hand side is strictly negative. Thus, every eigenfunction that does not change sign has to be necessarily associated to the first eigenvalue and this concludes the proof. $\square$

The following corollary is a direct consequence of Theorem 3.4.1 which states among other things that $\lambda_1 < \lambda_2$, and generalizes to $\mathcal{H}_p$ a well-known property of the eigenfunctions of the standard p-Laplacian (see e.g. [86, Cor. 3.6])

Corollary 3.4.2. *Any eigenvector associated to an eigenvalue different from λ_1 has at least two nodal domains.*

3.4.2 The largest variational eigenvalue

Opposite to the case of the first variational eigenvalue, the last variational eigenvalue realizes the maximum of the Rayleigh quotient:

$$\lambda_N = \max_{f \in \mathcal{S}_p} \mathcal{R}_{\mathcal{H}_p}(f)$$

and, following [3], one can provide an upper bound to the magnitude of λ_N in terms of μ, ν and the potential κ.

Proposition 3.4.3. *The largest variational eigenvalue λ_N of the generalized p-Laplacian operator $\mathcal{H}_p$ defined on a connected graph satisfies:*

$$|\lambda_N| \leq \max_{u \in V} \Big(2^{p-1} \sum_{v \sim u} \frac{\mu_{uv}}{\nu_u} + \frac{|\kappa_u|}{\nu_u} \Big) .$$

Proof. Let f_N be an eigenfunction associated to λ_N and let u_0 be a node where νf_N assumes the maximal absolute value $|\nu_{u_0} f_N(u_0)| = \max_{v \in V} |\nu_v f_N(v)|$. Then, from the eigenvalue equation, we have

$$\nu_{u_0} |\lambda_N| |f_N(u_0)|^{p-1} = \Big| \sum_{v \sim u_0} \mu_{u_0 v} \phi_p\big(f_N(u_0) - f_N(v)\big) + \kappa_{u_0} \phi_p\big(f_N(u_0)\big) \Big|$$

from which we obtain

$$|\lambda_N| \leq \sum_{v \sim u_0} \frac{\mu_{u_0 v}}{\nu_{u_0}} 2^{p-1} + \frac{|\kappa_{u_0}|}{\nu_{u_0}} \leq \max_{u \in V} \big(2^{p-1} \sum_{v \sim u} \frac{\mu_{uv}}{\nu_u} + \frac{|\kappa_u|}{\nu_u} \big). \qquad \square$$

As done for the first eigenfunction, we provide here a characterization of the sign pattern of the last (maximal) eigenfunction in the particular case of bipartite graphs. Our result extends to the generalized p-Laplacian the analogous results obtained in the linear case in [8, 72] and in the case of the p-Laplacian with Dirichlet boundary conditions in [54].

Theorem 3.4.4. *If $\mathcal{G}$ is a bipartite connected graph, then the largest eigenvalue λ_N of $\mathcal{H}_p$ is simple and the corresponding unique eigenfunction f_N is such that $f_N(u) f_N(v) < 0$, for any $u \sim v$.*

Proof. We start by proving that if $f \in \mathcal{S}_p$ is a maximizer of the Rayleigh quotient, necessarily $f(u)f(v) < 0$, $\forall u \sim v$. Indeed, since $\mathcal{G}$ is a bipartite graph we can decompose V into two subsets $V = V_1 \sqcup V_2$, such that if $u, v \in V_i$, $i = 1, 2$, then $u \not\sim v$. Thus, starting from f, we define f' such that $f'(u) = |f(u)|$, $\forall u \in V_1$ and $f'(u) = -|f(u)|$, $\forall u \in V_2$. Now observe that

$$\begin{aligned}\mathcal{R}_{\mathcal{H}_p}(f) &= \sum_{uv \in E} \mu_{uv}|f(u) - f(v)|^p + \sum_{u \in V} \kappa_u |f(u)|^p \\ &\leq \sum_{uv \in E} \mu_{uv}\big||f(u)| + |f(v)|\big|^p + \sum_{u \in V} \kappa_u |f(u)|^p = \mathcal{R}_{\mathcal{H}_p}(f')\end{aligned}$$

where the equality holds if and only if $f = \pm f'$. Since f is a maximal eigenfunction, then $f = f'$ up to a sign and thus $f(u)f(v) \leq 0$, $\forall u \sim v$. To conclude, if $f'(u) = 0$ then, for $u \in V_1$ we have $\lambda_n f'(u) = \mathcal{H}_p(f')(u) \leq 0$ and the equality holds only if $f'(v) = 0$ for every $v \sim u$. Since the graph is connected this would lead to the absurd $f' \equiv 0$. Thus, we have that $f'(u) \neq 0$, $\forall u$, implying $f(u)f(v) < 0$, $\forall u \sim v$.

We now prove uniqueness of the maximizer. Given two maximizers $f, g \in \mathcal{S}_p$ such that

$$\mathcal{R}_{\mathcal{H}_p}(f) = \lambda_n = \mathcal{R}_{\mathcal{H}_p}(g)\,,$$

up to a sign as above, f and g must be strictly greater than zero on V_1 and strictly smaller than zero on V_2. Then, similarly to the proof of Theorem 3.4.1, we first multiply the eigenvalue equations for f and g by $f(u) - |g(u)|^p/\phi_p(f(u))$ and $g(u) - |f(u)|^p/\phi_p(g(u))$, respectively. Then, we sum the two equations together and over all the nodes to obtain:

$$\begin{aligned}&\sum_{uv \in E} \mu_{uv}\bigg(|g(u) - g(v)|^p - \phi_p\big((f(u) - f(v)\big)\Big(\frac{|g(u)|^p}{\phi_p\big(f(u)\big)} - \frac{|g(v)|^p}{\phi_p\big(f(v)\big)}\Big)\bigg) + \\ &\quad \sum_{uv \in E} \mu_{uv}\bigg(|f(u) - f(v)|^p - \phi_p\big(g(u) - g(v)\big)\Big(\frac{|f(u)|^p}{\phi_p\big(g(u)\big)} - \frac{|f(v)|^p}{\phi_p\big(g(v)\big)}\Big)\bigg) = 0\end{aligned}$$

From Lemma B.0.1, both the sums above are smaller than zero unless $f = g$, thus showing uniqueness of the maximizer and hence of the maximal eigenfunction f_N. □

Corollary 3.4.5. *Consider a graph $\mathcal{G}$ and the generalized p-Laplacian operator $\mathcal{H}_p$. Then, the graph $\mathcal{G}$ is bipartite and connected if and only if the maximal eigenfunction f_N of $\mathcal{H}_p$ induces exactly N nodal domains.*

Proof. If the graph is bipartite, by Theorem 3.4.4 the N-th variational eigenfunction is unique and induces N nodal domains. Vice-versa, let f_N be an eigenfunction such that f_N induces exactly N nodal domains. Then, considering $V_1 = \{v | f_N(v) > 0\}$ and $V_2 = \{v | f_N(v) < 0\}$, we have $V = V_1 \sqcup V_2$ and each node in V_1 is connected only to nodes in V_2, showing that the graph is bipartite. □

3.4.3 Further properties of $\mathcal{H}_p$ and its eigenfunctions

Observe that, similarly to the linear Schrödinger operator and unlike the p-Laplacian case, the eigenvalues of the generalized p-Laplacian depend on the potential κ_u and may attain both positive and negative values. This follows directly from the eigenvalue equation (3.2) for (λ_1, f_1):

$$\sum_{v\sim u}\Big(\mu_{uv}|f_1(u)-f_1(v)|^{p-2}(f_1(u)-f_1(v))\Big)+\kappa_u f_1(u)^{p-1}=\lambda_1\,\nu_u f_1(u)^{p-1}\,.$$

In fact, summing over all the vertices $u \in V$ yields

$$\lambda_1=\frac{\sum_{u\in V}\kappa_u f_1(u)^{p-1}}{\sum_{u\in V}\nu_u f_1(u)^{p-1}}=\frac{\sum_{u\in V}\frac{\kappa_u}{\nu_u}\nu_u f_1(u)^{p-1}}{\sum_{u\in V}\nu_u f_1(u)^{p-1}}$$

which shows that λ_1 is in the convex hull of the coefficients $\{\frac{\kappa_u}{\nu_u}\}$ and, since κ_u may be negative, $\mathcal{H}_p$ may not be positive definite.

The next lemmas extend to the generalized p-Laplacian the results proved in [86] for the standard p-Laplacian, and provide partial orderings for the given eigenpairs. In particular, Lemma 3.4.6 below follows directly by replacing the standard p-Laplacian with the generalized operator $\mathcal{H}_p$ in the proof of [86, Lemma 3.8] and, for this reason, its proof is omitted.

Lemma 3.4.6. *If f is an eigenfunction relative to an eigenvalue λ and $A_1,\dots,A_m$ are the nodal domains of f, consider $f|_{A_i}$ the function that is equal to f on A_i and zero on $V\setminus A_i$. Then*

$$\max\Big\{\mathcal{R}_{\mathcal{H}_p}(f) : f\in \mathrm{span}\{f|_{A_1},\dots,f|_{A_m}\}\Big\}\le\lambda\,.$$

Corollary 3.4.7. *If f is an eigenfunction relative to an eigenvalue λ and f induces k nodal domains, then $\lambda\ge\lambda_k$.*

Proof. If $A_1,\dots,A_k$ are the nodal domains of f, consider $f|_{A_i}$ the function that is equal to f on A_i and zero on $V\setminus A_i$. If $\pi=\mathrm{span}\{f|_{A_1},\dots,f|_{A_k}\}$, then notice that the Krasnoselskii genus of $\mathcal{A}$ is k, i.e., $\gamma(\pi)=k$. Thus, from Lemma 3.4.6, we have that $\lambda_k=\min_{A\in\mathcal{F}_k}\max_{f\in A}\mathcal{R}_{\mathcal{H}_p}\le\max_{f\in\pi}\mathcal{R}_{\mathcal{H}_p}(f)\le\lambda$. □

We conclude by noticing that, combining Corollaries 3.4.2 and 3.4.7, one immediately obtains that, as for the standard p-Laplacian, the second variational eigenvalue λ_2 of the generalized p-Laplacian is the smallest eigenvalue larger than λ_1. Precisely, it holds:

Theorem 3.4.8.

$$\lambda_2=\min\{\lambda:\lambda>\lambda_1\ \textit{is an eigenvalue of}\ \mathcal{H}_p\}$$

3.5 Graph perturbations and Weyl's-like inequalities

In this section we show how to modify the graph and, consequently, the associated generalized p-Laplacian operator, maintaining eigenpairs. In particular, we will show how to remove edges and nodes obtaining a new generalized p-Laplacian operator on a simpler graph written as a "small" perturbation of the initial operator $\mathcal{H}_p$. For this perturbed operator, we will prove Weyl's like inequalities relating its variational eigenvalues to those of the starting operator.

3.5.1 Removing an edge

Consider a graph $\mathcal{G}$ and the generalized p-Laplacian operator $\mathcal{H}_p$ on $\mathcal{G}$. Let λ and f be an eigenvalue and a corresponding eigenfunction of $\mathcal{H}_p$ and let $e_0 = (u_0, v_0)$ be an edge of the graph such that $f(u_0)f(v_0) \neq 0$. We want to define a new generalized p-Laplacian operator $\mathcal{H}'_p$ on the graph $\mathcal{G}' := \mathcal{G} \setminus e_0$, such that (f, λ) is also an eigenpair of $\mathcal{H}'_p$.

Our strategy extends to the nonlinear case the work of [5], where the new operator $\mathcal{H}'_p$ is written as a rank-one variation of the starting Laplacian. To this end, we write $\mathcal{H}'_p = \mathcal{H}_p + \Xi_p$ where

$$(\Xi_p g)(u) = \begin{cases} 0 & \text{if } u \neq u_0, v_0 \\ \mu_{u_0v_0}\Big(\phi_p(1-\alpha)\phi_p\big(g(u_0)\big) - \phi_p\big(g(u_0) - g(v_0)\big)\Big) & \text{if } u = u_0 \\ \mu_{u_0v_0}\Big(\phi_p(1-\frac{1}{\alpha})\phi_p\big(g(v_0)\big) - \phi_p\big(g(v_0) - g(u_0)\big)\Big) & \text{if } u = v_0 \end{cases}, \tag{3.10}$$

$\alpha := f(v_0)/f(u_0)$ and $\phi_p(x) := |x|^{p-2}x$ as before. It can be easily proved that $\mathcal{H}'_p$ is obtained from $\mathcal{H}_p$ by considering the edge weights μ' given by $\mu'_{uv} = \mu_{uv}$ if $(uv) \neq (u_0v_0)$ and $\mu'_{u_0v_0} = 0$. Thus $\mathcal{H}'_p$ can be seen as a generalized p-Laplacian operator on a graph $\mathcal{G}'$ that is obtained from $\mathcal{G}$ by deleting the edge e_0 and that acts on the nodes that are not adjacent to e_0 exactly as $\mathcal{H}_p$ does. Observe that $\mathcal{H}'_p$ depends on the original eigenfunction f and a direct computation shows that (λ, f) is still an eigenpair of the new operator.

Now we want to compare the variational eigenvalues of $\mathcal{H}'_p$ with the ones of $\mathcal{H}_p$ with ordering purposes. We first write the Rayleigh quotient of the new operator $\mathcal{H}'_p$ as

$$\mathcal{R}_{\mathcal{H}'_p}(g) = \mathcal{R}_{\mathcal{H}_p}(g) + \mathcal{R}_{\Xi_p}(g)\,,$$

where, for $g \in \mathcal{S}_p$:

$$\frac{\mathcal{R}_{\Xi_p}(g)}{\mu_{u_0v_0}} = \left(\frac{|g(u_0)|^p}{\phi_p(f(u_0))} - \frac{|g(v_0)|^p}{\phi_p(f(v_0))}\right)\phi_p\big(f(u_0) - f(v_0)\big) \\ - \big(g(u_0) - g(v_0)\big)\phi_p\Big(g(u_0) - g(v_0)\Big)\,.$$

A direct application of Lemma B.0.1, shows that R_{Ξ_p} is positive if $\frac{f(v_0)}{f(u_0)}$ is negative and negative if $\frac{f(v_0)}{f(u_0)}$ is positive. Moreover, if we assume that $g(v_0)$ and $g(u_0)$ are non zero, we can write $\Xi_p g$ in the following equivalent way

$$(\Xi_p g)(u) = \begin{cases} 0 & \text{if } u \neq u_0, v_0 \\ \mu_{u_0 v_0}\phi_p\big(g(u_0)\big)\Big(\phi_p(1-\alpha) - \phi_p(1-\frac{g(v_0)}{g(u_0)}\Big) & \text{if } u = u_0 \\ \mu_{u_0 v_0}\phi_p\big(g(v_0)\big)\Big(\phi_p(1-\frac{1}{\alpha}) - \phi_p(1-\frac{g(u_0)}{g(v_0)})\Big) & \text{if } u = v_0 \end{cases} .$$

From this last equation we can easily see that, if $g(v_0) = \alpha g(u_0)$, then $\Xi_p g = \mathcal{R}_{\Xi_p}(g) = 0$.

To continue, we need the following lemma from [83], reported here without proof, which provides a bound on the Krasnoselskii genus of the intersection of different subsets.

Lemma 3.5.1. *[83, Prop. 4.4] Let X be a Banach space and $\mathcal{A}$ the class of the closed symmetric subsets of X. Given $A \in \mathcal{A}$, consider a Karsnoselskii test map $\varphi \in \Lambda_k(A)$ with $k < \gamma(A)$. Then, $\gamma(\varphi^{-1}(0)) \geq \gamma(A) - k$.*

Using the fact that R_{Ξ_p} is zero on the hyperplane $\pi = \{g : g(u_0)f(v_0) - g(v_0)f(u_0) = 0\}$, we obtain the following ordering of the k-th eigenvalue of $\mathcal{H}_p$ within the spectrum of $\mathcal{H}_p'$.

Lemma 3.5.2. *Assume that there exist an eigenfunction f of $\mathcal{H}_p$ and an edge $e_0 = (u_0, v_0)$ such that $f(u_0), f(v_0) \neq 0$ and consider the operator $\mathcal{H}_p' = \mathcal{H}_p + \Xi_p$, where Ξ_p is defined as in (3.10). Let η_k be the variational eigenvalues of $\mathcal{H}_p'$ and λ_k those of $\mathcal{H}_p$. The following inequalities hold:*

- *If $\frac{f(v_0)}{f(u_0)} < 0$, then $\eta_{k-1} \leq \lambda_k \leq \eta_k$;*
- *If $\frac{f(v_0)}{f(u_0)} > 0$, then $\eta_k \leq \lambda_k \leq \eta_{k+1}$.*

Proof. Let $\mathcal{F}_k$ be the Krasnoselskii family $\mathcal{F}_k = \{A \subseteq \mathcal{A} \cap \mathcal{S}_p | \gamma(A) \geq k\}$ as defined in Section 3.3. Let $A_k \in \mathcal{F}_k$ be such that $\lambda_k = \max_{f \in A_k} \mathcal{R}_{\mathcal{H}_p}(f)$, and let

$$\pi = \{g : g(u_0)f(v_0) - g(v_0)f(u_0) = 0\} .$$

Then $A_k \cap \pi = \phi|_{A_k}^{-1}(0)$, and from Lemma 3.5.1, since $\phi|_{A_k} \in \Lambda_1(A_k)$, we have

$$\gamma(A_k \cap \pi) \geq \gamma(A_k) - 1 \geq k - 1 .$$

Thus, $A_k \cap \pi \in \mathcal{F}_{k-1}$ and

$$\eta_{k-1} = \min_{A \in \mathcal{F}_{k-1}} \max_{f \in A} \mathcal{R}_{\mathcal{H}_p'}(f) \leq \max_{f \in A_k \cap \pi} \mathcal{R}_{\mathcal{H}_p'}(f) = \max_{f \in A_k \cap \pi} \mathcal{R}_{\mathcal{H}_p} + \mathcal{R}_{\Xi_p} \leq \lambda_k .$$

This implies that $\eta_{k-1} \leq \lambda_k$. Moreover, since $\mathcal{R}_{\Xi_p} \geq 0$ we have that $\mathcal{R}_{\mathcal{H}_p'}(f) \geq \mathcal{R}_{\mathcal{H}_p}$, which implies

$$\lambda_k = \min_{A \in \mathcal{F}_k} \max_{f \in A} \mathcal{R}_{\mathcal{H}_p}(f) \leq \min_{A \in \mathcal{F}_k} \max_{f \in A} \mathcal{R}_{\mathcal{H}_p'}(f) = \eta_k ,$$

and this concludes the proof of the first inequality. The second inequality can be proved analogously, by exchanging the roles of $\mathcal{H}_p'$ and $\mathcal{H}_p$. □

3.5.2 Removing a node

Consider a generalized p-Laplacian operator $\mathcal{H}_p$ defined on a graph $\mathcal{G}$ and let u_0 be a node of $\mathcal{G}$. We want to define a new operator $\mathcal{H}'_p$ on the graph $\mathcal{G}' := \mathcal{G} \setminus \{u_0\}$ that behaves on $\mathcal{G}'$ like $\mathcal{H}_p$ behaves on the hyperplane $\{f \,:\, f(u_0) = 0\}$. Note that, in the linear case, this operation is equivalent to considering the principal submatrix of the generalized Laplacian matrix obtained by removing the row and the column relative to u_0. If we remove a node u_0, we have to remove also all its incident edges from the graph $\mathcal{G}$. Thus, on the graph $\mathcal{G}'$ we can define the generalized p-Laplacian:

$$\mathcal{H}'_p(f)(u) := \sum_{v \in V'} \mu'_{uv} \phi_p(f(u) - f(v)) + \kappa'_u \phi_p(f(u)) \,,$$

where $V' = V \setminus \{u_0\}$, $\mu'_{uv} = \mu_{uv}$ and $\kappa'_u = \kappa_u + \mu_{uu_0}$.

Remark 3.5.3. *If f is an eigenfunction of the generalized p-Laplacian $\mathcal{H}_p$ on $\mathcal{G}$, with eigenvalue λ and such that $f(u_0) = 0$, then the restriction f' of f on the graph $\mathcal{G}' = \mathcal{G} \setminus \{u_0\}$ is automatically an eigenfunction of $\mathcal{H}'_p$ with eigenvalue λ. Indeed, for each $u \neq u_0$ we have*

$$\lambda \nu_u \phi_p(f(u)) = \sum_{v \neq u_0} \mu_{uv} \phi_p(f(u) - f(v)) + \mu_{uu_0} \phi_p(f(u)) + \kappa_u \phi_p(f(u)) = \mathcal{H}'_p(f)(u) \,.$$

Next, we provide an ordering for the variational eigenvalues of $\mathcal{H}'_p$, in comparison with those of $\mathcal{H}_p$, as stated in the following lemma.

Lemma 3.5.4. *Given a node u_0 of $\mathcal{G}$, let $\mathcal{H}_p$ and $\mathcal{H}'_p$ be generalized p-Laplacian operators defined on the graphs $\mathcal{G}$ and $\mathcal{G}' = \mathcal{G} \setminus \{u_0\}$, respectively, and let λ_k and η_k be the corresponding variational eigenvalues. Then:*

$$\lambda_k \leq \eta_k \leq \lambda_{k+1} \,.$$

Proof. Let $\mathcal{S}'_p = \{f : V' \to \mathbb{R} : \|f\|_p = 1\}$ and consider $A'_k \in \mathcal{F}_k(\mathcal{S}'_p)$ such that

$$\eta_k = \max_{f \in A'_k} \mathcal{R}_{\mathcal{H}'_p}(f) = \max_{f \in A'_k} \sum_{(uv) \in E'} w'(uv)|f(u) - f(v)|^p + \sum_{u \in V'} \kappa'_u |f(u)|^p \,,$$

where E' is the set of edges of $\mathcal{G}'$. Consider now A_k, the immersion of A'_k in the $N-1$ dimensional hyperplane $\pi = \{f : V \to \mathbb{R} : f(u_0) = 0\}$, i.e. the set of functions f that, when restricted to the nodes different from u_0, belong to A'_k and are such that $f(u_0) = 0$. Thus, A_k belongs to $\mathcal{F}_k(\mathcal{S}_p)$ since A_k and A'_k are homeomorphic, and we obtain:

$$\lambda_k = \min_{A \in \mathcal{F}_k} \max_{f \in A} \mathcal{R}_{\mathcal{H}_p}(f) \leq \max_{f \in A_k} \mathcal{R}_{\mathcal{H}_p}(f) = \max_{f \in A'_k} \mathcal{R}_{\mathcal{H}'_p}(f) = \eta_k \,.$$

To prove the other inequality, consider $A_{k+1} \in \mathcal{F}_{k+1}(\mathcal{S}_p)$ such that

$$\lambda_{k+1} = \max_{f \in A_{k+1}} \mathcal{R}_{\mathcal{H}_p}(f)\,.$$

Because of Lemma 3.5.1, we have that $\gamma(A_{k+1} \cap \{f : f(u_0) = 0\}) \geq k$, which implies that $A_{k+1} \cap \{f : f(u_0) = 0\} \in \mathcal{F}_k(\mathcal{S}_p)$. Thus

$$\eta_k \leq \max_{f \in A'_k} \mathcal{R}_{\mathcal{H}'_p}(f) = \max_{f \in A_{k+1} \cap \pi} \mathcal{R}_{\mathcal{H}_p}(f) \leq \max_{f \in A_{k+1}} \mathcal{R}_{\mathcal{H}_p}(f) = \lambda_{k+1}\,,$$

where A'_k is the set of functions $f' : V' \to \mathbb{R}$ obtained as the restriction of functions from $A_{k+1} \cap \pi$ to $\mathcal{G}'$, i.e., $f' \in A'_k$ if the lifting $f : V \to \mathbb{R}$ defined as $f(u_0) = 0$ and $f(u) = f'(u)$, $\forall u \neq u_0$, belongs to $A_{k+1} \cap \pi$. □

This result can be generalized by induction to the case of n removed nodes, obtaining the main theorem of this section.

Theorem 3.5.5. *Let $\mathcal{H}_p$ be the generalized p-Laplacian operator defined on the graph $\mathcal{G}$ and let $\mathcal{G}'$ be the graph obtained from $\mathcal{G}$ by deleting the n nodes $u_1, u_2, \ldots, u_n$. Consider the generalized p-Laplacian operator on $\mathcal{G}'$ defined as*

$$\mathcal{H}'_p(u) := \sum_{v \in V'} \mu'_{uv} \phi_p(f(u) - f(v)) + \kappa'_u \phi_p(f(u))\,,$$

where $V' = V \setminus \{u_1, \ldots, u_n\}$, $\mu'_{uv} = \mu_{uv}$ and $\kappa'_u = \kappa_u + \sum_{i=1}^n \mu_{uu_i}$. Let $\{\lambda_k\}$ denote the variational eigenvalues of $\mathcal{H}_p$ and $\{\eta_k\}$ those of $\mathcal{H}'_p$. Then

$$\lambda_k \leq \eta_k \leq \lambda_{k+n}\,,$$

for any $k \in \{1, \ldots, |V| - n\}$.

Proof. The proof follows directly from Lemma 3.5.4, removing recursively the nodes $u_1, \ldots, u_n$. □

3.6 Nodal domain count on trees

In this section we deal with the case in which $\mathcal{T} := \mathcal{G} = (V, E)$ is a tree and we provide proofs of the two Theorems 3.3.7 and 3.3.8. In particular, we will prove that the eigenvalues of the generalized p-Laplacian on a tree are all and only the variational ones. Moreover, again restricting ourselves to trees, we will show that, if an eigenfunction of the k-th variational eigenvalue is everywhere non zero, then it induces exactly k nodal domains. This generalizes to the nonlinear case a well-known result for the linear Shrödinger operator.

In the following, given a tree $\mathcal{T} = (V, E)$, we assume a root $r \in V$ is chosen arbitrarily. This provides a partial ordering of the nodes so that a precise root is automatically assigned to any subtree of $\mathcal{T}$. In particular, we write $v < u$ if v is a descendant of u and $v \prec u$ if v is a direct child of u. Moreover, for each node $u \in V$, we let $\mathcal{T}_u$ denote the subtree of $\mathcal{T}$ having u as root and formed by all the descendants of u. On this subtree we can define a new operator $\mathcal{H}_p^u$ obtained as follows: starting from $\mathcal{T}$, we remove all the nodes that do not belong to $\mathcal{T}_u$ and, for each deleted node, we modify the original operator $\mathcal{H}_p$ on $\mathcal{T}$ as in Section 3.5.2.

We also consider the operator $\mathcal{H}_p^{\widetilde{u}}$, obtained by removing from $\mathcal{T}_u$ also the root node u and by modifying $\mathcal{H}_p^u$ accordingly. This latter operator is defined on a subforest, $\mathcal{T}_{\widetilde{u}} = \sqcup_i \mathcal{T}_i$, that has as many connected components as the number of children of u. From the generalized Weyl's inequalities of Section 3.5, we have that

$$\cdots \leq \lambda_i(\mathcal{H}_p^u) \leq \lambda_i(\mathcal{H}_p^{\widetilde{u}}) \leq \lambda_{i+1}(\mathcal{H}_p^u) \leq \cdots$$

where $\lambda_i(\mathcal{H}_p^u)$ and $\lambda_i(\mathcal{H}_p^{\widetilde{u}})$ denote the i-th variational eigenvalue of $\mathcal{H}_p^u$ and $\mathcal{H}_p^{\widetilde{u}}$, respectively. Observe also that $\mathcal{H}_p^{\widetilde{u}} = \bigoplus_{v_i \prec u} \mathcal{H}_p(\mathcal{T}_i)$, where $\mathcal{H}_p(\mathcal{T}_i)$ is the generalized p-Laplacian of $\mathcal{T}_i$ and $v_i \prec u$ indicates that v_i is a direct child of u.

3.6.1 Generating functions

Consider now an eigenfunction f of $\mathcal{H}_p$ with eigenvalue λ and assume that $f \neq 0$ everywhere. For each u different from the root r, we denote by u_F the parent of u in $\mathcal{T}$. Then, the following quantity

$$g(u) := \frac{f(u_F)}{f(u)}$$

is well defined for all $u \neq r$ and we can rewrite the eigenvalue equation $\mathcal{H}_p(f)(u) = \lambda \nu_u \phi_p(f(u))$ as

$$\mu_{uu_F} \phi_p(1 - g(u)) = \lambda \nu_u - \kappa_u - \sum_{v \prec u} \mu_{uv} \phi_p\Big(1 - \frac{1}{g(v)}\Big), \tag{3.11}$$

for each $u \neq r$.

Now, if u is a leaf, Equation (3.11) allows us to write $g(u)$ explicitly as a function of λ:

$$g_u(\lambda) = 1 + \phi_p^{-1}\Big(\frac{\kappa_u - \nu_u \lambda}{\mu_{uu_F}}\Big). \tag{3.12}$$

Similarly, for a generic node u different from the root, we can use (3.11) to characterize $g(u)$ implicitly as a function of the variable λ:

$$g_u(\lambda) = 1 + \phi_p^{-1}\left(\frac{\kappa_u - \nu_u \lambda + \sum_{v \prec u} \mu_{uv} \phi_p\Big(1 - \frac{1}{g_v(\lambda)}\Big)}{\mu_{uu_F}}\right). \tag{3.13}$$

Finally, for the root $u = r$, we define

$$g_r(\lambda) := 1 + \phi_p^{-1}\Big(\kappa_r - 1 - \lambda\nu_r + \sum_{v \prec r} \mu_{rv}\phi_p\Big(1 - \frac{1}{g_v(\lambda)}\Big)\Big)\,. \tag{3.14}$$

Observe that, whenever $g_v(\lambda) = 0$ with $v \prec u$, the function g_u is not well defined in λ. In this case, we say that λ is a pole of g_u. However, these discontinuities do not affect the definitions of g_w with $u < w$. Indeed, given a pole λ of the function g_v with $v \prec u$, we can define, both in (3.13) and (3.14), $1/g_v(\lambda) = 0$. The following Lemma 3.6.4 proves that such definition makes $1/g_v$ continuous in the poles of g_v. We call the functions g_u defined in (3.12), (3.13), (3.14) the *generating functions* of the eigenfunction f. In fact, we will show in Section 3.6.2 that $g_u(\lambda)$ characterizes the ratio $f(u_F)/f(u)$ for any eigenfunction of λ such that $f(u) \neq 0$. To this end, we need a number of preliminary results to unveil several properties of the generating functions g_u.

First, observe that when $f \neq 0$ everywhere, the claimed characterizing property follows directly from the definition of g_u. We highlight this statement in the following remark.

Remark 3.6.1. *If λ is an eigenvalue of $\mathcal{H}_p^{u_0}$ for some $u_0 \in V$, and f is an associated eigenfunction such that $f(u) \neq 0$, $\forall u \in \mathcal{T}_{u_0}$, then by the definition of the functions $g_u(\lambda)$ one directly obtains that*

$$\frac{f(u_F)}{f(u)} = g_u(\lambda) \neq 0, \quad \forall u \in \mathcal{T} \setminus \{u_0\} \qquad \textit{and} \qquad g_{u_0}(\lambda) = 0\,.$$

On the other hand, it is not difficult to observe that also the opposite property holds, namely

Remark 3.6.2. *Assume that λ is a zero of $g_{u_0}(\lambda)$ and $g_u(\lambda) \neq 0$, for all $u < u_0$, i.e., for all the descendents of u_0 and not only the direct children. Then λ is an eigenvalue of $\mathcal{H}_p^{u_0}$ and a corresponding eigenfunction f can be defined on the subtree $\mathcal{T}_{u_0}$ by setting $f(u_0) = 1$ and $f(u) = \frac{f(u_F)}{g_u(\lambda)}$, for all $u < u_0$. Indeed, with these definitions,* (3.12) *and* (3.13) *imply that λ and f are solutions of the system of equations*

$$\begin{cases} \displaystyle\sum_{v \prec u_0} \mu_{u_0 v}\phi_p(f(u_0) - f(v)) + \mu_{u_0 u_{0,f}}\phi_p(f(u_0)) + \kappa_u\phi_p(f(u_0)) = \lambda\nu_u\phi_p(f(u_0))\,, \\ \displaystyle\sum_{v \in \mathcal{G}} \mu_{uv}\phi_p(f(u) - f(v)) + \kappa_u\phi_p(f(u)) = \lambda\nu_u\phi_p(f(u)) \qquad \forall u < u_0\,, \end{cases}$$

which shows that λ and f are an eigenvalue and an eigenfunction of $\mathcal{H}_p^{u_0}$.

We have observed already that it is possible to relate the eigenpairs of the subtrees of $\mathcal{T}$ with the values of the functions $g_u(\lambda)$. Then, we show that it is always possible to immerse the tree $\mathcal{T}$ in a larger tree for which the values of the functions $g_u(\lambda)$ do not change.

Remark 3.6.3. *Let $\mathcal{H}_p$ be the generalized p-Laplacian operator defined on a tree $\mathcal{T} = (V, E)$. We can always immerse $\mathcal{T}$ in a tree $\widetilde{\mathcal{T}}$ obtained adding a parent r_F to the root r. Next, we define the generalized p-Laplacian operator $\widetilde{\mathcal{H}}_p$ on $\widetilde{\mathcal{T}}$ by setting $\widetilde{\mu}_{uv} = \mu_{uv}$, $\forall (u, v) \in E$, $\widetilde{\mu}_{rr_F} = 1$, $\widetilde{\kappa}_u = \kappa_u$, $\forall u \in V \setminus \{r\}$ and $\widetilde{\kappa}_r = \kappa_r - 1$. Considering $\widetilde{\mathcal{T}}_{r_F}$ and $\widetilde{\mathcal{H}}_p^{r_F}$, the subtree and the operator obtained removing the root r_F from $\widetilde{\mathcal{T}}$, it is straightforward to observe that $\mathcal{T} = \widetilde{\mathcal{T}}_{r_F}$ and $\mathcal{H}_p = \widetilde{\mathcal{H}}_p^{r_F}$. Morover, working on $\widetilde{\mathcal{T}}$ and the associated operator $\widetilde{\mathcal{H}}_p$, it is possible to introduce the functions $\widetilde{g}_u(\lambda)$ as in* (3.12)*,* (3.13)*,* (3.14)*.*

$$g_u(\lambda) = \widetilde{g}_u(\lambda), \quad \forall u \in \mathcal{T}\,.$$

Thus, the generalized p-Laplacian eigenavalue problem on a tree can always be studied as the generalized p-Laplacian eigenvalue problem on a subtree of a suitable larger tree.

Finally, the following lemma summarizes several relevant structural properties of the functions $g_u(\lambda)$.

Lemma 3.6.4. *For each $u \in V$, consider the function $g_u(\lambda)$ defined as in* (3.12)*–*(3.14)*. Then:*

1. *the poles of $g_u(\lambda)$ are the zeros of the functions $\{g_v(\lambda)\}_{v \prec u}$;*
2. *g_u is strictly decreasing between each two consecutive poles;*
3. $\lim_{\lambda\to-\infty} g_u = +\infty$*,* $\lim_{\lambda\to+\infty} g_u = -\infty$*,* $\lim_{\lambda\to p^-} g_u = -\infty$*,* $\lim_{\lambda\to p^+} g_u = +\infty$ *where p is any of the poles.*

In particular, the number of zeros of the function g_u is equal to the number of distinct zeros of the functions $\{g_v\}_{v \prec u}$ plus one.

Proof. Let $u \in V$, if u is a leaf then the three properties follow immediately from (3.12). Otherwise, assume by induction the thesis holds for each $v \prec u$. From (3.13), it immediately follows that the poles of g_u are the zeros of $\{g_v\}_{v \prec u}$. To show that the function $\sum_{v \prec u} \mu_{uv} \phi_p\Big(1 - \frac{1}{g_v(\lambda)}\Big)$ is strictly decreasing between any couple of neighboring poles, observe that $x \mapsto \phi_p(x)$ is strictly increasing and, by induction, $\forall v \prec u$, $\lambda \mapsto g_v(\lambda)$ is strictly decreasing between any two of its zeros (i.e. the poles of g_u). Moreover, since $\lambda \mapsto -\nu_u \lambda$ is decreasing and ϕ_p^{-1} increasing, we can conclude that the function $\lambda \mapsto g_u(\lambda)$ is strictly decreasing between any two of its poles. Finally, the limits of $g_u(\lambda)$ for $\lambda \to p^{\pm}$ in the third statement follow as a consequence of the previous observations, while the limits for $\lambda \to \pm\infty$ can be proved directly by the induction assumption. □

3.6.2 Eigenfunction characterization via generating functions

The following result shows that the generating functions $g_u(\lambda)$ always characterize the eigenfunctions of λ, generalizing what observed earlier in Remark 3.6.1.

Theorem 3.6.5. *Let (f, λ) be an eigenpair of a generalized p-Laplacian operator, $\mathcal{H}_p$, defined on a tree $\mathcal{T} = (V, E)$ with root r. For any node $u \in V \setminus \{r\}$ such that $f(u) \neq 0$, it holds*

$$\frac{f(u_F)}{f(u)} = g_u(\lambda)\,,$$

where u_F is the parent of u in $\mathcal{T}$.

Proof. If $f(v) \neq 0$ $\forall v \in \mathcal{T}$ we have already observed in Remark 3.6.1 that the thesis holds. Assume thus that there exist $v_1, \dots, v_k \in V$ such that

$$f(v_i) = 0, \quad i = 1, \dots, k \qquad \text{and} \qquad f(u) \neq 0, \quad \forall u \notin \{v_i\}_{i=1}^k$$

and let $\mathcal{T}' = \sqcup_{i=1}^h \mathcal{T}_i$ and $\mathcal{H}'_p = \oplus_{i=1}^h \mathcal{H}_p(\mathcal{T}_i)$ be the forest and the corresponding operator obtained from $\mathcal{G}$ removing the nodes $v_1, \dots, v_k$ as in Section 3.5.2. From Remark 3.5.3, $\forall\, i = 1, \dots, h$, the pair $(f|_{\mathcal{T}_i}, \lambda)$ is an eigenpair of $\mathcal{H}_p(\mathcal{T}_i)$ such that $f|_{\mathcal{T}_i}(u) \neq 0$, $\forall\, u \in \mathcal{T}_i$. Denoting with r_i the root of $\mathcal{T}_i$ and using (3.12),(3.13), (3.14) and Remark 3.6.1, $\forall\, \mathcal{T}_i$, starting from the leaves, we can define functions $g_u^{\mathcal{T}_i}(\lambda)$ such that

$$\begin{cases} g_u^{\mathcal{T}_i}(\lambda) = \dfrac{f|_{\mathcal{T}_i}(u_F)}{f|_{\mathcal{T}_i}(u)} \neq 0 & \forall u \in \mathcal{T}_i \setminus \{r_i\} \\ g_{r_i}^{\mathcal{T}_i}(\lambda) = 0 & \end{cases}$$

We claim that $\forall i = 1, \dots, h$ and $\forall u \in \mathcal{T}_i$, then $g_u(\lambda) = g_u^{\mathcal{T}_i}(\lambda)$. The thesis follows directly from this claim since

$$g_u(\lambda) = g_u^{\mathcal{T}_i}(\lambda) = \frac{f(u_F)}{f(u)} \qquad \forall\, u \in \mathcal{T}_i\,.$$

To prove the claim, first we introduce a partial ordering on $\{\mathcal{T}_i\}_{i=1}^h$ and $\{v_j\}_{j=1}^k$ so that $\mathcal{T}_i \prec v_j$ if v_j is the parent of the root of $\mathcal{T}_i$, while $v_j \prec \mathcal{T}_i$ if v_j is the child of some node of $\mathcal{T}_i$. Then, if $v_j \prec \mathcal{T}_i$ there exists a subtree $\mathcal{T}_l \prec v_j$. In fact, considering the generalized p-Laplacian eigenvalue equation in v_j with $u_i = {v_j}_F \in \mathcal{T}_i$, we can write

$$\mu_{v_j u_i} \phi_p\Big(f(u_i)\Big) + \sum_{u \prec\, v_j} \mu_{v_j u} \phi_p\Big(f(u)\Big) = 0\,.$$

Since $f(u_i) \neq 0$, there exists a node $u_l \prec v_j$ such that $f(u_l) \neq 0$ i.e. $u_l \in \mathcal{T}_l \prec v_j$. Similarly, one observes that if $f(v_j) = 0$, and v_j is a leaf, then also $f({v_j}_F) = 0$. Because of these two facts, there exists some $\mathcal{T}_{i_0}$ in the set $\{\mathcal{T}_i\}_{i=1}^h$ such that a node v_j with $v_j \prec \mathcal{T}_{i_0}$ cannot exist. In addition, the leaves of $\mathcal{T}_{i_0}$ are all and only the leaves of $\mathcal{T}$ that are connected to $\mathcal{T}_{i_0}$. It is then easy to observe that, for any such $\mathcal{T}_{i_0}$, by definition, $g_u^{\mathcal{T}_{i_0}}(\lambda) = g_u(\lambda)\ \forall\, u \in \mathcal{T}_{i_0},\ u \neq r_{i_0}$. Moreover, when $u = r_{i_0} \prec v_j$ we have

$$g_{r_{i_0}}^{\mathcal{T}_{i_0}}(\lambda) := 1 + \phi_p^{-1}\left(\kappa'_{r_{i_0}} - 1 - \lambda\nu_{r_{i_0}} + \sum_{v \prec r_1} \mu_{r_{i_0}v}\phi_p\Big(1 - \frac{1}{g_v(\lambda)}\Big)\right) = 0 \quad , \tag{3.15}$$

which implies

$$\kappa_{r_{i_0}} + \mu_{r_{i_0}v_j} - \lambda\nu_{r_{i_0}} + \sum_{v \prec r_1} \mu_{r_{i_0}v}\phi_p\Big(1 - \frac{1}{g_v(\lambda)}\Big) = 0\,, \tag{3.16}$$

where, since v_j is one of the removed nodes, we have used the expression $\kappa'_{r_{i_0}} = \kappa_{r_{i_0}} + \mu_{r_{i_0}v_j}$ that we obtain when moving from $\mathcal{H}_p$ to $\mathcal{H}'_p$ as in Section 3.5.2.

Thus, (3.16) implies

$$\begin{aligned} g_{r_{i_0}}(\lambda) &= 1 + \phi_p^{-1}\left(\frac{\kappa_{r_{i_0}} - \nu_{r_{i_0}}\lambda_0 + \sum_{v \prec r_{i_0}} \mu_{r_{i_0}v}\phi_p\Big(1 - \frac{1}{g_v(\lambda)}\Big)}{\mu_{r_{i_0}v_j}}\right) \\ &= 1 + \phi_p^{-1}\left(\frac{-\mu_{r_{i_0}v_j}}{\mu_{r_{i_0}v_j}}\right) = 0\,, \end{aligned} \tag{3.17}$$

that is $g_{r_{i_0}}(\lambda) = g_{r_{i_0}}^{\mathcal{T}_{i_0}}(\lambda) = 0$ and λ is a pole of g_{v_j}, due to Lemma 3.6.4.

Now, given a general subtree $\mathcal{T}_{i_0}$, w.l.o.g. we can assume that the claim is true for any $\mathcal{T}_i \prec v_j \prec \mathcal{T}_{i_0}$. Then if u is a leaf of $\mathcal{T}_{i_0}$ that is also a leaf of $\mathcal{T}$, clearly

$$g_u^{\mathcal{T}_{i_0}}(\lambda) = g_u(\lambda)\,.$$

Consider now the case of a leaf, u, of $\mathcal{T}_{i_0}$ that is not a leaf of $\mathcal{T}$. Since u is not a leaf of $\mathcal{T}$, by construction, there exist some node $v_j \prec u$ and some subtree $\mathcal{T}_i \prec v \prec \mathcal{T}_{i_0}$. For any such v_j, by the inductive assumption, λ has to be a pole of the corresponding g_{v_j}, leading to the following equation:

$$\begin{aligned} g_u(\lambda) &= 1 + \phi_p^{-1}\left(\frac{\kappa_u - \nu_u\lambda + \sum_{v_j \prec u} \mu_{uv_j}\phi_p\Big(1 - \frac{1}{g_{v_j}(\lambda)}\Big)}{\mu_{uu_F}}\right) \\ &= 1 + \phi_p^{-1}\left(\frac{\kappa_u - \nu_u\lambda + \sum_{v_i \prec u} \mu_{uv_i}\phi_p(1)}{\mu_{uu_F}}\right) \\ &= 1 + \phi_p^{-1}\left(\frac{\kappa'_u - \nu_u\lambda}{\mu_{uu_F}}\right) = g_u^{\mathcal{T}_i}(\lambda)\,. \end{aligned}$$

Here we have used as before the fact $\kappa'_u = \kappa_u + \sum_{v_j \prec u} \mu_{uv_j}$, see Section 3.5.2.

The case of u a generic node of $\mathcal{T}_{i_0}$ can be proved analogously assuming, w.l.o.g., the claim true for any $w < u$, $w \in \mathcal{T}_{i_0}$. Indeed, recalling $\kappa'_u = \kappa_u +$

$\sum_{v_j \prec u} \mu_{uv_j}$ and that, by the inductive assumption, λ is a pole of g_{v_j} for any $v_j \prec u$, we get

$$\begin{aligned} g_u(\lambda) &= 1 + \phi_p^{-1}\left(\mu_{uu_F}^{-1}\left(\kappa_u - \nu_u\lambda_0 + \sum_{v_j \prec u} \mu_{uv_j} + \sum_{\substack{w \prec u \\ w \in \mathcal{T}_{i_0}}} \mu_{uw}\phi_p\Big(1 - \frac{1}{g_w(\lambda)}\Big)\right)\right) \\ &= 1 + \phi_p^{-1}\left(\mu_{uu_F}^{-1}\left(\kappa_u' - \nu_u\lambda + \sum_{\substack{w \prec u \\ w \in \mathcal{T}_{i_0}}} \mu_{uw}\phi_p\Big(1 - \frac{1}{g_w(\lambda)}\Big)\right)\right) = g_u^{\mathcal{T}_i}(\lambda) \end{aligned}$$

The case of $u = r_{i_0}$ can be finally dealt with as done in (3.15)(3.17), concluding the proof. □

Corollary 3.6.6. *Let (f, λ) be an eigenpair of $\mathcal{H}_p$, then, if $g_u(\lambda) = 0$, necessarily $f(u_F) = 0$.*

Proof. First, notice that Remark 3.6.3 allows us to assume that, given any $u \in \mathcal{T} \setminus r$, also the node u_F has a parent, since we can always think of $\mathcal{T}$ as immersed in a larger tree with a suitably defined generalized p-Laplacian. Assume by contradiction that $f(u_F) \neq 0$, then by Theorem 3.6.5 we would have that

$$\frac{f(u_{FF})}{f(u_F)} = g_{u_F}(\lambda)\,.$$

At the same time, Lemma 3.6.4 implies that λ is a pole of the function g_{u_F}, leading to a contradiction. □

3.6.3 Multiplicity via generating functions

Theorem 3.6.5 shows that given any eigenpair (f, λ), the generating functions $\{g_u(\lambda)\}_u$ characterize the value of f up to a scaling factor. In this section we observe that counting the number of generating functions that vanishes on the eigenvalue λ provides several insights about its multiplicity.

First, we obtain the following sufficient result for simple eigenvalues, which directly follows from Theorem 3.6.5.

Proposition 3.6.7. *Let $\mathcal{H}_p$ be the generalized p-Laplacian operator defined on a tree $\mathcal{T} = (V, E)$, and let $u_0 \in V$. If $g_u(\lambda) \neq 0$, $\forall u < u_0$ and $g_{u_0}(\lambda) = 0$, then λ is a simple eigenvalue of $\mathcal{H}_p^{u_0}$ associated to an everywhere nonzero eigenfunction.*

Proof. We have alredy observed in Remark 3.6.2 that such a non zero eigenfunction f exists. Assume by absurd that there exist also an eigenfunction f^* of $\mathcal{H}_p^{u_0}$ associated to λ with $f^* \neq cf$, $\forall c \in \mathbb{R}$. Then, due to Theorem 3.6.5, there has to exist a node v such that $f^*(v) = 0$. Since $f^*(v) = 0$ and for any node u such that $f^*(u) \neq 0$ it holds that $f^*(u_F) = f^*(u)g_u(\lambda) \neq 0$, then necessarily we get that $f^*(u) = 0$, $\forall u < v$. On the other hand, by the generalized p-Laplacian eigenvalue equation, if $f^*(v) = 0$ and $f^*(u) = 0$ $\forall u \prec v$, then we have in addition that $f^*(u_F) = 0$. Thus, if f^* is zero in some node then necessarily $f^* = 0$ everywhere, yielding a contradiction. $\square$

Next, in the following lemma, we establish a more general condition for λ to be an eigenvalue, counted with its γ-multiplicity, in terms of zeros of the generating functions $g_u(\lambda)$.

Lemma 3.6.8. *Let $\mathcal{H}_p$ be a generalized p-Laplacian on a forest $\mathcal{G}$ and, for any $u \in V$ and any tree of the forest, let $g_u(\lambda)$ be the function defined in* (3.12)(3.13)(3.14)*. Given λ, assume there exist $v_1, \dots, v_k \in V$ such that*

$$g_{v_i}(\lambda) = 0 \qquad \forall i = 1, \dots, k\,.$$

Let $\{u_j\}_{j=1}^h$ be the set of the parents of the nodes $\{v_i\}_{i=1}^k$, where roots do not have parents. Then, λ is an eigenvalue of $\mathcal{H}_p$ if and only if $k - h > 0$, and

$$\gamma\text{-mult}(\lambda) = k - h\,.$$

Proof. From Theorem 3.6.5 and Corollary 3.6.6 we know that if λ is an eigenvalue, any corresponding eigenfunction f is such that

$$f(u_j) = 0 \qquad \text{and} \qquad \frac{f(w_F)}{f(w)} = g_w(\lambda) \quad \text{if } f(w) \neq 0\,.$$

Following the strategy of Section 3.5.2, remove the nodes $\{u_j\}_{j=1}^h$ from the forest $\mathcal{G}$ ending with a forest $\mathcal{G}'$ and an associated operator $\mathcal{H}_p'$ of the form

$$\mathcal{G}' = \bigsqcup_{l=1}^{n} \mathcal{T}_l \qquad \mathcal{H}_p' = \bigoplus_{l=1}^{n} \mathcal{H}_p(\mathcal{T}_l)\,.$$

for some $n \geq 1$. Then, from Remark 3.5.3, any eigenfunction of $\mathcal{H}_p$ with eigenvalue λ corresponds to an eigenfunction of $\mathcal{H}_p'$. In particular, given any subtree $\mathcal{T}_l$ and corresponding operator $\mathcal{H}_p(\mathcal{T}_l)$ it is easy to observe that

$$g_u^{\mathcal{T}_l}(\lambda) = g_u(\lambda) \qquad u \in \mathcal{T}_l\,,$$

where $g_u^{\mathcal{T}_l}$ are the generating functions defined starting from $\mathcal{H}_p(\mathcal{T}_l)$ via equations (3.12),(3.13),(3.14) (see the proof of Theorem 3.6.5 for a similar construction). Among the $\{\mathcal{T}_l\}_{l=1}^n$, let $\{\mathcal{T}_i'\}_{i=1}^k$ be the subtrees with root $r_i = v_i$. Due to Proposition 3.6.7, for any such $\mathcal{T}_i'$ and corresponding $\mathcal{H}_p(\mathcal{T}_i')$ there exists a unique everywhere nonzero eigenfunction f_i' of $\mathcal{H}_p(\mathcal{T}_i')$ with eigenvalue λ whose ratios $f_i'(w_F)/f_i'(w)$ are induced by the functions $g_w(\lambda)$, $\forall w \in \mathcal{T}_i'$. Moreover, notice that for any f eigenfunction of $\mathcal{H}_p$ with eigenvalue λ, since f is also an eigenfunction of $\mathcal{H}_p'$, we have $f|_{\mathcal{T}_i'} = \alpha_i f_i'$, for some $\alpha_i \in \mathbb{R}$.

On the other hand, on the subtrees $\{\mathcal{T}_j''\}_{j=1}^{n-k}$ whose root r_j is such that $r_j \neq v_i$ $\forall i = 1, \dots, k$, since $g_w(\lambda) \neq 0$, $\forall w \in \mathcal{T}_j''$, any eigenfunction associated to λ of

$\mathcal{H}_p$ has to be such that $f|_{\mathcal{T}_j''}(w) = 0$ because of Theorem 3.6.5. Indeed, suppose by contradiction that f is an eigenfunction associated to λ such that $f|_{\mathcal{T}_j''} \neq 0$, then f should be an eigenfunction of $\mathcal{H}_p(\mathcal{T}_j'')$ with same eigenvalue λ. However, $g_w(\lambda) \neq 0$, $\forall w \in \mathcal{T}_j''$ implies that $f|_{\mathcal{T}_j''}(w) \neq 0$, $\forall w \in \mathcal{T}_j''$ and thus, by Remark 3.6.1, we would have that $g_{r_j}(\lambda) = 0$, which is absurd.

Now, let f_i be the immersion of f_i' into $\mathbb{R}^N$ such that $f_i|_{\mathcal{T}_i'} = f_i'$ and $f_i(w) = 0$ for all $w \notin \mathcal{T}_i'$. Define $\Omega := \mathrm{span}\{f_i\}_{i=1}^k$ the k-dimensional linear space spanned by the f_i. The observations above together with Corollary 3.6.6 imply that if f is an eigenfunction of $\mathcal{H}_p$ with eigenvalue λ, then $f \in \Omega$. Starting from Ω, we want to recover all the possible eigenfunctions of $\mathcal{H}_p$ relative to λ. To this end, we select among the functions $f \in \Omega$ all those functions that satisfy the eigenvalue equation for $\mathcal{H}_p$ also in the removed points $\{u_j\}_{j=1}^h$. For any node u_j, let $w_{i,j}$ be the node in the neighborhood of u_j such that $w_{i,j} \in \mathcal{T}_i'$. Then, the $\mathcal{H}_p$ eigenvalue equation on a node u_j reads

$$\begin{aligned}\Theta_j(f) &:= \sum_i \mu_{w_{i,j}u_j}\phi_p(\beta_{w_{i,j}})\phi_p(f_i(r_i)) \\ &= \sum_i \mu_{w_{i,j}u_j}\phi_p(f_i(w_{i,j})) = \Big(\lambda\nu_{u_j} - \kappa_{u_j}\Big)\phi_p(f(u_j) = 0 \qquad \forall j = 1,\dots,h\end{aligned}$$

where we have used the fact that on any $\mathcal{T}_i'$ the ratios between the components of f_i are fixed by the functions $g_w(\lambda)$, $w \in \mathcal{T}_i'$ and thus, for every $w \in \mathcal{T}_i'$ there exists $\beta_w \neq 0$ such that $f_i(w) = \beta_w f_i(r_i)$.

We continue by defining the set $A = \{f \,|\, \Theta_j(f) = 0,\ \forall\, j = 1,\dots,h\}$. It is clear that f is an eigenfunction of $\mathcal{H}_p$ relative to λ if and only if $f \in A \cap \Omega$. Thus, let us now study the genus of such a set. Observe that $\gamma(A \setminus \{0\}) = N - h$, since A is diffeomorphic to a linear subspace of dimension $N - h$ through the homeomorphism of $\mathbb{R}^N$ given by $x_i \mapsto \phi_p(x_i)$, $i = 1,\dots,N$ (the set of equations $\{\Theta_j(f) = 0\}$ is transformed into a set of h linearly independent equation by the change of variable $y_i := \phi_p(f_i(r_i))$). Thus, if $k > h$ then the intersection is always nonempty because of Lemma 3.5.1 and in particular

$$\gamma(A \cap \Omega \setminus \{0\}) \geq \gamma(A \setminus \{0\}) - (N - k) = N - h - N + k = k - h\,.$$

Now we claim that it is possible to define a function $\widetilde{\psi}$ in the set of Krasnoselskii test maps $\Lambda_{k-h}(\Omega \cap A \setminus \{0\})$ such that $0 \notin \widetilde{\psi}(\Omega \cap A \setminus \{0\})$. This implies $\gamma(\Omega \cap A \setminus \{0\}) \leq k - h$, from which the statement follows. To construct such $\widetilde{\psi}$, consider the function $\psi \in \Lambda_k(\Omega)$ given by:

$$f = \sum_{i=1}^k \alpha_i f_i \mapsto \psi(f) := \Big(f(r_1)\,,\dots,f(r_k)\Big) = \Big(\alpha_1 f_1(r_1),\dots,\alpha_k f_k(r_k)\Big)\,.$$

It is easy to verify that $0 \notin \psi(\Omega \setminus \{0\})$, as $f_i(r_i) \neq 0$, $\forall i$. Since we want to define the function $\widetilde{\psi}$ on $A \cap \Omega$, we define $\widetilde{\psi}$ as the restriction to $\mathbb{R}^{k-h}$ of ψ. To define such a restriction, note that among the $\{\mathcal{T}_i'\}$ it is possible to select h distinct subtrees $\{\mathcal{T}_{i_l}'\}_{l=1}^h$ such that any node u_j is incident to some $\mathcal{T}_{i_l}'$. As before, let $w_{i_l,j}$ be the neighbor of u_j in $\mathcal{T}_{i_l}'$. Then consider the function $\widetilde{\psi} : \Omega \cap A \to \mathbb{R}^{k-h}$, entrywise defined as

$$\Big(\widetilde{\psi}(f)\Big)_i = \Big(\psi(f)\Big)_i \qquad i \neq i_l, \;\; l = 1, \dots, h\,.$$

It is easily proved that $\widetilde{\psi} \in \Lambda_{k-h}(\Omega \cap A)$. Finally, we show that if $\widetilde{\psi}(f) = 0$, for some $f \in \Omega \cap A$, then necessarily $f = 0$. To this end, write $f = \sum_{i=1}^k \alpha_i f_i$. If $\widetilde{\psi}(f) = 0$, then (up to a reordering of the indices of the chosen subtrees)

$$\alpha_i f_i(r_i) = 0 \qquad \forall i \neq i_l,\, l = 1 \dots, h\,.$$

Thus, $f = \sum_{l=1}^h \alpha_{i_l} f_{i_l}$. Then, observe that, since $f \in A$, we have

$$\Theta_j(f) = \sum_l \mu_{w_{i_l,j} u_j} \phi_p(\alpha_{i_l}) \phi_p(\beta_{w_{i_l,j}}) \phi_p(f_{i_l}(r_{i_l})) = 0 \quad \forall j = 1, \dots, h\,. \tag{3.18}$$

Consider a node u_{j_0} that is incident only to one of the subtrees $\{\mathcal{T}_{i_l}'\}_{l=1}^h$, say $\mathcal{T}_{i_h}'$, observe that such a node necessarily exists because there are no loops in a forest. Then (3.18) for $j = j_0$ reads (up to a reordering of the indices)

$$\Theta_{j_0}(f) = \mu_{u_{j_0} w_{i_h,j_0}} \phi_p(\alpha_{i_h}) \phi_p(\beta_{w_{i_h}}) \phi_p(f_{i_h}(r_{i_h})) = 0\,.$$

This means that $\alpha_{i_h} = 0$, i.e. $f = \sum_{l=1}^{h-1} \alpha_{i_l} f_{i_l}$. Repeating this procedure for all the h nodes u_j, we obtain that all the α_i have to be zero. In particular, this implies that, if $k = h$, then all the α_i are zero and thus $A \cap \Omega = \{0\}$ i.e. λ is not an eigenvalue, thus concluding the proof. $\square$

To conclude this preparatory section needed to tackle the proofs of Theorems 3.3.7 and 3.3.8, we show in the next result how the eigenvalues and the corresponding γ-multiplicities change when moving from $\mathcal{H}_p^u$ to $\mathcal{H}_p^{\widetilde{u}}$ (recall that $\mathcal{H}_p^u$ is the operator obtained by removing all the nodes different from u and its descendants while $\mathcal{H}_p^{\widetilde{u}}$ is the one obtained by removing also the node u). We state this result as a corollary of the previous lemma, recalling that λ is not an eigenvalue if and only if γ-mult$(\lambda) = 0$.

Corollary 3.6.9.

1. *Let λ be such that $g_u(\lambda) = 0$, then λ is an eigenvalue of $\mathcal{H}_p^u$ and* γ-mult$_{\mathcal{H}_p^u}(\lambda) =$ γ-mult$_{\mathcal{H}_p^{\widetilde{u}}}(\lambda) + 1$.

2. *Let λ be an eigenvalue of $\mathcal{H}_p^{\widetilde{u}}$ such that $g_w(\lambda) \neq 0$ for all $w \prec u$ and $g_u(\lambda) \neq 0$, then λ is an eigenvalue of $\mathcal{H}_p^u$ such that* γ-mult$_{\mathcal{H}_p^u}(\lambda) = \gamma$-mult$_{\mathcal{H}_p^{\widetilde{u}}}(\lambda)$.

3. *Let λ be an eigenvalue of $\mathcal{H}_p^{\widetilde{u}}$ and assume there exist $w_1, \dots, w_h \prec u$ with $g_{w_i}(\lambda) = 0$, then* γ-mult$_{\mathcal{H}_p^u}(\lambda) = \gamma$-mult$_{\mathcal{H}_p^{\widetilde{u}}}(\lambda) - 1$.

Proof. From Lemma 3.6.8 we know that λ is an eigenvalue if and only if $k-h>0$, where k is the number of nodes v such that $g_v(\lambda)=0$ and h is the number of their parents. In particular, we have that $\gamma\text{-mult}_{\mathcal{H}_p}(\lambda)=k-h$. To prove point 1, we observe that, since u is the root of the subtree $\mathcal{T}_u$, u has no parents and thus necessarily $h<k$. Moreover, by Lemma 3.6.4, $g_v(\lambda)\neq 0$, for all $v\prec u$ implying that h does not change when moving from $\mathcal{T}_{\widetilde{u}}$ to $\mathcal{T}_u$, while k increases by one. This implies the statement. To prove point 2, it is enough to observe that the number $k-h$ does not change going from $\mathcal{T}_{\widetilde{u}}$ to $\mathcal{T}_u$. Finally, in order to prove point 3 observe that in this case, when moving from $\mathcal{T}_{\widetilde{u}}$ to $\mathcal{T}_u$, k does not change while h increases by one. $\square$

3.6.4 Proofs of Theorems 3.3.7 and 3.3.8

We are finally ready to prove the main Theorems 3.3.7 and 3.3.8.

Proof of Theorem 3.3.7. We first observe that if the thesis holds for trees, then it holds as well for forests. To prove this fact, we assume that all the eigenvalues on trees are variational and the multiplicity matches the γ-multiplicity. Then, we note that if $\mathcal{G}=\sqcup_i\mathcal{T}_i$, with $\mathcal{T}_i=(V_i,E_i)$ trees, then $\mathcal{H}_p=\oplus_i\mathcal{H}_p(\mathcal{T}_i)$, where $\mathcal{H}_p(\mathcal{T}_i)$ is a suitable generalized p-Laplacian operator defined on $\mathcal{T}_i$, and hence $\sigma(\mathcal{H}_p)=\cup_i\sigma(\mathcal{H}_p(\mathcal{T}_i))$. In other words, the spectrum of $\mathcal{H}_p$ is the union of the spectra of the operators defined on the trees forming $\mathcal{G}$. Next, we observe that, by the same assumption on trees, $\sigma(\mathcal{H}_p(\mathcal{T}_i))$ is formed only by variational eigenvalues and thus it contains at most $|V_i|$ distinct elements, implying that $\sigma(\mathcal{H}_p)$ is formed by at most N different eigenvalues. Now, let $\lambda\in\sigma(\mathcal{H}_p)$. By the previous assumption, λ is a variational eigenvalue of $\mathcal{H}_p(\mathcal{T}_i)$, for some $i\in\{1,\dots,k\}$ and $\text{mult}_{\mathcal{H}_p(\mathcal{T}_i)}(\lambda)=\gamma\text{-mult}_{\mathcal{H}_p(\mathcal{T}_i)}(\lambda)=m_i(\lambda)$. Then, for any $i\in\{1,\dots,k\}$ there exists $\varphi_i\in\Lambda_{m_i(\lambda)}(A^i_\lambda)$ s.t. $0\notin\varphi_i(A^i_\lambda\cap\mathcal{S}_p)$, where

$$A^i_\lambda=\{f:V_i\to\mathbb{R}\,|\,\mathcal{H}_p(\mathcal{T}_i)(f)=\lambda|f|^{p-2}f\}\,.$$

Let

$$A_\lambda=\{f:V\to\mathbb{R}\,|\mathcal{H}_p(f)=\lambda|f|^{p-2}f\}\,.$$

Then, we can consider the extensions of the functions φ_i to A_λ and, given $m(\lambda)=\sum_i m_i(\lambda)$, define the function $\varphi_\lambda\in\Lambda_{m(\lambda)}(A_\lambda)$ as a linear combination of φ_i such that $0\notin\varphi_\lambda(A_\lambda\cap\mathcal{S}_p)$. This implies that $\gamma\text{-mult}_{\mathcal{H}_p}(\lambda)\le m(\lambda)$. Noting that $N=\sum_\lambda\sum_i m_i(\lambda)$, we have $\sum_\lambda\gamma\text{-mult}_{\mathcal{H}_p}(\lambda)\le N$. Thus, by Corollary 3.3.6 we conclude that all the eigenvalues of $\mathcal{H}_p$ are variational and $\text{mult}_{\mathcal{H}_p}(\lambda)=\gamma\text{-mult}_{\mathcal{H}_p}(\lambda)=\sum_{i=1}^k\text{mult}_{\mathcal{H}_p(\mathcal{T}_i)}(\lambda)$.

Now, we address the proof of the assumption and consider the case in which $\mathcal{G}=\mathcal{T}$ is a tree. The proof proceeds by induction on the number of nodes N. If $N=1$, from (3.12) and Proposition 3.6.7, we can conclude that there exists only one eigenvalue, λ_1, with $\gamma\text{-mult}_{\mathcal{H}_p}(\lambda_1)=\text{mult}_{\mathcal{H}_p}(\lambda_1)=1$. Assume now that $N>1$ and that the theorem holds up to $N-1$. First note that the inductive assumption and the result derived in the previous paragraph imply that the thesis holds for any forest composed by trees, each one, with less than N nodes. Then, fix a root r for $\mathcal{T}$ and consider $\mathcal{T}_{\widetilde{r}} = \sqcup_{i=1}^n \mathcal{T}_i$ and $\mathcal{H}_p^{\widetilde{r}} = \oplus_i \mathcal{H}_p(\mathcal{T}_i)$. Proceed by dividing the eigenvalues of $\mathcal{H}_p^{\widetilde{r}}$ into two sets $\{\varsigma_j\}_{j=1}^k$ and $\{\xi_l\}_{l=1}^h$, where each ς_j is a zero of some g_v for some $v \prec r$, whereas ξ_l is not. By the inductive assumption, we have that

$$\sum_{j=1}^k \gamma\text{-mult}_{\mathcal{H}_p^{\widetilde{r}}}(\varsigma_j) + \sum_{l=1}^h \gamma\text{-mult}_{\mathcal{H}_p^{\widetilde{r}}}(\xi_l) = N-1\,.$$

Now, let us divide the eigenvalues of $\mathcal{H}_p$ in a similar way. Let $\{\eta_i\}_{i=1}^{k+1}$ be the eigenvalues that are zeros of g_r. By Lemma 3.6.4 we know that they are exactly $k+1$, where k is the number of eigenvalues of $\mathcal{H}_p^{\widetilde{r}}$ that are zeros of some function g_v with $v \prec r$ By Lemma 3.6.4 we know that they are exactly $k+1$. Lemma 3.6.8 ensures that all the other eigenvalues of $\mathcal{H}_p$ are also eigenvalues of $\mathcal{H}_p^{\widetilde{r}}$ and, in particular, they must be either in the set $\{\varsigma_j\}_{j=1}^k$ or in the set $\{\xi\}_{l=1}^h$. Moreover, from Lemma3.6.4 we deduce that $\{\varsigma_j\}_{j=1}^k \cap \{\eta_i\}_{i=1}^{k+1} = \emptyset$ while $\{\xi_l\}_{l=1}^h \cap \{\eta_i\}_{i=1}^{k+1}$ could be non empty. In particular, let us set

$$\{\xi\}_{l=1}^{h_1} = \{\xi_l\}_{l=1}^h \setminus \{\xi_l\}_{l=1}^h \cap \{\eta_i\}_{i=1}^{k+1}\,.$$

Then,

$$\begin{aligned}
&\sum_{i=1}^{k+1} \gamma\text{-mult}_{\mathcal{H}_p}(\eta_i) + \sum_{j=1}^{k} \gamma\text{-mult}_{\mathcal{H}_p}(\varsigma_j) + \sum_{l=1}^{h_1} \gamma\text{-mult}_{\mathcal{H}_p}(\xi_l)\\
=&\sum_{i=1}^{k+1} \Big(\gamma\text{-mult}_{\mathcal{H}_p^{\widetilde{r}}}(\eta_i)+1\Big) + \sum_{j=1}^{k} \Big(\gamma\text{-mult}_{\mathcal{H}_p^{\widetilde{r}}}(\varsigma_j)-1\Big) + \sum_{l=1}^{h_1} \gamma\text{-mult}_{\mathcal{H}_p^{\widetilde{r}}}(\xi_l)\\
=&k+1-k+\sum_{j=1}^{k} \gamma\text{-mult}_{\mathcal{H}_p^{\widetilde{r}}}(\varsigma_j) + \sum_{l=1}^{h} \gamma\text{-mult}_{\mathcal{H}_p^{\widetilde{r}}}(\xi_l) = N-1+1 = N
\end{aligned}$$

where we have used Corollary 3.6.9 and the fact that $\{\xi_l\}_{l=1}^h \subseteq (\{\xi_l\}_{l=1}^{h_1} \cup \{\eta_i\}_{i=1}^{k+1})$, with $\gamma\text{-mult}_{\mathcal{H}_p^{\widetilde{r}}}(\eta_i) = 0$ if $\eta_i \notin \{\xi_l\}_{l=1}^h$. Together with Corollary3.3.6, the latter equality concludes the proof. □

Before moving on to the proof of Theorem 3.3.8, several observations are in order.

Remark 3.6.10. *Suppose $\mathcal{G} = \sqcup_{i=1}^m \mathcal{T}_i$ is a forest and let $\mathcal{H}_p = \oplus_{i=1}^m \mathcal{H}_p(\mathcal{T}_i)$ as before. If we consider an eigenfunction f_k of $\mathcal{H}_p$ that is everywhere nonzero Theorem 3.3.7 ensures that the corresponding eigenvalue λ_k has multiplicity exactly equal to m. Indeed, necessarily $f_k|_{\mathcal{T}_i}$ is an eigenfunction of $\mathcal{H}_p(\mathcal{T}_i)$ and, since it is everywhere non-zero, its corresponding eigenvalue is simple because of Proposition 3.6.7.*

In addition, observe that $e_0 = (u_0, v_0)$ is an edge of some $\mathcal{T}_i$ such that $f_k(u_0)f_k(v_0) < 0$ if and only if e_0 separates two distinct nodal domains. This means that the number of nodal domains induced by f_k on $\mathcal{T}_i$ is equal to the number of edges where $f_k|_{\mathcal{T}_i}$ changes sign, plus one. Thus the total number of nodal domains induced by f_k on $\mathcal{G}$ is equal to m plus the total number of edges where f_k changes sign. Combining all these observations, we eventually obtain the following proof.

Proof of Theorem 3.3.8. First we note that, by the hypotheses and Remark 3.6.10, the eigenvalue $\lambda_k = \lambda_{k+m-1}$, which corresponds to an everywhere nonzero eigenfunction, has multiplicity exactly equal to m, i.e.:

$$\lambda_{k-1} < \lambda_k = \cdots = \lambda_{k+m-1} < \lambda_{k+m}\,.$$

Now we prove by induction on k that, if f_k is an eigenfunction everywhere nonzero associated to the multiple eigenvalue $\lambda_k = \cdots = \lambda_{k+m-1}$, then f_k changes sign on exactly $k-1$ edges, implying that f_k induces exactly $k-1+m$ nodal domains. If $k=1$, $f_1|_{\mathcal{T}_i}$ is an eigenfunction related to the first eigenvalue of each operator $\mathcal{H}_p(\mathcal{T}_i)$, $i = 1, \dots, m$. Thus, as a consequence of Theorem 3.4.1, f_1 is strictly positive or strictly negative on every tree $\mathcal{T}_i$ and overall it induces m nodal domains. Moreover, it does not change sign on any edge. Now we assume the statement to be true for every $h < k$ and prove it for $h = k$. If $k > 1$, then f_k cannot be a first eigenfunction on every tree $\mathcal{T}_i$. Then, by Theorem 3.4.1, there exists at least one edge $e_0 = (u_0, v_0)$ in some $\mathcal{T}_{i_0}$ such that $f_k(u_0)f_k(v_0) < 0$. Thus, we operate as in Section 3.5.1 and remove edge e_0 to disconnect $\mathcal{T}_{i_0}$ into the two subtrees $\mathcal{T}'_{i_0}$ and $\mathcal{T}''_{i_0}$, so that the reduced graph $\mathcal{G}'$ is the union of the $m+1$ subtrees:

$$\mathcal{G}' = \Big(\bigsqcup_{\substack{i=1\\ i\neq i_0}}^{m} \mathcal{T}_i \Big) \sqcup \mathcal{T}'_{i_0} \sqcup \mathcal{T}''_{i_0}\,.$$

Similarly, the new operator $\mathcal{H}'_p$, obtained after removing e_0, can be decomposed as:

$$\mathcal{H}'_p = \Big(\bigoplus_{\substack{i=1\\ i\neq i_0}}^{m} \mathcal{H}_p(\mathcal{T}_i) \Big) \oplus \mathcal{H}'_p(\mathcal{T}'_{i_0}) \oplus \mathcal{H}'_p(\mathcal{T}''_{i_0})\,.$$

Now we can compare the eigenvalues $\{\eta_k\}$ of $\mathcal{H}'_p$ with the ones $\{\lambda_k\}$ of $\mathcal{H}_p$. From Lemma 3.5.2 we have:

$$\eta_{k-1} \le \lambda_k \le \eta_k \le \cdots \le \lambda_{k+m-1} \le \eta_{k+m-1} \le \lambda_{k+m}\,.$$

Due to Remark 3.6.10 and Theorem 3.3.7, the multiplicity of η_k has to be exactly $m+1$ and, by assumption, $\lambda_{k-1} < \lambda_k = \cdots = \lambda_{k+m-1} < \lambda_{k+m}$, i.e. $\eta_{k-1} = \cdots = \eta_{k+m-1}$. Moreover, by the inductive assumption, f_k changes sign on $k-1$ edges of the graph $\mathcal{G}'$. Thus, on the original graph $\mathcal{G}$, f_k changes sign $k-1+1=k$ times, concluding the proof. $\square$

3.7 Nodal domain count on generic graphs

In this final section we prove Theorems 3.3.9 and 3.3.10, providing upper and lower bounds for the number of nodal domains of f and for the number of edges where f changes sign. To this end, we need first a few preliminary results.

Consider an eigenpair (f, λ) of the generalized p-Laplacian operator $\mathcal{H}_p$ on a generic graph $\mathcal{G}$. Suppose we remove from $\mathcal{G}$ an edge $e_0 = (u_0, v_0)$, obtaining the graph $\mathcal{G}' = \mathcal{G} \setminus \{e_0\}$. Modifying accordingly the generalized p-Laplacian $\mathcal{H}_p$ as in Section 3.5.1, the new operator $\mathcal{H}_p'$ on $\mathcal{G}'$ is such that the pair (f, λ), restricted to $\mathcal{G}'$, remains an eigenpair of $\mathcal{H}_p'$.

Let us denote by $\Delta l(e_0, f)$ the variation of the number of independent loops of constant sign, namely the difference between the number of loops of constant sign of f in $\mathcal{G}'$ minus the number of those in $\mathcal{G}$. Similarly, let $\Delta\mathcal{N}(e_0, f)$ be the variation between the number of nodal domains induced by f on $\mathcal{G}'$ and on $\mathcal{G}$. We can characterize the difference $\Delta\mathcal{N}(e_0, f) - \Delta l(e_0, f)$ in terms of $\mathrm{sign}_{e_0}(f) = f(u_0)f(v_0)$, i.e. whether or not f changes sign over e_0. In fact, note that, by definition, if $\mathrm{sign}_{e_0}(f) < 0$, then neither the number of loops of constant sign nor the number of nodal domains changes. If, instead, $\mathrm{sign}_{e_0}(f) > 0$, then either the number of independent loops decreases by one ($\Delta(e_0, f) = -1$) or the number of nodal domains increases by one ($\Delta(e_0, f) = +1$). Overall, we have

$$\Delta\mathcal{N}(e_0, f) - \Delta l(e_0, f) = \begin{cases} 0 & \mathrm{sign}_{e_0}(f) < 0 \\ 1 & \mathrm{sign}_{e_0}(f) > 0 \end{cases} . \tag{3.19}$$

Based on the above formula, the following lemma provides a relation between the number of nodal domains induced by an eigenfunction and the number of edges where the sign changes. It is a generalization of a result from [4], which was proved for linear Laplacians and for the case of everywhere nonzero functions.

Lemma 3.7.1. *Consider $f : V \to \mathbb{R}$, a function on the graph $\mathcal{G}$. Denote by $\zeta(f)$ the number of edges where f changes sign, by $z(f)$ he number of nodes where f is zero, by $l(f)$ the number of independent loops in $\mathcal{G}$ where f has constant sign, and by $|E_z|$ the number of edges incident to the zero nodes. Then*

$$\zeta(f) = |E| - |E_z| + z(f) - |V| + \mathcal{N}(f) - l(f) \leq |E| - |V| + \mathcal{N}(f) - l(f) .$$

Proof. Operating as in Section 3.5.2, we start by removing from $\mathcal{G}$ all the $z(f)$ nodes where f is zero, thus obtaining a new graph $\mathcal{G}'$ with the corresponding new generalized p-Laplacian $\mathcal{H}_p'$. Since $|E_z|$ is the number of edges incident to the zero nodes that have been removed, the number of edges in $\mathcal{G}'$ can be estimated as $|E'| = |E| - |E_z| \leq |E| - z(f)$. Moreover, the edges incident to the zero nodes neither connect different nodal domains nor belong to constant sign loops. Hence, the restriction of f to $\mathcal{G}'$ remains an eigenfunction of $\mathcal{H}_p'$ having the same number of nodal domains and the same number of constant sign loops as f.

Next, we proceed by removing from $\mathcal{G}'$ all the edges $e_1, \dots, e_{\tau(f)}$ where f does not change sign (i.e., such that $\mathrm{sign}_{e_i}(f) > 0$) and modify consequently the operator $\mathcal{H}'_p$ as in Section 3.5.1. We obtain a new graph $\mathcal{G}''$ and the corresponding new operator $\mathcal{H}''_p$ in such a way that (λ, f) remains an eigenpair. Since the number of edges where f does not change sign is $\tau(f) = |E'| - \zeta(f)$, thanks to (3.19), we have that

$$\sum_{i=1}^{h} \Big(\Delta\mathcal{N}(e_i, f) - \Delta l(e_i, f) \Big) = |E'| - \zeta(f) \leq |E| - z(f) - \zeta(f)\,. \tag{3.20}$$

In the final graph $\mathcal{G}''$, only edges connecting nodes of different sign are present, so that each node is a nodal domain of f. As a consequence, there are a total $|V| - z$ nodal domains of f and no loops with constant sign. Then:

$$\sum_{i=1}^{h} \Delta\mathcal{N}(e_i, f) = |V| - z(f) - \mathcal{N}(f) \qquad \text{and} \qquad \sum_{i=1}^{h} \Delta l(e_i, f) = -l(f)\,,$$

and, by (3.19), $\tau(f) = \sum_{i=1}^{h} \Delta\mathcal{N}(e_i, f) - \Delta\ l(e_i, f) = |V| - z(f) - \mathcal{N}(f) + l(f)$. Hence, using (3.20) and the fact that $\tau(f) = |E| - |E_z| - \zeta(f)$, we obtain

$$\zeta(f) = |E| - |E_z| + z(f) - |V| + \mathcal{N}(f) - l(f) \leq |E| - |V| + \mathcal{N}(f) - l(f)$$

thus concluding the proof. □

The above lemma allows us to prove our third and fourth main results given in Theorems 3.3.9 and 3.3.10, which provide new upper and lower bounds for the number of nodal domains of the eigenfunctions of the generalized p-Laplacian, extending and generalizing previous results for the standard p-Laplacian and the linear Schrödinger operators [5, 86, 87]. As the proof of the two claims in Theorem 3.3.10 requires different arguments, we subdivide it into two parts, each addressing one of the two points P1 and P2 in the statement.

Proof of Theorem 3.3.9. For a connected graph $\mathcal{G}$, let f be an eigenfunction of $\mathcal{H}_p$ relative to λ. Let $\mathcal{N}(f)$ denote the number of nodal domains of f and let $\mathcal{G}_1, \dots, \mathcal{G}_{\mathcal{N}(f)}$ be such domains. Furthermore, let $e_1, \dots, e_\zeta$ be the edges where f changes sign and $v_1, \dots, v_z$ the nodes where f is zero, with $z = z(f)$ the number of such nodes. The proof proceeds as follows.

According to Section 3.5.2, we start by removing the nodes $v_1, \dots, v_z$ from $\mathcal{G}$ obtaining a new graph $\mathcal{G}'$. Operator $\mathcal{H}_p$ is then modified to form the operator $\mathcal{H}'_p$ in such a way that the restriction of f to $\mathcal{G}'$ is an eigenfunction of $\mathcal{H}'_p$ with the same eigenvalue λ. Moreover, as all the zero nodes that are not part of any nodal domain are now removed, we observe that f restricted to $\mathcal{G}'$ has no zeros and induces the same nodal domains that f induces on $\mathcal{G}$. From Lemma 3.5.4, we conclude that $\lambda < \lambda_k \leq \lambda'_k$, where λ'_k denotes the k-th variational eigenvalue of $\mathcal{H}'_p$.

Then, operating as in Section 3.5.1, we remove from $\mathcal{G}'$ all the edges $e_1,\dots,e_\zeta$ obtaining a new graph $\mathcal{G}''$ and the new operator $\mathcal{H}''_p$ such that f restricted to $\mathcal{G}''$ is an eigenfunction of $\mathcal{H}''_p$ with the same eigenvalue λ. Notice that, since we removed only nodes where f is zero and edges where f changes sign, then $\mathcal{G}''$ can be written as the disjoint union of the nodal domains, namely $\mathcal{G}'' = \sqcup_{i=1}^{\mathcal{N}(f)} \mathcal{G}_i$ and, as a consequence, we have

$$\mathcal{H}''_p = \bigoplus_{i=1}^{\mathcal{N}(f)} \mathcal{H}''_p(\mathcal{G}_i)$$

where $\mathcal{H}''_p(\mathcal{G}_i)$ is the restriction of the generalized p-Laplacian operator onto $\mathcal{G}_i$. Hence, from Lemma 3.5.2 we have

$$\lambda < \lambda'_k \le \lambda''_k \,. \tag{3.21}$$

where λ''_k denotes the k-th variational eigenvalue of $\mathcal{H}''_p$. Note that the restriction $f|_{\mathcal{G}_i}$ to each of the nodal domains $\mathcal{G}_i$ of f has constant sign and it is then necessarily the first eigenpair of $\mathcal{H}''_p(\mathcal{G}_i)$ corresponding to λ (see Theorem 3.4.1) and Corollary 3.4.2). Hence, λ is also the first eigenvalue of $\mathcal{H}''_p$ and, as an eigenvalue of $\mathcal{H}''_p$, has multiplicity exactly equal to $\mathcal{N}(f)$. Indeed, defined $\pi = \mathrm{span}\{f|_{\mathcal{G}_i}\}$, since $\gamma(\pi \cap \mathcal{S}_p) = \dim(\pi) = \mathcal{N}(f)$, we observe

$$\lambda = \lambda''_1 \le \lambda''_{\mathcal{N}(f)} = \min_{A \in \mathcal{F}_{\mathcal{N}(f)}(\mathcal{S}_p)} \max_{f \in A} \mathcal{R}_{\mathcal{H}''_p}(f) \le \max_{f \in \pi \cap \mathcal{S}_p} \mathcal{R}_{\mathcal{H}''_p}(f) = \lambda,$$

which yields $\lambda''_{\mathcal{N}(f)} = \lambda$. Moreover since π corresponds to the set of eigenfunctions of $\mathcal{H}''_p$ associated to λ and $\gamma(\pi \cap \mathcal{S}_p) = \dim(\pi) = \mathcal{N}(f)$, from Lemma 3.3.5 we conclude $\mathrm{mult}_{\mathcal{H}''_p}(\lambda) \le \mathcal{N}(f)$. We deduce that $\lambda = \lambda''_1 = \cdots = \lambda''_{\mathcal{N}(f)}$ which, combined with (3.21), implies $k > \mathcal{N}(f)$, thus concluding the proof. $\square$

Proof of P1 in Theorem 3.3.10. Using the same notation of the proof of Theorem 3.3.9 above, suppose that $\lambda > \lambda_k$. Then Lemmas 3.5.2 and 3.5.4 imply that

$$\lambda > \lambda_k \ge \lambda'_{k-z} \ge \lambda''_{k-z-\zeta}$$

where we define $\lambda'_h = \lambda''_h = -\infty$ for $h \le 0$ and $\lambda'_h = \lambda''_h = +\infty$ for $h \ge N - z(f) + 1$. As observed above, λ is also the first eigenvalue of $\mathcal{H}''_p$. Thus, the above inequality can hold only if $k - z(f) - \zeta \le 0$. Using Lemma 3.7.1 we obtain $k - z(f) - |E| + |E_z| - z(f) + |V| - \mathcal{N}(f) + l(f) \le 0$, with $l(f)$ being the number of independent loops in $\mathcal{G}$ where f has constant sign. This implies

$$\mathcal{N}(f) \ge k - z(f) - (|E| - |E_z|) + (|V| - z(f)) + l(f) = k + l(f) - \beta'(f) - z(f) + c(f)$$

where $c(f)$ is the number of connected components of $\mathcal{G}'$ and $\beta'(f) := (|E| - |E_z|) - (|V| - z(f)) + c(f)$ the number of independent loops in $\mathcal{G}'$. $\square$

Next, we provide a proof for P2, Theorem 3.3.10. The idea is similar to the one used in the proof of P1. In the latter, we reduced the starting graph to the disjoint union of the nodal domains of an eigenfunction and doing so we knew that the corresponding eigenvalue would become the first variational one on the reduced graph. Now, instead, we reduce the graph to a forest where we know from Theorem 3.3.7 that our eigenvalue becomes a variational one and we know by Theorem 3.3.8 how to relate the nodal domains induced by the eigenfunction to the index of the eigenvalue.

Proof of P2 in Theorem 3.3.10. Let $\mathcal{H}_p$ be a generalized p-Laplacian operator defined on a connected graph, $\mathcal{G}$ and assume

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{k-1} < \lambda_k = \cdots = \lambda_{k+m-1} < \lambda_{k+m} \leq \cdots \leq \lambda_N \,,$$

to be the variational spectrum of $\mathcal{H}_p$ and f to be an eigenfunction relative to $\lambda = \lambda_k = \cdots = \lambda_{k+m-1}$. Additionally, denote by $\mathcal{N}(f)$ the number of nodal domains of f, by $l(f)$ the number of independent loops where f has constant sign, and by $v_1, \ldots, v_z$ the nodes where f is zero, with $z = z(f)$ the number of such nodes.

Using the results of Section 3.5.2, we start by removing the nodes $v_1, \ldots, v_z$ from $\mathcal{G}$ and accordingly modifying the operator $\mathcal{H}_p$, obtaining a graph $\mathcal{G}'$ and an operator $\mathcal{H}_p'$, such that the restriction of f to $\mathcal{G}'$ is an eigenfunction of $\mathcal{H}_p'$ with the same eigenvalue λ. Observe that, since we have removed all and only the nodes of $\mathcal{G}$ where f is zero, f restricted to $\mathcal{G}'$ has no zeros and induces the same nodal domains and constant sign loops induced on $\mathcal{G}$. From Lemma 3.5.4, we have that

$$\lambda'_{k+m-1-z} \leq \lambda \leq \lambda'_{k+m-1} \,, \tag{3.22}$$

where $\{\lambda'_k\}$ denote the variational eigenvalues of $\mathcal{H}_p'$. In particular λ is an eigenvalue of $\mathcal{H}_p'$ i.e. $\lambda \in [\lambda'_1, \lambda'_{N-z(f)}]$ ($N - z(f)$ the number of nodes of $\mathcal{G}'$). Thus, since the variational eigenvalues of $\mathcal{H}_p'$ split its spectrum in intervals, there has to exist and index, h with $\lambda'_h < \lambda'_{h+1}$ such that $\lambda \in [\lambda'_h, \lambda'_{h+1})$ where $\lambda'_h = \infty$ if $h > N - z(f)$. Morover from (3.22) we can state

$$h \geq k + m - z(f) - 1 \,. \tag{3.23}$$

Now observe that if $c(f)$ is the number of connected componets of $\mathcal{G}'$ and $\beta'(f) = |E'| - |V'| + c(f)$ the number of independent loops of $\mathcal{G}'$, we can remove $\beta'(f)$ edges from $\mathcal{G}'$ to obtain a forest $\mathcal{T}$ with the same number of connected components of $\mathcal{G}'$. Every time we remove an edge, we modify the operator $\mathcal{H}_p'$ as in Section 3.5.1 so that the pair (f, λ) remains an eigenpair of the resulting operator. At each step, denote by e_0 the edge we are removing, and by $\widetilde{\mathcal{G}}'$, $\widetilde{\mathcal{H}}_p'$ and $\widetilde{\mathcal{G}}''$, $\widetilde{\mathcal{H}}_p''$ the graph and the corresponding generalized p-Laplace operator before and after cutting e_0. Denote by $\{\widetilde{\lambda}'_k\}$ and $\{\widetilde{\lambda}''_k\}$ the variational spectra of $\widetilde{\mathcal{H}}_p'$ and $\widetilde{\mathcal{H}}_p''$. Letting $\lambda \in [\widetilde{\lambda}'_\ell, \widetilde{\lambda}'_{\ell+1})$ (always with the assumption $\widetilde{\lambda}'_\ell = \infty$ if $\ell > |V(\widetilde{\mathcal{G}}')|$), due to Lemma 3.5.2, we can bound λ in terms of the spectrum of the new operator as:

$$\widetilde{\lambda}''_{\ell-1} \leq \lambda < \widetilde{\lambda}''_{\ell+2} \,.$$

Now, define the two counting functions $\Delta n(e_0, f)$ and $M(e_0, f)$. The first one counts how the variational interval in which λ is contained changes when moving from $\widetilde{\mathcal{G}}'$ to $\widetilde{\mathcal{G}}''$, namely:

$$\Delta n(e_0, f) = \begin{cases} -1 & \lambda < \widetilde{\lambda}''_\ell \\ +1 & \lambda \geq \widetilde{\lambda}''_{\ell+1} \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $\Delta n(e_0, f) = -1$ implies that $\lambda \in [\widetilde{\lambda}''_{\ell-1}, \widetilde{\lambda}''_\ell)$, $\Delta n(e_0, f) = 1$ implies that $\lambda \in [\widetilde{\lambda}''_{\ell+1}, \widetilde{\lambda}''_{\ell+2})$, and $\Delta n(e_0, f) = 0$ implies that $\lambda \in [\widetilde{\lambda}''_\ell, \widetilde{\lambda}''_{\ell+1})$.

The second counting function, $M(e_0, f)$, takes into account the sign of f on the removed edge e_0. Recall that, if $\operatorname{sign}_{e_0}(f) < 0$ then from Section 3.5.1 it follows that $\lambda \in [\widetilde{\lambda}''_{\ell-1}, \widetilde{\lambda}''_{\ell+1})$, otherwise we have $\lambda \in [\widetilde{\lambda}''_\ell, \widetilde{\lambda}''_{\ell+2})$. Thus, we define

$$M(e_0, f) := \begin{cases} \begin{cases} -1 & \lambda < \widetilde{\lambda}''_\ell \\ 0 & \text{otherwise} \end{cases} & \text{if} \quad \operatorname{sign}_{e_0}(f) < 0\,, \\ \begin{cases} 0 & \lambda \geq \widetilde{\lambda}''_{\ell+1} \\ -1 & \text{otherwise} \end{cases} & \text{if} \quad \operatorname{sign}_{e_0}(f) > 0\,. \end{cases}$$

It follows by their definition and from Lemma 3.5.2 that

$$\Delta n(e_0, f) - M(e_0, f) = \begin{cases} 0 & \operatorname{sign}_{e_0}(f) < 0 \\ 1 & \operatorname{sign}_{e_0}(f) > 0 \end{cases}.$$

Thus, thanks to (3.19), every time we cut an edge e_0 and modify consequently the operator $\mathcal{H}'_p$, we have the following identity

$$\Delta n(e_0, f) - M(e_0, f) = \Delta\mathcal{N}(e_0, f) - \Delta l(e_0, f)\,, \tag{3.24}$$

where $\Delta\mathcal{N}(e_0, f)$ and $\Delta l(e_0, f)$ are the difference between the number of nodal domains and the number of constant sign loops induced by f in $\widetilde{\mathcal{G}}'$ and in $\widetilde{\mathcal{G}}''$, respectively.

After β' steps, (f, λ) will be an eigenpair of a generalized p-Laplacian operator $\mathcal{H}''_p$ defined on the forest $\mathcal{T}$, such that $f(u) \neq 0$ $\forall u \in \mathcal{G}''$. We have proved in Theorem 3.3.7 that $\mathcal{H}''_p$ has only variational eigenvalues, so, w.l.o.g., we can assume that λ has became the s-th variational eigenvalue of $\mathcal{H}''_p$. Note that, thanks to Theorem 3.3.7 and Remark 3.6.10 we have that $\operatorname{mult}_{\mathcal{H}''_p}(\lambda) = c(f)$, that is

$$\lambda''_{s-c(f)} < \lambda = \lambda''_{s-c(f)+1} = \cdots = \lambda''_s < \lambda''_{s+1}.$$

Moreover, because of Theorem 3.3.8, we know that f induces s nodal domains on the forest $\mathcal{T}$. Thus, using (3.24) and the equality $\sum_{i=1}^{\beta'} \Delta\mathcal{N}(e_i, f) = s - \mathcal{N}(f)$, the number of nodal domains, $\mathcal{N}(f)$, induced on the original graph $\mathcal{G}$ by f (which is the same as the one induced on $\mathcal{G}'$), can be written as

$$\mathcal{N}(f) = s - \sum_{i=1}^{\beta'} \Delta\mathcal{N}(e_i, f) = s - \sum_{i=1}^{\beta'} \Delta n(e_i, f) - \sum_{i=1}^{\beta'} \Delta l(e_i, f) + \sum_{i=1}^{\beta'} M(e_i, f)\,.$$

Finally, observe that, by definition of Δn, it holds

$$\sum_{i=1}^{\beta'} \Delta n(e_i, f) = s - h, \qquad \text{and} \qquad \sum_{i=1}^{\beta'} \Delta l(e_i, f) = -l(f)$$

because we have removed all the loops, while $\sum_{i=1}^{\beta'} M(e_i, f) \geq -\beta'(f)$ (note that the equality holds if and only if $M(e_i, f) = -1$, $\forall i$). Hence, using inequality (3.23), we obtain

$$\mathcal{N}(f) \geq s-s+h+l(f)-\beta'(f) = h+l(f)-\beta'(f) \geq k+m-1-z(f)+l(f)-\beta'(f)\,,$$

which concludes the proof. $\square$

4 Reformulation of the p-Laplacian eigenvalue problem

4.1 Introduction

In this chapter we address the problem of computing the p-Laplacian eigenpairs when p is greater than 2, i.e. solutions of the nonlinear eigenequation:

$$\Delta_p f = \lambda |f|^{p-2} f \,.$$

Recall that, in the general case, the number of p-Laplacian eigenpairs is unknown, but it is always possible to select an ordered set, $\{\lambda_k\}_{k=1}^N$ whose cardinality is equal to the dimension of the graph $\mathcal{G}$. The remaining eigenvalues are usually classified depending on their position in the variational spectrum, $\lambda_k \leq \lambda < \lambda_{k+1}$. The computation of the p-Laplacian eigenpairs can be addressed by tackling two distinct problems:

- The development of effective numerical algorithms converging toward solutions of the eigenequation.
- The classification of the numerically found eigenpairs in terms of the variational spectrum.

Despite different algorithms have been proposed in the last few years [15, 50, 88], to the best of our knowledge, no methods exist that are capable of accomplishing both of the above tasks. In [88], the authors propose a numerical method capable, in principle, to compute a sequence of N eigenpairs, where given the space, L, spanned by the first $k-1$ computed eigenfunctions ($L := \text{span}\{\widetilde{f}_1, \dots, \widetilde{f}_{k-1}\}$), the k-th eigenpair is found performing

$$\widetilde{\lambda}_k = \min_{g \perp L} \operatorname*{local\,max}_{\widetilde{f} \in span\{g,L\}} \mathcal{R}_p(\widetilde{f}) \,.$$

If the so computed $\widetilde{f}_k \notin L$, the authors show that $(\widetilde{f}_k, \widetilde{\lambda}_k)$ is a p-Laplacian eigenpair and that, assuming the local differentiability of $g \to \operatorname*{local\,max}_{\widetilde{f} \in span\{g,L\}} \mathcal{R}_p(\widetilde{f})$, $\widetilde{f}_k$ has local minmax index of order $k-1$, where the local minmax index is the number of local strict decreasing directions of the p-Rayleigh quotient, $\mathcal{R}_p(f) = \|\nabla f\|_p^p / \|f\|_P^P$, in $\widetilde{f}_k$. However note that there is no evidence that the eigenpair sequence so computed exists. Indeed except for the smallest and the biggest variational eigenvalues all the other ones could not be local maximal values of the p-Rayleigh quotient on the linear subspaces spanned by the corresponding eigenfunction and some other eigenfunctions with smaller eigenvalues

On the other hand for the nonlinear power method introduced in [50] and the gradient flow method from [15], both thought to compute the extremal eigenvalues, it is always possible to prove the convergence toward some eigenpair but no information about its position in the spectrum is available. Moreover we highlight that both of these methods are not suited to compute a full sequence of eigenpairs.

In the following, ispired by the Dynamical-Monge-Kantorovich method introduced in [41, 43], we propose the reformulation of the p-Laplacian eigenproblem in terms of a constrained linear eigenproblem. In particular we show that any p-Laplacian eigenpair, (f, λ), can also be regarded as a weighted linear eigenpair and that the index of the variational eigenvalue can be bounded using the index assigned to λ as a linear eigenvalue. Moreover, based on this analogy, we introduce and discuss numerical algorithms to compute a class of p-Laplacian eigenpairs. The strength of our method is the ability to calculate nonlinear eigenpairs by means of linear ones, but the proof of convergence of our method is not complete and deserves a future in depth study.

4.2 Notation

To ensure self-consistency of the chapter, we begin by recalling the basic definitions and notation that will be used in this chapter. Let $\mathcal{G} = (E, V, \omega)$ be a graph, where E are the edges, V the nodes and ω is a weight on the edges such that $\omega_{uv} = \omega_{vu}$. Given a function $f : V \to \mathbb{R}$ and a function $G : E \to \mathbb{R}$ define

$$\begin{aligned} \nabla :\ &\mathcal{H}(V) \longrightarrow \mathcal{H}(E) \\ &f \mapsto \nabla f(u, v) = \omega_{uv}\big(f(v) - f(u)\big) \end{aligned}$$

$$\begin{aligned} \operatorname{div} = -\frac{1}{2}\nabla^T :\ &\mathcal{H}(E) \longrightarrow \mathcal{H}(V) \\ &G \mapsto \operatorname{div} G(u) = \frac{1}{2}\sum_{v\sim u} \omega_{uv}\big(G(u, v) - G(v, u)\big) \end{aligned}$$

where $v \sim u$ means that $(u, v) \in E$. Throughout the whole chapter, if not otherwise specified, we use capital letters to denote edge functions and lowercase letters to denote node functions. Recall also the definition of the p-Laplacian operator

$$(\Delta_p f)(u) := -\operatorname{div}(|\nabla f|^{p-2}\nabla f) = \sum_{v\sim u} \omega_{uv} |\nabla f(v, u)|^{p-2}\big(\nabla f(v, u)\big)\,.$$

Finally, we say that (f, λ) is a p-Laplacian eigenpair with homogeneous Dirichlet boundary conditions assigned on a subset of the nodes $B \subset V$, if it solves the following equation

$$\begin{cases} \Delta_p f(u) = \sum_{v\sim u} \omega_{uv} |\nabla f(v,u)|^{p-2} \nabla f(v,u) = \lambda |f(u)|^{p-2} f(u) & \forall u \in V \setminus B \\ f(u) = 0 & \forall u \in B \end{cases} \tag{4.1}$$

i.e., if and only if (f, λ) is a critical point/value of the p-Rayleigh quotient on $\mathcal{H}_0(V)$:

$$\mathcal{R}_p(f) = \frac{\|\nabla f\|_p^p}{\|f\|_p^p} = \frac{\frac{1}{2}\sum_{(u,v)\in E} |\nabla f(u,v)|^p}{\sum_{u\in V} |f(u)|^p}$$

where $\mathcal{H}_0(V) = \{f : V \to \mathbb{R} \mid f(u) = 0\ \forall u \in B\}$.

4.3 An equivalent formulation of the p-Laplacian eigenvalue problem

In this section we consider a trivial reformulation of the p-Laplacian eigenvalue problem in terms of a constrained weighted Laplacian eigenvalue problem. Using such an equivalence, since the eigenvalues of the corresponding weighted Laplacian are finite, it is possible to assign to every p-Laplacian eigenvalue, λ, an index which is the index of λ regarded as a linear eigenavalue. We prove that this index, which is theoretically computable, matches the Morse index of $\mathcal{R}_p$ in f, where f is the eigenfunction corresponding to λ. We would like to stress the fact that here we are assuming $p > 2$.

4.3.1 Weigthed Laplacian equivalence

Recalling the p-Laplacian eigenvalue problem (4.1), it is easy to observe that (f, λ) is an eigenpair of the p-Laplacian if and only if (f, λ) is an eigenpair of the following constrained weighted Laplacian Dirichlet problem:

$$\begin{cases} \Delta_{\mu_0} f(u) = \sum\limits_{v\sim u} \mu_{0uv} \omega_{uv} \nabla f(v,u) = \lambda \nu_{0u} f(u) & \forall u \in V \setminus B \\ f(u) = 0 & \forall u \in B \\ \mu_{0uv} = |\nabla f(u,v)|^{p-2} & \forall (u,v) \in E \\ \nu_{0u} = |f(u)|^{p-2} & \forall u \in V \setminus B \end{cases}, \tag{4.2}$$

moreover $\mu \in \mathcal{M}^+(E)$ and $\nu \in \mathcal{M}^+(V)$ where $\mathcal{M}^+(E)$ and $\mathcal{M}^+(V)$ denote the positive measures space on the edges and on the nodes

Definition 4.3.1.

$$\mathcal{M}^+(E) = \{\mu \mid \mu_{uv} \geq 0\ \forall (u,v) \in E\} \quad \textit{and} \quad \mathcal{M}^+(V) = \{\nu \mid \nu_u \geq 0\ \forall u \in V\}.$$

Let (f, λ) be an eigenpair of the p-Laplacian and let

$$\nu_u := |f(u)|^{p-2} \quad \text{and} \quad \mu_{uv} := |\nabla f(u,v)|^{p-2} .$$

Introducing the quantities

$$\|g\|_{2,\nu}^2 = \sum_u \nu_u |g(u)|^2 \quad \text{and} \quad \|\nabla g\|_{2,\mu}^2 = \frac{1}{2} \sum_{(u,v)\in E} \mu_{uv} |\nabla g(u,v)|^2 ,$$

the 2-Rayleigh quotient weighted in μ, ν is given by:

$$\mathcal{R}_{2,\mu,\nu}(g) = \|\nabla g\|_{2,\mu}^2 / \|g\|_{2,\nu}^2 .$$

Clearly the critical points and values of $\mathcal{R}_{\mu,\nu}$ are the eigenpairs of the linear generalized eigenvalue problem

$$\Delta_\mu f = -\mathrm{div}(\mu \nabla f) = \lambda \nu f .$$

Since the last is a linear eigenvalue problem its eigenpairs can be numbered and, using an increasing ordering for the eigenvalues, we denote by $\big(f_k(\mu,\nu), \lambda_k(\mu,\nu)\big)$ the k-th eigenpair.

Observe that by definition of ν, if $\|f\|_p = 1$, then also $\|f\|_{2,\nu} = 1$. We now introduce the two spheres

$$S_p := \{g \mid \|g\|_p = 1\} \quad \text{and} \quad S_{2,\nu} := \{g \mid \|g\|_{2,\nu} = 1\} ,$$

and denote their tangent spaces in the point f by $T_f(S_p)$ and $T_f(S_{2,\nu})$. Then, it is not difficult to observe that

$$\{\xi \mid \langle \xi, |f|^{p-2} f\rangle = 0\} = T_f(S_p) = T_f(S_{2,\nu}) = \{\xi \mid \langle \xi, \nu f\rangle = 0\} .$$

Moreover, considered $\mathcal{R}_p$ and $\mathcal{R}_{2,\mu,\nu}$ as functions defined on the manifolds S_p and $S_{2,\nu}$ respectively, in the next Lemma we show that it is possible to compare the Morse indices of the functions $\mathcal{R}_p$ and $\mathcal{R}_{2,\mu,\nu}$ in the point f, $\mathcal{MI}_f(\mathcal{R}_p)$ and $\mathcal{MI}_f(\mathcal{R}_{2,\mu,\nu})$. In particular this allows to relate $\mathcal{MI}_f(\mathcal{R}_p)$ to the linear index of λ, i.e., the position of λ in the spectrum of the associated linear eigenvalue problem, $\Delta_\mu f = \lambda \nu f$. We recall that the Morse index of a function ϕ at a point x is essentially the number of local decreasing directions of ϕ in x, $\mathcal{MI}_x(\phi)$. More precisely, it is the number of negative eigenvalues of the associated Hessian matrix, $\partial^2\phi/\partial x^2$, see [71]. Before enunciating the Lemma observe that the μ-ν weighted Laplacian eigenvalue problem can be degenerate in case $\mathrm{Ker}\big(\mathrm{diag}(\nu)\big)$ is non empty. In this case, indeed there would be only $N - \dim\Big(\mathrm{Ker}\big(\mathrm{diag}(\nu)\big)\Big)$ well defined eigenpairs, nevertheless with a small abuse of notation we can always complete the set of eigenpairs to a base of the space. We have to differentiate between two cases, first if f is in $\mathrm{Ker}\big(\mathrm{diag}(\nu)\big)$ but not in $\mathrm{Ker}\big(\Delta_\mu\big)$, we can say that f is an eigenfunction of eigenvalue λ equal to infinity, f indeed would be a eigenfunction of zero eigenvalue of the inverse eigenvalue problem

$$\mathrm{diag}(\nu)f = \lambda\Delta_\mu f\,.$$

If instead we have a function f such that $f \in \mathrm{Ker}\big(\mathrm{diag}(\nu)\big)\bigcap\mathrm{Ker}\big(\Delta_\mu\big)$ then f satisfies the μ-ν Laplacian eigenvalue problem for any eigenvalue λ, in this case we can thus say that f is an eigenfunction for an arbitrarily chosen eigenvalue λ.

Lemma 4.3.2. *Let (f,λ) be an eigenpair of the p-Laplacian, and $\nu = |f|^{p-2}$, $\mu = |\nabla f|^{p-2}$. Assume that h is the linear index of λ, i.e. $(f,\lambda) = \big(f_h(\mu,\nu),\lambda_h(\mu,\nu)\big)$, where $f_h(\mu,\nu)$ and $\lambda_h(\mu,\nu)$ are the h-th eigenfunction and eignvalue of the μ-ν weighted Laplacian problem given in* (4.2) *and are such that:*

$$\lambda_{h+m}(\mu,\nu) > \lambda_{h+m-1}(\mu,\nu) = \lambda_h(\mu,\nu) > \lambda_{h-1}(\mu,\nu)\,.$$

Here in the multiplicity of λ_h we count also the dimension of $\mathrm{Ker}(\Delta_\mu)\cap\mathrm{Ker}\big(\mathrm{diag}(\nu)\big)$, *i.e., whenever f is such that $f \in \mathrm{Ker}(\Delta_\mu)\cap\mathrm{Ker}\big(\mathrm{diag}(\nu)\big)$, we artificially impose that the corresponding eigenvalue λ satisfies $\lambda = \lambda_h$. Instead, for every $f \in \mathrm{Ker}\big(\mathrm{diag}(\nu)\big)$ but $f \notin \mathrm{Ker}(\Delta_\mu)$, with a small abuse of notation, we write that f is an eigenfunction associated to the eigenvalue $\lambda = \infty$. Then*

$$\mathcal{MI}_f(\mathcal{R}_p) = \mathcal{MI}_f(\mathcal{R}_{2,\mu,\nu}) = h-1\,.$$

$$\mathcal{MI}_f(-\mathcal{R}_p) = \mathcal{MI}_f(-\mathcal{R}_{2,\mu,\nu}) = N-h-m+1\,.$$

Proof. To prove the lemma we first show that $\forall\xi \in T_f(S_p) = T_f(S_\nu)$ we have:

$$\frac{\partial^2}{\partial\epsilon^2}\left(\frac{\|\nabla(f+\epsilon\xi)\|_p^p}{\|f+\epsilon\xi\|_p^p}\right)\bigg|_{\epsilon=0} = \frac{p(p-1)}{2}\frac{\partial^2}{\partial\epsilon^2}\left(\frac{\|\nabla(f+\epsilon\xi)\|_{2,\mu}^2}{\|f+\epsilon\xi\|_{2,\nu}^2}\right)\bigg|_{\epsilon=0}.$$

The first derivative is zero as f is a critical point for both the Rayleigh quotients because of the equivalence of the p-Laplacian and weighted Laplacian eigenvalue problems in f.

$$\begin{aligned}
\frac{\partial}{\partial\epsilon}\left(\frac{\|\nabla(f+\epsilon\xi)\|_p^p}{\|f+\epsilon\xi\|_p^p}\right)\bigg|_{\epsilon=0} &= \frac{p}{\|f\|_p^p}\left(\langle|\nabla f|^{p-2}\nabla f,\nabla\xi\rangle - \frac{\|\nabla f\|_p^p}{\|f\|_p^p}\langle|f|^{p-2}f,\xi\rangle\right)\\
\frac{\partial}{\partial\epsilon}\left(\frac{\|\nabla(f+\epsilon\xi)\|_{2,\mu}^2}{\|f+\epsilon\xi\|_{2,\nu}^2}\right)\bigg|_{\epsilon=0} &= \frac{2}{\|f\|_{2,\nu}^2}\left(\langle\mu\nabla f,\nabla\xi\rangle - \frac{\|\nabla f\|_{2,\mu}^2}{\|f\|_{2,\nu}^2}\langle\nu f,\xi\rangle\right)\\
&= \frac{2}{\|f\|_p^p}\left(\langle|\nabla f|^{p-2}\nabla f,\nabla\xi\rangle - \frac{\|\nabla f\|_p^p}{\|f\|_p^p}\langle|f|^{p-2}f,\xi\rangle\right)
\end{aligned} \tag{4.3}$$

Then, recall first that $\xi \in T_f(S_p) = T_f(S_\nu)$, which means

$$\frac{\partial}{\partial\epsilon}\|f+\epsilon\xi\|_p^p = \frac{\partial}{\partial\epsilon}\|f+\epsilon\xi\|_{2,\nu}^2 = C\langle|f|^{p-2}f,\xi\rangle = C\langle\nu f,\xi\rangle = 0\,,$$

second that

$$\frac{\partial}{\partial\epsilon}\left(\frac{\|\nabla(f+\epsilon\xi)\|_p^p}{\|f+\epsilon\xi\|_p^p}\right)\Big|_{\epsilon=0}=\frac{\partial}{\partial\epsilon}\left(\frac{\|\nabla(f+\epsilon\xi)\|_{2,\mu}^2}{\|f+\epsilon\xi\|_{2,\nu}^2}\right)\Big|_{\epsilon=0}=0,$$

and last that

$$\frac{\partial|x+\epsilon y|^{p-2}(x+\epsilon y)}{\partial\epsilon}|_{\epsilon=0}=(p-2)|x|^{p-3}\frac{(x)^2}{|x|}y+|x|^{p-2}y=(p-1)|x|^{p-2}y\,.$$

Then, using the last remarks, differentiate the first derivatives (4.3) , where we recall the $|\nabla(f+\epsilon\xi)|^{p-2}\big(\nabla(f+\epsilon\xi)\big)$ and $|f+\epsilon\xi|^{p-2}(f+\epsilon\xi)$ are entrywise products:

$$\begin{aligned}\frac{\partial^2}{\partial\epsilon^2}\left(\frac{\|\nabla(f+\epsilon\xi)\|_p^p}{\|f+\epsilon\xi\|_p^p}\right)\Big|_{\epsilon=0}&=\frac{p(p-1)}{\|f\|_p^p}\left(\langle|\nabla f|^{p-2}\nabla\xi,\nabla\xi\rangle-\frac{\|\nabla f\|_p^p}{\|f\|_p^p}\langle|f|^{p-2}\xi,\xi\rangle\right)\\ \frac{\partial^2}{\partial\epsilon^2}\left(\frac{\|\nabla(f+\epsilon\xi\|_{2,\mu}^2}{\|f+\epsilon\xi\|_{2,\nu}^2}\right)\Big|_{\epsilon=0}&=\frac{2}{\|f\|_{2,\nu}^2}\left(\langle\mu\nabla\xi,\nabla\xi\rangle-\frac{\|\nabla f\|_{2,\mu}^2}{\|f\|_{2,\nu}^2}\langle\nu\xi,\xi\rangle\right)\\ &=\frac{2}{\|f\|_p^p}\left(\langle|\nabla f|^{p-2}\nabla\xi,\nabla\xi\rangle-\frac{\|\nabla f\|_p^p}{\|f\|_p^p}\langle|f|^{p-2}\xi,\xi\rangle\right)\end{aligned}\tag{4.4}$$

and this yelds to the desired equality.

Next observe that $T_f(S_{2,\nu})=\mathrm{span}\{f_i(\mu,\nu)\}_{i\neq h}$, indeed $Tg_f(S_{2,\nu})=\{\xi,|,\langle\nu f,\xi\rangle=0\}$, and $\{f_i(\mu,\nu)\}_i$ is a ν-othogonal basis of the space. Hence, the following implications hold:

$$\begin{aligned}\frac{\partial^2}{\partial\epsilon^2}\left(\frac{\|\nabla(f+\epsilon\xi)\|_{2,\mu}^2}{\|f+\epsilon\xi\|_{2,\nu}^2}\right)\Big|_{\epsilon=0}<0\quad&\Longleftrightarrow\quad\xi\in\mathrm{span}\{f_i(\mu,\nu)|\,\lambda_i(\mu,\nu)<\lambda_h(\mu,\nu)\}\,,\\ \frac{\partial^2}{\partial\epsilon^2}\left(\frac{\|\nabla(f+\epsilon\xi)\|_{2,\mu}^2}{\|f+\epsilon\xi\|_{2,\nu}^2}\right)\Big|_{\epsilon=0}>0\quad&\Longleftrightarrow\quad\xi\in\mathrm{span}\{f_i(\mu,\nu)|\,\lambda_i(\mu,\nu)>\lambda_h(\mu,\nu)\}\,.\end{aligned}$$

To prove the last statement, let $\xi=\sum_{i\neq h}\alpha_i f_i(\mu,\nu)$ and recall that if $i\neq j$, $\langle\mu\nabla f_i,\nabla f_j\rangle=0$ and $\langle\nu f_i,f_j\rangle=0$. Hence, using (4.4), we can provide the following equality that allows easily to conclude the proof of the lemma:

$$\begin{aligned}\frac{\partial^2}{\partial\epsilon^2}\left(\frac{\|\nabla(f+\epsilon\xi\|_{2,\mu}^2}{\|f+\epsilon\xi\|_{2,\nu}^2}\right)\Big|_{\epsilon=0}&=\frac{2}{\|f\|_{2,\nu}^2}\sum_{i\neq h}\sum_{j\neq h}\alpha_i\alpha_j\Big(\langle\mu\nabla f_i,\nabla f_j\rangle-\lambda_h\langle\nu f_i,f_j\rangle\Big)\\ &=\frac{2}{\|f\|_{2,\nu}^2}\sum_{i\neq h}\alpha_i^2\Big(\langle\mu\nabla f_i,\nabla f_i\rangle-\lambda_h\langle\nu f_i,f_i\rangle\Big)\end{aligned}$$

In the last equality observe that if f_i is an eigenfunction corresponding to an eigenvalue λ_i with $f_i\notin\mathrm{Ker}\big(\mathrm{diag}(\nu)\big)$, then

$$\Big(\langle\mu\nabla f_i,\nabla f_i\rangle-\lambda_h\langle\nu f_i,f_i\rangle\Big)=\|f_i\|_{2,\nu}^2\big(\lambda_i-\lambda_h\big)\,,$$

i.e., f_i is an increasing or a decreasing direction of $\mathcal{R}_{2,\mu,\nu}$ in f according to the inequalities $\lambda_i > \lambda_h$ or $\lambda_i < \lambda_h$. Moreover if $f_i \in \mathrm{Ker}(\Delta_\mu) \cap \mathrm{Ker}\big(\mathrm{diag}(\nu)\big)$ by definition we have $\lambda_i = \lambda_h$ which corresponds to observing that f_i is neither an increasing nor a decreasing direction of $\mathcal{R}_{2,\mu,\nu}$ in f

$$\Big(\langle \mu \nabla f_i, \nabla f_i\rangle - \lambda_h \langle \nu f_i, f_i\rangle\Big) = 0\,.$$

Finally if $f_i \in \mathrm{Ker}\big(\mathrm{diag}(\nu)\big)$ but $f_i \notin \mathrm{Ker}(\Delta_\mu)$, then by definition we have $\lambda_i = \infty$ which corresponds to observing that f_i is an increasing direction of $\mathcal{R}_{2,\mu,\nu}$ in f, indeed:

$$\Big(\langle \mu \nabla f_i, \nabla f_i\rangle - \lambda_h \langle \nu f_i, f_i\rangle\Big) = \Big(\langle \mu \nabla f_i, \nabla f_i\rangle\Big) > 0\,.$$

Observe that the last equalities proves that the tangent directions corresponding to smaller(higher) eigenfunctions of the μ, ν weighted Laplacian problem correspond to decreasing(increasing) directions of $\mathcal{R}_{2,\mu,\nu}$ and thus of $\mathcal{R}_p$. □

4.4 Variational Eigenvalues

We devote this section to show that for certain families of variational eigenvalues of the p-Laplacian we can compare their index with the corresponding index of the same eigenvalues seen as eigenvalues of the corresponding linear weighted Laplacian. We begin this section recalling that since the p-Laplacian operator is nor linear we do not know a priori how many eigenvalues exist. Nevertheless it is always possible to select N of them as representants of the whole spectrum. Since the usual selection procedure is performed by means of variational methods the selected eigenvalues are named variational eigenvalues. The most classical variational eigenvalues are defined using the Lusternik-Schnirelman theory and a min-max method based on the definition of Krasnoselskii genus of a set [46, 47, 63, 85]. Nevertheless this is not the only possibility and using the same tools it is possible to select different families of variational eigenvalues more suitable to our scope.

We start recalling the definition of the Krasnoselskii variational eigenvalues. The domain of definition of the Krasnoselskii genus is the family of the closed and symmetric subsets of $\mathbb{R}^n$:

$$\mathcal{A} = \{A \subseteq \mathbb{R}^N |\ A \ \mathit{closed}\,,\ \ A = -A\}\,.$$

For any $A \in \mathcal{A}$, the Krasnoselskii genus of A is then defined as the number

$$\gamma(A) = \begin{cases} \inf\{k \in \mathbb{N}\,:\, \exists\, \varphi \in C(A, \mathbb{R}^k \setminus \{0\}) \ \ s.t. \ \ \varphi(x) = -\varphi(-x)\} & \\ \infty & \text{if } \nexists\, k \text{ as above} \\ 0 & \text{if } A = \emptyset \end{cases},$$

where $C(A,\mathbb{R}^k\setminus\{0\}$ is the space of continuous functions on A with values in $\mathbb{R}^k\setminus\{0\}$. In other words, the genus of A can be seen as the smallest k such that A can be continuously mapped on the sphere of dimension $k-1$ preserving the symmetry. Consider now the family $\mathcal{F}_k(S_p\cap\mathcal{D}_0)=\{A\subseteq\mathcal{A}\cap S_p\cap\mathcal{D}_0\mid\gamma(A)\geq k\}$, the Krasnoselskii variational eigenvalues, $\{\lambda_k^{\mathcal{K}}\}_{k=1}^N$, of Δ_p are defined by the following expression

$$\lambda_k^{\mathcal{K}}=\inf_{A\in\mathcal{F}_k}\sup_{f\in A}\mathcal{R}_p(f)\,. \tag{4.5}$$

It is easy to prove that such values are actually eigenvalues (i.e. critical points of the Rayleigh quotient) see [46, 47, 63, 85]. Moreover, since we are working in a finite dimensional space, we can prove that the inf sup is actually a min max.

Lemma 4.4.1. *Let $\lambda_k^{\mathcal{K}}$ be the k-th variational eigenvalue*

$$\lambda_k^{\mathcal{K}}=\inf_{A\in\mathcal{F}_k}\sup_{f\in A}\mathcal{R}_p(f)\,.$$

Then there exists $A_0\in\mathcal{F}_k$ s.t.

$$\lambda_k^{\mathcal{K}}=\max_{f\in A_0}\mathcal{R}_p(f)$$

Proof. Since any $A\in\mathcal{F}_k$ is compact it is clear that we can replace the sup by a max. On the other hand, we need to do some more work to work out of the inf condition. Assume $A_n\in\mathcal{F}_k$ to be a minimizing sequence, i.e.

$$\lambda_k^{\mathcal{K}}\leq\max_{f\in A_n}\mathcal{R}_p(f)\leq\lambda_k^{\mathcal{K}}+\frac{1}{n}\,, \tag{4.6}$$

and define

$$A_0=\bigcap_{m\in\mathbb{N}}\big(\overline{\bigcup_{n>m}A_n}\big)\,, \tag{4.7}$$

A_0 is a nonempty symmetric closed set as it is the limit of a sequence of symmetric closed subsets of a compact set, S_p. Now we study the genus of A_0, if $\gamma(A_0)=h<k$, then there exists an odd function f_0 continuous in A_0 and with values in $\mathbb{R}^h\setminus\{0\}$, $f_0\in C(A_0,\mathbb{R}^h\setminus\{0\})$. Thanks to the Tietze extension theorem it is possible to find an odd function $\widetilde{f_0}\in C(\mathbb{R}^N,\mathbb{R}^h)$ such that $\widetilde{f_0}|_{A_0}=f_0$. The preimage $U_0=\widetilde{f_0}^{-1}(\mathbb{R}^h\setminus 0)$ is a neighborhood of A_0, thus from (4.7), there exist n_0 such that

$$A_n\subset U_0\,,\qquad\forall n>n_0\,.$$

Observe that this leads to an absurd because, considering $\widetilde{f_0}|_{A_n}$ as a Krasnoselskii test function for A_n, it implies that $\forall n>n_0$, $\gamma(A_n)\leq h<k$. Thus, necessarily, $\gamma(A_0)\geq k$ and, since $A_0\subseteq A_n$ $\forall n\in\mathbb{N}$, thanks to (4.6) and the definition of $\lambda_k^{\mathcal{K}}$

$$\lambda_k^{\mathcal{K}}\leq\max_{f\in A_0}\mathcal{R}_p(f)\leq\lambda_k^{\mathcal{K}}\,,$$

concluding the proof. □

We refer to the previous chapter 3.4 for an in depth study of such eigenvalues, in particular we remind the following result from section3.4.

Theorem 4.4.2. *Let $\lambda_1^{\mathcal{K}}$ and $\lambda_2^{\mathcal{K}}$ be the first two Krasnoselskii variational eigenvalues. Then first $\lambda_1^{\mathcal{K}}$ is simple, meaning that there exists a unique eigenfunction f_1 associated to $\lambda_1^{\mathcal{K}}$. Second f_1 is the only strictly positive eigenfunction, i.e. if f is an eigenfunction of Δ_p and $f(v) > 0$ for all $v \in V \setminus B$, then*

$$f = f_1 \,.$$

Third, there are no eigenvalues between $\lambda_1^{\mathcal{K}}$ and $\lambda_2^{\mathcal{K}}$, i.e.

$$\lambda_2^{\mathcal{K}} = \min\{\lambda \mid \lambda \textit{ eigenvalue of } \Delta_p \textit{and } \lambda > \lambda_1^{\mathcal{K}}\} \,.$$

The definition of the variational eigenvalues of the p-Laplacian using the Krasnoselskii theory is widely used in literature, but not the only one. For example, from [32, 33] a "Drabek" family of variational eigenvalues, say $\{\lambda_k^D\}_k$ can be defined as follows

$$\lambda_k^D := \inf_{A \in \Lambda_k^D} \sup_{f \in A} \mathcal{R}_{\Delta_p}(f) \,, \tag{4.8}$$

where

$$\mathcal{F}_k^D := \{A \subset S_p \cap \mathcal{D}_0 \mid \exists h \in C(S^{k-1}, A), \ h \text{ odd and surjective}\}, .$$

Observe that, since $\mathcal{F}_k^D \subset \mathcal{F}_k$, necessarily $\lambda_k \le \lambda_k^D$. In [32, 33] it is also proved that $\lambda_1 = \lambda_1^D$ and $\lambda_2 = \lambda_2^D$ whereas the equality of the upper variational eigenvalues so defined remains an open problem.

Inspired by these works, below we present a new class of variational eigenvalues whose definition allows a comparison between their variational index and their index as eigenvalues of a weighted Laplacian. The families of subsets suitable to define a sequence of variational eigenvalues can be characterized as the families of subsets stable with respect to appropriate deformations of the domain.

To describe our new definition, we first introduce some classical notation. We say that c is a critical value of $\mathcal{R}_p$ if there exist $f \in S_p \cap \mathcal{R}_p^{-1}(c)$ s.t. $\partial_f \mathcal{R}_p(f) = 0$ and in this case f is said a critical point. Any $c \in \mathbb{R}$ that is not a critical value is said a regular value. We denote by C the closed, and hence compact, set of critical points of $\mathcal{R}_p$ on $S_p \cap \mathcal{D}_0$ and by $\mathcal{R}_p(C)$ the compact set of the critical values. Finally, for any $x \in \mathbb{R}$, we denote by $\mathcal{R}_p^x = \mathcal{R}_p^{-1}(-\infty, x]$. The following Lemma is a classical results from critical point theory [75, 85] and it is here presented in the version that better suits to our scope. Given a family of subsets $\mathcal{F}$ and a functional identified with $\mathcal{R}_p$, the Lemma describes some sufficient conditions that guarantee that the min max, or the max min, over $\mathcal{F}$ of $\mathcal{R}_p$ is a critical point.

Lemma 4.4.3. *Assume $\mathcal{F}$ to be a family of subsets of $S_p \cap \mathcal{D}_0$ such that for any regular value $c \in \mathbb{R}$ of $\mathcal{R}_p$, there exist $\epsilon > 0$ and continuous deformations of the domain $\phi : [0,1] \times S_p \to S_p$, $\psi : [0,1] \times S_p \to S_p$ such that:*

$$\begin{cases} \phi(0,\cdot) = id_{S_p}(\cdot) \\ \phi(1, \mathcal{R}^{c+\epsilon}) \subset \mathcal{R}^{c-\epsilon} \\ \phi(t, A) \in \mathcal{F}\,,\ \forall A \in \mathcal{F}\,,\ \forall t \in [0,1] \end{cases} \qquad \begin{cases} \psi(0,\cdot) = id_{S_p}(\cdot) \\ \psi(1, S_p \setminus \mathcal{R}^{c-\epsilon}) \subset S_p \setminus \mathcal{R}^{c+\epsilon} \\ \psi(t, A) \in \mathcal{F}\,,\ \forall A \in \mathcal{F}\,,\ \forall t \in [0,1]. \end{cases}$$

Then

$$\lambda := \inf_{A\in\mathcal{F}} \sup_{f\in A} \mathcal{R}_p(f)\,, \qquad \eta := \sup_{A\in\mathcal{F}} \inf_{f\in A} \mathcal{R}_p(f)$$

are critical values of $\mathcal{R}_p$.

Proof. Assume by absurd that λ is a regular value and consider a deformation of the domain ϕ_0 as in the hypotheses such that $\phi(1, \mathcal{R}_p^{c+\epsilon}) \subset \mathcal{R}_p^{c-\epsilon}$. Then consider a subset, $A_\epsilon \in \mathcal{F}$, such that

$$\sup_{f\in A_\epsilon} \mathcal{R}_p(f) < \lambda + \epsilon\,,$$

by hypotheses $\phi(1, A_\epsilon) \in \mathcal{F}$ and

$$\sup_{f\in \phi(1,A_\epsilon)} \mathcal{R}_p(f) < \lambda - \epsilon$$

which is an absurd because of the definition of λ as an inf. The proof for η is similar. □

The next step is to show that for any regular value c there exists a deformation of the domain which perturbs the set $\mathcal{R}_p^{c+\epsilon}$ into the set $\mathcal{R}_p^{c-\epsilon}$ for a suitable ϵ. Then we will be able to choose a family $\mathcal{F}$ that is invariant with respect to such deformations. The following lemma is a p-Laplacian adapted version of a family of deformation lemmas suitable to work in more general settings, see [47, 75, 85].

Lemma 4.4.4 (Deformation Lemma)**.** *Let $p > 2$ and assume c to be a regular value of $\mathcal{R}_p$. Then there exists $\epsilon > 0$ and a C^1 family of $\phi \in C^1([-1,1] \times S_p, S_p)$ such that*

1. *$\phi(t,\cdot)$ is a C^2 odd homeomorphism for any $t \in [-1,1]$,*
2. *$\phi(1, \mathcal{R}^{c+\epsilon}) \subset \mathcal{R}_p^{c-\epsilon}$ and $\phi(-1, S_p \cap \mathcal{D}_0 \setminus \mathcal{R}^{c-\epsilon}) \subset S_p \cap \mathcal{D}_0 \setminus \mathcal{R}^{c+\epsilon}$,*

where $\mathcal{H}_0(V) = \{f : V \to \mathbb{R} \mid f(u) = 0\ \forall u \in B\}$.

Proof. Let $B_\eta(\mathcal{R}^{-1}_{\Delta_p}(c))$ and $B_{2\eta}(\mathcal{R}^{-1}_{\Delta_p}(c))$ be two symmetric open neighborhoods of $\mathcal{R}^{-1}_{\Delta_p}(c)$ in $S_p \cap \mathcal{D}_0$ such that $B_\eta(\mathcal{R}^{-1}_{\Delta_p}(c)) \subset B_{2\eta}(\mathcal{R}^{-1}_{\Delta_p}(c)) \subset S_p \setminus K$. Consider $\xi \in C^\infty(S_p, \mathbb{R}^+)$ a symmetric cutoff function such that

$$\xi|_{B_\eta} = 1\,, \qquad \xi|_{S_p\setminus B_{2\eta}} = 0\,.$$

Now introduce the deformation function $\phi : \mathbb{R} \times S_p \to S_p$ that solves the following Cauchy problem

$$\begin{cases} \partial t \phi(t, f) = -\xi(\phi(t, f)) \partial f \mathcal{R}_p(\phi(t, f)) \\ \phi(0, f) = f \end{cases}$$

Since it is easy to observe that $\partial f\big(\mathcal{R}_p(f)\big)$ is an odd C^1 function, $\partial f\big(\mathcal{R}_p(f)\big) \in C^1(S_p \cap \mathcal{D}_0, \mathbb{R}^M)$ and $\partial f\big(\mathcal{R}_p(f)\big) = -\partial f\big(\mathcal{R}_p(-f)\big)$, for any $t \in [-1, 1]$ and $p \geq 1$ we have that $\phi(t, f)$ is a C^2 odd function, $\phi(t, \cdot) \in C^2(S_p \cap \mathcal{D}_0, S_p \cap \mathcal{D}_0)$ and $\phi(t, f) = -\phi(t, -f)$. Moreover $\phi(\cdot, t + s) = \phi(\cdot, t) \circ \phi(\cdot, s)$ and thus $\phi(\cdot, t) = \phi^{-1}(\cdot, -t)$. Finally observe that

$$\frac{\partial}{\partial t}(\mathcal{R}_p(\phi(f, t)) = -\xi(\phi(f, t)) \Big\| \frac{\partial}{\partial f} \mathcal{R}_p(\phi(f, t) \Big\|^2 \leq 0 \,.$$

Since c is a regular value and $\mathcal{R}_p^{-1}(c)$ is a compact set, $\|\partial/\partial f\big(\mathcal{R}_p(f)\big)\|$ admits a minimum greater than zero on $\mathcal{R}_p^{-1}(c)$ and so, for any $t > 0$ there exists an $\epsilon > 0$ such that

$$\mathcal{R}_p(\phi(f, t)) < c - \epsilon \quad \forall f \in \mathcal{R}_p^{-1}(c) \,.$$

To conclude, by the continuity of ϕ, there exists a neighborhood U_ϵ (that we can assume w.l.o.g to be $\mathcal{R}_p^{c+\epsilon}$) of $\mathcal{R}_p^{-1}(c)$ such that

$$\mathcal{R}_p(\phi(f, t)) < c - \epsilon \quad \forall f \in U_\epsilon \,.$$

Moreover, since $\phi(\cdot, -1) = \phi^{-1}(\cdot, 1)$, necessarily $\phi(-1, S_p \setminus \mathcal{R}_p^{c-\epsilon}) \subset S_p \setminus \mathcal{R}_p^{c+\epsilon}$. $\square$

It is now possible to introduce another choice of variational eigenvalues in addition to the Krasnoselskii and Drabek definitions for which, given a variational p-Laplacian eigenvalue λ, it is possible to compare the variational index of λ with its linear index, i.e. the index of λ as eigenvalue of the linear problem (4.2). To this aim, for any $p \geq 2$, we introduce the family of embedded C^2 k-spheres in S_p given by:

$$\mathcal{F}'_k = \{A \subset S_p \cap \mathcal{D}_0 |\, \exists \varphi \in C^2(S^{k-1}, A), \varphi^{-1} \in C^2(A, S^{k-1})\, \}$$

, where S^{k-1} is the $k-1$-dimensional sphere. Observe that, thanks to Lemma4.4.3, the family $\mathcal{F}'_k$, for any k, satisfies the hypotheses of Lemma4.4.3 and thus we can define the following family of variational eigenvalues

$$\lambda'_k = \inf_{A \in \mathcal{F}'_k} \max_{f \in A} \mathcal{R}_p(f) \qquad \lambda''_k = \sup_{B \in \mathcal{F}'_{N-k+1}} \min_{f \in B} \mathcal{R}_p(f) \,. \tag{4.9}$$

Observe that since $\mathcal{F}'_k \subset \mathcal{F}^D_k \subset \mathcal{F}_k$ the eigenvalues λ'_k satisfy the following inequalities with respect to the Krasnoselskii (4.5) and the Drabek (4.8) variational eigenvalues:

$$\lambda^{\mathcal{K}}_k \leq \lambda^D_k \leq \lambda'_k \,. \tag{4.10}$$

Oserve also that for given two symmetric subsets, $A_k \subset S_p$ and $B_k \subset S_p$, respectively homeomorphic to a k-sphere and to a $N-k+1$-sphere we have that $A_k \cap B_k \neq \emptyset$ (see Lemma 3.5.1), thus $\forall \epsilon$ we have the following :

$$\lambda''_k - \epsilon = \min_{f \in B^\epsilon_k} \mathcal{R}_{\Delta_p}(f) \le \min_{f \in B^\epsilon_k \cap A^\epsilon_k} \mathcal{R}_{\Delta_p}(f) \le \max_{f \in A^\epsilon_k \cap B^\epsilon_k} \mathcal{R}_{\Delta_p}(f) \le \max_{f \in A^\epsilon_k} \mathcal{R}_{\Delta_p}(f) \le \lambda'_k + \epsilon\,.$$

Hence we can establish the following inequality between the max min and min max variational inequalities:

$$\lambda''_k \le \lambda'_k\,. \tag{4.11}$$

In addition we can prove the following equality about the first variational eigenvalues:

Lemma 4.4.5.

$$\lambda^{\mathcal{K}}_1 = \lambda^D_1 = \lambda'_1 = \lambda''_1 \quad \textit{and} \quad \lambda^{\mathcal{K}}_2 = \lambda^D_2 = \lambda'_2 = \lambda''_2\,.$$

Proof. The case $k = 1$ is trivially proved since all $\lambda^K_1, \lambda^D_1, \lambda'_1$ and λ''_1 are defined as pointwise minimizers of $\mathcal{R}_p$.

To prove the second sequence of equalities, let $A_2 \in \mathcal{F}_2$ be such that

$$\max_{f \in A_2} \mathcal{R}_p(f) = \lambda^{\mathcal{K}}_2\,.$$

From the definition of Krasnoselskii genus, A_2 must be necessarily a closed, symmetric, and connected subset of S_p. This fact implies the existence of a continuous closed curve $\delta : [0,1] \mapsto A_2$ such that

$$\max_{f \in \delta(t)} \mathcal{R}_p(f) = \lambda^{\mathcal{K}}_2\,.$$

Now observe that the curve $\delta(t)$ can be approximated by C^2 curves $\delta^\epsilon(t) \in \Lambda'_2$ with $d\big(\delta(t), \delta^\epsilon(t)\big) < \epsilon$, thus using the continuity of $\mathcal{R}_p$ we easily get that

$$\lambda'_2 \le \lambda^{\mathcal{K}}_2\,, \tag{4.12}$$

which, together with (4.10), implies that equality holds. Last observe that from (4.11),

$$\lambda''_2 \le \lambda'_2\,. \tag{4.13}$$

However we recall from 4.4.2 that there are no eigenvalues of the p-Laplacian between $\lambda^{\mathcal{K}}_1$ and $\lambda^{\mathcal{K}}_2$. Hence, from (4.10), (4.13) and (4.12) it follows

$$\lambda^K_2 = \lambda^D_2 = \lambda'_2 = \lambda''_2\,.$$

□

Now, thanks to Lemma 4.3.2, given a variational p-Laplacian eigenvalue, λ, we are able to compare its position in the variational spectrum with its index as an eigenvalue of the corresponding weighted Laplacian.

Lemma 4.4.6. *Let (λ, f) be a p-Laplacian eigenpair such that*

$$\lambda = \lambda'_k.$$

Define $\mu = |\nabla f|^{p-2}$ and $\nu = |f|^{p-2}$ and assume $\lambda = \lambda_{h-m+1}(\mu,\nu) = \cdots = \lambda_h(\mu,\nu)$, where $\lambda_j(\mu,\nu)$ are the eigenvalues of the weighted Laplacian as given in Lemma (4.3.2) and m is their multiplicity. Then

$$k \le h.$$

Moreover, if $\lambda''_k = \lambda$, then

$$h - m + 1 \le k\,.$$

Proof. We first assume $k > h$. Then by characterization (4.9), for any $\epsilon > 0$ there exists $f_\epsilon \in S_p \cap \mathcal{D}_0$ and a subset $A^\epsilon_{k-1} \in S_p \cap \mathcal{D}_0$, C^2-diffeomorphic to a $(k-1)$-dimensional sphere such that

$$\lambda + \epsilon = \mathcal{R}_p(f_\epsilon) = \max_{f \in A^\epsilon_{k-1}} \mathcal{R}_p(f)\,.$$

Then for any $\xi \in T_{f_\epsilon}(S_p \cap A^\epsilon_{k-1})$ and any curve, $\gamma(t)$, in A^ϵ_{k-1} such that $\gamma(0) = f_\epsilon$ and $\gamma'(0) = \xi$, we have

$$\frac{d^2}{dt^2}\mathcal{R}_p\big(\gamma(t)\big)\Big|_{t=0} = \langle \xi, D^2\mathcal{R}_p(f_\epsilon)\xi\rangle + \langle \nabla\mathcal{R}_p(f_\epsilon), \gamma''(t)\rangle \le 0\,. \tag{4.14}$$

Then assuming, up to subsequences, that $f = \lim_{\epsilon\to 0} f_\epsilon$ and $\pi := \lim_{\epsilon\to 0} Tg_{f_\epsilon}(S_p \cap A^\epsilon_{k-1}) \in T_f(S_p \cap \mathcal{D}_0)$, recalling that $\partial f\big(\mathcal{R}_{\Delta_p}(f)\big) = 0$, for any $\xi \in \pi$, from (4.14), we have

$$\frac{\partial^2}{\partial\epsilon^2}\left(\frac{\|\nabla(f+\epsilon\xi)\|_p^p}{\|f+\epsilon\xi\|_p^p}\right)\Bigg|_{\epsilon=0} = \frac{p(p-1)}{2}\frac{\partial^2}{\partial\epsilon^2}\left(\frac{\|\nabla(f+\epsilon\xi)\|_\mu^2}{\|f+\epsilon\xi\|_\nu^2}\right)\Bigg|_{\epsilon=0} \le 0. \tag{4.15}$$

Moreover, if $\lambda = \lambda_h(\mu,\nu)$, thanks to lemma 4.3.2, we know that if $\xi \in T_f(S_\nu) = T_f(S_p)$ and $\xi \in \operatorname{span}\{f_j(\mu,\nu)\}_{j>h}$ then

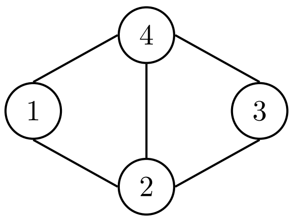
$$\frac{\partial^2}{\partial\epsilon^2}\left(\frac{\|\nabla(f+\epsilon\xi)\|_p^p}{\|f+\epsilon\xi\|_p^p}\right)\Bigg|_{\epsilon=0} = \frac{p(p-1)}{2}\frac{\partial^2}{\partial\epsilon^2}\left(\frac{\|\nabla(f+\epsilon\xi)\|_\mu^2}{\|f+\epsilon\xi\|_\nu^2}\right)\Bigg|_{\epsilon=0} > 0 \tag{4.16}$$

Now observe that $\dim\big(\pi \cap T_f(S_p)\big) = k-1$ and $\dim\big(\operatorname{span}\{f_j(\mu,\nu)\}_{j>h}\big) = N-h > N-k$. Thus

$$T_f(S_p \cap A_k) \cap \operatorname{span}\{f_j(\mu,\nu)\}_{j>h} \neq \emptyset$$

which is a contradiction by (4.16) and (4.15).
In the second case, if $k < h - m + 1$, by (4.9), we can state that there exists a subset π of $T_f(S_p \cap \mathcal{D}_0)$, of dimension $N-k$ such that for any $\xi \in \pi$

$$\frac{\partial^2}{\partial\epsilon^2}\left(\frac{\|\nabla(f+\epsilon\xi)\|_p^p}{\|f+\epsilon\xi\|_p^p}\right)\Bigg|_{\epsilon=0} = \frac{p(p-1)}{2}\frac{\partial^2}{\partial\epsilon^2}\left(\frac{\|\nabla(f+\epsilon\xi)\|_\mu^2}{\|f+\epsilon\xi\|_\nu^2}\right)\Bigg|_{\epsilon=0} \ge 0.$$



1. $f = (1,1,1,1), \quad \lambda = 0$
2. $f = (1,0,-1,0), \quad \lambda = 2$
3. $f = (0,1,0,-1), \quad \lambda = 2 + 2^{p-1}$
4. $f = (1,0,1,-2^{\frac{1}{p-1}}),\ (1,-2^{\frac{1}{p-1}},1,0)$
   $\lambda = 1 + \big(1 + 2^{\frac{1}{p-1}}\big)^{p-1}$
5. $f = (1,-1,1,-1), \quad \lambda = 2^p$

Figure 4.1: Left: Example graph in which the corresponding p-Laplacian Δ_p with $\omega_{uv} = 1\ \forall(u,v) \in E$, has more eigenvalues then the dimesion of the space. Right: Set of five eigenvalues and corresponding eigenfunctions.

Moreover, again from 4.3.2, we know also that $\forall \xi \in \operatorname{span}\{f_j(\mu,\nu)\}_{j<h-m+1}$

$$\frac{\partial^2}{\partial\epsilon^2}\left(\frac{\|\nabla(f+\epsilon\xi)\|_p^p}{\|f+\epsilon\xi\|_p^p}\right)\bigg|_{\epsilon=0} = \frac{p(p-1)}{2}\frac{\partial^2}{\partial\epsilon^2}\left(\frac{\|\nabla(f+\epsilon\xi)\|_\mu^2}{\|f+\epsilon\xi\|_\nu^2}\right)\bigg|_{\epsilon=0} < 0$$

leading to an absurd since

$$\dim\big(Tg_f(S_p \cap A_{N-k})\big) + \dim\big(\operatorname{span}\{f_j(\mu,\nu)\}_{j<h-m+1}\big) = N - k + h - m > N - 1.$$

$\square$

Observe that if $\lambda'_k = \lambda''_k$, as in the case of $k = 1, 2$, Lemma 4.4.6 implies that λ is the k-th eigenvalue of the corresponding weighted Laplacian.

Remark 4.4.7. *Observe also that the last Lemma 4.4.6 provides a novel tool in the study of the variational eigenvalues. Recall, for example, the graph presented in the introduction with the corresponding eigenpairs of the p-Laplacian (see Figure 4.1) Then, since the number of eigenpairs is greater than the dimension of the space, at least one of the eigenvalues is a non variational one. However, the only definition of the variational eigenvalues does not help to identify which eigenvalue is variational and which one is not. Differently the last lemma allows us to conclude that the eigenvalue $\lambda = 2 + 2^{p-1}$ is not a variational eigenvalue. Indeed the gradient of $f = (0,1,0,-1)$ is everywhere different from zero, which implies that:*

$$\operatorname{Ker}(\Delta_\mu) \cap \operatorname{Ker}(\nu) = \emptyset\,,$$

where $\mu = |\nabla f|^{p-2}$ and $\nu = |f|^{p-2}$. Moreover, it is easy to prove that $(f,\lambda) = \Big(\big(0,1,0,-1\big), 2+2^{p-1}\Big)$ is the second eigenpair of the eigenvalue problem

$$\Delta_\mu f = \lambda \nu f\,,$$

and that λ is a simple eigenvalue of the (μ,ν)-weighted linear eigenvalue problem, i.e.:

$$\lambda_1(\mu,\nu) < \lambda = \lambda_2(\mu,\nu) < \lambda_3(\mu,\nu) = \lambda_4(\mu,\nu) = \infty\,.$$

Thus, if λ was a variational eigenvalue $\lambda = \lambda'_k(\Delta_p)$, Lemma 4.4.6 would yield, $k \le 2$. However from the characterization of the second eigenvalue in the previous chapter (see Theorem 3.4.8) and Lemma 4.4.5 we know that λ can not be the second variational eigenvalues. Otherwise λ should be the smallest eigenvalue among the eigenvalues that are strictly bigger that the eigenvalue 0, nonetheless the eigenvalue 2 is always between 0 and λ see Fig.4.1. Hence we can conlcude that $\lambda = 2 + 2^{p-1}$ is surely not a variational eigenvalue.

4.5 p-Laplacian eigenpairs as critical points of a class of energy functions

In this section, using the equivalence of the p-Laplacian eigenvalue problem and the constrained weighted Laplacian eigenvalue problem (4.2), we characterize a set of p-Laplacian eigenpairs in terms of critical points of a family of energy functions whose variables are densities on the edges and on the nodes of the graph. To formalize some proofs, before considering the classical p-Laplacian eigenvalue problem, we need to introduce the weighted $(p,2)$-Laplacian eigenvalue problem that can be of independent interest [15, 45, 74].

4.5.1 The $(p,2)$-Laplacian eigenvalue problem

Let $\nu : V \to \mathbb{R}$ be a density on the node space of a graph $\mathcal{G}$ and consider the $(p,2)$-Rayleigh quotient,

$$\mathcal{R}_{p,2,\nu}(f) = \frac{\|\nabla f\|_p^p}{\|f\|_{2,\nu}^p} = \frac{\frac{1}{2}\sum_{(u,v)\in E} |\nabla f(uv)|^p}{\left(\sum_{u\in V} \nu_u |f(u)|^2\right)^{\frac{p}{2}}},$$

Assuming $\mathcal{R}_{p,2,\nu}$ to be defined on the domain $\mathcal{D}_0$ we call its critical point equation, the $(p,2)$-Laplacian eigenvalue equation weighted in ν:

$$\begin{cases} (\Delta_p f)(u) = \lambda\, \nu_u\, \|f\|_{2,\nu}^{p-2} f(u) & \forall u \in V \setminus B \\ f(u) = 0 & \forall u \in B \end{cases}, \tag{4.17}$$

and we refer with $\lambda(p,2,\nu)$ to indicate the $(p,2)$-eigenvalues.

Observe that if f solves (4.17), we have the following equation $\forall u \in V \setminus B$:

$$\sum_{\substack{v\in V\setminus B\\ v\sim u}} \omega_{uv} |\nabla f(v,u)|^{p-2}(\nabla f(v,u)) + \Big(\sum_{\substack{v\in B\\ v\sim u}} \omega_{uv}^p\Big)|f(u)|^{p-2} f(u) = \lambda\, \nu_u\, \|f\|_{2,\nu}^{p-2} f(u)\,. \tag{4.18}$$

Moreover, as done for the p-Laplacian, we can define the $(p,2)$-variational eigenvalues as follows. Let $\mathcal{S}_2 = \{f : \|f\|_{2,\nu} = 1\}$ and for any $1 \leq k \leq n$ consider the Krasnoselskii family $\mathcal{F}_k(\mathcal{S}_2 \cap \mathcal{D}_0) = \{A \subseteq \mathcal{A} \cap \mathcal{S}_2 \cap \mathcal{D}_0 \,|\, \gamma(A) \geq k\}$, then

$$\lambda_k(p,\nu) = \min_{A \in \mathcal{F}_k} \max_{f \in A} \mathcal{R}_{p,2,\nu}(f)$$

defines the k-th variational eigenvalue of the $(p,2)$-Laplacian. Next we provide a quite classical charaterization of the first eigenpair $\big(f_1, \lambda_1(p,\nu)\big)$ of the $(p,2)$-Laplacian. The idea is very similar to the one used in [54] for the classical p-Laplacian eigenvalue problem. We recall first the following maximum principle from [77] that we use in the proof, whose proof is reported, for completeness, in the appendix ThmA.0.2.

Theorem 4.5.1. *Suppose that f and g satisfies*

$$\Delta_p f(u) + r(u)|f(u)|^{p-2} f(u) \geq \Delta_p g(u) + r(u)|g(u)|^{p-2} g(u),$$

where $r(u) \geq 0$ for any u in V. Then $f(u) \geq g(u)$ for any $u \in V$.

Then we can prove that the first eigenvalue is simple and positive and the corresponding unique first eigenfunctions is the only eigenfunction that is strictly greater than zero on all internal nodes. This is summerized in the following theorem:

Theorem 4.5.2. *Let $\mathcal{G} = (E, V, \omega)$ be a connected graph with boundary, B. Assume (λ_1, f_1) to be a first eigenpair of the $(p,2)$-Laplacian with Dirichlet boundary conditions. Then $\lambda_1 \geq 0$ and $f_1(u) > 0 \quad \forall u \in V$. Moreover λ_1 is simple and f_1 is the unique eigenfunction strictly greater than zero on every internal node.*

Proof. Let f_1 be a minimizer of $\mathcal{R}_{p,2,\nu}$ such that $\|f_1\|_{2,\nu} = 1$. Observe that $\mathcal{R}_{p,2,\nu}(|f|) \leq \mathcal{R}_{p,2,\nu}(f)$ with equality if and only if $f = |f|$, implies that f_1 satisfies $f_1(u) \geq 0 \; \forall u \in V \setminus B$. Moreover if there exists $u \in V \setminus B$ such that $f_1(u) = 0$ then, from (4.18), $f_1(v) = 0$ for any $v \sim u$ and the connectedness of the graph implies $f_1 = 0$ which is a contradiction.

Now we can prove the second part of the theorem. Assume that there exists a positive eigenfunction $f_2 > 0$ such that $\mathcal{R}_{p,2,\nu}(f_2) = \lambda_2 > \lambda_1$. Take $t > 0$ such that

$$\lambda_2 f_2(u) > t\lambda_1 f_1(u) \; \forall u \in V \setminus B \quad \text{and} \quad \exists u_0 \in V \setminus B \text{ s.t. } t f_1(u_0) > f_2(u_0) \; .$$

Applying Theorem 4.5.1 to the functions tf_1 and f_2, we get a contradiction, proving that positive eigenfunctions have to be associated to the first eigenvalue. We are left to prove that λ_1 is simple, i.e., the uniqueness of the corresponding eigenfunction f_1. Assume that there exist two positive eigenfunctions f_1 and f_2 relative to λ_1 with $\|f_1\|_{2,\nu} = \|f_2\|_{2,\nu} = 1$. Then, the function

$$g(u) = \left(f_1^2(u) + f_2^2(u)\right)^{\frac{1}{2}} ,$$

has 2-norm given by $\|g\|_{2,\nu}^p = 2^{\frac{p}{2}}$, and its gradient satisfies:

$$\|\nabla g\|_p^p \le 2^{\frac{p-2}{2}} \big(\|\nabla f_1\|_p^p + \|\nabla f_2\|_p^p\big)$$

with equality holding if and only if $\nabla f_1(u,v) = \nabla f_2(u,v)\ \forall (u,v) \in E$. To prove the last inequality, consider an edge (u,v) and use first the Cauchy Schwarz inequality applied to the two vectors $\big(f_1(u), f_2(u)\big)$ $\big(f_1(v), f_2(v)\big)$ and then the Jensen inequality applied to the function $x \mapsto |x|^{\frac{p}{2}}$:

$$\begin{aligned}
|\nabla g(v,u)|^p &= \omega_{uv}^p \Big| \big(f_1(u)^2 + f_2(u)^2\big)^{\frac{1}{2}} - \big(f_1(v)^2 + f_2(v)^2\big)^{\frac{1}{2}} \Big|^p \\
&\le \omega_{uv}^p \Big| \big(f_1(u) - f_1(v)\big)^2 + \big(f_2(u) - f_2(v)\big)^2 \Big|^{\frac{p}{2}} \\
&\le \omega_{uv}^p 2^{\frac{p-2}{2}} \Big(\big|f_1(u) - f_1(v)\big|^p + \big|f_2(u) - f_2(v)\big|^p \Big) \\
&= 2^{\frac{p-2}{2}} \big(|\nabla f_1(v,u)|^p + |\nabla f_2(v,u)|^p\big)
\end{aligned}$$

where, by convexity of the function $|x|^{\frac{p}{2}}$, we have equality if and only if $f_1(u) - f_1(v) = f_2(u) - f_2(v)$. Then we have

$$\lambda_1 2^{\frac{p}{2}} = \lambda_1 \|g\|_{2,\nu}^p \le \|\nabla g\|_p^p \le 2^{\frac{p-2}{2}} \big(\|\nabla f_1\|_p^p + \|\nabla f_2\|_p^p\big) = \lambda_1 2^{\frac{p}{2}}$$

implying that for any edge $f_1(u) - f_1(v) = f_2(u) - f_2(v)$ and thus, since $\|f_1\|_{2,\nu} = \|f_2\|_{2,\nu}$, $f_1 = f_2$. □

4.5.2 Weigthed Laplacian equivalence

Similarly to the p-Laplacian eigenvalue problem discussed in Section 4.3.1, the $(p,2)$-Laplacian eigenvalue problem can be reformulated in terms of a constrained weighted Laplacian eigenvalue problem. To this aim, we first look at the eigenvalue equation (4.18)

$$\sum_{\substack{v \in V \setminus B \\ v \sim u}} \omega_{uv} |\nabla f(v,u)|^{p-2} (\nabla f(v,u)) + \Big(\sum_{v \in B} \omega_{uv}^p\Big) |f(u)|^{p-2} f(u) = \lambda\, \nu_u \|f\|_{2,\nu}^{p-2} f(u)\,.$$

Dividing both the terms by $\|f\|_{2,\nu}^{p-2}$, it is straightforward to observe that (f,λ) is an eigenpair of the $(p,2)$-Laplacian if and only if (f,λ) is an eigenpair of the constrained weighted Laplacian problem

$$\begin{cases}
\Delta_\mu f(u) := \sum_{v \sim u} \omega_{uv} \mu_{uv} \nabla f(v,u) = \lambda \nu_u f(u) & \forall u \in V \setminus B \\
f(u) = 0 & \forall u \in B \\
\mu_{uv} = \frac{|\nabla f(u,v)|^{p-2}}{\|f\|_{2,\nu}^{p-2}} \ge 0 & \forall\, (u,v) \in E
\end{cases} \tag{4.19}$$

4.5.3 Energy function of the first eigenpair of the $(p,2)$-Laplacian

Given a density $\nu \in \mathcal{M}^+(\nu)$ on the node space, we introduce a convex energy function whose minimum can be proved to correspond to the unique first eigenapair of the $(p,2)$ eigenvalue problem weighted in ν. The results and the techniques presented in this section are the starting point for the next paragraphs that address the classical p-Laplacian eigenvalue problem.

Consider the energy function, written here for a fixed $\nu \in \mathcal{M}^+(V)$:

$$\begin{aligned}\mathcal{L}_{1,E}(\mu,\nu) &= \frac{1}{\lambda_1(\mu,\nu)} + \mathrm{M}_{E,p}(\mu) = \sup_f \frac{\|f\|_{2,\nu}^2}{\|\nabla f\|_{2,\mu}^2} + \frac{p-2}{2p}\sum_{(u,v)\in E}\mu_{uv}^{\frac{p}{p-2}} \\ &= \sup_f \frac{2\sum_{u\in V}\nu_u f(u)^2}{\sum_{(u,v)\in E}\mu_{uv}|\nabla f(uv)|^2} + \frac{p-2}{2p}\sum_{uv\in E}\mu_{uv}^{\frac{p}{p-2}}\end{aligned} \tag{4.20}$$

where $\mathrm{M}_{E,p}(\mu) := \frac{p-2}{2p}\sum_{(u,v)\in E}\mu_{uv}^{\frac{p}{p-2}}$. Since in this section we are assuming ν to be fixed, to simplify the notation, we will write $\lambda(\mu)$ in place of $\lambda(\mu,\nu)$ and $\mathcal{L}_{1,E}(\mu)$ in place of $\mathcal{L}_{1,E}(\mu,\nu)$.

In what follows we will be using the following result about the differentiability of the first eigenvalue of the Laplacian operator. This result is well-known [61] and we report here only the special case of a generalized weighted Laplacian matrix.

Lemma 4.5.3. *Assume $\mu_0 \in \mathcal{M}^+(E)$ and $\nu \in \mathcal{M}^+(V)$ to be positive densities on the edges and on the nodes (see Def4.3.1) and let Δ_{μ_0} to be the weighted generalized Laplacian matrix associated to the homogeneous Dirichlet boundary problem on the graph $\mathcal{G} = (V,E,\omega)$ with boundary B. Define $\lambda_1(\mu_0)$ to be the 1-st eigenvalue of the eigenvalue problem $\Delta_{\mu_0} f = \lambda \nu f$, and assume Δ_{μ_0} to be not degenerate, then the subgradient of the function $\mu \mapsto \lambda_1^{-1}(\mu)$ in μ_0 is given by:*

$$\partial\Big(\lambda_1^{-1}(\mu_0)\Big) = Co\Big\{ -\frac{|\nabla f|^2}{2\lambda_1(\mu_0)^2\|f\|_{2,\nu}^2} \mid \Delta_{\mu_0} f = \lambda_1(\mu_0)\nu f\Big\}$$

Where $Co\{\cdot\}$ denotes the convex hull.

Proof. The proof is a trivial consequence of the forumla for the subgradient of the supremum of a family of convex functions (see Thm 4.4.2 [51]) which states that, given $\phi(x) = \sup_{\alpha\in A}\phi_\alpha(x)$ with ϕ_α convex, A a compact and $\alpha \to \phi_\alpha(x)$ an upper semi continuous for any x in a neighborhood of x_0, then

$$\partial\big(\phi(x_0)\big) = Co\{\partial\phi_\alpha(x_0) \mid \alpha \ s.t. \ \phi_\alpha(x_0) = \phi(x_0)\}\,. \tag{4.21}$$

In our case we have

$$\lambda_1^{-1}(\mu) = \sup_{\|f\|_{2,}=1} \frac{\|f\|_{2,\nu}^2}{\|\nabla f\|_{2,\mu}^2}$$

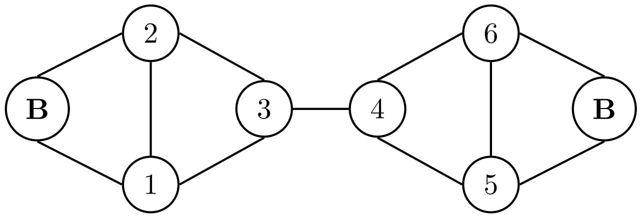


Figure 4.2: A graph with non-simple first eigenvalue. Assume $\nu_u = 1\ \forall u \in V \setminus B$, then the graph is symmetric and the first eigenfunction of Δ_p, $f_1(\Delta_p, \nu)$, is unique and necessarily agrees with the symmetry of the graph. This means that $\nabla f_1(3,4) = 0$ and thus the density $\mu_0 = |\nabla f_1|^{p-2}$ of eq.(4.19) is also zero on the edge $(3,4)$ and splits $\mathcal{G}$ in two connected components. As a result $\lambda_1(\mu_0, \nu)$ is not simple.

with $\mu \mapsto \frac{\|f\|^2_{2,\nu}}{\|\nabla f\|^2_{2,\mu}}$ differentiable in μ_0 and

$$\partial_\mu \frac{\|f\|^2_{2,\nu}}{\|\nabla f\|^2_{2,\mu}}(\mu_0) = -\frac{|\nabla f|^2 \|f\|^2_{2,\nu}}{2\|\nabla f\|^4_{2,\mu_0}}\,.$$

Thus from (4.21), recalling that $\frac{\|f\|_2^4}{\|\nabla f\|^4_{2,\mu_0}} = \lambda^{-2}(\mu_0)$, we get the desired result:

$$\partial\Big(\lambda_1^{-1}(\mu_0)\Big) = Co\bigg\{ -\frac{|\nabla f|^2}{2\lambda_1(\mu_0)^2\|f\|^2_{2,\nu}} \mid\ \Delta_{\mu_0} f = \lambda_1(\mu_0)\nu f\bigg\}\,.$$

$\square$

We recall in particular that if the graph is connected and $\mu > 0$ then $\lambda_1(\mu)$ is simple (see Theorem A.0.1). Otherwise the multiplicity of $\lambda_1(\mu)$ can grow up to the number of connected components of $\mathcal{G} = \cup_{i=1}^M \mathcal{G}_i$ and the eigenspace of $\lambda_1(\mu)$ is the span of the eigenvectors relative to every connected component. In other words, if $\Delta_\mu f = \lambda_1(\mu)\nu f$,

$$f \in \mathrm{span}\{f_{1,i}\ i = 1,\dots,M,\ s.t.\ \Delta_\mu f_{1,i}|_{\mathcal{G}_i} = \lambda_1(\mu)\nu f_{1,i}|_{\mathcal{G}_i}\,,\ f_{1,i}(u) = 0\ \forall u \in \mathcal{G}\backslash\mathcal{G}_i\}$$

We would like to remark that an edge with zero density can be removed from the graph without changing the eigenfunction when we consider the constrained weighted-Laplacian eigenvalue problem equivalent to the p-Laplacian eigenvalue problem (4.19). Hence, the first eigenvalue may not be simple even for a connected graph if $\mu \geq 0$. Figure 4.2 reports a simple example where this situation occurs.

The above result of Lemma 4.5.3 can be extended also to the higher eignvalues by means of the Clarke subdifferential of a locally Lipschitz function, see [25, 27, 52]. Here, however, we limit our study of the higher eigenvalues to the differentiable case. Assuming $\lambda_k(\mu)$ to be differentiable in a point μ_0, in the next Lemma we provide a trivial characterization of its derivative in the variable μ.

Lemma 4.5.4. *Let $\lambda_k(\mu)$ to be the k-th eigenvalue of Δ_μ, the weighted generalized Laplacian matrix associated to the homogeneous Dirichlet boundary problem on the graph $\mathcal{G} = (V, E, \omega)$ with boundary B. If $\lambda_k(\mu)$ is differentiable in μ_0, then*

$$\partial_\mu\Big(\lambda_k^{-1}(\mu)\Big)(\mu_0) = -\frac{|\nabla f_k|^2}{2\lambda_k(\mu_0)^2\|f_k\|_{2,\nu}^2}$$

Proof. Observe first of all that if $\lambda_k(\mu)$ is differentiable in μ_0 then $\lambda_k(\mu_0)$ is simple and thus f_k is uniquely defined, see [61].By the chain rule it is enough to show that

$$\partial_{\mu_{uv}}\lambda_k(\mu) = \frac{\partial\mu_{uv}\Big(f_k^T\nabla^T\mathrm{diag}(\mu)\nabla f_k\Big)}{2\|f_k\|_{2,\nu}^2} = \frac{|\nabla f_k(u,v)|^2}{2\|f_k\|_{2,\nu}^2}\,.$$

To prove the last equality, we differentiate both the terms of the eigenvalue equation with respect to μ_{uv}:

$$\partial\mu_{uv}\Big(\Delta_\mu f_k\Big) = \partial\mu_{uv}\Big(\lambda_k\mathrm{diag}(\nu)f_k\Big)$$
$$\partial\mu_{uv}\big(\Delta_\mu\big)f_k + \Delta_\mu\partial\mu_{uv}\big(f_k\big) = \partial\mu_{uv}\big(\lambda_k\big)\mathrm{diag}(\nu)f_k + \lambda_k\mathrm{diag}(\nu)\partial\mu_{uv}\big(f_k\big)\,.$$

Then multiply both terms by f_k and remember $\Delta_\mu f_k = \Delta_\mu^T f_k = \lambda_k f_k$ and $\Delta_\mu = \frac{1}{2}\nabla^T\mathrm{diag}(\mu)\nabla$

$$f_k^T\partial\mu_{uv}\big(\Delta_\mu\big)f_k + \lambda_k f_k^T\partial\mu_{uv}\big(f_k\big) = \partial\mu_{uv}\big(\lambda_k\big)f_k^T\mathrm{diag}(\nu)f_k + \lambda_k f_k^T\mathrm{diag}(\nu)\partial\mu_{uv}\big(f_k\big)$$
$$\frac{1}{2}f_k^T\nabla^T\mathrm{diag}(e_{uv})\nabla f_k = \partial\mu_{uv}\big(\lambda_k\big)f_k^T\mathrm{diag}(\nu)f_k$$

where e_{uv} is the characteristic function of the edge (u, v), this concludes the proof. □

Now we have all the instruments to study the critical points of the function $\mathcal{L}_{1,E}(\mu)$ (4.20), we will show that it admits a unique minimum, μ^*, and that the first eigenfunction of Δ_{μ^*} corrsponds to the unique first eigenpair of the $(p, 2)$-Laplacian. Indeed, that the function $\mathcal{L}_{1,E}(\mu)$ (4.20) is a convex function on the convex cone $\mathcal{M}^+(E)$, hence it admits a unique minimum that can be characterized by means of Lemma 4.5.3.

Theorem 4.5.5. *Given $\nu_u \in \mathcal{M}^+(V)$ such that $\nu_u > 0\ \forall u \in V \setminus B$, let μ^* be a minimizer of $\mathcal{L}_{1,E}(\mu)$ on $\mathcal{M}^+(E)$. Then there exists $f_1(\mu^*)$ such that*

$$\begin{cases} \Delta_{\mu^*}f_1(\mu^*) = \lambda_1(\mu^*)\nu f_1(\mu^*) \\ \\ \mu^* = \dfrac{|\nabla f_1(\mu^*)|^{p-2}}{\lambda_1^{p-2}(\mu^*)\|f_1(\mu^*)\|_{2,\nu}^{p-2}}\,. \end{cases}$$

In particular $\big(f_1(\mu^*), \lambda_1^{p-1}(\mu^*)\big)$, is the first $(p, 2)$-eigenpair, $\big(f_1(\mu^*), \lambda_1^{p-1}(\mu^*)\big) = \big(f_1(p, \nu), \lambda_1(p, \nu)\big)$. Moreover

$$\mathcal{L}_{1,E}(\mu^*) = \frac{2p-2}{p}\lambda_1^{-\frac{1}{p-1}}(p,\nu)$$

Proof. We first observe that, necessarily, Δ_{μ^*} is a non singular matrix. Indeed, if this was the case we would have $\lambda_1(\mu^*) = 0$ and thus $\mathcal{L}_{\mu^*} = \infty$, implying that μ^* is not a minimizer. Then, from [24], we know that the minimizer μ^* of the constrained problem has to satisfy the following KKT system of equations

$$\begin{cases} 0 \in \partial^C \mathcal{L}_{1,E}(\mu^*) - \sum_{uv} c_{uv} e_{uv} & \\ c_{uv}\mu^*_{uv} = 0 & \forall (u,v) \in E \\ c_{uv} \geq 0 & \forall (u,v) \in E \end{cases}$$

where e_{uv} is the characteristic function of the edge (u,v) and $\{c_{uv}\}$ is a family of edge-wise Lagrange multipliers. Using lemma 4.5.3 we get

$$\begin{cases} 0 \in Co\Big\{ -\dfrac{|\nabla f(u,v)|^2}{2\lambda_1^2(\mu^*)\|f\|^2_{2,\nu}} \Big\} + \dfrac{{\mu^*_{uv}}^{\frac{2}{p-2}}}{2} - \displaystyle\sum_{uv} c_{uv} e_{uv} & \\ \Delta_{\mu^*} f = \lambda_1(\mu^*)\nu f & \\ c_{uv}\mu^*_{uv} = 0 & \forall (u,v) \in E \\ c_{uv} \geq 0 & \forall (u,v) \in E \end{cases} \tag{4.22}$$

In the case in which μ^* does not disconnect the graph, $\lambda_1(\mu)$ is differentiable in μ^*, and thus the proof trivially follows from Lemma 4.5.3 since

$$Co\Big\{ -\frac{|\nabla f(u,v)|^2}{2\lambda_1^2(\mu^*)\|f\|^2_{2,\nu}} \;\Big|\; \Delta_{\mu^*} f = \lambda_1(\mu^*)\nu f \Big\} = -\frac{|\nabla f_1^*(u,v)|^2}{2\lambda_1^2(\mu^*)\|f_1^*\|^2_{2,\nu}}$$

where f_1^* is the only eigenvector associated to $\lambda_1(\mu^*)$. Assume now that the graph has been disconnected by the zeros of μ^* and consider an edge, $(u_0, v_0) \in E$, such that $\mu^*_{u_0 v_0} = 0$. Let $G : E \to \mathbb{R}^-$ and $c : E \to \mathbb{R}^+$ be, respectively, the element in the subgrandient of $\lambda_1^{-1}(\mu^*)$ and the Lagrange multiplier that satisfies (4.22). Then we have $G(u_0, v_0) = c_{u_0 v_0} = 0$. Now we claim that if G is zero on all the edges where μ^* is zero, necessarily G is an extremal point of the convex set $\partial^C(\lambda_1^{-1}(\mu^*))$, i.e., there exists an eigenfunction $f_1(\mu^*)$ such that

$$G = -\frac{|\nabla f_1(\mu^*)|^2}{2\lambda_1(\mu^*)\|f_1(\mu^*)\|_{2,\nu}}$$

The last claim follows by recalling that, if the graph has been disconnected by the zeros of μ^*, $\mathcal{G} = \cup_i \mathcal{G}_{i=1}^m$ and the multiplicity of $\lambda_1(\mu^*)$ can increase up to the number, m, of connected components of $\mathcal{G}$. Moreover, the corresponding eigenspace is generated by the eigenvectors relative to every connected component. This means that if $\Delta_\mu g = \lambda_1(\mu)\nu g$,

$$g \in \operatorname{span}\{f_{1,i}\ i = 1, \dots, m,\ s.t.\ \Delta_\mu f_{1,i}|_{\mathcal{G}_i} = \lambda_1(\mu)\nu f_{1,i}|_{\mathcal{G}_i}\,,\ f_{1,i}(u) = 0\ \forall u \in \mathcal{G}\backslash\mathcal{G}_i\}$$

Now, (4.22) ensures that for any $\mu^*_{uv} \neq 0$ $|\nabla f(u,v)|^2 \neq 0$, which implies that $f_{1,i} \neq 0\ \forall i = 1, \dots, m$, i.e., the multiplicity of $\lambda_1(\mu^*)$ is exactly equal to m. Thus, up to rescaling, there can exist only one linear combination $f_1 \in \text{span}\{f_{1,i}\}$ such that $\nabla f_1(u,v) = 0$ for all edges (u,v) where $\mu^*_{uv} = 0$. This in particular means that if

$$G(u,v) \in Co\Big\{ -\frac{|\nabla f(u,v)|^2}{2\lambda_1^2(\mu^*)\|f\|^2_{2,\nu}} \,\Big|\, \Delta_{\mu^*} f = \lambda_1(\mu^*)\nu f \Big\}$$

and $G(u,v) = 0 \;\; \forall (u,v)$ s.t. $\mu^*_{uv} = 0$, necessarily

$$G = -\frac{|\nabla f_1(u,v)|^2}{2\lambda_1^2(\mu^*)\|f_1\|^2_{2,\nu}}\,,$$

concluding the proof of the claim. It follows that equation (4.22) can be written as:

$$\begin{cases} \mu^* = \dfrac{|\nabla f_1|^{p-2}}{\lambda_1^{p-2}(\mu^*)\|f_1\|^{p-2}_{2,\nu}} \\ \\ \Delta_{\mu^*} f_1 = \lambda_1(\mu^*)\nu f_1 \end{cases}$$

that reads

$$\sum_{\substack{v\in V\\ v\sim u}} \omega_{uv} |\nabla f_1(v,u)|^{p-2}\big(\nabla f(v,u)\big) = \lambda_1(\mu^*)^{p-1}\|f_1\|^{p-2}_{2,\nu}\nu f_1(u) \qquad \forall u \in V \quad (4.23)$$

From equation (4.23) it is straightforward to observe that the pair $\big(f_1(\mu^*), \lambda_1^{p-1}(\mu^*)\big)$, associated to the critical weight μ^*, corresponds to an eigenpair of the $(p,2)$-Laplacian. Moreover

$$f_1(\mu^*) > 0 \qquad \forall u \in V \setminus B$$

beacause it is the first eigenfunction of a Laplacian opertator. Thus, following Theorem 4.5.2, $\big(f_1(\mu^*), \lambda_1^{p-1}(\mu^*)\big)$ is necessarily the first eigenpair of the $(p,2)$-Laplacian, $\lambda_1^{p-1}(\mu^*) = \lambda_1(p,\nu)$. To conclude, we observe that

$$\begin{aligned} \mathcal{L}_{1,E}(\mu^*) &= \lambda_1^{-\frac{1}{p-1}}(p,\nu) + \frac{p-2}{2p}\sum_{uv\in E} {\mu^*_{uv}}^{\frac{p}{p-2}} \\ &= \frac{1}{\lambda_1^{\frac{1}{p-1}}(p,\nu)} + \frac{p-2}{p}\frac{\lambda_1(p,\nu)}{\lambda_1^{\frac{p}{p-1}}(p,\nu)} = \frac{2p-2}{p}\lambda_1^{-\frac{1}{p-1}}(p,\nu)\,. \end{aligned}$$

□

4.5.4 The p-Laplacian eigenvalue problem

In this section we discuss the more classical (p,p)-Laplacian eigenvalue problem as presented in section 4.2 (we will refer to it just as the p-Laplacian eigenvalue problem). We recall that it corresponds to the study of the critical points and values of the p-Rayleight quotient:

$$\mathcal{R}_p(f) = \frac{\|\nabla f\|_p^p}{\|f\|_p^p} = \frac{\frac{1}{2}\sum_{(u,v)\in\omega} |\nabla f(u,v)|^p}{\sum_{v\in V} |f(v)|^p},$$

or, in other terms, the p-Laplacian eigenvalue equation reads:

$$\begin{cases} \Delta_p f(u) = \sum_{v\sim u} \omega_{uv} |\nabla f(v,u)|^{p-2} \nabla f(v,u) = \lambda |f(u)|^{p-2} f(u) & \forall u \in V \setminus B \\ f(u) = 0 & \forall u \in B \end{cases}. \tag{4.24}$$

As in the case of the $(p,2)$-Laplacian eigenvalue problem, the same characterization of the first eigenpair, $(f_1(p), \lambda_1(p))$, carries over directly to the (p,p) case, see Theorem 4.4.2. Thus $\lambda_1(p)$ is a simple eigenvalue and $f_1(p)$ is the only p-Laplacian eigenfunction to be strictly positive on all the internal nodes of the graph.

Analogously, the equivalence of the p-Laplacian eigenvalue problem with a generalized linear eigenvalue problem carries over directly, see 4.3. Thus we say that (f,λ) is an eigenpair of the p-Laplacian, i.e., satisfies eq.(4.24), if and only if (f,λ) is an eigenpair of the constrained weighted Laplacian Dirichlet problem

$$\begin{cases} \Delta_\mu f(u) = \sum\limits_{v\sim u} \mu_{uv}\omega_{uv} \nabla f(v,u) = \lambda \nu_u f(u) & \forall u \in V \setminus B \\ f(u) = 0 & \forall u \in B \\ \mu_{uv} = |\nabla f(u,v)|^{p-2} & \forall (u,v) \in E \\ \nu_u = |f(u)|^{p-2} & \forall u \in V \setminus B \end{cases}$$

Observe that, alternatively, we can write (4.24) as a constrained weighted $(p,2)$-Laplacian eigenvalue problem, halving the number of free variables:

$$\begin{cases} \Delta_p f(u) = \lambda \nu_{0u} \|f\|_{2,\nu}^{p-2} f(u) & \forall u \in V \setminus B \\ \nu_u = \dfrac{|f(u)|^{p-2}}{\|f\|_{2,\nu_0}^{p-2}} & \forall u \in V \setminus B \\ f(u) = 0 & \forall u \in B \end{cases}.$$

4.5.5 Energy function of the first eigenpair of the p-Laplacian

In section 4.5.3 we have proved that, given a weight function on the nodes ν, it is possible to characterize the first eigenpair of the $(p,2)$-Laplacian eigenvalue problem weighted in ν by the minimizer μ^*_ν of the function $\mathcal{L}_{1,E}(\mu)$ (see eq. (4.20)). We can analogously introduce an energy function only of the variable ν as follows:

$$\mathcal{L}_{1,V}(\nu) = \frac{2(p-1)}{p} \lambda_1^{-\frac{1}{p-1}}(p,\nu) - \frac{p-2}{p} \sum_{u\in V\setminus B} \nu_u^{\frac{p}{p-2}}.$$

Observe that for any non singular ν, from Theorem 4.5.5, we have the following equality:

$$\mathcal{L}_{1,V}(\nu) = \mathcal{L}_{1,E}(\mu^*_\nu, \nu) - \mathrm{M}_{V,p}(\nu)$$

where $\mathrm{M}_{V,p}(\nu) := \frac{p-2}{p}\sum_{u\in V\setminus B}\nu_u^{\frac{p}{p-2}}$. We want to show that the only critical point of this function corresponds to the first eigenpair of the p-Laplacian. However, before taking derivatives, we have to prove sufficient regularity of the function $\nu \mapsto \lambda_1(p,\nu)$. Similar results have been proved for the regularity of the first p-Laplacian eigenfunction with respect to perturbations of the domain (see [64]).

Lemma 4.5.6. *The function $\lambda_1 : \nu \mapsto \lambda_1(p,\nu)$ is continuous together with its first derivativer, i.e., $\lambda_1(p,\cdot) \in C^1(\mathcal{M}^+(V),\mathbb{R})$, where $\mathcal{M}^+(V) = \{\nu : V\setminus B \to \mathbb{R} | \nu_u \geq 0\}$. Moreover*

$$\frac{\partial\lambda_1}{\partial\nu}(p,\nu_0) = -\frac{p}{2}\frac{\lambda_1(p,\nu_0)|f_0|^2}{\|f_0\|^2_{2,\nu}}$$

Proof. In the proof we write $\lambda_1(\nu)$ to denote $\lambda_1(p,\nu)$. Consider the function

$$\mathcal{R}(f,\nu) := \frac{\|\nabla f\|_p^p}{\|f\|^p_{2,\nu}} = \frac{\frac{1}{2}\sum_{uv\in E}|\nabla f(uv)|^p}{\left(\sum_{u\in V\setminus B}\nu_u|f(u)|^2\right)^{\frac{p}{2}}}$$

Recall that, given ν, the fist eigenvalue is characterized by

$$\lambda_1(\nu) := \min_f \mathcal{R}(f,\nu) = \mathcal{R}(f_\nu,\nu)\,.$$

The function that associates to a density ν the corresponding first eigenfunction, , f_ν, of the $(p,2)$-Laplacian weighted in ν, with $\|f_\nu\|_{2,\nu} = 1$ is well defined by Theorem 4.5.2 and continuous by the continuity of the minimizers. Observe also that given a density ν_0 and considered the corresponding first eigenfunction f_{ν_0}, from Theorem 4.5.2 we know that $f_{\nu_0}(u) > 0\ \forall\, u \in V\setminus B$. Now consider the variation of λ_1 near a point ν_0, where we use the notation $f_0 := f_{\nu_0}$

$$\begin{aligned}\lambda_1(\nu_0) - \lambda_1(\nu) &= \mathcal{R}(f_0,\nu_0) - \mathcal{R}(f_\nu,\nu)\\ &\leq \mathcal{R}(f_\nu,\nu_0) - \mathcal{R}(f_\nu,\nu) = \partial_\nu\mathcal{R}(f_\nu,\nu_0)(\nu_0-\nu) + o\big(\|\nu_0-\nu\|\big)\end{aligned}$$

which means

$$\begin{aligned}&\Rightarrow \limsup_{\nu\to\nu_0}\Big(\lambda_1(\nu_0) - \lambda_1(\nu) - \partial_\nu\mathcal{R}(f_0,\nu_0)(\nu_0-\nu)\Big)\\ &\qquad\qquad \leq \limsup_{\nu\to\nu_0}\Big(\partial_\nu\mathcal{R}(f_\nu,\nu_0) - \partial_\nu\mathcal{R}(f_0,\nu_0)\Big)(\nu_0-\nu) = 0\end{aligned}$$

Similarly observe that

$$\begin{aligned}\lambda_1(\nu_0) - \lambda_1(\nu) &= \mathcal{R}(f_0,\nu_0) - \mathcal{R}(f_\nu,\nu)\\ &\geq \mathcal{R}(f_0,\nu_0) - \mathcal{R}(f_0,\nu) = \partial_\nu\mathcal{R}(f_0,\nu_0)(\nu_0-\nu) + o\big(\|\nu_0-\nu\|\big)\end{aligned}$$

i.e.

$$\Rightarrow \liminf_{\nu\to\nu_0} \lambda_1(\nu_0) - \lambda_1(\nu) - \partial_\nu \mathcal{R}(f_0,\nu_0)(\nu_0-\nu) \geq 0\,.$$

Thus we get

$$\partial_\nu \lambda_1(\nu_0) = \partial_\nu \mathcal{R}(f_0,\nu_0) = -\frac{p}{2}\frac{\lambda_1(\nu_0)|f_0|^2}{\|f_0\|^2_{2,\nu}}$$

□

Because of lemma 4.5.6 we know that $\mathcal{L}_{1,V} \in C^1(\mathcal{M}^+(\nu),\mathbb{R})$ and hence we can study its critical points. It turns out that, if μ^*, maximizer of $\mathcal{L}_{1,V}(\nu)$, is nonzero, then there is only one critical point, as the following theorem asserts

Theorem 4.5.7. *Let ν^* be a maximizer of the function $\mathcal{L}_{1,V}(\nu)$ and $\big(\lambda_1(p,\nu^*), f_{\nu^*}\big)$ be the first eigenpair of the correspondingg weighted $(p,2)$-Laplacian. Then $\big(\lambda_1^{\frac{p}{2(p-1)}}(p,\nu^*), f_{\nu^*}\big)$ is the first eigenpair of the p-Laplacian. Moreover ν^* belongs to the interior of $\mathcal{M}^+(V)$, i.e.,*

$$\nu^* \in \{\nu : V\setminus B \to \mathbb{R} \mid \nu_u > 0\;\forall\, u \in V\setminus B\}\,,$$

and not other internal critical points of the function $\mathcal{L}_{1,V}(\nu)$ exist.

Proof. Thanks to Lemma 4.5.6 we have

$$\frac{\partial \mathcal{L}_{1,V}}{\partial \nu}(\nu) = \lambda^{-\frac{1}{p-1}}(p,\nu)\frac{|f_\nu|^2}{\|f_\nu\|^2_{2,\nu}} - \nu^{\frac{2}{p-2}}$$

Then considering the KKT conditions for the maximum constrained problem we get that, if ν^* is a maximizer, there exist a family of Lagrange multiplier $\{c_u\}_{u\in V}$ such that

$$\begin{cases} \lambda_1^{-\frac{1}{p-1}}(p,\nu^*)\dfrac{|f_{\nu^*}(u)|^2}{\|f_{\nu^*}\|^2_{2,\nu^*}} - \nu_u^{*\,\frac{2}{p-2}} + c_u = 0 & \forall u \in V \\ c_u\nu^*_u = 0 & \forall u\in V \\ c_u \geq 0 & \forall u \in V \\ \Delta_p f_{\nu^*} = \lambda_1(p,\nu^*)\|f_{\nu^*}\|^{p-2}_{2,\nu^*}\nu f_{\nu^*} \end{cases} \tag{4.25}$$

Observe that the first three equations necessarily imply

$$\nu^*_u = \lambda_1^{-\frac{p-2}{2(p-1)}}(p,\nu^*)\frac{|f_{\nu^*}(u)|^{p-2}}{\|f_{\nu^*}\|^{p-2}_{2,\nu^*}}\,. \tag{4.26}$$

Replacing (4.26) in the last of (4.25), we obtain

$$\Delta_p f_{\nu^*} = \lambda_1(p,\nu^*)\|f_{\nu^*}\|^{p-2}_{2,\nu^*}\lambda_1^{-\frac{(p-2)}{2(p-1)}}(p,\nu^*)\frac{|f_{\nu^*}|^{p-2}}{\|f_{\nu^*}\|^{p-2}_{2,\nu^*}}f_{\nu^*} = \lambda_1(p,\nu^*)^{\frac{p}{2(p-1)}}|f_{\nu^*}|^{p-2}f_{\nu^*}\,. \tag{4.27}$$

Observe now that, since f_{ν^*} is the first $(p,2)$-Laplacian eigenfunction, Theorem 4.5.2 ensures that $f_{\nu^*}(u) > 0$ for all $u \in V\setminus B$, and thus, by (4.26), ν^* is an internal point of the domain $\mathcal{M}^+(V)$. Moreover, from Theorem 4.4.2 and (4.27), $\big(\lambda_1^{\frac{1}{2(p-1)}}(p,\nu^*), f_\nu^*\big)$ is necessarily the first p-Laplacian eigenpair. We conclude by observing that, other internal critical points of the function $\mathcal{L}_{1,V}(\nu)$ would correspond to other critical points of the p-Rayleigh quotient. This would correspond to a first eigenvalue of the p-Laplacian with multiplicity greater than one, contraddicting the result of Theorem 4.4.2. □

We would like to remark that, by Theorems 4.5.7 and 4.5.5, the point (μ^*,ν^*) satisfies:

$$(\mu^*_{\nu^*},\nu^*) = \underset{\nu\in \mathrm{Int}(\mathcal{M}^+(\nu))}{\arg\max}\ \underset{\mu\in\mathcal{M}^+(\mu)}{\arg\min}\ \frac{1}{\lambda_1(\mu,\nu)} + \mathrm{M}_p(\mu) - \mathrm{M}_p(\nu)$$

Thus $(\mu^*_{\nu^*},\nu^*)$ is the only, possibly non-differentiable, saddle point of the function $\mathcal{E}_1 \in C(\mathrm{Int}(\mathcal{M}^+(V))\times\mathcal{M}^+(E),\mathbb{R})$

$$\mathcal{E}_1(\mu,\nu) := \frac{1}{\lambda_1(\mu,\nu)} + \mathrm{M}_{E,p}(\mu) - \mathrm{M}_{V,p}(\nu)\,. \tag{4.28}$$

In particular we can state the following

Theorem 4.5.8. *Assume (μ^*,ν^*) to be a saddle point of the energy function $\mathcal{E}_1(\mu,\nu)$ and let $\big(\lambda_1(\mu^*,\nu^*), f_1(\mu^*,\nu^*)\big)$ be the first eigenpair of the Laplacian eigenvalue problem weighted in (μ^*,ν^*), then $\big(\lambda_1^{\frac{p}{2}}(\mu^*,\nu^*), f_1(\mu^*,\nu^*)\big)$ is the first p-Laplacian eigenpair.*

Proof. The proof follows directly from Theorems 4.5.5 and 4.5.7. □

4.5.6 The other eigenpairs

Recalling (4.5.4), it is clear that any eigenpair of the p-Laplacian can be seen as an eigenpair of a constrained weighted linear Laplacian. Hence to any p-Laplacian eigenvalue we can associate the index of the corresponding linear eigenvalue. Then, it is natural to consider energy functions similar to (4.28) but with eigenvalues of higher order in place of $\lambda_1(\mu,\nu)$, i.e.:

$$\mathcal{E}_k^p(\mu,\nu) := \frac{1}{\lambda_k(\mu,\nu)} + \mathrm{M}_{E,p}(\mu) - \mathrm{M}_{V,p}(\nu)\,, \tag{4.29}$$

where $\lambda_k(\mu,\nu)$ is the k-th eigenvalue of the weighted laplacian eigenvalue problem

$$\begin{cases} \Delta_{\mu_0} f_k(u) = \sum_{v\sim u} \mu_{0uv}\omega_{uv}\nabla f_k(v,u) = \lambda_k(\mu,\nu)\nu_{0u} f_k(u) & \forall u\in V\setminus B \\ f_k(u) = 0 & \forall u \in B \end{cases},$$

and we recall the definitions

$$\mathrm{M}_{V,p}(\nu) := \frac{p-2}{p}\sum_{u\in V\setminus B}\nu_u^{\frac{p}{p-2}}, \quad \text{and} \quad \mathrm{M}_{E,p}(\mu) := \frac{p-2}{2p}\sum_{(u,v)\in E}\mu_{uv}^{\frac{p}{p-2}}.$$

In the following of the chapter, since we are not interested in varying p, we omit the superscript p in the definition of $\mathcal{E}_k^p$.

For the eigenpairs beyond the first one there are no results similar to the one provided by Theorem 4.5.8. However, under the assumption of regularity and differentiability we can prove that saddle points of the energy functions in equation (4.29) correspond to eigenpairs beyond the first one. Indeed, observe that the energy functoins in equation (4.29) for $k > 1$ are continuous in $\Omega = \mathrm{Int}(\mathcal{M}^+(E)) \times \mathcal{M}^+(V) \cup \mathcal{M}^+(E) \times \mathrm{Int}(\mathcal{M}^+(V))$ but may loose continuity in both $\mu \in \partial\mathcal{M}^+(E)$ and $\nu \in \partial\mathcal{M}^+(V)$ (where $\partial\mathcal{M}^+(E)$ and $\partial\mathcal{M}^+(V)$ denote the boundary of $\mathcal{M}^+(E)$ and $\mathcal{M}^+(E)$) as in this case the eigenvalues may no longer be continuous [53]. Moreover the functions $\mathcal{E}_k(\mu,\nu)$ are not differentiable whenever $\lambda_k(\mu,\nu)$ is not simple. Avoiding these degenerate situations it is possible to prove that any smooth saddle point of $\mathcal{E}_k(\mu,\nu)$ corresponds to a p-Laplacian eigenpair. We collect this result in the form of a theorem as follows.

Theorem 4.5.9. *Let $(\mu^*,\nu^*) \in \Omega$ be a smooth saddle point of the function $\mathcal{E}_k(\mu,\nu)$. Then $\big(\lambda_k^{\frac{p}{2}}(\mu^*,\nu^*), f_k(\mu^*,\nu^*)\big)$ is a p-Laplacian eigenpair.*

Proof. To simplify the notation in this proof we denote $\big(\lambda_k(\mu^*,\nu^*), f_k(\mu^*,\nu^*)\big)$ by (λ, f). The saddle point equation of the energy function $\mathcal{E}_k(\mu,\nu)$, thanks to Lemma 4.5.4, reads:

$$\begin{cases} -\dfrac{|\nabla f(u,v)|^2}{2\lambda^2(\mu^*,\nu^*)\|f\|^2_{2,\nu^*}} + \dfrac{{\mu^*_{uv}}^{\frac{2}{p-2}}}{2} - c_{uv} = 0 & \forall (u,v) \in E \\ \dfrac{|f(v)|^2}{\|\nabla f\|^2_{2,\mu^*}} - {\nu^*_v}^{\frac{2}{p-2}} + s_v = 0 & \forall v \in V \\ c_{uv}\mu^*_{uv} = 0 & \forall (u,v) \in E \\ c_{uv} \geq 0 & \forall (u,v) \in E \\ s_v \nu^*_u = 0 & \forall v \in V \\ s_v \geq 0 & \forall v \in V \\ \Delta_{\mu^*} f = \lambda \nu^* f \end{cases} \tag{4.30}$$

where $\{c_{uv}\}_{(u,v)\in E}$ and $\{s_v\}_{v\in V}$ are suitable families of Lagrange multipliers. observe that if $\mu^*_{uv} = 0$, since $c_{uv} \geq 0$, the equation

$$-\frac{|\nabla f(u,v)|^2}{2\lambda^2(\mu^*,\nu^*)\|f\|^2_{2,\nu^*}} - c_{uv} = 0$$

admits only the solution $\nabla f(u,v) = 0$, $c_{uv} = 0$. Analogously $\nu^*_v = 0$ implies

$f(v) = s_v = 0$. Hence equation (4.30) yields:

$$\begin{cases} \mu^* = \dfrac{|\nabla f|^{p-2}}{\lambda^{p-2}\|f\|_{2,\nu^*}^{p-2}} \\ \nu^* = \dfrac{|f|^{p-2}}{\|\nabla f\|_{2,\mu^*}^{p-2}} \\ \Delta_{\mu^*} f = \lambda \nu^* f \end{cases} . \tag{4.31}$$

Now we can write:

$$\mu^* = c_\mu |\nabla f_1|^{p-2} \qquad \nu^* = c_\nu |f_1|^{p-2}\,, \tag{4.32}$$

and, from (4.31) we immediately obtain

$$\begin{cases} c_\mu = \lambda_1^{2-p}\|f\|_{2,\nu^*}^{2-p} \\ c_\nu = \|\nabla f\|_{2,\mu^*}^{2-p} \end{cases} . \tag{4.33}$$

Finally, divide the second equation in (4.33) by the first one to yield

$$\frac{c_\nu}{c_\mu} = \lambda_1^{p-2}\Big(\frac{\|f_1\|_{2,\nu^*}^2}{\|\nabla f_1\|_{2,\mu^*}^2}\Big)^{\frac{p-2}{2}} = \lambda_1^{\frac{p-2}{2}}\,. \tag{4.34}$$

Replacing (4.32) in the last equation of (4.31), dividing by c_μ, and using (4.34), we obtain

$$\sum_{v\sim u} \omega_{uv} |\nabla f(v,u)|^{p-2} \nabla f(v,u) = \lambda^{\frac{p}{2}} |f(u)|^{p-2} f(u)\,.$$

That concludes the proof. □

4.5.7 Numerical Aspects

In this final section of the chapter we discuss how the results of the above Theorems 4.5.9 and 4.5.8 can be used to effective numerical algorithms to compute p-Laplacian eigenpairs. The key idea is to define gradient flows for the functions $\mathcal{E}_k(\mu,\nu)$. However, we face the problem of the lack of regularity of the functions $\mathcal{E}_k(\mu,\nu)$ in case of eigenvalues with multiplicity greater than 1. Nevertheless, our preliminary numerical results show that the developed schemes actually deliver acceptable results in most situations. In the following we describe the methods and discuss some benefits and drawbacks.

4.5.8 Gradient flows

We would like to start this section by noticing that computing the saddle points of the functions $\mathcal{E}_k(\mu,\nu)$ is a constrained critical point problem. To avoid adding the positivity constraints our numerical schemes we perform the change of variable

$\mu = \sigma_1^2$ and $\nu = \sigma_2^2$. Using the new variables, the functions $\mathcal{E}_k(\sigma_1^2, \sigma_2^2)$ become well defined everywhere in $\mathbb{R}^{|E|} \times \mathbb{R}^{|V|}$. We thus define a dynamics for the variables (μ, ν) as the gradient flow in the variables σ_1 and σ_2 i.e.

$$\dot{\mu} = 2\sigma_1\dot{\sigma_1} = -2\sigma_1 \frac{\partial \mathcal{E}_K(\sigma_1^2, \sigma_2^2)}{\partial \sigma_1} = -4\sigma_1^2 \frac{\partial \mathcal{E}_K(\mu, \nu)}{\partial \mu} = -4\mu \frac{\partial \mathcal{E}_K(\mu, \nu)}{\partial \mu}$$

and analogously

$$\dot{\nu} = 4\nu \frac{\partial \mathcal{E}_K(\mu, \nu)}{\partial \nu} .$$

Writing explicitly the partial derivatives and neglecting constant multiplicative factors we end up with the following gradient flow system:

$$\begin{cases} \dot{\mu} = \mu\big(\frac{|\nabla f|^2}{\lambda_k(\mu,\nu)\|f\|_\nu^2} - \mu^{\frac{2}{p-2}}\big) \\ \dot{\nu} = \nu\Big(\frac{|f|^2}{\|\nabla f\|_\mu^2} - \nu^{\frac{2}{p-2}}\Big) \\ \Delta_\mu f = \lambda_k(\mu, \nu) f \end{cases} .$$

The system of algebraic-differential equations is discretized by means of a simple explicit Euler method with an empirically-determined constant time step size, t. The third (purely algebraic) equation is solved by diagonalizing the μ-weighted linear Laplacian by means of standard blas-like methods. Given the value of k and the initial values μ_k^0 and ν_k^0, for $n = 0, 1, \ldots$ the final scheme takes on the form:

$$\begin{aligned} \Delta_{\mu_k^n} f &= \lambda_k^n(\mu_k^n, \nu_k^n) f \\ \mu_k^{n+1} &= \mu_k^n + t\mu_k^n \left(\frac{|\nabla f|^2}{(\lambda_k^n)^2 \|f\|_{\nu_k^n}^2} - (\mu_k^n)^{\frac{2}{p-2}} \right) \\ \nu_k^{n+1} &= \nu_k^n + t\nu_k^n \left(\frac{|f|^2}{\|\nabla f\|_{\mu_k^n}^2} - (\nu_k^n)^{\frac{2}{p-2}} \right) . \end{aligned}$$

Convergence towards equilibrium is considered achived when the error

$$\text{err} = \|\Delta_p f_k(t) - \lambda_k^{\frac{p}{2}}(t) |f_k(t)|^{p-2} f_k(t)\|_\infty \tag{4.35}$$

is below a given tolerance.

Figure 4.3 shows the experimental results obtained on a graph of 49 vertices with weights randomly chosen between 0.1 and 1.1. The graph is plotted by distributing the nodes randomly in space with edge lengths equal to the reciprocal of the weights. The results are relative to a value of $p = 6$. The first 6 eigenfunctions (left panels) and relative convergence behaviour are reported. We note that convergence towards equilibrium for $k = 1$ and $k = 2$ is smooth and fast. However, for $k = 3$ strong oscillations when the error reaches 10^{-4} appear and convergence is completely absent. For $k > 3$ the initial oscillations disappear

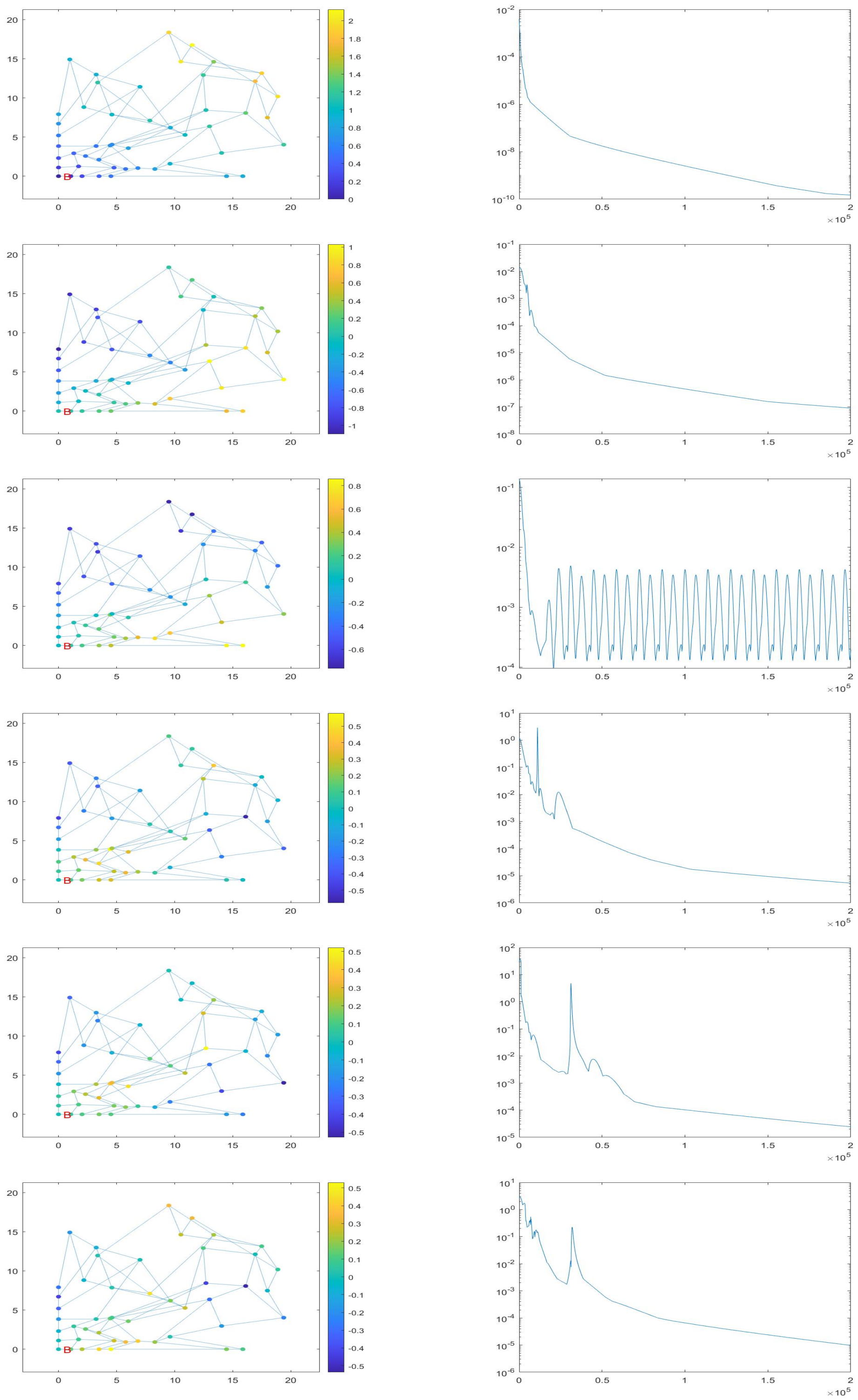


Figure 4.3: Left panel: first six eigenfunctions as calculated by the proposed method for $p = 6$. The graph nodes are randomly distributed with edge lengths equal to the reciprocal of the weights. The nodal values of the eigenfunctions are plotted with the color-code shown on the right of the figure for $k = 1, \ldots, 6$ (top to bottom). For each k the right panel reports the behavior of the error defined in eq.(4.35) as a function of time steps (iterations) n.

quickly and convergence of the discrete gradient flow proceeds smoothly after that.

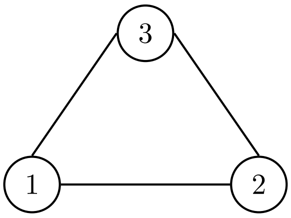
We must recall here that for $k = 1$ Theorem 4.5.8 ensures that the energy function $\mathcal{E}_1$ has only one saddle point and the proposed algorithm is expected to converge. However, for $k > 1$ nothing is known. In particular, if the eigenvalues are not simple, the energy function loses continuity, and the ODE trajectories identified by the gradient flow intersect, potentially leading to an oscillatory behaviour of the discrete method.

For $k = 3$ the initial oscillations clearly noticeable in the convergence profile are due to the jumping back and forth between energy levels relative to different values of k of the numerically calculated trajectories. In this case the gradient flow stagnates. In other cases we observe experimentally an oscillatory behaviour which actually converges towards stationarity. This behaviour can be justified empirically postulating that the time step becomes large enough to jump over the discontinuity point and, by chance the numerical scheme picks an appropriate trajectory and carries the calculations to convergence. However, unlike in the linear ($p = 2$) case, we have no means at the moment to identify the position in the spectrum towards which we converge.

4.5.9 Final Considerations

We have observed in section 4.3 that every p-Laplacian eigenpair can be considered as a linear eigenpair of a properly weighted Laplacian eigenproblem. Using such a characterization in this section we have introduced a class of energy functions whose "smooth" saddle points correspond to p-Laplacian eigenpairs. Nevertheless, except for the first eigenpair of Δ_p (Theorem 4.5.8), not all of the other eigenpairs can be found as saddle points of one of these functions. Indeed, first of all, there exist p-Laplacian eigenvalues that are not simple eigenvalues of the corresponding weighted Laplacian eigenvalue problem, see Fig 4.2. In addition there are also p-Laplacian eigenpairs that correspond to smooth critical points of the relevant energy function without, however, being saddle points, i.e. minima in μ and maxima in ν. Considering the proof of Theorem 4.5.9 it is indeed clear that if (μ^*, ν^*) is a smooth critical point of $\mathcal{E}_k$ such that $\mu^*_{uv} > 0$ $\forall (u, v) \in E$ and $\nu^*_u > 0$ $\forall u \in V$, then $\big(\lambda_k^{\frac{p}{2}}(\mu^*, \nu^*), f_k(\mu^*, \nu^*)\big)$ is a p-Laplacian eigenpair. On the other hand, in the case of a partial degenerate μ^*, ν^*, the KKT conditions for extremal values that are not min max do not necessarily imply that the optimizer corresponds to a p-Laplacian eigenpair (but they do not exclude this possibility either).

We devote the remaining part of this paragraph to show that such situations can actually occur. To simplify the problem, let us fix the ν-variable and reduce the problem to finding the minimum in μ i.e. consider the functions $\mathcal{L}_{k,E}(\mu) = \frac{1}{\lambda_3(\mu,\nu)} + \mathrm{M}_p(\mu)$. Their smooth minimizers correspond to eigenvalues of the $(p, 2)$-Laplacian weighted in ν. In particular, as for $\mathcal{L}_{1,E}$, it can be proved that, given μ^*



- $f^* = (1, 1, -2^{\frac{1}{p-1}})$, $\lambda^* = (1 + 2^{\frac{1}{p-1}})^{p-1}$

Figure 4.4: Simple example of a case where the p-Laplacian eigenpair does not correspond to a saddle point of any of the energy functions $\mathcal{E}_k, k = 1, \ldots, n$. Here the lengths are all unitary, i.e.: $\omega_{uv} = 1\ \forall (u, v) \in E$

a smooth minimizer of $\mathcal{L}_{k,E}$, the pair $\big(\lambda_k^{\frac{1}{p-1}}(\mu^*, \nu), f_K(\mu^*, \nu)\big)$ is a $(p, 2)$-eigenpair, (simply repeat the proof of Theorem 4.5.9 letting ν to be fixed).

Consider now the graph and the eigenpair in Figure 4.4, and let

$$\mu = \frac{\left|\nabla f^*\right|^{p-2}}{\|f^*\|_p^{p-2} \lambda^{*\frac{p-2}{p-1}}}, \quad \text{and} \quad \nu = \frac{|f^*|^{p-2}}{\|f^*\|_p^{p-2}},$$

A simple calculation shows that $(f^*, \lambda^{*\frac{1}{p-1}})$ satisfies the eigenvalue problem

$$\Delta_\mu f = \lambda \nu f\,.$$

Moreover from a numerical computation we obtain that $(\lambda^*)^{\frac{1}{p-1}} = \lambda_3(\mu, \nu)$ is a simple eigenvalue. Thus, we deduce that $(f^*, \lambda^{*\frac{1}{p-1}})$ should correspond to a critical point of the energy function $\mathcal{L}_{3,E}$.

However, we can prove that ir is not a minimizer of $\mathcal{L}_{2,E}$. To this aim, we compute the Hessian matrix of $\mathcal{L}_{3,E}$. The gradient of $\mathcal{L}_{3,E}$ (Lemma 4.5.4) is given by

$$\frac{\partial\Big(\mathcal{L}_{3,E}(\mu)\Big)}{\partial \mu_{uv}} = -\frac{|\nabla f_3(u, v)|^2}{2\lambda_k(\mu, \nu)^2 \|f_3\|_{2,\nu}^2} + \frac{1}{2}\mu_{uv}^{\frac{2}{p-2}}\,.$$

Now before going into the computation of the Hessian observe that we need the derivative of f_k with respect to μ. Start from the derivative of the eigenvalue equation

$$\frac{\partial(\Delta_\mu)}{\partial \mu} f_3 + \Delta_\mu \frac{\partial(f_3)}{\partial \mu} = \frac{\partial(\Delta_\mu f_3)}{\partial \mu} = \nu \frac{\partial(\lambda_3 f_3)}{\partial \mu} = \frac{\partial \lambda_3}{\partial \mu} \nu f_3 + \lambda_3 \nu \frac{\partial f_3}{\partial \mu}\,,$$

then recall that the (μ, ν)-eigenfunctions are a basis of $\mathbb{R}^{|V|}$ and thus we can express $\partial f_3/\partial\mu = \sum_{i=1}^3 \alpha_i f_i$ as a linear combination of the eigenfunctions. Assuming $\|f_i\|_\nu = 1\ \forall i$, multiply both the terms by a generic eigenfucntion f_i to yield:

$$\Big\langle f_i, \frac{\partial \Delta_\mu}{\partial \mu} f_3 \Big\rangle + \Big\langle f_i, \Delta_\mu \frac{\partial f_3}{\partial \mu} \Big\rangle = \frac{\partial \lambda_3}{\partial \mu} \Big\langle f_i, \nu f_3 \Big\rangle + \lambda_3 \Big\langle f_i, \nu \frac{\partial f_3}{\partial \mu} \Big\rangle\,.$$

Recalling that Δ_μ is a symmetric matrix, $\langle f_i, \nu f_3\rangle = \delta_{i,3}$, we obtain

$$\begin{aligned}
&(\lambda_i - \lambda_3)\Big\langle f_i, \nu\frac{\partial f_3}{\partial \mu}\Big\rangle = -\Big\langle \nabla f_i, \frac{\partial\big(diag(\mu)\big)}{\partial \mu}\nabla f_3\Big\rangle \qquad i \neq 3.\\
&\Big\langle f_3, \nu\frac{\partial f_3}{\partial \mu}\Big\rangle = 0
\end{aligned}$$

i.e.

$$\frac{\partial f_3}{\partial \mu_{uv}} = \sum_{i=1}^{2} \frac{\big(\nabla f_i(u,v), \nabla f_3(u,v)\big)}{(\lambda_3 - \lambda_i)} f_i\,.$$

Using the last expression for the derivative of the eigenvectors we can derive the expression of the Hessian matrix of $\mathcal{L}_{3,E}$,

$$\begin{aligned}
\frac{\partial^2 \mathcal{L}_{3,E}}{\partial \mu_{e_1}\partial \mu_{e_2}}(\mu) =& \frac{|\nabla f_3(e_1)|^2|\nabla f_3(e_2)|^2}{\lambda_3^3}\\
&- \frac{1}{\lambda_3^2}\sum_{i\neq 1}^{2} \frac{\big(\nabla f_i(e_1)\nabla f_3(e_1)\big)\big(\nabla f_i(e_2)\nabla f_3(e_2)\big)}{\lambda_3 - \lambda_i} + \frac{1}{p-2}\mu_{e_1}^{\frac{4-p}{p-2}}\delta_{e_1,e_2}
\end{aligned}$$

where e_1 and e_2 denote two edges (u_1, u_2) and (v_1, v_2) and we are assuming $\|f_i\|_\nu = 1\ \forall i = 1, \ldots, 3$.

Now, if we consider the case $p = 4$, a trivial numerical computation of the eigenvalues of $\frac{\partial^2 \mathcal{L}_{3,E}}{\partial \mu_{e_1}\partial \mu_{e_2}}(\mu)$ with $f_3 = f^*/\|f^*\|_\nu$ returns the values

$$\lambda_1 = -0.5874, \qquad \lambda_2 = 1, \qquad \lambda_3 = 3$$

revealing the nature of saddle point and not minimizer of the point μ.

5 The Graph Infinity Laplacian Eigenvalue Problem

5.1 Introduction

In this chapter we discuss the infinity Laplacian eigenvalue problem. The topic has been largely addressed in the continuous setting [9, 39, 57, 58, 68, 89]. However, to the best of our knowledge, the discrete case has never been discussed before. The interest to the study of this problem is twofold, on one hand we remind that solutions of p-Laplacian equations, for large or infinite values of p, play a fundamental role in various applications like L^1 optimal transport problems [40], semisupervised learning [36, 82] and image manipulation and [1, 38]. On the other hand the results proved in the continuous setting show that the infinity eigenpairs encode topological information about the domain, it is thus natural to investigate analogous results in the discrete setting where such results could find applications in data analysis and machine learning. The main difficulties in the study of the infinity Laplacian eigenvalue problem are due to the lack of differentiability of the ∞-norm. This fact makes it necessary to generalize to the nonsmooth case the approaches and methods used to study the p-Laplacian eigenpairs. If we remind that the p-Laplacian eigenpairs are defined as the critical points/values of the Rayleigh quotient $\mathcal{R}_p(f) = \|\nabla f\|_\infty / \|f\|_\infty$, it is clear that also defining the ∞-eigenpairs needs some work and open some problems. Indeed, different generalizations of the notion of p-eigenpair lead to different notions of ∞-eigenpairs and this fact opens the problem of comparing these formulations and discussing the pros and cons of each one of them. For example, in the continuous setting, the study of the infinity eigenpairs, started from Lindqvist, Juutinen et al. [57, 58], is based on the study of the solutions of the limiting p-Laplacian eigenvalue equation when p goes to infinity. This, however, is completely different from the approach used to study the 1-Laplacian eigenvalue problem [20, 50] which is based on the idea of a generalized critical point theory for nonsmooth functionals as $\mathcal{R}_1$. Considering this second approach with the functional $\mathcal{R}_\infty$ we get a completely different notion of infinity eigenpairs defined as the generalized critical points and values of $\mathcal{R}_\infty$. This second approach has been recently addressed in [16, 17] to study the minimizers of $\|f\|_\infty$ in L^∞ and in L^2. In this chapter we face both the approaches in the discrete setting. In particular in section 5.3, using the first approach we define the limiting eigenpairs and we extend to the discrete case the results obtained by Lindqvist, Juutinen et al. Section 5.4, instead, is devoted to the discussion of the second approach, here we define the subgradient infinity eigenpairs and the variational infinity eigenpairs which are a subset of the subgradient infinity eigenvalues. Moreover in section 5.4 we provide a comparison between the two different formulations of ∞-eigenapairs in the graph setting. In particular we prove, first of all, that the variational infinity eigenpairs satisfy the same approximation properties owned by the limiting variational eigenpairs [57, 58] and secondly that the ∞-limit eigenvalue problem, section 5.3, is "stronger" than the generalized critical point theory for $\mathcal{R}_\infty$, section 5.4, in the sense that any limiting eigenpair is also subgradient infinity eigenpair. Finally, we devote section 5.5 to the reformulation of the generalized critical point problem of $\mathcal{R}_\infty$ in terms of a constrained linear weighted Laplacian eigenvalue problem and finally to the characterization of the first infinite eigenpairs as saddle points of a smooth energy function. From this discussion we observe that, on one hand the infinity limit eigenvalue problem seems more interesting than the subgradient infinity eigenvalue problem, indeed it is "stronger" and encodes the same geometrical informations about the graph. On the other hand, the subgradient infinity eigenvalue problem allows a reformulation in terms of a constrained linear eigenvalue problem which is more easily feasible to be faced from a numerical point of view.

In more detail in section 5.3 we consider the limit of p-Laplacian eigenpairs and we prove that any accumulation point (f, Λ) solves the system of equations

$$0 = \begin{cases} \min\{\|(\nabla_\omega f)(u)\|_\infty - \Lambda f(u)\ ,\ \Delta_\infty f(u)\} & \text{if } f(u) > 0 \\ \Delta_\infty f(u) = 0 & \text{if } f(u) = 0\ , \\ \max\{-\|(\nabla_\omega f)(u)\|_\infty - \Lambda f(u)\ ,\ \Delta_\infty f(u)\} & \text{if } f(u) < 0 \end{cases} \tag{5.1}$$

while it is not in general true the inverse. Moreover we prove that if (f_k, Λ_k) is an accumulation point of the sequence of the sets of the $k-th$ variational eigenpairs of the p-Laplacian, we have that

$$\Lambda_k \leq \frac{1}{R_k}\,,$$

where R_k is the maximal radius that allows to inscribe k distinct balls in the graph (following [49], R_k can be named the k-th packing radii of the graph). Moreover, denoting by $\mathcal{N}(f_k)$ the number of nodal domains induced by f_k, we can prove that

$$\frac{1}{R_{\mathcal{N}(f_k)}} \leq \Lambda_k\,,$$

leading to the identity:

$$\Lambda_k = \frac{1}{R_k} \quad if\ k = 1, 2\,.$$

In section 5.4, we consider the subgradient eigenvalue problem

$$0 \in \partial\|\nabla f\|_\infty \cap \Lambda\partial\|f\|_\infty$$

to define the ∞-variational eigenvalues Λ_k and we again prove that

$$\Lambda_k \leq \frac{1}{R_k} \quad and \quad \Lambda_k = \frac{1}{R_k} \; if\; k = 1, 2\,.$$

Then, we compare the two formulations and, using a geometrical characterization of the eigenvalues and eigenfunctions, we prove that the first formulation is stronger than the second. In addition, we can prove that any solution of the second formulation, up to considering a subgraph, solves the limit equation (5.1). Finally in section 5.5 we propose a characterization of the ∞-eigenpairs and their subgradients in terms of constrained linear weighted Laplacian eigenpairs and we propose a method to compute the first eigenpair.

5.2 Notation

5.2.1 Graph setting and p-Laplacian operators

We start by recalling our basic definitions in the discrete setting. Let $\mathcal{G} = (V, E, \omega)$, be our graph. The weight $\omega : E \to \mathbb{R}$ is such that $\omega_{uv} = \omega_{vu}$ and ω_{uv} represents the inverse of the edge length. This implies that, given two nodes u and v, we can define their distance as the length of the shortest path that connects them, i.e.

$$d(u, v) = \min\Big\{\sum_{i=1}^{n} \omega(v_{i-1}, v_i)^{-1} \;: n \in \mathbb{N},\; u = v_0 \sim \cdots \sim v_n = v\Big\} \tag{5.2}$$

where $v \sim u$ means that $(u, v) \in E$ and thus a sequence $\{u = v_0 \sim v_1 \sim \cdots \sim v_n = v\}$ represents a path connecting u and v. In particular given a path $\Gamma = \{v_0 \sim v_1 \sim \cdots \sim v_n\}$ we define its length as:

$$\mathrm{length}(\Gamma) = \sum_{i=0}^{n-1} \frac{1}{\omega_{v_i v_{i+1}}}\,.$$

Now, we remind the definition of the differential operators. Given a function on the nodes, $f : V \to \mathbb{R}$, define its gradient on an edge (u, v) by

$$\nabla_\omega f(u, v) = \omega_{uv}\big(f(v) - f(u)\big)\,.$$

The local gradient of f at a node u is the set of the gradients defined on the edges outgoing from u

$$\nabla_\omega f(u) = \{\nabla_\omega f(u, v) \mid v \sim u\}\,.$$

We define the p-Laplacian operator as

$$\Delta_p f(u) := -\mathrm{div}(|\nabla f|^{p-2}\nabla f) = \sum_{v\in V}\omega_{uv}^p |f(u)-f(v)|^{p-2}\big(f(u)-f(v)\big)\,,$$

where given an edge function $G: E \to \mathbb{R}$, its divergence is defined on the nodes as follows:

$$\mathrm{div}G(u) = -\frac{1}{2}\big(\nabla^T\big)G(u) = \frac{1}{2}\sum_{v\sim u}\omega_{uv}G(u,v) - G(v,u)$$

and satisfies the following "integration by parts" formula

$$\langle G, \nabla f\rangle_E = \frac{1}{2}\sum_{(u,v)\in E} G(u,v)\nabla f(u,v) = \langle -\mathrm{div}G, f\rangle_V = \sum_{u\in V} -\mathrm{div}\, G(u)f(u)\,.$$

Throughout the chapter, if not otherwise specified, we use capital letters (latin or greek) to denote edge functions and lowercase letters to denote node functions. Observe that from the definiton of the scalar products we can define the 2-norms (and more generally the p norms) on the edge and node spaces as:

$$\|G\|_{p,E}^p = \frac{1}{2}\sum_{(u,v)\in E}|G(u,v)|^p \qquad \|f\|_{p,V}^p = \sum_{u\in V}|f(u)|^p\,.$$

Now, assign a subset of the nodes, $B \subset V$, that we call boundary, and consider the set of the functions that are zero on it $\mathcal{H}_0(V) := \{f \mid f(u) = 0\ \forall u \in B\}$. Then, from the study of the critical points of $\mathcal{R}_p(f) = \|\nabla f\|_p/\|f\|_p$ in the domain $S_p \cap \mathcal{H}_0(V)$ we derive the diiscrete equivalent of the p-Laplacian eigenvalue equation with homogeneous Dirichlet boundary conditions, i.e.

$$\begin{cases}\Delta_p f(u) = \lambda |f|^{p-2}(u)f(u) & u \in V\setminus B\\ f(u) = 0 & u \in B\end{cases} \tag{5.3}$$

If $B \neq \emptyset$ we also introduce the distance from the boundary defined as follows

$$d_B(u) = \min_{v\in B} d(u,v)\,. \tag{5.4}$$

Now we remind the definition of the p-Laplacian variational eigenvalues:

$$\lambda_k(\Delta_p) = \min_{A\in\mathcal{F}_k(S_p)\cap\mathcal{H}_0(V)}\ \max_{f\in A}\mathcal{R}_p(f)\,,$$

where $\mathcal{F}_k(S_p\cap\mathcal{H}_0(V))$ is the set of the closed and symmetric subsets of $\mathcal{H}_0(V)\cap S_p := \{f \| \|f\|_{p,E} = 1\ \&\ f(v) = 0\ \forall v \in B\}$ with Krasnoselskii genus greater or equal than k. We remind the definition of Krasnoselskii grenus of a closed symmetric set A:

$$\gamma(A) = \begin{cases}\inf\{h\in\mathbb{N} : \exists\varphi \in C(A, \mathbb{R}^h\setminus\{0\})\ \ s.t.\ \ \varphi(x) = -\varphi(-x)\}\\ \infty & \text{if } \nexists\, h \text{ as above}\\ 0 & \text{if } A = \emptyset\end{cases}\,. \tag{5.5}$$

Finally given the infinity norm of a function f defined on the nodes,

$$\|f\|_\infty = \max_{u \in V}\{|f(u)|\}\,,$$

and of its gradient defined on the edges,

$$\|\nabla_\omega f\|_\infty = \max_{(u,v)\in V}\{|\nabla_\omega f(u,v)|\},$$

we define the maximal sets

Definition 5.2.1.

$$V_{max}(f) = \{u \in V \mid |f(u)| = \|f\|_\infty\} \qquad E_{max}(f) = \{(u,v) \in E \mid |\nabla_\omega f(uv)| = \|\nabla_\omega f\|_\infty\}$$

5.2.2 The ∞-Laplacian

In this paragraph, we introduce the ∞-Laplacian operator. We start from the well-studied continuous case and use this as a guide to define analogous operators in the discrete case, we refer to [37, 38, 70] for a detailed exposition.

Continuous setting: In the continuos setting the ∞-Laplacian operator is defined as the second order partial differential operator:

$$\Delta_\infty f := \sum_{i,j}^{n} \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\frac{\partial^2 f}{\partial x_i \partial x_j} = (\nabla f)^T H(f)\nabla f\,,$$

where $H(f)$ denotes the Hessian matrix. It is worth mentioning, see [70], that given a bounded domain, $\Omega \in \mathbb{R}^n$, and a Lipschitz continuous function, $g \in W^{1,\infty}(\bar{\Omega})$, a viscosity solution, f, of the problem

$$\begin{cases} \Delta_\infty f = 0 & in\ \Omega \\ f = g & on\ \partial\Omega \end{cases}$$

can be built taking the limit, for $p \to +\infty$ of a sequence of functions $\{f_p\}_p$ such that, for any p,

$$\begin{cases} \Delta_p f_p = 0 & in\ \Omega \\ f_p = g & on\ \partial\Omega\,. \end{cases}$$

Discrete setting: Moving to the discrete setting, the discrete ∞-Laplacian operator is defined in such a way to recover analogous results to the one that hold in the continuous setting. First note that, contrary to the most classical definition of continuous p-Laplacian operator (i.e. $\mathrm{div}(|\nabla f|^{p-2}\nabla f$), our discrete

p-Laplacian, given 5.2.1, is positive definite. Moreover, as observed in [37], the discrete p-Laplacian can be rewritten in the following way

$$\begin{aligned}(\Delta_p f)(u) :=& \sum_{v\in V} \omega_{uv}^p |f(u)-f(v)|^{p-2}\big(f(u)-f(v)\big)\\ =&\Big(\sum_{v\in V}\omega_{uv}^p\big|\big(f(u)-f(v)\big)^+\big|^{p-1} - \sum_{v\in V}\omega_{uv}^p\big|\big(f(u)-f(v)\big)^-\big|^{p-1}\Big)\\ =&\Big(\|(\nabla_{\omega'} f)^-(u)\|_{p-1}^{p-1} - \|(\nabla_{\omega'} f)^+(u)\|_{p-1}^{p-1}\Big),\end{aligned} \tag{5.6}$$

where $x^+ := \max\{x, 0\}$, $x^- := \max\{-x, 0\}$ and $\omega'_{uv} = \omega_{uv}^{\frac{p}{p-1}}$. Because of (5.6), in [1, 37, 38] the discrete ∞-Laplacian is defined as

$$\Delta_\infty(f)(u) = \Big(\|(\nabla_\omega f)^-(u)\|_\infty - \|(\nabla_\omega f)^+(u)\|_\infty\Big).$$

By consistency with the continuous case if f is the limit of solutions of $\Delta_p f_p = 0$ with prescribed values on certain nodes, it must satisfy $\Delta_\infty f = 0$.

5.3 The Infinity limit eigenvalue problem

Now we discuss the ∞-eigenvalue problem, again starting from the continuous setting as introduced in [58], [57]. The idea is to build the ∞-eigenfunctions as the limit of the eigenfunctions of the p-Laplacian operator. In particular, the limit of appropriate subsequences of the k-th variational eigenfunctions of the p-Laplacian leads to an ∞-eigenfunction whose corresponding eigenvalue can be expressed formally as:

$$\Lambda_k = \text{“}\lim_{p\to\infty}\big(\lambda_k(\Delta_p)\big)^{\frac{1}{p}}\text{”}. \tag{5.7}$$

Observe that both in the continuous and in the discrete cases we know that the set $\{\lambda_k^{\frac{1}{p}}(\Delta_p)\}_p$ is bounded [31], thus we can extrapolate at least one convergent subsequence. Nevertheless, for general k, there are no results about the convergence of the whole sequence. Hence, Λ_k in (5.7) is not uniquely defined and can be understood as any accumulation point of the sequence $\{\lambda_k^{\frac{1}{p}}(\Delta_p)\}_p$.

5.3.1 The infinity limit eigenvalue equation

Continuous setting: In the continuous setting, the ∞-eigenvalue equation for a general eigenvalue Λ, thought as the limit of p-Laplacian eigenvalue equations, takes the form [57]

$$0 = \begin{cases}\min\{|\nabla f| - \Lambda f, -\Delta_\infty f\} & \text{if } f > 0\\ -\Delta_\infty f & \text{if } f = 0\\ \max\{-|\nabla f| - \Lambda f, -\Delta_\infty f\} & \text{if } f < 0\end{cases}, \tag{5.8}$$

where the above equation has to be understood in the sense of viscosity solutions (see [58] for a short and clear discussion about viscosity solutions in this setting). It is possible to show that if the pair (f, Λ) solves eq.(5.8), then

$$\Lambda = \frac{\|\nabla f\|_\infty}{\|f\|_\infty}\,.$$

Moreover, in [9], the authors prove that the above equation can be reformulatad more compactly as

$$\min\{|\nabla f| - \Lambda f, -\Delta_\infty f\} + \max\{-|\nabla f| - \Lambda f, -\Delta_\infty f\} + \Delta_\infty(f) = 0\,.$$

Discrete setting In the discrete setting, we can formulate the discrete analogue of the ∞-eigenvalue equation as follows. First, we write the eigenvalue equation of the p-Laplacian given in (5.3) in full form

$$\sum_{v\sim u} \omega_{uv}^p |f(u) - f(v)|^{p-2}\big(f(u) - f(v)\big) = \lambda |f(u)|^{p-2} f(u) \qquad \forall u \in V \setminus B\,. \tag{5.9}$$

Assume $\{(\lambda_p, f_p)\}_p$ to be a sequence of p-Laplacian eigenpair and remember that we want to compute an eigenvalue of the ∞-Laplacian as the limit of $\lambda_p^{\frac{1}{p}}$, i.e:

$$\Lambda = \lim_{p\to\infty} \left(\lambda_p\right)^{\frac{1}{p}}\,.$$

Hence, by means of the equality in (5.6), we rewrite (5.9) as below (as in (5.6), $\omega'_{uv} = \omega_{uv}^{\frac{p}{p-1}}$).

$$\begin{cases} \begin{cases} \left(\|(\nabla_{\omega'} f)^-(u)\|_{p-1}^{p-1} - \|(\nabla_{\omega'} f)^+(u)\|_{p-1}^{p-1}\right)^{\frac{1}{p-1}} = \lambda^{\frac{1}{p-1}} f(u) \\ \|(\nabla f)^-(u)\|_{p-1} - \|(\nabla f)^+(u)\|_{p-1} > 0 \end{cases} & \text{if } f(u) > 0 \\ \begin{cases} \left(\|(\nabla_{\omega'} f)^+(u)\|_{p-1}^{p-1} - \|(\nabla_{\omega'} f)^-(u)\|_{p-1}^{p-1}\right)^{\frac{1}{p-1}} = -\lambda^{\frac{1}{p-1}} f(u) \\ \|(\nabla f)^-(u)\|_{p-1} - \|(\nabla f)^+(u)\|_{p-1}^{p-1} < 0 \end{cases} & \text{if } f(u) < 0 \\ \|(\nabla_{\omega'} f)^-(u)\|_{p-1} - \|(\nabla_{\omega'} f)^+(u)\|_{p-1} = 0 & \text{if } f(u) = 0\,. \end{cases} \tag{5.10}$$

Studying the limit for $p \to \infty$ we can explicitate the ∞-eigenvalue equation as in the following theorem

Theorem 5.3.1. *Let (f_{p_j}, λ_{p_j}) be a sequence of p-Laplacian eigenparis such that $\lim_{j\to\infty} \lambda_{p_j}^{\frac{1}{p_j}} = \Lambda$ and $\lim_{j\to\infty} f_{p_j} = f$. Then (f, Λ) satisfies the following set of* *equations*

$$0 = \begin{cases} \min\{\|\nabla_\omega f(u)\|_\infty - \Lambda_k f(u)\ ,\ \Delta_\infty f(u)\} & \text{if } u \in V \setminus B \text{ and } f(u) > 0 \\ \Delta_\infty f(u) = 0 & \text{if } u \in V \setminus B \text{ and } f(u) = 0 \\ \max\{-\|\nabla_\omega f(u)\|_\infty - \Lambda_k f(u)\ ,\ \Delta_\infty f(u)\} & \text{if } u \in V \setminus B \text{ and } f(u) > 0 \\ f(u) & \text{if } u \in B \end{cases} \tag{5.11}$$

Proof. To prove the thesis, consider the case $f(u) > 0$. Then for any j large enough, by (5.10), we can assume

$$\begin{cases} \|(\nabla_{\omega'} f_{p_j})^-(u)\|_{p_j-1}\left(1 - \left(\frac{\|(\nabla_{\omega'} f_{p_j})^+(u)\|_{p_j-1}}{\|(\nabla_{\omega'} f_{p_j})^-(u)\|_{p_j-1}}\right)^{p_j-1}\right)^{\frac{1}{p_j-1}} = \lambda_{p_j}^{\frac{1}{p_j-1}} f_{p_j}(u) \\ \|(\nabla f_{p_j})^-(u)\|_{p_j-1} > \|(\nabla f_{p_j})^+(u)\|_{p_j-1}\,. \end{cases}$$

Taking the limit we have

$$\begin{cases} \|\nabla_\omega f(u)\|_\infty \lim_{j\to\infty}\left(1 - \left(\frac{\|(\nabla_{\omega'} f_{p_j})^+(u)\|_{p_j-1}}{\|(\nabla_{\omega'} f_{p_j})^-(u)\|_{p_j-1}}\right)^{p_j-1}\right)^{\frac{1}{p_j-1}} = \Lambda f(u) \\ \|(\nabla f)^-(u)\|_\infty \geq \|(\nabla f)^+(u)\|_\infty \end{cases}.$$

Now observe that

$$\lim_{j\to\infty}\left(1 - \left(\frac{\|(\nabla_{\omega'} f_{p_j})^+(u)\|_{p_j-1}}{\|(\nabla_{\omega'} f_{p_j})^-(u)\|_{p_j-1}}\right)^{p_j-1}\right)^{\frac{1}{p_j-1}} \leq 1\,,$$

and, if $\|(\nabla f)^-(u)\|_\infty \gneqq \|(\nabla f)^+(u)\|_\infty$,

$$\lim_{j\to\infty}\left(1 - \left(\frac{\|(\nabla_{\omega'} f_{p_j})^+(u)\|_{p_j-1}}{\|(\nabla_{\omega'} f_{p_j})^-(u)\|_{p_j-1}}\right)^{p_j-1}\right)^{\frac{1}{p_j-1}} = 1\,.$$

This implies

$$\|\nabla_\omega f(u)\|_\infty = \Lambda f(u)\,.$$

Viceversa, if $\lim_{j\to\infty}\left(1 - \left(\frac{\|(\nabla_{\omega'} f_{p_j})^+(u)\|_{p_j-1}}{\|(\nabla_{\omega'} f_{p_j})^-(u)\|_{p_j-1}}\right)^{p_j-1}\right)^{\frac{1}{p_j-1}} < 1$, necessarily we have

$$\|(\nabla f)^-(u)\|_\infty = \|(\nabla f)^+(u)\|_\infty\,,$$

showing that (5.11) is satisfied. □

Observe that equation (5.11) is similar to eq.(5.8), where the absolute value is replaced by the infinity norm of the local gradient.
Now we can enunciate the following proposition which tells us that if (f, Λ) satisfies (5.11), Λ is exactly the value of $\mathcal{R}_\infty(f)$.

Proposition 5.3.2. *Assume that* (Λ, f) *satifies equation* (5.11) *and* f *is not a constant function. Considered* $u \in V_{max}(f)$*, it holds* $\|\nabla_\omega f\|_\infty = \|\nabla_\omega f(u)\|_\infty = \Lambda|f(u)| = \Lambda\|f\|_\infty$*, i.e.*

$$\Lambda = \frac{\|\nabla_\omega f\|_\infty}{\|f\|_\infty}$$

Proof. If u is such that $|f(u)| = \|f\|_\infty$, then, assuming w.l.o.g $f(u) > 0$,

$$\|\nabla_\omega f(u)\|_\infty \geq \Lambda f(u) > 0$$

Moreover since u is a maximizer, $\|(\nabla f)^+_\omega(u)\|_\infty = 0$ and we get

$$\Delta_\infty f(u) > 0$$

Thus, by eq (5.11), necessarily we have

$$\|\nabla_\omega f(u)\|_\infty - \Lambda f(u) = 0\,.$$

The last equality allows us to observe that, if the thesis holds for some $w \in V_{max}(f)$, it holds as well for each $u \in V_{max}(f)$, since

$$\frac{\|\nabla_\omega f(u)\|_\infty}{|f(u)|} = \frac{\|\nabla_\omega f(u)\|_\infty}{\|f\|_\infty} = \Lambda = \frac{\|\nabla_\omega f\|_\infty}{\|f\|_\infty}$$

implies $\|\nabla_\omega f(u)\|_\infty = \|\nabla_\omega f\|_\infty$. To prove the existence of such a node w, let u_0 be such that there exists an edge (v, u_0) with $|\nabla_\omega f(vu_0)| = \|\nabla f\|_\infty$. If $\Delta_\infty f(u) \neq 0$ then

$$\Lambda = \frac{\|\nabla_\omega f(u_0)\|_\infty}{|f(u_0)|} = \frac{\|\nabla_\omega f\|_\infty}{|f(u_0)|} \geq \frac{\|\nabla_\omega f(u)\|_\infty}{|f(u)|} = \frac{\|\nabla_\omega f(u)\|_\infty}{\|f\|_\infty} = \Lambda\,.$$

Thus necessairly $|f(u_0)| = \|f\|_\infty$. If instead $\Delta_\infty f(u_0) = \Delta_\infty f(v) = 0$, assuming without loss of generality $f(u_0) > f(v)$, there exist $u_1 \sim u_0$ such that $f(u_1) > f(u_0)$ and

$$\omega(u_0u_1)\big(f(u_1) - f(u_0)\big) = \|(\nabla_\omega f)^-(u_0)\|_\infty = \|(\nabla_\omega f)^+(u_0)\|_\infty = \omega(u_0v)\big(f(u_0) - f(v)\big) = \|\nabla_\omega f\|_\infty$$

Thus there exists another edge (u_0, u_1) such that $|\nabla_\omega f(u_0u_1)| = \|\nabla f\|_\infty$ and $f(u_1) > f(u_0) > f(v)$. Iterating this procedure, by the finiteness of the graph and the previous argument, there must exist a node u_k such that

$$\|\nabla f(u_k)\|_\infty = \|\nabla f\|_\infty = \Lambda|f(u_k)| = \Lambda\|f\|_\infty\,.$$

□

5.3.2 Geometrical Properties of the ∞-eigenpairs

In what follows, we use ∇f to denote $\nabla_\omega f$, unless otherwise specified. As we see in the next proposition the distance function introduced in (5.2) is a significant tool in the study of the infinite eigenpairs interpreted as solutions of (5.11)

Proposition 5.3.3. *Let* (Λ, f) *satisfy* (5.11), *f not a constant function. Then for any $u \in V_{max}(f)$ there exist a path of edges* $\Gamma = \{(u_i, u_{i+1})\}_{i=1}^{n-1}$ *such that*

1. $u_1 = u$.
2. $\nabla f(u_i, u_{i+1}) = \|\nabla f\|_\infty$ *and f is monotone along* Γ.
3. $\begin{cases} u_n \in B \\ \Lambda = \dfrac{1}{\text{length}(\Gamma)} \end{cases}$ *or* $\begin{cases} f(u_n) = -f(u) \\ \Lambda = \dfrac{2}{\text{length}(\Gamma)} \end{cases}$
4. $\Lambda = \min\left\{ \dfrac{1}{d_B(u)}, \min\limits_{\{v \,|\, f(v) = -f(u)\}} \dfrac{2}{d(u,v)} \right\}$

Proof. The first two points and the fact that u_n has to belong to one of the two subsets in point 3 follow from Proposition 5.3.2 and equation (5.11). Indeed given $u_0 = u$, by Proposition 5.3.2, there exists $u_1 \sim u_0$ such that $|\nabla f(u_0, u_1)| = \|\nabla f\|_\infty$. Then from eq (5.11) we conclude that

$$f(u_1) = -f(u_0) \qquad or \qquad u_1 \in B \qquad or \qquad |f(u_1)| < |f(u_0)|\,.$$

In the last case, since $\|\nabla f(u_1)\|_\infty = \|\nabla f\|_\infty$, necessarily $\Delta_\infty f(u_1) = 0$, i.e. there exists $u_2 \sim u_1$ such that $|\nabla f(u_0, u_1)| = \|\nabla f\|_\infty$ and, assuming w.l.o.g. $f(u_0) > f(u_1)$, it follows $f(u_1) > f(u_2)$. Iterating this procedure, by the finiteness of the graph, there exists a path Γ such that $u_n \in B$ or $f(u_n) = -f(u_0)$. Moreover, from the equalities

$$\frac{f(u_i) - f(u_{i+1})}{\|\nabla f\|_\infty} = \frac{1}{\omega_{u_i, u_{i+1}}} \ \forall i, \qquad \Lambda = \frac{\|\nabla f\|_\infty}{|f(u)|}, \qquad \text{length}(\Gamma) = \sum_i \frac{1}{\omega_{u_i, u_{i+1}}}$$

we easily get to the expressions $\Lambda = 1/\text{length}(\Gamma)$ if $u_n \in B$ and $\Lambda = 2/\text{length}(\Gamma)$ if $f(u_n) = -f(u)$. To conclude observe that, if $u_n \in B$, necessarily

$$\text{length}(\Gamma) = d_B(u) \qquad and \qquad \text{length}(\Gamma) \le 2d(u,v) \ \ \forall v \ s.t.\ f(v) = -f(u)$$

Indeed, assume by contradiction that there exists a path $\Gamma' = \{(v_i, v_{i+1})\}_{i=0}^{m-1}$ with $v_0 = u$, $v_m \in B$, and $\sum_{i=0}^{m-1} \omega_{v_i v_{i+1}}^{-1} < \sum_{i=0}^{n-1} \omega_{u_i u_{i+1}}^{-1}$. Then, by construction

$$|f(u)| = \|\nabla f\|_\infty \sum_{i=0}^{n-1} \omega_{u_i u_{i+1}}^{-1}\,.$$

Then, a contradiction arises, since $\omega_{v_i v_{i+1}}|f(v_i) - f(v_{i+1})| = |\nabla f(v_i, v_{i+1})| < \|\nabla f\|_\infty$, by the triangular inequality

$$|f(u)| \leq \|\nabla f\|_\infty \sum_{i=0}^{m-1} \omega_{v_i v_{i+1}}^{-1} < \|\nabla f\|_\infty \sum_{i=0}^{n-1} \omega_{u_i u_{i+1}}^{-1} = |f(u)| \,.$$

Observe that the inequality $\text{length}(\Gamma) \leq 2d(u,v) \;\; \forall v$ such that $f(v) = -f(u)$, can be proved with a similar argument, assuming the existence of a path Γ', from u to v, with $f(v) = -f(u)$, such that $\text{length}(\Gamma') < 2\,\text{length}(\Gamma)$. Finally the case $f(u_n) = -f(u)$ can be treated analogously, concluding the proof. $\square$

The study of the accumulation points of the sequences of the variational eigenvalues of the p-Laplacian with homogeneous boundary conditions has received particular attention in the continuous setting. In the remaining part of this section we will recall the main results and extend them to graphs.

Continuous setting: Given a bounded domain $\Omega \in \mathbb{R}^n$, for any integer k introduce the maximal radius which allows us to inscribe k distinct balls in Ω, i.e.

$$R_k := \sup\{r \;\text{ s.t. }\; \exists B_r(x_1), \ldots, B_k(x_k) \subset \Omega, \;\; B_r(x_i) \cap B_r(x_j) = \varnothing \;\; \forall i,j = 1,\ldots,k\} \,.$$

Then, given Λ_k, an accumulation point of the sequence of the k-th variational eigenvalues of the p-Laplacian, it is possible to relate such value to R_k (see [57, 58]) getting the following inequality:

$$\Lambda_k \leq \frac{1}{R_k} \,.$$

The cases $k = 1, 2$ deserve a particular attention, indeed in these cases it holds also the opposite inequality, leading to the convergence of the sequences $\{\lambda_{1,2}(\Delta_p)^{\frac{1}{p}}\}_{p=2}^{\infty}$ and to the equalities

$$\Lambda_1 = \frac{1}{\|d_B\|_\infty} \,, \qquad \Lambda_2 = \frac{1}{R_2} \,.$$

In the end particular attention is also dedicated to the study of the ∞-eigenfunctions associated to Λ_1 (see [55, 59, 70, 89]). Indeed, despite the node function d_B is always a minimizer of $\mathcal{R}_\infty$, it is not always also a first eigenfunction, meaning that (d_B, Λ), depending on Ω, may not solve equation (5.8). Moreover the first eigenfunction may not be unique and there may be first eigenfunctions that solve the ∞-limit eigenvalue equation without beeing limits of eigenfunctions of the p-Laplacian.

Graph setting: In the following we investigate analogue problems on graphs. First of all, observe that the minimum of the ∞-Rayleigh quotient

$$\mathcal{R}_\infty(f) = \frac{\|\nabla f\|_\infty}{\|f\|_\infty}$$

can be easily computed, as in the continuous case, considering the distance from the boundary, d_B (5.4), see [17]. To prove it, observe the following trivial inequality: Assume the distance between the nodes u and v is realized along the path $\Gamma = \{(v_i, v_{i+1})\}_{i=1}^{n-1}$, with $v_1 = u$ and $v_n = v$, i.e., $d(u,v) = \sum_{i=1}^m 1/\omega_{v_i v_{i+1}}$. Then, we get the inequality:

$$\begin{aligned} |f(u) - f(v)| &\le |f(v_1) - f(v_2)| + |f(v_2) - f(v_3)| + \cdots + |f(v_m) - f(v_{m+1})| \\ &\le \|\nabla f\|_\infty \sum_{i=1}^m \frac{1}{\omega_{v_i v_{i+1}}} = \|\nabla f\|_\infty d(u,v) \end{aligned} \tag{5.12}$$

Proposition 5.3.4. *The boundary distance function d_B defined in* (5.4) *realizes the minimunm of the ∞-Rayleigh quotient:*

$$d_B \in \arg\min_{f \in \mathcal{H}_0(V)} \frac{\|\nabla f\|_\infty}{\|f\|_\infty} ,$$

Proof. It is easy to prove that

$$\|\nabla d_B\|_\infty \le 1$$

indeed, given an edge (u,v) and considered the gradient of the node function d_B on it, we have

$$(\nabla d_B)(u,v) = \omega_{uv}\big(d_b(v) - d_b(u)\big) \le \omega_{uv}(\omega_{uv})^{-1} = 1$$

Thus, to conlude it is sufficient to show that the boundary distance function satisfies

$$d_B \in \arg\max_{f \in \mathcal{F} : \|\nabla f\|_\infty \le 1} \|f\|_\infty$$

To this end, for any function f with $f \in \mathcal{F}$ and $\|\nabla f\|_\infty \le 1$ and for any node $u \in V$, let $v \in B$ be the boundary node such that $d(u,v) = d_B(u)$. Then, using (5.12) we can write:

$$|f(u)| = |f(u) - f(v)| \le d(u,v) = d_B(u)$$

The last equation clearly implies

$$d_B \in \arg\max_{f \in \mathcal{H}_0(V) : \|\nabla f\|_\infty \le 1} \|f\|_\infty .$$

$\square$

Corollary 5.3.5.
$$\min_{f \in \mathcal{H}_0(V)} \frac{\|\nabla f\|_\infty}{\|f\|_\infty} = \frac{1}{R_1}\,,$$
where R_1 is defined as
$$R_1 = \sup_{v \in V}\{d_B(v)\}\,.$$

Proof. Because of Proposition 5.3.4, it is enough to compute $\|\nabla d_B\|_\infty$ and $\|d_B\|_\infty$. Trivially $\|d_B\|_\infty = R_1$. From the proof of Proposition 5.3.4, we already know that $\big|\nabla d_B(u,v)\big| \leq 1\ \ \forall (u,v) \in E$. Moreover, as it has been proved in [17],
$$|\big(\nabla d_B\big)(u,v)| = 1 \quad \Longleftrightarrow \quad v \in \Gamma(u,B)\ \text{ or } u \in \Gamma(v,B)\,,$$
where $\Gamma(u,B)$ is the shortest path from the node u to the boundary. Thus necessarily $\|\nabla d_B\|_\infty = 1$ and we can conclude. □

Now, given (f,Λ), a solution to the ∞-limit eigenvalue equation, consider the nodal domains induced by f, i.e. the maximal connected subgraphs where f is strictly positive or negative. In the next theorem we show that Λ can be regarded as a radius that allows us to inscribe as many balls in the graph as the number of nodal domains induced by f.

Theorem 5.3.6. *Let (f,Λ) be an eignepair that satisfies equation* (5.11)*, and assume that f induces k nodal domains. Then there exist $v_1,\ldots,v_k \in V$ and $r_k > 0$ such that*
$$d(v_i,v_j) \geq 2r_k\ \ \forall i \neq j \qquad d_B(v_i) \geq r_k\ \ \forall i = 1,\ldots,k\,.$$
Moreover
$$\Lambda = \frac{1}{r_k}$$

Proof. Consider first the case of f strictly positive. Then given $u_1 \in V_{max}(f)$, from Proposition 5.3.3 there exist a path $\Gamma = \{(u_i,u_{i+1})\}_{i=1}^{n-1}$ such that $u_1 = v_1$, $v_{n-1} \in B$ and length$(\Gamma) = d_B(v_1) = \Lambda^{-1}$. Thus the node v_1 and the radius $r_1 = \Lambda^{-1}$ satisfy the thesis. Consider now the case of a generic function f and assume $\{A_i\}$, $\{B_j\}$ to be the nodal domains induced by f such that
$$f(u) > 0 \quad \forall u \in \cup_i A_i\,, \qquad f(u) < 0 \quad \forall u \in \cup_j B_j\,.$$
Starting from the original graph $\mathcal{G}$ and the function f, we can define a new disconnected graph $\mathcal{G}' = \cup_h \mathcal{G}_h$ with as many connected components, $\mathcal{G}_h$, as the nodal domains of f and such that, for any h, $(f|_{\mathcal{G}_h},\Lambda)$ satisfy the ∞-limit eigenvalue equation (5.11) on $\mathcal{G}_h$ and $f|_{\mathcal{G}_h}$ is strictly positive or strictly negative. Indeed, we can consider all the nodes u such that $f(u) = 0$ as boundary nodes, $u \in B(\mathcal{G}')$. Furthermore, for any edge (u,v) such that $f(u)f(v) < 0$, we add a boundary node $w \in B(\mathcal{G}')$ and replace the edge (u,v) by the two edges (u,w) and (w,v)

with weights $\omega_{uw} = \omega_{uv}\big(1 - f(v)/f(u)\big)$ and $\omega_{wv} = \omega_{uv}\big(1 - f(u)/f(v)\big)$. Observe that

$$\nabla f(u,v) = \omega_{uv}\big(f(v) - f(u)\big) = -\omega_{uv}\Big(1 - \frac{f(v)}{f(u)}\Big)f(u) = \omega_{uw}\big(0 - f(u)\big) = \nabla f(u,w)\,,$$
$$\nabla f(v,u) = \omega_{uv}\big(f(u) - f(v)\big) = -\omega_{uv}\Big(1 - \frac{f(u)}{f(v)}\Big)f(v) = \omega_{vw}\big(0 - f(v)\big) = \nabla f(v,w)\,, \tag{5.13}$$

and

$$\frac{1}{\omega_{uw}} + \frac{1}{\omega_{vw}} = \frac{1}{\omega_{uv}}\,. \tag{5.14}$$

Moreover, the internal nodes of the new graph $\mathcal{G}'$ are the nodes of $\mathcal{G}$ where f was non zero, $\mathcal{G}' = \cup_{h=1}^{n}\mathcal{G}_h$ where $\big(\mathcal{G}_{h_1} \cap \mathcal{G}_{h_2}\big) \setminus B(\mathcal{G}') = \emptyset \ \ \forall h_1 \neq h_2 = 1,\dots,n$ and every $\mathcal{G}_h$ matches one of the nodal domains induced by f. To conclude, observe that for any internal node u, from (5.13), the local gradient, $(\nabla f)(u)$ is unchanged, implying that $(f|_{\mathcal{G}_h}, \Lambda)$ satisfies the ∞-limit equation (5.11) with respect to the graph $\mathcal{G}_h$. Thus, since $f|_{\mathcal{G}_h}$ is strictly positive or negative, from the first part of the proof we get that, for any h, there exists $v_h \in \mathcal{G}_h$ such that

$$\Lambda = \frac{1}{d_{B(\mathcal{G}')}(v_h)}\,.$$

Conclude observing that, because of the above construction, it holds:

$$\frac{2}{\Lambda} = d_{B(\mathcal{G}')}(v_{h_1}) + d_{B(\mathcal{G}')}(v_{h_2}) \leq d(v_{h_1}, v_{h_2})\,.$$

To prove it, let Γ be the shortest path that joins v_{h_1} and v_{h_2} in $\mathcal{G}$, i.e. $d(v_{h_1}, v_{h_2}) = \text{length}(\Gamma)$. Then, recall that $\mathcal{G}_{h_1} \cap \mathcal{G}_{h_2} \setminus B(\mathcal{G}') = \emptyset$ and that, by construction (5.14), we have not changed distances among internal nodes. Thus, in $\mathcal{G}'$, the path Γ has necessarily been replaced by some path Γ' that crosses $B(\mathcal{G}')$ and such that $\text{length}(\Gamma') = \text{length}(\Gamma)$, which concludes the proof. □

Next, we study the relationships between the limits of the p-Laplacian variational eigenvalues, $\{\Lambda_k\}_k$, and the packing radii of $\mathcal{G}$ [49], $\{R_k\}_k$, i.e. the maximal radiuses that allow one to inscribe a prescribed number of disjoint balls in the graph. We define the k-th packing radius of the graph $\mathcal{G}$ as

Definition 5.3.7.

$$R_k := \max\{r \ \text{ s.t. } \ \exists v_1,\dots,v_k \text{ s.t. } d(v_i,v_j) \geq 2r,\ d(v_i,B) \geq r \ \forall i,j = 1,\dots,k\}\,.$$

First of all, from the previous Theorem 5.3.6, we trivially get the following corollary

Corollary 5.3.8. *Let* (f,Λ) *be an eigenpair that satisfies equation* (5.11) *and such that* f *induces* k *nodal domains, then*

$$\Lambda \geq \frac{1}{R_k}$$

Proof. The proof is a trivial consequence of Theorem 5.3.6. Indeed from this it follows that $\Lambda = 1/r_k$ with $r_k \leq R_k$. □

Next recalling that Λ_k can be used to denote any accumulation point of the sequence $\lambda_k(\Delta_p)^{\frac{1}{p}}$, we can prove the following upper bound for Λ_k.

Proposition 5.3.9. *Let $\lambda_k(\Delta_p)$ be the k-th variational eigenvalue of the p-Laplacian on a graph $\mathcal{G}$ and let R_k be the k-th packing radius of $\mathcal{G}$. Then,*

$$\limsup_{p\to\infty} \lambda_k(\Delta_p)^{\frac{1}{p}} \leq \frac{1}{R_k}$$

Proof. Let $u_1, \ldots, u_k$ be k nodes as in the definition 5.3.7 of R_k , i.e.

$$d(u_i, u_j) \geq 2R_k\,, \qquad d_B(u_i) \geq R_k \quad \forall i, j = 1, \ldots, k\,.$$

Then, define the k linearly independent functions

$$f_j(u) = \max\{R_k - d(u, u_j), 0\}\,,$$

and the set $A_k = span\{f_j\}_{j=1}^k$, as $\dim(A_K) = k$, also its Krasnoselskii genus will be equal to k,

$$\gamma(A_k) = k\,.$$

By definition,

$$\lambda_k(\Delta_p) \leq \max_{f\in A_k} \mathcal{R}_{\Delta_p}(f) = \max_{f\in A_k} \frac{\sum_{(u,v)\in E} \omega_{uv}^p |f(u) - f(v)|^p}{2\sum_{u\in V} |f(u)|^p}\,. \tag{5.15}$$

Consider a function f in A_k, $f = \sum_{i=1}^k \alpha_i f_i$. Then we have

$$\frac{\sum_{(u,v)\in E} \omega_{uv}^p |f(u) - f(v)|^p}{2\sum_{u\in V} |f(u)|^p} = \frac{\sum_{(u,v)\in E} \omega_{uv}^p \big|\sum_i \alpha_i f_i(u) - \sum_i \alpha_i f_i(v)\big|^p}{2\sum_{i=1}^k \big(|\alpha_i|^p \sum_{d(u,u_i)<R_k} |f_i(u)|^p\big)}\,. \tag{5.16}$$

Let us analyze first the numerator. If u and v are such that both $d(u, u_i), d(v, u_i) < R_k$, then

$$\begin{aligned}\omega_{uv}|f(u) - f(v)| &= \omega_{uv}|\alpha_i||f_i(u) - f_i(v)\big| = \omega_{uv}|\alpha_i|\big|d(u, u_i) - d(v, u_i)\big| \\ &\leq \omega_{uv}|\alpha_i|d(u, v) \leq \omega_{uv}|\alpha_i|\frac{1}{\omega_{uv}} \leq |\alpha_i|\,.\end{aligned}$$

If instead $d(u, u_i) < R_k$ and $d(v, u_j) < R_k$ with $i \neq j$, then

$$\begin{aligned}\omega_{uv}|f(u) - f(v)| &= \omega_{uv}|\alpha_i\, f_i(u) - \alpha_j\, f_j(v)| \\ &\leq \omega_{uv}\ \max\{|\alpha_i|,\, |\alpha_j|\}\, \big(R_k - d(u, u_i) + R_k - d(v, u_j)\big) \\ &\leq \omega_{uv}\ \max\{|\alpha_l|\}_{l=1}^k \big(2R_k - d(u_i, u_j) + d(u, v)\big) \\ &\leq \omega_{uv}\ \max\{|\alpha_l|\}_{l=1}^k \big(2R_k - 2R_k + d(u, v)\big) \\ &\leq \omega_{uv}\ \max\{|\alpha_l|\}_{l=1}^k d(u, v) \\ &\leq \max\{|\alpha_l|\}_{l=1}^k\,.\end{aligned}$$

Last, if $d(u,u_i) < R_k$ and $d(v,u_j) \geq R_k \; \forall j$, we have

$$\begin{aligned}\omega_{uv}|f(u)-f(v)| =& \omega_{uv}|\alpha_i|\big(R_k - d(u,u_i)\big)\\ &= \omega_{uv}|\alpha_i|\big(R_k - d(u,u_i) + d(u,v) - d(u,v)\big)\\ &\leq \omega_{uv}|\alpha_i|\big(d(u,v) + R_k - d(u_i,v)\big)\\ &\leq \omega_{uv}|\alpha_i| d(u,v)\\ &\leq |\alpha_i|\,.\end{aligned}$$

Inserting the above inequalities in (5.16), we can write

$$\frac{\sum_{(u,v)\in E}\omega_{uv}^p\big|\sum \alpha_i f_i(u) - \sum \alpha_i f_i(v)\big|^p}{2\sum_{i=1}^k\big(|\alpha_i|^p\sum_{d(u,u_i)<R_k}|f_i(u)|^p\big)} \leq \frac{\sum_{(u,v)\in E}\max_i|\alpha_i|^p}{2\sum_{i=1}^k\big(|\alpha_i|^p\sum_{d(u,u_i)<R_k}|f_i(u)|^p\big)}\,.$$

Now, using (5.15), we obtain

$$\Lambda_k \leq \limsup_{p\to\infty}\lambda_k^{\frac{1}{p}} \leq \limsup_{p\to\infty}\left(\frac{\sum_{(u,v)\in E}\max_i|\alpha_i|^p}{2\sum_{i=1}^k\big(|\alpha_i|^p\sum_{d(u,u_i)<R_k}|f_i(u)|^p\big)}\right)^{\frac{1}{p}} = \frac{\max_i|\alpha_i|}{\max_i|\alpha_i f_i(u)|}\,.$$

Finally, observe that since $f_i(u_i) = R_k$, $\max_i|\alpha_i f_i(u)| = R_k\max_i|\alpha_i|$, which implies

$$\Lambda_k \leq \frac{1}{R_k}\,.$$

□

From Proposition 5.3.9 and Corollary 5.3.5, it is straightforward to deduce that the sequence $\{\lambda_1(\Delta_p)^{\frac{1}{p}}\}_p$ converges to Λ_1, with

$$\Lambda_1 = \frac{1}{R_1}$$

Besides the characterization of the first eigenvalue, in the next theorem, we also prove the convergence of the sequence of the second variational eiganvalues of the p-Laplacian, providing, in addition, a geometrical characterization of Λ_2.

Theorem 5.3.10. *Assume $\mathcal{G}$ to be a connected graph and let $\Lambda_2 := \lim_{p\to\infty}\lambda_2(\Delta_p)^{\frac{1}{p}}$. Then*

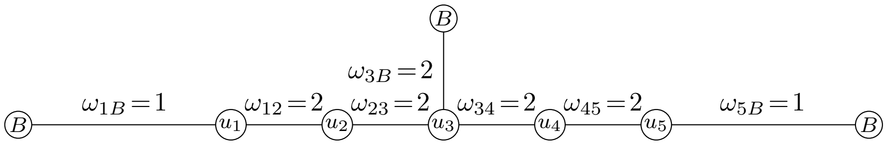
$$\Lambda_2 = \frac{1}{R_2}\,.$$

Proof. From Proposition 5.3.9, we know that

$$\Lambda_2 \leq \frac{1}{R_2}\,.$$

Now consider a sequence of convergent eigenpairs

$$\big(f_2(\Delta_{p_h}), \lambda_2(\Delta_{p_h})\big) \to (f, \Lambda_2)\,.$$



From [31, 86], we know that, for any p_h, $f_2(\Delta_{p_h})$ has at least two nodal domains, A_{p_h} and B_{p_h}, such that $f_2(\Delta_{p_h})(u) > 0 \ \forall u \in A_{p_h}$ and $f_2(\Delta_{p_h})(u) < 0 \ \forall u \in B_{p_h}$. The sets

$$A = \cap_n \cup_{p_h > n} A_{p_h} \quad \text{and} \quad B = \cap_n \cup_{p_h > n} B_{p_h}$$

are both non empty and such that $f(u) \geq 0$ for any $u \in A$ and $f(u) \leq 0$ for any $u \in B$. If by contradiction $\{u \mid f(u) > 0\} = \emptyset$, since $f \neq 0$, there has to exist a node u with $f(u) = 0$ that is connected to a node $v \sim u$ such that $f(v) < 0$ that means

$$\Delta_\infty f(u) = \|(\nabla f)^-(u)\|_\infty - \|(\nabla f)^+(u)\|_\infty = \|(\nabla f)^-(u)\|_\infty > 0\,.$$

But this is an absurd because f has to satisfy eq (5.11), implying that f must induce at least two nodal domains. Then, thanks to Corollary 5.3.8, we get

$$\Lambda_2 = \frac{1}{r_2} \geq \frac{1}{R_2}$$

which concludes the proof. □

We conclude this section by producing some examples that show that the solution of (5.11) is not always well defined and that not any solution of (5.11) is also the limit of p-Laplacian eigenfunctions.

Example 5.3.11. *Consider the following graph:*

Some easy computations show that both

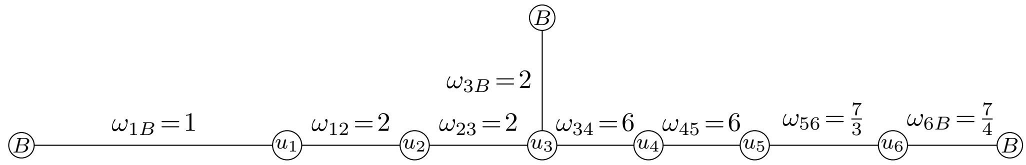
$$f = \left(1, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}, 1\right)$$

and

$$g = \left(1, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}, \frac{4}{9}\right)$$

satisfy equation (5.11) *with* $\Lambda = 1$. *Nevertheless,* g *cannot be the limit of first eigenfunctions of the* p*-Laplacian. These indeed, because of their uniqueness, have to be symmetric for any* p *on the above graph.*

Example 5.3.12. *Consider the following graph:*



Similarly to the previous example, it is not difficult to see that

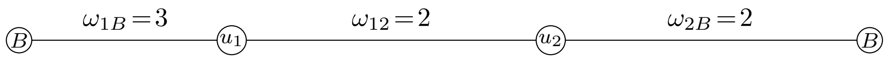
$$f = \left(1, \frac{2}{3}, \frac{1}{3}, \frac{2}{5}, \frac{7}{15}, \frac{4}{15}\right)$$

and

$$g = \left(1, \frac{2}{3}, \frac{1}{3}, \frac{2}{5}, \frac{12}{35}, \frac{48}{245}\right)$$

satisfy equation (5.11) *with* $\Lambda = 1$. *Thus the solution of* (5.11) *is neither uniquely defined on* $V \setminus V_{max}(f)$

Example 5.3.13. *Consider now the distance function on th following graph:*



$$d_B(u_1, u_2) = \left(\frac{1}{3}, \frac{1}{2}\right).$$

Then

$$\Delta_\infty d_B(u_1) = 1 - 2(\frac{1}{2} - \frac{1}{3}) = 1 - \frac{1}{3} \gneq 0$$

and

$$\|\nabla d_B(u_1)\|_\infty - \Lambda_1 d_B(u_1) = 1 - \frac{2}{3} \gneq 0\,.$$

Thus d_B *does not satisfy the* ∞*-limit eigenvalue equation* (5.11), *meaning that it can not be the limit of the first eigenfunctions of the* p*-Laplacian.*

Remark 5.3.14. *It is worthwhile to spend a short remark about the non-boundary case, that is analogous to the homogeneous Neumann case in the continuum setting [39]. In this case, we can think of a graph with the boundary set,* B*, formed by nodes* v *at an infinite distance from any internal node so that the edge weights* $\omega_{uv} = 0\ \forall\, u \in V \setminus B\,,\ v \in B$. *Then it is clear that in the non-boundary case* $\Lambda_1 = 0$ *and*

$$\Lambda_2 = \frac{1}{R_2} := 1/\sup\{r \text{ s.t. } \exists v_1, v_2 \text{ s.t. } d(v_1, v_2) \geq 2r\}\,,$$

i.e. the half of the diameter of the graph. Moreover, for the higher eigenvalues, we can similarly state from Proposition 5.3.9 that

$$\Lambda_k^{-1} \geq \sup\{r \text{ s.t. } \exists v_1, \ldots, v_k \text{ s.t. } d(v_i, v_j) \geq 2r\ \forall i, j = 1, \ldots, k\}\,.$$

5.4 The subdifferential infinity eigenvalue equation

In this section we develop a different approach to the ∞-Laplacian eigenvalue problem. Instead of thinking about the infinity eigenpairs as solutions of the limit p-Laplacian eigenvalue equation, it is possibile to define the infinity eigenpairs as the "critical points" of the Rayleigh quotient:

$$\mathcal{R}_\infty(f) = \frac{\|\nabla f\|_\infty}{\|f\|_\infty}\,.$$

Recall that the p-Laplacian eigenpairs can be seen as critical points/values of the functional $\|\nabla f\|_p$ on the manifold $\|f\|_p = 1$, i.e. the points in which the differential of $\|\nabla f\|_p$ is normal to S_p. Obviously, $\|f\|_\infty$ is not a differentiable operator and S_∞ is not a smooth manifold. Nevertheless, following [22], since $\|\nabla f\|_\infty$ and $\|f\|_\infty$ are convex functions, we can generalize the notion of critical point

Definition 5.4.1. *We say that (Λ, f) is a generalized ∞-eigenpair if and only if*

$$0 \in \partial\|\nabla f\|_\infty \cap \Lambda\partial\|f\|_\infty\,.$$

where $\partial\|\nabla f\|_\infty$ and $\partial\|f\|_\infty$ are the subgradients of the functions $(f \mapsto \|\nabla f\|_\infty)$ and $(f \mapsto \|f\|_\infty)$, respectively.

From [22], we observe that 5.4.1 can be considered as the generalized critical point equation of the functional $(f \mapsto \|\nabla f\|_\infty)$ on S_∞ since $\partial\|\nabla f\|_\infty$ is a generalization of the differential of $\|\nabla f\|_p$ when $p = \infty$, while, from Lemma 4.2 and 4.3 of [22], (see also Lemma 2.2.7) the external cone to S_∞ in a point f, i.e.,

$$C_{Ext}(f) = \{\xi|\, \langle \xi, g - f\rangle \;\forall g \in S_\infty\}$$

can be characterized as

$$C_{Ext}(f) = \bigcup_{\lambda \geq 0} \lambda\partial\|f\|_\infty\,.$$

Moreover we point out that, from Theorem 5.8 of [22] (see also Lemma 2.2.9), it is possible to introduce the family of Krasnoselskii ∞-variational eigenvalues.

Definition 5.4.2.

$$\Lambda_k^{\mathcal{K}} = \min_{A \in \mathcal{F}_k(S_\infty)} \max_{f \in A} \mathcal{R}_\infty(f)$$

Where, recall $\mathcal{F}_k(S_\infty)$ is the family of the closed symmetric subsets of S_∞ with Krasnoselskii genus greater or equal than k (see (5.5)*).*

Similar kinds of approaches have been used to study the 1-Laplacian eigenpairs [20, 50], and recently to study minimizers of $\|\nabla f\|_\infty$ in L^2 and L^∞ spaces, [16, 17].

In the next theorem we show that, as the limiting variational eigenvalues $\{\Lambda_k\}$ defined in Section 5.3, also the variational eigenvalues Λ_k^G, can be related to the radii of inscribed balls, R_k, as defined in Definition 5.3.7.

Theorem 5.4.3. *Let Λ_k^G be defined as in Definition 5.4.2 and R_k as in Definition 5.3.7. Then,*

$$\Lambda_k^G \leq \frac{1}{R_k} \qquad \forall k = 1, \dots, |V|$$

and the equality holds for $k = 1, 2$.

Proof. For the first part we can follow the proof of Prop 5.3.9. Hence, consider $u_1, \dots, u_k$ such that $d(u_i, u_j) \geq 2r,\ d(u_i, B) \geq r\ \forall i, j = 1, \dots, k$ and define the k linearly independent functions

$$f_j(u) = \max\{R_k - d(u, u_j), 0\}\,, \quad j = 1, \dots, k\,.$$

The set $A_k = span\{f_j\}_{j=1}^k$ has Krasnoleskii genus equal to k, $\gamma(A_k) = k$, and thus:

$$\Lambda_k^G \leq \max_{f \in A_k} \mathcal{R}_\infty(f)\,.$$

Repeating all the computations as in the proof of Proposition 5.3.9, it is easy to prove that

$$\max_{f \in A_k} \mathcal{R}_\infty(f) \leq \frac{1}{R_k}\,.$$

Thus, because of Corollary 5.3.5, we are left to prove that $\Lambda_2^G \geq R_2^{-1}$. To do this, we first observe that any $A \in \mathcal{F}_2(S_\infty)$ necessarily contains a symmetric closed and connected subset of S_∞. Consider the function

$$\begin{aligned} \psi : S_\infty &\longrightarrow \ \mathbb{R} \\ f &\longrightarrow \|f^+\|_\infty - \|f^-\|_\infty\,. \end{aligned}$$

Then, we can state that for all $A \in \mathcal{F}_2(S_\infty)$, there esists $f_A \in A$ such that $\psi(f_A) = 0$ or in other words, there exist $u^+, u^- \in V$ such that

$$\|f\|_\infty = f_A(u^+) = -f_A(u^-)\,.$$

To conclude, from (5.12), we observe that

$$d(u^\pm, B) \geq \frac{\|f_A\|_\infty}{\|\nabla f_A\|_\infty}\,, \qquad d(u^+, u^-) \geq 2\frac{\|f_A\|_\infty}{\|\nabla f_A\|_\infty}\,,$$

i.e. :

$$\frac{\|f_A\|_\infty}{\|\nabla f_A\|_\infty} \leq R_2 \qquad \Rightarrow \qquad \Lambda_2^G \geq \min_{A \in \mathcal{F}_2} \frac{\|\nabla f_A\|_\infty}{\|f_A\|_\infty} \geq \frac{1}{R_2}\,.$$

□

In the following example we show that the result presented in the last Theorem 5.4.3 is a sharp result. In particular we show that there exist graphs where the equality $\Lambda_k^G = 1/R_k$ is not achieved for $k > 2$.

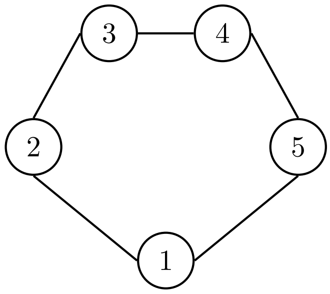


Figure 5.1: Cycle graph with 5 nodes

Example 5.4.4. *Consider the cycle graph with five nodes $\mathcal{C}_5$, Fig 5.1, with all edges of length 1, $\omega_{uv} = 1\ \forall (u,v) \in E$. Then we easily observe that $R_2 = 1$ and $R_3 = 1/2$, which means that*

$$1 = \Lambda_2 \leq \Lambda_3 \leq 2$$

We aim to prove that the inequality $\Lambda_3 \leq 1/R_3 = 2$ is strict. To this end, we show the existence of a submanifold $\mathcal{S}$, of $S_\infty = \{f \in \mathbb{R}^5 | \|f\|_\infty = 1\}$, such that $\mathcal{S}$ is homeomorphic to a 2-dimensional sphere and $\mathcal{R}_\infty(f) \leq 1$ for any $f \in \mathcal{S}$. Observe that the existence of such $\mathcal{S}$ implies

$$\Lambda_3 = \min_{\gamma(A)\geq 3} \max_{f\in A} \mathcal{R}_\infty(f) \leq \max_{f\in\mathcal{S}} \mathcal{R}_\infty(f) \leq 1\,.$$

To build such $\mathcal{S}$ consider the decomposition as CW-simplex of S_∞ induced by the ordering of the pairs of disjoint subsets of V [90]:

$$\mathcal{P}_2(V) = \{(A,B) : A \cap B = \emptyset, A \cup B \neq \emptyset, A, B \subset V\},$$

and $(A,B) < (A',B')$ if $A \subset A'$ and $B \subset B'$. The maximal simplices of such decomposition are given by $\{\Delta_{A,\sigma} : A \subset V, \sigma$ permutation on $V\}$, where given $A \subset V$ and $\sigma : \{1,\dots,n\} \to \{1,\dots,n\}$ permutation

$$\Delta_{A,\sigma} = conv\Big(1_A - 1_{V\setminus A} \cup \Big\{1_{A\setminus\{\sigma(1),\dots,\sigma(i)\}} - 1_{(V\setminus A)\setminus\{\sigma(1),\dots,\sigma(i)\}} \Big|\ i = 1,\dots,n-1\Big\}\Big)\,.$$

Then, any maximal subcomplex correspond to the functions, f, with a prescribed order of the values of f on the vertices of the graph, i.e.,

$$\Delta_{A,\sigma} = \left\{ f \;\middle|\; \begin{array}{l} f(v) \geq 0\ \forall v \in A, f(v) \leq 0\ \forall v \in V\setminus A, \\ \big|f\big(\sigma(v_i)\big)\big| \leq \big|f\big(\sigma(v_{i+1})\big)\big|\ \ \forall i = 1,\dots n-1 \end{array} \right\}$$

Thus, for any $\Delta_{A,\sigma}$ there exist $v_1, v_2 \in V$ and a sign ± 1 such that for any $f \in \Delta_{A,\sigma}$ it holds $\|\nabla f\|_\infty = \pm\big(f(v_1) - f(v_2)\big)$. This means that $\mathcal{R}_\infty$ is linear on the obtained subcomplexes of S_∞. Then it is trivial to observe that a linear function on a simplicial complex must assume its maximal value in one of the vertices of the complex.

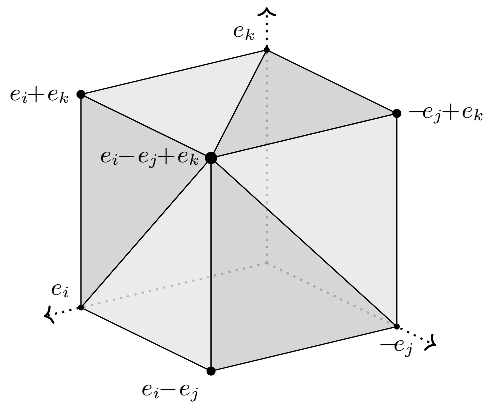


Figure 5.2: Visualization of the subcomplex S_f, where $f = e_i - e_j + e_k$

Now, given a node of the complex, $f \in S_\infty$, let S_f be the subcomplex of S_∞ induced by f and the nodes $g < f$. Here "$<$" has to be understood with respect to the ordering given to the pairs of disjoint subsets, taking into account the equivalence between functions and pairs of disjoint subsets:

$$h \sim (A_h, B_h) \quad A_h = \{v \in V \,|\, h(v) > 0\},\ B_h = \{v \in V \,|\, h(v) < 0\}$$

Let $\{e_i\}_{i=1}^5$ be the canonical basis of $\mathbb{R}^5$ and consider some $f = +e_i - e_j + e_k$ with $i, j, k \in \{1, \ldots, 5\}$ one different from each other, then we give a visualization of S_f in figure 5.2.

Next, we consider the following 8 vertices:

$$\begin{array}{lll} f_1 = (1,0,-1,-1,0); & f_2 = (1,0,-1,0,1); & f_3 = (1,1,0,-1,0); \\ f_4 = (0,1,1,1,0); & f_5 = (0,-1,-1,0,1); & f_6 = (0,-1,0,1,1); \\ f_7 = (0,1,0,1,1); & f_8 = (1,1,0,0,1); & \end{array}$$

and their symmetric versions $f_{-i} := -f_i$ for any $i = 1, \ldots, 8$. Considered the subcomplexes induced by such vertices, it is not difficult to observe that

$$\max_{f \in S_{f_i}} \mathcal{R}_\infty(f) \le 1 \qquad \forall i = -8, \ldots, -1, 1, \ldots, 8\,,$$

indeed it is enough to prove that such inequality holds in the node f_i and in any node $g < f_i$. Moreover, we say that S_{f_i} and S_{f_j} are adjacent, $S_{f_j} \leftrightarrow S_{f_i}$, if they share at least one 1-dimensional simplex. Observe that if S_{f_i} and S_{f_j} are adjacent, there exist two nodes of the CW-simplex, g_1 and g_2, such that both $g_1, g_2 < f_i, f_j$. In particular, since $f_i \neq f_j$, g_1 and g_2 have the shape $g_1 = \pm e_h \pm e_k$ and $g_2 = \pm e_h$ for some h, k and sign, $+$ or $-$, associated to h and k, look at figure 5.2 for an easy visualization. In particular, observe that, necessarily, also the node $g_3 = \pm e_k$ belongs to both S_{f_1} and S_{f_2}. Thus, if S_{f_1} and S_{f_2} are adjacent, they share exactly *two 1-dimensional simplices, whose union corresponds to the subcomplex S_{g_1} for some $g_1 < f_1, f_2$, with $g_1 = \pm e_h \pm e_k$, $h, k = 1, \ldots, 5$, $h \neq k$. We write*

$$S_{f_i} \underset{g}{\longleftrightarrow} S_{f_j},$$

to say that S_{f_1} and S_{f_2} are adjacent along the subcomplex S_g. In particular we observe the following all and only adjacency relations between the subcomplexes $\{S_{f_i}\}_{i=-8,\ldots,-1}^{1,\ldots,8}$.

$$
\begin{array}{lll}
S_{f_1} \underset{(1,0,-1,0,0)}{\longleftrightarrow} S_{f_2}\,, & S_{f_1} \underset{(1,0,0,-1,0)}{\longleftrightarrow} S_{f_3}\,, & S_{f_1} \underset{(0,0,-1,-1,0)}{\longleftrightarrow} S_{f_{-4}} \\
S_{f_2} \underset{(1,0,-1,0,0)}{\longleftrightarrow} S_{f_1}\,, & S_{f_2} \underset{(0,0,-1,0,1)}{\longleftrightarrow} S_{f_5}\,, & S_{f_2} \underset{(1,0,0,0,1)}{\longleftrightarrow} S_{f_8} \\
S_{f_3} \underset{(1,0,0,-1,0)}{\longleftrightarrow} S_{f_1}\,, & S_{f_3} \underset{(0,1,0,-1,0)}{\longleftrightarrow} S_{f_{-6}}\,, & S_{f_3} \underset{(1,1,0,0,0)}{\longleftrightarrow} S_{f_8} \\
S_{f_4} \underset{(0,0,1,1,0)}{\longleftrightarrow} S_{f_{-1}}\,, & S_{f_4} \underset{(0,1,1,0,0)}{\longleftrightarrow} S_{f_{-5}}\,, & S_{f_4} \underset{(0,1,0,1,0)}{\longleftrightarrow} S_{f_7} \\
S_{f_5} \underset{(0,0,-1,0,1)}{\longleftrightarrow} S_{f_2}\,, & S_{f_5} \underset{(0,-1,-1,0,0)}{\longleftrightarrow} S_{f_{-4}}\,, & S_{f_5} \underset{(0,-1,0,0,1)}{\longleftrightarrow} S_{f_6} \\
S_{f_6} \underset{(0,-1,0,1,0)}{\longleftrightarrow} S_{f_{-3}}\,, & S_{f_6} \underset{(0,-1,0,0,1)}{\longleftrightarrow} S_{f_5}\,, & S_{f_6} \underset{(0,0,0,1,1)}{\longleftrightarrow} S_{f_7} \\
S_{f_7} \underset{(0,1,0,1,0)}{\longleftrightarrow} S_{f_4}\,, & S_{f_7} \underset{(0,0,0,1,1)}{\longleftrightarrow} S_{f_6}\,, & S_{f_7} \underset{(0,1,0,0,1)}{\longleftrightarrow} S_{f_8} \\
S_{f_8} \underset{(1,0,0,0,1)}{\longleftrightarrow} S_{f_2}\,, & S_{f_8} \underset{(1,1,0,0,0)}{\longleftrightarrow} S_{f_3}\,, & S_{f_8} \underset{(0,1,0,0,1)}{\longleftrightarrow} S_{f_7}
\end{array}
$$

Now, for any $i = -8, \ldots, -1, 1, \ldots, 8$, we note that S_{f_i} is a surface with boundary and that its boundary is composed by 6 1-simplices. Moreover from the above adjacency relations we observe that any S_{f_i} is adjacent along any pair of boundary simplices to one and only one other S_{f_j}. Thus, if we glue all these sublcomplexes along the shared simplices we obtain a closed surface without boundary, $\mathcal{S}$:

$$\mathcal{S} = \left(\bowtie_{i=-8}^{-1} S_{f_i} \right) \bowtie \left(\bowtie_{i=1}^{8} S_{f_i} \right).$$

Observe that, because of the above adjacencies, the following paths are contained in $\mathcal{S}$,

$$f_1 \to f_2 \to f_8 \to f_7 \to f_4 \to f_{-1} \qquad \text{and} \qquad f_1 \to f_3 \to f_{-6} \to f_{-5} \to f_{-2} \to f_{-1}$$

which means that $\mathcal{S}$ is a connected surface. Let us now compute the Poincarè-Euler characteristic of $\mathcal{S}$ to show that it is homeomorphic to a 2-dimensional sphere. The number of faces of $\mathcal{S}$ is given by 6×16, indeed every S_{f_i} is composed by 6 triangles, see Fig 5.2. The number of edges of $\mathcal{S}$ is given by $6 \times 16 + 6 \times (16/2)$, indeed every S_{f_i} has 6 "internal" edges and 6 shared edges (each shared edge belongs exactly to 2 different S_{f_i}). Finally, concerning the number of nodes, note that every S_{f_i} has one internal node, 3 nodes shared with only one other S_{f_j} (the nodes with two non zero entries) and 3 nodes with only one non zero entry.

The total number of nodes with only 1 *nonzero entry, since we are in* $\mathbb{R}^5$*, is equal to* 10 *and it is not difficult to verify that any such node belongs to some* S_{f_i} *with* $i = -8, \dots, -1, 1, \dots, 8$*. Thus the number of nodes of* $\mathcal{S}$ *is given by* $16 + 3 \times (16/2) + 10$ *and we obtain*

$$\chi(\mathcal{S}) = 6 \times 16 - 6 \times 16 - 6 \times 8 + 16 + 3 \times 8 + 10 = 2\,,$$

which is the Euler-characteristic of the sphere. It follows that $\mathcal{S}$ *has genus* 3*,* $\gamma(\mathcal{S}) = 3$*, and thus we can conclude with the desired inequality:*

$$\Lambda_3 = \min_{\gamma(A)\geq 3} \min_{f\in A} \mathcal{R}_\infty(f) \leq \min_{f\in\mathcal{S}} \mathcal{R}_\infty(f) \leq 1\,,$$

where the last inequality is a consequence of the fact that $\max_{f\in S_{f_i}} \mathcal{R}_\infty(f) \leq 1\ \forall\, i = -8, \dots, -1, 1, \dots, 8$.

Next, we study the structure of the two sets $\partial\|\nabla f\|_\infty$, $\partial\|f\|_\infty$. As done in [17], because of its homogeneity, it is easy to derive the following characterization, (see also [18] or section 2.2.1):

$$\partial\big(f \mapsto \|f\|_\infty\big) = \{\xi \mid \|g\|_\infty \geq \langle \xi, g\rangle\, \forall g : V \mapsto \mathbb{R},\ \|f\|_\infty = \langle \xi, f\rangle\}$$

i.e.

$$\partial\|f\|_\infty := \left\{ \xi \;\middle|\; \begin{array}{l} \|\xi\|_{1,V} = 1,\ \xi(u) = 0\ \ \forall u \in V \setminus V_{max}(f)\,, \\ |\,\xi(u)|\,|f(u)| = \xi(u) f(u)\ \ \forall u \in V_{max}(f) \end{array} \right\}. \tag{5.17}$$

Moreover, from [80] we can use the subdifferential chain rule for linear transformations $\Big(\partial\big(x \mapsto \phi(Ax)\big) = A^T \partial\big(y \mapsto \phi(y)\big)|_{y=Ax}\Big)$ to characterize $\partial(f \mapsto \|\nabla f\|_\infty)$ i.e.

$$\partial\|\nabla f\|_\infty := \left\{ -\operatorname{div}\Xi \;\middle|\; \begin{array}{l} \|\Xi\|_{1,E} = 1,\ \Xi(u,v) = 0\ \ \forall (u,v) \in E \setminus E_{max}(f)\,, \\ |\,\Xi(u,v)\,|\,|\nabla f(u,v)| = \Xi(u,v)\nabla f(u,v)\ \ \forall (u,v) \in E_{max}(f) \end{array} \right\} \tag{5.18}$$

where we recall the definition of the divergence operator

$$-\operatorname{div}\Xi(u) = \frac{1}{2}\sum_{v\sim u} \omega_{uv}\Big(\Xi(v,u) - \Xi(u,v)\Big) = \frac{1}{2}\nabla^T\Xi(u)\,,$$

of the norms

$$\|f\|_{1,V} = \sum_{u\in V} |f(u)|\,, \qquad \|G\|_{1,E} = \frac{1}{2}\sum_{(u,v)\in E} |G(u,v)|,$$

and of the maximal sets:

$$\begin{aligned} E_{max}(f) &= \{(u,v) \in E \mid |\nabla f(uv)| = \|\nabla f\|_\infty\} \\ V_{max}(f) &= \{u \in V \mid |f(u)| = \|f\|_\infty\}\,. \end{aligned}$$

Now we give the following definition of generalized ∞-eigenpair.

Definition 5.4.5. (f, Λ) *is a generalized* ∞*-eigenpair if and only if there exist* $\xi \in \partial\|f\|_\infty$ *and* Ξ *with* $-\mathrm{div}(\Xi) \in \partial\|\nabla f\|_\infty$ *such that*

$$-\mathrm{div}\,\Xi = \Lambda\xi\,.$$

Note that, putting together the above definition of generalized eigenpair and the characterization of the subgradients $\partial\|\nabla f\|_\infty$ (eq. (5.17)) and $\partial\|f\|_\infty$ (eq.(5.18)), we have that

Proposition 5.4.6. (f, Λ) *is an* ∞*-eigenpair if and only if there exist* $\xi : V \to \mathbb{R}$ *and* $\Xi : E \to \mathbb{R}$ *such that*

$$\begin{cases} -\mathrm{div}(\Xi) = \Lambda\xi, & \\ \|\Xi\|_{1,E} = 1 & \\ \|\xi\|_{1,V} = 1 & \\ |f(u)| = \|f\|_\infty & if\ \xi(u) \neq 0 \\ |\nabla f(u,v)| = \|\nabla f\|_\infty & if\ \Xi(u,v) \neq 0 \\ \mathrm{sign}\big(\Xi(u,v)\big) = \mathrm{sign}\big(\nabla f(u,v)\big) & if\ \Xi(u,v) \neq 0 \\ \mathrm{sign}\big(\xi(\mathrm{u})\big) = \mathrm{sign}\big(\mathrm{f(u)}\big) & if\ \xi(u) \neq 0 \end{cases} \tag{5.19}$$

Moreover, up to redefining $\Xi(u,v) = \Big(\Xi(u,v) - \Xi(v,u)\Big)/2$*, we can assume* $\Xi(u,v) = -\Xi(v,u)$*.*

Remark 5.4.7. *We would like to observe that since* $\mathcal{R}_\infty$ *is a locally Lipschitz function of* $\mathbb{R}^n\backslash\{0\}$*, the notion of critical point can also be generalized considering the Clarke subderivative* $\partial^{Cl}\mathcal{R}_\infty$*, see [25], i.e.* f *is a Clarke* ∞*-eigenpairs iff*

$$0 \in \partial^{Cl}\mathcal{R}_\infty(f) \tag{5.20}$$

Also in this case, by a classical argument, (see the Deformation Lemma in [19]), considering $\mathcal{R}_\infty(f)$ *on the sphere* $S_2 = \{f|\|f\|_2 = 1\}$*, we can introduce the following family of generalized critical points of* $\mathcal{R}_\infty$*:*

$$\Lambda_k^{Cl} = \min_{A\in\mathcal{F}_k(S_2)} \max_{f\in A} \mathcal{R}_\infty(f)\,,$$

where, as usual, $\mathcal{F}_k(S_2) := \{A \subseteq S_2 \mid A \text{ closed}\,,\ A = -A\,,\ \gamma(A) \geq k\}$ *and* $\gamma(A)$ *is the Krasnoselskii genus of* A*. By repeating the proof of Theorem 5.4.3, it is possible to prove that also the Clarke eigenpairs can be estimated using inscribed ball radii:*

$$\Lambda_k^{Cl} \leq \frac{1}{R_k}\ \forall k = 1, \ldots, |V|\,, \qquad \Lambda_k^{Cl} = \frac{1}{R_k}\ k = 1, 2\,.$$

We conclude this remak by recalling that we can write [25]

$$\partial^{Cl}\mathcal{R}_\infty(f) \subseteq \frac{\partial\|\nabla f\|_\infty\|f\|_\infty - \partial\|f\|_\infty\|\nabla f\|_\infty}{\|f\|_\infty^2}$$

As a consequene, we can state that the notion of generalized ∞-eigenpair, see Definition (5.4.5)*, generalizes the notion of Clarke-eigenapair. Nevertheless, the definition in Definition* (5.4.5) *provides some practical advantages with respect to the one in* (5.20)*, since, differently from $\partial^{Cl}\mathcal{R}_\infty(f)$, both $\partial\|f\|_\infty$ and $\partial\|\nabla f\|_\infty$ can be explicitly identified.*

5.4.1 Geometrical characterization

Next we deal with a geometrical characterization of the generalized ∞-eigenfunctions similar to the one proved in Proposition 5.3.3. In both the characterizations, given an eigenpair, (f, λ), we prove the existence of "good" paths that connect points in $V_{max}(f) \cup B$ and whose length matches the value of the eigenvalue. Nevertheless, differently from Proposition 5.3.3, where we proved that for any extremal point $v \in V_{max}(f)$ there exist a "good" path Γ, in the case of generalized critical points there could exist extremal point that do not correspond to any "good" path. Moreover, in the case of limiting eigenpairs the existence of such good paths was only a necessary condition, instead, in the case of generalized eigenpairs, the existence of "good" paths is also a sufficient condition.

Proposition 5.4.8. (f, Λ) *is a generalized ∞-eigenpair if and only if there exists a path $\Gamma = \{(u_i, u_{i+1})\}_{i=0}^{n-1}$ such that*

1. $u_i \in V_{max}(f) \cup B \quad i = 0, n\,.$
2. $(u_i, u_{i+1}) \in E_{max}(f)\ \ \forall i = 1, \dots, n$
3. $f(u_i) > f(u_{i+1})$
4. *Assuming w.l.o.g. that $f(u_0) > 0$, if $u_n \in B$ then $\Lambda = \frac{1}{\mathrm{length}(\Gamma)}$, while if $f(u_n) = -f(u_0)$ then $\Lambda = \frac{2}{\mathrm{length}(\Gamma)}$. Moreover*

$$\frac{1}{\Lambda} = \min\left\{ \min_{\{v|f(v)=-\|f\|_\infty\}} \frac{d(u_0, v)}{2}, d_B(u_0) \right\}.$$

Proof. Assume that (f, Λ) is a generalized ∞-eigenpair and let (ξ, Ξ) be the two subgradients as in Definition (5.19), i.e., such that

$$-\mathrm{div}(\Xi) = \Lambda\xi\,, \tag{5.21}$$

and $\Xi(u, v) = -\Xi(v, u)$. Let $\xi(u_0) \neq 0$ and w.l.o.g. assume $f(u_0) > 0$. Then from equation (5.19) there has to exist an edge $(u_0, u_1) \in E_{max}(f)$ such that $\Xi(u_0, u_1) \neq 0$, i.e. $f(u_1) = f(u_0) - \|\nabla f\|_\infty < f(u_0)$. Let us focus now on the node u_1. Since $f(u_1) < f(u_0)$, if $u_1 \notin B$ and $f(u_1) \neq -f(u_0)$, we have that $\xi(u_1) = 0$. Moreover $\Xi(u_0, u_1) < 0$ and Ξ has to satisfy (5.21):

$$-\mathrm{div}(\Xi)(u_1) = \sum_{v \sim u_1} \omega_{vu_1}\Xi(v, u_1) = \Lambda\xi(u_1) = 0\,,$$

hence, there must exist an edge $(u_2, u_1) \in E_{max}(f)$ such that $\Xi(u_1, u_2) > 0$, i.e., $f(u_2) = f(u_1) - \|\nabla f\|_\infty < f(u_1)$. Thus we can define a path $\Gamma = \{u_i, u_{i+1}\}_{i=0}^{n-1}$ such that $u_n \in B \cup V_{max}(f)$, $(u_i, u_{i+1}) \in E_{max}(f)$ and $f(u_i) > f(u_{i+1})$. Furthermore, as in the proof of Proposition 5.3.3, it is easy to see that, given a function f and a path Γ that satisfy the first three items of the thesis, necessarily we have that, if $u_n \in B$, $\Lambda = \frac{1}{\text{length}(\Gamma)}$ while, if $f(u_n) = -f(u_0)$, $\Lambda = \frac{2}{\text{length}(\Gamma)}$ and

$$\frac{1}{\Lambda} = \min\left\{ \min_{f(v)=-\|f\|_\infty} \frac{d(u_0, v)}{2}, d_B(u_0) \right\}.$$

Proceeding by absurd, if one of the last equalities is not true then an edge (v_1, v_2) with $|\nabla f(v_1, v_2)| > \|\nabla f\|_\infty$ must exist.

To prove the opposite inclusion, we assume that, given (f, Λ), there exists a path Γ that satisfies the thesis and concentrate on the case $u_n \in \{v \mid f(v) = -f(u_0)\}$ with $\Lambda = \frac{2}{d(u,u_n)} = \frac{2}{\text{length}(\Gamma)} < d_B(u_0)$ (the other case can be proved analogously). Given the two functions

$$\Xi(u, v) := \frac{\delta_\Gamma(u, v)}{\omega_{uv}} \frac{\text{sign}\big(\nabla f(u, v)\big)}{\text{length}(\Gamma)}$$

$$\xi(v) := \frac{\delta_{u_0}(v)\text{sign}\big(f(u_0)\big) + \delta_{u_n}(v)\text{sign}\big(f(u_n)\big)}{2},$$

where δ_Γ δ_{u_i} are the delta functions ($\delta_\Gamma(u, v) = 1$ if $(u, v) \in \Gamma$, zero otherwise and analogously for δ_{u_i}). Observe that ξ and $-\text{div}\Xi$ belong respectively to the two subgradients $\partial\|f\|_\infty$ and $\partial\|\nabla f\|_\infty$. Moreover, for any node $v \notin \Gamma$ as well as for any node $v \in \Gamma$, $v \neq u_0, u_n$

$$-\text{div}\Xi(v) = 0 = \Lambda\, \xi(v).$$

Finally, in the case $v = u_0$ (or analogously $v = u_n$), we have

$$-\text{div}\Xi(u_0) = \frac{\omega_{u_0 u_2}}{2}\Big(\Xi(u_2, u_0) - \Xi(u_0, u_2)\Big) = \frac{\text{sign}\big((\nabla f(u_2, u_0)\big)}{\text{length}(\Gamma)} = \frac{\Lambda}{2} = \Lambda\xi(u_0),$$

where we have used that by hypotheses $\text{sign}\big(\nabla f(u_2, u_0)\big) > 0$. This concludes the proof.

□

As a corollary of Proposition 5.4.8 and Proposition 5.3.3 it is easy to prove that any eigenpair that satisfies the limiting eigenvalue equation (5.11) is also a generalized ∞-eigenpair.

Corollary 5.4.9. *Let (f, Λ) satisfy the limiting eigenvalue equation* (5.11)*, then (f, Λ) is also a generalized ∞-eigenpair according to Definition 5.4.5.*

Proof. The proof is a consequence of the fact that, from Propositon 5.3.3, for any eigenpair that satisfies the limiting eigenvalue equation (5.11), there exists a path Γ that satisfies the hypotheses of Proposition 5.4.8. $\square$

Next we deal with the opposite problem and pose the question if any generalized ∞-eigenvalue, see Definition 5.4.1, can be associated to an eigenfunction that solves (5.11). As we prove in Example 5.4.12, the answer is negative in general. Nevertheless as we see in Lemma 5.4.11 the statement can always be proved to hold, up to considering a subgraph of $\mathcal{G}$. Before this, let us observe that the ∞-eigenvalue problem can be reformualted in terms of a constrained weighted Laplacian eigenvalue problem. Indeed, from the characterizations of the subgradient equations (5.18) and (5.17), it is possible to reformulate the system (5.19) as in the following proposition.

Proposition 5.4.10. *The pair (f, Λ) is a generalized ∞-eigenpair if and only if there exist two admissible densities $\nu : V \to \mathbb{R}_+$, and $\mu : E \to \mathbb{R}_+$, with $\mu_{uv} = \mu_{vu}$ such that:*

$$\begin{cases} -\operatorname{div}(\mu \nabla f)(u) = \Lambda \nu_u f(u) & \forall u \in V \\ |\nabla f(u,v)| = \|\nabla f(u,v)\|_\infty & \textit{if } \mu_{uv} > 0 \\ |f(u)| = \|f(u)\|_\infty & \textit{if } \nu_u > 0 \\ \|\mu \nabla f\|_{1,E} = 1 \\ \|\nu f\|_{1,V} = 1 \end{cases} \tag{5.22}$$

Proof. Straightforward substitution into equation (5.19) shows that the following quantities:

$$\nu_u := \frac{|\xi(u)|}{2\|f\|_\infty} \qquad \mu := \frac{|\Xi(u,v)| + |\Xi(v,u)|}{\|\nabla f\|_\infty}$$

are the desired admissible densities. The inverse follows by inverting the above definitions of ν and μ. $\square$

Now, let (f, Λ) denote an ∞-eigenpair and assume (μ, ν) to satisfy the condition of Proposition 5.4.10. We say that a node $u \in V$ is supported by μ, ($u \in V_\mu$), if there exists an edge $(u, v) \in E$ such that $\mu_{uv} > 0$. Observe that if $u \notin V_\mu$, then necessarily $\nu_u = 0$ (recall that if $\nu_u \neq 0$, $|f(u)| = \|f\|_\infty$). We write $u \in supp(V)$, if there exist at least one (μ, ν) as in Proposition 5.4.10, such that $u \in V_\mu$.
Now we can prove that any generalized ∞-eigenpair can be regarded as a limiting ∞-eigenpair up to considering a proper subraph of $\mathcal{G}$.

Lemma 5.4.11. *Assume (f, Λ) to be an ∞-eigenpair as in Definition 5.4.1. If $u \in supp(V)$ then f satisfies the limiting eigenvalue equation* (5.11) *in u.*

Proof. Assume $f(u) > 0$ and let (μ, ν) be admissible densities such that $u \in V_\mu$. The weighted eigenvalue equation $-\operatorname{div}(\mu \nabla f)(u) = \Lambda \nu_u f(u)$ reads

$$-\|\nabla f\|_\infty \sum_{v \sim u} \mu_{uv}\, \omega_{uv} \operatorname{sign}\big(\nabla f(u,v)\big) = \Lambda \nu_u f(u) = \frac{\|\nabla f\|_\infty}{\|f\|_\infty} \nu_u f(u)\,. \tag{5.23}$$

We first consider the case $f(u) < \|f\|_\infty$. Since necessarily $\nu_u = 0$, we get that $\|\nabla f(u)^-\|_\infty = \|\nabla f(u)^+\|_\infty$, implying $\Delta_\infty f(u) = 0$. Moreover

$$\|\nabla f(u)\|_\infty - \Lambda f(u) = \|\nabla f\|_\infty - \frac{\|\nabla f\|_\infty}{\|f\|_\infty} f(u) > 0\,.$$

If instead $f(u) = \|f\|_\infty$, for any $v \sim u$ we get $\nabla f(u,v) \leq 0$ and thus $\nu_u \neq 0$ (otherwise, by hypotheses, (5.23) could not be satisfied). Then (5.23) reads

$$\|\nabla f\|_\infty \Big(\sum_{v \sim u} \mu_{uv} \omega_{uv} \Big) = \frac{\|\nabla f\|_\infty}{\|f\|_\infty} \nu_u \|f\|_\infty\,,$$

and we get $\sum_{v \sim u} \mu_{uv}\omega_{uv} = \nu_u$, Replacing this last equality again in (5.23) we find

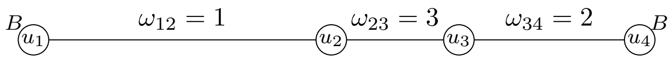
$$\|\nabla f(u)\|_\infty = \|\nabla f\|_\infty = \Lambda f(u)$$

$$\Delta_\infty f(u) = \|\nabla f(u)^-\|_\infty = \|\nabla f\|_\infty > 0\,.$$

The cases $f(u) < 0$ and $f(u) = 0$ can be proved analogously. □

We conclude this section by proving that there exist ∞-eigenpairs that are critical points of $\mathcal{R}_\infty$ but that do not satisfy the ∞-eigenvalue equation (5.11). By the same example we also prove that there exists ∞-eigenvalues between the first and second variational one.

Example 5.4.12. *Consider the following graph:*



The node farther from the boundary is u_2 and $d(u_2, B) = \frac{1}{2} + \frac{1}{3} = \frac{5}{5}$. Then the pair

$$f_1(u_2,\, u_3) = \left(\frac{5}{6},\, \frac{1}{2}\right)\ ,\ \Lambda_1 = \frac{6}{5}$$

is an infinite eigenpair with

$$(\nu_2,\, \nu_3) = \left(\frac{6}{5},\, 0\right)\ ,\ (\mu_{12},\, \mu_{23},\, \mu_{34}) = \left(0,\, \frac{2}{5},\, \frac{3}{5}\right)\,.$$

However, it is easy to verify that the following are also eignepair :

$$f_2(u_2,\, u_3) = \left(\frac{1}{6},\, -\frac{1}{6}\right)\ ,\ \Lambda_2 = 6 \quad (\nu_2,\, \nu_3) = (3,\, 3)\ ,\ (\mu_{12},\, \mu_{23},\, \mu_{34}) = (0,\, 1,\, 0)\ .,$$

$$f(u_2,\, u_3) = \left(*,\, \frac{1}{2}\right)\ ,\ \Lambda = 2 \quad (\nu_2,\, \nu_3) = (0,\, 2)\ ,\ (\mu_{12},\, \mu_{23},\, \mu_{34}) = (0,\, 0,\, 1)\ \ * \in \left[\frac{1}{6},\frac{1}{2}\right]\,.$$

Note that $\Lambda_2 = R_2 = 6$ is the second variational eigenvalue while $\Lambda_1 < \Lambda < \Lambda_2$. Moreover, it is worth noting that (f, Λ) does not satisfy (5.11). *Indeed, to get a solution of* (5.11) *$f(u_2)$ should be determined in such a way to get*

$$\Delta_\infty f(u_2) = 0, \quad \|\nabla f(u_2)\|_\infty - 2f(u_2) \geq 0$$

or

$$\Delta_\infty f(u_2) \geq 0, \quad \|\nabla f(u_2)\|_\infty - 2f(u_2) = 0$$

But, in the first case, $\Delta_\infty f(u_2) = 0$ implies

$$f_2(u_2) = \frac{3}{8} \;\Rightarrow\; \|\nabla f_2(u_2)\|_\infty - 2f(u_2) = \frac{3}{8} - 2\frac{3}{8} < 0\,.$$

In the second case, the equality $\|\nabla f(u_2)\|_\infty - 2f(u_2) = 0$ implies

$$f_2(u_2) = \frac{3}{10} \;\Rightarrow\; \Delta_\infty f_2(u_2) = \frac{3}{10} - 3\frac{1}{5} < 0\,.$$

5.5 A variational characterization of the first Infinity Eigenpair

In this section we discuss a characterization of the ∞-eigenpairs based on Proposition 5.4.10. Such characterization offers the possibility of computing families of ∞-Laplacian eigenpairs as limiting points of sequences of linear Laplacian eigenvalues. We will briefly investigate the numerical alorithms in the following section.

Consider the following class of energy functions, where k varies from 1 to $N = |V|$:

$$\begin{aligned}\mathcal{E}_k(\mu,\nu) &= \frac{1}{\lambda_k(\mu,\nu)} + \mathrm{M}_E(\mu) - \mathrm{M}_V(\nu)\\ &= \frac{1}{\lambda_k(\mu,\nu)} + \frac{1}{2}\sum_{(u,v)\in E}\mu_{uv} - \sum_{v\in V}\nu_v\,, \qquad (\mu,\nu) \in \big(\mathcal{M}^+(E), \mathcal{M}^+(V)\big)\end{aligned} \tag{5.24}$$

here $\lambda_k(\mu,\nu)$ is the k-th eigenvalue of the weighted linear eigenvalue problem

$$-\mathrm{div}(\mu\nabla f) = \lambda\nu f\,,$$

and the following definitions hold

$$\mathrm{M}_E(\mu) := \frac{1}{2}\sum_{(u,v)\in E}\mu_{uv}\,, \qquad \mathrm{M}_V(\nu) := -\sum_{v\in V}\nu_v\,,$$

$$\mathcal{M}^+(V) := \{\nu : V \to \mathbb{R}_{\geq 0}\}\,, \quad \mathcal{M}^+(E) := \{\mu : E \to \mathbb{R}_{\geq 0}\}\,.$$

Assuming the differentiability in (μ, ν), the K-K-T conditions for the saddle points (minimum in μ, maximum in ν), of the above functions read

$$\begin{cases} -\mathrm{div}(\mu \nabla f) = \lambda_k \nu f \\ -\dfrac{|\nabla f(e)|^2}{\lambda_k^2 \|f\|_{2,\nu}^2} + 1 - m(e) = 0 & \forall e \in E \\ -\dfrac{|f(v)|^2}{\|\nabla f\|_{2,\mu}^2} + 1 - n(u) = 0 & \forall v \in V \\ m(e) \geq 0 & \forall e \in E \\ n(v) \geq 0 & \forall v \in V \\ m(e) = 0 & \forall e \ s.t. \ \mu_e > 0 \\ n(v) = 0 & \forall v \ s.t. \ \nu_v > 0 \end{cases} \tag{5.25}$$

where $\{m(e)\}_{e \in E}$ and $\{n(v)\}_{v \in V}$ are suitable families of Lagrange multipliers.

These equations, as we show in Lemma 5.5.1, imply that $(f, \Lambda) := (f, \sqrt{\lambda})$ is an ∞-eigenpair as in Definition 5.4.1, with $(f, \sqrt{\lambda})$ the weighted 2-Laplacian eigenpair that satisfies (5.25). The challenge of this approach is that the generalized eigenvalues are in genereal not continuous with respect to perturbations of the matrices in the case in which both matrices are singular (see for an example [53]). Moreover, the differentiability of an eigenvalue is not realized for multiplicities greater than 1 (see [61]). Observe also that from (5.18) and (5.17), the two densities μ and ν in (5.25) are generally "almost everywhere zero" making the two matrices Δ_μ and $\mathrm{diag}(\nu)$ singular. Studying saddle points of $\mathcal{E}_k$ is thus generally impossible. However, this approach can be pursued for the first eigenpair ($k = 1$). The rigorous proof for $k > 1$ and more in general that a class of ∞-eigenpairs can be computed by means of generalized weighted Laplacian eigenepairs remains an open problem.

Lemma 5.5.1. *Let the pairs (μ, ν) and (f, λ) satisfy the system of equations (5.25). Then $(f, \Lambda) := (f, \sqrt{\lambda})$ is a generalized ∞-eigenpair.*

Proof. Observe first of all that (5.25) implies the following

$$\begin{cases} -\mathrm{div}(\mu \nabla f) = \lambda \nu f \\ \|\nabla f\|_\infty = |\nabla f(e)| = \lambda \|f\|_{2,\nu} & \forall e \ s.t. \ \mu_e > 0 \\ \|f\|_\infty = |f(u)| = \|\nabla f\|_{2,\mu} & \forall v \ s.t. \ \nu_v \geq 0\,. \end{cases} \tag{5.26}$$

Now define

$$\bar{\mu} = \frac{\mu}{\|f\|_{2,\nu}} \quad \text{and} \quad \bar{\nu} = \frac{\sqrt{\lambda}\nu}{\|f\|_{2,\nu}}\,.$$

We show now that $\bar{\mu}$ and $\bar{\nu}$ satisfy equation (5.22). Indeed the first three equations of (5.22) are trivially satisfied, thus we have only to show that

$$\|\bar{\mu} \nabla f\|_1 = 1\,, \quad \|\bar{\nu} f\|_1 = 1\,. \tag{5.27}$$

Before analyzing the 1-norms, recalling that $|\nabla f|$ and $|f|$ are constant on the supports of μ and ν, we derive the following expression involving the total masses of μ and ν:

$$\lambda = \frac{1}{2}\frac{\sum_{e\in E}\mu_e|\nabla f(e)|^2}{\sum_{v\in V}\nu_v|f(v)|^2} = \frac{\|\nabla f\|_{2,\mu}^2}{\|\nabla f\|_{2,\mu}^2\sum_{v\in V}\nu_v} = \frac{\lambda^2\|f\|_{2,\nu}^2\sum_{e\in E}\mu_e}{2\|f\|_{2,\nu}^2},$$

which implies

$$\sum_{v\in V}\nu_v = \frac{1}{\lambda} = \frac{1}{2}\sum_{e\in E}\mu_e\,.$$

Now, to prove the result in equation (5.27), we start from $\|\bar\mu\nabla f\|_1$. Using (5.26), we can write

$$\|\bar\mu\nabla f\|_1 = \frac{1}{2}\sum_{e\in E}\bar\mu_e|\nabla f(e)| = \frac{\|\nabla f\|_\infty}{2}\sum_{e\in E}\bar\mu_e = \frac{2\lambda\|f\|_{2,\nu}}{2\lambda\|f\|_{2,\nu}} = 1\,.$$

For $\|\bar\nu f\|_1$, analogously we have:

$$\|\bar\nu f\|_1 = \sum_{v\in V}\bar\nu_v|f(v)| = \|f\|_\infty\sum_{v\in V}\bar\nu_v = \frac{\sqrt{\lambda}\|\nabla f\|_{2,\mu}}{\lambda\|f\|_{2,\nu}} = \frac{\lambda\|f\|_{2,\nu}}{\lambda\|f\|_{2,\nu}} = 1\,,$$

which concludes the proof. □

Remark 5.5.2. *Before discussing the function $\mathcal{E}_1$, we recall that the (μ,ν)-weighted linear Laplacian problem, with $\mu\in\mathcal{M}^+(V)$ and $\nu\in\mathcal{M}^+(V)$,*

$$\Delta_\mu(f) = -\mathrm{div}(\mu\nabla f) = \lambda\nu f$$

can be partially degenerate. Indeed, the number of well defined eignvalues is equal to $|V| - \dim\Big(\mathrm{Ker}(\mathrm{diag}(\nu))\Big)$. Indeed any $f\in\mathrm{Ker}(\mathrm{diag}(\nu))$ with $f\notin\mathrm{Ker}(\Delta_\mu)$ can be regarded as an eigenfunction with eigenvalue "$\lambda=\infty$". Otherwise, whenever $f\in\mathrm{Ker}(\mathrm{diag}(\nu))\cap\mathrm{Ker}(\Delta_\mu)$, f satisfies the eigenavlue equation with an indefinite eigenvalue. Considering only the well defined eigenvalues, we write

$$\lambda_1(\mu,\nu) = \inf_{f\in S_{2,\nu}}\frac{\|\nabla f\|_{2,\mu}^2}{\|f\|_{2,\nu}^2}\,, \tag{5.28}$$

where $S_{2,\nu} = \{f\in\mathcal{H}_0(V)\mid \|f\|_{2,\nu}=1\}$ is the $(2,\nu)$-sphere, and the two norms are defined as:

$$\|f\|_{2,\nu}^2 = \sum_{u\in V\setminus B}\nu_u|f(u)|^2\,, \qquad \|\nabla f\|_{2,\mu}^2 = \frac{1}{2}\sum_{(u,v)\in E}\mu_{uv}|\nabla f(u,v)|^2\,.$$

Note also that whenever both μ and ν are degenerate, then the eigenfunctions are not uniquely defined. Indeed, let $\mathcal{G}_\nu$ be the subgraph induced by the nodes v

such that $\nu_v \neq 0$ *and let* $\mathcal{G}_\mu$ *be the subgraph induced by the edges* (u, v)*, such that* $\mu_{uv} \neq 0$*. Then, given an eigenvalue* λ*, for any node* $u \in \mathcal{G} \setminus \{\mathcal{G}_\mu \cup \mathcal{G}_\nu\}$ *the* (μ, ν) *eigenvalue equation reads*

$$0 = \sum_{v \sim u} \mu_{uv} \nabla f(v, u) = \Delta_\mu f(v) = \lambda \nu_v f(v) = 0\,, \tag{5.29}$$

where we have used that $\nu_u = 0$ *and* $\mu_{uv} = 0\ \forall\, v \sim u$*. In particular,* (5.29) *shows that* f *is not well defined in the node* u*. Hence, whenever we have partially degenerate densities* μ *and* ν*, we can consider the subgraph* $\mathcal{G}_{\mu,\nu}$ *induced by the nonzero entries of* μ *and* ν*:*

$$u \in \mathcal{G}_{\mu,\nu} \text{ if and only if } \nu_u \neq 0 \text{ and/or } \mu_{uv} \neq 0 \text{ for some } v \in V$$
$$(u, v) \in \mathcal{G}_{\mu,\nu} \text{ if and only if } \mu_{uv} \neq 0\,.$$

Then, we can identify the eigenpairs of the degenerate eigenvalue problem 5.5.2 on the graph $\mathcal{G}$ *with the* (μ, ν)*-Laplacian eigenpairs of the subgraph* $\mathcal{G}_{\mu,\nu}$ *.*

Now we discuss the particular case $k = 1$ of (5.24). In such a case, using the definition (5.28), the function $\mathcal{E}_1(\mu, \nu)$ can be written as

$$\mathcal{E}_1(\mu, \nu) = \sup_{f \in S_{2,\nu}} \frac{\|f\|_{2,\nu}^2}{\|\nabla f\|_{2,\mu}^2} + \frac{1}{2} \sum_{e \in E} \mu_e - \sum_{v \in \nu} \nu_v\,.$$

Moreover, the results related to the study of minimizers of $\frac{\|\nabla f\|_\infty}{\|f\|_2}$ reported in [17] give us enough regularity to prove the following theorem.

Theorem 5.5.3. *Let*

$$\mathcal{E}_1(\mu, \nu) := \frac{1}{\lambda_1(\mu, \nu)} + \frac{1}{2} \sum_{e \in E} \mu_e - \sum_{v \in V} \nu_v,$$

Then the functions

$$\mu \mapsto \frac{1}{\lambda_1(\mu, \nu)} + \frac{1}{2} \sum_{e \in E} \mu_e \quad \text{and} \quad \nu \mapsto \min_\mu \frac{1}{\lambda_1(\mu, \nu)} + \frac{1}{2} \sum_{e \in E} \mu_e - \sum_{v \in V} \nu_v\,,$$

are respectively convex in μ *and concave in* ν *and*

$$\max_\nu \min_\mu \mathcal{E}_1(\mu, \nu) = \big(\min_f \mathcal{R}_\infty(f)\big)^{-2} = \Lambda_1^{-2}\,.$$

Moreover, if $(\nu^*, \mu^*) := \arg\max_\nu \arg\min_\mu \mathcal{E}_1(\nu, \mu)$*, then there exists* (f, λ_1) *a first eigenpair of the generalizzed eigenvalue problem* $\Delta_{\mu^*} f = \lambda \nu^* f$*, such that* $(f, \Lambda) := (f, \sqrt{\lambda_1})$ *is a first eigenpair of the generalized* ∞*-eigenvalue problem (see Definition 5.4.1).*

Proof. Start by noting that, by means of the expression of $\lambda_1(\mu,\nu)$ as a minimum (5.28) and the convexity of the function $(x \mapsto 1/x)$ on the positive semiaxis, it is trivial to prove that

$$\mu \mapsto \frac{1}{\lambda_1(\mu,\nu)} + \frac{1}{2}\sum_{e\in E}\mu_e = \sup_{f\in S_{2,\nu}} \frac{\|f\|_{2,\nu}^2}{\|\nabla f\|_{2,\mu}^2} + \frac{1}{2}\sum_{e\in E}\mu_e$$

is a convex function. Next we study the minimizer of such function:

$$\min_{\mu}\sup_{f\in S_{2,\nu}} \frac{\|f\|_{2,\nu}^2}{\|\nabla f\|_{2,\mu}^2} + \frac{1}{2}\sum_{e\in E}\mu_e\,.$$

Exchanging $\min_\mu \sup_f$ with $\sup_f \min_\mu$, by the max-min inequality we have:

$$\max_{f\in S_{2,\nu}}\min_{\mu} \frac{\|f\|_{2,\nu}^2}{\|\nabla f\|_{2,\mu}^2} + \frac{1}{2}\sum_{e\in E}\mu_e \le \min_{\mu}\max_{f\in S_{2,\nu}} \frac{\|f\|_{2,\nu}^2}{\|\nabla f\|_{2,\mu}^2} + \frac{1}{2}\sum_{e\in E}\mu_e\,. \tag{5.30}$$

Now observe that from Lemma B.0.2 we can state that, given any admissible f, there exists μ_f such that

$$\mu_f \in \arg\min_{\mu} \frac{\|\nabla f\|_{2,\nu}^2}{\|f\|_{2,\mu}^2} + \frac{1}{2}\sum_{e\in E}\mu_e = \frac{2\|f\|_{2,\nu}}{\|\nabla f\|_\infty} \qquad \text{and} \qquad f\mu_f \in \partial\|\nabla f\|_\infty\,,$$

which, together with (5.30) , yields:

$$\min_{\mu}\sup_{f} \frac{\|f\|_{2,\nu}^2}{\|\nabla f\|_{2,\mu}^2} + \sum_{e\in E}\mu_e \ge \sup_f 2\frac{\|f\|_{2,\nu}}{\|\nabla f\|_\infty}$$

Next, we want to prove the opposite inequality. Given an admissible f, from (5.12), we know $|f(u)| \le d_B(u)\|\nabla f\|_\infty$. Hence, any minimizer f^* of the above (∞,ν)-Rayleigh quotient

$$f^* \in \arg\min_f \frac{\|\nabla f\|_\infty}{\|f\|_{2,\nu}} := \Lambda_1(\infty,2,\nu)\,, \tag{5.31}$$

satisfies the following properties:

$$f^*(u) = \|\nabla f\|_\infty d_B(u) \quad \forall u \;\; s.t. \;\; \nu_u > 0 \tag{5.32}$$

The last eq. (5.31), joint with the characterization of the subgradient of $\|\nabla f\|_\infty$ (5.18), imply that for any minimizer f^*, there exists μ_{f^*} such that

$$-\mathrm{div}(\mu_{f^*}\nabla f^*) \in \partial\|\nabla f^*\|_\infty \cap \Lambda_1(\infty,2,\nu)\partial_f\|f^*\|_{2,\nu}\,.$$

In particular, from the characterization of the subgradient, the pair (f^*, μ_{f^*}) satisfies

$$\begin{cases} \Delta_{\mu_{f^*}} f^* = -\mathrm{div}(\mu_{f^*} \nabla f^*) = \Lambda_1(\infty, 2, \nu)\nu f^* \\ \|\mu_{f^*} \nabla f^*\|_1 = 1 \\ |\nabla f^*| = \|\nabla f^*\|_\infty \quad if\ \mu_{f^*} > 0 \end{cases} \tag{5.33}$$

Note that as a consequence of (5.33) and (5.32), the support of the density ν is necessarily contained in the subgraph induced by μ_{f^*}, i.e. $V_{max}(\nu) \subseteq V_{\mu_{f^*}}$. The last inclusion and remark 5.5.2, show that the eigenpairs of the (μ_{f^*}, ν) eigenvalue problem are not well defined outside the subgraph $\mathcal{G}_{\mu_{f^*}}$ induced by the non zero entries of μ_{f^*}. In particular, we can identify the eigenpairs of the (μ_{f^*}, ν)-Laplacian eigenvalue problem on $\mathcal{G}$ with the (μ_{f^*}, ν)-Laplacian eigenpairs on the graph $\mathcal{G}_{\mu_{f^*}}$, see remark 5.5.2.

Moreover, we recall that the first eigenfunction of a weighted Laplacian on a connected graph is characterized as the only everywhere-positive eigenfunction Theorem A.0.1. Hence we have that $f^*|_{\mathcal{G}_{\mu_{f^*}}}$ is necessarily a first eigenfunction of the $(\mu_{f^*}, \nu)|_{\mathcal{G}_{\mu_{f^*}}}$-Laplacian eigenvalue problem on the graph $\mathcal{G}_{\mu_{f^*}}$, i.e.,

$$f^* \in \arg\max_f \frac{\|f\|_{2,\nu}}{\|\nabla f\|_{2,\mu_{f^*}}} = \arg\max_f \frac{\|f\|_{2,\nu}}{\|\nabla f\|_\infty} . \tag{5.34}$$

Finally, (5.34) yields the desired inequality:

$$\min_\mu \max_f \frac{\|f\|^2_{2,\nu}}{\|\nabla f\|^2_{2,\mu}} + \sum_{e\in E} \mu_e \leq \max_f \frac{\|f\|^2_{2,\nu}}{\|\nabla f\|^2_{2,\mu_{f^*}}} + \sum_{e\in E} \mu_e = \max_f 2\frac{\|f\|_{2,\nu}}{\|\nabla f\|_\infty} = \frac{\|f^*\|_{2,\nu}}{\|\nabla f^*\|_\infty} .$$

We have thus proved that

$$\min_\mu \max_f \frac{\|f\|^2_{2,\nu}}{\|\nabla f\|^2_{2,\mu}} + \sum_{e\in E} \mu_e = \max_f 2\frac{\|f\|_{2,\nu}}{\|\nabla f\|_\infty} .$$

The concavitiy of $\nu \mapsto \min_\mu \frac{1}{\lambda_1(\mu,\nu)} + \frac{1}{2}\sum_{e\in E} \mu_e - \sum_{v\in V} \nu_v$ follows directly, since, using (5.32), we can write:

$$\nu \mapsto \min_\mu \frac{1}{\lambda_1(\mu, \nu)} + \sum_{e\in E} \mu_e - \sum_{v\in V} \nu_v = \max_f 2\frac{\|f\|_{2,\nu}}{\|\nabla f\|_\infty} - \sum_{v\in V} \nu_v = 2\|d_B\|_{2,\nu} - \sum_{v\in V} \nu_v .$$

Then, switch $\max_\nu$ with $\max_f$ and observe that, for any f, if ν_f is a maximizer of $2\big(\|f\|_{2,\nu}/\|f\|_\infty\big) - \sum_{v\in V} \nu_v$, there exist a family of Lagrange multipliers $\{n(v)\}_{v\in V}$ such that

$$\begin{cases} \dfrac{|f(v)|^2}{\|f\|_{2,\nu_f}\|\nabla f\|_\infty} - 1 + n(v) = 0 & \forall v \in V \\ n(v) \geq 0 & \forall v \in V \\ n(v) = 0 & \forall v\ s.t.\ \nu_{f_v} > 0 , \end{cases} \tag{5.35}$$

yielding:

$$|f(v)|^2 = \|f\|_\infty^2 = \|\nabla f\|_\infty \|f\|_{2,\nu_f} \qquad \forall v \; s.t. \; \nu_{f_v} > 0$$

In particular we obtain the following expression for the mass of the density ν_f,

$$\sum_{v \in V} (\nu_f)_v = \frac{\|f\|_\infty^2}{\|\nabla f\|_\infty^2} \,,$$

that means

$$\begin{aligned}
\max_\nu \min_\mu \max_f \frac{\|f\|_{2,\nu}^2}{\|\nabla f\|_{2,\mu}^2} + \sum_{e \in E} \mu_e - \sum_{v \in \nu} \nu_v = \max_\nu \max_f \min_\mu \frac{\|f\|_{2,\nu}^2}{\|\nabla f\|_{2,\mu}^2} + \sum_{e \in E} \mu_e - \sum_{v \in \nu} \nu_v = \\
\max_\nu \max_f 2 \frac{\|f\|_{2,\nu}^2}{\|\nabla f\|_\infty^2} - \sum_{v \in \nu} \nu_v = \max_f \max_\nu 2 \frac{\|f\|_{2,\nu}^2}{\|\nabla f\|_\infty^2} - \sum_{v \in \nu} \nu_v = \\
\max_f 2 \frac{\|f\|_\infty^2}{\|\nabla f\|_\infty^2} - \frac{\|f\|_\infty^2}{\|\nabla f\|_\infty^2} = \max_f \frac{\|f\|_\infty^2}{\|\nabla f\|_\infty^2}
\end{aligned} \tag{5.36}$$

concluding the first part of the proof. Moreover, assuming (ν^*, μ^*) to be a max-min of $\mathcal{E}_1$, then, from (5.36) there exists f^* such that

$$f^* \in \arg\min_f \|\nabla f\|_{2,\mu^*}^2 / \|f\|_{2,\nu^*}^2 = \min_f \mathcal{R}_\infty^2(f) \,.$$

Defined $\mu^* = \mu_{f^*}$ $\nu^* = \nu_{f^*}$, Lemma B.0.2 and (5.35) yields that (ν^*, μ^*, f^*) satisfy the set of equations (5.25). Finally the second part of the Theorem follows directly from Lemma 5.5.1. □

5.5.1 Gradient flows

We have observed in the previous section that, by means of the unique saddle point of the energy function $\mathcal{E}_1 : \mathcal{M}^+(E) \times \mathcal{M}^+(V) \to \mathbb{R}$, it is possible to characterize the first ∞-eigenpair and the corresponding admissible densities as in Proposition 5.4.10, see Theorem 5.5.3. Such result extends to the infinty case a result previously proved in Chapter 4 for the case $p \in (2, \infty)$, see Theorem 4.5.8. In Chapter 4, besides the study of the first p-Laplacian eigenpair, it is also proved that any smooth saddle point of a function $\mathcal{E}_k^p$ (4.29) corresponds to a p-Laplacian eigenpair different from the first variational eigenpair, see Theorem 4.5.9. As a consequence of the discussion in the beginning of section 5.5, the same result could be proved for saddle points of the functions $\mathcal{E}_k = \mathcal{E}_k^\infty$. However, as now we show, the assumption of smoothness, in the case $p = \infty$, seems too restrictive. Indeed, from the characterization of the ∞-eigenpairs as linear eigenpairs, see Proposition 5.4.10, and from the charaterization of the subgradients (5.18) (5.17), we recall that any ∞-eigenpair (f, Λ) corresponds to a linear eigenpair of a singular generalized eigenvalue problem. In particular, we have that (f, Λ) satisfies the eigenvalue equation

$$\Delta_\mu f = \Lambda \nu f, \tag{5.37}$$

for some degenerate ν that is supported on the subgraph induced by μ. In addition, from Remark 5.5.2, we recall that the number of well defined eigenvalues of the generalized eigenvalue problem (5.37) is equal to the number of nonzero entries of ν. Assume now that (f, Λ) corresponds to the k-th eigenpair of (5.37) and consider the density $\nu' = \nu + \epsilon e_v$, where e_v is the density equal to zero everywhere except that on the node v, and v is some node not included in the subgraph induced by μ. Then we easily observe that if ϵ is different from zero, the set of the (μ, ν') eigenvalues is given by zero plus the (μ, ν)-eigenvalues. This means that λ_k is not continuous in (μ, ν). Indeed we have

$$\Lambda = \lambda_k(\mu, \nu + 0\, e_v) \qquad \Lambda = \lambda_{k+1}(\mu, \nu') = \lambda_{k+1}(\mu, \nu + \epsilon\, e_v)\ \forall \epsilon > 0\,.$$

Despite such problems of continuity, in the following of this section we present a preliminary discussion and the results of some numerical integrations of gradients flows for the functions $\mathcal{E}_k$.

In particular, for any k we consider the gradient flow obtained as an extension to the case $p = \infty$ of the gradient flows presented in Chapter 4:

$$\begin{cases} \dot{\mu} = \mu\Big(\frac{|\nabla f|^2}{\lambda_k(\mu,\nu)\|f\|_\nu^2} - 1\Big) \\ \dot{\nu} = \nu\Big(\frac{|f|^2}{\|\nabla f\|_\mu^2} - 1\Big) \\ \Delta_\mu f = \lambda_k(\mu, \nu) f \end{cases}. \tag{5.38}$$

Note that, given two strictly positive initial densities μ_0 and ν_0, the gradient flow (5.38) corresponds to the following dynamics extended to the space of signed measures on edges and nodes:

$$\begin{cases} \dot{\mu} = \mu\Big(\min_{\Phi \in \operatorname{sign}(\mu)} \big(\frac{|\nabla f|^2}{\lambda_k(\mu,\nu)\|f\|_\nu^2} - \Phi\big)\Big) \\ \dot{\nu} = \nu\Big(\min_{\phi \in \operatorname{sign}(\nu)} \big(\frac{|f|^2}{\|\nabla f\|_\mu^2} - \phi\big)\Big) \\ \Delta_\mu f = \lambda_k(\mu, \nu) f \end{cases}, \tag{5.39}$$

where

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{if } x > 0 \\ [-1, 1] & \text{if } x = 0 \\ -1 & \text{if } x > 0 \end{cases}.$$

Observe that Φ and ϕ in (5.39) are chosen, respectively, in the subgradients of $\|\mu\|_1 = \sum_{uv \in E} |\mu_{uv}|$ and $\|\nu\|_1 = \sum_{u \in V \setminus B} |\nu_u|\,.$

The correspondence between the gradient flows (5.39) and (5.38), when starting from two strictly positive initial densities μ_0 and ν_0, can be proved observing that as long as $\nu(t)$ and $\mu(t)$ are everywhere nonzero $\phi = 1$ and $\Phi = 1$. Moreover, if at some time t_0, $\nu_u(t_0)$ or $\mu_{uv}(t_0)$ become zero for some $u \in V \setminus B$ or $(u, v) \in E$, then $\dot{\nu}_u(t)$ and $\dot{\mu}_{uv}(t)$ are zero for any $t > t_0$ and the dynamics cannot exit anymore from the subspaces $\{\nu | \nu_u = 0\}$, $\{\mu | \mu_{uv} = 0\}$. This means that the value of ϕ_u and Φ_{uv} is irrelevant whenever $\nu_u(t_0)$ or $\mu_{uv}(t_0)$ is zero. Then, we note that the dynamics (5.39) can be considered as a regularization of the following dynamics, whose importance is pointed out in the next Lemma 5.5.4

$$\begin{cases} \dot{\mu} = \min_{\Phi \in \mathrm{sign}(\mu)} \left(\frac{|\nabla f|^2}{\lambda_k(\mu,\nu) \|f\|_\nu^2} - \Phi \right) \\ \dot{\nu} = \min_{\phi \in \mathrm{sign}(\nu)} \left(\frac{|f|^2}{\|\nabla f\|_\mu^2} - \phi \right) \\ \Delta_\mu f = \lambda_k(\mu, \nu) f \end{cases} . \tag{5.40}$$

The regularization is represented by the multiplicative factors ν and μ in (5.39). Indeed, such factors slow down the velocities $\dot{\mu}$ and $\dot{\nu}$ when the densities μ and ν approach zero on some edge or node. From the discussion at the beginning of this section, we comprehend the importance of such regularization. Indeed, we have observed that, when (μ, ν) become zero somewhere, it is possible to experience a lack of continuity of the eigenvalue $\lambda_k(\mu, \nu)$. Thus, differently from the dynamics (5.40) which can pass through different discontinuities of the eigenvalue $\lambda_k(\mu, \nu)$, the dynamics (5.39) is expected to experience a lack of continuity of the eigenvalue $\lambda_k(\mu, \nu)$ only in the limiting point. Now we highlight the importance of the dynamics (5.40) showing that its equilibrium points, if exist, correspond to ∞-eigenpairs.

Lemma 5.5.4. *Suppose (μ_0, ν_0) to be a equilibrium point of the dynamics* (5.40)*. Then $(f, \Lambda) = \big(f, \sqrt{\lambda_k(\mu_0, \nu_0)}\big)$ is an ∞-eigenpair. Moreover the support of μ_0 and ν_0 matches respectively the support of some $\Xi \in \partial \|\nabla f\|_\infty$ and $\xi \in \partial \|f\|_\infty$ such that $-\mathrm{div}(\Xi) = \Lambda \xi$.*

Proof. Oberve that if (μ_0, ν_0) is an equilibrium point, then

$$\begin{aligned} 0 &= \min_{\Phi \in \mathrm{sign}(\mu_0)} \Big(\frac{|\nabla f|^2}{\lambda_k(\mu_0, \nu_0) \|f\|_\nu^2} - \Phi \Big) \\ 0 &= \min_{\phi \in \mathrm{sign}(\nu_0)} \Big(\frac{|f|^2}{\|\nabla f\|_{\mu_0}^2} - \phi \Big) \end{aligned} \tag{5.41}$$

where (f, λ_k) is the k-th (μ_0, ν_0)-eigenpair. Thus, given Φ and ϕ minimizers in (5.41), they necessarily satisfy

$$\begin{aligned} &0 \le \Phi_{uv} \le 1 \;\forall\, (u, v) \in E \qquad \Phi_{uv} = 1 \;\forall\, (u, v) \text{ s.t. } \mu_{0uv} \neq 0 \\ &0 \le \phi_u \le 1 \;\forall\, u \in V \setminus B \qquad \phi_u = 1 \;\forall\, u \text{ s.t. } \nu_{0u} \neq 0. \end{aligned}$$

If we set $m = 1 - \phi$ and $n = 1 - \phi$, it is easy to observe that (μ_0, ν_0) and (f, λ_k) satisfy the set of equations (5.25). The thesis follows applying Lemma 5.5.1 and Proposition 5.4.10. $\square$

Now we discuss the results of the numerical integration of the dynamics (5.38). The system of algebraic-differential equations is discretized by means of a simple explicit Euler method with an empirically-determined constant time step size, t. The third (purely algebraic) equation is solved by the QZ algorithm. Given the value of k and the initial values μ_k^0 and ν_k^0, for $n = 0, 1, \ldots$ the final scheme takes on the form:

$$\begin{aligned}
\Delta_{\mu_k^n} f &= \lambda_k^n(\mu_k^n, \nu_k^n) f \\
\mu_k^{n+1} &= \mu_k^n + t\mu_k^n \left(\frac{|\nabla f|^2}{(\lambda_k^n)^2 \|f\|^2_{\nu_k^n}} - 1 \right) \\
\nu_k^{n+1} &= \nu_k^n + t\nu_k^n \left(\frac{|f|^2}{\|\nabla f\|^2_{\mu_k^n}} - 1 \right).
\end{aligned}$$

Convergence towards equilibrium is considered achived when the error

$$\mathrm{err} = \left| \sqrt{\lambda} - \frac{\|\nabla f\|_\infty}{\|f\|_\infty} \right| \tag{5.42}$$

is below a given tolerance.

Figure 5.3 shows the experimental results obtained on a graph of 49 vertices obtained gridding uniformly the square. The nodes on the edges of the square are considered as boundary nodes subject to Dirichlet boundary conditions. In addition, we impose uniform weights on the edges. The first 4 eigenfunctions (left panels), the relative convergence behaviour (central panel) and the plot of the values of the corresponding μ and ν are reported. The value of μ on any edge is represented using a proportional thickness of the edge. Similarly, the value of ν on any internal node is represented by both plotting the node with a proportional dimension and giving to the node the appropriate color in accordance to the color-code on the right of the figure.

We must recall that, even if the eigenvalue $\lambda_k(\mu, \nu)$ is expected to be continuous along the trajectory of (μ, ν) in (5.38), the same is not in general true for the eigenfunction f. Indeed whenever the eigenvalue $\lambda_k(\mu, \nu)$ is not simple, the eigenfunction f is not uniquely defined and hence also the system of algebraic differential equations (5.38) is not well defined. From a numerical point of view we can suppose that the discrete time step allows to jump over the discontinuity points of the eigenfunction. Nevertheless, when we overcome a point in which the k-th eigenvalue is not simple, the trajectory of the k-th eigenpair can be exchanged with the trajectory of the $k-1$ or $k+1$ eigenpair, and the discontinuity in the eigenfunction is reflected in a discontinuity in the error plot. Such behavior is reflected in several tests, see for example $k = 3, 4$ Figure 5.3. We highlight that

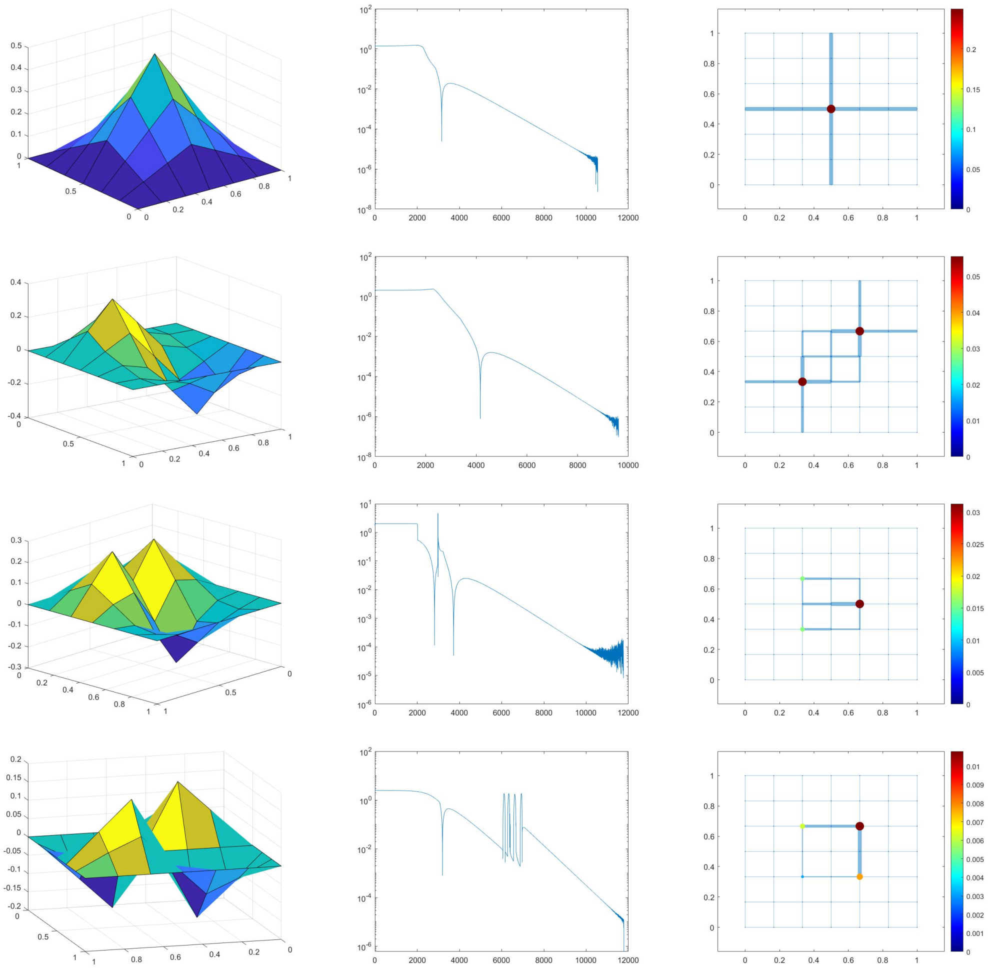


Figure 5.3: Left panel: four eigenfunctions as calculated by the proposed method with $k = 1, 2, 3, 4$ and uniform initial densities ν_0 and μ_0. The edge length is uniform on all the edges and equal to the reciprocal of the edge length. For each k the central panel reports the behavior of the error defined in eq.(5.42) as a function of time steps (iterations) n. The Right panel show the values of μ and ν. The edge values of μ are plotted with the proportional thickness of the edge. The nodal values of ν are plotted with the thickness of the node and the color-code shown on the right of the figure for $k = 1, \ldots, 4$ (top to bottom)

in all the reported numerical tests, at convergence, the behaviour of the densities μ and ν reflects the behavior proved in Proposition 5.4.10 and Proposition 5.4.8. So, ν is supported on extremal points (maxima or minima) of the function f, and μ is supported on the shortest paths that join extremal point of different sign or extramal points to the boundary.

Now we discuss the results of the same numerical integration performed for $k = 5, 6$ (Fig 5.4 and Fig 5.5). In these cases, when the error of the eigenvalue is approximately 10^{-6}, we experience some convergence of the densities μ and ν. However it is easy to observe that the graph induced by the limiting densities is symmetric and disconnected. This fact implies that the limiting eigenvalues are not simple, and the uncertainty in the choice of the correct eigenfunction leads to escape from an equilibrium and to converge toward a different one, see Fig 5.4 and Fig 5.5.

Moreover, taking for example $k = 6$, we note that in the final equilibria of the dynamics, the density ν is supported only on one node, meaning that there is a unique well defined eigenpair. The index of the limiting eigenvalue is thus 1 and not 6 as along the whole trajectory. This leads to a discontinuity of the eigenvalue in the limiting point. Such discontinuity is not reflected in the numerical tests because the numerical integration is stopped when the error is sufficiently small and, at this point, the densities μ and ν are still not exactly equal to zero on the nodes and edges that are not supported by the limiting densities.

We conclude observing that, interestingly, all the presented numerical tests converge toward ∞-eigenpair. However, we are not able to provide any information about the position of the computed eigenvalues in the variational spectrum. Moreover, the above discussion and the problems of continuity of the eigenpairs, make the theoretical study of convergence of the above algorithm particularly complicated, especially when $k > 1$. The necessity to overcome the problems of continuity of the eigenpairs suggests us future investigations of similar dynamics in which the index of the eigenvalue is not fixed a priori. Indeed, the possibility to have an eigenvalue index that change along the trajectory could allow us to obtain a smoothly varying eigenfunction, which, in turn, would result in smooth trajectories of λ, μ and ν.

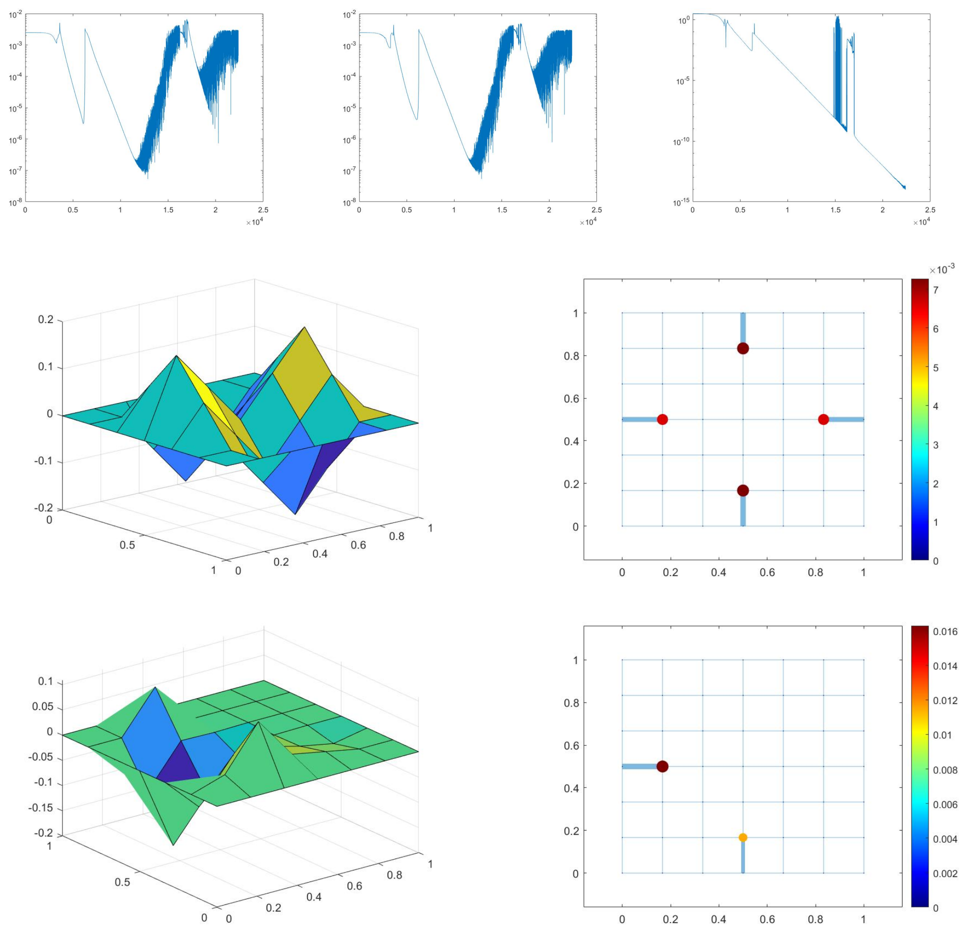


Figure 5.4: Proposed method performed for $k = 5$ and uniform initial densities ν_0 and μ_0. The edge length is uniform on all the edges and equal to the reciprocal of the edge length. The top panel reports from left to right the relative error of μ, the relative error of ν and the error of the eigenvalue (5.42). The left panel reports the plot of the eigenfunction f, on the top when the error of the eigenvalue is smallear than 10^{-6}, at the bottom when the error of the eiegnvalue is smaller than 10^{-14}. The right panel reports the corresponding densities μ and ν when the error of the eiegnvalue is smaller than 10^{-6} (top) and 10^{-14} (bottom). The edge values of μ are plotted with the proportional thickness of the edge. The nodal values of ν are plotted with the thickness of the node and the color-code shown on the right of the figure.

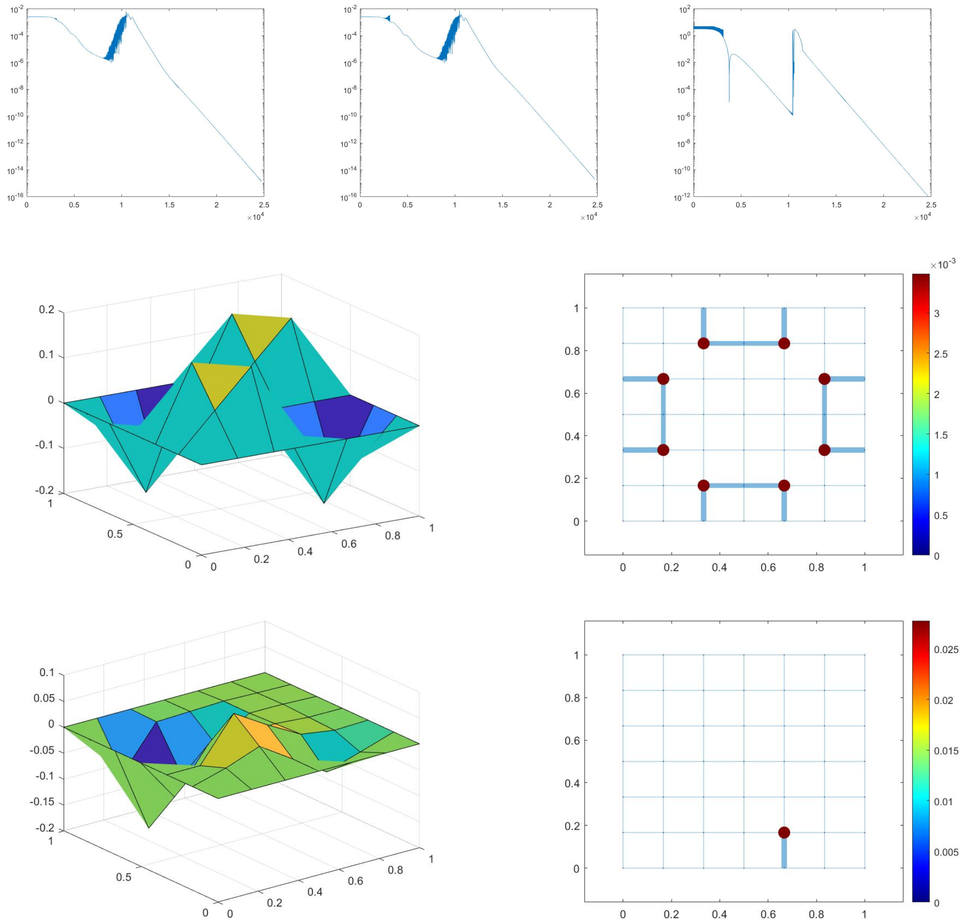


Figure 5.5: Proposed method performed for $k = 6$ and uniform initial densities ν_0 and μ_0. The edge length is uniform on all the edges and equal to the reciprocal of the edge length. The top panel reports from left to right the relative error of μ, the relative error of ν and the error of the eigenvalue (5.42). The left panel reports the plot of the eigenfunction f, on the top when the error of the eigenvalue is smallear than 10^{-5}, at the bottom when the error of the eiegnvalue is smaller than 10^{-12}. The right panel reports the corresponding densities μ and ν when the error of the eiegnvalue is smaller than 10^{-5} (top) and 10^{-12} (bottom). The edge values of μ are plotted with the proportional thickness of the edge. The nodal values of ν are plotted with the thickness of the node and the color-code shown on the right of the figure.

A

This appendix is devoted to recall some classical technical results of crucial importance to prove some of our results. For the sake of completness we report these results with their proof.

Theorem A.0.1. *Let $\mathcal{G} = (V, E, \omega)$ be a graph with boundary B, $\mu : E \to \mathbb{R}^{>+}$ a strictly positive weight on the edges and $\nu : V \setminus B \to \mathbb{R}^+$ a positive weight on the nodes such that is not everywhere zero $\nu \neq 0$. Define $\Delta_\mu f(u) = \nabla^T \mathrm{diag}(\mu) \nabla f(u) + r_u f(u)$ to be the μ-weighted Laplacian matrix associated to the homogeneous Dirichlet boundary conditions, i.e.*

$$\Delta_\mu f(u) = \sum_{\substack{v \sim u \\ v \in V \setminus B}} \omega_{uv} \mu_{uv} \Big(f(u) - f(v) \Big) + \sum_{v \in B} \omega_{uv} \mu_{uv} f(u) \,.$$

Consider the generalized eigenvalue problem

$$\Delta_\mu f = \lambda \nu f \,. \tag{A.1}$$

Then,

$$\lambda_1(\mu, \nu) = \min_f \frac{\|\nabla f\|_\mu^2}{\|f\|_\nu^2} = \min_f \frac{\sum_{(u,v) \in E} \mu_{uv} |\nabla f(u,v)|^2}{\sum_{u \in V \setminus B} \nu_u |f(u)|^2}$$

is a simple eigenvalue, meaning that there exists a unique eigenfunction f_1 such that $\Delta_\mu f_1 = \lambda_1 \nu f_1$. Moreover f_1 is the only eigenfunction such that $f_1(u) > 0$ for any internal node $u \in V \setminus B$.

Proof. First observe that if the boundary is empty, the Laplacian matrix Δ_μ has non-empty kernel which given by the only constant vector. Moreover note that the constant vector is supported everywhere and thus also on the support of ν. Second consider the case of an non-empty boundary, then the Laplacian matrix Δ_μ is not singular meaning the eigenvalue problem (A.1) admits a number of eigenvalues equal to $N - \dim\big(\mathrm{Ker}(\mathrm{diag}\nu)\big)$, where $N = |V \setminus B|$. Moreover each eigenvalue is such that the corresponding eigenfunction is in $\mathbb{R}^N \setminus \mathrm{Ker}(\mathrm{diag}\nu)$.

Hence w.l.o.g. we can consider a minimizer of $\mathcal{R}_{p,2,\nu}$, f_1, such that $\|f_1\|_{2,\nu} = 1$. Note that by the triangular inequalty,

$$|f_1(u) - f_1(v)| \geq \big|\, |f_1(u)| - |f_1(v)| \,\big| \,,$$

where the equality holds if and only if $f = \pm|f|$. Moreover since $\|f_1\|_\nu = \|\,|f_1|\,\|_\nu$, we derive the following inequality:

$$\frac{\|\nabla|f_1|\,\|_\mu^2}{\||f_1|\|_\nu^2} \leq \frac{\|\nabla f_1\|_\mu^2}{\|f_1\|_\nu^2} = \min_f \frac{\|\nabla f\|_\mu^2}{\|f\|_\nu^2}\,.$$

The last inequlality proves that, since the the graph is connected, we can consider f_1 such that $f_1(u) \geq 0\; \forall u \in V \setminus B$.

Moreover if there exists $u \in V \setminus B$ such that $f_1(u) = 0$, using the fact that $f_1(v) \geq 0$ for any $v \sim u$ we observe that the eigenvalue equation (A.1) reads:

$$0 \geq \Delta_\mu f(u) = 0,,$$

with equality if and only if $f_1(v) = 0\; \forall\; v \sim u$. Hence, by the connctedness of the graph, if $f_1(u) = 0$ for some $u \in V \setminus B$, then $f_1 = 0$ everywhere, which is a contradiction of the hypothesis $\|f_1\|_\nu = 0$. We have proved that any first eigenfunction, up to multiplicative factors, is strictly positive.

Now we can prove the second part of the theorem. Assume that there exists a positive eigenfunction $f_2 > 0$ such that

$$\frac{\|\nabla f_2\|_\mu^2}{\|f_2\|_\nu^2} = \lambda_2 > \lambda_1\,.$$

Then, there exists some $t > 0$ such that

$$\lambda_2 f_2(u) > t\lambda_1 f_1(u)\; \forall u \in V \setminus B \quad \text{and} \quad \exists u_0 \in V \setminus B \text{ s.t. } tf_1(u_0) > f_2(u_0)\ .$$

Applying Theorem A.0.2 to the functions tf_1 and f_2, we get a contradiction. Hence we have proved that any positive eigenfunction is necessarily associated to the first eigenvalue.

We are left to prove that λ_1 is simple, i.e., the uniqueness of the corresponding eigenfunction f_1. Assume that there exist two positive eigenfunctions f_1 and f_2 relative to λ_1 with $\|f_1\|_\nu = \|f_2\|_\nu = 1$. Then, the function

$$g(u) = \left(f_1^2(u) + f_2^2(u)\right)^{\frac{1}{2}},$$

has ν-norm given by $\|g\|_\nu^2 = 2$, and its gradient satisfies:

$$\|\nabla g\|_\mu^2 \leq \left(\|\nabla f_1\|_\mu^2 + \|\nabla f_2\|_\mu^2\right)$$

with equality holding if and only if $\nabla f_1(u,v) = \nabla f_2(u,v)\; \forall (u,v) \in E$. To prove the last inequality, consider an edge (u,v) and use the Cauchy Schwarz inequality applied to the two vectors $\big(f_1(u), f_2(u)\big)\; \big(f_1(v), f_2(v)\big)$:

$$\begin{aligned}
|\nabla g(v,u)|^2 &= \omega_{uv}^2 \Big|\left(f_1(u)^2 + f_2(u)^2\right)^{\frac{1}{2}} - \left(f_1(v)^2 + f_2(v)^2\right)^{\frac{1}{2}}\Big|^2 \\
&\leq \omega_{uv}^2 \Big|\big(f_1(u) - f_1(v)\big)^2 + \big(f_2(u) - f_2(v)\big)^2\Big| \\
&= \left(|\nabla f_1(v,u)|^2 + |\nabla f_2(v,u)|^2\right)
\end{aligned}$$

Then we have

$$2\lambda_1 = \lambda_1 \|g\|_\nu^2 \leq \|\nabla g\|_\mu^2 \leq \big(\|\nabla f_1\|_\mu^2 + \|\nabla f_2\|_\mu^2\big) = 2\lambda_1$$

implying that for any edge $f_1(u) - f_1(v) = f_2(u) - f_2(v)$ and thus, since $\|f_1\|_\nu = \|f_2\|_\nu$ and both f_1 and f_2 are positive, necessarily $f_1 = f_2$. Hence, we have proved that the first eigenfunction is simple, concluding the proof. □

Theorem A.0.2. *[77] Suppose that f and g satisfies*

$$\Delta_p f(u) + r(u)|f(u)|^{p-2} f(u) \geq \Delta_p g(u) + r(u)|g(u)|^{p-2} g(u),$$

where $r(u) \geq 0$ for any u in V. Then $f(u) \geq g(u)$ for any $u \in V$.

Proof. Let $S = \{u \in V | g(u) > f(u)\}$ and $x^+ = \max\{x, 0\}$, then we have:

$$\sum_{u \in V} \Big(\Delta_p f(u) + r(u)|f(u)|^{p-2} f(u) - \Delta_p g(u) - r(u)|g(u)|^{p-2} g(u)\Big) \big(g(u) - f(u)\big)^+ \geq 0\,.$$

Exploiting the last equation and the expression of $\Delta_p f(u)$ and $\Delta_p g(u)$ we obtain:

$$\begin{aligned}
0 \leq & \sum_{u \in S} r(u)\big(|f(u)|^{p-2} f(u) - |g(u)|^{p-2} g(u)\big)\big(g(u) - f(u)\big) \\
& + \sum_{u \in S} \sum_{\substack{v \sim u \\ v \in S}} \omega_{uv} \Big(|f(u) - f(v)|^{p-2}\big(f(u) - f(v)\big) - |g(u) - g(v)|^{p-2}\big(g(u) - g(v)\big)\Big)\big(g(u) - f(u)\big) \\
& + \sum_{u \in S} \sum_{\substack{v \sim u \\ v \in V \setminus S}} \omega_{uv} \Big(|f(u) - f(v)|^{p-2}\big(f(u) - f(v)\big) - |g(u) - g(v)|^{p-2}\big(g(u) - g(v)\big)\Big)\big(g(u) - f(u)\big)
\end{aligned} \tag{A.2}$$

However observe first that:

$$r(u)\big(|f(u)|^{p-2} f(u) - |g(u)|^{p-2} g(u)\big)\big(g(u) - f(u)\big) \leq 0 \quad \forall u \in S\,.$$

Second observe that if $u \in S$ and $v \in V \setminus S$, $f(u) < g(u)$ and $f(v) \geq g(v)$ meaning that

$$|g(u) - g(v)|^{p-2}\big(g(u) - g(v)\big) > |f(u) - f(v)|^{p-2}\big(f(u) - f(v)\big)\,,$$

i.e. $\forall u \in S$ $v \in V \setminus S$:

$$\omega_{uv}\Big(|f(u) - f(v)|^{p-2}\big(f(u) - f(v)\big) - |g(u) - g(v)|^{p-2}\big(g(u) - g(v)\big)\Big)\big(g(u) - f(u)\big) < 0\,.$$

Third observe that, since $|a|^p + |b|^p - ab(|a|^{p-2} + |b|^{p-2}) \geq 0$

$$\sum_{u \in S} \sum_{\substack{v \sim u \\ v \in V \setminus S}} \omega_{uv} \Big(|f(u) - f(v)|^{p-2} \big(f(u) - f(v)\big) - |g(u) - g(v)|^{p-2} \big(g(u) - g(v)\big) \Big) \big(g(u) - f(u)\big)$$

$$= \sum_{\substack{u,v \in S \\ u \sim v}} -\Big(|f(u) - f(v)|^p + |g(u) - g(v)|^p \Big)$$

$$+ \sum_{\substack{u,v \in S \\ u \sim v}} \Big(|f(u) - f(v)|^{p-2} + |g(u) - g(v)|^{p-2} \Big) \big(f(u) - f(v)\big) \big(g(u) - g(v)\big) \leq 0$$

Hence, since all the terms in (A.2) are smaller than zero, with the second being strictly smaller, we conclude that necessarily $S = \emptyset$. $\square$

B

We devote this appendix to present some novel technical results which are necessary to prove some of our main results.

Lemma B.0.1. *Consider the function*

$$R(\beta_1,\beta_2) = \Big(\frac{|\beta_1|^p}{\phi_p(\alpha_1)} - \frac{|\beta_2|^p}{\phi_p(\alpha_2)}\Big)\phi_p(\alpha_1-\alpha_2) - (\beta_1-\beta_2)\phi_p\Big(\beta_1-\beta_2\Big)\,, \tag{B.1}$$

where $\phi_p(x) = |x|^{p-2}x$, $\alpha_1, \alpha_2, \beta_1, \beta_2$ are real numbers, and $\alpha = \alpha_2/\alpha_1$. Then $R(\beta_1,\beta_2)$ is positive if α is negative and negative if α is positive. Moreover $R(\beta_1,\beta_2) = 0$ if and only if $\beta = \beta_1/\beta_2 = \alpha_1/\alpha_2$.

Proof. We first consider the special cases where either β_1 or β_2 are zero or $\alpha = 1$. When $\beta_2 = 0$ (B.1) becomes

$$R(\beta_1, 0) = |\beta_1|^p\Big(\phi_p(1-\alpha) - 1\Big),$$

and a simple computation shows that $\big(\phi_p(1-\alpha)-1\big) \geq 0$ if and only if $\alpha < 0$. The case with $\beta_1 = 0$ is similar, since $R(0,\beta_2) = |\beta_2|^p\Big(\phi_p\big(1-\frac{1}{\alpha}\big)-1\Big)$. Next, consider the case $\alpha = 1$. In this case (B.1) simplifies to $R(\beta_1,\beta_2) = -|\beta_1-\beta_2|^p \leq 0$ and one easily sees that the equality holds if and only if $\beta_1 = \beta_2$.

Consider now the case where both β_1 and β_2 are different from zero and $\alpha \neq 1$. Equation (B.1) can be written as

$$R(\beta_1,\beta_2) = |\beta_1|^p\Big(\phi_p(1-\alpha) - \phi_p\Big(1-\frac{\beta_2}{\beta_1}\Big)\Big) + |\beta_2|^p\Big(\phi_p\Big(1-\frac{1}{\alpha}\Big) - \phi_p\Big(1-\frac{\beta_1}{\beta_2}\Big)\Big). \tag{B.2}$$

Dividing (B.2) by $|\beta_2|^p$ and letting $\beta = \beta_1/\beta_2$ we get the chain of inequalities

$$\begin{aligned}
&|\beta|^p\phi_p(1-\alpha) + \phi_p\Big(1-\frac{1}{\alpha}\Big) \geq |\beta|^p\phi_p\Big(1-\frac{1}{\beta}\Big) + \phi_p\Big(1-\beta\Big)\\
\iff\quad &\frac{|\beta(1-\alpha)|^p}{(1-\alpha)} + \frac{|1-\frac{1}{\alpha}|^p}{1-\frac{1}{\alpha}} \geq |\beta-1|^{p-2}(\beta^2-\beta) + |\beta-1|^{p-2}(1-\beta)\\
\iff\quad &\frac{|\beta(1-\alpha)|^p}{(1-\alpha)} + \frac{|1-\frac{1}{\alpha}|^p}{1-\frac{1}{\alpha}} \geq |\beta-1|^p\,.
\end{aligned} \tag{B.3}$$

Now, if $1 < \alpha < 0$, then $0 < \frac{1}{(1-\alpha)} < 1$ and $\frac{1}{(1-\alpha)} + \frac{1}{1-\frac{1}{\alpha}} = 1$, so we can use the convexity of $x \mapsto |x|^p$ to obtain

$$\frac{|\beta(1-\alpha)|^p}{(1-\alpha)} + \frac{|1-\frac{1}{\alpha}|^p}{1-\frac{1}{\alpha}} \geq |\beta - 1|^p .$$

Since $x \mapsto |x|^p$ is strictly convex for $p > 1$, the equality in the expression above holds if and only if $\beta(1-\alpha) = \frac{1}{\alpha} - 1$ showing that $R(\beta_1, \beta_2)$ is positive if α is negative.

To face the case $\alpha > 0$, consider again equation (B.2). We can assume without loss of generality that $0 < \alpha < 1$. Indeed, if $\alpha > 1$, we can divide (B.2) by $|\beta_1|^p$ to obtain an equation like (B.3) where $1/\alpha$ is used in place of α and the proof would follow from the argument above. Returning to the case $0 < \alpha < 1$, from (B.2) and the following sequence of inequalities can be obtained following the same steps as above:

$$\begin{aligned} & |\beta|^p \phi_p(1-\alpha) + \phi_p\Big(1 - \frac{1}{\alpha}\Big) \leq |\beta - 1|^p \\ \iff \quad & |\beta|^p \leq \frac{|\beta-1|^p}{\phi_p(1-\alpha)} + \frac{\phi_p\Big(\frac{1-\alpha}{\alpha}\Big)}{\phi_p(1-\alpha)} \\ \iff \quad & |\beta|^p \leq \Big|\frac{\beta - 1}{(1-\alpha)}\Big|^p (1-\alpha) + \Big|\frac{1}{\alpha}\Big|^p \alpha \end{aligned}$$

Note that, as before, the last inequality holds due to the convexity of $x \mapsto |x|^p$ and thus equality holds if and only if $\frac{\beta-1}{(1-\alpha)} = \frac{1}{\alpha}$ which implies $\beta = \frac{1}{\alpha}$, concluding the proof. □

Lemma B.0.2. *Let $\mathcal{M}(X)$ be the space of the finite signed measures on a measurable space X with $\mathcal{M}^+(X)$ being the cone of the positive measures. Given a measurable function $f : X \to \mathbb{R}$*

$$\inf_{\eta \in \mathcal{M}^+(X)} \frac{1}{\|f\|_{2,\eta}^2} + |\eta| = \frac{2}{\|f\|_\infty}$$

Where $|\eta| (:= \|\eta\|_1)$ denotes the total variation of η and $\|f\|_{2,\eta}$ denotes the 2-norm of f with respect to η. Moreover, if $|f|$ admits a maximum, there exists a η^* that realizes the minimum and $f\eta^* \in \partial \|f\|_\infty$.

Proof. Observe first of all that the function

$$\begin{aligned} \Xi : \mathcal{M}^+ &\longrightarrow \mathbb{R}^+ \\ \eta &\mapsto \frac{1}{\|f\|_{2,\eta}^2} + |\eta| \end{aligned}$$

is a convex function on the convex cone of the positive measures. Then, fixed a measure η, the function

$$\begin{aligned}\Xi_\eta : \mathbb{R}^+ &\longrightarrow \mathbb{R}^+\\ c &\mapsto \Xi(c\eta)\end{aligned}$$

is clearly a convex function on $\mathbb{R}^+$, and differentiating in c, we can compute its minimizer c_η by solving:

$$0 = \frac{d\Xi_\eta}{dc}(c_\eta) = -\frac{1}{c_\eta^2 \|f\|_{2,\eta}^2} + |\eta|\,,$$

yielding

$$c_\eta^{-2} = |\eta| \|f\|_{2,\eta}^2\,. \tag{B.4}$$

Then, we can write

$$\inf_{\eta\in\mathcal{M}^+(X)} \Xi(\eta) = \inf_{\substack{\eta\in\mathcal{M}^+(X)\\|\eta|=1}} \Xi(c_\eta\eta) = \inf_{\substack{\eta\in\mathcal{M}^+(X)\\|\eta|=1}} 2\sqrt{\frac{|\eta|}{\|f\|_{2,\eta}^2}} = \inf_{\substack{\eta\in\mathcal{M}^+(X)\\|\eta|=1}} 2\sqrt{\frac{1}{\|f\|_{2,\eta}^2}} = \frac{2}{\|f\|_\infty}\,,$$

where the last equality is easily obtained considering any probability measure on the subsets $\{x||f(x)| > \|f\|_\infty - \epsilon\}$, and letting ϵ go to zero. Moreover if $|f|$ admits maximum, the inf is easily proved to be a minimum. Denoting with η^* the minimizer, from (B.4) we have:

$$\|f\|_{2,\eta^*}^2 = \frac{1}{|\eta^*|} = \|f\|_\infty$$

from which we can easily see that $f\eta^* \in \partial\|f\|_\infty\,.$ □

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