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Curvature evolution of nonconvex lens-shaped domains

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Abstract. We study the curvature flow of planar nonconvex lens-shaped domains, considered as special symmetric networks with two triple junctions. We show that the evolving domain becomes convex in finite time; then it shrinks homothetically to a point, as proved in [22]. Our theorem is the analog of the result of Grayson [13] for curvature flow of closed planar embedded curves.

1. Introduction

Mean curvature flow of partitions, in particular of planar networks, has been considered by various authors, see for instance [20], [5], [6], [8], [19], [10], [21]. Such a geometric flow is a generalization of mean curvature flow, when more than two phases are present. The main difficulties are due to the presence of multiple junctions, typically triple points in the planar case.

In this paper we consider the curvature flow of a lens-shaped network, that is, of a particular planar network symmetric with respect to the first coordinate axis, and having there two triple junctions. If the bounded region enclosed by the network is convex, it is proved in [22] that the evolution remains convex and shrinks to a point in finite time, while its shape approaches a unique profile γ^{h}, corresponding to a homothetically shrinking solution (see [22], Figure 1). This is the precise analog of the well-known result of Gage and Hamilton [11], which shows that a closed convex planar curve evolving by curvature shrinks to a point in finite time, approaching a circle. This result has been generalized by Grayson [13] who showed that a closed nonconvex initial embedded curve has no singularities before the extinction, it becomes convex and eventually shrinks to a point. A different proof of Grayson's theorem was given by Huisken in [17].

Our aim is to study the long time curvature evolution of a general (not necessarily convex) lens-shaped network. We will show that such a network becomes convex in finite time and eventually shrinks homothetically to a point, as described in [22]. Our result is, therefore, the analog of the result of Grayson, but in the context of curvature flow of networks. Our proof is based on the classification of all possible singularities, in analogy to the proof given in [17] for curvature flow of curves. We point out that in the evolution considered here we are able to overcome the technical difficulties which prevented in [19] the complete analysis of type II singularities.

The main result of the present paper, which is a consequence of Theorems 3.1, 4.2 and 5.1, reads as follows:

Theorem 1.1. *Assume that the initial curve* $\bar{\gamma} : [0,1] \to \mathbb{R}^2$ *satisfies the regularity and compatibility conditions listed in assumption* (A) (*Section* 2.2) *and is embedded* (*hypothesis* (2.11)). *Then there exist* $T \in (0, +\infty)$ *and a solution* $\gamma \in \mathscr{C}^{2,1}\big([0,1] \times [0,T)\big)$ *of the evolution problem* (2.1) *expressing the curvature flow of a symmetric network with two triple junctions, such that*

$$L\big(\gamma(t)\big) \leqq C\sqrt{2(T-t)}, \quad t \in [0,T),$$

$$\|\kappa_{\gamma(t)}\|_{L^\infty([0,1])} \leqq \frac{C}{\sqrt{2(T-t)}}, \quad t \in [0,T),$$

where $L\big(\gamma(t)\big)$ *and* $\kappa_{\gamma(t)}$ *denote the length and the curvature of* $\gamma(t)$ *respectively, and* C *is an absolute positive constant. Moreover, there exists* $\bar{t} \in [0,T)$ *such that the region* $E\big(\gamma(t)\big)$ *enclosed by the corresponding network is uniformly convex for all* $t \in [\bar{t}, T)$, *and* T *is the extinction time of the evolution, i.e.*

$$\lim_{t \to T^-} L\big(\gamma(t)\big) = \lim_{t \to T^-} \big|E\big(\gamma(t)\big)\big| = 0.$$

Finally, a suitable rescaled and translated version of $\gamma(t)$ *converges in* $\mathscr{C}^2([0,1];\mathbb{R}^2)$ *to* γ^{h} *as* $t \to T^-$.

We note that to prove Theorem 1.1 the only result needed from [22] is the uniqueness of γ^{h}.

In the last section of the paper we exhibit two examples of singularities appearing *before* the extinction time. In Example 1 we show the formation of a singularity, starting from a suitable immersed initial datum $\bar{\gamma}$ (see Figure 5); in this case the L^∞-norm of the curvature of $\gamma(t)$ blows up at $t = T$, and T is smaller than the extinction time. In Example 2, starting from an embedded double-bubble shaped $\bar{\gamma}$ as in Figure 6 (hence with different Neumann boundary conditions with respect to the ones in Theorem 1.1) we show that the singularity appears at $t = T$ before the extinction time, due to the collision of the two triple junctions.

We conclude this introduction by mentioning that a general analysis of curvature flow of planar networks has been recently announced by Tom Ilmanen [18].

2. Notation

Given $T > 0$ and a map $\gamma = (\gamma_1, \gamma_2) : [0,1] \times [0,T) \to \mathbb{R}^2$, for $t \in [0,T)$ we set $\gamma(t) : [0,1] \to \mathbb{R}^2$, $\gamma(t)(x) := \gamma(x,t)$. If $\gamma \in \mathscr{C}^{2,1}\big([0,1] \times [0,T); \mathbb{R}^2\big)$, we introduce the following notation:

• $L(\gamma(t)) := \int_0^1 |\gamma_x(x,t)|\,dx$ is the length of $\gamma(t)$, where γ_x denotes the derivative with respect to x.

• $s \in I(t) := [0, L(\gamma(t))]$ is the (time dependent) arclength parameter of $\gamma(t)$, and $\partial_s := \frac{\partial_x}{|\gamma_x|}$ denotes the derivative with respect to s.

• $\tau_{\gamma(t)} = \tau(t) = (\tau_1(t), \tau_2(t)) := \gamma_s(t)$ is the unit tangent vector to $\gamma(t)$, and $\tau(t)(x) := \tau(x,t)$.

• $v_{\gamma(t)} = v(t) := (-\tau_2(t), \tau_1(t))$ is the normal vector to $\gamma(t)$ obtained by rotating $\tau(t)$ counterclockwise of $\pi/2$, and $v(t)(x) := v(x,t)$.

• $\kappa_{\gamma(t)} := \langle \tau_s(t), v(t) \rangle = \left\langle \frac{\gamma_{xx}(t)}{|\gamma_x(t)|^2}, v(t) \right\rangle$ is the curvature of $\gamma(t)$, and

$$\kappa_\gamma(x,t) := \kappa_{\gamma(t)}(x).$$

• $\gamma_t := \partial_t \gamma$ denotes the derivative of γ with respect to t.

We denote by $|E|$ the Lebesgue measure of a measurable set $E \subseteq \mathbb{R}^2$.

2.1. The geometric evolution equation. We are concerned with the following geometric evolution problem:

$$\begin{cases} \gamma_t = \dfrac{\gamma_{xx}}{|\gamma_x|^2} & \text{in } (0,1) \times (0,T), \\ \gamma_2(0,t) = \gamma_2(1,t) = 0, & t \in (0,T), \\ \tau(0,t) = \left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right), & t \in (0,T), \\ \tau(1,t) = \left(\dfrac{1}{2}, -\dfrac{\sqrt{3}}{2}\right), & t \in (0,T), \\ \gamma(0) = \bar{\gamma} & \text{in } (0,1), \end{cases} \tag{2.1}$$

where the initial curve $\bar{\gamma} = (\bar{\gamma}_1, \bar{\gamma}_2) \in \mathscr{C}^2([0,1]; \mathbb{R}^2)$ satisfies

$$|\bar{\gamma}_x(x)| \neq 0, \quad x \in [0,1], \tag{2.2}$$

and the compatibility conditions

$$\bar{\gamma}_2(0) = \bar{\gamma}_2(1) = 0, \quad \frac{\bar{\gamma}_x(0)}{|\bar{\gamma}_x(0)|} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \quad \frac{\bar{\gamma}_x(1)}{|\bar{\gamma}_x(1)|} = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right). \tag{2.3}$$

System (2.1) corresponds to motion by curvature (first equation) of a planar curve with the extremal points $\gamma(0,t)$, $\gamma(1,t)$ sliding on the first coordinate axis (second equation), and satisfying the following Neumann boundary conditions (third and fourth equation):

$$\text{angle between } e_1 \text{ and } \tau(t) = \begin{cases} \pi/3 \text{ at } \gamma(0,t) = \bigl(\gamma_1(0,t),0\bigr), \\ -\pi/3 \text{ at } \gamma(1,t) = \bigl(\gamma_1(1,t),0\bigr), \end{cases} \tag{2.4}$$

where $e_1 := (1,0)$.

2.2. Definitions of γ^{sp} and λ. For $t \in [0,T)$ we define the "specular" curve $\gamma^{\mathrm{sp}} := (\gamma_1, -\gamma_2)$. The corresponding network mentioned in the Introduction is the one formed by $\gamma([0,1],t) \cup \gamma^{\mathrm{sp}}([0,1],t)$ and by the two horizontal half lines $\bigl(-\infty, \gamma_1(0,t)\bigr)$ and $\bigl(\gamma_1(1,t), +\infty\bigr)$ lying on the first coordinate axis.

In the following, we let the function $\lambda_\gamma = \lambda : [0,1] \times [0,T) \to \mathbb{R}$ be such that

$$\gamma_t = \lambda\tau + \kappa\nu. \tag{2.5}$$

Note that

$$\lambda = \langle \gamma_t, \tau \rangle = \left\langle \frac{\gamma_{xx}}{|\gamma_x|^2}, \tau \right\rangle. \tag{2.6}$$

Formally differentiating in time the boundary conditions in (2.1) (second equation) and using (2.5) we have at $(0,t)$ and $(1,t)$ the relation $0 = \partial_t \gamma_2 = \lambda\tau_2 + \kappa_\gamma \nu_2$, which gives

$$\kappa_\gamma(0,t) = -\sqrt{3}\lambda(0,t), \quad \kappa_\gamma(1,t) = \sqrt{3}\lambda(1,t), \tag{2.7}$$

where we make use of the third and fourth equations in (2.1). Moreover, recalling from [19], formula (2.4), that $\tau_t = (\partial_s \kappa_\gamma + \lambda\kappa_\gamma)\nu$, we find

$$\partial_s \kappa_\gamma(0,t) + \lambda(0,t)\kappa_\gamma(0,t) = \partial_s \kappa_\gamma(1,t) + \lambda(1,t)\kappa_\gamma(1,t) = 0. \tag{2.8}$$

Notice that (2.8) and (2.7) imply

$$\begin{aligned} \partial_s \kappa_\gamma(0,t) &= -\lambda(0,t)\kappa_\gamma(0,t) = \frac{\kappa_\gamma(0,t)^2}{\sqrt{3}} \geqq 0, \\ \partial_s \kappa_\gamma(1,t) &= -\lambda(1,t)\kappa_\gamma(1,t) = -\frac{\kappa_\gamma(1,t)^2}{\sqrt{3}} \leqq 0 \end{aligned} \tag{2.9}$$

for all $t \in (0,T)$. In particular, the function $\kappa_{\gamma(t)}$ can never attain its maximum at $x = 0$ unless $\kappa_\gamma(0,t) = \partial_s \kappa_\gamma(0,t) = 0$; similarly $\kappa_{\gamma(t)}$ can never attain its maximum at $x = 1$ unless $\kappa_\gamma(1,t) = \partial_s \kappa_\gamma(1,t) = 0$.

From now on we will always make the following assumption (A) on $\bar{\gamma}$:

(A) $\bar\gamma \in \mathscr{C}^2([0,1];\mathbb{R}^2)$ satisfies (2.2), (2.3) and the second order compatibility conditions

$$\langle \bar\gamma_{xx}(0), \bar\nu(0)\rangle = -\sqrt{3}\langle \bar\gamma_{xx}(0), \bar\tau(0)\rangle, \quad \langle \bar\gamma_{xx}(1), \bar\nu(1)\rangle = -\sqrt{3}\langle \bar\gamma_{xx}(1), \bar\tau(1)\rangle, \tag{2.10}$$

where

$$\bar\tau = \frac{\bar\gamma_x}{|\bar\gamma_x|} = (\bar\tau_1, \bar\tau_2) \quad \text{and} \quad \bar\nu := (-\bar\tau_2, \bar\tau_1).$$

Note that under the sole assumption (A) the set $\bar\gamma([0,1],t)$ may have self-intersections, see Figure 1.

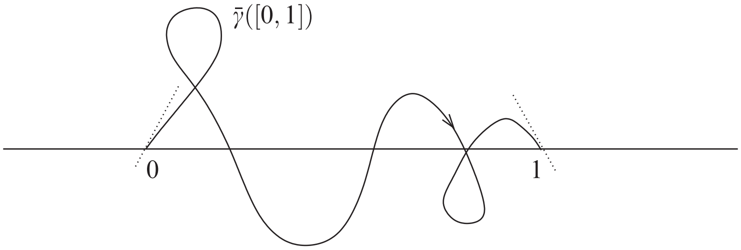

Figure 1. An immersed initial datum $\bar\gamma$ satisfying assumption (A).

Definition 2.1. We will refer to the embedded case, provided

$$\bar\gamma \text{ is injective and } \bar\gamma_2(x) > 0 \text{ for all } x \in (0,1). \tag{2.11}$$

In the embedded case $\bar\gamma([0,1])$ is not necessarily a graph with respect to the first coordinate axis. However, we can speak of the connected bounded plane region $E(\bar\gamma)$ in between $\bar\gamma([0,1])$ and $\bar\gamma^{\mathrm{sp}}([0,1])$, see Figure 2.

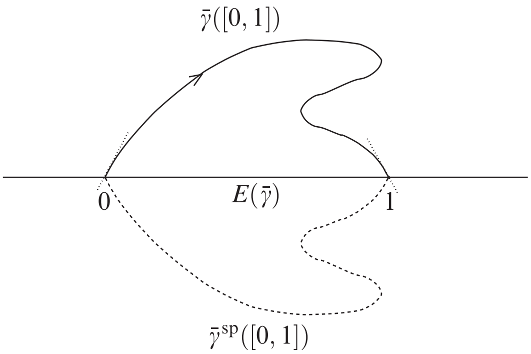

Figure 2. An embedded initial datum $\bar\gamma$, with its specular one (dotted curve) and the region $E(\bar\gamma)$ enclosed between the two curves. The points $(0,0)$ and $(1,0)$ are the two triple junctions, if one imagines to add to the curves the horizontal half lines on the left of $(0,0)$ and on the right of $(1,0)$.

We will refer to the *convex* case, provided

$$\bar\gamma((0,1)) \text{ is the graph of a positive concave function.}$$

The convex case is in particular embedded, and has been studied in [22], where it is proven that $\gamma(t)$ remains concave. Therefore, the plane region $E\big(\gamma(t)\big)$ between $\gamma([0,1],t)$ and $\gamma^{\mathrm{sp}}([0,1],t)$ is still well defined, it is a convex lens-shaped domain evolving by curvature, and having the two singular points $\gamma(0,t)$, $\gamma(1,t)$ in its boundary.

Remark 2.2. With our convention, in the convex case $\kappa_{\gamma(t)}$ is negative, since $\gamma(t)$ is parametrized in such a way that $E\big(\gamma(t)\big)$ lies locally on the right of $\gamma(t)$.

2.3. The homothetically shrinking solution γ^{h}. In [7], [22] it is proven that there exists a unique embedding $\gamma^{\mathrm{h}} \in \mathscr{C}^\infty([0,1];\mathbb{R}^2)$ which satisfies

$$\gamma_2^{\mathrm{h}}(0) = \gamma_2^{\mathrm{h}}(1) = 0, \quad \frac{\gamma_x^{\mathrm{h}}(0)}{|\gamma_x^{\mathrm{h}}(0)|} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \quad \frac{\gamma_x^{\mathrm{h}}(1)}{|\gamma_x^{\mathrm{h}}(1)|} = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right),$$

which gives raise to a homothetically shrinking curvature evolution, namely

$$\kappa_{\gamma^{\mathrm{h}}} + \langle \gamma^{\mathrm{h}}, \nu_{\gamma^{\mathrm{h}}} \rangle = 0 \quad \text{in } (0,1). \tag{2.12}$$

Moreover

$$\inf_{x\in(0,1)} \kappa_{\gamma^{\mathrm{h}}}(x) > 0.$$

3. Immersed initial data

In the next theorem $\gamma([0,1],t)$ is allowed to have self-intersections.

Theorem 3.1. *Assume that $\bar\gamma$ satisfies* (A). *Then problem* (2.1) *has a unique solution*

$$\gamma \in \mathscr{C}^\infty\big([0,1]\times(0,T);\mathbb{R}^2\big) \cap \mathscr{C}^{2,1}\big([0,1]\times[0,T);\mathbb{R}^2\big),$$

defined on a maximal time interval $[0,T)$, *and* $T < +\infty$. *Moreover*

$$\limsup_{t\to T^-} \|\kappa_{\gamma(t)}\|_{L^2([0,1])} = +\infty. \tag{3.1}$$

Proof. All assertions but $T < +\infty$ follow from [19], Theorems 3.1, 3.18 and Remark 3.24. Let us show that $T < +\infty$. Take an initial open convex bounded lens-shaped domain $E(\bar\eta)$ with

$$E(\bar\eta) \supset \bar\gamma([0,1]),$$

whose boundary is given by $\bar\eta([0,1]) \cup \bar\eta^{\mathrm{sp}}([0,1])$, where $\bar\eta : [0,1] \to \mathbb{R}^2$ gives raise to a homothetically shrinking curvature evolution $\eta : [0,1]\times[0,t^*) \to \mathbb{R}^2$, $t^* < +\infty$, with the same boundary conditions as γ, i.e.,

$$\eta_2(0,t) = \eta_2(1,t) = 0, \quad \frac{\eta_x(0,t)}{|\eta_x(0,t)|} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \quad \frac{\eta_x(1,t)}{|\eta_x(1,t)|} = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right), \tag{3.2}$$

see Figure 3 and Section 2.3.

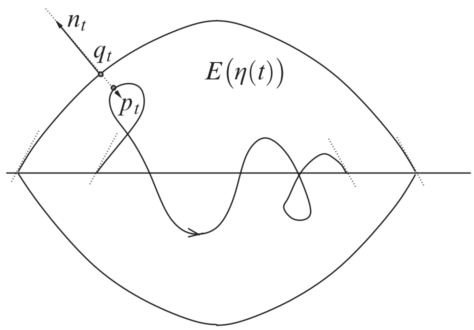

Figure 3. The inner curve is $\gamma(t)$, the outer curve is $\eta(t) \cup \eta^{\mathrm{sp}}(t)$, bounding the self-similar shrinking convex set $E(\eta(t))$.

We claim that the following comparison principle holds:

$$E(\eta(t)) \supset \gamma([0,1],t), \tag{3.3}$$

for all times $t \in [0, t^{\#})$, where $t^{\#} := \min(t^*, T)$.

Since the proof of this comparison result differs slightly from the standard comparison proof for curvature flow, we indicate here the main steps. Define

$$\delta(t) := \operatorname{dist}\big(\partial E(\eta(t)), \gamma([0,1],t)\big), \quad t \in [0, t^{\#}).$$

To prove (3.3), it is enough to show that

$$\lim_{h \to 0^+} \frac{\delta(t+h) - \delta(t)}{h} \geqq 0, \quad t \in (0, t^{\#}). \tag{3.4}$$

For any $(x, \xi, t) \in [0,1]^2 \times [0, T)$ set

$$u(x,\xi,t) := |\eta(x,t) - \gamma(\xi,t)|, \quad v(x,\xi,t) := |\eta^{\mathrm{sp}}(x,t) - \gamma(\xi,t)|.$$

It is well-known (see for instance [14]) that

$$\lim_{h \to 0^+} \frac{\delta(t+h) - \delta(t)}{h} = \min\big(U(t), V(t)\big)$$

where

$$U(t) := \min\left\{\frac{\partial u}{\partial t}(x,\xi,t) : (x,\xi) \in [0,1] \times [0,1], \delta(t) = u(x,\xi,t)\right\},$$

$$V(t) := \min\left\{\frac{\partial v}{\partial t}(y,\eta,t) : (y,\eta) \in [0,1] \times [0,1], \delta(t) = v(y,\eta,t)\right\}.$$

Given $t \in (0, t^{\#})$, we denote by $x^t, \xi^t \in [0,1]$ two parameters for which either

$$\delta(t) = u(x^t, \xi^t, t), \quad \text{or} \quad \delta(t) = v(x^t, \xi^t, t).$$

Without loss of generality, we assume $\delta(t) = u(x^t, \xi^t, t)$, and we set

$$q^t := \eta(x^t, t), \quad p^t := \gamma(\xi^t, t),$$

see Figure 3. Note that

$$q^t \notin \{\eta(0,t), \eta(1,t)\}. \tag{3.5}$$

Indeed if by contradiction we have for instance $q^t = \eta(0,t)$ then, in view of the Neumann boundary conditions in (2.1) and (3.2), the distance between p^t and a point q on $\partial E(\eta(t))$ would decrease when q moves from q^t sliding slightly either on $\eta([0,1],t)$ or on $\eta^{\mathrm{sp}}([0,1],t)$.

We now distinguish two cases.

Case 1. $p^t \notin \{\gamma(0,t), \gamma(1,t)\}$, see Figure 3. In this case, thanks to (3.5), we are reduced to the standard situation of curvature flow (see for instance [3]), and (3.4) follows:

Case 2. $p^t \in \{\gamma(0,t), \gamma(1,t)\}$. Without loss of generality, we can assume that $p^t = \gamma(1,t)$, and that the second component of q^t is positive. Let $n^t := \dfrac{q^t - p^t}{|q^t - p^t|}$. Then it is not difficult to see that n^t equals the unit normal to $\partial E(\eta(t))$ at q^t pointing out of $E(\eta(t))$. Let $K := \{(\cos\theta, \sin\theta) : \theta \in [0, \pi/6]\}$. If $n^t \in \partial K$ then again (3.4) follows in a standard way. On the other hand, we cannot have $n^t = (\cos\theta^t, \sin\theta^t)$ with $\theta^t \in [0, \pi/6)$, since this contradicts the Neumann boundary conditions in (3.2) and the convexity of $\eta(t)$.

The proof of (3.3) is concluded, and in particular $T \leqq t^*$. □

Note that the smoothness of γ implies that $\|\kappa_{\gamma(t)}\|_{L^\infty([0,1])}$ is finite for all $t \in [0,T)$. On the other hand, from (3.1) we deduce that

$$\limsup_{t \to T^-} \|\kappa_{\gamma(t)}\|_{L^\infty([0,1])} = +\infty. \tag{3.6}$$

Proposition 3.2. *There exists a constant $c > 0$ independent of γ such that*

$$L(\gamma(t)) \leqq cL(\bar{\gamma}), \quad t \in [0,T). \tag{3.7}$$

Proof. Since $\gamma_t(0,t)$ and $\gamma_t(1,t)$ are horizontal, it follows from (2.6) that $\lambda(0,t) = \partial_t \gamma_1(0,t)/2$, and $\lambda(1,t) = \partial_t \gamma_1(1,t)/2$. Observing (see [19], Proposition 3.2) that the time-derivative of the measure ds is given by

$$(\lambda_s - \kappa_\gamma^2)\, ds, \tag{3.8}$$

we have

$$\begin{aligned} \frac{d}{dt} L(\gamma(t)) &= \lambda|_{x=0}^{x=1} - \int_{I(t)} \kappa_{\gamma(t)}^2\, ds = \frac{1}{2}\big(\partial_t \gamma_1(1,t) - \partial_t \gamma_1(0,t)\big) - \int_{I(t)} \kappa_{\gamma(t)}^2\, ds \\ &\leqq \frac{1}{2}\big(\partial_t \gamma_1(1,t) - \partial_t \gamma_1(0,t)\big). \end{aligned} \tag{3.9}$$

Hence

$$(3.10)\qquad L(\gamma(t)) \leqq L(\bar{\gamma}) - \frac{1}{2}(\gamma_1(1,0) - \gamma_1(0,0)) + \frac{1}{2}(\gamma_1(1,t) - \gamma_1(0,t)).$$

Therefore, to conclude the proof it is enough to show that $\gamma_1(1,t) - \gamma_1(0,t)$ is bounded by $cL(\bar{\gamma})$, where $c > 0$ is an absolute constant independent of γ. This assertion can be proved by a comparison argument as in the proof of Theorem 3.1: taking a lens-shaped convex domain as in Theorem 3.1, it follows that the horizontal length $\gamma_1(1,t) - \gamma_1(0,t)$ cannot be larger than the corresponding horizontal length of $E(\eta(t))$, which can be bounded by an absolute constant times $L(\bar{\gamma})$. □

Following [16] and recalling (3.6), we say that:

- γ develops a *type* I *singularity* at $t = T$ if there exists $C > 0$ such that

$$(3.11)\qquad \|\kappa_{\gamma(t)}\|_{L^\infty([0,1])} \leqq \frac{C}{\sqrt{2(T-t)}}, \quad t \in [0,T).$$

- γ develops a *type* II *singularity* at $t = T$ if

$$\limsup_{t \to T^-} \sqrt{2(T-t)}\|\kappa_{\gamma(t)}\|_{L^\infty([0,1])} = +\infty.$$

Before passing to the next result, we recall from [19], equation (2.6), that the evolution equation for κ reads as follows:

$$(3.12)\qquad \partial_t \kappa_\gamma = \partial_{ss}\kappa_\gamma + \lambda \partial_s \kappa_\gamma + \kappa_\gamma^3.$$

Note that this equation, being local, is valid under the sole assumption (A).

The next observation is used to prove Proposition 3.4, which in turn will be used to prove Theorem 5.1.

Remark 3.3. The solution γ of (2.1) is analytic in $(0,1) \times (0,T)$; in particular, for a given $t \in (0,T)$, the set

$$z(t) := \{x \in [0,1] : \kappa_{\gamma(t)}(x) = 0\}$$

is finite.

Proposition 3.4. *For any $t \in [0,T)$ we have*

$$(3.13)\qquad \frac{d}{dt}\int_{I(t)} |\kappa_{\gamma(t)}|\,ds = -2\sum_{x \in z(t)} |\partial_s \kappa_\gamma(x,t)| \leqq 0.$$

Proof. Using Remark 3.3, (3.8) and (3.12) we compute

$$\frac{d}{dt}\int_{I(t)}|\kappa_{\gamma(t)}|\,ds = \int_{I(t)}\left[\frac{\kappa_\gamma}{|\kappa_\gamma|}\partial_t\kappa_\gamma + (\lambda_s - \kappa_\gamma^2)|\kappa_\gamma|\right]ds \tag{3.14}$$

$$= \int_{I(t)}\left[\frac{\kappa_\gamma}{|\kappa_\gamma|}\partial_{ss}\kappa_\gamma + (\lambda|\kappa_\gamma|)_s\right]ds.$$

Integrating by parts we have

$$\int_{I(t)}\frac{\kappa_\gamma}{|\kappa_\gamma|}\partial_{ss}\kappa_\gamma\,ds = \frac{\kappa_\gamma}{|\kappa_\gamma|}\partial_s\kappa_\gamma\Big|_{x=0}^{x=1} - \int_{I(t)}\left(\frac{\kappa_\gamma}{|\kappa_\gamma|}\right)_s\partial_s\kappa_\gamma\,ds. \tag{3.15}$$

Moreover

$$\int_{I(t)}\left(\frac{\kappa_\gamma}{|\kappa_\gamma|}\right)_s\partial_s\kappa_\gamma\,ds = 2\sum_{x\in z(t)}|\partial_s\kappa_\gamma(x,t)|. \tag{3.16}$$

Hence from (3.14), (3.15) and (3.16) we deduce

$$\frac{d}{dt}\int_{I(t)}|\kappa_{\gamma(t)}|\,ds = -2\sum_{x\in z(t)}|\partial_s\kappa_\gamma(x,t)| + \frac{\kappa_\gamma}{|\kappa_\gamma|}(\partial_s\kappa_\gamma + \lambda\kappa_\gamma)\Big|_{x=0}^{x=1} \tag{3.17}$$

$$= -2\sum_{x\in z(t)}|\partial_s\kappa_\gamma(x,t)| \leqq 0. \quad \square$$

4. Embedded nonconvex initial data: type I singularities

In this section, as well as in Section 5, we consider the embedded case. We begin to show that embeddedness is a property which is preserved by the evolution.

Proposition 4.1. *Assume that $\bar\gamma$ satisfies* (A) *and* (2.11). *Then*:

(i) *For any $t \in [0,T)$*

$$\gamma(t) \text{ is injective and } \gamma_2(x,t) > 0 \text{ for all } x \in (0,1). \tag{4.1}$$

(ii) *For any $t \in [0,T)$*

$$|E(\gamma(t))| = -\frac{4\pi}{3}t + |E(\bar\gamma)|. \tag{4.2}$$

Proof. Let $\delta := \sup\{t \in [0,T) : \gamma(t) \text{ is injective for } t \in [0,\delta)\}$. By (2.11) and the smoothness of the evolution it follows that $\delta > 0$. Given $(x,y,t) \in [0,1]^2 \times [0,\delta)$ with $x < y$, let $S(x,y,t)$ be the relatively open segment connecting $\gamma(x,t)$ with $\gamma(y,t)$. Provided $S(x,y,t) \cap \gamma([x,y],t) = \emptyset$, we let $A^\gamma(x,y,t)$ be the subset of $\mathbb{R}^2$ bounded by $\gamma([x,y],t)$ and $S(x,y,t)$.

Given $x, y \in [0,1]$ and $t \in [0,\delta)$, let also $\Sigma(x,y,t)$ be the relatively open segment connecting $\gamma(x,t)$ with $\gamma^{\mathrm{sp}}(y,t)$. Provided $\Sigma(x,y,t) \cap \partial E(\gamma(t)) = \emptyset$, we have that either $E(\gamma(t)) \setminus \Sigma(x,y,t)$ is the union of two connected regions, or $(\mathbb{R}^2 \setminus E(\gamma(t))) \setminus \Sigma(x,y,t)$ is the union of two connected regions. We denote by $A^{\gamma}_{\min}(x,y,t)$ the region of minimal area among these two regions.

We define the function $g_\gamma : [0,\delta) \to [0,+\infty)$ as follows: For $t \in [0,\delta)$,

$$g_\gamma(t) := \min\left(Q_1^\gamma(t), Q_2^\gamma(t)\right), \tag{4.3}$$

where

$$Q_1^\gamma(t) := \inf_{x,y\in[0,1],\, x<y,\, S(x,y,t)\cap\gamma([x,y],t)=\emptyset} \frac{|\gamma(x,t)-\gamma(y,t)|^2}{|A^\gamma(x,y,t)|}, \tag{4.4}$$

$$Q_2^\gamma(t) := \inf_{x,y\in[0,1],\, \Sigma(x,y,t)\cap\partial E(\gamma(t))=\emptyset} \frac{|\gamma^{\mathrm{sp}}(x,t)-\gamma(y,t)|^2}{|A^\gamma_{\min}(x,y,t)|}. \tag{4.5}$$

Note that g_γ is invariant under rescalings of γ, i.e.,

$$\vartheta > 0 \Rightarrow g_{\vartheta\gamma}(t) = g_\gamma(t), \quad t \in [0,\delta). \tag{4.6}$$

By assumption (2.11) it follows that

$$g_\gamma(0) > 0. \tag{4.7}$$

From [19], Proposition 4.4 it follows that g_γ is increasing in every time subinterval of $[0,\delta)$ where it is strictly less than $4\sqrt{3}$. In particular (4.7) implies

$$g_\gamma(t) \geqq \min\left(g_\gamma(0), 4\sqrt{3}\right), \quad t \in [0,\delta). \tag{4.8}$$

From (4.8) it follows that $\delta = T$, and (i) is proved.

Finally

$$\frac{1}{2}\frac{d}{dt}\left|E(\gamma(t))\right| = \int_{I(t)} \kappa_{\gamma(t)}\, ds = -\frac{2}{3}\pi, \tag{4.9}$$

which gives (4.2). □

4.1. Type I singularities. As usual in the blow-up analysis of type I singularities, let us define the parameter t as

$$t(\mathrm{t}) := T - e^{-2\mathrm{t}}, \quad \mathrm{t} \in \left[-\frac{1}{2}\log T, +\infty\right).$$

Given a point $p = (p_1, p_2) \in \mathbb{R}^2$ set also

$$\tilde{\gamma}^p(\mathrm{t}) := \frac{\gamma(t(\mathrm{t})) - p}{\sqrt{2(T - t(\mathrm{t}))}}, \quad \mathrm{t} \in \left[-\frac{1}{2}\log T, +\infty\right).$$

We let $\tilde{I}(\mathrm{t}) := [0, L(\tilde{\gamma}(\mathrm{t}))]$,

$$\tilde{\tau}(\mathrm{t}) := \tilde{\gamma}_s(\mathrm{t}), \quad \tilde{v}(\mathrm{t}) := \big(-\tilde{\tau}_2(\mathrm{t}), \tilde{\tau}_1(\mathrm{t})\big) = v_{\tilde{\gamma}(\mathrm{t})}, \quad \kappa_{\tilde{\gamma}(\mathrm{t})} := \left\langle \frac{\tilde{\gamma}_{xx}(\mathrm{t})}{|\tilde{\gamma}_x(\mathrm{t})|^2}, \tilde{v}(\mathrm{t}) \right\rangle, \tag{4.10}$$

$\tilde{\kappa}(x, \mathrm{t}) = \kappa_{\tilde{\gamma}(\mathrm{t})}(x)$, and

$$\tilde{\lambda}(\mathrm{t}) := \left\langle \frac{\tilde{\gamma}_{xx}(\mathrm{t})}{|\tilde{\gamma}_x(\mathrm{t})|^2}, \tilde{\tau}(\mathrm{t}) \right\rangle. \tag{4.11}$$

Notice that $\tilde{\gamma}$ satisfies the forced curvature flow equation

$$\tilde{\gamma}_{\mathrm{t}} = \sqrt{2\big(T - t(\mathrm{t})\big)}\gamma_t + \tilde{\gamma} = \tilde{\kappa}\tilde{v} + \tilde{\lambda}\tilde{\tau} + \tilde{\gamma}, \tag{4.12}$$

coupled with the boundary conditions $\tilde{\gamma}_2(0) = \tilde{\gamma}_2(1) = \dfrac{-p_2}{\sqrt{2\big(T - t(\mathrm{t})\big)}}$, and the usual Neumann boundary conditions

$$\frac{\tilde{\gamma}_x(0)}{|\tilde{\gamma}_x(0)|} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \quad \frac{\tilde{\gamma}_x(1)}{|\tilde{\gamma}_x(1)|} = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right). \tag{4.13}$$

As a consequence, by a direct computation (see [19], Formulae (2.7), (65), (66)) and using (4.12) we get

$$\begin{aligned} \tilde{\kappa}_{\mathrm{t}} &= \tilde{\kappa}_{ss} + \tilde{\lambda}\tilde{\kappa}_s + (\tilde{\kappa}^2 - 1)\tilde{\kappa}, \\ \tilde{\lambda}_{\mathrm{t}} &= \tilde{\lambda}_{ss} - \tilde{\lambda}\tilde{\lambda}_s - 2\tilde{\kappa}\tilde{\kappa}_s + (\tilde{\kappa}^2 - 1)\tilde{\lambda}. \end{aligned} \tag{4.14}$$

Therefore, letting $\tilde{w} := \tilde{\kappa}^2 + \tilde{\lambda}^2$, we find

$$\begin{aligned} \tilde{w}_{\mathrm{t}} &= \tilde{w}_{ss} - \tilde{\lambda}\tilde{w}_s + 2(\tilde{\kappa}^2 - 1)\tilde{w} - 2(\tilde{\kappa}_s^2 + \tilde{\lambda}_s^2) \\ &\leqq \tilde{w}_{ss} - \tilde{\lambda}\tilde{w}_s + 2(\tilde{\kappa}^2 - 1)\tilde{w}. \end{aligned} \tag{4.15}$$

In this section we prove the following result, whose mainly follows the lines in [19] (given for one triple junction only), except for the arguments in Step 8.

Theorem 4.2. *Assume that $\bar{\gamma}$ satisfies* (A) *and* (2.11). *If γ develops a type* I *singularity at $t = T$, then*

$$T = \frac{3|E(\bar{\gamma})|}{4\pi}, \quad \lim_{t \to T^-} |E(\gamma(t))| = 0, \tag{4.16}$$

and

$$\lim_{t \to T^-} L(\gamma(t)) = 0, \tag{4.17}$$

so that T is the extinction time of the evolution. Moreover:

- *There exists* $t_c \in (0, T)$ *such that* $\gamma(t)$ *is uniformly convex in* $[0, 1]$ *for any* $t \in [t_c, T)$.
- *There exists* $p \in \mathbb{R}^2$ *such that*

$$\lim_{\mathrm{t} \to +\infty} \|\tilde{\gamma}^p(\mathrm{t}) - \gamma^{\mathrm{h}}\|_{\mathscr{C}^2([0,1];\mathbb{R}^2)} = 0. \tag{4.18}$$

Proof. Let us assume that (3.11) holds. From [19], Theorem 6.23, it follows that, if we assume (2.11) and if in addition $\inf_{t \in [0,T)} L(\gamma(t)) > 0$, then γ cannot develop type I singularities at $t = T$. Therefore

$$\liminf_{t \to T^-} L(\gamma(t)) = 0. \tag{4.19}$$

Using (4.19) and the fact that $t \in [0, T) \to |E(\gamma(t))|$ is decreasing (see Proposition 4.1(ii)) it follows that $\lim_{t \to T^-} |E(\gamma(t))| = 0$. In particular, from (4.2) we have $T \leqq \frac{3|E(\bar{\gamma})|}{4\pi}$, and the equality holds if and only if $\lim_{t \to T^-} |E(\gamma(t))| = 0$. To prove (4.17), we observe that, as in the proof of Proposition 3.2 and since the constant c in that statement is independent of γ, given $a, b \in (0, T)$ with $a < b$, we have $L(\gamma(b)) \leqq cL(\gamma(a))$, with $c > 0$ independent of a and b. This observation, coupled with (4.19), proves (4.17).

From (4.17) and recalling the comparison argument used in the proof of Theorem 3.1, we deduce that for any $x \in [0, 1]$ there exists the limit $\lim_{t \to T^-} \gamma(x, t) \in \mathbb{R}^2$. Moreover, by (4.17) such a limit is independent of x. We can therefore define

$$p := \lim_{t \to T^-} \gamma(x, t) \in \mathbb{R}^2. \tag{4.20}$$

Set

$$\tilde{\gamma} := \tilde{\gamma}^p.$$

Recalling the notation in (4.10), thanks to (3.11)

$$|\tilde{\kappa}(x, \mathrm{t})| = \sqrt{2(T - t(\mathrm{t}))}\,|\kappa_\gamma(x, t(\mathrm{t}))| \leqq C, \quad \mathrm{t} \in \left[-\frac{1}{2}\log T, +\infty\right), \quad x \in [0, 1]. \tag{4.21}$$

We now divide the proof of the theorem into seven steps.

Step 1. We have

$$\tilde{\gamma}(0, \mathrm{t}), \tilde{\gamma}(1, \mathrm{t}) \in B_{\frac{2C}{\sqrt{3}}}(p), \quad \mathrm{t} \in \left[-\frac{1}{2}\log T, +\infty\right), \tag{4.22}$$

where $B_{\frac{2C}{\sqrt{3}}}(p)$ is the ball of radius $\frac{2C}{\sqrt{3}}$ centered at p.

Indeed, since $-\kappa_\gamma(0,\sigma) = \frac{\sqrt{3}}{2}|\gamma_t(0,\sigma)|$ for any $\sigma \in (0,T)$, using (4.21) we have

$$|\tilde\gamma(0,\mathrm{t})| = \frac{1}{\sqrt{2(T-t(\mathrm{t}))}}\left|\int_{t(\mathrm{t})}^{T}\gamma_t(0,\sigma)\,d\sigma\right|$$
$$\leqq \frac{2}{\sqrt{3}\sqrt{2(T-t(\mathrm{t}))}}\int_{t(\mathrm{t})}^{T}|\kappa_\gamma(0,\sigma)|\,d\sigma$$
$$\leqq \frac{2C}{\sqrt{3}\sqrt{2(T-t(\mathrm{t}))}}\int_{t(\mathrm{t})}^{T}\frac{1}{\sqrt{2(T-\sigma)}}\,d\sigma = \frac{2C}{\sqrt{3}}.$$

Since the same estimate holds for $|\tilde\gamma(1,\mathrm{t})|$, Step 1 is proved.

Step 2. We have

$$|E(\tilde\gamma(\mathrm{t}))| = \frac{4\pi}{3}, \quad \mathrm{t} \in \left[-\frac{1}{2}\log T, +\infty\right). \tag{4.23}$$

Indeed, from (4.2) and (4.16) it follows that $|E(\gamma(t))| = \frac{4\pi}{3}(T-t)$, and therefore (4.23) follows from the definition of $\tilde\gamma$.

Without loss of generality, from now on we assume $p = (0,0)$. We recall the so-called *rescaled monotonicity formula* (see [16], [19], Proposition 6.7):

$$\frac{d}{d\mathrm{t}}\int_{\tilde I(\mathrm{t})} e^{-\frac{|\tilde\gamma(\mathrm{t})|^2}{2}}\,ds = -\int_{\tilde I(\mathrm{t})} e^{-\frac{|\tilde\gamma(\mathrm{t})|^2}{2}}|\kappa_{\tilde\gamma(\mathrm{t})} + \langle\tilde\gamma(\mathrm{t}), \nu_{\tilde\gamma(\mathrm{t})}\rangle|^2\,ds =: -f(\mathrm{t}) \leqq 0. \tag{4.24}$$

Integrating (4.24) on $\left[-\frac{1}{2}\log T, +\infty\right)$ we get

$$\int_{-\frac{1}{2}\log T}^{+\infty} f(\mathrm{t})\,d\mathrm{t} = \int_{\tilde I(-\frac{1}{2}\log T)} e^{-\frac{|\tilde\gamma(-\frac{1}{2}\log T)|^2}{2}}\,ds = \frac{1}{\sqrt{2T}}\int_{I(0)} e^{-\frac{|\gamma(0)|^2}{4T}}\,ds < +\infty.$$

As a consequence, the nonnegative function f belongs to $L^1\left(\left[-\frac{1}{2}\log T, +\infty\right)\right)$. Since $\sum_{j=1}^{+\infty}\frac{1}{j} = +\infty$, we then have that for any sequence $\{\mathrm{t}_n\} \subset \left(-\frac{1}{2}\log T, +\infty\right)$ converging to $+\infty$, there exist a subsequence $\{\mathrm{t}_{n_j}\}$ and times $r_j \in [\mathrm{t}_{n_j}, \mathrm{t}_{n_j} + 1/j]$ such that

$$\lim_{j\to+\infty} f(r_j) = 0. \tag{4.25}$$

Assume now that

$$\sup_{\mathrm{t}\in[-\frac{1}{2}\log T, +\infty)} L(\tilde\gamma(\mathrm{t})) < +\infty. \tag{4.26}$$

Step 3. Weak convergence to γ^∞ in $W^{2,\infty}$ along a subsequence $\{r_{j_k}\}$.

From (4.21) and assumption (4.26) we have that

$$\sup_j \big[L\big(\tilde{\gamma}(r_j)\big) + \|\kappa_{\tilde{\gamma}(r_j)}\|_{L^\infty([0,1])}\big] < +\infty.$$

It follows that there exist a subsequence $\{r_{j_k}\}$ and a map

$$\gamma^\infty \in W^{2,\infty}([0,1];\mathbb{R}^2), \tag{4.27}$$

such that $\tilde{\gamma}(r_{j_k})$ converges to γ^∞ weakly in $W^{2,\infty}([0,1];\mathbb{R}^2)$ as $k \to +\infty$. In particular

$$\lim_{k\to+\infty} \|\tilde{\gamma}(r_{j_k}) - \gamma^\infty\|_{\mathscr{C}^1([0,1];\mathbb{R}^2)} = 0, \tag{4.28}$$

and

$$\lim_{k\to+\infty} \tilde{\gamma}_{xx}(r_{j_k}) = \gamma^\infty_{xx} \quad \text{weakly in } L^2([0,1];\mathbb{R}^2). \tag{4.29}$$

Hence from Steps 1, 2, and 3 and (4.26) it follows that:

(i) $\gamma^\infty(0), \gamma^\infty(1) \in B_{\frac{2C}{\sqrt{3}}}(0)$, and $\gamma^\infty(0)$, $\gamma^\infty(1)$ belong to the first coordinate axis.

(ii) $\dfrac{\gamma^\infty_x(0)}{|\gamma^\infty_x(0)|} = \left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$, $\dfrac{\gamma^\infty_x(1)}{|\gamma^\infty_x(1)|} = \left(\dfrac{1}{2}, -\dfrac{\sqrt{3}}{2}\right)$.

(iii) $|E(\gamma^\infty)| = \dfrac{2\pi}{3}$.

(iv) $L(\gamma^\infty) < +\infty$.

Moreover, as a consequence of (iii), and respectively of (ii), (iv) and (4.27), we have:

(v) $L(\gamma^\infty) > 0$.

(vi) κ_{γ^∞} is not identically zero.

Step 4. We have:

(vii) $\gamma^\infty_2(x) > 0$ for any $x \in (0,1)$.

(viii) γ^∞ is injective.

Indeed, from (4.6) and (4.8) we have

$$g_{\tilde{\gamma}}(\mathrm{t}) = g_\gamma\big(t(\mathrm{t})\big) \geqq \min\big(g_\gamma(0), 4\sqrt{3}\big), \quad \mathrm{t} \in \left[-\frac{1}{2}\log T, +\infty\right). \tag{4.30}$$

Moreover, since $g_{\tilde{\gamma}}(t)$ is defined as an infimum, it is upper semicontinuous, in the sense that

$$(4.31)\qquad \lim_{k\to+\infty}\|\tilde{\gamma}(r_{j_k})-\gamma^\infty\|_{\mathscr{C}^1([0,1];\mathbb{R}^2)}=0 \quad\Rightarrow\quad g_{\gamma^\infty}\geqq \limsup_{k\to+\infty} g_{\tilde{\gamma}}(r_{j_k}),$$

where g_{γ^∞} is (the constant) defined as in (4.3), where we substitute $\gamma(\cdot,t)$ with $\gamma^\infty(\cdot)$ on the right-hand side of (4.4). From (4.30) and (4.31) it follows that $g_{\gamma^\infty}\geqq \min\bigl(g_\gamma(0),4\sqrt{3}\bigr)$, and this implies (vii) and (viii).

As a consequence of (ii) and (viii) we have:

(ix) $\gamma^\infty(0)\neq\gamma^\infty(1)$.

Step 5. We have

$$(4.32)\qquad \kappa_{\gamma^\infty}+\langle\gamma^\infty,v_{\gamma^\infty}\rangle=0\quad\text{a.e. in }[0,1].$$

Indeed, from Fatou's Lemma and (4.25) we have

$$(4.33)\qquad \int_0^1 \liminf_{j\to+\infty}\Bigl[e^{-\frac{|\tilde{\gamma}(r_j)|^2}{2}}|\kappa_{\tilde{\gamma}(r_j)}+\langle\tilde{\gamma}(r_j),v_{\tilde{\gamma}(r_j)}\rangle|^2|\tilde{\gamma}_x(r_j)|\Bigr]\,dx\leqq \lim_{j\to+\infty}f(r_j)=0.$$

On the other hand, by (4.28) and (4.29), the left-hand side of (4.33) equals

$$(4.34)\qquad \int_0^{L(\gamma^\infty)} e^{-\frac{|\gamma^\infty|^2}{2}}|\kappa_{\gamma^\infty}+\langle\gamma^\infty,v_{\gamma^\infty}\rangle|^2\,ds,$$

and (4.32) follows.

By elliptic regularity [12] it follows that $\kappa_{\gamma^\infty}\in\mathscr{C}^0([0,1])$, hence $\gamma^\infty\in\mathscr{C}^2([0,1];\mathbb{R}^2)$, and (4.32) is valid everywhere in classical sense in $[0,1]$. Recalling Section 2.3, we deduce by uniqueness that

$$(4.35)\qquad \gamma^\infty=\gamma^{\mathrm{h}}.$$

Note that from (4.35) it follows that γ^∞ is independent of the subsequence $\{j_k\}$, hence (4.28) is valid for the whole sequence $\{r_j\}$, i.e.

$$(4.36)\qquad \lim_{j\to+\infty}\|\tilde{\gamma}(r_j)-\gamma^\infty\|_{\mathscr{C}^1([0,1];\mathbb{R}^2)}=0.$$

Step 6. We have

$$(4.37)\qquad \lim_{j\to+\infty}\|\tilde{\gamma}(\mathrm{t}_{n_j})-\gamma^{\mathrm{h}}\|_{\mathscr{C}^1([0,1];\mathbb{R}^2)}=0.$$

From Step 3 applied to the sequence $\{\tilde{\gamma}(\mathrm{t}_{n_j})\}$ in place of $\{\tilde{\gamma}(r_j)\}$, it follows that there exist a map

$$\tilde{\gamma}^\infty\in W^{2,\infty}([0,1];\mathbb{R}^2),$$

and a subsequence $\{n_{j_h}\}$ such that

$$\lim_{h \to +\infty} \|\tilde{\gamma}(\mathrm{t}_{n_{j_h}}) - \tilde{\gamma}^\infty\|_{\mathscr{C}^1([0,1];\mathbb{R}^2)} = 0, \tag{4.38}$$

and such that $\tilde{\gamma}^\infty$ satisfies properties (i)–(ix) listed in Steps 3 and 4.

In order to show (4.37), it is enough to prove that

$$\tilde{\gamma}^\infty = \gamma^{\mathrm{h}}. \tag{4.39}$$

Using (4.38), (4.36) and the inequality

$$\begin{aligned} \|\tilde{\gamma}^\infty - \gamma^{\mathrm{h}}\|_{\mathscr{C}^0([0,1];\mathbb{R}^2)} \leqq \|\tilde{\gamma}^\infty - \tilde{\gamma}(\mathrm{t}_{n_{j_h}})\|_{\mathscr{C}^0([0,1];\mathbb{R}^2)} + \|\tilde{\gamma}(\mathrm{t}_{n_{j_h}}) - \tilde{\gamma}(r_{j_h})\|_{\mathscr{C}^0([0,1];\mathbb{R}^2)} \\ + \|\tilde{\gamma}(r_{j_h}) - \gamma^{\mathrm{h}}\|_{\mathscr{C}^0([0,1];\mathbb{R}^2)}, \end{aligned}$$

to prove (4.39) it is sufficient to show that

$$\lim_{j \to +\infty} \|\tilde{\gamma}(r_j) - \tilde{\gamma}(\mathrm{t}_{n_j})\|_{\mathscr{C}^0([0,1];\mathbb{R}^2)} = 0. \tag{4.40}$$

In order to prove (4.40), we recall that $\tilde{\kappa}(x,\mathrm{t})$ is uniformly bounded for all (x,t) by (4.21) and, as a consequence, $\tilde{\lambda}(x,\mathrm{t})$ is also uniformly bounded by (4.14) and (4.15) as in [19], p. 264. Hence, using also (4.12) and (4.24),

$$\begin{aligned} \|\tilde{\gamma}(r_j) - \tilde{\gamma}(\mathrm{t}_{n_j})\|_{\mathscr{C}^0([0,1];\mathbb{R}^2)} \leqq \int_{\mathrm{t}_{n_j}}^{r_j} \int_0^1 |\tilde{\gamma}_{\mathrm{t}}|\, dx \leqq \int_{\mathrm{t}_{n_j}}^{r_j} \int_0^1 (|\tilde{\kappa}| + |\tilde{\lambda}| + |\tilde{\gamma}|)\, dx \\ \leqq C|r_j - \mathrm{t}_{n_j}| \leqq \frac{C}{j}, \end{aligned}$$

which gives (4.40) and proves Step 6.

From (4.37) and [19], Proposition 6.16, we have the improved convergence

$$\lim_{j \to +\infty} \|\tilde{\gamma}(\mathrm{t}_{n_j}) - \gamma^{\mathrm{h}}\|_{\mathscr{C}^2([0,1];\mathbb{R}^2)} = 0. \tag{4.41}$$

Since the sequence $\{\mathrm{t}_n\}$ is arbitrary we deduce

$$\lim_{\mathrm{t} \to +\infty} \|\tilde{\gamma}(\mathrm{t}) - \gamma^{\mathrm{h}}\|_{\mathscr{C}^2([0,1];\mathbb{R}^2)} = 0. \tag{4.42}$$

Eventually, we observe that, since γ_2^∞ is uniformly concave in $[0,1]$ (see Section 2.3), from (4.41) we deduce that $\gamma(t_c)$ becomes uniformly convex for some $t_c \in (0,T)$. From the results proved in [22], Lemma 3.3, it follows that $\gamma(t)$ remains uniformly convex in $[t_c, T)$ (this last assertion also follows from (3.12) and (2.8) using the maximum principle).

Step 7. Assume now that (4.26) does not hold, that is, there exists a sequence $\{\mathrm{t}_n\}$ converging to $+\infty$ such that

$$\lim_{n\to+\infty} L\big(\tilde{\gamma}(\mathrm{t}_n)\big) = +\infty. \tag{4.43}$$

Reasoning as in Step 1, there exist a subsequence $\{\mathrm{t}_{n_j}\}$ and times $r_j \in [\mathrm{t}_{n_j}, \mathrm{t}_{n_j} + 1/j]$ such that (4.25) holds. Moreover from (4.21) and $r_j - \mathrm{t}_{n_j} \leqq 1/j$, and from (3.9) and (4.43) we obtain

$$\lim_{j\to+\infty} L\big(\tilde{\gamma}(r_j)\big) = +\infty. \tag{4.44}$$

If we parametrize $\tilde{\gamma}(r_j)$ by arclength on $\big[0, L\big(\tilde{\gamma}(r_j)\big)\big]$, and we pass to the limit as in Step 3 as $j \to +\infty$, we get that there exists a subsequence $\{r_{j_k}\}$ such that $\{\tilde{\gamma}(r_{j_k})\}$ converges weakly in $W^{2,2}_{\mathrm{loc}}\big([0,+\infty);\mathbb{R}^2\big)$ (so that (4.28) and (4.29) hold with $\mathscr{C}^1_{\mathrm{loc}}\big([0,+\infty);\mathbb{R}^2\big)$ in place of $\mathscr{C}^1([0,1];\mathbb{R}^2)$ and $L^2_{\mathrm{loc}}\big([0,+\infty);\mathbb{R}^2\big)$ in place of $L^2([0,1];\mathbb{R}^2)$ respectively) to a curve γ^∞ of infinite length which, arguing as in Steps 4 and 5, has the following properties:

(a) $\gamma_2^\infty(0) = 0$, $\gamma_2^\infty(s) > 0$ for any $s \in (0,+\infty)$.

(b) $\gamma_s^\infty(0) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

(c) γ^∞ is injective (by using (4.30)).

(d) γ^∞ solves (4.32) almost everywhere in $[0,+\infty)$.

By elliptic regularity $\gamma^\infty \in \mathscr{C}^\infty\big([0,+\infty);\mathbb{R}^2\big)$ and solves (4.32) in classical sense. Then by the results in [7] and [22] it follows that $\gamma^\infty([0,+\infty)$ is contained in a curve of Abresch–Langer [1]. In view of the Neumann condition (b) and the properties of the curves of Abresch–Langer, it then follows that

$$\gamma^\infty(s) = \left(\frac{s}{2}, \frac{\sqrt{3}s}{2}\right), \quad s \in [0,+\infty).$$

Similarly, if we parametrize $\tilde{\gamma}(r_j)$ by arclength on $\big[-L\big(\tilde{\gamma}(r_j)\big), 0\big]$, we find a subsequence $\{\tilde{\gamma}(r_{j_{k_\ell}})\}$ of $\{\tilde{\gamma}(r_{j_k})\}$ converging to a curve $\gamma^\infty \in \mathscr{C}^\infty\big((-\infty,0];\mathbb{R}^2\big)$ of infinite length satisfying (a), (c), (d), $\gamma_s^\infty(0) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, and contained in a curve of Abresch–Langer. Hence necessarily

$$\gamma^\infty(s) = \left(\frac{s}{2}, -\frac{\sqrt{3}s}{2}\right), \quad s \in (-\infty,0].$$

We now reach a contradiction since, being the convergence of $\{\tilde{\gamma}(r_{j_{k_\ell}})\}$ in $\mathscr{C}^1_{\mathrm{loc}}$, it follows that $\tilde{\gamma}(r_{j_{k_\ell}})$ is not injective for ℓ sufficiently large. Indeed, provided $\ell \in \mathbb{N}$ is such that

$$\|\tilde{\gamma}(s, r_{j_{k_\ell}}) - \gamma^\infty(s)\|_{\mathscr{C}^1([0,1];\mathbb{R}^2)} + \big\|\tilde{\gamma}\big(L\big(\tilde{\gamma}(r_{j_{k_\ell}})\big) - s, r_{j_{k_\ell}}\big) - \tilde{\gamma}^\infty(-s)\big\|_{\mathscr{C}^1([0,1];\mathbb{R}^2)} \leqq \frac{1}{2},$$

recalling the boundary conditions in (4.13), we have that there exist $s_1, s_2 \in [0,1]$ such that

$$\tilde{\gamma}(s_1, r_{j_{k_\ell}}) = \tilde{\gamma}\big(L\big(\tilde{\gamma}(r_{j_{k_\ell}})\big) - s_2, r_{j_{k_\ell}}\big).$$

Hence (4.26) necessarily holds, and the proof of the theorem is complete. □

5. Embedded non-convex initial data: type II singularities

This section is devoted to the proof of the following result:

Theorem 5.1. *Assume that $\bar{\gamma}$ satisfies* (A) *and* (2.11). *Then γ cannot develop type* II *singularities at $t = T$.*

Proof. Let us assume by contradiction that γ develops a type II singularity at $t = T$. We employ a rescaling procedure originally due to R. Hamilton (see [2]). Let us choose as in [19], Section 7.1, a sequence $\{(x_n, t_n)\} \subset [0,1] \times [0,T)$ satisfying the following properties:

- $t_n \in [0, T - 1/n)$ and $t_n < t_{n+1}$ for any $n \in \mathbb{N}$.
- Letting

$$\mu_n := |\kappa_\gamma(x_n, t_n)|, \quad n \in \mathbb{N},$$

we have $0 < \mu_n < \mu_{n+1}$ and $\lim_{n\to+\infty} \mu_n = +\infty$.

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$$\lim_{n\to+\infty} \mu_n \sqrt{T - 1/n - t_n} = +\infty, \tag{5.1}$$

and for any $n \in \mathbb{N}$

$$\mu_n \sqrt{T - 1/n - t_n} = \max_{t\in[0,T-1/n]} \big(\|\kappa_{\gamma(t)}\|_{L^\infty([0,1])} \sqrt{T - 1/n - t}\big). \tag{5.2}$$

Note that the maximum in (5.2) is attained in $[0, T - 1/n)$ by (5.1). Note also that

$$\lim_{n\to+\infty} -\mu_n^2 t_n = -\infty, \quad \lim_{n\to+\infty} \mu_n^2 (T - t_n) = +\infty. \tag{5.3}$$

Let us define the parameter t as

$$t(\mathrm{t}) := t_n + \mathrm{t}/\mu_n^2, \quad \mathrm{t} \in \big[-\mu_n^2 t_n, \mu_n^2 (T - t_n)\big),$$

and the curves γ_n as

$$\gamma_n(x, \mathrm{t}) := \mu_n\big(\gamma\big(x, t(\mathrm{t})\big) - \gamma(x_n, t_n)\big), \quad x \in [0,1],\ \mathrm{t} \in \big[-\mu_n^2 t_n, \mu_n^2 (T - t_n)\big).$$

We have

$$\gamma_n(x_n, 0) = (0,0), \quad |\kappa_{\gamma_n}(x_n, 0)| = 1, \quad n \in \mathbb{N}. \tag{5.4}$$

From (5.2) it follows as in [19], Section 7, that for every $\varepsilon, \omega > 0$ there exists $\bar{n} \in \mathbb{N}$ such that

$$\|\kappa_{\gamma_n(\mathrm{t})}\|_{L^\infty([0,1])} \leqq 1 + \varepsilon, \quad n \geqq \bar{n},\ \mathrm{t} \in [-\mu_n^2 t_n, \omega]. \tag{5.5}$$

We now divide the proof of the theorem into nine steps.

Step 1. We have

$$\lim_{n \to +\infty} L(\gamma_n(\mathrm{t})) = +\infty, \quad \mathrm{t} \in \mathbb{R}. \tag{5.6}$$

Indeed, this is obvious if T is not the extinction time, since in that case $\inf_{t \in [0,T)} L(\gamma(t)) > 0$. If T is the extinction time, namely $0 = \lim_{t \to T^-} L(\gamma(t)) = \lim_{t \to T^-} |E(\gamma(t))|$, by the isoperimetric inequality and taking into account that γ satisfies (2.3), it follows that there exists an absolute constant $c > 0$ such that $L(\gamma(t)) \geqq c\sqrt{|E(\gamma(t))|}$ for all $t \in [0, T)$. Hence, to prove (5.6) it is enough to show that

$$\lim_{n \to +\infty} |E(\gamma_n(\mathrm{t}))| = +\infty, \quad \mathrm{t} \in \mathbb{R}. \tag{5.7}$$

Recalling (4.2), we have

$$|E(\gamma_n(\mathrm{t}))| = \mu_n^2 |E(\gamma(t(\mathrm{t})))| = \frac{4}{3}\pi\mu_n^2 (T - t(\mathrm{t})), \quad \mathrm{t} \in [-\mu_n^2 t_n, \mu_n^2 (T - t_n)).$$

In particular $|E(\gamma_n(0))| = (4/3)\pi\mu_n^2(T - t_n)$, hence $\lim_{n \to +\infty} |E(\gamma_n(0))| = +\infty$ by (5.3). Then Step 1 follows, since $|E(\gamma_n(\mathrm{t}))| = |E(\gamma_n(0))| - (4/3)\pi\mathrm{t}$ for any $\mathrm{t} \in [-\mu_n^2 t_n, \mu_n^2(T - t_n))$.

Before passing to the next step we need some preparation. Given

$$\mathrm{t} \in [-\mu_n^2 t_n, \mu_n^2(T - t_n)),$$

we now reparametrize the curves $\gamma_n(\mathrm{t})$ by arclength and, performing a suitable translation in the parameter space, we obtain curves

$$\hat{\gamma}_n(\mathrm{t}) : [a_n(\mathrm{t}), b_n(\mathrm{t})] \to \mathbb{R}^2,$$

with $a_n(\mathrm{t}) \leqq 0 \leqq b_n(\mathrm{t})$, and $b_n(\mathrm{t}) - a_n(\mathrm{t}) = L(\gamma_n(\mathrm{t}))$.

Thanks to (5.6), we have

$$\lim_{n \to +\infty} (b_n(\mathrm{t}) - a_n(\mathrm{t})) = +\infty, \quad \mathrm{t} \in \mathbb{R}. \tag{5.8}$$

Without loss of generality we assume

$$\hat{\gamma}_n(0,0) = \gamma_n(x_n, 0) = (0,0). \tag{5.9}$$

We can also assume that there exists a subsequence $\{n_j\}$ such that

$$\lim_{j\to+\infty} a_{n_j}(0) =: a_\infty \in [-\infty, 0], \quad \lim_{j\to+\infty} b_{n_j}(0) =: b_\infty \in [0, +\infty]. \tag{5.10}$$

Note that by (5.8) we have that if $a_\infty \in (-\infty, 0]$ (resp. $b_\infty \in [0, +\infty)$) then $b_\infty = +\infty$ (resp. $a_\infty = -\infty$).

We now choose the starting point of the reparametrization (still keeping the notation $\hat{\gamma}_n$) as follows: If $b_\infty = +\infty$ we set $a_{n_j}(\mathrm{t}) := a_{n_j}(0)$ for any $\mathrm{t} \in \mathbb{R}$; if $b_\infty \in [0, +\infty)$ we set $b_{n_j}(\mathrm{t}) := b_{n_j}(0)$ for any $\mathrm{t} \in \mathbb{R}$. Hence in both cases

$$\lim_{j\to+\infty} a_{n_j}(\mathrm{t}) =: a_\infty, \quad \lim_{j\to+\infty} b_{n_j}(\mathrm{t}) =: b_\infty, \quad \mathrm{t} \in \mathbb{R}. \tag{5.11}$$

If $a_\infty \in (-\infty, 0]$ (resp. $b_\infty \in [0, +\infty)$) we set $I_\infty := [a_\infty, +\infty)$ (resp. $I_\infty := (-\infty, b_\infty]$); if $|a_\infty| = b_\infty = +\infty$ we set $I_\infty := \mathbb{R}$. Observe that $0 \in I_\infty$.

Exploiting also (5.9), the proof of the next step is the same as in [19], Proposition 7.1, using also (5.8), (5.5) and (5.4).

Step 2. The sequence $\{\hat{\gamma}_{n_j}\}$ admits a subsequence $\{\hat{\gamma}_{n_{j_h}}\}$ converging in $\mathscr{C}^2_{\mathrm{loc}}(I_\infty \times \mathbb{R}; \mathbb{R}^2)$ to an embedded curvature evolution $\gamma_\infty \in \mathscr{C}^\infty(I_\infty \times \mathbb{R}; \mathbb{R}^2)$ with

$$\begin{gathered} L\big(\gamma_\infty(\mathrm{t})\big) = +\infty, \quad \mathrm{t} \in \mathbb{R}, \\ \gamma_\infty(0,0) = (0,0), \\ \|\kappa_{\gamma_\infty}\|_{L^\infty(I_\infty\times\mathbb{R})} = 1 = |\kappa_{\gamma_\infty}(0,0)|. \end{gathered} \tag{5.12}$$

Moreover:

- If $I_\infty = [a_\infty, +\infty)$ then $\gamma_{\infty s}(a_\infty, \mathrm{t}) = (1/2, \sqrt{3}/2)$ for all $\mathrm{t} \in \mathbb{R}$, and

$$\gamma_{\infty 2}(s, \mathrm{t}) \geqq \gamma_{\infty 2}(a_\infty, \mathrm{t}), \quad s \in I_\infty, \ \mathrm{t} \in \mathbb{R}.$$

- If $I_\infty = (-\infty, b_\infty]$ then $\gamma_{\infty s}(b_\infty, \mathrm{t}) = (1/2, -\sqrt{3}/2)$ for all $\mathrm{t} \in \mathbb{R}$, and

$$\gamma_{\infty 2}(s, \mathrm{t}) \geqq \gamma_{\infty 2}(b_\infty, \mathrm{t}), \quad s \in I_\infty, \ \mathrm{t} \in \mathbb{R}.$$

Note that the $\mathscr{C}^2_{\mathrm{loc}}(I_\infty \times \mathbb{R}; \mathbb{R}^2)$-convergence can be improved to $\mathscr{C}^\infty_{\mathrm{loc}}(I_\infty \times \mathbb{R}; \mathbb{R}^2)$ [11], since the curves $\hat{\gamma}_n$ evolve by curvature and have a uniform L^∞-bound on their curvature.

Step 3. For all $\mathrm{t} \in \mathbb{R}$ we have $\kappa_{\gamma_\infty}(s, \mathrm{t}) \neq 0$ for all $s \in I_\infty$.

We follow [2], Theorem 7.7. Write for simplicity

$$J_h(\mathrm{t}) := [a_{n_{j_h}}(\mathrm{t}), b_{n_{j_h}}(\mathrm{t})], \quad \hat{\kappa}_h(s,\mathrm{t}) = \kappa_{\hat{\gamma}_{n_{j_h}}}(s,\mathrm{t}), \quad z_h(\mathrm{t}) := \{s \in J_h(\mathrm{t}) : \hat{\kappa}_h(s,\mathrm{t}) = 0\}.$$

For all $M > 0$, recalling (3.13), we have

$$-2\int_{-M}^{M} \sum_{s\in z_h(\mathrm{t})} |\partial_s \hat{\kappa}_h|\, d\mathrm{t} = \int_{-M}^{M} \frac{d}{d\mathrm{t}} \int_{J_h(\mathrm{t})} |\hat{\kappa}_h|\, ds\, d\mathrm{t} \tag{5.13}$$

$$= \int_{J_h(M)} |\hat{\kappa}_h(s,M)|\, ds - \int_{J_h(-M)} |\hat{\kappa}_h(s,-M)|\, ds.$$

Using the invariance of $\int_{I(t)} |\kappa_\gamma(\cdot,t)|\, ds$ under rescalings and writing

$$\gamma_h := \gamma_{n_{j_h}}, \quad t_h := t_{n_{j_h}}, \quad \mu_h := \mu_{n_{j_h}},$$

from (5.13) we then obtain

$$-2\int_{-M}^{M} \sum_{s\in z_h(\mathrm{t})} |\partial_s \hat{\kappa}_h|\, d\mathrm{t} \tag{5.14}$$

$$= \int_{I\left(t_h+\frac{M}{\mu_h^2}\right)} |\kappa_{\gamma_h}(s, t_h + M/\mu_h^2)|\, ds - \int_{I\left(t_h-\frac{M}{\mu_h^2}\right)} |\kappa_{\gamma_h}(s, t_h - M/\mu_h^2)|\, ds.$$

In view of Proposition 3.4 the function $t \to \int_{I(t)} |\kappa_{\gamma(t)}|\, ds$ is nonincreasing, hence it admits a finite limit as $t \to T^-$. In particular,

$$\lim_{h\to+\infty} \int_{I\left(t_h+\frac{M}{\mu_h^2}\right)} |\kappa_{\gamma_h}(s, t_h + M/\mu_h^2)|\, ds = \lim_{h\to+\infty} \int_{I\left(t_h-\frac{M}{\mu_h^2}\right)} |\kappa_{\gamma_h}(s, t_h - M/\mu_h^2)|\, ds.$$

It then follows from (5.14) that

$$\lim_{h\to+\infty} \int_{-M}^{M} \sum_{s\in z_h(\mathrm{t})} |\partial_s \hat{\kappa}_h|\, d\mathrm{t} = 0. \tag{5.15}$$

From (5.15) and Fatou's Lemma we deduce that

$$0 = \liminf_{h\to+\infty} \sum_{s\in z_h(\mathrm{t})} |\partial_s \hat{\kappa}_h(s,\mathrm{t})| \quad \text{for a.e. } \mathrm{t} \in [-M, M]. \tag{5.16}$$

Since (5.16) holds for any $M > 0$, and all quantities involved are continuous with respect to t, we obtain

$$0 = \liminf_{h\to+\infty} \sum_{s\in z_h(\mathrm{t})} |\partial_s \hat{\kappa}_h(s,\mathrm{t})|, \quad \mathrm{t} \in \mathbb{R}. \tag{5.17}$$

On the other hand, the $\mathscr{C}^2_{\mathrm{loc}}(I_\infty \times \mathbb{R}; \mathbb{R}^2)$-convergence of $\hat{\gamma}_h$ to γ_∞ given in Step 2 implies that

$$\liminf_{h\to+\infty} \sum_{s\in z_h(\mathrm{t})} |\partial_s \hat{\kappa}_h(s,\mathrm{t})| \geqq \sum_{s\in I_\infty : \kappa_{\gamma_\infty}(s,\mathrm{t})=0} |\partial_s \kappa_{\gamma_\infty}(s,\mathrm{t})|, \quad \mathrm{t} \in \mathbb{R}. \tag{5.18}$$

Since the right-hand side of (5.18) is nonnegative, from (5.17) we deduce

$$0 = \sum_{s \in I_\infty : \kappa_{\gamma_\infty}(s, t) = 0} |\partial_s \kappa_{\gamma_\infty}(s, t)|, \quad t \in \mathbb{R}.$$

It follows that for any $t \in \mathbb{R}$ we have

$$\{s \in I_\infty : \kappa_{\gamma_\infty}(s, t) = 0, \partial_s \kappa_{\gamma_\infty}(s, t) \neq 0\} = \emptyset.$$

On the other hand, γ_∞ evolves by curvature (see Step 2), and therefore, from the results of [4], if there exists $(s, t) \in I_\infty \times \mathbb{R}$ such that $\kappa_{\gamma_\infty}(s, t) = 0$ and $\partial_s \kappa_{\gamma_\infty}(s, t) = 0$, then $\gamma_\infty(\cdot, t)$ is linear, hence $\gamma_\infty(\cdot, \cdot)$ is linear. Since this is in contradiction with (5.12), the proof of Step 3 is concluded.

Step 4. $I_\infty \neq \mathbb{R}$.

Indeed, assume by contradiction that $I_\infty = \mathbb{R}$. From Step 3, reasoning as in [2], pp. 512–513, it follows that γ_∞ is the so-called *grim reaper*. For the grim reaper the function $Q_1^{\gamma_\infty} : \mathbb{R} \to (0, +\infty)$ defined on the right-hand side of (4.4) (with $[0, 1]$ replaced by I_∞) is identically zero. On the other hand, from (4.6) and arguing as in Step 4 of the proof of Theorem 4.2 we have that $g_{\hat{\gamma}_h} : \left[-\mu_h^2 t_h, \mu_h^2 (T - t_h)\right) \to (0, +\infty)$ is bounded from below by a positive constant uniformly with respect to $h \in \mathbb{N}$. Recall now that the sequence $\{\hat{\gamma}_h\}$ converges in $\mathscr{C}^2_{\mathrm{loc}}(I_\infty \times \mathbb{R}; \mathbb{R}^2)$ to γ_∞ and that we have (similarly to the inequality in (4.31))

$$(5.19) \qquad Q_1^{\gamma_\infty}(t) \geqq \limsup_{h \to +\infty} Q_1^{\hat{\gamma}_h}(t) \geqq \limsup_{h \to +\infty} g_{\hat{\gamma}_h}(t), \quad t \in \mathbb{R}.$$

Then (5.19) is in contradiction with $Q_1^{\gamma_\infty} \equiv 0$, and the proof of Step 4 is concluded.

Thanks to Step 3 we can consider only two cases: either $\kappa_{\gamma_\infty}(s, t) < 0$ for any $(s, t) \in I_\infty \times \mathbb{R}$, or $\kappa_{\gamma_\infty}(s, t) > 0$ for any $(s, t) \in I_\infty \times \mathbb{R}$. Let us first assume

$$(5.20) \qquad \kappa_{\gamma_\infty}(s, t) < 0, \quad (s, t) \in I_\infty \times \mathbb{R}.$$

Recalling our conventions (see Remark 2.2), inequality (5.20) implies that $\gamma_\infty(\cdot, t)$ is a convex curve.

From Step 4 we have that either a_∞ is finite or b_∞ is finite. We assume that $a_\infty \in (-\infty, 0]$, the case $b_\infty \in [0, +\infty)$ being analogous. Therefore we have

$$I_\infty = [a_\infty, +\infty).$$

Observe that from (5.11) we have

$$(5.21) \qquad \gamma_{\infty 2}(a_\infty, t) = \gamma_{\infty 2}(a_\infty, 0), \quad t \in \mathbb{R}.$$

Recall also (see Step 2) that

$$(5.22) \qquad \partial_s \gamma_\infty(a_\infty, t) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \quad t \in \mathbb{R}.$$

Step 5. We have

$$\int_{I_\infty} \kappa_{\gamma_\infty}(s,\mathrm{t})\,ds \in [-\pi/3, 0), \quad \mathrm{t} \in \mathbb{R}. \tag{5.23}$$

Indeed, if by contradiction there exists $\mathrm{t} \in \mathbb{R}$ such that the left-hand side of (5.23) is less than $-\pi/3$, then thanks to (5.20) and the Neumann boundary condition (5.22), the curve $\gamma_\infty(\cdot,\mathrm{t})$ has another intersection (different from $\gamma_\infty(a_\infty,0)$) with the horizontal axis ℓ passing from $\gamma_\infty(a_\infty,0)$. This implies $Q_2^{\gamma_\infty} \equiv 0$, where $Q_2^{\gamma_\infty}$ is defined as in (4.4) (with $[0,1]$ replaced by I_∞, and $\gamma_\infty^{\mathrm{sp}}$ is now the specular of γ_∞ with respect to ℓ). This leads to a contradiction, as in Step 4.

In particular, the convex curve $\gamma_\infty(\cdot,\mathrm{t})$ can be written as the graph of a strictly concave smooth function $y = y(x,\mathrm{t})$, where $(x,\mathrm{t}) \in [\gamma_{\infty 1}(a_\infty,\mathrm{t}), +\infty) \times \mathbb{R}$.

Let $\theta(x,\mathrm{t}) := \tan^{-1}(y_x(x,\mathrm{t})) \in (0,\pi/3]$ be the angle that the tangent vector to $\gamma_\infty(\cdot,\mathrm{t})$ makes with the first coordinate axis.

Step 6. We have

$$\partial_\mathrm{t}\kappa_{\gamma_\infty}(s,\mathrm{t}) \leqq 0, \quad (s,\mathrm{t}) \in I_\infty \times \mathbb{R}. \tag{5.24}$$

Write for simplicity

$$\kappa_{\gamma_\infty} = \kappa. \tag{5.25}$$

Recalling that γ_∞ evolves by curvature, the evolution of κ in the (θ,t)-coordinates reads as follows (see [11]):

$$\partial_\mathrm{t}\kappa = \kappa^2\kappa_{\theta\theta} + \kappa^3. \tag{5.26}$$

Let $\mathrm{t}_1 \in \mathbb{R}$ and define $h := \kappa + 2(\mathrm{t}-\mathrm{t}_1)\partial_\mathrm{t}\kappa$. We have $h(\theta,\mathrm{t}_1) < 0$ for any $\theta \in (0,\pi/3]$, and

$$h_\mathrm{t} = \kappa^2 h_{\theta\theta} + \left(\kappa^2 + \frac{2\partial_\mathrm{t}\kappa}{\kappa}\right)h. \tag{5.27}$$

Moreover, from $\partial_s = \kappa\partial_\theta$ and (2.8) we have that h satisfies the boundary condition

$$h_\theta\left(\frac{\pi}{3},\mathrm{t}\right) = \frac{1}{\sqrt{3}}h\left(\frac{\pi}{3},\mathrm{t}\right), \quad \mathrm{t} \in \mathbb{R}. \tag{5.28}$$

We now observe that the remaining Dirichlet boundary condition for h reads as

$$h(0,\mathrm{t}) = 0, \quad \mathrm{t} \in \mathbb{R}. \tag{5.29}$$

Indeed, from (5.20) and (5.23) and the Lipschitz continuity of κ in s, which is uniform with respect to t (this follows from (5.12) and the interior regularity estimates in [9]), we have

$$\lim_{\theta\to 0^+} \kappa(\theta,\mathrm{t}) = 0, \quad \mathrm{t} \in \mathbb{R}. \tag{5.30}$$

Using again [9] we deduce

$$\lim_{\theta\to 0^+} \kappa_\theta(\theta, \mathrm{t}) = \lim_{\theta\to 0^+} \kappa_{\theta\theta}(\theta, \mathrm{t}) = 0, \quad \mathrm{t} \in \mathbb{R}. \tag{5.31}$$

Then (5.29) follows from (5.30) and (5.31).

By (5.27), (5.28), (5.29) and the maximum principle it then follows $h(\theta, \mathrm{t}) \leqq 0$ for all $\theta \in (0, \pi/3]$ and $\mathrm{t} \geqq \mathrm{t}_1$, hence

$$\partial_\mathrm{t}\kappa \leqq -\frac{\kappa}{2(\mathrm{t} - \mathrm{t}_1)}, \quad \mathrm{t} > \mathrm{t}_1,$$

which implies (5.24), by letting $\mathrm{t}_1 \to -\infty$.

Step 7. We have

$$\partial_\mathrm{t}\kappa_{\gamma_\infty}(s, \mathrm{t}) = 0, \quad (s, \mathrm{t}) \in I_\infty \times \mathbb{R}. \tag{5.32}$$

Let us adopt the notation in (5.25), and define $Z(\mathrm{t}) := \int_0^{\pi/3} \partial_\mathrm{t}\big(\log(-\kappa)\big)\, d\theta$. Notice that $Z \geqq 0$ since $\partial_\mathrm{t}\kappa \leqq 0$ by Step 6 and $\kappa < 0$ by (5.20). Step 7 will be proved if we show that

$$Z \equiv 0. \tag{5.33}$$

Following [2], Section 8, we compute

$$\kappa_{\mathrm{tt}} = (\kappa^2\kappa_{\theta\theta} + \kappa^3)_\mathrm{t} = \kappa^2(\kappa_{\theta\theta\mathrm{t}} + \kappa_t) + 2\frac{(\kappa_\mathrm{t})^2}{\kappa}. \tag{5.34}$$

Using (5.34) and integrating by parts we get

$$\begin{aligned} Z'(\mathrm{t}) &= \int_0^{\pi/3} \partial_\mathrm{t}\left(\frac{\kappa_\mathrm{t}}{\kappa}\right) d\theta \\ &= \int_0^{\pi/3} \frac{\kappa_{\mathrm{tt}}}{\kappa}\, d\theta - \int_0^{\pi/3} \frac{\kappa_\mathrm{t}(\kappa^2\kappa_{\theta\theta} + \kappa^3)}{\kappa^2}\, d\theta \\ &= \int_0^{\pi/3} \kappa(\kappa_{\theta\theta\mathrm{t}} + \kappa_\mathrm{t}) + 2\frac{(\kappa_\mathrm{t})^2}{\kappa^2}\, d\theta - \int_0^{\pi/3} \frac{\kappa_\mathrm{t}(\kappa^2\kappa_{\theta\theta} + \kappa^3)}{\kappa^2}\, d\theta \\ &= \int_0^{\pi/3} \kappa\kappa_{\theta\theta\mathrm{t}} - \kappa_\mathrm{t}\kappa_{\theta\theta} + 2\frac{(\kappa_\mathrm{t})^2}{\kappa^2}\, d\theta \\ &= \kappa(\pi/3, \mathrm{t})\kappa_{\theta\mathrm{t}}(\pi/3, \mathrm{t}) - \kappa_\theta(\pi/3, \mathrm{t})\kappa_\mathrm{t}(\pi/3, \mathrm{t}) + 2\int_0^{\pi/3} \frac{(\kappa_\mathrm{t})^2}{\kappa^2}\, d\theta. \end{aligned} \tag{5.35}$$

We now observe that from $\kappa_s = \kappa\kappa_\theta$ and from (2.9) we have

$$\kappa_\theta(\pi/3, \mathrm{t}) = \frac{\kappa(\pi/3, \mathrm{t})}{\sqrt{3}}, \quad \mathrm{t} \in \mathbb{R}.$$

Differentiating this relation with respect to t we obtain

$$\kappa(\pi/3, \mathrm{t})\kappa_{\theta\mathrm{t}}(\pi/3, \mathrm{t}) = \kappa_\theta(\pi/3, \mathrm{t})\kappa_\mathrm{t}(\pi/3, \mathrm{t}), \quad \mathrm{t} \in \mathbb{R}. \tag{5.36}$$

From (5.35), (5.36) and the Schwarz inequality we deduce

$$Z'(\mathrm{t}) = 2\int_0^{\pi/3} \frac{(\kappa_\mathrm{t})^2}{\kappa^2}\,d\theta = 2\int_0^{\pi/3} \big(\partial_\mathrm{t}\big(\log(-\kappa)\big)\big)^2\,d\theta \geqq \frac{6Z^2(\mathrm{t})}{\pi}.$$

Assume now that $Z(\mathrm{t}_1) > 0$ for some $\mathrm{t}_1 \in \mathbb{R}$. It follows that $Z(\mathrm{t}) \geqq Z(\mathrm{t}_1) > 0$ for all $\mathrm{t} \geqq \mathrm{t}_1$, which implies

$$Z(\mathrm{t}_1) \leqq \frac{1}{\dfrac{1}{Z(\mathrm{t}_2)} + \dfrac{6}{\pi}(\mathrm{t}_2 - \mathrm{t}_1)} \leqq \frac{\pi}{6(\mathrm{t}_2 - \mathrm{t}_1)}$$

for all $\mathrm{t}_2 \geqq \mathrm{t}_1$. Letting $\mathrm{t}_2 \to +\infty$ we get $Z(\mathrm{t}_1) \leqq 0$, a contradiction. Hence (5.33) follows, and the proof of Step 7 is concluded.

Step 8. Assume now that

$$\kappa_{\gamma_\infty}(s, \mathrm{t}) > 0, \quad (s, \mathrm{t}) \in I_\infty \times \mathbb{R}. \tag{5.37}$$

Reasoning as in Step 5 we have

$$\int_{I_\infty} \kappa_{\gamma_\infty}(s, \mathrm{t})\,ds \in (0, 2\pi/3], \quad \mathrm{t} \in \mathbb{R}. \tag{5.38}$$

Note that in this case the image of $\gamma_\infty(\cdot, \mathrm{t})$ is not necessarily a graph, but still the function θ is well-defined, thanks to (5.37), and takes values in $[\pi/3, \pi)$. Reasoning as in Steps 6 and 7, using the boundary conditions (5.28) and

$$h(0, \mathrm{t}) = \pi, \quad \mathrm{t} \in \mathbb{R},$$

and the choice $Z(\mathrm{t}) := \int_{\pi/3}^{\pi} \partial_\mathrm{t}(\log\kappa)\,d\theta$, we deduce that (5.32) is still valid.

Step 9. γ_∞ is one of the two specific pieces of the grim reaper depicted in Figure 4.

Figure 4. Two pieces of the grim reaper, with the given $\pi/3$-Neumann boundary condition.

From Step 7 and (5.26) we have $\partial_{\theta\theta}\kappa_{\gamma_\infty} + \kappa_{\gamma_\infty} = 0$. By direct integration and using (5.22), it follows that γ_∞ is a one-parameter family of pieces of grim reapers (the parameter being for instance the horizontal velocity of translation), see Figure 4. As in Step 5, we have $Q_2^{\gamma_\infty} \equiv 0$, which gives a contradiction. This shows that γ cannot develop type II singularities, and concludes the proof of the theorem. □

6. Examples

In the first example we show a graph-like initial datum $\bar\gamma$ which develops a type II singularity: differently from Section 5 (see (2.11)), in this case $\bar\gamma_2$ changes sign.

6.1. Example 1. For $x \in [0,1]$ let $\bar\gamma(x) := (x, \bar f(x))$ where $\bar f$ is a smooth function the graph of which satisfies the Neumann boundary conditions (2.4) at $x = 0$ and $x = 1$, with the property that there exist $x_1, x_2 \in (0,1)$, $x_1 < x_2$, such that $\bar f > 0$ on $(0, x_1) \cup (x_2, 1)$, and $\bar f < 0$ on (x_1, x_2) (see Figure 5). Set

$$\int_0^{x_1} \bar f(x)\,dx =: \varepsilon > 0, \qquad \int_{x_1}^{x_2} \bar f(x)\,dx =: -c < 0.$$

Then the image of $\gamma(t)$ can be written as the graph, over a smoothly variable interval $[a(t), b(t)]$, of a smooth function $f(\cdot,t) : [a(t), b(t)] \to \mathbb{R}$, for $t \in [0, T)$, which solves the problem

$$\begin{cases} f_t = \dfrac{f_{xx}}{1 + (f_x)^2} & \text{in } \big(a(t), b(t)\big) \times (0,T), \\ f\big(a(t),t\big) = f\big(b(t),t\big) = 0, & t \in (0,T), \\ f_x\big(a(t),t\big) = \sqrt{3}, & t \in (0,T), \\ f_x\big(b(t),t\big) = -\sqrt{3}, & t \in (0,T), \\ a(0) = 0, & \\ b(0) = 1, & \\ f(\cdot,0) = \bar f(\cdot) & \text{in } (0,1), \end{cases} \tag{6.1}$$

where, for notational simplicity, we still denote by x the first variable in $\mathbb{R}^2$.

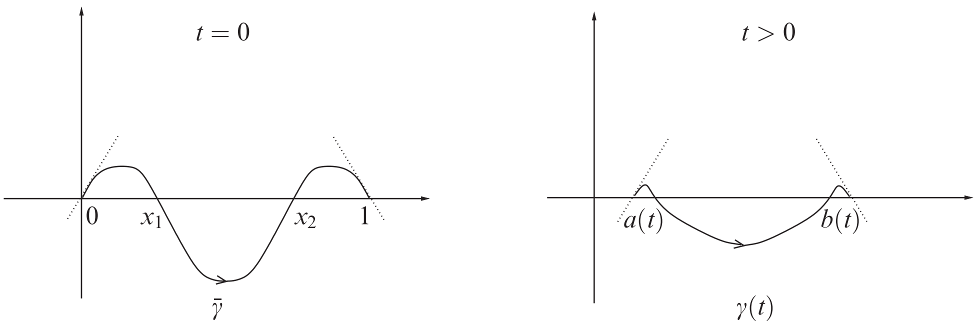

Figure 5. Example 1: The initial datum (left) and its evolution (right), which develops a type II singularity before the extinction.

By the maximum principle for f_x (see [22]) the functions $f(\cdot,t)$ are Lipschitz continuous, with a Lipschitz constant which is uniform with respect to $t \in [0,T)$. By the smoothness of the flow, there exist $t_s \in (0,T]$ and two continuous functions $x_1, x_2 : [0,t_s) \to \mathbb{R}$, with $a(t) < x_1(t) < x_2(t) < 1$ for any $t \in [0,t_s)$, such that $x_i(0) = x_i$, $i = 1,2$, $f(\cdot,t) > 0$ on $\big(a(t), x_1(t)\big) \cup \big(x_2(t), 1\big)$, and $f(\cdot,t) < 0$ on $\big(x_1(t), x_2(t)\big)$. Define, for any $t \in (0,t_s)$, the nonnegative functions

$$V^+(t) := \int_{a(t)}^{x_1(t)} f(x,t)\,dx, \quad V^-(t) := -\int_{x_1(t)}^{x_2(t)} f(x,t)\,dx.$$

By a direct computation, we get

$$\frac{d}{dt}V^+(t) \leqq -\frac{\pi}{3}, \quad \frac{d}{dt}V^-(t) \geqq -\pi,$$

so that

$$V^+(t) \leqq \varepsilon - \frac{\pi}{3}t, \quad V^-(t) \geqq c - \pi t, \quad t \in (0,t_s). \tag{6.2}$$

Observe that if there exists $\bar{t} \in (0,t_s]$ such that $V^+ > 0$ in $[0,\bar{t})$, $V^+(\bar{t}) = 0$ (hence $a(\bar{t}) = x_1(\bar{t})$) and $V^- > 0$ in $[0,\bar{t}]$, then $\bar{t}$ is a singularity time due to the boundary conditions (and $\bar{t}$ is not the extinction time). Hence, from (6.2) it follows that if ε is small enough, i.e. $c - 3\varepsilon > 0$, a singularity occurs *before* the extinction of the evolution. It follows that $t_s = T \leqq 3\varepsilon/\pi$.

Reasoning as in Theorem 4.2, we can exclude that $\gamma(t)$ develops type I singularities at $t = T$: indeed, developing a type I singularity at $t = T$ would imply a nontrivial homothetic solution obtained as a blow up, which (thanks to the boundary conditions) is unique, and would correspond to the extinction at $t = T$, which contradicts $\liminf_{t\to T^-} V^-(t) > 0$. It follows that $\gamma(t)$ develops a type II singularity at $t = T$. Arguing as in the proof of Theorem 5.1, a suitable rescaled and translated version of $\gamma(t)$ converges either to a grim reaper or to a piece of the grim reaper with a boundary point. In fact, we can rule out the first possibility, since the grim reaper cannot be written as the graph of a Lipschitz function. We conclude that if $\epsilon < c/3$, a type II singularity (the blow-up of which is as in Figure 4) must occur before the extinction time.

In the next example we show a singularity due to collision of the boundary points, happening before the extinction time.

6.2. Example 2. Let us consider an evolution similar to (2.1), where we substitute the boundary conditions on $\tau(0,t)$ and $\tau(1,t)$ with

$$\tau(0,t) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \quad \tau(1,t) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right), \tag{6.3}$$

so that the angle between e_1 and $\tau(t)$ equals $2\pi/3$ at $\gamma(0,t) = \big(\gamma_1(0,t), 0\big)$, and equals $-2\pi/3$ at $\gamma(1,t) = \big(\gamma_1(1,t), 0\big)$.

We still assume that $\bar{\gamma}$ is smooth and embedded, with $\bar{\gamma}_2 > 0$ in $(0,1)$ as in Sections 4 and 5 (see Figure 6). At the singular time $t = T$ either (3.1) holds or the curvature stays bounded but there is a collision of the boundary points, i.e.

$$\liminf_{t \to T^-} |\gamma_1(1,t) - \gamma_1(0,t)| = 0. \tag{6.4}$$

Notice that this is impossible for the solutions of (2.1), due to the boundary conditions.

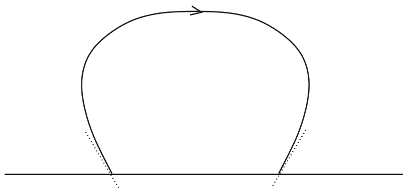

Figure 6. The initial datum in Example 2.

Since Theorem 4.2 applies also to this situation, we can exclude the formation of type I singularities before the extinction time. Moreover, since $\bar{\gamma}$ is embedded and $\bar{\gamma}_2$ is positive in $(0,1)$, we can also exclude type II singularities, reasoning exactly as in Section 5.

Assume now that T is the extinction time of the evolution, and that the evolution develops a type I singularity at $t = T$. By the analysis in Section 4, it follows that the evolution converges, after rescaling, to a homothetic solution. However there are no such solutions compatible with the boundary conditions (6.3), see [7], [15]. Hence T is not the extinction time of the evolution and (6.4) necessarily holds. A collision of the boundary points occurs as $t \to T^-$, while the curvature remains bounded.

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