KOHNEN'S FORMULA AND A CONJECTURE OF DARMON AND TORNARÍA

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ABSTRACT. We generalize a result of W. Kohnen in [9] to explicit Waldspurger lifts constructed by E. M. Baruch and Z. Mao in [1]. As an application, we prove a conjecture formulated by H. Darmon and G. Tornaría in [6].

1. Introduction

The aim of this note is to extend a result of Kohnen [9, Thm. 3] to Waldspurger lifts of elliptic modular forms constructed in [1] and use this formula to prove a conjecture of H. Darmon and G. Tornaría in [6].

Let us first explain the generalization of Kohnen's formula we are interested in. Suppose that $f = \sum_{n\geq 1} a_n q^n$ is a modular form of weight $2k$ for $\Gamma_0(M)$, the usual congruence subgroup of level M in $\mathrm{SL}_2(\mathbb{Z})$, where $M \geq 1$ is a square free odd integer. Thanks to the work of Baruch and Mao [1] one attaches to f and a divisor $M' \mid M$ a form $g = \sum_{n\geq 1} c_n q^n$ of weight $k+1/2$ and with respect to the congruence group $\Gamma_1(4MM')$. Let s_0 be the cardinality of the set S_0 of primes dividing M'. Let $\mathcal{D}(f, S_0)$ be the set of fundamental discriminants D such that $\left(\frac{D}{\ell}\right) = -w_\ell$ if $\ell \mid M'$ and $\left(\frac{D}{\ell}\right) = +w_\ell$ if $\ell \mid M/M'$, where w_ℓ is the sign of the Atkin-Lehner involution acting on f. We are interested in fundamental discriminants satisfying the following condition

$$(*) \qquad D \in \mathcal{D}(f, S_0) \text{ and } (-1)^{s_0+k} = \mathrm{sgn}(D).$$

Suppose D_1 and D_2 are fundamental discriminant satisfying $(*)$. Kohnen's formula relates the product $c_{|D_1|} \cdot \overline{c_{|D_2|}}$ to certain linear combinations of explicit Shintani integrals, namely, integrals of the differential form $f(z)dz$ along geodesic cycles in the upper half plane. The main result is Theorem 2.3 below, which is a generalization of [9, Thm. 3]. However, the proof of this result is not a direct generalization of the proof of *loc. cit.*, which has a more combinatoric flavour. Instead, our proof is based on methods from [1] and [15], working in the context of automorphic forms. Finally, let us point out that the above result is proved in the more general setting of automorphic forms over totally real number fields (in Proposition 2.7 below), although for the application to Darmon-Tornaría conjecture we only need the case of rational numbers.

We now briefly explain the application to elliptic curves, and the content of the Darmon-Tornaría conjecture. Let $E/\mathbb{Q}$ be an elliptic curve of conductor $N = Mp$, where $M > 1$ is an odd square free positive integer and $p \nmid 2M$ is a prime number. Fix as above a divisor $M' \mid M$ and let g be the generalized Kohnen-Waldspurger lift in [1] of the modular form f attached to E. This is a modular form of weight $3/2$ for $\Gamma_1(4NM')$ with Fourier expansion $\sum_{n\geq 1} c_n q^n$. This expansion is only well-defined up to (non-zero) scalar, and therefore we may form the quotients

$\tilde{c}_n := c_n/c_{n_0}$, where $c_{n_0} \neq 0$ (the existence of such an integer n_0 follows from the main result of Baruch-Mao, combined with standard non-vanishing results for L-series). We show that these coefficients can be seen as the value at 1 of rigid analytic functions $\tilde{c}_n(k)$, defined on a neighborhood $\mathcal{U}$ of 1 in a suitable weight space, which incorporate similarly defined quotients of Fourier coefficients of the generalized Kohnen-Waldspurger lifts of classical even weight modular forms in the Hida family passing through f. We fix a fundamental discriminant D satisfying the following modification of the condition $(*)$, where as above w_ℓ for $\ell \mid N$ is the sign of the Atkin-Lehner involution on E (recall that s_0 is the cardinality of the fixed set S_0):

$$(\dagger\dagger)\quad \left(\frac{D}{\ell}\right) = -w_\ell \text{ if } \ell \mid M'p;\ \left(\frac{D}{\ell}\right) = +w_\ell \text{ if } \ell \mid (M/M') \text{ and } (-1)^{s_0} = -\mathrm{sgn}(D).$$

It turns out that $(\dagger\dagger)$ implies $\tilde{c}_{|D|}(1) = 0$ ([1, Thm. 1.1]), although the function $\tilde{c}_D(k)$ is not identically zero in $\mathcal{U}$ because D satisfies $(*)$ with respect to all the newforms of level $\Gamma_0(M)$ appearing in the Hida family; (actually in the Hida family the associated p-stabilized forms of level $\Gamma_0(N)$ appear). Our main result (Theorem 4.5 below, a consequence of Theorem 4.4) is the following: There exists a family of points $P_D \in E(\bar{\mathbb{Q}}) \otimes_{\mathbb{Z}} \bar{\mathbb{Q}}$, one for each D as above, which is non-zero if and only if $L'(E, \chi_D, 1) \neq 0$, and such that

$$(1)\qquad \log_E(P_D) = \left(\frac{d}{dk}\tilde{c}_{|D|}(k)\right)_{|k=1}.$$

Further, if $D < 0$, then we may take $P_D \in E(\sqrt{D}) \otimes_{\mathbb{Z}} \bar{\mathbb{Q}}$. Here $\log_E$ is the formal group logarithm on E (see §4.1 for a precise definition). Further, the point P_D arises from Darmon's theory of his eponymous points, introduced [5] and developed by several authors (see for example [3], [4], [7], [18], [22], [12], [13]). With $S = \emptyset$, Theorem 4.4 corresponds to [6, Thm. 1.5]. We finally use this result to prove Theorem 4.6, a conjecture of Darmon and Tornaría, [6, Conj. 5.3]. This conjecture predicts that the Fourier coefficients $c_{|D|}$ of g (where recall that g is the generalized Kohnen-Waldspurger lift of the modular form f attached to the elliptic curve E) for D a fundamental discriminant satisfying the following condition

$$(\dagger)\quad \left(\frac{D}{\ell}\right) = -w_\ell \text{ if } \ell \mid M';\ \left(\frac{D}{\ell}\right) = +w_\ell \text{ if } \ell \mid (Mp/M');\text{ and } (-1)^{s_0} = -\mathrm{sgn}(D)$$

(which are not necessarily zero by [1, Thm. 1.1]), are encoded by certain p-adic Shintani integrals, denoted $\vartheta(f, D, D')$ and introduced in §4.1, depending on the auxiliary choice of an auxiliary fundamental discriminant D' satisfying $(\dagger\dagger)$. Further, the common coefficient of proportionality between the Fourier coefficients $c_{|D|}$ of g and $\vartheta(f, D, D')$ is $\log_E(P_{D'})$, and thus is non zero if and only if $L'(E, \chi_{D'}, 1) \neq 0$. For $S = \emptyset$, this is [6, Thm 5.1], from which [6, Conj. 5.3] is inspired.

The authors acknowledge H. Darmon, G. Tornaría and J. Hakim for useful discussions. M. L. is partly supported by PRIN 2010-11 and PRAT 2013. Z. M. is partly supported by NSF DMS 1400063.

2. Kohnen's formula

2.1. **Generalized Kohnen-Shintani correspondence.** Let $f = \sum_{n\geq 1} a_n q^n$ be a newform of even integral weight $2k$, square-free odd level M and trivial character.

We let S be the set of primes dividing M, whose cardinality we denote $s = |S|$. Fix a subset $S_0 \subset S$, write M' for the product of the primes in S_0, and let $s_0 = |S_0|$ be the cardinality of S_0. Let $\mathcal{D}(f, S_0)$ be the set of fundamental discriminants D defined in the Introduction. We also denote χ_D the quadratic character $a \mapsto \left(\frac{D}{a}\right)$ attached to the fundamental discriminant D. Fix a Dirichlet character χ' of $(\mathbb{Z}/(4M))^\times$ such that $\chi'_\ell = 1$ if $\ell \mid (M/M')$, $\chi'_\ell(-1) = -1$ if $\ell \mid M'$ and $\chi'(-1) = 1$. We can consider χ' as a character of $(\mathbb{Z}/(4MM'))^\times$. Attached to f and the choice of the auxiliary character χ', we may consider the explicit Waldspurger's lift of Baruch-Mao relative to S_0 ([1], [26], [28], [27]),

$$g = \sum_{n\geq 1} c_n q^n \in S_{k+1/2}(\Gamma_0(4MM', \chi'))$$

which satisfies the following properties (see [1, Thm. 1.1]):

(a) g is a Shimura lift of f.
(b) g belongs to the Kohnen's plus space: $c_n = 0$ if $(-1)^{s_0+k} n \equiv 2, 3 \mod 4$.
(c) $c_{|D|} = 0$ if $(-1)^{s_0+k} D > 0$ and $D \notin \mathcal{D}(f, S_0)$.
(d) If D satisfies $(*)$ in the Introduction, then

$$\frac{|c_{|D|}|^2}{\langle g, g\rangle} = \frac{L(f, \chi_D, k)}{\langle f, f\rangle} \cdot \frac{2^{|S|} \cdot |D|^{k-1/2} \cdot (k-1)!}{\pi^k} \cdot \prod_{\ell | S_0} \frac{\ell}{\ell+1}. \tag{2}$$

Otherwise, if $D \in \mathcal{D}(f, S_0)$ and $(-1)^{s_0+k} \neq \mathrm{sgn}(D)$, then $L(f, \chi_D, k) = 0$.

2.2. **Kohnen's formula.** Let $K = \mathbb{Q}(\sqrt{\Delta})$ be a real quadratic field of fundamental discriminant Δ such that all primes dividing M are split in K.

Fix $\tau = \left(\begin{smallmatrix} \tau_1 & \tau_3 \\ \tau_2 & -\tau_1 \end{smallmatrix}\right)$ such that $\tau_1^2 + \tau_2\tau_3 = \Delta$ and $2M \mid \tau_2$, $2 \mid \tau_3$. Let $\mathcal{F}_\Delta$ denote the set of binary integral primitive quadratic forms

$$Q(x, y) = Ax^2 + Bxy + Cy^2$$

of discriminant Δ satisfying the following properties: (1) $M \mid A$ and (2) $B \equiv \tau_1$ modulo M. The group $\Gamma_0(M)$ acts on $\mathcal{F}_\Delta$ from the right via

$$(Q|\gamma)(x, y) := Q(ax + by, cx + dy)$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

Recall that canonical projection induces a bijection between $\mathcal{F}_\Delta/\Gamma_0(M)$ and the group of $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of integral primitive binary quadratic forms of discriminant Δ equipped with Gaussian composition law. This is identified, by class field theory, with the Galois group $G_K^+ = \mathrm{Gal}(H_K^+/K)$ of the strict Hilbert class field H_K^+ of K.

Let $r + s\sqrt{\Delta}$ be fundamental unit of norm 1 in $\mathcal{O}_\Delta = \mathbb{Z}[(\Delta + \sqrt{\Delta})/2]$ normalized with $r > 0$ and $s > 0$ and define $\gamma_Q := \left(\begin{smallmatrix} r+sB & 2Cs \\ -2As & r-sB \end{smallmatrix}\right)$, an element in $\Gamma_0(M)$. Define Shintani integrals attached to f and $Q \in \mathcal{F}_\Delta$ as

$$r(f, Q) := \int_\tau^{\gamma_Q(\tau)} f(z) Q(z, 1)^{k-1} dz \tag{3}$$

Fix a genus character χ_{D_1,D_2} of K, attached to the pair of Dirichlet characters (χ_{D_1}, χ_{D_2}), where $\Delta = D_1 \cdot D_2$, and $(D_1, D_2) = 1$. Set as in [20]

$$r(f; D_1, D_2; \tau) := \sum_{[Q]\in\mathcal{F}_\Delta/\Gamma_0(M)} \chi_{D_1,D_2}(Q) \cdot r(f, Q)$$

Remark 2.1. The value of $(r(f;D_1,D_2;\tau))^2$ does not depend on the choice of τ. Let χ be the character of $(\mathbb{Z}/M')^\times$ such that $\chi_\ell=\chi'_\ell$ when $\ell\mid M'$. Then

$$r_\chi(f;D_1,D_2):=\chi(\tau_1)r(f;D_1,D_2;\tau)$$

is independent of the choice of τ.

Remark 2.2. When $D_1=D_2$ and $\tau=\left(\begin{smallmatrix} & D_1\\ -D_1 & \end{smallmatrix}\right)$, we denote $r(f;D_1,D_2;\tau)$ by $r(f;D_1,D_1)$.

Theorem 2.3. *Suppose $\Delta=D_1\cdot D_2$ with D_1, D_2 satisfying $(*)$ above and D_1 odd. Then*

$$\frac{c_{|D_2|}\overline{c_{|D_1|}}}{\langle g,g\rangle}=(-2i)^k\cdot 2^{|S|}\cdot\chi(|D_1|)^{-1}\cdot\prod_{\ell\in S_0}\frac{\ell}{1+\ell}\cdot\frac{r_\chi(f;D_2,D_1)}{\langle f,f\rangle}.$$

Moreover $r_\chi(f;D_2,D_1)=r_\chi(f;D_1,D_2)$.

Remark 2.4. The difference in constant (a factor of $2^{|S|}$) between the above Theorem and [9, Theorem 3] lies in the difference of $r_\chi(f;D_2,D_1)$ (which is defined through a sum of *oriented* optimal embeddings in [20]) and $r_{k,N}(f;D_1,D_2)$ in [9] (which is a sum over non-oriented optimal embeddings). [1]

Remark 2.5. A combination of (2), [3, Eq. (28)] and [3, Eq. (29)] already shows, at least in the cases of weight $2k>2$ which will be relevant for the following sections, that the square norm of the above formula is true.

The proof of the theorem is based on results in [1] and [15]. Before proving the Theorem, we first give a generalization of Kohnen's formula in the setting of automorphic forms over a totally real number field.

2.3. **Theta correspondence.** From this subsection to the end of §2.7 we will use a notation different from to the other sections of the paper. Thus, some symbols used here (e.g., $\tau,\Delta,D_1,D_2,S,S_0$) will have a different meaning with respect to the other parts of the paper.

Let F be a totally real number field, $\mathbb{A}$ its adele ring. Fix an additive character ψ on $\mathbb{A}/F$ which is nontrivial. We will recall now the theta correspondence (Shimura correspondence) studied by Waldspurger [28].

Let M be the space of 2×2 matrices, and M^0 be the subspace consisting of matrices with trace 0. Let Φ be in $\mathcal{S}(\mathrm{M}^0(\mathbb{A}))$ the space of Schwartz functions on $\mathrm{M}^0(\mathbb{A})$. Let ω_ψ be the Weil representation of $\mathrm{PGL}_2\times\widetilde{\mathrm{SL}}_2$ associated to ψ, (see for example [1] for definition). We can construct a theta function Θ_Φ^ψ on $\mathrm{PGL}_2\times\widetilde{\mathrm{SL}}_2$:

$$\Theta_\Phi^\psi(g,h)=\sum_{x\in\mathrm{M}^0(F)}\omega_\psi(g,h)\Phi(x),\quad g\in\mathrm{PGL}_2(\mathbb{A}),\ h\in\widetilde{\mathrm{SL}}_2(\mathbb{A}).$$

Then for any cusp form φ on PGL_2 and $\Phi\in\mathcal{S}(\mathrm{M}^0(\mathbb{A}))$ define:

$$\theta_\Phi^\psi(\varphi)(h)=\int_{\mathrm{PGL}_2(F)\backslash\mathrm{PGL}_2(\mathbb{A})}\Theta_\Phi^\psi(g,h)\varphi(g)\,dg.$$

For irreducible cuspidal representations π of PGL_2, the space

$$\{\theta_\Phi^\psi(\varphi):\varphi\in\pi,\Phi\in\mathcal{S}(\mathrm{M}^0(\mathbb{A}))\}$$

[1] A similar factor should also appear in [6, Theorem 2.1].

is an irreducible cuspidal representation $\tilde{\pi}$ (which could be trivial). We denote $\tilde{\pi} = \theta^{\psi}(\pi)$ and call it the theta correspondence of π.

2.4. **Transition of periods.** Let $\tau \in \mathrm{GL}_2(F)$ such that $\tau^2 = \left(\begin{smallmatrix} D & \\ & D \end{smallmatrix}\right)$, $D \in F^\times$. Let T_τ be the centralizer of τ in PGL_2. We fix measure on $T_{\tau,v}$ over a local place F_v as follows. Let $K_v = F_v(\tau)$ be a quadratic algebra over F_v, take the measure on F_v to be self dual with respect to ψ_v, the measure on K_v to be self dual with respect to $\psi_v \circ tr_{K_v/F_v}$, and on $K_v^\times$ to be $\zeta_{K_v}(1)\frac{dx}{|N_{K_v/F_v}x|_v}$, on $F_v^\times$ to be $\zeta_{F_v}(1)\frac{dx}{|x|_v}$.(Here ζ_{F_v} and ζ_{K_v} are the $L-$functions associated to F_v and K_v when v is finite place, and set to 1 when v is infinite place). The measure on $T_{\tau,v}$ is just the quotient measure on $F_v^\times \backslash K_v^\times$. The global measure is taken to be the product of local measures. The choice of measure on PGL_2 is not important for the discussion below.

Define

$$P_\tau(\varphi) = \int_{T_\tau(F)\backslash T_\tau(\mathbb{A})} \varphi(t)\ dt,\ \ \varphi \in \pi$$

and

$$\tilde{W}^D(\tilde{\varphi}) = \int_{F\backslash\mathbb{A}} \tilde{\varphi}(\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right))\psi(-Dx)\ dx,\ \ \tilde{\varphi} \in \tilde{\pi}.$$

The following Lemma is the analogue of [15, Proposition 2.1], we will skip the proof as it is identical to the proof given in ibid.

Lemma 2.6. *If* $\tilde{\varphi} = \theta_\Phi^\psi(\varphi)$*, then*

$$\tilde{W}^D(\tilde{\varphi}) = P_\tau(f_{\Phi,\tau} * \varphi) \tag{4}$$

where $f_{\Phi,\tau}$ *is a function on* PGL_2 *satisfying*

$$\int_{T_\tau(\mathbb{A})} f_{\Phi,\tau}(tg)\ dt = \Phi(g^{-1}\tau g)$$

and

$$f_{\Phi,\tau} * \varphi = \int_{\mathrm{PGL}_2(\mathbb{A})} f_{\Phi,\tau}(g)\varphi(\cdot g)\ dg.$$

Of course $f_{\Phi,\tau}$ is not uniquely determined; the discussion below applies to any choice of $f_{\Phi,\tau}$. For the special case $\tau = \left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right)$, we write $f_\Phi := f_{\Phi,\tau}$, $T := T_\tau$ and $P := P_\tau$.

2.5. **A generalization of Kohnen's formula.** Let π be such that $L(\pi, \frac{1}{2}) \neq 0$. Then it is well known that $\tilde{\pi} = \theta^\psi(\pi) \neq 0$.

For $\varphi \in \pi$, define the corresponding Whittaker function

$$W_\varphi(g) = \int_{F\backslash\mathbb{A}} \varphi(\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) g)\psi(-x)\ dx.$$

Let $W(\varphi) = W_\varphi(e)$. By uniqueness of Whittaker model, if $\varphi = \otimes_{v\in S}\varphi_v \otimes_{v\notin S} \varphi_{v,0}$, where $\varphi_{v,0}$ is a fixed unramified vector in π_v, we can write for $g \in F_S = \otimes_{v\in S}F_v$, $W_\varphi(g) = \prod_{v\in S} W_v(g)$ for a compatible choice of functions W_v in the local Whittaker spaces of π_v.

On the other hand, from [28] we get locally the space of $T_{\tau,v}$ invariant forms on π_v is of dimension at most one. Any such invariant form (on the Whittaker space of π_v) is a scalar multiple of

$$P_\tau(W_v) = \int_{T_{\tau,v}} W_v(t)\ dt.$$

(The above integral converges as π_v is a unitary representation). Thus there is a constant $c_{\pi,S}$ depending on π and S (and not on φ) such that

$$P_\tau(\varphi) = c_{\pi,S} \prod_{v\in S} P_\tau(W_v).$$

By [15, Lemma 3.1], if Φ_v and τ_v are unramified, in the sense that Φ_v is the characteristic function of $\mathrm{M}^0(R_v)$ where R_v is the ring of integers in F_v and $\tau_v \in \mathrm{GL}_2(R_v)$, we can let $f_{\Phi,\tau,v} = f_0$ the characteristic function of $\mathrm{PGL}_2(R_v)$. Thus for S large enough

$$\tilde{W}^D(\theta^\psi_\Phi(\varphi)) = c_{\pi,S} \prod_{v\in S} P_\tau(f_{\Phi,\tau,v} * W_v).$$

Apply the discussion to the special case $\tau = \left(\begin{smallmatrix}1 & \\ & -1\end{smallmatrix}\right)$; we have for some constant $c^1_{\pi,S}$

$$\tilde{W}(\theta^\psi_\Phi(\varphi)) = c^1_{\pi,S} \prod_{v\in S} P(f_{\Phi,v} * W_v).$$

It is well known that $c^1_{\pi,S} = L^S(\pi, \frac{1}{2})$ where $L^S(\pi, s)$ is the partial $L-$function of π.

Let $\tilde{\varphi} = \theta^\psi_\Phi(\varphi)$ be such that $\tilde{W}(\tilde{\varphi}) \neq 0$, we get

$$\tilde{W}^D(\tilde{\varphi})\overline{\tilde{W}(\tilde{\varphi})} = \frac{c_S \prod_{v\in S} P_\tau(f_{\Phi,\tau,v} * W_v)}{c^1_S \prod_{v\in S} P(f_{\Phi,v} * W_v)} |\tilde{W}(\tilde{\varphi})|^2.$$

Recall the (special case of) main formula in [1]:

$$\frac{|\tilde{W}(\tilde{\varphi})|^2}{\|\tilde{\varphi}\|^2} = \frac{|W(\varphi)|^2}{\|\varphi\|^2} L(\pi, \frac{1}{2}) \prod_{v\in S} E_v(\varphi_v, \tilde{\varphi}_v, \psi), \tag{5}$$

where $E_v(\varphi_v, \tilde{\varphi}_v, \psi)$ are local constants defined in [1]; $\|\tilde{\varphi}\|^2$ and $\|\varphi\|^2$ are Peterson norm with respect to a pair of compatible measures on $\widetilde{\mathrm{SL}}_2$ and PGL_2, fixed in [1]. We get from the above discussion:

Proposition 2.7. *Let $\Phi = \otimes\Phi_v \in \mathcal{S}(\mathrm{M}^0(\mathbb{A}))$, $\phi = \otimes\phi_v \in \pi$ and $\tilde{\varphi} = \theta^\psi_\Phi(\varphi)$ be such that $\tilde{W}(\tilde{\varphi}) \neq 0$. Let S be large enough so that for $v \notin S$, $\psi_v, \phi_v, \Phi_v, \tau_v$ are unramified, and assume $W_\varphi = \prod_{v\in S} W_v$ (over F_S) be such that $P_\tau(W_v) \neq 0$. Then:*
(6)

$$\frac{\tilde{W}^D(\tilde{\varphi})\overline{\tilde{W}(\tilde{\varphi})}}{\|\tilde{\varphi}\|^2} = P_\tau(\varphi)\frac{|W(\varphi)|^2}{\|\varphi\|^2} \prod_{v\in S}\left(\frac{P_\tau(f_{\Phi,\tau,v} * W_v)}{P_\tau(W_v)P(f_{\Phi,v} * W_v)} L(\pi_v, \frac{1}{2}) E_v(\varphi_v, \tilde{\varphi}_v, \psi)\right).$$

The above equation gives a relation between the product of distinct Whittaker functionals of $\tilde{\varphi}$ and the period P_τ of φ, up to some local factors. We can consider it as a generalization of Kohnen's formula in [9, Theorem 3].

We remark that the same argument as above gives

$$\tilde{W}^D(\tilde{\varphi}')\overline{\tilde{W}(\tilde{\varphi})} = P_\tau(\varphi)(c^1_S)^{-1} \prod_{v\in S} \frac{P_\tau(f_{\Phi',\tau,v} * W_v)}{P_\tau(W_v)P(f_{\Phi,v} * W_v)} |\tilde{W}(\tilde{\varphi})|^2 \tag{7}$$

where $\tilde{\varphi}' = \theta^\psi_{\Phi'}(\varphi)$. We will derive Theorem 2.3 from (7) and (5).

2.6. **Specification of the formula** (7)**.** Now consider the case $F = \mathbb{Q}$. Let M be a square free odd number. Let π' be associated to a new form f of weight $2k$ and level M. Let D be a positive fundamental discriminant such that ℓ splits in $\mathbb{Q}(\sqrt{D})$ for all primes $\ell \mid M$. Assume $D = D_1 D_2$ where D_1 and D_2 are coprime, moreover we assume D_1 is odd and D_1 satisfies conditions $(*)$, then D_2 also satisfies conditions $(*)$.

Let $\pi = \pi' \otimes \chi_{D_1}$. (We use as above χ_a to denote the quadratic character of $\mathbb{A}$ attached to $a \in \mathbb{Q}^\times$.) Let $\psi(x) = \psi_0(|D_1|x)$ where $\psi_0(x)$ is fixed as follows: $\psi_{0,v}(x) = e^{2\pi i x}$ if v is finite and $e^{-2\pi i x}$ if $v = \infty$. For convenience, we fix the local measures with respect to ψ_0 instead of ψ. This does not change the global measure, thus the statements of our global results.

We assume $L(f, D_1, 1) \neq 0$, then $\tilde{\pi} = \theta^\psi(\pi) \neq 0$. Moreover with our restrictions on D_1 (that it satisfies the condition $(*)$), the representation $\tilde{\pi}$ is independent of our choice of parameter D_1, (see [28], summarized in [1, Theorem 3.2]).

We will calculate the local constant

$$C(\varphi_v, \Phi_v, \Phi'_v) := \frac{P_\tau(f_{\Phi',\tau,v} * W_v)}{P_\tau(W_v) P(f_{\Phi,v} * W_v)}$$

for some specific choice of $\varphi_v, \Phi_v, \Phi'_v$. With our specific choices, we will check in most cases

$$f_{\Phi',\tau,v} * W_v = \alpha'_v W_v, f_{\Phi,v} * W_v = \alpha_v W_v,\ \alpha'_v \alpha_v \neq 0. \tag{8}$$

Thus the above constant is just

$$C(\varphi_v, \Phi_v, \Phi'_v) = \frac{\alpha'_v}{\alpha_v P(W_v)}. \tag{9}$$

We take $\varphi' = \otimes \varphi'_v$ be the vector in π' such that φ'_v is the new vector at all finite places v and φ'_∞ is the lowest weight vector. Let $\varphi(g) = \varphi'(g) \cdot \chi_{D_1}(\det g)$.

Then S could be taken to be $\{\infty, 2\} \cup \{l : l|DM\}$. Recall for $g \in \mathbb{Q}_S$, $W_\varphi = \prod_{v \in S} W_v(g_v)$.

It is convenient to use the following notations: for $\alpha \in \mathrm{PGL}_2$, $\alpha * W(g) := W(g\alpha)$; for $\tilde{\alpha} \in \widetilde{\mathrm{SL}}_2$, $\tilde{\alpha} * \tilde{W}(g) = \tilde{W}(g\tilde{\alpha})$ and $\tilde{\alpha} * \Phi = \omega_\psi(\tilde{\alpha})\Phi$; $\underline{\tilde{a}} = (\left(\begin{smallmatrix} a & \\ & a^{-1} \end{smallmatrix}\right), 1)$. Recall the local theta correspondence defines a map from $\pi_v \otimes \mathcal{S}(\mathrm{M}^0(F_v))$ to $\tilde{\pi}_v$, we denote it by $\tilde{\varphi}_v = \theta(\varphi_v, \Phi_v, \psi_v) \in \tilde{\pi}_v$, (this is only defined up to a scalar multiple). Sometimes we also denote the absolute value $|D_1| \in \mathbb{Q}^+$ by $D_1^\sharp$, to distinguish from $|D|_v$.

Case (1) $v = l | D_2$ is odd. Then π is unramified and φ_l is the unramified vector. We take $\Phi_l = \Phi'_l$ to be the characteristic function of $\mathrm{M}^0(\mathbb{Z}_l)$. Then $\tilde{\varphi}_l$ is an unramified vector in $\tilde{\pi}$. As $\tau = k^{-1} \left(\begin{smallmatrix} & D \\ 1 & \end{smallmatrix}\right) k$ for an element $k \in \mathrm{PGL}_2(\mathbb{Z}_l)$, by [15, Lemma 3.1], we can take $f_{\Phi,l} = 1_{\mathrm{PGL}_2(\mathbb{Z}_l)}$ and $f_{\Phi',\tau,l} = |D|_l^{-\frac{1}{2}} 1_{\mathrm{PGL}_2(\mathbb{Z}_l)}$. Thus (8) holds with $\alpha'_l = |D|_l^{-\frac{1}{2}} \mathrm{vol}(\mathrm{PGL}_2(\mathbb{Z}_l))$ and $\alpha_l = \mathrm{vol}(\mathrm{PGL}_2(\mathbb{Z}_l))$. The local constant is

$$C(\varphi_l, \Phi_l, \Phi'_l) = \frac{|D|_l^{-\frac{1}{2}}}{P(W_l)}.$$

Case (2) $l | D_1$. Then π' is unramified and φ'_l is the unramified vector. We take Φ'_l to be as chosen in [15, Lemma 3.2]. Namely $\Phi'_l(\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right))$ is 0 if one of a, b, c is not integral, or both b and c are prime in $\mathbb{Z}_l$, or $a^2 + bc$ is in $\mathbb{Z}_l^\times$. Otherwise, we set $\Phi'_l(\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)) = \chi_{D_1}(c)$ if $c \in \mathbb{Z}_l^\times$ or $\chi_{D_1}(-b)$ if $b \in \mathbb{Z}_l^\times$. We let $\Phi_l = \underline{\tilde{D_1^\sharp}} * \Phi'_l$. Explicitly

$\Phi_l(X) = \Phi'_l(D_1^\sharp X)|D_1|_l^{\frac{3}{2}}\mathfrak{g}$ where $\mathfrak{g}$ is a root of unity, equal to $l^{-\frac{1}{2}}$ times Gaussian sum associated to the quadratic character on the finite field F_l. Then $\tilde{\varphi} = \widetilde{\underline{D_1^\sharp}} * \tilde{\varphi}'$ is unramified (see [21, Proposition 3.4]). (In [21] it is shown that $\theta(\varphi_l, \overline{\Phi'_l}, \psi'_l)$ where $\psi'_l(x) := e^{2\pi i x/D_1^\sharp}$ is unramified. This easily implies that $\tilde{\varphi} = \theta(\varphi_l, \Phi_l, \psi_l)$ is unramified.)

By [15, Lemma 3.2], $f_{\Phi',\tau,l} = 1_{\mathrm{PGL}_2(\mathbb{Z}_l)}|D_1|_l^{-\frac{1}{2}}\chi_{D_1}(\det\cdot)$. (Note in [15, Lemma 3.2] $L(\chi_D, 1) = 1$). On the other hand

$$f_{\Phi,l} = \mathfrak{g}|D_1|^{\frac{1}{2}}(1-l^{-1})1_{\mathrm{PGL}_2(\mathbb{Z}_l)}\left(\begin{pmatrix} 1 & D_1^\sharp \\ 1 & -D_1^\sharp \end{pmatrix}\cdot\right)\chi_{D_1}(\det\cdot).$$

(Note the volume of the intersection of $\mathrm{PGL}_2(\mathbb{Z}_l)$ with the centralizer of $\begin{pmatrix} & l \\ l^{-1} & \end{pmatrix}$ is $(l-1)^{-1}$). Thus $\alpha'_l = \mathrm{vol}(\mathrm{PGL}_2(\mathbb{Z}_l))|D_1|_l^{-\frac{1}{2}}$ while

$$f_{\Phi,l} * W_l(\cdot) = \mathfrak{g}|D_1|^{\frac{1}{2}}(1-l^{-1})\,\mathrm{vol}(\mathrm{PGL}_2(\mathbb{Z}_l))W_l(\cdot\begin{pmatrix} 1 & D_1^\sharp \\ 1 & -D_1^\sharp \end{pmatrix}).$$

Using the Iwasawa decomposition $\begin{pmatrix} 1 & D_1^\sharp \\ 1 & -D_1^\sharp \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} D_1^\sharp & \\ & 1 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 1 & -D_1^\sharp \end{pmatrix}$, we get

$$P(W_l(\cdot\begin{pmatrix} 1 & D_1^\sharp \\ 1 & -D_1^\sharp \end{pmatrix})) = \int_{\mathbb{Q}_l^*}\psi_l(t)W_l\left(\begin{pmatrix} tD_1^\sharp & \\ & 1 \end{pmatrix}\right)\, d^*t.$$

It is clear that $W(\begin{pmatrix} a & \\ & 1 \end{pmatrix}) = 0$ if $a \notin l^{-1}\mathbb{Z}_l$. Thus we can restrict the above integration to $|t|_l \leq l^2$. On the other hand over the domain $|t|_l = c$ where $c \leq l$ is fixed, we have $\psi_l(t) = 1$ and the integrand is a constant multiple of $\chi_{D_1}(t)$. Thus the integration over $|t|_l = c \leq l$ is 0. We are left with

$$\begin{aligned} P(W_l(\cdot\begin{pmatrix} 1 & D_1^\sharp \\ 1 & -D_1^\sharp \end{pmatrix})) &= \int_{|t|_l=l^2}\psi_l(t)W_l(\begin{pmatrix} tD_1^\sharp & \\ & 1 \end{pmatrix})\, d^*t \\ &= (1-l^{-1})^{-1}W_l(\begin{pmatrix} (D_1^\sharp)^{-1} & \\ & 1 \end{pmatrix})|D_1|^{\frac{1}{2}}\mathfrak{g}^{-1}. \end{aligned}$$

The local constant is

$$C(\varphi_l, \Phi_l, \Phi'_l) = |D_1|_l^{-\frac{3}{2}}(W_l(\begin{pmatrix} (D_1^\sharp)^{-1} & \\ & 1 \end{pmatrix}))^{-1}.$$

Case (3) $l|M$ and $l \notin S_0$. In this case with our assumption on D_1, $\epsilon(\pi_l, \frac{1}{2}) = 1$. Take $\Phi = \Phi'_l$ the characteristic function of $\{\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\}$ where $c \in l\mathbb{Z}_l$ and $a, b \in \mathbb{Z}_l$. Then $\tilde{\varphi} = \tilde{\varphi}'$ is the vector in $\tilde{\pi}_l$ of the lowest level, i.e. a multiple of the vector $\tilde{\varphi}$ appearing in [1, Lemma 8.3]; the representation $\tilde{\pi}_l$ is a special representation. Let $K_{0,l} := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, d \in \mathbb{Z}_l^\times, b \in \mathbb{Z}_l, c \in l\mathbb{Z}_l\}$, $w_l := \begin{pmatrix} & 1 \\ l & \end{pmatrix}$. We get $f_{\Phi',\tau,l} = f_{\Phi,l} = 1_{K_{0,l} \cup w_l K_{0,l}}$. Since $\epsilon(\pi_l, \frac{1}{2}) = 1$, we have $\pi(w_l)W_l = W_l$. Thus $f_{\Phi',\tau,l} * W_l = f_{\Phi,l} * W_l = 2\,\mathrm{vol}(K_{0,l})W_l$ and

$$C(\varphi_l, \Phi_l, \Phi'_l) = \frac{1}{P(W_l)}.$$

Case (4) $l \in S_0$. Then $\epsilon(\pi_v, \frac{1}{2}) = -1$. Take $\Phi_l = \Phi'_l$ whose value at $\{\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\}$ is zero unless $a \in \mathbb{Z}_l^\times$, $b \in \mathbb{Z}_l$ and $c \in l\mathbb{Z}_l$. Otherwise it is $\chi_l(a)$ where χ_l is an odd character on $\mathbb{Z}_l^\times/(1 + l\mathbb{Z}_l)$. Then $\tilde{\varphi} = \tilde{\varphi}'$ is the vector described by [1, Proposition 8.5]. It is still the vector of lowest level in the space of $\tilde{\pi}_l$, this time however $\tilde{\pi}_l$ is a supercuspidal representation.

We get $f_{\Phi',\tau,l} = (1_{K_{0,l}} - 1_{w_l K_{0,l}})\chi_l(\tau_1)$ and $f_{\Phi,l} = 1_{K_{0,l}} - 1_{w_l K_{0,l}}$. In this case $\pi(w_l)W_l = -W_l$. Thus $f_{\Phi,l} * W_l = 2\operatorname{vol}(K_{0,l})W_l$ and

$$f_{\Phi',\tau,l} * W_l = 2\operatorname{vol}(K_{0,l})\chi_l(\tau_1)W_l.$$

We get

$$C(\varphi_l, \Phi_l, \Phi_l') = \frac{\chi_l(\tau_1)}{P(W_l)}.$$

Case (5) $l = 2$. Take $\Phi_l = \Phi_l'$ whose value at $\{\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)\}$ is zero unless $a \in \mathbb{Z}_l$ and $b, c \in 2\mathbb{Z}_l$. Then $\tilde{\varphi} = \tilde{\varphi}'$ is the Kohnen vector described by [1, (9.4)]. By [15, Lemma 3.4], $f_{\Phi',\tau,2} = |D|_2^{-\frac{1}{2}} 1_{\mathrm{PGL}_2(\mathbb{Z}_2)}$ and $f_{\Phi,2} = 1_{\mathrm{PGL}_2(\mathbb{Z}_2)}$. We get $\alpha' = |D|_2^{-\frac{1}{2}} \operatorname{vol}(\mathrm{PGL}_2(\mathbb{Z}_2))$ and $\alpha = \operatorname{vol}(\mathrm{PGL}_2(\mathbb{Z}_2))$. The local constant is

$$C(\varphi_2, \Phi_2, \Phi_2') = \frac{|D|_2^{-\frac{1}{2}}}{P(W_2)}.$$

Case (6) $v = \infty$. Take Φ_∞ to be the function in [21, p.544]. It follows from [24, Remark 2.1] that $\tilde{\varphi}$ is the lowest weight vector in $\tilde{\pi}_\infty$. Recall $D > 0$. Then there exists an element $\gamma \in \mathrm{PGL}_2(F_\infty)$ such that

$$\gamma^{-1}\tau_\infty\gamma = \sqrt{D}\left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right) \in \mathrm{GL}_2(F_\infty).$$

Let $\tilde{t}_{D,\infty} = (\begin{pmatrix} \sqrt{D} & \\ & \sqrt{D}^{-1} \end{pmatrix}, 1) \in \widetilde{\mathrm{SL}}_2(F_\infty)$.

Lemma 2.8. *Let $\Phi' = \tilde{t}_{D,\infty}^{-1} * \Phi$, then*

$$P_\tau(f_{\Phi',\tau,\infty} * W_\infty) = |D|_\infty^{-\frac{3}{4}} P_\tau(\gamma * (f_{\Phi_\infty,\infty} * W_\infty)).$$

Proof. First observe

$$\begin{aligned} P_\tau(f_{\Phi',\tau,\infty} * W_\infty) &= \int_{T_{\tau,\infty}} \int_{\mathrm{PGL}_2(F_\infty)} f_{\Phi_\infty',\tau,\infty}(g) W_\infty(tg)\, dg\, dt \\ = \int_{T_{\tau,\infty}} \int_{\mathrm{PGL}_2(F_\infty)} & f_{\Phi_\infty',\tau,\infty}(tg) W_\infty(g)\, dg\, dt = \int_{\mathrm{PGL}_2(F_\infty)} \Phi_\infty'(g^{-1}\tau g) W_\infty(g)\, dg \\ &= \int_{\mathrm{PGL}_2(F_\infty)} \Phi_\infty'(g^{-1}\gamma\sqrt{D}\left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right)\gamma^{-1} g) W_\infty(g)\, dg \\ &= \int_{\mathrm{PGL}_2(F_\infty)} \Phi_\infty'(g^{-1}\sqrt{D}\left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right) g) W_\infty(\gamma g)\, dg. \end{aligned}$$

Now use the fact that $\tilde{t}_{D,\infty} * \Phi_\infty'(X) = |D|_\infty^{\frac{3}{4}} \Phi_\infty'(\sqrt{D}X)$. The above becomes

$$\begin{aligned} &|D|_\infty^{-\frac{3}{4}} \int_{\mathrm{PGL}_2(F_\infty)} \Phi_\infty(g^{-1}\left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right) g) W_\infty(\gamma g)\, dg \\ &\quad = |D|_\infty^{-\frac{3}{4}} \int_{T(F_\infty)} f_{\Phi_\infty,\infty} * W_\infty(\gamma t)\, dt = |D|_\infty^{-\frac{3}{4}} \int_{T_\tau(F_\infty)} f_{\Phi_\infty,\infty} * W_\infty(t\gamma)\, dt \end{aligned}$$

where the last identity comes from the Jacobian for the change of variable $t \mapsto \gamma^{-1}t\gamma$ is 1. □

It follows from the proof of [21, Proposition 3.5] that $f_{\Phi_\infty,\infty} * W_\infty = \alpha W_\infty$ for a nonzero scalar α. Then

$$P_\tau(\varphi)P_\tau(f_{\Phi_\infty',\tau,\infty} * W_\infty) = \alpha|D|_\infty^{-\frac{3}{4}} P_\tau(\varphi)P_\tau(\gamma * W_\infty) = \alpha|D|_\infty^{-\frac{3}{4}} P_\tau(\gamma * \varphi)P_\tau(W_\infty).$$

Thus the local constant is:

$$C(\varphi_\infty, \Phi_\infty, \Phi'_\infty) = \frac{|D|_\infty^{-\frac{3}{4}} P_\tau(\gamma * \varphi)}{P_\tau(\varphi) P(W_\infty)}.$$

Note (for some nonzero α) $W_\infty(\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)) = \alpha a^k e^{-2\pi D_1^\sharp a}$ when $a > 0$ and 0 otherwise. Thus

$$\begin{aligned} P(W_\infty) &= \alpha \int_0^\infty a^{k-1} e^{-2\pi D_1^\sharp a}\, da \\ &= \alpha (2\pi D_1^\sharp)^{-k} \Gamma(k) = \frac{(k-1)!}{\pi^k} 2^{-k} e^{2\pi} W_l\left(\left(\begin{smallmatrix} (D_1^\sharp)^{-1} & \\ & 1 \end{smallmatrix}\right)\right). \end{aligned}$$

2.7. **Summary.** It is known that if $D_1 \in \mathbb{Z}_l^\times$, then

$$P(W_l) = L(\pi_l, \frac{1}{2}) W_l(e) = L(\pi_l, \frac{1}{2}) W_l\left(\left(\begin{smallmatrix} (D_1^\sharp)^{-1} & \\ & 1 \end{smallmatrix}\right)\right).$$

When $l | D_1$, $L(\pi, s) \equiv 1$. So (7) becomes

(10)

$$\tilde{W}^D(\tilde{\varphi}') \overline{\tilde{W}(\tilde{\varphi})} = D^{-\frac{1}{4}} D_1^\sharp \chi(\tau_1) \frac{\pi^k}{(k-1)!} 2^k e^{-2\pi} \frac{P_\tau(\gamma * \varphi)}{L^f(\pi, \frac{1}{2}) W_\varphi(\left(\begin{smallmatrix} (D_1^\sharp)^{-1} & \\ & 1 \end{smallmatrix}\right))} |\tilde{W}(\tilde{\varphi})|^2.$$

Here $L^f(\pi, \cdot)$ is the finite part of the $L-$function $L(\pi, \cdot)$; χ is an odd character on $(\mathbb{Z}/\prod_{l \in S_0} l)^\times$ associated to $\{\chi_l : l \in S_0\}$, $\tilde{\varphi}$ is the vector corresponding to the half integral weight form in $S_{k+\frac{1}{2}}(4M \prod_{l \in S_0} l, \chi)$ defined in [1, Theorem 10.1], while $\tilde{\varphi}' = \chi(D_1^\sharp)(\underline{\sqrt{\frac{D_1}{D_2}}}_\infty \underline{D_1^\sharp}^{-1}) * \tilde{\varphi}$, (note that $\underline{D_1^\sharp}_l * \tilde{\varphi}_l = \tilde{\varphi}_l$ when $D_1 \in \mathbb{Z}_l^\times$ and $l \notin S_0$; when $l \in S_0$, $\underline{D_1^\sharp}_l * \tilde{\varphi}_l = \chi_l(D_1^\sharp) \tilde{\varphi}_l$). Thus

$$\tilde{W}^D(\tilde{\varphi}') = \chi(D_1^\sharp) \tilde{W}_{\tilde{\varphi}}^{D/D_1^2}(\underline{\sqrt{\frac{D_1}{D_2}}}_\infty).$$

2.8. **Proof of Theorem 2.3.** We note $P_\tau(\gamma * \varphi)$ is the integral $\ell(\varphi)$ defined in [20, (6.1.2)]. (It is easy to check that $\chi_{D_1} \circ \det$ when restricted to T_τ is a genus character on T_τ). From [20, (6.1.7)], it equals:

$$\Delta^{-\frac{k}{2}} i^{-k} r(f; D_2, D_1; \tau).$$

(Note the measure chosen by [20] differs from ours by a factor of $\Delta^{\frac{1}{2}}$: see [20, p. 830]). We observe here $\chi_{D_1} \circ \det = \chi_{D_2} \circ \det$ on T_τ, thus the relation:

$$r(f; D_2, D_1; \tau) = r(f; D_1, D_2; \tau).$$

By [27, Lemme 3], we have

$$\tilde{W}_{\tilde{\varphi}}^\xi(\underline{\sqrt{a}}_\infty) = a^{\frac{k}{2}+\frac{1}{4}} e^{-2\pi a \xi |D_1|} c(\xi |D_1|)$$

if $g(z) = \sum_{n \geq 1} c_n e^{2\pi i n z}$ is the half integral weight form corresponding to $\tilde{\varphi}$ by the recipe of [27]. Also $L^f(\pi, \frac{1}{2}) = L(f, D_1, k)$, and $W_\varphi\left(\left(\begin{smallmatrix} (|D_1|)^{-1} & \\ & 1 \end{smallmatrix}\right)\right) = e^{-2\pi}$, (recall $\psi(x) = \psi_0(|D_1| x)$). Thus (10) becomes:

$$(\frac{D_1}{D_2})^{\frac{k}{2}+\frac{1}{4}} c(|D_2|) \overline{c(|D_1|)} = \Delta^{-\frac{1}{4}-\frac{k}{2}} |D_1| i^{-k} \chi(\frac{\tau_1}{|D_1|}) \frac{\pi^k}{(k-1)!} 2^k \frac{r(f; D_2, D_1; \tau)}{L(f, D_1, k)} |c(|D_1|)|^2.$$

Now we apply [1, Theorem 10.1], and get:

$$\frac{|c_{|D_1|}|^2}{\langle g,g\rangle}=\frac{L(f,D_1,k)}{\langle f,f\rangle}|D_1|^{k-\frac{1}{2}}\frac{(k-1)!}{\pi^k}2^{|S|}\prod_{\ell\in S_0}\frac{\ell}{1+\ell}.$$

Our Theorem follows from the above two equations.

3. Families of modular forms

We keep from now on the following notation: f is a weight 2 new form of level N, square free and odd, trivial character and rational Fourier coefficients, corresponding to an elliptic curve E. Fix a prime number $p\mid N$ and put $M:=N/p$.

Choose an embedding $\bar{\mathbb{Q}}\hookrightarrow\bar{\mathbb{Q}}_p$. We will then identify algebraic numbers with p-adic numbers by means of this embedding without making this explicit in the following.

3.1. **Hida families.** Let f_∞ be the Hida family passing through f, the weight 2 modular form attached to E by Taylor-Wiles's modularity theorem. More precisely, and to fix notations, there exists a compact open neighborhood $\mathcal{U}$ of 1 in $\mathbb{Z}_p$, contained in the residue class of 1 modulo $p-1$, and a formal series expansion

$$f_\infty(k)=\sum_{n\geq1}a_n(k)q^n$$

where $a_n(k)$ are $\mathbb{C}_p$-valued rigid analytic functions on $\mathcal{U}$ (and $\mathbb{C}_p$ is the completion of a fixed algebraic closure of $\mathbb{Q}_p$), such that:

1. For any integer $k\geq1$ in $\mathcal{U}$, $f_k:=f_\infty(k)$ is the q-expansion of a p-ordinary cusp form of weight $2k$, level $\Gamma_0(N)$ and trivial character, which is an eigenform for all Hecke operators;
2. $f_1=f$.

For integers $k>1$ in $\mathcal{U}$, f_k is not p-new, and we let $f_k^\sharp=\sum_{n\geq1}a_n^\sharp(k)q^n$ be the weight $2k$ cusp form of level $\Gamma_0(M)$ and trivial character whose p-stabilization is f_k.

3.2. **Analytic continuation of generalized Kohnen's lift.** Fix a set of divisors S_0 of M and let M' be the product of the prime numbers in S_0. Let $g=\sum_{n\geq1}c_nq^n$ and $g_k^\sharp=\sum_{n\geq1}c_n^\sharp(k)q^n$ be the lifts of f and $f_k^\sharp$, respectively, relative to this choice of S_0 and the choice of an auxiliary character χ' as in Sec. 2.1. Recall that s_0 is the cardinality of S_0 and χ is a character of $(\mathbb{Z}/M')^*$ determined by χ'.

Definition 3.1. Let D be a fundamental discriminant of a quadratic field such that

1. $\left(\frac{D}{\ell}\right)=w_\ell$ if $\ell\mid(M/M')$;
2. $\left(\frac{D}{\ell}\right)=-w_\ell$ if $\ell\mid M'$;
3. $(-1)^{s_0+1}=\operatorname{sgn}(D)$.

We say that D is of *Type I* or *Type II* if

(I) $\left(\frac{D}{p}\right)=w_p$;

(II) $\left(\frac{D}{p}\right)=-w_p$.

So, discriminants of type I (resp. II) are those satisfying (†) (resp. (††)) of the Introduction. Thus, $L(f, \chi_D, 1) = 0$ and $c_{|D|} = 0$ for all D of type II, while non-vanishing results for L-functions show that there are infinitely many fundamental discriminant D_0 of type I such that $L(f, \chi_{D_0}, 1) \neq 0$ (cf. [17, Cor. 2], for example), and consequently we also have $c_{|D_0|}(1) \neq 0$. We fix such a choice of D_0 from now on.

Lemma 3.2. *There exists a neighborhood $\mathcal{U}$ of 1 in $\mathbb{Z}_p$ such that the coefficients $c^\sharp_{|D_0|}(k)$ do not vanish for all $k \in \mathcal{U}$.*

Proof. By (2), this is equivalent to show that the same is true for the values $L(f^\sharp_k, \chi_{D_0}, k)$.

We begin by fixing for each integer $k > 1$ in $\mathcal{U}$, Shimura periods $\Omega^\pm_{f^\sharp_k}$ satisfying the additional property that

$$I^\pm(f^\sharp_k, P, r, s) := \frac{\int_r^s f^\sharp_k(z)P(z)dz \pm \overline{\int_r^s f^\sharp_k(z)P(z)dz}}{2\Omega^\pm_{f^\sharp_k}}$$

belongs to the field $K_{f^\sharp_k}$ generated over $\mathbb{Q}$ by the Fourier coefficients of $f^\sharp_k$, for all polynomials P of degree at most $k-2$ and all $r, s \in \mathbb{P}^1(\mathbb{Q})$, where $\xi \mapsto \bar{\xi}$ is complex conjugation. Define the algebraic part of the special values of the relevant L-functions to be

$$L^*(f^\sharp_k, \chi_{D_0}, k) := \frac{(k-1)! \cdot \tau(\chi_{D_0})}{(-2\pi i)^{k-1} \cdot \Omega^{w_\infty}_{f^\sharp_k}} \cdot L(f^\sharp_k, \chi_{D_0}, k),$$

and, in weight 2,

$$L^*(f, \chi_{D_0}, 1) := \frac{\tau(\chi_{D_0})}{\Omega^{w_\infty}_f} \cdot L(f, \chi_{D_0}, 1),$$

where $\tau(\chi_{D_0})$ is the Gauss sum ([3, §3.1], for example). These are algebraic numbers, which we can see as p-adic numbers by the fixed embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. Then, one can equivalently show that the values $L^*(f^\sharp_k, \chi_{D_0}, k)$ do not vanish in a neighborhood of 1.

Recall the Mazur-Kitagawa p-adic L-function $L_p^{\mathrm{MK}}(f_\infty, \chi_{D_0}, k, s)$, in two variables k and s for which we use the notation in [3, Sec. 3] (except that here the weight variable is $2k$ instead of k in *loc. cit.*; to avoid confusion, we require that, for a fixed $k = k_0 \in \mathbb{Z} \cap \mathcal{U}$, $k_0 \geq 1$, the function $L_p^{\mathrm{MK}}(f_\infty, \chi_{D_0}, k_0, s)$ is the cyclotomic p-adic L-function of f_{2k_0} instead of f_{k_0} as in [3]). Its definition requires the choice of a sign at infinity w_∞ corresponding to the choice of the w_∞ eigencomponent for the action of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ on modular symbols, and the choice of the corresponding Shimura period $\Omega^{w_\infty}_{f^\sharp_k}$ and $\Omega^{w_\infty}_f$ (cf. [3, §1.1]). We make the choice of w_∞ so that (note k is odd)

$$\chi_{D_0}(-1) = (-1)^{k-1} w_\infty = w_\infty.$$

Then, by [3, Theorem 3.1] we have

$$L_p^{\mathrm{MK}}(f_\infty, \chi_{D_0}, 1, 1) = (1 - \chi_{D_0}(p)a_p^{-1}) \cdot L^*(f, \chi_{D_0}, 1) \tag{11}$$

and, by [3, Corollary 2.3], we have

$$L_p^{\mathrm{MK}}(f_\infty, \chi_{D_0}, k, k) = \lambda(k) \cdot (1 - \chi_{D_0}(p)a_p^{-1}(k)p^{k-1})^2 \cdot L^*(f^\sharp_k, \chi_{D_0}, k).$$

The $\bar{\mathbb{Q}}_p$-valued function $\lambda \mapsto \lambda(k)$ is non-zero in a neighborhood of 1, by [2, Proposition 1.7]. The choice of D_0 implies that $\chi_{D_0}(p) = w_p$, and since $a_p = -w_p$, we see that $1 - \chi_{D_0}(p) a_p^{-1} \neq 0$. Thus, since also $L(f, \chi_{D_0}, 1) \neq 0$, the Mazur-Kitagawa p-adic L-function does not vanish at $(1,1)$; since it is p-adic analytic in a neighborhood of $(1,1)$, it follows then that there exists a neighborhood of 1 where it is non-zero, which proves the non vanishing of the algebraic parts of the L-functions of $f_k^\sharp$ in a neighborhood of 1. □

Fix $\mathcal{U}$ as in Lemma 3.2. Define for each $k \in \mathcal{U} \cap \mathbb{Z}$, $k > 1$, and D a fundamental discriminant prime to D_0, the normalized Fourier coefficients

$$\tilde{c}_{|D|}(k) := \frac{1 - \chi_D(p) \cdot p^{k-1} \cdot a_p^{-1}(k)}{1 - \chi_{D_0}(p) \cdot p^{k-1} \cdot a_p^{-1}(k)} \cdot \frac{c^\sharp_{|D|}(k)}{c^\sharp_{|D_0|}(k)}. \tag{12}$$

Let $\mathbb{Q}(\chi)$ be the subfield of $\bar{\mathbb{Q}}$ generated by the values of χ. Via our fixed embedding, we will also view $\mathbb{Q}(\chi)$ as a subfield of $\bar{\mathbb{Q}}_p$.

Proposition 3.3. *Let D be a fundamental discriminant of Type I or II. After replacing $\mathcal{U}$ by a smaller neighborhood of 1 in $\mathbb{Z}_p$, we can insure that the normalized coefficients $\tilde{c}_{|D|}(k)$ extend to a p-adic analytic function on $\mathcal{U}$, whose value at 1 is*

$$\tilde{c}_{|D|}(1) = \frac{c_{|D|}}{c_{|D_0|}}.$$

Finally, $\tilde{c}_{|D|}(1)$ belongs to $\mathbb{Q}(\chi)$.

Proof. Fix a neighborhood as in Lemma 3.2 to start with. Note that

$$\frac{c^\sharp_{|D|}(k)}{c^\sharp_{|D_0|}(k)} = \frac{c^\sharp_{|D|(k)} \cdot \overline{c^\sharp_{|D_0|}(k)}}{|c^\sharp_{|D_0|}(k)|^2}.$$

Assume first that $(D, D_0) = 1$. Combining (2) and Theorem 2.3 we find:

$$\tilde{c}_{|D|}(k) = \frac{1 - \chi_D(p) \cdot p^{k-1} \cdot a_p^{-1}(k)}{1 - \chi_{D_0}(p) \cdot p^{k-1} \cdot a_p^{-1}(k)} \cdot \frac{(-2\pi i)^k \cdot \chi(|D|)^{-1}}{|D_0|^{k-1/2} \cdot (k-1)!} \cdot \frac{r_\chi(f_k^\sharp, D, D_0)}{L(f_k^\sharp, \chi_{D_0}, k)}.$$

Using the expression in the proof of the lemma above for $L(f_k^\sharp, \chi_{D_0}, k)$ in terms of the Mazur-Kitagawa p-adic L-function, we find

$$\tilde{c}_{|D|}(k) = \frac{-\tau(\chi_{D_0}) \cdot \chi(|D|)^{-1}}{|D_0|^{k-1/2}} \cdot \frac{1}{L_p^{\mathrm{MK}}(f_\infty, \chi_{D_0}, k, k)} \cdot \left(1 - \chi_D(p) \cdot p^{k-1} \cdot a_p^{-1}(k)\right) \cdot$$
$$\cdot \left(1 - \chi_{D_0}(p) \cdot p^{k-1} \cdot a_p^{-1}(k)\right) \cdot \frac{\lambda(k) \cdot (2\pi i) \cdot r_\chi(f_k^\sharp, D, D_0)}{\Omega_{f_k^\sharp}^{\omega_\infty}}.$$

Here, as in the proof of the above lemma, we make the choice of $w_\infty = \chi_{D_0}(-1)$.

Suppose that D is of Type II and $(D, D_0) = 1$. Since D_0 and D are of different types, we have

$$\left(1 - \chi_D(p) \cdot p^{k-1} \cdot a_p^{-1}(k)\right) \cdot \left(1 - \chi_{D_0}(p) \cdot p^{k-1} \cdot a_p^{-1}(k)\right) = 1 - a_p^{-2}(k) p^{2k-2}.$$

One observes now that

$$\lambda(k) \cdot \left(1 - a_p^{-2}(k) p^{2k-2}\right) \cdot \frac{(2\pi i) \cdot r_\chi(f_k^\sharp, D, D_0)}{\Omega_{f_k^\sharp}^{\omega_\infty}} = \mathcal{L}_p^{\mathrm{BD}}(f_\infty / \mathbb{Q}(\sqrt{D \cdot D_0}), \chi_{D, D_0}, k)$$

where the RHS is (up to a constant multiple) the p-adic L-function defined by Bertolini and Darmon in [3, Definition 3.4, (1)] (note that the prime p is inert in $\mathbb{Q}(\sqrt{D\cdot D_0})$, and all primes dividing M are split; also note the usual shift of notation in the weight, so actually, and more precisely, by $\mathcal{L}_p^{\mathrm{BD}}(f_\infty/\mathbb{Q}(\sqrt{D\cdot D_0}),\chi_{D,D_0},k)$ we mean the function $\mathcal{L}_p(f_\infty/\mathbb{Q}(\sqrt{D\cdot D_0}),\chi_{D,D_0},2k)$ in *loc. cit.*). Therefore, we can express $\tilde{c}_{|D|}(k)$ as a product of factors, each of them extending to a p-adic analytic function in a neighborhood of 1, and therefore the extension of normalized coefficients follows. Further, $L_p^{\mathrm{BD}}(f_\infty/\mathbb{Q}(\sqrt{D\cdot D_0}),\chi_{D,D_0},1)=0$ (cf. [3, Sec. 4]) and, since $c_{|D|}=0$, we have the claimed equality

$$\tilde{c}_{|D|}(1)=c_{|D|}=\frac{c_{|D|}}{c_{|D_0|}}.$$

Suppose now that D is of Type I and $(D,D_0)=1$. Then all primes dividing $N=Mp$ are split in $\mathbb{Q}(\sqrt{D\cdot D_0})$. Since D_0 and D are of the same type, we have

$$\left(1-\chi_D(p)\cdot p^{k-1}\cdot a_p^{-1}(k)\right)\cdot\left(1-\chi_{D_0}(p)\cdot p^{k-1}\cdot a_p^{-1}(k)\right)=\left(1-\chi_{D_0}a_p^{-1}(k)p^{k-1}\right)^2.$$

In this case, we have

$$\lambda(k)\cdot\left(1-\chi_{D_0}a_p^{-1}(k)p^{k-1}\right)^2\cdot\frac{(2\pi i)\cdot r_\chi(f_k^\sharp,D,D_0)}{\Omega_{f_k^\sharp}^{\omega_\infty}}=$$
$$=\mathcal{L}_p^{\mathrm{Sh}}(f_\infty/\mathbb{Q}(\sqrt{D\cdot D_0}),\chi_{D,D_0},k)$$

where the RHS is the p-adic analytic function (up to a constant multiple) defined in S. Shahabi's thesis [23, §. 3.2], and the above formula is [23, Prop. 3.3.1], except for the usual shift of weight. A similar construction, working more generally for rational quaternion algebras which are split at the Archimedean prime, is described in [16, §5.1] and [14, §4.5]. The extension of the normalized coefficients follows. Its value at 1 is given by

$$\frac{-\tau(\chi_{D_0})\cdot\chi(|D|)^{-1}}{|D_0|^{1/2}}\cdot\frac{\mathcal{L}_p^{\mathrm{Sh}}(f_\infty/\mathbb{Q}(\sqrt{D\cdot D_0}),\chi_{D,D_0},1)}{L_p^{\mathrm{MK}}(f_\infty,\chi_{D_0},1,1)}.$$

By [16, Prop. 90], or [14, Prop. 4.24], we have

$$\mathcal{L}_p^{\mathrm{Sh}}(f_\infty/\mathbb{Q}(\sqrt{D\cdot D_0}),\chi_{D,D_0},1)=2\cdot\frac{(2\pi i)\cdot r_\chi(f,D,D_0)}{\Omega_f^{w_\infty}}$$

and, by (11),

$$L_p^{\mathrm{MK}}(f_\infty,\chi_{D_0},1,1)=2\cdot\frac{\tau(\chi_{D_0})}{\Omega_f^{w_\infty}}\cdot L(f,\chi_{D_0},1)$$

and therefore the value $\tilde{c}_{|D|}(1)$ is given by

$$\frac{-2\pi i\cdot\chi(|D|)^{-1}}{|D_0|^{1/2}}\cdot\frac{r_\chi(f,D,D_0)}{L(f,\chi_{D_0},1)}.$$

This is equal to $c_{|D|}/c_{|D_0|}$ again by a combination of (2) and Theorem 2.3, and the statement follows.

The rationality of $c_{|D|}/c_{|D_0|}$ follows from results on the rationality of the factors appearing in the above factorizations. More precisely, one may choose Shimura periods $\Omega_f^{w_\infty}$ satisfying $\frac{2\pi\cdot r_\chi(f,D,D_0)}{\Omega_f^{w_\infty}} \in \mathbb{Q}(\chi)$ because f has integer Fourier coefficients (see [25], for example), and then, with this choice of Shimura periods, $\frac{\tau(\chi_{D_0})\cdot L(f,\chi_{D_0},1)}{\Omega_f^{w_\infty}\cdot|D_0|^{1/2}}$ belongs to $\mathbb{Q}(\chi)$ (see [17], for example).

We finally deal with the remaining case of $(D, D_0) \neq 1$. One chooses an auxiliary discriminant D_0', prime to both D and D_0, satisfying the same conditions of D_0 (this is possible by [17]). Then, express $\tilde{c}_{|D|}(k)$ as the product

$$\left(\frac{1-\chi_D(p)\cdot p^{k-1}\cdot a_p^{-1}(k)}{1-\chi_{D_0'}(p)\cdot p^{k-1}\cdot a_p^{-1}(k)}\cdot\frac{c^\sharp_{|D|}(k)}{c^\sharp_{|D_0'|}(k)}\right)\cdot\left(\frac{1-\chi_{D_0'}(p)\cdot p^{k-1}\cdot a_p^{-1}(k)}{1-\chi_{D_0}(p)\cdot p^{k-1}\cdot a_p^{-1}(k)}\cdot\frac{c^\sharp_{|D_0'|}(k)}{c^\sharp_{|D_0|}(k)}\right)$$

and repeat the above argument to each of the two factors in parenthesis appearing above, using the previously proved cases. This concludes the proof. □

Thus, for D_1 of type I and D_2 of type II, we have $\tilde{c}_{D_2}(1) = 0$ even if the function $k \mapsto \tilde{c}_{D_2}(k)$ is not identically zero on the neighborhood $\mathcal{U}$ of 1 where it is defined (this follows because $\mathcal{L}_p^{\mathrm{BD}}(f_\infty/\mathbb{Q}(\sqrt{D_1\cdot D_2}),\chi_{D_1,D_2},k)$ is not necessarily zero). It is naturally of interest to investigate then the value at 1 of its p-adic derivative, $\left(\frac{d}{dk}\tilde{c}_{D_2}(k)\right)_{|k=1}$.

Remark 3.4. It might be interesting to prove Proposition 3.3 directly, in a way similar to [6, Prop. 1.3], using arguments borrowed from the proof of [25, Thm. 5.5] (and its sequels [19], [10], [11]). Formally, our proof makes a systematic recourse to p-adic L-functions instead; however, note that the principle of our proof (i.e., the construction of p-adic L-functions) and the proof of [25, Thm. 5.5] share the same fundamental tool, namely, the p-adic interpolation of complex integrals (which are Shintani integrals for p-adic L-functions over real quadratic extensions), and originated from the seminal paper [8].

We finally need to understand the action of complex conjugation on these normalized coefficients. Let

$$i : \mathbb{Q}(\chi) \hookrightarrow \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$$

be obtained by composition with the fixed embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. Let $c \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ be a fixed complex conjugation. The composition of c on $\mathbb{Q}(\chi)$, viewed as a subfield of $\bar{\mathbb{Q}}$, with i gives rise to a second embedding

$$i^* := i\circ c : \mathbb{Q}(\chi) \hookrightarrow \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$$

(use that $\mathbb{Q}(\chi)$ is Galois over $\mathbb{Q}$). For each integer $k \in \mathcal{U}$, and each fundamental discriminant D prime to D_0, we thus have p-adic numbers

$$\overline{\tilde{c}_{|D|}(k)} := i^*\left(\tilde{c}_{|D|}(k)\right).$$

The map $i(\mathbb{Q}(\chi)) \to i^*(\mathbb{Q}(\chi))$ defined by $i(a) \mapsto i^*(a)$ clearly extends on the p-adic completions, and we denote the resulting map with the symbol $x \mapsto \bar{x}$. Thus, from Prop. 3.3, the function $k \mapsto \overline{\tilde{c}_{|D|}(k)}$ extends to a p-adic analytic function on $\mathcal{U}$, denoted by the same symbol, whose value at 1 is $\overline{\tilde{c}_{|D|}(1)} = \frac{\overline{c_{|D|}}}{\overline{c_{|D_0|}}}$, which belongs to $i^*(\mathbb{Q}(\chi)) \subseteq \bar{\mathbb{Q}}_p$.

4. Darmon-Tornaría Conjecture

We keep the notation of Sec. 3: f is a weight 2 new form of level N, square free and odd, trivial character and rational Fourier coefficients, corresponding to an elliptic curve E; $p \mid N$ is a prime and put $M := N/p$.

4.1. **Rational points on elliptic curves.** Let $K = \mathbb{Q}(\sqrt{\Delta})$ be a quadratic imaginary field where all primes dividing M are split and the prime p is inert. Fix Shimura's periods $\Omega^{\pm}_{f_k^\sharp}$ as in Lemma 3.2 for each integer $k > 1$ in $\mathcal{U}$. Recall the definition of Shintani integrals $r(f_k^\sharp, Q)$ in (3) attached to $f_k^\sharp$ and the quadratic form $Q \in \mathcal{F}_\Delta$, and put

$$\tilde{r}^{\pm}(k,Q) := \left(1 - \frac{p^{2k-2}}{a_p(k)^2}\right) \frac{(r(f_k^\sharp, Q) \pm \overline{r(f_k^\sharp, Q)})}{2\Omega^{\pm}_{f_k^\sharp}}.$$

Then these values belong to $K_{f_k^\sharp}$. Similarly, let $\tau_Q := \frac{-B+\sqrt{\Delta}}{2A}$ and define

$$I^{\pm}(k,Q) := \left(1 - \frac{p^{2k-2}}{a_p(k)^2}\right) I^{\pm}(f_k^\sharp, (z-\tau_Q)^{k-2}, \tau, \gamma_Q\tau)$$

which belong again to $K_{f_k^\sharp}$ (this is again independent on the choice of τ and only depends on the $\Gamma_0(M)$-equivalence class of Q).

Fix an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. Then τ_Q, as well as elements in $K_{f_k^\sharp}$, can be alternatively viewed in $\bar{\mathbb{Q}}$ or $\bar{\mathbb{Q}}_p$. We collect here the relevant facts about these integrals (see [6, Sec. 3] for proofs):

(a) The functions $k \mapsto r^{\pm}(k,Q)$ and $k \mapsto I^{\pm}(k,Q)$, defined for integers $k > 1$ in $\mathcal{U}$, extend to p-adic analytic functions on $\mathcal{U}$, taking values in $\bar{\mathbb{Q}}_p$, which we denote by the same symbols $r^{\pm}(k,Q)$ and $I^{\pm}(k,Q)$.

(b) We have $I^{\pm}(1,Q) = 0$.

(c) If we denote by $\xi \mapsto \varsigma(\xi)$ the non-trivial automorphism of the quadratic unramified extension $\mathbb{Q}_{p^2}$ of $\mathbb{Q}_p$, and we define

$$\vartheta^{\pm}(f,Q) := \left(\frac{d}{dk} I^{\pm}(k,Q)\right)_{|k=1}$$

then $\vartheta^{\pm}(f,Q)$ belongs to $\mathbb{Q}_{p^2}$ and

$$\vartheta^{\pm}(f,Q) + \varsigma(\vartheta^{\pm}(f,Q)) = 2\left(\frac{d}{dk}\tilde{r}^{\pm}(k,Q)\right)_{|k=1}. \tag{13}$$

Let χ_{D_1,D_2} be the genus character associated to a factorization $\Delta = D_1 \cdot D_2$ of the discriminant of K, as in Sec. 2. Recall that χ_{D_1,D_2} is associated with the pair of Dirichlet characters (χ_{D_1}, χ_{D_2}), with associated quadratic fields $K_i = \mathbb{Q}(\sqrt{D_i})$, $i = 1, 2$. Let $\epsilon = +1$ if the K_i's are both real and $\epsilon = -1$ if the K_i's are both imaginary. Following the terminology in [3, Def. 3.4], the claracter χ_{D_1,D_2} is said to be *even* in the first case, and *odd* in the second. Since $\chi_{D_1} \cdot \chi_{D_2} = \chi_K$, which is the quadratic character associated with K, then $\chi_{D_1}(\ell) = \chi_{D_2}(\ell)$ for all $\ell \mid M$, while $\chi_{D_1}(p) = -\chi_{D_2}(p)$.

Define the following p-adic number (in $\mathbb{Q}_{p^2}$):

$$\vartheta(f, D_1, D_2; \tau) := \sum_{[Q] \in \mathcal{F}_\Delta/\Gamma_0(M)} \chi_{D_1,D_2}(Q)\vartheta(f,Q)$$

and let H_{D_1,D_2} be the quadratic extension of K determined by χ_{D_1,D_2}. Also, denote by

$$\log_E : E(\bar{\mathbb{Q}}_p) \longrightarrow \bar{\mathbb{Q}}_p$$

the p-adic formal logarithm defined by $\log_E(P) := \log_q(\Phi_{\mathrm{Tate}}^{-1}(P))$, where q is Tate's period of E at p, $\log_q$ is the branch of the p-adic logarithm satisfying $\log_q(q) = 0$ and Φ_{Tate} is Tate's uniformization of the elliptic curve. The following result is [3, Thm. 4.3].

Theorem 4.1 (Bertolini-Darmon). *Suppose that (χ_{D_1}, χ_{D_2}) satisfies*

$$\chi_{D_1}(-M) = \chi_{D_2}(-M) = -w_M, \qquad \chi_{D_2}(p) = -\chi_{D_1}(p) = -w_p.$$

There exists a point

$$P_{D_1,D_2} \in (E(H_{D_1,D_2}) \otimes_{\mathbb{Z}} \mathbb{Q})^{\chi_{D_1,D_2}},$$

in the subspace of $E(H_{D_1,D_2}) \otimes_{\mathbb{Z}} \mathbb{Q}$ where the Galois group $\mathrm{Gal}(H_{D_1,D_2}/K)$ *acts via the character χ_{D_1,D_2}, such that:*

1. $\log_E(P_{D_1,D_2}) = \vartheta(f, D_1, D_2; \tau)$;
2. *P_{D_1,D_2} is non-zero if and only if $L'(E, \chi_{D_2}, 1) \neq 0$.*

Remark 4.2. As in Remark 2.1, $\vartheta_\chi(f, D_1, D_2) := \chi(\tau_1)\vartheta(f, D_1, D_2; \tau)$ is independent of the choice of τ.

4.2. **Darmon points and generalized Kohnen lifts.** We fix the sign ϵ as in Sec. 4.1 taking $\epsilon = 1$ for χ_{D_1,D_2} even and $\epsilon = -1$ for χ_{D_1,D_2} odd.

Recall the choice of the periods $\Omega^{\pm}_{f_k^\sharp}$ made in Sec. 4 and the fundamental discriminant D_0 of type I chosen in Sec. 2. By [6, Lemma 3.3], these periods can be chosen so that, after replacing $\mathcal{U}$ by a smaller neighborhood, the following equality holds:

$$\Omega^{\epsilon}_{f_k^\sharp} = \left(1 - w_p \frac{p^{k-1}}{a_p(k)}\right)^2 r(f_k^\sharp, D_0, D_0). \tag{14}$$

We will assume to have done this choice from now on.

Recall that g is the generalized Kohnen-Waldspurger lift in [1] and $\tilde{c}_{|D|}(k)$ are the normalized coefficients introduced in (12).

Proposition 4.3. *Let D_1 (reps. D_2) be of type I (resp. type II). Then*

$$\chi(|D_1|) \cdot \overline{\tilde{c}_{|D_1|}(1)} \cdot \left(\frac{d}{dk}\tilde{c}_{|D_2|}(k)\right)_{|k=1} = \vartheta_\chi(f, D_1, D_2).$$

Proof. Notice that, for all integers $k > 1$ in $\mathcal{U}$, we have

$$\overline{\tilde{c}_{|D_1|}(k)} \cdot \tilde{c}_{|D_2|}(k) = \frac{\left(1 - \frac{p^{2k-2}}{a_p(k)^2}\right) \overline{c_{|D_1|}(k)} c_{|D_2|}(k)}{\left(1 - w_p \frac{p^{k-1}}{a_p(k)}\right)^2 |c_{|D_0|}(k)|^2}.$$

By Theorem 2.3, we have then

$$\overline{\tilde{c}_{|D_1|}(k)} \cdot \tilde{c}_{|D_2|}(k) = \chi(|D_1|)^{-1} \cdot \frac{\left(1 - \frac{p^{2k-2}}{a_p(k)^2}\right) r_\chi(f_k^\sharp, D_1, D_2)}{\left(1 - w_p \frac{p^{k-1}}{a_p(k)}\right)^2 r(f_k^\sharp, D_0, D_0)}.$$

Therefore, using (14),

$$\overline{\tilde{c}_{|D_1|}(k)} \cdot \tilde{c}_{|D_2|}(k) = \chi(|D_1|)^{-1} \cdot \left(1 - \frac{p^{2k-2}}{a_p(k)^2}\right) \frac{r_\chi(f_k^\sharp, D_1, D_2)}{\Omega^\epsilon_{f_k^\sharp}}. \tag{15}$$

With the above choice of ϵ, we have $r(f_k^\sharp, Q) = \tilde{r}^\epsilon(k, Q)$ (cf. [3, eq. (27)], or [16, Lemma 3.4], [14, §4.3]). Combining (13) and [6, eq. (16)] we get

$$\vartheta_\chi(f, D_1, D_2) = \chi(|D_1|) \cdot \frac{d}{dk}\left(\overline{\tilde{c}_{|D_1|}(k)} \cdot \tilde{c}_{|D_2|}(k)\right)_{|k=1}. \tag{16}$$

Differentiating (15), using that $\tilde{c}_{D_2}(1) = 0$ because D_2 is of type II, and substituting (16) we get the result. □

We now apply Theorem 4.1 in this situation. Before doing this, we observe that, for fundamental discriminants D_1 and D_2 of type I and II respectively, the condition

$$\chi_{D_2}(p) = -\chi_{D_1}(p) = -w_p$$

is (I) and (II) in Def. 3.1, respectively, while the condition

$$\chi_{D_1}(-M) = \chi_{D_2}(-M) = -w_M$$

is equivalent to

$$\chi_{D_1}(-1) = \chi_{D_2}(-1) = (-1)^{s_0+1},$$

where recall that s_0 is the cardinality of the set S_0 and M' is the product of the primes in S_0: this is because $\chi_{D_1}(\ell) = \chi_{D_2}(\ell) = w_\ell$ for all primes $\ell \mid (M/M')$ (by (1) in Def. 3.1) and $\chi_{D_1}(\ell) = \chi_{D_2}(\ell) = -w_\ell$ for all primes dividing M' (by (2) in Def. 3.1). Thus, $\mathbb{Q}(\sqrt{D_1})$ and $\mathbb{Q}(\sqrt{D_2})$ are both real or imaginary, accordingly with the parity of s_0: odd in the first case, even in the second, and this is precisely condition (3) in Def. 3.1 (which, of course, agrees with $(*)$ required in the introduction of the paper).

Theorem 4.4. *Let D_1 be of type I and D_2 of type II. Also assume that $c_{|D_1|} \neq 0$. There exists a point*

$$\mathbf{P} \in (E(H_{D_1,D_2}) \otimes_{\mathbb{Z}} \mathbb{Q}(\chi))^{\chi_{D_1,D_2}}$$

such that:

1. $\log_E(\mathbf{P}) = \left(\frac{d}{dk}\tilde{c}_{|D_2|}(k)\right)_{|k=1}$*;*
2. $\mathbf{P}$ *is non-zero if and only if* $L'(E, \chi_{D_2}, 1) \neq 0$*.*

Proof. Combining Prop. 4.3 and Thm. 4.1, we see that, for $P = P_{D_1,D_2}$ as in Thm. 4.1,

$$\log_E(P) = \chi\left(\frac{|D_1|}{\tau_1}\right) \cdot \overline{\tilde{c}_{|D_1|}(1)} \cdot \left(\frac{d}{dk}\tilde{c}_{|D_2|}(k)\right)_{|k=1}.$$

Since $\tilde{c}_{|D_1|}(1)$ belongs to $\mathbb{Q}(\chi)$, assertion (1) follows with

$$\mathbf{P} = P \otimes \left(\chi\left(\frac{|D_1|}{\tau_1}\right) \cdot \overline{\tilde{c}_{|D_1|}(1)}\right).$$

Finally, Prop. 3.3 shows that $\tilde{c}_{|D_1|}(1) \neq 0$ if and only if $c_{|D_1|} \neq 0$, so the second part of Theorem 4.1 shows that this point is non-zero if and only $L'(E, \chi_{D_2}, 1) \neq 0$, thus showing assertion (2) and finishing the proof. □

We close this section with another application, which establishes Equation (1) of the Introduction.

Theorem 4.5. *Let D be a fundamental discriminant of type II. There exists a point $P_D \in E(\bar{\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathbb{Q}(\chi)$, which is non-zero if and only if $L'(E, \chi_D, 1) \neq 0$, such that*

$$\log_E(P_D) = \left(\frac{d}{dk} \tilde{c}_{|D|}(k) \right)_{|k=1}.$$

Further, if $D < 0$, then we may take $P_D \in E(\sqrt{D}) \otimes_{\mathbb{Z}} \mathbb{Q}(\chi)$.

Proof. Put $D_2 := D$. Fix a discriminant D_1 of type I such that $(D_1, D_2) = 1$ and $c_{|D_1|}$. Let $\Delta := D_1 \cdot D_2$ be the discriminant of the totally real field $K = \mathbb{Q}(\sqrt{\Delta})$. Then one can apply Theorem 4.6 and obtain the first part of the statement for $P_D = \mathbf{P}$. The second part follows from the proof of [3, Theorem 4.3] because if $D < 0$ then the point P_D actually belongs to the imaginary quadratic field $\mathbb{Q}(\sqrt{D})$. □

4.3. **Generating series and a conjecture of Darmon-Tornaría.** Based on the work accomplished thus far, we are in position to address the conjecture of Darmon and Tornaría in [6, Conj. 5.3, Case 1]. Let D_1 (resp. D_2) be of type I (resp. type II), coprime to each other with D_1 and D_2 both positive (resp. negative) if s_0 is odd (resp. even). Put

$$\eta_\chi(f, D_1, D_2) := \chi^{-1}(|D_1|) \cdot \vartheta_\chi(f, D_1, D_2)$$

for D_1 of type I and D_2 of type II. Let g be given as in Sec. 2.1 and $g^* := \sum_n \overline{c_n} q^n$ be the form obtained from $g = \sum_n c_n q^n$ by applying the complex conjugation.

Theorem 4.6. *The coefficients $\eta_\chi(f, D_1, D_2)$ for D_1 of type I are proportional to the $|D_1|$-th coefficient of g^*, and they do not vanish identically if and only if $L'(E, \chi_{D_2}, 1) \neq 0$.*

Proof. Combining Prop. 4.3 and Prop. 3.3, and using that $c_{D_0} \neq 0$, we have

$$\vartheta_\chi(f, D_1, D_2) = \frac{\chi(|D_1|) \cdot \left(\frac{d}{dk} \tilde{c}_{|D_2|}(k) \right)_{|k=1}}{\overline{c_{|D_0|}}} \overline{c_{|D_1|}}.$$

Thm. 4.4 shows that the coefficient of proportionality $\frac{\left(\frac{d}{dk} \tilde{c}_{|D_2|}(k) \right)_{|k=1}}{c_{|D_0|}}$ is non-zero if and only if $L'(E, \chi_{D_2}, 1) \neq 0$. □

Remark 4.7. We can remove the dependence on the character χ in the above Theorem. Fix a congruence class m in $\mathbb{Z}/M'\mathbb{Z}$ and let $g_m := \sum_n c_n^{(m)} q^n$ where $c_n^{(m)} = c_n$ if $n = m \in \mathbb{Z}/M'\mathbb{Z}$ and $c_n^{(m)} = 0$ if $n \neq m \in \mathbb{Z}/M'\mathbb{Z}$. Then g_m is a half integral weight form with respect to the congruence group $\Gamma_1(4MM')$. We get from the above theorem that $\vartheta(f, D_1, D_2, \tau)$ is proportional to $|D_1|$-th coefficient of g_m^* whenever D_1 is of type I and $|D_1| = m \in \mathbb{Z}/M'\mathbb{Z}$. They do not vanish identically if and only if $L'(E, \chi_{D_2}, 1) \neq 0$.

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