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Tangential derivatives and higher-order regularizing properties of the double layer heat potential

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Abstract: We prove an explicit formula for the tangential derivatives of the double layer heat potential. By exploiting such a formula, we prove the validity of a regularizing property for the integral operator associated to the double layer heat potential in spaces of functions with high-order derivatives in parabolic Hölder spaces defined on the boundary of parabolic cylinders which are unbounded in the time variable.

Keywords: Integral operators in parabolic Schauder spaces, double layer heat potential

MSC 2010: 31B10

1 Introduction

This paper is mainly devoted to continuity and regularizing properties of the integral operator associated to the double layer heat potential on the boundary of parabolic cylinders which are unbounded in the time variable, and is intended as a continuation of [17]. Throughout the paper, we assume that

$$n \in \mathbb{N} \setminus \{0, 1\},$$

where $\mathbb{N}$ denotes the set of natural numbers including 0. Let $\alpha \in]0, 1[$, $m \in \mathbb{N} \setminus \{0\}$, $T \in]-\infty, +\infty]$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Let $\nu \equiv (\nu_l)_{l=1,\dots,n}$ denote the external unit normal to $\partial\Omega$. Let Φ_n denote the fundamental solution of the heat equation in $\mathbb{R}^n$ (cf. (2.3)). Then we introduce the double layer heat potential

$$w[\partial_T\Omega, \varphi](t, x) \equiv \int_{-\infty}^{t} \int_{\partial\Omega} \frac{\partial}{\partial\nu(y)} \Phi_n(t - \tau, x - y)\varphi(\tau, y)\, d\sigma_y\, d\tau$$

for all $(t, x) \in \overline{]-\infty, T[} \times \mathbb{R}^n$, where the density (or moment) φ is a function from $\overline{]-\infty, T[} \times \partial\Omega$ to $\mathbb{C}$.

The analysis of the properties of the integral operator associated to the double layer heat potential is a classical topic. Indeed, the double layer heat potential has been systematically exploited in the analysis of boundary value problems for the heat equation (cf., e.g., Ladyženskaja, Solonnikov and Ural'ceva [16], Pogrozelski [22], Miranda [21] and Baderko [1]).

A first systematic treatment of the properties of layer heat potentials can be found in the works of Gevrey [7, 8], where the author has studied the properties of heat potentials in the case $n = 1$.

Then Van Tun [24–26] has developed the work of Gevrey in a series of papers and has obtained some results on the Schauder regularity of heat potentials. In particular, Van Tun has proved that the integral operator associated to the double layer heat potential defined on the boundary of a parabolic cylinder has a regularizing effect, more precisely, that the Hölder exponent of the integral operator associated to the dou-

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ble layer heat potential on the boundary of a parabolic cylinder equals the sum of the Hölder exponent of the density and of $\frac{1}{2}$.

In case $m \in \mathbb{N}$ and Ω is of class $C^{m,\alpha}$, it has long been known that if the density φ is of class

$$C^{\frac{m+\alpha}{2};m+\alpha}(\overline{]-\infty, T[} \times \partial\Omega),$$

then the restriction of the double layer potential to the set $\overline{]-\infty, T[} \times \Omega$ can be extended to a function of $C^{\frac{m+\alpha}{2};m+\alpha}(\overline{]-\infty, T]} \times \mathrm{cl}\,\Omega)$ (cf., e.g., Ladyženskaja, Solonnikov and Ural'ceva [16]).

In case $m \in \mathbb{N}$ and Ω is of class $C^{m+2,\alpha}$, Kamynin [11–14] has proved that the integral operator associated to the double layer heat potential is bounded from the Schauder space $C^{\frac{m+\alpha}{2};m+\alpha}([0, T] \times \partial\Omega)$ to $C^{\frac{m+1+\alpha'}{2};m+1+\alpha'}([0, T] \times \partial\Omega)$ for $\alpha' \in]0, \alpha[$, $T < +\infty$.

Then Costabel [2] has considered the case of anisotropic Sobolev spaces and has proved some mapping property of heat potentials in Sobolev spaces on Lipschitz domains. We also mention the work of Lewis and Murray [18] and Hofmann and Lewis [10] for time dependent Lipschitz domains.

Finally, we mention the work of Konënkov [15], where the author has studied the mapping properties of the double layer heat potential in Zygmund spaces and has applied such results to the Dirichlet problem for the heat equation in Zygmund space.

In this paper, we are interested in the regularizing property of the integral operator $w[\partial_T\Omega, \cdot\,]$ in parabolic Schauder spaces defined on the boundary of parabolic cylinders under the assumption that Ω is bounded and of class $C^{m,\alpha}$. Thus we plan to prove, in a parabolic setting, the corresponding results of [4] for the double layer potential associated to the fundamental solution of an arbitrary second-order elliptic operator with constant coefficients. For references to previous contributions on the double layer potential for second-order elliptic operators, we refer to [4]. In particular, we do not flatten the boundary with parametrization functions as done by other authors, but we prove our statements exploiting tangential derivatives, time derivatives and an inductive argument to reduce the problem to the case $m = 0$, which we have already considered in [17]. The tangential derivatives of $f \in C^1(\partial\Omega)$ are defined by the equality

$$M_{ij}[f] \equiv \nu_i \frac{\partial \tilde{f}}{\partial x_j} - \nu_j \frac{\partial \tilde{f}}{\partial x_i} \quad \text{on } \partial\Omega \quad \text{for } i, j \in \{1, \dots, n\}.$$

Here $\tilde{f}$ denotes a C^1-extension of f to an open neighborhood of $\partial\Omega$, and one can easily verify that $M_{ij}[f]$ is independent of the specific choice of the extension $\tilde{f}$ of f. Then we prove an explicit formula for

$$M_{ij}[w[\partial_T\Omega, \varphi]](t, x) - w[\partial_T\Omega, M_{ij}[\varphi]](t, x)$$

for all $(t, x) \in \overline{]-\infty, T[} \times \partial\Omega$ and for all $\varphi \in C^{\frac{1}{2};1}(\overline{]-\infty, T[} \times \partial\Omega)$ (see formula (4.1)). Formula (4.1) involves auxiliary integral operators, which we analyze in Section 3, where we prove some mapping properties of such auxiliary operators in Schauder spaces.

In Section 4, we can prove that, for all $\beta \in]0, \alpha[$, the operator $w[\partial_T\Omega, \cdot\,]$ is linear and continuous from $C^{\frac{m}{2};m}(\overline{]-\infty, T[} \times \partial\Omega)$ to $C^{\frac{m+\beta}{2};m+\beta}(\overline{]-\infty, T[} \times \partial\Omega)$ and from $C^{\frac{m+\beta}{2};m+\beta}(\overline{]-\infty, T[} \times \partial\Omega)$ to $C^{\frac{m+\alpha}{2};m+\alpha}(\overline{]-\infty, T[} \times \partial\Omega)$ by exploiting formula (4.1) and the results contained in [17]. In particular, the double layer heat potential has a regularizing effect on the boundary of $\overline{]-\infty, T[} \times \Omega$ when Ω is of class $C^{m,\alpha}$.

2 Notation and preliminaries

We denote the norm on a normed space $\mathcal{X}$ by $\|\cdot\|_{\mathcal{X}}$. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. We endow the space $\mathcal{X} \times \mathcal{Y}$ with the norm defined by $\|(x, y)\|_{\mathcal{X}\times\mathcal{Y}} \equiv \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{Y}}$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$, while we use the Euclidean norm for $\mathbb{R}^n$. For standard definitions of calculus in normed spaces, we refer to Deimling [3]. Let $\mathbb{D} \subseteq \mathbb{R}^n$. Then $\mathrm{cl}\,\mathbb{D}$ denotes the closure of $\mathbb{D}$, $\partial\mathbb{D}$ denotes the boundary of $\mathbb{D}$ and $\mathrm{diam}(\mathbb{D})$ denotes the diameter of $\mathbb{D}$. However, if $t_1 \in [-\infty, +\infty[$, $t_2 \in]-\infty, +\infty]$, $t_1 \le t_2$, then $\overline{]t_1, t_2[}$ denotes the closure of $]t_1, t_2[$. The symbol $|\cdot|$ denotes the Euclidean modulus in $\mathbb{R}^n$ or in $\mathbb{C}$. For all $R \in]0, +\infty[$, $x \in \mathbb{R}^n$, x_j denotes the j-th coordinate of x for all $j \in \{1, \dots, n\}$, and $\mathbb{B}_n(x, R)$ denotes the ball $\{y \in \mathbb{R}^n : |x - y| < R\}$. The space of bounded and continuous functions from $\mathbb{D}$ to $\mathcal{X}$ is denoted by $B(\mathbb{D}, \mathcal{X})$ and $C^0(\mathbb{D}, \mathcal{X})$, respectively.

We endow $B(\mathbb{D},\mathcal{X})$ with the sup-norm, and we set $C^0_b(\mathbb{D},\mathcal{X}) \equiv C^0(\mathbb{D},\mathcal{X}) \cap B(\mathbb{D},\mathcal{X})$. Let Ω be an open subset of $\mathbb{R}^n$. Then we set $\Omega^- \equiv \mathbb{R}^n \setminus \mathrm{cl}\,\Omega$. The space of m-times continuously differentiable complex-valued functions on Ω is denoted by $C^m(\Omega,\mathbb{C})$, or more simply, by $C^m(\Omega)$. Let $f \in C^m(\Omega)$. Then Df denotes the Jacobian matrix of f. Let $\eta \equiv (\eta_1,\dots,\eta_n) \in \mathbb{N}^n$, $|\eta| \equiv \eta_1 + \dots + \eta_n$. Then $D^\eta f$ denotes

$$\frac{\partial^{|\eta|} f}{\partial x_1^{\eta_1} \dots \partial x_n^{\eta_n}}.$$

The subspace of $C^m(\Omega)$ of those functions f whose derivatives $D^\eta f$ of order $|\eta| \le m$ can be extended with continuity to $\mathrm{cl}\,\Omega$ is denoted $C^m(\mathrm{cl}\,\Omega)$.

The subspace of $C^m(\mathrm{cl}\,\Omega)$ whose derivatives up to order m are bounded is denoted $C^m_b(\mathrm{cl}\,\Omega)$. Then $C^m_b(\mathrm{cl}\,\Omega)$ endowed with the norm $\|f\|_{C^m_b(\mathrm{cl}\,\Omega)} \equiv \sum_{|\eta|\le m} \sup_{\mathrm{cl}\,\Omega} |D^\eta f|$ is a Banach space. If Ω is bounded, then $C^m_b(\mathrm{cl}\,\Omega) = C^m(\mathrm{cl}\,\Omega)$. Now let ω be a function of $]0,+\infty[$ to itself such that

$$\omega \text{ is increasing}, \quad \lim_{r\to 0^+} \omega(r) = 0,$$

$$\sup_{(a,r)\in[1,+\infty[\times]0,+\infty[} \frac{\omega(ar)}{a\omega(r)} < +\infty \quad \text{and} \quad \sup_{r\in]0,1[} \omega^{-1}(r)r < +\infty.$$

If f is a function from a subset $\mathbb{D}$ of $\mathbb{R}^n$ to a normed space $\mathcal{X}$, then we set

$$|f : \mathbb{D}|_{\omega(\cdot)} \equiv \sup\left\{ \frac{\|f(x) - f(y)\|_{\mathcal{X}}}{\omega(|x-y|)} : x, y \in \mathbb{D},\ x \neq y \right\}.$$

If $|f : \mathbb{D}|_{\omega(\cdot)} < +\infty$, we say that f is $\omega(\cdot)$-Hölder continuous. Sometimes, we simply write $|f|_{\omega(\cdot)}$ instead of $|f : \mathbb{D}|_{\omega(\cdot)}$. The subspace of $C^0(\mathbb{D},\mathcal{X})$ whose functions are $\omega(\cdot)$-Hölder continuous is denoted $C^{0,\omega(\cdot)}(\mathbb{D},\mathcal{X})$.

The space $C^{0,\omega(\cdot)}_b(\mathbb{D},\mathcal{X}) \equiv C^{0,\omega(\cdot)}(\mathbb{D},\mathcal{X}) \cap B(\mathbb{D},\mathcal{X})$ endowed with the norm

$$\|f\|_{C^{0,\omega(\cdot)}_b(\mathbb{D},\mathcal{X})} \equiv \sup_{\mathbb{D}} \|f\|_{\mathcal{X}} + |f : \mathbb{D}|_{\omega(\cdot)}$$

is a Banach space. If $\mathcal{X} = \mathbb{C}$, we simply write $C^0(\mathbb{D})$, $C^{0,\omega(\cdot)}(\mathbb{D})$, $C^{0,\omega(\cdot)}_b(\mathbb{D})$ instead of $C^0(\mathbb{D},\mathbb{C})$, $C^{0,\omega(\cdot)}(\mathbb{D},\mathbb{C})$, $C^{0,\omega(\cdot)}_b(\mathbb{D},\mathbb{C})$, respectively.

Particularly important is the case in which $\omega(\cdot)$ is the function r^α for some fixed $\alpha \in]0,1]$. In this case, we simply write $C^{0,\alpha}(\mathbb{D})$, $C^{0,\alpha}_b(\mathbb{D})$, $|\cdot : \mathbb{D}|_\alpha$ instead of $C^{0,r^\alpha}(\mathbb{D})$, $C^{0,r^\alpha}_b(\mathbb{D})$, $|\cdot : \mathbb{D}|_{r^\alpha}$, respectively. Let $m \in \mathbb{N}$. For the definition of open subsets of $\mathbb{R}^n$ of class C^m and $C^{m,\alpha}$ for some $\alpha \in]0,1]$, and function spaces $C^{m,\alpha}(\mathrm{cl}\,\Omega)$, $C^{m,\alpha}(\partial\Omega)$ for some $\alpha \in]0,1]$, we refer to Gilbarg and Trudinger [9, pp. 52, 95]. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class C^1. We denote by $\nu \equiv (\nu_l)_{l=1,\dots,n}$ the external unit normal to $\partial\Omega$. Then we have the following well-known extension result. For a proof, we refer to Troianiello [23, Theorem 1.3, Lemma 1.5].

Lemma 2.1. *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$, $j \in \{0,\dots,m\}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Let $R \in]0,+\infty[$ be such that $\mathrm{cl}\,\Omega \subseteq \mathbb{B}_n(0,R)$. Then there exists a linear and continuous extension operator "$\tilde{\cdot}$" of $C^{j,\alpha}(\partial\Omega)$ to $C^{j,\alpha}(\mathrm{cl}\,\mathbb{B}_n(0,R))$, which takes $\mu \in C^{j,\alpha}(\partial\Omega)$ to a map $\tilde{\mu} \in C^{j,\alpha}(\mathrm{cl}\,\mathbb{B}_n(0,R))$ such that $\tilde{\mu}_{|\partial\Omega} = \mu$ and such that the support of $\tilde{\mu}$ is compact and contained in $\mathbb{B}_n(0,R)$. The same statement holds by replacing $C^{m,\alpha}$ and $C^{j,\alpha}$ by C^m and C^j, respectively.*

Let Ω be a bounded open subset of $\mathbb{R}^n$ of class C^1. Let $i, j \in \{1,\dots,n\}$. Let $f \in C^1(\partial\Omega)$. Then the M_{ij}-tangential derivative of f is defined as

$$M_{ij}[f] \equiv \nu_i \frac{\partial \tilde{f}}{\partial x_j} - \nu_j \frac{\partial \tilde{f}}{\partial x_i} \quad \text{on } \partial\Omega,$$

and the tangential gradient $D_{\partial\Omega} f$ of f is defined as

$$D_{\partial\Omega} f \equiv D\tilde{f} - (\nu \cdot D\tilde{f})\nu \quad \text{on } \partial\Omega, \tag{2.1}$$

where $\tilde{f}$ is an extension of f of class C^1 in an open neighborhood of $\partial\Omega$ as in Lemma 2.1. It is easy to verify that $M_{ij}[f]$ is independent on the specific choice of the extension $\tilde{f}$ of f. Moreover, we have the following formula for the r-th component of $D_{\partial\Omega} f$:

$$\frac{\partial \tilde{f}}{\partial x_r} - (\nu \cdot D\tilde{f})\nu_r = \sum_{l=1}^n M_{lr}[f]\nu_l \quad \text{on } \partial\Omega \quad \text{for all } r \in \{1,\dots,n\}. \tag{2.2}$$

We also need the following well-known consequence of the divergence theorem.

Lemma 2.2. *Let Ω be a bounded open subset of $\mathbb{R}^n$ of class C^1. If $\varphi, \psi \in C^1(\partial\Omega)$, then*

$$\int_{\partial\Omega} M_{lj}[\varphi]\psi \, d\sigma = -\int_{\partial\Omega} \varphi M_{lj}[\psi] \, d\sigma \quad \text{for all } l, j \in \{1, \dots, n\}.$$

Next we introduce the parabolic Hölder spaces on cylindrical domains. If $T \in]-\infty, +\infty]$ and if $\mathbb{D}$ is a subset of $\mathbb{R}^n$, then we set

$$\mathbb{D}_T \equiv \overline{]-\infty, T[} \times \mathbb{D}, \quad \partial_T\mathbb{D} \equiv (\partial\mathbb{D})_T = \overline{]-\infty, T[} \times \partial\mathbb{D}.$$

Clearly, $\overline{]-\infty, T[} =]-\infty, T]$ if $T \in \mathbb{R}$, and $\overline{]-\infty, T[} =]-\infty, +\infty[$ if $T = +\infty$. We also note that $(\mathrm{cl}\,\mathbb{D})_T = \mathrm{cl}\,\mathbb{D}_T$.

Remark 2.3. As is well known, the map Ξ from the vector space $\mathbb{C}^{\mathbb{D}_T}$ of functions from $\mathbb{D}_T$ to $\mathbb{C}$ to the vector space $(\mathbb{C}^{\mathbb{D}})^{\overline{]-\infty,T[}}$ of functions from $\overline{]-\infty, T[}$ to $\mathbb{C}^{\mathbb{D}}$, which takes a function f to the function Ξf from $\overline{]-\infty, T[}$ to $\mathbb{C}^{\mathbb{D}}$, which takes t to $f(t, \cdot)$, is an isomorphism. As a rule, we omit to write the canonical identification map Ξ.

Then we have the following:

Definition 2.4. Let $\alpha', \alpha'' \in]0, 1]$, $T \in]-\infty, +\infty]$. Let $\mathbb{D}$ be a subset of $\mathbb{R}^n$. Then $C^{0,\frac{\alpha'}{2};0,\alpha''}(\mathbb{D}_T)$ denotes the space of bounded functions u from $\mathbb{D}_T$ to $\mathbb{C}$ such that

$$\|u\|_{C^{0,\frac{\alpha'}{2};0,\alpha''}(\mathbb{D}_T)} \equiv \sup_{\mathbb{D}_T}|u| + \sup_{\substack{t_1,t_2\in\overline{]-\infty,T[},\\ t_1\neq t_2}} \frac{\|u(t_1,\cdot) - u(t_2,\cdot)\|_{C^0_b(\mathbb{D})}}{|t_1 - t_2|^{\frac{\alpha'}{2}}} + \sup_{t\in\overline{]-\infty,T[}} |u(t,\cdot) : \mathbb{D}|_{\alpha''} < +\infty.$$

We now define spaces of functions with higher-order derivatives in parabolic Hölder spaces, i.e., parabolic Schauder spaces. For this purpose, we set

$$C^{0,0;0,0}(\mathbb{D}_T) \equiv C^0_b(\mathbb{D}_T), \quad C^{0,\alpha';0,0}(\mathbb{D}_T) \equiv C^{0,\alpha'}_b(\overline{]-\infty, T[}, C^0(\mathbb{D})),$$
$$C^{0,0;0,\alpha''}(\mathbb{D}_T) \equiv C^0_b(\overline{]-\infty, T[}, C^{0,\alpha''}(\mathbb{D}))$$

for all subsets $\mathbb{D}$ of $\mathbb{R}^n$.

Definition 2.5. Let $\alpha', \alpha'' \in [0, 1]$, $T \in]-\infty, +\infty]$. Let Ω be an open subset of $\mathbb{R}^n$. Then $C^{0,\frac{1+\alpha'}{2};1,\alpha''}(\mathrm{cl}\,\Omega_T)$ denotes the space of bounded continuous functions from $\mathrm{cl}\,\Omega_T$ to $\mathbb{C}$ such that $\partial_{x_i}u$ is a bounded continuous function from Ω_T to $\mathbb{C}$, which admits a continuous extension to $\mathrm{cl}\,\Omega_T$ for all $i \in \{1, \dots, n\}$, and

$$\|u\|_{C^{0,\frac{1+\alpha'}{2};1,\alpha''}(\mathrm{cl}\,\Omega_T)} \equiv \sup_{\mathrm{cl}\,\Omega_T}|u| + \sum_{i=1}^n \|\partial_{x_i}u\|_{C^{0,\frac{\alpha'}{2};0,\alpha''}(\mathrm{cl}\,\Omega_T)} + \sup_{\substack{t_1,t_2\in\overline{]-\infty,T[},\\ t_1\neq t_2}} \frac{\|u(t_1,\cdot) - u(t_2,\cdot)\|_{C^0(\mathrm{cl}\,\Omega)}}{|t_1 - t_2|^{\frac{1+\alpha'}{2}}} < +\infty.$$

In particular, $u \in C^{0,\frac{1+\alpha'}{2};1,\alpha''}(\mathrm{cl}\,\Omega_T)$ if and only if

$$\partial_{x_i}u \in C^{0,\frac{\alpha'}{2};0,\alpha''}(\mathrm{cl}\,\Omega_T) \quad \text{for all } i \in \{1, \dots, n\} \quad \text{and} \quad u \in C^{0,\frac{1+\alpha'}{2}}(\overline{]-\infty, T[}, C^0(\mathrm{cl}\,\Omega)).$$

Let $m \in \mathbb{N} \setminus \{0, 1\}$. We denote by $[h]$ and $\{h\}$ the integer and the fractional part of a real number $h \in \mathbb{R}$, respectively. Then $C^{[\frac{m}{2}],\{\frac{m}{2}\}+\frac{\alpha'}{2};m,\alpha''}(\mathrm{cl}\,\Omega_T)$ denotes the space of bounded continuous functions from $\mathrm{cl}\,\Omega_T$ to $\mathbb{C}$ such that $\partial_{x_i}u$ and $\partial_t u$ are bounded continuous from Ω_T to $\mathbb{C}$ and admit a continuous extension to $\mathrm{cl}\,\Omega_T$ and

$$\partial_{x_i}u \in C^{[\frac{m-1}{2}],\{\frac{m-1}{2}\}+\frac{\alpha'}{2};m-1,\alpha''}(\mathrm{cl}\,\Omega_T), \quad \partial_t u \in C^{[\frac{m-2}{2}],\{\frac{m-2}{2}\}+\frac{\alpha'}{2};m-2,\alpha''}(\mathrm{cl}\,\Omega_T) \quad \text{for all } i \in \{1, \dots, n\}.$$

Then we set

$$\|u\|_{C^{[\frac{m}{2}],\{\frac{m}{2}\}+\frac{\alpha'}{2};m,\alpha''}(\mathrm{cl}\,\Omega_T)} \equiv \sup_{\mathrm{cl}\,\Omega_T}|u| + \sum_{i=1}^n \|\partial_{x_i}u\|_{C^{[\frac{m-1}{2}],\{\frac{m-1}{2}\}+\frac{\alpha'}{2};m-1,\alpha''}(\mathrm{cl}\,\Omega_T)}$$
$$+ \|\partial_t u\|_{C^{[\frac{m-2}{2}],\{\frac{m-2}{2}\}+\frac{\alpha'}{2};m-2,\alpha''}(\mathrm{cl}\,\Omega_T)}.$$

It is well known that

$$\left(C^{[\frac{m}{2}],\{\frac{m}{2}\}+\frac{\alpha'}{2};m,\alpha''}(\mathrm{cl}\,\Omega_T), \|\cdot\|_{C^{[\frac{m}{2}],\{\frac{m}{2}\}+\frac{\alpha'}{2};m,\alpha''}(\mathrm{cl}\,\Omega_T)}\right)$$

is a Banach space.

We note that if $m \in \mathbb{N} \setminus \{0\}$ and if Ω is a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$, then one can define the space $C^{[\frac{m}{2}],\{\frac{m}{2}\}+\frac{\alpha'}{2};m,\alpha''}(\partial_T\Omega)$ by means of local parametrizations, and the corresponding norm can be proved to be equivalent to the norm

$$\|u\|_{C^{0,\frac{1+\alpha'}{2};1,\alpha''}(\partial_T\Omega)} \equiv \sup_{\partial_T\Omega}|u| + \sum_{i,j=1}^{n} \|M_{ij}[u]\|_{C^{0,\frac{\alpha'}{2};0,\alpha''}(\partial_T\Omega)} + \sup_{\substack{t_1,t_2\in]-\infty,T[,\\ t_1\neq t_2}} \frac{\|u(t_1,\cdot)-u(t_2,\cdot)\|_{C^0(\partial\Omega)}}{|t_1-t_2|^{\frac{1+\alpha'}{2}}} \quad \text{for } m=1$$

and

$$\|u\|_{C^{[\frac{m}{2}],\{\frac{m}{2}\}+\frac{\alpha'}{2};m,\alpha''}(\partial_T\Omega)} \equiv \sup_{\partial_T\Omega}|u| + \sum_{i,j=1}^{n} \|M_{ij}[u]\|_{C^{[\frac{m-1}{2}],\{\frac{m-1}{2}\}+\frac{\alpha'}{2};m-1,\alpha''}(\partial_T\Omega)} + \|\partial_t u\|_{C^{[\frac{m-2}{2}],\{\frac{m-2}{2}\}+\frac{\alpha'}{2};m-2,\alpha''}(\partial_T\Omega)} \quad \text{for } m \in \mathbb{N}\setminus\{0,1\}.$$

For the sake of brevity, if $\alpha \in [0,1[$, we set

$$C^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\Omega_T) \equiv C^{[\frac{m}{2}],\{\frac{m}{2}\}+\frac{\alpha}{2};m,\alpha}(\mathrm{cl}\,\Omega_T) \;\left(= C^{[\frac{m+\alpha}{2}],\{\frac{m+\alpha}{2}\};m,\alpha}(\mathrm{cl}\,\Omega_T)\right),$$
$$C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega) \equiv C^{[\frac{m}{2}],\{\frac{m}{2}\}+\frac{\alpha}{2};m,\alpha}(\partial_T\Omega) \;\left(= C^{[\frac{m+\alpha}{2}],\{\frac{m+\alpha}{2}\};m,\alpha}(\partial_T\Omega)\right).$$

Remark 2.6. Let $\alpha', \alpha'' \in [0,1]$, $T \in]-\infty,+\infty]$, $m \in \mathbb{N}\setminus\{0\}$. Let Ω be a bounded open subset of $\mathbb{R}^n$. We note that, from the definition of parabolic Schauder spaces, it follows that, for all $i,j \in \{1,\dots,n\}$, the operators

$$\partial_{x_i}\colon C^{[\frac{m}{2}],\{\frac{m}{2}\}+\frac{\alpha'}{2};m,\alpha''}(\mathrm{cl}\,\Omega_T) \to C^{[\frac{m-1}{2}],\{\frac{m-1}{2}\}+\frac{\alpha'}{2};m-1,\alpha''}(\mathrm{cl}\,\Omega_T) \quad \text{if } m \geq 1,$$
$$\partial_t\colon C^{[\frac{m}{2}],\{\frac{m}{2}\}+\frac{\alpha'}{2};m,\alpha''}(\mathrm{cl}\,\Omega_T) \to C^{[\frac{m-2}{2}],\{\frac{m-2}{2}\}+\frac{\alpha'}{2};m-2,\alpha''}(\mathrm{cl}\,\Omega_T) \quad \text{if } m \geq 2$$

are linear and continuous. Moreover, if Ω is of class $C^{m,\alpha}$, then the operators

$$M_{ij}\colon C^{[\frac{m}{2}],\{\frac{m}{2}\}+\frac{\alpha'}{2};m,\alpha''}(\partial_T\Omega) \to C^{[\frac{m-1}{2}],\{\frac{m-1}{2}\}+\frac{\alpha'}{2};m-1,\alpha''}(\partial_T\Omega) \quad \text{if } m \geq 1,$$
$$\partial_t\colon C^{[\frac{m}{2}],\{\frac{m}{2}\}+\frac{\alpha'}{2};m,\alpha''}(\partial_T\Omega) \to C^{[\frac{m-2}{2}],\{\frac{m-2}{2}\}+\frac{\alpha'}{2};m-2,\alpha''}(\partial_T\Omega) \quad \text{if } m \geq 2$$

are linear and continuous.

Remark 2.7. Let $T \in]-\infty,+\infty]$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class C^1. We note that, for parabolic Schauder spaces, the same embeddings as in the classical case hold true. That is, for all $m', m'' \in \mathbb{N}$, $\alpha, \beta \in [0,1[$ such that $m'+\alpha \geq m''+\beta$, $C^{\frac{m'+\alpha}{2};m'+\alpha}(\mathrm{cl}\,\Omega_T)$ is continuously embedded into $C^{\frac{m''+\beta}{2};m''+\beta}(\mathrm{cl}\,\Omega_T)$. The same embeddings hold true for parabolic Schauder spaces on $\partial_T\Omega$ provided that Ω is of class $C^{m',\alpha}$.

The function Φ_n from $\mathbb{R}\times\mathbb{R}^n\setminus\{(0,0)\}$ to $\mathbb{R}$ defined by

$$\Phi_n(t,x) \equiv \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} & \text{if } (t,x) \in]0,+\infty[\times\mathbb{R}^n, \\ 0 & \text{if } (t,x) \in (]-\infty,0]\times\mathbb{R}^n)\setminus\{(0,0)\} \end{cases} \tag{2.3}$$

is well known to be the fundamental solution for the heat operator $\partial_t - \Delta$ in $\mathbb{R}^{1+n}$. It is well known that $\Phi_n \in C^\infty(\mathbb{R}^{1+n}\setminus\{(0,0)\})$ (cf., e.g., Ladyženskaja, Solonnikov and Ural'ceva [16, p. 274]).

We now collect some known properties about the double and single layer heat potentials in the following statements.

Theorem 2.8 (Properties of the double layer heat potential). *Let $\alpha \in]0,1[$, and let $T \in]-\infty,+\infty]$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Then the following statements hold.*

(i) *If $\mu \in C^0_b(\partial_T\Omega)$, then the function $w[\partial_T\Omega,\mu]$ from $(\mathbb{R}^n)_T$ to $\mathbb{C}$ defined by*

$$w[\partial_T\Omega,\mu](t,x) \equiv \int_{-\infty}^{t}\int_{\partial\Omega} \frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y\,d\tau \quad \text{for all } (t,x)\in(\mathbb{R}^n)_T$$

solves the heat equation in $(\mathbb{R}^n\setminus\partial\Omega)_T$. The restriction $w[\partial_T\Omega,\mu]_{|\Omega_T}$ can be extended uniquely to a continuous function $w^+[\partial_T\Omega,\mu]$ from $\mathrm{cl}\,\Omega_T$ to $\mathbb{C}$. The restriction $w[\partial_T\Omega,\mu]_{|\Omega^-_T}$ can be extended uniquely to a continuous function $w^-[\partial_T\Omega,\mu]$ from $\mathrm{cl}\,\Omega^-_T$ to $\mathbb{C}$. Moreover, the following jump formula holds for all $(t,x)\in\partial_T\Omega$:

$$w^{\pm}[\partial_T\Omega,\mu](t,x) = \mp\frac{1}{2}\mu(t,x) + \int_{-\infty}^{t}\int_{\partial\Omega}\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y\,d\tau. \tag{2.4}$$

(ii) *Let $m\in\mathbb{N}\setminus\{0\}$. Let Ω be of class $C^{m,\alpha}$. Let $R\in\,]0,+\infty[$ such that $\mathrm{cl}\,\Omega\subseteq\mathbb{B}_n(0,R)$. Then the map from the space $C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$ to $C^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\Omega_T)$, which takes μ to $w^+[\partial_T\Omega,\mu]$, is linear and continuous, and the map from the space $C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$ to the space $C^{\frac{m+\alpha}{2};m+\alpha}((\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\Omega)_T)$, which takes μ to $w^-[\partial_T\Omega,\mu]_{|(\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\Omega)_T}$, is linear and continuous.*

Theorem 2.9 (Properties of the single layer heat potential). *Let $\alpha\in\,]0,1[$, and let $T\in\,]-\infty,+\infty]$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $R\in\,]0,+\infty[$ such that $\mathrm{cl}\,\Omega\subseteq\mathbb{B}_n(0,R)$. Then the following statements hold.*

(i) *Let $n\geq 3$. Let $\mu\in C^0_b(\partial_T\Omega)$. Then the function $v[\partial_T\Omega,\mu]$ from $(\mathbb{R}^n)_T$ to $\mathbb{C}$ defined by*

$$v[\partial_T\Omega,\mu](t,x) \equiv \int_{-\infty}^{t}\int_{\partial\Omega}\Phi_n(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y\,d\tau$$

for all $(t,x)\in(\mathbb{R}^n)_T$, i.e., the (n-dimensional) single layer potential, is continuous in $(\mathbb{R}^n)_T$ and solves the heat equation in $(\mathbb{R}^n\setminus\partial\Omega)_T$.
Let $n=2$. Let $\mu\in C^0_b(\partial_T\Omega)$. Let $x_0\in\Omega$. Then the function $v[\partial_T\Omega,\mu]$ from $(\mathbb{R}^n)_T$ to $\mathbb{C}$ defined by

$$v[\partial_T\Omega,\mu](t,x) \equiv \int_{-\infty}^{+\infty}\int_{\partial\Omega}(\Phi_n(t-\tau,x-y)-\Phi_n(0-\tau,x_0-y))\mu(\tau,y)\,d\sigma_y\,d\tau$$

for all $(t,x)\in(\mathbb{R}^2)_T$, i.e., the (2-dimensional) single layer potential, is continuous in $(\mathbb{R}^2)_T$ and solves the heat equation in $(\mathbb{R}^2\setminus\partial\Omega)_T$.
Both in case $n\geq 3$ and in case $n=2$, we denote by $v^+[\partial_T\Omega,\mu]$ and $v^-[\partial_T\Omega,\mu]$ the restriction of $v[\partial_T\Omega,\mu]$ to $\mathrm{cl}\,\Omega_T$ and $\mathrm{cl}\,\Omega^-_T$, respectively.

(ii) *Let $m\in\mathbb{N}\setminus\{0\}$ and $r\in\{1,\dots,n\}$. Let Ω be of class $C^{m,\alpha}$. Then the map from the space*

$$C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega) \quad \text{to} \quad C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\mathrm{cl}\,\Omega_T),$$

which takes μ to the function $\frac{\partial}{\partial x_r}v^+[\partial_T\Omega,\mu]$, is linear and continuous, and the map from the space

$$C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega) \quad \text{to} \quad C^{\frac{m-1+\alpha}{2};m-1+\alpha}((\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\Omega)_T),$$

which takes μ to the function $\frac{\partial}{\partial x_r}v^-[\partial_T\Omega,\mu]_{|(\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\Omega)_T}$, is linear and continuous.

(iii) *Let $m\in\mathbb{N}\setminus\{0\}$. Let Ω be of class $C^{m,\alpha}$. Then the map from the space*

$$C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega) \quad \text{to} \quad C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\mathrm{cl}\,\Omega_T),$$

which takes μ to $\frac{\partial}{\partial t}v^+[\partial_T\Omega,\mu]$, is linear and continuous, and the map from the space

$$C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega) \quad \text{to} \quad C^{\frac{m-1+\alpha}{2};m-1+\alpha}((\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\Omega)_T),$$

which takes the function μ to the restriction $\frac{\partial}{\partial t}v^-[\partial_T\Omega,\mu]_{|(\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\Omega)_T}$, is linear and continuous.

(iv) *Let $n \geq 3$, $m \in \mathbb{N} \setminus \{0\}$. Let Ω be of class $C^{m,\alpha}$. Then the map from the space*

$$C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega) \quad to \quad C^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\Omega_T),$$

which takes μ to $v^+[\partial_T\Omega,\mu]$, is linear and continuous, and the map from the space

$$C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega) \quad to \quad C^{\frac{m+\alpha}{2};m+\alpha}((\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\Omega)_T),$$

which takes μ to $v^-[\partial_T\Omega,\mu]_{|(\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\Omega)_T}$, is linear and continuous.

(v) *Let $\mu \in C^{\frac{\alpha}{2};\alpha}(\partial_T\Omega)$. Let $r \in \{1,\dots,n\}$. Then the following jump relations hold for all $(t,x) \in \partial_T\Omega$:*

$$\frac{\partial}{\partial\nu(x)}v^\pm[\partial_T\Omega,\mu](t,x) = \pm\frac{1}{2}\mu(t,x) + \int_{-\infty}^{t}\int_{\partial\Omega}\frac{\partial}{\partial\nu(x)}\Phi_n(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y\,d\tau, \tag{2.5}$$

$$\frac{\partial}{\partial x_r}v^\pm[\partial_T\Omega,\mu](t,x) = \pm\frac{1}{2}\mu(t,x)\nu_r(x) + \int_{-\infty}^{t}\int_{\partial\Omega}\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y\,d\tau.$$

For a proof of statement (i) of Theorem 2.8 and of statements (i), (v) of Theorem 2.9, we refer to Friedman [6, Theorem 1, p. 137], Ladyženskaja, Solonnikov and Ural'tseva [16, p. 404, 407], Watson [27, Lemma 2.7, p. 41]. A proof of the Schauder regularity properties for the double layer potential of Theorem 2.8 (ii) and a proof of the Schauder regularity properties for the single layer potential of Theorem 2.9 (ii), (iii) can be effected by following the arguments of Ladyženskaja, Solonnikov and Ural'tseva in [16, Chapter 4.2], who prove the same properties in case the set Ω is replaced by a half space (see also [20, Appendix B]).

Remark 2.10. The above definition of single layer heat potential $v[\partial_T\Omega,\mu]$ in the case $n = 2$ clearly depends on the choice of $x_0 \in \Omega$. Indeed, a different choice would define a single layer which differs from that with x_0 by a constant. However, we note that if $T \in]0,+\infty]$ and $\mathrm{supp}\,\mu \subseteq \overline{[0,T[} \times \partial\Omega$ (and this is the case when one considers an initial-boundary value problem in $\overline{[0,T[} \times \mathrm{cl}\,\Omega$), then the single layer potential $v[\partial_T\Omega,\mu]$ no longer depends on x_0, and

$$v[\partial_T\Omega,\mu](t,x) \equiv \int_0^t\int_{\partial\Omega}\Phi_n(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y\,d\tau \quad \text{for all } (t,x) \in \overline{[0,T[} \times \mathbb{R}^2,$$

which is the classical definition of single layer heat potential (cf., e.g., Friedman [6, p. 136]).

Then we have the following lemma, which we later need to compute the time derivative of a single layer potential in the case the density does not depend upon time.

Lemma 2.11. *Let $T \in]-\infty,+\infty]$. Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. If $\mu \in C^0(\partial\Omega)$, then*

$$\int_{-\infty}^{t}\int_{\partial\Omega}\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\mu(y)\,d\sigma_y\,d\tau = 0 \quad \textit{for all } (t,x) \in \Omega_T.$$

Proof. First we note that

$$\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y) = \frac{e^{-\frac{|x-y|^2}{4(t-\tau)}}\big(-\frac{n}{2}+\frac{|x-y|^2}{4(t-\tau)}\big)}{(4\pi)^{\frac{n}{2}}(t-\tau)^{\frac{n}{2}+1}}$$

for all $(t,x) \in \Omega_T$, $(\tau,y) \in \partial_T\Omega$, $\tau < t$. Then, by the change of variable $u|x-y|^2 = 4(t-\tau)$, we have

$$\begin{aligned}\int_{-\infty}^{t}\int_{\partial\Omega}\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\mu(y)\,d\sigma_y\,d\tau &= \int_{\partial\Omega}\frac{1}{|x-y|^n}\int_0^{+\infty}\Big(-\frac{n}{2}\frac{e^{-\frac{1}{u}}}{\pi^{\frac{n}{2}}u^{\frac{n}{2}+1}} + \frac{e^{-\frac{1}{u}}}{\pi^{\frac{n}{2}}u^{\frac{n}{2}+2}}\Big)\mu(y)\,du\,d\sigma_y\\ &= \frac{1}{\pi^{\frac{n}{2}}}\int_{\partial\Omega}\frac{1}{|x-y|^n}\Big(-\frac{n}{2}\Gamma\Big(\frac{n}{2}\Big)+\Gamma\Big(\frac{n}{2}+1\Big)\Big)\mu(y)\,d\sigma_y = 0\end{aligned}$$

for all $(t,x) \in \Omega_T$, where Γ is the Euler Γ-function (cf., e.g., Folland [5, p. 58]). □

3 Auxiliary integral operators

In order to compute the tangential derivatives of the double layer heat potential, we now introduce some auxiliary integral operators, and we analyze the corresponding properties.

Lemma 3.1. *Let $T \in]-\infty, +\infty]$. Let $r \in \{1, \dots, n\}$. Then the following statements hold.*

(i) *Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Let $\theta \in]0, 1]$. If $(f, \mu) \in C^{0,\theta}(\mathrm{cl}\,\Omega) \times L^\infty(\partial_T\Omega)$, then the function defined by*

$$Q_r^\sharp[f, \mu](t, x) \equiv \int_{-\infty}^{t} \int_{\partial\Omega} (f(x) - f(y)) \frac{\partial}{\partial x_r} \Phi_n(t - \tau, x - y) \mu(\tau, y)\, d\sigma_y\, d\tau$$

for all $(t, x) \in \mathrm{cl}\,\Omega_T$ is continuous and bounded.

(ii) *Let $\alpha, \beta, \theta \in]0, 1[$. Let $m \in \mathbb{N} \setminus \{0\}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Then the map $Q_r^\sharp[\cdot, \cdot]$ from*

$$C^{m-1,\theta}(\mathrm{cl}\,\Omega) \times C^{\frac{m-1+\beta}{2}; m-1+\beta}(\partial_T\Omega) \quad \text{to} \quad C^{\frac{m-1+\min\{\theta,\alpha,\beta\}}{2}; m-1+\min\{\theta,\alpha,\beta\}}(\mathrm{cl}\,\Omega_T),$$

which takes (f, μ) to $Q_r^\sharp[f, \mu]$, is bilinear and continuous.

Proof. By [17, Remark 3 (iii)], the kernel $\frac{\partial}{\partial x_r}\Phi(t - \tau, x - y)$ satisfies the assumptions of [17, Lemma 8.1], which imply the validity of statement (i). We now consider statement (ii). By separately treating the cases $(t, x) \in \partial_T\Omega$ and $(t, x) \in \Omega_T$, by the classical jump relations of the derivatives of a single layer potential and by classical differentiation theorems for integrals depending on a parameter, we have

$$Q_r^\sharp[f, \mu](t, x) = f(x) \frac{\partial}{\partial x_r} v^+[\partial_T\Omega, \mu](t, x) - \frac{\partial}{\partial x_r} v^+[\partial_T\Omega, f\mu](t, x)$$

for all $(f, \mu) \in C^{m-1,\theta}(\mathrm{cl}\,\Omega) \times C^{\frac{m-1+\beta}{2}; m-1+\beta}(\partial_T\Omega)$ and for all $(t, x) \in \mathrm{cl}\,\Omega_T$ (cf. Theorem 2.9 (v)). Then the statement follows by Theorem 2.9 (ii) and by the continuity of the pointwise product in Hölder spaces. □

Lemma 3.2. *Let $T \in]-\infty, +\infty]$. Then the following statements hold.*

(i) *Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Let $\theta \in]0, 1]$. If (f, μ) belongs to*

$$C^{0,\theta}(\mathrm{cl}\,\Omega) \times C_b^{0,\frac{1}{2}}(\overline{]-\infty, T[}, C^0(\partial\Omega)),$$

then the function defined by

$$\tilde{Q}_t^\sharp[f, \mu](t, x) \equiv \int_{-\infty}^{t} \int_{\partial\Omega} (f(x) - f(y)) \frac{\partial}{\partial t} \Phi_n(t - \tau, x - y)(\mu(\tau, y) - \mu(t, y))\, d\sigma_y\, d\tau$$

for all $(t, x) \in \mathrm{cl}\,\Omega_T$ is continuous and bounded.

(ii) *Let $\alpha, \beta, \theta \in]0, 1[$. Let $m \in \mathbb{N} \setminus \{0, 1\}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Then the map $\tilde{Q}_t^\sharp[\cdot, \cdot]$ from*

$$C^{m-1,\theta}(\mathrm{cl}\,\Omega) \times C^{\frac{m+\beta}{2}; m+\beta}(\partial_T\Omega) \quad \text{to} \quad C^{\frac{m-1+\min\{\theta,\alpha,\beta\}}{2}; m-1+\min\{\theta,\alpha,\beta\}}(\mathrm{cl}\,\Omega_T),$$

which takes (f, μ) to $\tilde{Q}_t^\sharp[f, \mu]$, is bilinear and continuous.

Proof. By [17, Remark 3 (ii)], the kernel $\frac{\partial}{\partial t}\Phi(t - \tau, x - y)$ satisfies the assumptions of [17, Lemma 8.3], which imply the validity of statement (i). Now we consider statement (ii). Let $(t, x) \in \Omega_T$. Then, by Lemma 2.11, we have

$$\tilde{Q}_t^\sharp[f, \mu](t, x) = f(x) \frac{\partial}{\partial t} v^+[\partial_T\Omega, \mu](t, x) - \frac{\partial}{\partial t} v^+[\partial_T\Omega, f\mu](t, x)$$

for all $(f, \mu) \in C^{m-1,\theta}(\mathrm{cl}\,\Omega) \times C^{\frac{m+\beta}{2}; m+\beta}(\partial_T\Omega)$. By Theorem 2.9 (iii), the right-hand side defines a continuous function of $(t, x) \in \mathrm{cl}\,\Omega_T$. By statement (i), the left-hand side is also continuous in $(t, x) \in \mathrm{cl}\,\Omega_T$. Hence the above equality holds for all $(t, x) \in \mathrm{cl}\,\Omega_T$, and the statement follows by Theorem 2.9 (iii) and by the continuity of the pointwise product in Hölder spaces. □

We now collect three statements of theorems that have been proved in [17, Theorem 8.2, Theorem 10.2, Theorem 10.3].

Theorem 3.3. *Let $T \in]-\infty, +\infty]$. Let $\theta \in]0,1]$, $\theta_1 \in]0,\theta[$. Let $r \in \{1,\dots,n\}$. Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Then the map $Q[\partial_{x_r}\Phi_n(t-\tau,x-y),\cdot,\cdot]$ from $C^{0,\theta}(\partial\Omega)\times L^\infty(\partial_T\Omega)$ to $C^{\frac{\theta_1}{2},\theta_1}(\partial_T\Omega)$, which takes (g,μ) to the function defined by*

$$Q[\partial_{x_r}\Phi_n(t-\tau,x-y),g,\mu](t,x) \equiv \int_{-\infty}^{t}\int_{\partial\Omega}(g(x)-g(y))\partial_{x_r}\Phi_n(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y\,d\tau \tag{3.1}$$

for all $(t,x)\in\partial_T\Omega$, is bilinear and continuous.

Theorem 3.4. *Let $T \in]-\infty, +\infty]$. Let $\alpha \in]0,1[$, $\beta \in]0,\alpha[$. Let $r \in \{1,\dots,n\}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Then the following statements hold.*

(i) *The map $Q[\partial_{x_r}\Phi_n(t-\tau,x-y),\cdot,\cdot]$ from*

$$C^{0,\alpha}(\partial\Omega)\times C^{\frac{\beta}{2};\beta}(\partial_T\Omega) \quad \text{to} \quad C^{\frac{\alpha}{2};\alpha}(\partial_T\Omega),$$

which takes (g,μ) to $Q[\partial_{x_r}\Phi_n(t-\tau,x-y),g,\mu]$ defined in (3.1), is bilinear and continuous.

(ii) *The map $Q[\partial_{x_r}\Phi_n(t-\tau,x-y),\cdot,\cdot]$ from*

$$C^{0,\alpha}(\partial\Omega)\times C_b^{0,\frac{1+\beta}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega)) \quad \text{to} \quad C_b^{0,\frac{1+\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega)),$$

which takes (g,μ) to $Q[\partial_{x_r}\Phi_n(t-\tau,x-y),g,\mu]$, is bilinear and continuous.

(iii) *The map $Q[\partial_{x_r}\Phi_n(t-\tau,x-y),\cdot,\cdot]$ from*

$$C^{0,\alpha}(\partial\Omega)\times C_b^{0,\frac{1}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega)) \quad \text{to} \quad C_b^{0,\frac{1+\beta}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega)),$$

which takes (g,μ) to $Q[\partial_{x_r}\Phi_n(t-\tau,x-y),g,\mu]$, is bilinear and continuous.

Theorem 3.5. *Let $T \in]-\infty, +\infty]$. Let $\alpha \in]0,1[$, $\beta \in]0,\alpha[$. Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Then the following statements hold.*

(i) *The map $\tilde{Q}[\partial_t\Phi_n(t-\tau,x-y),\cdot,\cdot]$ from $C^{0,\alpha}(\partial\Omega)\times C^{0,\frac{1+\beta}{2};0,1}(\partial_T\Omega)$ to the space $C^{\frac{\alpha}{2};\alpha}(\partial_T\Omega)$, which takes (g,φ) to the function*

$$\tilde{Q}[\partial_t\Phi_n(t-\tau,x-y),g,\varphi](t,x) \equiv \int_{-\infty}^{t}\int_{\partial\Omega}(g(x)-g(y))\partial_t\Phi_n(t-\tau,x-y)(\varphi(\tau,y)-\varphi(t,y))\,d\sigma_y\,d\tau \tag{3.2}$$

for all $(t,x)\in\partial_T\Omega$, is bilinear and continuous.

(ii) *The map $\tilde{Q}[\partial_t\Phi_n(t-\tau,x-y),\cdot,\cdot]$ is bilinear and continuous from*

$$C^{0,\alpha}(\partial\Omega)\times C^{0,\frac{1}{2};0,1}(\partial_T\Omega) \quad \text{to} \quad C^{\frac{\beta}{2};\beta}(\partial_T\Omega).$$

(iii) *The map $\tilde{Q}[\partial_t\Phi_n(t-\tau,x-y),\cdot,\cdot]$ is bilinear and continuous from*

$$C^{0,1}(\partial\Omega)\times C^{0,1;0,1}(\partial_T\Omega) \quad \text{to} \quad C_b^{0,\frac{1+\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega)).$$

We now present explicit formulas for time and tangential derivatives of the operators

$$Q[\partial_{x_r}\Phi_n(t-\tau,x-y),\cdot,\cdot] \quad \text{and} \quad \tilde{Q}[\partial_t\Phi_n(t-\tau,x-y),\cdot,\cdot].$$

From now on, we use the abbreviations

$$\begin{aligned} Q_r[\cdot,\cdot] &\equiv Q[\partial_{x_r}\Phi_n(t-\tau,x-y),\cdot,\cdot] \quad \text{for all } r\in\{1,\dots,n\},\\ \tilde{Q}_t[\cdot,\cdot] &\equiv \tilde{Q}[\partial_t\Phi_n(t-\tau,x-y),\cdot,\cdot]. \end{aligned}$$

Clearly, $Q_r^\sharp[\cdot,\cdot] = Q_r[\cdot,\cdot]$, $\tilde{Q}_t^\sharp[\cdot,\cdot] = \tilde{Q}_t[\cdot,\cdot]$ on $\partial_T\Omega$.

Lemma 3.6. *Let $\alpha \in]0,1[$, $T \in]-\infty,+\infty]$. Let $r \in \{1,\dots,n\}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{2,\alpha}$. Let $g \in C^{1,\alpha}(\partial\Omega)$ and $\varphi \in C^{\frac{1}{2};1}(\partial_T\Omega)$. Then $Q_r[g,\varphi](t,\cdot) \in C^1(\partial\Omega)$ for all $t \in \overline{]-\infty,T[}$. Moreover,*

$$\begin{aligned}
M_{ij}[Q_r[g,\varphi]](t,x) &= \nu_i(x)Q_r[(D_{\partial\Omega}g)_j,\varphi](t,x) - \nu_j(x)Q_r[(D_{\partial\Omega}g)_i,\varphi](t,x)\\
&\quad + \nu_i(x)Q_r\Big[g,\sum_{h=1}^n M_{hj}[\nu_h\varphi]\Big](t,x) - \nu_j(x)Q_r\Big[g,\sum_{h=1}^n M_{hi}[\nu_h\varphi]\Big](t,x)\\
&\quad + \nu_i(x)\Big\{\sum_{s=1}^n Q_s[\nu_j, M_{sr}[g]\varphi](t,x) + \sum_{s=1}^n Q_s[g, M_{sr}[\nu_j\varphi]](t,x)\Big\}\\
&\quad - \nu_j(x)\Big\{\sum_{s=1}^n Q_s[\nu_i, M_{sr}[g]\varphi](t,x) + \sum_{s=1}^n Q_s[g, M_{sr}[\nu_i\varphi]](t,x)\Big\}\\
&\quad + \nu_i(x)\tilde{Q}_t[g,\nu_r\nu_j\varphi](t,x) - \nu_j(x)\tilde{Q}_t[g,\nu_r\nu_i\varphi](t,x),
\end{aligned} \tag{3.3}$$

holds for all $(t,x) \in \partial_T\Omega$ and for all $i,j \in \{1,\dots,n\}$; see (2.1).

Proof. Let $R \in]0,+\infty[$ be such that $\mathrm{cl}\,\Omega \subseteq \mathbb{B}_n(0,R)$. Let "$\tilde{\cdot}$" be an extension operator as in Lemma 2.1, defined on $C^{1,\alpha}(\partial\Omega)$.

First we fix $\beta \in]0,\alpha[$, and we prove formula (3.3) under the assumption that $\varphi \in C^{\frac{1+\beta}{2};1+\beta}(\partial_T\Omega)$. By Lemma 3.1 (ii), we already know that $Q_r^\sharp[\tilde{g},\varphi](t,\cdot)$ belongs to $C^1(\mathrm{cl}\,\Omega)$ for all $t \in \overline{]-\infty,T[}$. Then we find it convenient to introduce the notation

$$M_{ij}^\sharp[f](x) \equiv \tilde{\nu}_i(x)\frac{\partial f}{\partial x_j}(x) - \tilde{\nu}_j(x)\frac{\partial f}{\partial x_i}(x) \tag{3.4}$$

for all $f \in C^1(\mathrm{cl}\,\Omega)$ and for all $x \in \mathrm{cl}\,\Omega$, $i,j \in \{1,\dots,n\}$. If necessary, we write $M_{ij,x}^\sharp$ to emphasize that we are taking x as variable of the differential operator $M_{ij}^\sharp$. Next we fix $(t,x) \in \Omega_T$, and we compute

$$M_{ij}^\sharp[Q_r^\sharp[\tilde{g},\varphi]](t,x) = \tilde{\nu}_i(x)\frac{\partial}{\partial x_j}Q_r^\sharp[\tilde{g},\varphi](t,x) - \tilde{\nu}_j(x)\frac{\partial}{\partial x_i}Q_r^\sharp[\tilde{g},\varphi](t,x).$$

By differentiation theorems for integrals depending on a parameter, we have

$$\begin{aligned}
\frac{\partial}{\partial x_i}Q_r^\sharp[\tilde{g},\varphi](t,x) &= \int_{-\infty}^t\int_{\partial\Omega}\frac{\partial\tilde{g}}{\partial x_i}(x)\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\,d\sigma_y\,d\tau\\
&\quad + \int_{-\infty}^t\int_{\partial\Omega}(\tilde{g}(x)-\tilde{g}(y))\frac{\partial^2}{\partial x_i\partial x_r}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\,d\sigma_y\,d\tau.
\end{aligned}$$

Since $\sum_{h=1}^n \nu_h^2 = 1$ on $\partial\Omega$, we have

$$\begin{aligned}
\frac{\partial}{\partial x_i}Q_r^\sharp[\tilde{g},\varphi](t,x) &= \int_{-\infty}^t\int_{\partial\Omega}\frac{\partial\tilde{g}}{\partial x_i}(x)\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\,d\sigma_y\,d\tau\\
&\quad - \int_{-\infty}^t\int_{\partial\Omega}(\tilde{g}(x)-\tilde{g}(y))\sum_{h=1}^n \nu_h^2(y)\frac{\partial}{\partial y_i}\Big(\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\Big)\varphi(\tau,y)\,d\sigma_y\,d\tau\\
&= \int_{-\infty}^t\int_{\partial\Omega}\frac{\partial\tilde{g}}{\partial x_i}(x)\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\,d\sigma_y\,d\tau\\
&\quad - \int_{-\infty}^t\int_{\partial\Omega}(\tilde{g}(x)-\tilde{g}(y))\sum_{h=1}^n\Big[\Big(\nu_h(y)\frac{\partial}{\partial y_i} - \nu_i(y)\frac{\partial}{\partial y_h}\Big)\Big(\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\Big)\Big]\\
&\qquad\qquad \times \varphi(\tau,y)\nu_h(y)\,d\sigma_y\,d\tau\\
&\quad - \int_{-\infty}^t\int_{\partial\Omega}(\tilde{g}(x)-\tilde{g}(y))\sum_{h=1}^n\Big[\nu_h(y)\frac{\partial}{\partial y_h}\Big(\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\Big)\Big]\\
&\qquad\qquad \times \nu_i(y)\varphi(\tau,y)\,d\sigma_y\,d\tau.
\end{aligned} \tag{3.5}$$

By Lemma 2.2, the second integral on the right-hand side of formula (3.5) takes the form

$$\begin{aligned}
&\int_{-\infty}^{t}\int_{\partial\Omega}(\tilde g(x)-\tilde g(y))\sum_{h=1}^{n}\Big[\Big(\nu_h(y)\frac{\partial}{\partial y_i}-\nu_i(y)\frac{\partial}{\partial y_h}\Big)\Big(\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\Big)\Big]\varphi(\tau,y)\nu_h(y)\,d\sigma_y\,d\tau\\
&\quad=-\sum_{h=1}^{n}\int_{-\infty}^{t}\int_{\partial\Omega}M_{hi,y}[(\tilde g(x)-\tilde g(y))\varphi(\tau,y)\nu_h(y)]\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\,d\sigma_y\,d\tau\\
&\quad=\sum_{h=1}^{n}\int_{-\infty}^{t}\int_{\partial\Omega}M_{hi,y}[\tilde g(y)]\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\nu_h(y)\,d\sigma_y\,d\tau\\
&\qquad-\sum_{h=1}^{n}\int_{-\infty}^{t}\int_{\partial\Omega}(\tilde g(x)-\tilde g(y))\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)M_{hi,y}[\varphi(\tau,y)\nu_h(y)]\,d\sigma_y\,d\tau.
\end{aligned}$$

Accordingly, we have

$$\begin{aligned}
\frac{\partial}{\partial x_i}Q_r^\sharp[\tilde g,\varphi](t,x)=\frac{\partial\tilde g}{\partial x_i}(x)&\int_{-\infty}^{t}\int_{\partial\Omega}\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\,d\sigma_y\,d\tau\\
&-\sum_{h=1}^{n}\int_{-\infty}^{t}\int_{\partial\Omega}M_{hi,y}[\tilde g(y)]\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\nu_h(y)\,d\sigma_y\,d\tau\\
&+\sum_{h=1}^{n}\int_{-\infty}^{t}\int_{\partial\Omega}(\tilde g(x)-\tilde g(y))\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)M_{hi,y}[\varphi(\tau,y)\nu_h(y)]\,d\sigma_y\,d\tau\\
&-\int_{-\infty}^{t}\int_{\partial\Omega}(\tilde g(x)-\tilde g(y))\frac{\partial}{\partial x_r}\Big(\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y)\Big)\nu_i(y)\varphi(\tau,y)\,d\sigma_y\,d\tau.
\end{aligned}$$

Then we have

$$\begin{aligned}
M_{ij}^\sharp[Q_r^\sharp[\tilde g,\varphi]](t,x)=M_{ij}^\sharp[\tilde g](x)&\int_{-\infty}^{t}\int_{\partial\Omega}\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\,d\sigma_y\,d\tau\\
&+\sum_{h=1}^{n}\int_{-\infty}^{t}\int_{\partial\Omega}\{-\tilde\nu_i(x)M_{hj,y}[\tilde g(y)]+\tilde\nu_j(x)M_{hi,y}[\tilde g(y)]\}\\
&\qquad\times\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\nu_h(y)\,d\sigma_y\,d\tau\\
&+\sum_{h=1}^{n}\int_{-\infty}^{t}\int_{\partial\Omega}(\tilde g(x)-\tilde g(y))\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\\
&\qquad\times\{\tilde\nu_i(x)M_{hj,y}[\varphi(\tau,y)\nu_h(y)]\\
&\qquad\qquad-\tilde\nu_j(x)M_{hi,y}[\varphi(\tau,y)\nu_h(y)]\}\,d\sigma_y\,d\tau\\
&+\int_{-\infty}^{t}\int_{\partial\Omega}(\tilde g(x)-\tilde g(y))\frac{\partial}{\partial x_r}\Big(\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y)\Big)\\
&\qquad\times\{\tilde\nu_j(x)\nu_i(y)-\tilde\nu_i(x)\nu_j(y)\}\varphi(\tau,y)\,d\sigma_y\,d\tau.
\end{aligned}\tag{3.6}$$

We now consider the first two terms on the right-hand side of formula (3.6). By the identity

$$M_{ij}^\sharp[\tilde g]=\tilde\nu_i\Big(\frac{\partial\tilde g}{\partial x_j}-(D\tilde g\cdot\tilde\nu)\tilde\nu_j\Big)-\tilde\nu_j\Big(\frac{\partial\tilde g}{\partial x_i}-(D\tilde g\cdot\tilde\nu)\tilde\nu_i\Big)\quad\text{in }\operatorname{cl}\Omega\tag{3.7}$$

and by the identity

$$\begin{aligned}\sum_{h=1}^{n}\{-\tilde{\nu}_i(x)M_{hj,y}[\tilde{g}(y)]+\tilde{\nu}_j(x)M_{hi,y}[\tilde{g}(y)]\}\nu_h(y)\\
=-\tilde{\nu}_i(x)\Big(\frac{\partial\tilde{g}}{\partial x_j}(y)-(D\tilde{g}(y)\cdot\tilde{\nu}(y))\tilde{\nu}_j(y)\Big)+\tilde{\nu}_j(x)\Big(\frac{\partial\tilde{g}}{\partial x_i}(y)-(D\tilde{g}(y)\cdot\tilde{\nu}(y))\tilde{\nu}_i(y)\Big)\end{aligned}\tag{3.8}$$

for all $y\in\partial\Omega$, which follows by formula (2.2), we rewrite the sum of the first two terms on the right-hand side of (3.6) in the form

$$\begin{aligned}
&M^{\sharp}_{ij}[\tilde{g}](x)\int_{-\infty}^{t}\int_{\partial\Omega}\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\,d\sigma_y\,d\tau\\
&\quad+\sum_{h=1}^{n}\int_{-\infty}^{t}\int_{\partial\Omega}\{-\tilde{\nu}_i(x)M_{hj,y}[\tilde{g}(y)]+\tilde{\nu}_j(x)M_{hi,y}[\tilde{g}(y)]\}\\
&\qquad\times\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\nu_h(y)\,d\sigma_y\,d\tau\\
&=\int_{-\infty}^{t}\int_{\partial\Omega}\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\,d\sigma_y\,d\tau\,\tilde{\nu}_i(x)\Big(\frac{\partial\tilde{g}}{\partial x_j}(x)-(D\tilde{g}(x)\cdot\tilde{\nu}(x))\tilde{\nu}_j(x)\Big)\\
&\quad-\int_{-\infty}^{t}\int_{\partial\Omega}\frac{\partial}{\partial x_r}\Phi(t-\tau,x-y)\varphi(\tau,y)\,d\sigma_y\,d\tau\,\tilde{\nu}_j(x)\Big(\frac{\partial\tilde{g}}{\partial x_i}(x)-(D\tilde{g}(x)\cdot\tilde{\nu}(x))\tilde{\nu}_i(x)\Big)\\
&\quad-\tilde{\nu}_i(x)\int_{-\infty}^{t}\int_{\partial\Omega}\Big(\frac{\partial\tilde{g}}{\partial x_j}(y)-(D\tilde{g}(y)\cdot\tilde{\nu}(y))\tilde{\nu}_j(y)\Big)\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\,d\sigma_y\,d\tau\\
&\quad+\tilde{\nu}_j(x)\int_{-\infty}^{t}\int_{\partial\Omega}\Big(\frac{\partial\tilde{g}}{\partial x_i}(y)-(D\tilde{g}(y)\cdot\tilde{\nu}(y))\tilde{\nu}_i(y)\Big)\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\,d\sigma_y\,d\tau\\
&=\tilde{\nu}_i(x)\int_{-\infty}^{t}\int_{\partial\Omega}\Big[\Big(\frac{\partial\tilde{g}}{\partial x_j}(x)-(D\tilde{g}(x)\cdot\tilde{\nu}(x))\tilde{\nu}_j(x)\Big)-\Big(\frac{\partial\tilde{g}}{\partial x_j}(y)-(D\tilde{g}(y)\cdot\tilde{\nu}(y))\tilde{\nu}_j(y)\Big)\Big]\\
&\qquad\times\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\,d\sigma_y\,d\tau\\
&\quad-\tilde{\nu}_j(x)\int_{-\infty}^{t}\int_{\partial\Omega}\Big[\Big(\frac{\partial\tilde{g}}{\partial x_i}(x)-(D\tilde{g}(x)\cdot\tilde{\nu}(x))\tilde{\nu}_i(x)\Big)-\Big(\frac{\partial\tilde{g}}{\partial x_i}(y)-(D\tilde{g}(y)\cdot\tilde{\nu}(y))\tilde{\nu}_i(y)\Big)\Big]\\
&\qquad\times\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\,d\sigma_y\,d\tau\\
&=\tilde{\nu}_i(x)Q^{\sharp}_r\Big[\frac{\partial\tilde{g}}{\partial x_j}-(D\tilde{g}\cdot\tilde{\nu})\tilde{\nu}_j,\varphi\Big](t,x)-\tilde{\nu}_j(x)Q^{\sharp}_r\Big[\frac{\partial\tilde{g}}{\partial x_i}-(D\tilde{g}\cdot\tilde{\nu})\tilde{\nu}_i,\varphi\Big](t,x).
\end{aligned}\tag{3.9}$$

Now we consider the third term on the right-hand side of formula (3.6), namely,

$$\begin{aligned}
&\sum_{h=1}^{n}\int_{-\infty}^{t}\int_{\partial\Omega}(\tilde{g}(x)-\tilde{g}(y))\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\\
&\qquad\times\{\tilde{\nu}_i(x)M_{hj,y}[\varphi(\tau,y)\nu_h(y)]-\tilde{\nu}_j(x)M_{hi,y}[\varphi(\tau,y)\nu_h(y)]\}\,d\sigma_y\,d\tau\\
&\quad=\tilde{\nu}_i(x)Q^{\sharp}_r\Big[\tilde{g},\sum_{h=1}^{n}M_{hj}[\nu_h\varphi]\Big](t,x)-\tilde{\nu}_j(x)Q^{\sharp}_r\Big[\tilde{g},\sum_{h=1}^{n}M_{hi}[\nu_h\varphi]\Big](t,x).
\end{aligned}\tag{3.10}$$

Next we consider the last integral on the right-hand side of formula (3.6). Since Φ_n solves the heat equation in $\mathbb{R}\times\mathbb{R}^n\setminus\{(0,0)\}$, if $(t,x)\in\Omega_T$ is fixed as above and $(\tau,y)\in\partial_T\Omega$, we have

$$\frac{\partial}{\partial x_r}\Big(\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y)\Big)=\sum_{s=1}^{n}\Big(\nu_s(y)\frac{\partial}{\partial y_r}-\nu_r(y)\frac{\partial}{\partial y_s}\Big)\frac{\partial}{\partial x_s}\Phi_n(t-\tau,x-y)$$
$$-\nu_r(y)\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y). \tag{3.11}$$

Then Lemma 2.2 and Lemma 2.11 imply

$$\begin{aligned}
&\int_{-\infty}^{t}\int_{\partial\Omega}(\tilde g(x)-\tilde g(y))\frac{\partial}{\partial x_r}\Big(\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y)\Big)\{\tilde\nu_j(x)\nu_i(y)-\tilde\nu_i(x)\nu_j(y)\}\varphi(\tau,y)\,d\sigma_y\,d\tau\\
&\quad=-\int_{-\infty}^{t}\int_{\partial\Omega}(\tilde g(x)-\tilde g(y))\Big[\sum_{s=1}^{n}\Big(\nu_s(y)\frac{\partial}{\partial y_r}-\nu_r(y)\frac{\partial}{\partial y_s}\Big)\frac{\partial}{\partial x_s}\Phi_n(t-\tau,x-y)\Big]\\
&\qquad\qquad\times\{\tilde\nu_i(x)(\nu_j(y)-\tilde\nu_j(x))+\tilde\nu_j(x)(\tilde\nu_i(x)-\nu_i(y))\}\varphi(\tau,y)\,d\sigma_y\,d\tau\\
&\qquad+\int_{-\infty}^{t}\int_{\partial\Omega}(\tilde g(x)-\tilde g(y))\nu_r(y)\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\{\tilde\nu_i(x)\nu_j(y)-\tilde\nu_j(x)\nu_i(y)\}\varphi(\tau,y)\,d\sigma_y\,d\tau\\
&\quad=\tilde\nu_i(x)\Big\{\sum_{s=1}^{n}Q_s^\sharp[\tilde\nu_j,M_{sr}[g]\varphi](t,x)+\sum_{s=1}^{n}Q_s^\sharp[\tilde g,M_{sr}[\nu_j\varphi]](t,x)\Big\}\\
&\qquad-\tilde\nu_j(x)\Big\{\sum_{s=1}^{n}Q_s^\sharp[\tilde\nu_i,M_{sr}[g]\varphi](t,x)+\sum_{s=1}^{n}Q_s^\sharp[\tilde g,M_{sr}[\nu_i\varphi]](t,x)\Big\}\\
&\qquad+\tilde\nu_i(x)\tilde Q_t^\sharp[\tilde g,\nu_r\nu_j\varphi](t,x)-\tilde\nu_j(x)\tilde Q_t^\sharp[\tilde g,\nu_r\nu_i\varphi](t,x).
\end{aligned}\tag{3.12}$$

Then, by combining formulas (3.6), (3.9), (3.10) and (3.12), we obtain

$$\begin{aligned}
M_{ij}^\sharp[Q_r^\sharp[\tilde g,\varphi]](t,x)&=\tilde\nu_i(x)Q_r^\sharp\Big[\frac{\partial\tilde g}{\partial x_j}-(D\tilde g\cdot\tilde\nu)\tilde\nu_j,\varphi\Big](t,x)-\tilde\nu_j(x)Q_r^\sharp\Big[\frac{\partial\tilde g}{\partial x_i}-(D\tilde g\cdot\tilde\nu)\tilde\nu_i,\varphi\Big](t,x)\\
&\quad+\tilde\nu_i(x)Q_r^\sharp\Big[\tilde g,\sum_{h=1}^{n}M_{hj}[\nu_h\varphi]\Big](t,x)-\tilde\nu_j(x)Q_r^\sharp\Big[\tilde g,\sum_{h=1}^{n}M_{hi}[\nu_h\varphi]\Big](t,x)\\
&\quad+\tilde\nu_i(x)\Big\{\sum_{s=1}^{n}Q_s^\sharp[\tilde\nu_j,M_{sr}[g]\varphi](t,x)+\sum_{s=1}^{n}Q_s^\sharp[\tilde g,M_{sr}[\nu_j\varphi]](t,x)\Big\}\\
&\quad-\tilde\nu_j(x)\Big\{\sum_{s=1}^{n}Q_s^\sharp[\tilde\nu_i,M_{sr}[g]\varphi](t,x)+\sum_{s=1}^{n}Q_s^\sharp[\tilde g,M_{sr}[\nu_i\varphi]](t,x)\Big\}\\
&\quad+\tilde\nu_i(x)\tilde Q_t^\sharp[\tilde g,\nu_r\nu_j\varphi](t,x)-\tilde\nu_j(x)\tilde Q_t^\sharp[\tilde g,\nu_r\nu_i\varphi](t,x)
\end{aligned}\tag{3.13}$$

for all $(t,x)\in\Omega_T$. Now, under our assumptions, the first argument of the terms $Q_r^\sharp$, $Q_s^\sharp$, $\tilde Q_t^\sharp$, which appear on the right-hand side of formula (3.13), belongs to $C^{0,\alpha}(\mathrm{cl}\,\Omega)$, and the second argument of the terms $Q_r^\sharp$, $Q_s^\sharp$, which appear on the right-hand side of formula (3.13), belongs to $L^\infty(\partial_T\Omega)$, and the second argument of the term $\tilde Q_t^\sharp$, which appears on the right-hand side of formula (3.13), belongs to $C^{\frac{1+\beta}{2};1+\beta}(\partial_T\Omega)$. Then Lemma 3.1 (i) and Lemma 3.2 (i) imply that the right-hand side of formula (3.13) defines a continuous function of the variable $(t,x)\in\mathrm{cl}\,\Omega_T$. Since Ω is of class $C^{2,\alpha}$ and $\tilde g\in C^{1,\alpha}(\mathrm{cl}\,\Omega)$, and since we are assuming $\varphi\in C^{\frac{1+\beta}{2};1+\beta}(\partial_T\Omega)$, Lemma 3.1 (ii) implies $M_{ij}^\sharp[Q_r^\sharp[\tilde g,\varphi]]\in C^0(\mathrm{cl}\,\Omega_T)$. Hence the equality of formula (3.13) must hold for all $(t,x)\in\mathrm{cl}\,\Omega_T$ and thus, in particular, for all $(t,x)\in\partial_T\Omega$. Since $M_{ij}^\sharp[\cdot]=M_{ij}[\cdot]$, $Q_r^\sharp[\cdot,\cdot]=Q_r[\cdot,\cdot]$, $\tilde Q_t^\sharp[\cdot,\cdot]=\tilde Q_t[\cdot,\cdot]$ on $\partial_T\Omega$, we conclude that equality (3.3) holds under the assumption $\varphi\in C^{\frac{1+\beta}{2};1+\beta}(\partial_T\Omega)$ (see also (2.1)).

Now let $\varphi\in C^{\frac12;1}(\partial_T\Omega)$. We consider only the case $T=+\infty$. Indeed, the case $T<+\infty$ can be treated similarly. We fix $t\in\overline{]-\infty,T[}$, and we consider $\eta_1,\eta_2,\eta_3\in C_b^\infty(\mathbb{R})$ such that $\sum_{i=1}^3\eta_i=1$, $0\le\eta_i\le1$ for all $i=1,2,3$, $\eta_2(\tau)=1$ for all $\tau\in[t-1,t+1]$ and

$$\operatorname{supp}\eta_2\subseteq\,]t-2,t+2[,\quad \operatorname{supp}\eta_1\subseteq\,]-\infty,t-1[,\quad \operatorname{supp}\eta_3\subseteq\,]t+1,+\infty[.$$

Then we set $\varphi_i(\tau, x) = \varphi(\tau, x)\eta_i(\tau)$ for all $(\tau, x) \in \partial_T\Omega$ and for all $i = 1, 2, 3$. Clearly,

$$\varphi(\tau, x) = \varphi_1(\tau, x) + \varphi_2(\tau, x) + \varphi_3(\tau, x) \quad \text{for all } (\tau, x) \in \partial_T\Omega.$$

We denote by $P_{ijr}[g, \varphi]$ the right-hand side of (3.3). We now show that the weak M_{ij}-derivative of $Q_r[g, \varphi_2]$ coincides with $P_{ijr}[g, \varphi_2]$. Since φ_2 has compact support, by considering an extension of φ_2 of class $C^{\frac{1}{2};1}$ with compact support in $\mathbb{R}^{n+1}$ (see Ladyženskaja, Solonnikov and Ural'ceva [16, Chapter 1.1, pp. 9–10]), by considering a sequence of mollifiers of such an extension and by taking the restriction to $\partial_T\Omega$, we conclude that there exists a sequence of functions $\{\varphi_{2l}\}_{l\in\mathbb{N}}$ in $C^{1;2}(\partial_T\Omega)$ such that φ_{2l} converges to φ_2 in $C^{\frac{1}{2};1}(\partial_T\Omega)$. Next we note that Remarks 2.6, 2.7 and Theorems 3.3, 3.5 (ii) and the continuity of the pointwise product in Schauder spaces imply that the operators $Q_r[g, \cdot\,]$ and $P_{ijr}[g, \cdot\,]$ are continuous from $C^{\frac{1}{2};1}(\partial_T\Omega)$ to $C^0_b(\partial_T\Omega)$. Moreover, we note that (3.3) holds for φ_{2l}. If $\mu \in C^1(\partial\Omega)$, then we have

$$\begin{aligned}
\int_{\partial\Omega} Q_r[g, \varphi_2](t, y)M_{ij}[\mu](y)\,d\sigma_y &= \lim_{l\to\infty} \int_{\partial\Omega} Q_r[g, \varphi_{2l}](t, y)M_{ij}[\mu](y)\,d\sigma_y \\
&= -\lim_{l\to\infty} \int_{\partial\Omega} M_{ij}[Q_r[g, \varphi_{2l}]](t, y)\mu(y)\,d\sigma_y \\
&= -\lim_{l\to\infty} \int_{\partial\Omega} P_{ijr}[g, \varphi_{2l}](t, y)\mu(y)\,d\sigma_y \\
&= -\int_{\partial\Omega} P_{ijr}[g, \varphi_2](t, y)\mu(y)\,d\sigma_y.
\end{aligned}$$

Hence $P_{ijr}[g, \varphi_2](t, \cdot)$ coincides with the weak M_{ij}-derivative of $Q_r[g, \varphi_2](t, \cdot)$ for all $i, j \in \{1, \dots, n\}$. Since both $P_{ijr}[g, \varphi_2]$ and $Q_r[g, \varphi_2]$ are continuous functions, it follows that $Q_r[g, \varphi_2](t, \cdot) \in C^1(\partial\Omega)$ and that $M_{ij}[Q_r[g, \varphi_2]](t, \cdot) = P_{ijr}[g, \varphi_2](t, \cdot)$ classically in $\partial\Omega$.

Moreover, $M_{ij}[Q_r[g, \varphi_1]](t, \cdot) = P_{ijr}[g, \varphi_1](t, \cdot)$ in $\partial\Omega$. Indeed, $\varphi_1(\tau, \cdot) = 0$ for all $\tau \in \,]t - 1, +\infty[$. Then the integral operators involved show no singularities, and then formulas (3.5)–(3.13) hold with φ replaced with φ_1 by classical differentiation theorems for integrals depending on a parameter.

Finally, since $\varphi_3(\tau, \cdot) = 0$ for all $\tau \in \,]-\infty, t + 1[$, the definition of Q_r and $\tilde{Q}_t$ implies

$$M_{ij}[Q_r[g, \varphi_3]](t, \cdot) = P_{ijr}[g, \varphi_3](t, \cdot) = 0 \quad \text{on } \partial\Omega.$$

Then $M_{ij}[Q_r[g, \varphi]] = P_{ijr}[g, \varphi]$ on $\partial_T\Omega$. □

Then we have the following:

Lemma 3.7. *Let $\alpha \in \,]0, 1[$, and let $T \in \,]-\infty, +\infty]$. Let $r \in \{1, \dots, n\}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class C^2. Let $g \in C^{0,\alpha}(\partial\Omega)$ and $\varphi \in C^{1;2}(\partial_T\Omega)$. Then $Q_r[g, \varphi]$ is continuously differentiable with respect to t, and*

$$\frac{\partial}{\partial t} Q_r[g, \varphi](t, x) = Q_r[g, \partial_t\varphi](t, x) \tag{3.14}$$

holds for all $(t, x) \in \partial_T\Omega$.

Proof. We consider $x \in \partial\Omega$ fixed, and we note that

$$\begin{aligned}
Q_r[g, \varphi](t, x) &= \int_{-\infty}^{t} \int_{\partial\Omega} (g(x) - g(y)) \frac{\partial}{\partial x_r} \Phi_n(t - \tau, x - y)\varphi(\tau, y)\,d\sigma_y\,d\tau \\
&= \int_{0}^{+\infty} \int_{\partial\Omega} (g(x) - g(y)) \frac{\partial}{\partial x_r} \Phi_n(\tau, x - y)\varphi(t - \tau, y)\,d\sigma_y\,d\tau \quad \text{for all } t \in \overline{]-\infty, T[}.
\end{aligned}$$

By the definition of Φ_n, we have

$$C_{0,1,\partial\Omega} \equiv \sup_{\substack{(t,x)\in]0,+\infty[\times\mathbb{R}^n,\\ |x|\le\operatorname{diam}(\partial\Omega)}} |D_x\Phi_n(t, x)| \frac{t^{\frac{n}{2}+1}}{|x|} e^{\frac{|x|^2}{4t}} < +\infty$$

(see also [17, Lemma 4.2 (i)]), and thus

$$|(g(x)-g(y))\tfrac{\partial}{\partial x_r}\Phi_n(\tau,x-y)\partial_t\varphi(t-\tau,y)| \le \|g\|_{C^{0,\alpha}(\partial\Omega)}\|\partial_t\varphi\|_{C^0_b(\partial_T\Omega)}C_{0,1,\partial\Omega}|x-y|^{1+\alpha}\tau^{-\frac{n}{2}-1}e^{-\frac{|x-y|^2}{4\tau}}$$

for all $t \in \overline{]-\infty,T[}$, $(\tau,y) \in]0,+\infty[\times\partial\Omega$. Moreover, there exists $c'_{\Omega,n-1-\alpha} > 0$ such that

$$\int_0^{+\infty}\int_{\partial\Omega}|x-y|^{1+\alpha}\tau^{-\frac{n}{2}-1}e^{-\frac{|x-y|^2}{4\tau}}\,d\sigma_y\,d\tau = 4^{\frac{n}{2}}\int_0^{+\infty}u^{-\frac{n}{2}-1}e^{-\frac{1}{u}}\,du\int_{\partial\Omega}\frac{1}{|x-y|^{n-1-\alpha}}\,d\sigma_y \le 4^{\frac{n}{2}}\Gamma\Big(\frac{n}{2}\Big)c'_{\Omega,n-1-\alpha}$$

for all $x \in \partial\Omega$ (cf. [17, Lemma 3.3 (i)]). Then the statement follows by classical differentiation theorems for integrals depending on a parameter. □

Then we have the following formula for the tangential derivatives of the operator $\tilde{Q}_t$, which we have introduced in Theorem 3.5.

Lemma 3.8. *Let $\alpha \in]0,1[$, and let $T \in]-\infty,+\infty]$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{2,\alpha}$. Let $g \in C^{1,\alpha}(\partial\Omega)$ and $\varphi \in C^{1;2}(\partial_T\Omega)$. Then $\tilde{Q}_t[g,\varphi](t,\cdot) \in C^1(\partial\Omega)$ for all $t \in \overline{]-\infty,T[}$. Moreover,*

$$\begin{aligned}
M_{ij}[\tilde{Q}_t[g,\varphi]](t,x) &= \nu_i(x)\tilde{Q}_t[(D_{\partial\Omega}g)_j,\varphi](t,x) - \nu_j(x)\tilde{Q}_t[(D_{\partial\Omega}g)_i,\varphi](t,x)\\
&\quad + \nu_i(x)\tilde{Q}_t\Big[g,\sum_{h=1}^n \nu_h M_{hj}[\varphi]\Big](t,x) - \nu_j(x)\tilde{Q}_t\Big[g,\sum_{h=1}^n \nu_h M_{hi}[\varphi]\Big](t,x)\\
&\quad - \nu_i(x)g(x)\tilde{Q}_t\Big[\sum_{h=1}^n M_{hj}[\nu_h],\varphi\Big](t,x) + \nu_j(x)g(x)\tilde{Q}_t\Big[\sum_{h=1}^n M_{hi}[\nu_h],\varphi\Big](t,x)\\
&\quad + \nu_i(x)\tilde{Q}_t\Big[\sum_{h=1}^n M_{hj}[\nu_h],g\varphi\Big](t,x) - \nu_j(x)\tilde{Q}_t\Big[\sum_{h=1}^n M_{hi}[\nu_h],g\varphi\Big](t,x)\\
&\quad + \nu_i(x)\sum_{h=1}^n M_{hj}[\nu_h](x)\tilde{Q}_t[g,\varphi](t,x) - \nu_j(x)\sum_{h=1}^n M_{hi}[\nu_h](x)\tilde{Q}_t[g,\varphi](t,x)\\
&\quad - \nu_i(x)\sum_{s=1}^n Q_s\Big[g,\nu_s\nu_j\frac{\partial\varphi}{\partial t}\Big](t,x) + \nu_j(x)\sum_{s=1}^n Q_s\Big[g,\nu_s\nu_i\frac{\partial\varphi}{\partial t}\Big](t,x)
\end{aligned} \tag{3.15}$$

holds for all $(t,x) \in \partial_T\Omega$ and for all $i,j \in \{1,\dots,n\}$; see (2.1), (3.2).

Proof. Let $R \in]0,+\infty[$ be such that $\mathrm{cl}\,\Omega \subseteq \mathbb{B}_n(0,R)$. Let "$\tilde{\cdot}$" be an extension operator as in Lemma 2.1, defined on $C^{1,\alpha}(\partial\Omega)$.

First we fix $\beta \in]0,\alpha[$, and we prove formula (3.15) under the assumption that $\varphi \in C^{\frac{2+\beta}{2};2+\beta}(\partial_T\Omega)$. By Lemma 3.2 (ii), we already know that $\tilde{Q}^\sharp_t[\tilde{g},\varphi](t,\cdot)$ belongs to $C^1(\mathrm{cl}\,\Omega)$ for all $t \in \overline{]-\infty,T[}$. Next we fix $(t,x) \in \Omega_T$, and we compute (see (3.4))

$$M^\sharp_{ij}[\tilde{Q}^\sharp_t[\tilde{g},\varphi]](t,x) = \tilde{\nu}_i(x)\frac{\partial}{\partial x_j}\tilde{Q}^\sharp_t[\tilde{g},\varphi](t,x) - \tilde{\nu}_j(x)\frac{\partial}{\partial x_i}\tilde{Q}^\sharp_t[\tilde{g},\varphi](t,x).$$

First we note that Lemma 2.11 implies

$$\int_{-\infty}^t\int_{\partial\Omega}(\tilde{g}(x)-\tilde{g}(y))\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\varphi(t,y)\,d\sigma_y\,d\tau = 0.$$

Then, by differentiation theorems for integrals depending on a parameter, we have

$$\begin{aligned}
\frac{\partial}{\partial x_i}\tilde{Q}^\sharp_t[\tilde{g},\varphi](t,x) &= \int_{-\infty}^t\int_{\partial\Omega}\frac{\partial\tilde{g}}{\partial x_i}(x)\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\,d\sigma_y\,d\tau\\
&\quad + \int_{-\infty}^t\int_{\partial\Omega}(\tilde{g}(x)-\tilde{g}(y))\frac{\partial^2}{\partial x_i\partial t}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\,d\sigma_y\,d\tau.
\end{aligned} \tag{3.16}$$

Since $\sum_{h=1}^n \nu_h^2 = 1$ on $\partial\Omega$, we have

$$\begin{aligned}
\frac{\partial}{\partial x_i}\tilde{Q}_t^\sharp[\tilde{g},\varphi](t,x) &= \int_{-\infty}^{t}\int_{\partial\Omega} \frac{\partial \tilde{g}}{\partial x_i}(x)\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\,d\sigma_y\,d\tau \\
&\quad - \int_{-\infty}^{t}\int_{\partial\Omega} (\tilde{g}(x)-\tilde{g}(y))\sum_{h=1}^n \nu_h^2(y)\frac{\partial}{\partial y_i}\Big(\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\Big)\varphi(\tau,y)\,d\sigma_y\,d\tau \\
&= \int_{-\infty}^{t}\int_{\partial\Omega} \frac{\partial \tilde{g}}{\partial x_i}(x)\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\,d\sigma_y\,d\tau \\
&\quad - \int_{-\infty}^{t}\int_{\partial\Omega} (\tilde{g}(x)-\tilde{g}(y))\sum_{h=1}^n \Big[\Big(\nu_h(y)\frac{\partial}{\partial y_i} - \nu_i(y)\frac{\partial}{\partial y_h}\Big)\Big(\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\Big)\Big] \\
&\qquad \times \varphi(\tau,y)\nu_h(y)\,d\sigma_y\,d\tau \\
&\quad - \int_{-\infty}^{t}\int_{\partial\Omega} (\tilde{g}(x)-\tilde{g}(y))\sum_{h=1}^n \Big[\nu_h(y)\frac{\partial}{\partial y_h}\Big(\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\Big)\Big] \\
&\qquad \times \nu_i(y)\varphi(\tau,y)\,d\sigma_y\,d\tau.
\end{aligned}$$

By Lemma 2.2, the second integral on the right-hand side takes the form

$$\begin{aligned}
&\int_{-\infty}^{t}\int_{\partial\Omega} (\tilde{g}(x)-\tilde{g}(y))\sum_{h=1}^n \Big[\Big(\nu_h(y)\frac{\partial}{\partial y_i} - \nu_i(y)\frac{\partial}{\partial y_h}\Big)\Big(\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\Big)\Big]\varphi(\tau,y)\nu_h(y)\,d\sigma_y\,d\tau \\
&\quad = \int_{-\infty}^{t}\int_{\partial\Omega} (\tilde{g}(x)-\tilde{g}(y))\sum_{h=1}^n M_{hi,y}\Big[\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\Big]\varphi(\tau,y)\nu_h(y)\,d\sigma_y\,d\tau \\
&\quad = \sum_{h=1}^n \int_{-\infty}^{t}\int_{\partial\Omega} M_{hi,y}[\tilde{g}(y)]\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\nu_h(y)\,d\sigma_y\,d\tau \\
&\qquad - \sum_{h=1}^n \int_{-\infty}^{t}\int_{\partial\Omega} (\tilde{g}(x)-\tilde{g}(y))\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)M_{hi,y}[\varphi(\tau,y)\nu_h(y)]\,d\sigma_y\,d\tau.
\end{aligned}$$

Accordingly, we have

$$\begin{aligned}
\frac{\partial}{\partial x_i}\tilde{Q}_t^\sharp[\tilde{g},\varphi](t,x) &= \frac{\partial \tilde{g}}{\partial x_i}(x)\int_{-\infty}^{t}\int_{\partial\Omega} \frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\,d\sigma_y\,d\tau \\
&\quad - \sum_{h=1}^n \int_{-\infty}^{t}\int_{\partial\Omega} M_{hi,y}[\tilde{g}(y)]\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\nu_h(y)\,d\sigma_y\,d\tau \\
&\quad + \sum_{h=1}^n \int_{-\infty}^{t}\int_{\partial\Omega} (\tilde{g}(x)-\tilde{g}(y))\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)M_{hi,y}[\varphi(\tau,y)\nu_h(y)]\,d\sigma_y\,d\tau \\
&\quad - \int_{-\infty}^{t}\int_{\partial\Omega} (\tilde{g}(x)-\tilde{g}(y))\frac{\partial}{\partial t}\Big(\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y)\Big)\nu_i(y)\varphi(\tau,y)\,d\sigma_y\,d\tau.
\end{aligned}$$

Then we have

$$
\begin{aligned}
M^{\sharp}_{ij}[\tilde{Q}^{\sharp}_t[\tilde{g},\varphi]](t,x) = M^{\sharp}_{ij}[\tilde{g}](x) \int_{-\infty}^{t}\int_{\partial\Omega} \frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\,d\sigma_y\,d\tau \\
+ \sum_{h=1}^{n}\int_{-\infty}^{t}\int_{\partial\Omega} \{-\tilde{\nu}_i(x)M_{hj,y}[\tilde{g}(y)] + \tilde{\nu}_j(x)M_{hi,y}[\tilde{g}(y)]\} \\
\times \frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\nu_h(y)\,d\sigma_y\,d\tau \\
+ \sum_{h=1}^{n}\int_{-\infty}^{t}\int_{\partial\Omega} (\tilde{g}(x)-\tilde{g}(y))\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y) \\
\times \{\tilde{\nu}_i(x)M_{hj,y}[\varphi(\tau,y)\nu_h(y)] \\
- \tilde{\nu}_j(x)M_{hi,y}[\varphi(\tau,y)\nu_h(y)]\}\,d\sigma_y\,d\tau \\
+ \int_{-\infty}^{t}\int_{\partial\Omega} (\tilde{g}(x)-\tilde{g}(y))\frac{\partial}{\partial t}\Big(\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y)\Big) \\
\times \{\tilde{\nu}_j(x)\nu_i(y) - \tilde{\nu}_i(x)\nu_j(y)\}\varphi(\tau,y)\,d\sigma_y\,d\tau. \qquad (3.17)
\end{aligned}
$$

By the identities (3.7), (3.8), we rewrite the sum of the first two terms on the right-hand side of (3.17) in the form

$$
\begin{aligned}
& M^{\sharp}_{ij}[\tilde{g}](x) \int_{-\infty}^{t}\int_{\partial\Omega} \frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\,d\sigma_y\,d\tau \\
&\quad + \sum_{h=1}^{n}\int_{-\infty}^{t}\int_{\partial\Omega} \{-\tilde{\nu}_i(x)M_{hj,y}[\tilde{g}(y)] + \tilde{\nu}_j(x)M_{hi,y}[\tilde{g}(y)]\} \\
&\qquad \times \frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\nu_h(y)\,d\sigma_y\,d\tau \\
&= \int_{-\infty}^{t}\int_{\partial\Omega} \frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\,d\sigma_y\,d\tau\,\tilde{\nu}_i(x)\Big(\frac{\partial\tilde{g}}{\partial x_j}(x) - (D\tilde{g}(x)\cdot\tilde{\nu}(x))\tilde{\nu}_j(x)\Big) \\
&\quad - \int_{-\infty}^{t}\int_{\partial\Omega} \frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\,d\sigma_y\,d\tau\,\tilde{\nu}_j(x)\Big(\frac{\partial\tilde{g}}{\partial x_i}(x) - (D\tilde{g}(x)\cdot\tilde{\nu}(x))\tilde{\nu}_i(x)\Big) \\
&\quad - \tilde{\nu}_i(x)\int_{-\infty}^{t}\int_{\partial\Omega} \Big(\frac{\partial\tilde{g}}{\partial x_j}(y) - (D\tilde{g}(y)\cdot\tilde{\nu}(y))\tilde{\nu}_j(y)\Big)\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\,d\sigma_y\,d\tau \\
&\quad + \tilde{\nu}_j(x)\int_{-\infty}^{t}\int_{\partial\Omega} \Big(\frac{\partial\tilde{g}}{\partial x_i}(y) - (D\tilde{g}(y)\cdot\tilde{\nu}(y))\tilde{\nu}_i(y)\Big)\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\,d\sigma_y\,d\tau \\
&= \tilde{\nu}_i(x)\int_{-\infty}^{t}\int_{\partial\Omega} \Big[\Big(\frac{\partial\tilde{g}}{\partial x_j}(x) - (D\tilde{g}(x)\cdot\tilde{\nu}(x))\tilde{\nu}_j(x)\Big) - \Big(\frac{\partial\tilde{g}}{\partial x_j}(y) - (D\tilde{g}(y)\cdot\tilde{\nu}(y))\tilde{\nu}_j(y)\Big)\Big] \\
&\qquad \times \frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\,d\sigma_y\,d\tau \\
&\quad - \tilde{\nu}_j(x)\int_{-\infty}^{t}\int_{\partial\Omega} \Big[\Big(\frac{\partial\tilde{g}}{\partial x_i}(x) - (D\tilde{g}(x)\cdot\tilde{\nu}(x))\tilde{\nu}_i(x)\Big) - \Big(\frac{\partial\tilde{g}}{\partial x_i}(y) - (D\tilde{g}(y)\cdot\tilde{\nu}(y))\tilde{\nu}_i(y)\Big)\Big] \\
&\qquad \times \frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\,d\sigma_y\,d\tau \\
&= \tilde{\nu}_i(x)\tilde{Q}^{\sharp}_t\Big[\frac{\partial\tilde{g}}{\partial x_j} - (D\tilde{g}\cdot\tilde{\nu})\tilde{\nu}_j,\varphi\Big](t,x) - \tilde{\nu}_j(x)\tilde{Q}^{\sharp}_t\Big[\frac{\partial\tilde{g}}{\partial x_i} - (D\tilde{g}\cdot\tilde{\nu})\tilde{\nu}_i,\varphi\Big](t,x). \qquad (3.18)
\end{aligned}
$$

Now we consider the third term on the right-hand side of formula (3.17):

$$\begin{aligned}\sum_{h=1}^{n}\int_{-\infty}^{t}\int_{\partial\Omega}&(\tilde{g}(x)-\tilde{g}(y))\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\\ &\times\{\tilde{\nu}_i(x)M_{hj,y}[\varphi(\tau,y)\nu_h(y)]-\tilde{\nu}_j(x)M_{hi,y}[\varphi(\tau,y)\nu_h(y)]\}\,d\sigma_y\,d\tau\\ &=\tilde{\nu}_i(x)\tilde{Q}_t^\sharp\Big[\tilde{g},\sum_{h=1}^{n}M_{hj}[\nu_h\varphi]\Big](t,x)\\ &\quad-\tilde{\nu}_j(x)\tilde{Q}_t^\sharp\Big[\tilde{g},\sum_{h=1}^{n}M_{hi}[\nu_h\varphi]\Big](t,x).\end{aligned}\tag{3.19}$$

Next we consider the first term on the right-hand side of (3.19), and we note that

$$\begin{aligned}\tilde{\nu}_i(x)\tilde{Q}_t^\sharp\Big[\tilde{g},\sum_{h=1}^{n}M_{hj}[\nu_h\varphi]\Big](t,x)&=\tilde{\nu}_i(x)\tilde{Q}_t^\sharp\Big[\tilde{g},\sum_{h=1}^{n}M_{hj}[\nu_h]\varphi\Big](t,x)\\ &\quad+\tilde{\nu}_i(x)\tilde{Q}_t^\sharp\Big[\tilde{g},\sum_{h=1}^{n}M_{hj}[\varphi]\nu_h\Big](t,x)\\ &\quad-\tilde{\nu}_i(x)\sum_{h=1}^{n}M_{hj}^\sharp[\tilde{\nu}_h](x)\tilde{Q}_t^\sharp[\tilde{g},\varphi](t,x)\\ &\quad+\tilde{\nu}_i(x)\sum_{h=1}^{n}M_{hj}^\sharp[\tilde{\nu}_h](x)\tilde{Q}_t^\sharp[\tilde{g},\varphi](t,x),\end{aligned}$$

and that, by definition of $\tilde{Q}_t^\sharp$,

$$\begin{aligned}\tilde{\nu}_i(x)\tilde{Q}_t^\sharp\Big[\tilde{g},\sum_{h=1}^{n}M_{hj}[\nu_h]\varphi\Big](t,x)&-\tilde{\nu}_i(x)\sum_{h=1}^{n}M_{hj}^\sharp[\tilde{\nu}_h](x)\tilde{Q}_t^\sharp[\tilde{g},\varphi](t,x)\\ &=-\tilde{\nu}_i(x)\tilde{g}(x)\tilde{Q}_t^\sharp\Big[\sum_{h=1}^{n}M_{hj}^\sharp[\tilde{\nu}_h],\varphi\Big](t,x)+\tilde{\nu}_i(x)\tilde{Q}_t^\sharp\Big[\sum_{h=1}^{n}M_{hj}^\sharp[\tilde{\nu}_h],g\varphi\Big](t,x),\end{aligned}$$

and that the corresponding equalities can be written for the second term on the right-hand side of (3.19). Hence we deduce that the right-hand side of formula (3.19) equals

$$\begin{aligned}\tilde{\nu}_i(x)\tilde{Q}_t^\sharp\Big[\tilde{g},\sum_{h=1}^{n}\nu_hM_{hj}[\varphi]\Big](t,x)&-\tilde{\nu}_j(x)\tilde{Q}_t^\sharp\Big[\tilde{g},\sum_{h=1}^{n}\nu_hM_{hi}[\varphi]\Big](t,x)\\ &-\tilde{\nu}_i(x)\tilde{g}(x)\tilde{Q}_t^\sharp\Big[\sum_{h=1}^{n}M_{hj}^\sharp[\tilde{\nu}_h],\varphi\Big](t,x)+\tilde{\nu}_j(x)\tilde{g}(x)\tilde{Q}_t^\sharp\Big[\sum_{h=1}^{n}M_{hi}^\sharp[\tilde{\nu}_h],\varphi\Big](t,x)\\ &+\tilde{\nu}_i(x)\tilde{Q}_t^\sharp\Big[\sum_{h=1}^{n}M_{hj}^\sharp[\nu_h],g\varphi\Big](t,x)-\tilde{\nu}_j(x)\tilde{Q}_t^\sharp\Big[\sum_{h=1}^{n}M_{hi}^\sharp[\tilde{\nu}_h],g\varphi\Big](t,x)\\ &+\tilde{\nu}_i(x)\sum_{h=1}^{n}M_{hj}^\sharp[\tilde{\nu}_h](x)\tilde{Q}_t^\sharp[\tilde{g},\varphi](t,x)-\tilde{\nu}_j(x)\sum_{h=1}^{n}M_{hi}^\sharp[\tilde{\nu}_h](x)\tilde{Q}_t^\sharp[\tilde{g},\varphi](t,x).\end{aligned}\tag{3.20}$$

Next we consider the last integral on the right-hand side of formula (3.17). By integration by parts, we have

$$\begin{aligned}\int_{-\infty}^{t}\int_{\partial\Omega}(\tilde{g}(x)-\tilde{g}(y))\frac{\partial}{\partial t}\Big(\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y)\Big)&\{\tilde{\nu}_j(x)\nu_i(y)-\tilde{\nu}_i(x)\nu_j(y)\}\varphi(\tau,y)\,d\sigma_y\,d\tau\\ &=\tilde{\nu}_j(x)\sum_{s=1}^{n}Q_s^\sharp[\tilde{g},\nu_s\nu_i\partial_t\varphi](t,x)-\tilde{\nu}_i(x)\sum_{s=1}^{n}Q_s^\sharp[\tilde{g},\nu_s\nu_j\partial_t\varphi](t,x).\end{aligned}\tag{3.21}$$

Then, by combining formulas (3.17), (3.18), (3.20) and (3.21), we obtain

$$
\begin{aligned}
M^\sharp_{ij}[\tilde{Q}^\sharp_t[\tilde{g},\varphi]](t,x) &= \tilde{\nu}_i(x)\tilde{Q}^\sharp_t\Big[\frac{\partial\tilde{g}}{\partial x_j} - (D\tilde{g}\cdot\tilde{\nu})\tilde{\nu}_j, \varphi\Big](t,x) - \tilde{\nu}_j(x)\tilde{Q}^\sharp_t\Big[\frac{\partial\tilde{g}}{\partial x_i} - (D\tilde{g}\cdot\tilde{\nu})\tilde{\nu}_i, \varphi\Big](t,x) \\
&\quad + \tilde{\nu}_i(x)\tilde{Q}^\sharp_t\Big[\tilde{g}, \sum_{h=1}^n \nu_h M_{hj}[\varphi]\Big](t,x) - \tilde{\nu}_j(x)\tilde{Q}^\sharp_t\Big[\tilde{g}, \sum_{h=1}^n \nu_h M_{hi}[\varphi]\Big](t,x) \\
&\quad - \tilde{\nu}_i(x)\tilde{g}(x)\tilde{Q}^\sharp_t\Big[\sum_{h=1}^n M^\sharp_{hj}[\tilde{\nu}_h], \varphi\Big](t,x) + \tilde{\nu}_j(x)\tilde{g}(x)\tilde{Q}^\sharp_t\Big[\sum_{h=1}^n M^\sharp_{hi}[\tilde{\nu}_h], \varphi\Big](t,x) \\
&\quad + \tilde{\nu}_i(x)\tilde{Q}^\sharp_t\Big[\sum_{h=1}^n M^\sharp_{hj}[\tilde{\nu}_h], g\varphi\Big](t,x) - \tilde{\nu}_j(x)\tilde{Q}^\sharp_t\Big[\sum_{h=1}^n M^\sharp_{hi}[\tilde{\nu}_h], g\varphi\Big](t,x) \\
&\quad + \tilde{\nu}_i(x)\sum_{h=1}^n M^\sharp_{hj}[\tilde{\nu}_h](x)\tilde{Q}^\sharp_t[\tilde{g},\varphi](t,x) - \tilde{\nu}_j(x)\sum_{h=1}^n M^\sharp_{hi}[\tilde{\nu}_h](x)\tilde{Q}^\sharp_t[\tilde{g},\varphi](t,x) \\
&\quad - \tilde{\nu}_i(x)\sum_{s=1}^n Q^\sharp_s[\tilde{g}, \nu_s\nu_j\partial_t\varphi](t,x) + \tilde{\nu}_j(x)\sum_{s=1}^n Q^\sharp_s[\tilde{g}, \nu_s\nu_i\partial_t\varphi](t,x)
\end{aligned}
\tag{3.22}
$$

for all $(t,x) \in \Omega_T$. Now, under our assumptions, the first argument of the terms $\tilde{Q}^\sharp_t$, $Q^\sharp_s$, which appear on the right-hand side of formula (3.22), belongs to $C^{0,\alpha}(\mathrm{cl}\,\Omega)$, and the second argument of the term $Q^\sharp_s$, which appears on the right-hand side of formula (3.22), belongs to $C^{\frac{\beta}{2};\beta}(\partial_T\Omega)$, and the second argument of the term $\tilde{Q}^\sharp_t$, which appears on the right-hand side of formula (3.13), belongs to $C^{\frac{1+\beta}{2};1+\beta}(\partial_T\Omega)$. Then Lemma 3.1 (i) and Lemma 3.2 (i) imply that the right-hand side of formula (3.22) defines a continuous function of the variable $(t,x) \in \mathrm{cl}\,\Omega_T$. Since Ω is of class $C^{2,\alpha}$ and $\tilde{g} \in C^{1,\alpha}(\mathrm{cl}\,\Omega)$, and since we are assuming that $\varphi \in C^{\frac{2+\beta}{2};2+\beta}(\partial_T\Omega)$, Lemma 3.2 (ii) implies $M^\sharp_{ij}[\tilde{Q}^\sharp_t[\tilde{g},\varphi]] \in C^0(\mathrm{cl}\,\Omega_T)$. Hence the equality of formula (3.22) must hold for all $(t,x) \in \mathrm{cl}\,\Omega_T$ and thus, in particular, for all $(t,x) \in \partial_T\Omega$. Since $M^\sharp_{ij}[\,\cdot\,] = M_{ij}[\,\cdot\,]$, $Q^\sharp_r[\,\cdot\,,\cdot\,] = Q_r[\,\cdot\,,\cdot\,]$, $\tilde{Q}^\sharp_t[\,\cdot\,,\cdot\,] = \tilde{Q}_t[\,\cdot\,,\cdot\,]$ on $\partial_T\Omega$, we conclude that (3.15) holds under the assumption $\varphi \in C^{\frac{2+\beta}{2};2+\beta}(\partial_T\Omega)$ (see also (2.1)).

Now let $\varphi \in C^{1;2}(\partial_T\Omega)$. We consider only the case $T = +\infty$. Indeed, the case $T < +\infty$ can be treated similarly. We fix $t \in \overline{]-\infty, T[}$, and we consider $\eta_1, \eta_2, \eta_3 \in C^\infty_b(\mathbb{R})$ such that $\sum_{i=1}^3 \eta_i = 1$, $0 \le \eta_i \le 1$ for all $i = 1,2,3$, $\eta_2(\tau) = 1$ for all $\tau \in [t-1, t+1]$ and

$$\operatorname{supp}\eta_2 \subseteq \,]t-2, t+2[, \quad \operatorname{supp}\eta_1 \subseteq \,]-\infty, t-1[, \quad \operatorname{supp}\eta_3 \subseteq \,]t+1, +\infty[.$$

Then we set $\varphi_i(\tau,x) = \varphi(\tau,x)\eta_i(\tau)$ for all $(\tau,x) \in \partial_T\Omega$ and for all $i = 1,2,3$. Clearly,

$$\varphi(\tau,x) = \varphi_1(\tau,x) + \varphi_2(\tau,x) + \varphi_3(\tau,x) \quad \text{for all } (\tau,x) \in \partial_T\Omega.$$

We denote by $\tilde{P}_{ijt}[g,\varphi]$ the right-hand side of (3.15). We now show that the weak M_{ij}-derivative of $\tilde{Q}_t[g,\varphi_2]$ coincides with $\tilde{P}_{ijt}[g,\varphi_2]$. Since φ_2 has compact support, by considering an extension of φ_2 of class $C^{1;2}$ with compact support in $\mathbb{R}^{n+1}$ (see Ladyženskaja, Solonnikov and Ural'ceva [16, Chapter 1.1, pp. 9–10]), by considering a sequence of mollifiers of such an extension and by taking the restriction to $\partial_T\Omega$, we conclude that there exists a sequence of functions $\{\varphi_{2l}\}_{l\in\mathbb{N}}$ in $C^{\frac{2+\alpha}{2};2+\alpha}(\partial_T\Omega)$ such that φ_{2l} converges to φ_2 in $C^{1;2}(\partial_T\Omega)$. Next we note that Remarks 2.6, 2.7 and Theorems 3.3, 3.5 (ii) and the continuity of the pointwise product in Schauder spaces imply that the operators $\tilde{Q}_t[g,\cdot\,]$ and $\tilde{P}_{ijt}[g,\cdot\,]$ are continuous from $C^{1;2}(\partial_T\Omega)$ to $C^0_b(\partial_T\Omega)$. Moreover, we note that (3.15) holds for φ_{2l}. If $\mu \in C^1(\partial\Omega)$, then we have

$$
\begin{aligned}
\int_{\partial\Omega} \tilde{Q}_t[g,\varphi_2](t,y)M_{ij}[\mu](y)\,d\sigma_y &= \lim_{l\to\infty}\int_{\partial\Omega} \tilde{Q}_t[g,\varphi_{2l}](t,y)M_{ij}[\mu](y)\,d\sigma_y \\
&= -\lim_{l\to\infty}\int_{\partial\Omega} M_{ij}[\tilde{Q}_t[g,\varphi_{2l}]](t,y)\mu(y)\,d\sigma_y \\
&= -\lim_{l\to\infty}\int_{\partial\Omega} \tilde{P}_{ijt}[g,\varphi_{2l}](t,y)\mu(y)\,d\sigma_y \\
&= -\int_{\partial\Omega} \tilde{P}_{ijt}[g,\varphi_2](t,y)\mu(y)\,d\sigma_y.
\end{aligned}
$$

Hence $\tilde{P}_{ijt}[g,\varphi_2](t,\cdot)$ coincides with the weak M_{ij}-derivative of $\tilde{Q}_t[g,\varphi_2](t,\cdot)$ for all $i,j\in\{1,\dots,n\}$. Since both $\tilde{P}_{ijt}[g,\varphi_2]$ and $\tilde{Q}_t[g,\varphi_2]$ are continuous functions, it follows that $\tilde{Q}_t[g,\varphi_2](t,\cdot)\in C^1(\partial\Omega)$ in $\partial\Omega$ and that $M_{ij}[\tilde{Q}_t[g,\varphi_2]](t,\cdot)=\tilde{P}_{ijt}[g,\varphi_2](t,\cdot)$ classically in $\partial\Omega$.

Moreover, $M_{ij}[\tilde{Q}_t[g,\varphi_1]](t,\cdot)=\tilde{P}_{ijt}[g,\varphi_1](t,\cdot)$ in $\partial\Omega$. Indeed, $\varphi_1(\tau,\cdot)=0$ for all $\tau\in\,]t-1,+\infty[$. Then the integral operators involved show no singularities, and then formulas (3.16)–(3.22) hold with φ replaced with φ_1 by classical differentiation theorems for integrals depending on a parameter.

Finally, since $\varphi_3(\tau,\cdot)=0$ for all $\tau\in\,]-\infty,t+1[$, the definition of Q_r and $\tilde{Q}_t$ implies

$$M_{ij}[\tilde{Q}_t[g,\varphi_3]](t,\cdot)=\tilde{P}_{ijt}[g,\varphi_3](t,\cdot)=0\quad\text{on }\partial\Omega.$$

Then $M_{ij}[\tilde{Q}_t[g,\varphi]]=\tilde{P}_{ijt}[g,\varphi]$ in $\partial_T\Omega$. □

Then we have the following:

Lemma 3.9. *Let $\alpha\in\,]0,1[$, and let $T\in\,]-\infty,+\infty]$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class C^3. Let $g\in C^{0,\alpha}(\partial\Omega)$ and $\varphi\in C^{\frac{3}{2};3}(\partial_T\Omega)$. Then $\tilde{Q}_t[g,\varphi]$ is continuously differentiable with respect to t, and*

$$\frac{\partial}{\partial t}\tilde{Q}_t[g,\varphi](t,x)=\tilde{Q}_t[g,\partial_t\varphi](t,x)\tag{3.23}$$

holds for all $(t,x)\in\partial_T\Omega$.

Proof. We consider $x\in\partial\Omega$ fixed, and we note that

$$\begin{aligned}\tilde{Q}_t[g,\varphi](t,x)&=\int_{-\infty}^{t}\int_{\partial\Omega}(g(x)-g(y))\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)(\varphi(\tau,y)-\varphi(t,y))\,d\sigma_y\,d\tau\\&=\int_{0}^{+\infty}\int_{\partial\Omega}(g(x)-g(y))\frac{\partial}{\partial t}\Phi_n(\tau,x-y)(\varphi(t-\tau,y)-\varphi(t,y))\,d\sigma_y\,d\tau\quad\text{for all }t\in\overline{]-\infty,T[}.\end{aligned}$$

By the definition of Φ_n, we have

$$C_{1,0,\partial\Omega}\equiv\sup_{\substack{(t,x)\in]0,+\infty[\times\mathbb{R}^n,\\|x|\le\operatorname{diam}(\partial\Omega)}}|\partial_t\Phi_n(t,x)|t^{\frac{n}{2}+1}e^{\frac{|x|^2}{8t}}<+\infty,$$

and thus

$$\begin{aligned}&|(g(x)-g(y))\tfrac{\partial}{\partial t}\Phi_n(\tau,x-y)(\partial_t\varphi(t-\tau,y)-\partial_t\varphi(t,y))|\\&\quad\le\|g\|_{C^{0,\alpha}(\partial\Omega)}\|\partial_t\varphi\|_{C^{\frac{1}{2};1}(\partial_T\Omega)}C_{1,0,\partial\Omega}|x-y|^{\alpha}\tau^{-\frac{n+1}{2}}e^{-\frac{|x-y|^2}{8\tau}}\end{aligned}$$

for all $t\in\overline{]-\infty,T[}$, $(\tau,y)\in\,]0,+\infty[\,\times\partial\Omega$. Moreover, there exists $c'_{\Omega,n-1-\alpha}>0$ such that

$$\begin{aligned}\int_{0}^{+\infty}\int_{\partial\Omega}|x-y|^{\alpha}\tau^{-\frac{n+1}{2}}e^{-\frac{|x-y|^2}{8\tau}}\,d\sigma_y\,d\tau&=8^{\frac{n-1}{2}}\int_{0}^{+\infty}u^{-\frac{n+1}{2}}e^{-\frac{1}{u}}\,du\int_{\partial\Omega}\frac{1}{|x-y|^{n-1-\alpha}}\,d\sigma_y\\&\le8^{\frac{n-1}{2}}\Gamma\Big(\frac{n-1}{2}\Big)c'_{\Omega,n-1-\alpha}\end{aligned}$$

for all $x\in\partial\Omega$ (cf. [17, Lemma 3.3 (i)]). Then the statement follows by classical differentiation theorems for integrals depending on a parameter. □

Exploiting formulas (3.3), (3.14), (3.15) and (3.23), we can prove the following:

Theorem 3.10. *Let $\alpha\in\,]0,1[$, $\beta\in\,]0,\alpha[$, $m\in\mathbb{N}\setminus\{0\}$, $T\in\,]-\infty,+\infty]$. Let $r\in\{1,\dots,n\}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Then the following statements hold.*

(i) *The bilinear operator Q_r from the space $C^{m-1,\alpha}(\partial\Omega)\times C^{\frac{m-1}{2};m-1}(\partial_T\Omega)$ to $C^{\frac{m-1+\beta}{2};m-1+\beta}(\partial_T\Omega)$, which takes (g,φ) to $Q_r[g,\varphi]$, is continuous.*
(ii) *The bilinear operator $\tilde{Q}_t$ from the space $C^{m-1,\alpha}(\partial\Omega)\times C^{\frac{m}{2};m}(\partial_T\Omega)$ to the space $C^{\frac{m-1+\beta}{2};m-1+\beta}(\partial_T\Omega)$, which takes (g,φ) to $\tilde{Q}_t[g,\varphi]$, is continuous.*
(iii) *The bilinear operator Q_r from the space $C^{m-1,\alpha}(\partial\Omega)\times C^{\frac{m-1+\beta}{2};m-1+\beta}(\partial_T\Omega)$ to $C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$, which takes (g,φ) to $Q_r[g,\varphi]$, is continuous.*

(iv) *The bilinear operator* $\tilde{Q}_t$ *from the space* $C^{m-1,\alpha}(\partial\Omega) \times C^{\frac{m+\beta}{2};m+\beta}(\partial_T\Omega)$ *to* $C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$, *which takes* (g,φ) *to* $\tilde{Q}_t[g,\varphi]$, *is continuous.*

Proof. We first prove both statement (i) and statement (ii) at the same time. We proceed by induction on m. The case $m = 1$ for the operator Q_r follows from Theorem 3.3, and for the operator $\tilde{Q}_t$, it follows from Theorem 3.5 (ii).

We now consider the case $m = 2$. The continuity of Q_r follows by the continuity of Q_r with values into $C_b^0(\partial_T\Omega)$, which follows by the case $m = 1$, by the continuity of the operator Q_r with values in

$$C_b^{0,\frac{1+\beta}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega)),$$

which follows by Theorem 3.4 (iii), by the continuity of $M_{ij}[Q_r[\cdot,\cdot]]$ with values in $C^{\frac{\beta}{2};\beta}(\partial_T\Omega)$, which follows by formula (3.3) of Lemma 3.6, by the case $m = 1$ for Q_r and $\tilde{Q}_t$, by the continuity of the pointwise product in Schauder spaces and by Remarks 2.6 and 2.7.

By the same arguments, the continuity of the pointwise product in Schauder spaces, Remarks 2.6 and 2.7, Theorem 3.5 (iii), formula (3.15) of Lemma 3.8 and the case $m = 1$ imply the validity of the statement for the operator $\tilde{Q}_t$.

We now prove that if statements (i) and (ii) hold for all $m' \leq m$ and $m \geq 2$, then they hold for $m + 1$. It suffices to prove that the following three statements hold.

(j) Q_r is continuous from the space $C^{m,\alpha}(\partial\Omega) \times C^{\frac{m}{2};m}(\partial_T\Omega)$ to $C_b^0(\partial_T\Omega)$, and $\tilde{Q}_t$ is continuous from the space $C^{m,\alpha}(\partial\Omega) \times C^{\frac{m+1}{2};m+1}(\partial_T\Omega)$ to $C_b^0(\partial_T\Omega)$.

(jj) $M_{ij}[Q_r]$ is continuous from the space $C^{m,\alpha}(\partial\Omega) \times C^{\frac{m}{2};m}(\partial_T\Omega)$ to $C^{\frac{m-1+\beta}{2};m-1+\beta}(\partial_T\Omega)$, and $M_{ij}[\tilde{Q}_t]$ is continuous from the space $C^{m,\alpha}(\partial\Omega) \times C^{\frac{m+1}{2};m+1}(\partial_T\Omega)$ to $C^{\frac{m-1+\beta}{2};m-1+\beta}(\partial_T\Omega)$ for all $i,j \in \{1,\dots,n\}$.

(jjj) $\frac{\partial}{\partial t}Q_r$ is continuous from the space $C^{m,\alpha}(\partial\Omega) \times C^{\frac{m}{2};m}(\partial_T\Omega)$ to $C^{\frac{m-2+\beta}{2};m-2+\beta}(\partial_T\Omega)$, and $\frac{\partial}{\partial t}\tilde{Q}_t$ is continuous from the space $C^{m,\alpha}(\partial\Omega) \times C^{\frac{m+1}{2};m+1}(\partial_T\Omega)$ to $C^{\frac{m-2+\beta}{2};m-2+\beta}(\partial_T\Omega)$.

Statement (j) holds by statements (i), (ii) with $m = 1$ and by Remark 2.7. We now consider statement (jj). The continuity of the pointwise product in Schauder spaces, Remarks 2.6 and 2.7, Lemma 3.6 and the inductive assumption imply the validity of statement (jj) for the operator Q_r. By the same argument, the continuity of the pointwise product in Schauder spaces, Remarks 2.6 and 2.7, Lemma 3.8 and the inductive assumption imply the validity of statement (jj) for the operator $\tilde{Q}_t$.

Next we consider statement (jjj). Remarks 2.6 and 2.7, Lemma 3.7 and the inductive assumption imply the validity of statement (jjj) for the operator Q_r.

By the same argument, Remarks 2.6 and 2.7, Lemma 3.9 and the inductive assumption imply the validity of statement (jjj) for the operator $\tilde{Q}_t$. Accordingly, the proof of statements (i) and (ii) is complete.

The proof of statements (iii) and (iv) follows along the lines of the proof of statements (i) and (ii), by replacing the use of Theorems 3.3, 3.4 (iii), 3.5 (ii) by that of Theorems 3.4 (i), 3.4 (ii), 3.5 (i), respectively. □

4 Tangential derivatives and regularizing properties of the double layer heat potential

In this section, we prove a formula for the tangential derivatives of the double layer heat potential. Then we show that such a formula implies the validity of regularizing properties for the integral operator associated to the double layer heat potential on the boundary of parabolic cylinders. We start by stating two theorems on the double layer heat potential (for a proof, we refer to [17, Theorem 8.1, Theorem 10.1]).

Theorem 4.1. *Let* $\alpha \in]0,1[$, $\beta \in]0,\alpha[$, $T \in]-\infty,+\infty]$. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of class* $C^{1,\alpha}$.

(i) *If* $\mu \in L^\infty(\partial_T\Omega)$, *then the double layer heat potential on* $\partial_T\Omega$, *i.e.,*

$$w[\partial_T\Omega,\mu](t,x) = \int_{-\infty}^{t}\int_{\partial\Omega} \frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y\,d\tau,$$

for all $(t,x)\in\partial_T\Omega$, *belongs to* $B(\overline{]-\infty,T[},C^{0,\alpha}(\partial\Omega))$. *Moreover, the operator from*

$$L^\infty(\partial_T\Omega) \quad \text{to} \quad B(\overline{]-\infty,T[},C^{0,\alpha}(\partial\Omega)),$$

which takes μ *to the map* $w[\partial_T\Omega,\mu]$, *is linear and continuous.*

(ii) *The operator from*

$$L^\infty(\partial_T\Omega) \quad \text{to} \quad C_b^{0,\frac{\beta}{2}}(\overline{]-\infty,T[},C^{0}(\partial\Omega)),$$

which takes μ *to the map* $w[\partial_T\Omega,\mu]$, *is linear and continuous.*

(iii) *The operator from*

$$L^\infty(\partial_T\Omega) \quad \text{to} \quad C^{\frac{\beta}{2};\beta}(\partial_T\Omega),$$

which takes μ *to the map* $w[\partial_T\Omega,\mu]$, *is linear and continuous.*

Theorem 4.2. *Let* $\alpha\in\,]0,1[$, $\beta\in\,]0,\alpha[$, $T\in\,]-\infty,+\infty]$. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of class* $C^{1,\alpha}$.

(i) *If* $\mu\in C_b^{0,\frac{\beta}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))$, *then* $w[\partial_T\Omega,\mu]\in C_b^{0,\frac{\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))$. *Moreover, the operator from the space* $C_b^{0,\frac{\beta}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))$ *to the space* $C_b^{0,\frac{\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))$, *which takes* μ *to* $w[\partial_T\Omega,\mu]$, *is linear and continuous.*

(ii) *If* $\mu\in C^{\frac{\beta}{2};\beta}(\partial_T\Omega)$, *then* $w[\partial_T\Omega,\mu]\in C^{\frac{\alpha}{2};\alpha}(\partial_T\Omega)$. *Moreover, the operator from the space* $C^{\frac{\beta}{2};\beta}(\partial_T\Omega)$ *to the space* $C^{\frac{\alpha}{2};\alpha}(\partial_T\Omega)$, *which takes* μ *to* $w[\partial_T\Omega,\mu]$, *is linear and continuous.*

(iii) *If* $\mu\in C_b^{0,\frac{1+\beta}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))$, *then* $w[\partial_T\Omega,\mu]\in C_b^{0,\frac{1+\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))$. *Moreover, the operator from the space* $C_b^{0,\frac{1+\beta}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))$ *to the space* $C_b^{0,\frac{1+\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))$, *which takes* μ *to* $w[\partial_T\Omega,\mu]$, *is linear and continuous.*

(iv) *If* $\mu\in C_b^{0,\frac{1}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))$, *then* $w[\partial_T\Omega,\mu]\in C_b^{0,\frac{1+\beta}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))$. *Moreover, the linear operator from the space* $C_b^{0,\frac{1}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))$ *to the space* $C_b^{0,\frac{1+\beta}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))$, *which takes* μ *to* $w[\partial_T\Omega,\mu]$, *is continuous.*

Now we prove a formula for the tangential derivatives of the integral operator associated to the double layer heat potential. We do so by means of the following:

Theorem 4.3. *Let* $\alpha\in\,]0,1[$, *and let* $T\in\,]-\infty,+\infty]$. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of class* $C^{1,\alpha}$. *Let* $\varphi\in C^{\frac{1}{2};1}(\partial_T\Omega)$. *Then* $w[\partial_T\Omega,\varphi](t,\cdot)\in C^1(\partial\Omega)$ *for all* $t\in\overline{]-\infty,T[}$. *Moreover,*

$$\begin{aligned}M_{ij}[w[\partial_T\Omega,\varphi]](t,x) = \sum_{r=1}^n &\{Q_r[\nu_i,M_{jr}[\varphi]](t,x) - Q_r[\nu_j,M_{ir}[\varphi]](t,x)\}\\ &+\nu_i(x)\tilde{Q}_t[\nu_j,\varphi](t,x)-\nu_j(x)\tilde{Q}_t[\nu_i,\varphi](t,x)+w[\partial_T\Omega,M_{ij}[\varphi]](t,x)\end{aligned} \tag{4.1}$$

for all $(t,x)\in\partial_T\Omega$ *and for all* $i,j\in\{1,\dots,n\}$.

Proof. We fix $\beta\in\,]0,\alpha[$, and we first consider the case in which $\varphi\in C^{\frac{1+\beta}{2};1+\beta}(\partial_T\Omega)$. Let $R\in\,]0,+\infty[$ be such that $\mathrm{cl}\,\Omega\subseteq\mathbb{B}_n(0,R)$. Let "$\tilde{\cdot}$" be an extension operator as in Lemma 2.1, defined on $C^{1,\beta}(\partial\Omega)$. By Theorem 2.8 (ii), we have $w^+[\partial_T\Omega,\varphi]\in C^{\frac{1+\beta}{2};1+\beta}(\mathrm{cl}\,\Omega_T)$. First we fix $(t,x)\in\Omega_T$, and we compute

$$M_{ij}^\sharp[w^+[\partial_T\Omega,\varphi]](t,x)=\tilde{\nu}_i(x)\frac{\partial}{\partial x_j}w^+[\partial_T\Omega,\varphi](t,x)-\tilde{\nu}_j(x)\frac{\partial}{\partial x_i}w^+[\partial_T\Omega,\varphi](t,x).$$

By formula (3.11), by Lemma 2.2 and by classical differentiation theorems for integrals depending on a parameter, we have

$$\frac{\partial}{\partial x_i}w^+[\partial_T\Omega,\varphi](t,x)=\sum_{r=1}^n\frac{\partial}{\partial x_r}v^+[\partial_T\Omega,M_{ir}[\varphi]](t,x)-\frac{\partial}{\partial t}v^+[\partial_T\Omega,\nu_i\varphi](t,x).$$

Then, by the obvious identity

$$-\tilde{\nu}_i(x)\nu_j(y)+\tilde{\nu}_j(x)\nu_i(y)=\tilde{\nu}_i(x)(\tilde{\nu}_j(x)-\nu_j(y))-\tilde{\nu}_j(x)(\tilde{\nu}_i(x)-\nu_i(y))$$

for all $y\in\partial\Omega$ and by Lemma 2.11, we have

$$\begin{aligned}M_{ij}^\sharp[w^+[\partial_T\Omega,\varphi]](t,x)=\sum_{r=1}^n&\left\{\tilde{\nu}_i(x)\frac{\partial}{\partial x_r}v^+[\partial_T\Omega,M_{jr}[\varphi]](t,x)-\tilde{\nu}_j(x)\frac{\partial}{\partial x_r}v^+[\partial_T\Omega,M_{ir}[\varphi]](t,x)\right\}\\ &+\tilde{\nu}_i(x)\tilde{Q}_t^\sharp[\tilde{\nu}_j,\varphi](t,x)-\tilde{\nu}_j(x)\tilde{Q}_t^\sharp[\tilde{\nu}_i,\varphi](t,x).\end{aligned} \tag{4.2}$$

Now, under our assumptions, Theorems 2.8 (ii), 2.9 (ii), (iii) and Lemma 3.2 (i) imply that both sides of (4.2) define continuous functions in $\mathrm{cl}\,\Omega_T$, and then (4.2) must hold for all $(t,x) \in \mathrm{cl}\,\Omega_T$. In particular, (4.2) holds for all $(t,x) \in \partial_T\Omega$. Now we fix $(t,x) \in \partial_T\Omega$. Then the jump relation (2.4) for the double layer heat potential and equality (4.2) in $\partial_T\Omega$ imply

$$\begin{aligned} M_{ij}[w[\partial_T\Omega,\varphi]](t,x) &= \frac{1}{2}M_{ij}[\varphi](t,x) + M_{ij}[w^+[\partial_T\Omega,\varphi]](t,x) \\ &= \frac{1}{2}M_{ij}[\varphi](t,x) + \sum_{r=1}^{n}\Big\{\nu_i(x)\frac{\partial}{\partial x_r}v^+[\partial_T\Omega, M_{jr}[\varphi]](t,x) \\ &\qquad - \nu_j(x)\frac{\partial}{\partial x_r}v^+[\partial_T\Omega, M_{ir}[\varphi]](t,x)\Big\} \\ &\quad + \nu_i(x)\tilde{Q}_t[\nu_j,\varphi](t,x) - \nu_j(x)\tilde{Q}_t[\nu_i,\varphi](t,x). \end{aligned}$$

Then the jump relation (2.5) for the single layer heat potential implies

$$\begin{aligned} M_{ij}[w[\partial_T\Omega,\varphi]](t,x) &= \frac{1}{2}M_{ij}[\varphi](t,x) + \frac{1}{2}\Big\{\sum_{r=1}^{n} M_{jr}[\varphi](t,x)\nu_r(x)\nu_i(x) - \sum_{r=1}^{n} M_{ir}[\varphi](t,x)\nu_r(x)\nu_j(x)\Big\} \\ &\quad + \sum_{r=1}^{n}\Big\{\nu_i(x)\int_{-\infty}^{t}\int_{\partial\Omega}\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)M_{jr}[\varphi](\tau,y)\,d\sigma_y\,d\tau \\ &\qquad - \nu_j(x)\int_{-\infty}^{t}\int_{\partial\Omega}\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)M_{ir}[\varphi](\tau,y)\,d\sigma_y\,d\tau\Big\} \\ &\quad + \nu_i(x)\tilde{Q}_t[\nu_j,\varphi](t,x) - \nu_j(x)\tilde{Q}_t[\nu_i,\varphi](t,x). \end{aligned} \tag{4.3}$$

We now consider the second term on the right-hand side of formula (4.3):

$$\begin{aligned} &\sum_{r=1}^{n} M_{jr}[\varphi](t,x)\nu_r(x)\nu_i(x) - \sum_{r=1}^{n} M_{ir}[\varphi](t,x)\nu_r(x)\nu_j(x) \\ &\quad = \sum_{r=1}^{n}\Big[\nu_j(x)\frac{\partial\tilde{\varphi}}{\partial x_r}(t,x)\nu_r(x)\nu_i(x) - \nu_r^2(x)\frac{\partial\tilde{\varphi}}{\partial x_j}(t,x)\nu_i(x) \\ &\qquad - \nu_i(x)\frac{\partial\tilde{\varphi}}{\partial x_r}(t,x)\nu_r(x)\nu_j(x) + \nu_r^2(x)\frac{\partial\tilde{\varphi}}{\partial x_i}(t,x)\nu_j(x)\Big] \\ &\quad = -M_{ij}[\varphi](t,x). \end{aligned} \tag{4.4}$$

We now consider the third term on the right-hand side of formula (4.3). First we observe that

$$\begin{aligned} \nu_i(x)M_{jr}[\varphi](\tau,y) - \nu_j(x)M_{ir}[\varphi](\tau,y) &= \big(\nu_i(x)M_{jr}[\varphi](\tau,y) - \nu_i(y)M_{jr}[\varphi](\tau,y)\big) \\ &\quad + \big(\nu_i(y)M_{jr}[\varphi](\tau,y) - \nu_j(y)M_{ir}[\varphi](\tau,y)\big) \\ &\quad + \big(\nu_j(y)M_{ir}[\varphi](\tau,y) - \nu_j(x)M_{ir}[\varphi](\tau,y)\big), \end{aligned}$$

for all $(\tau,y) \in \partial_T\Omega$. Moreover,

$$\begin{aligned} \nu_i(y)M_{jr}[\varphi](\tau,y) - \nu_j(y)M_{ir}[\varphi](\tau,y) &= \nu_i(y)\nu_j(y)\frac{\partial\tilde{\varphi}}{\partial y_r}(\tau,y) - \nu_i(y)\nu_r(y)\frac{\partial\tilde{\varphi}}{\partial y_j}(\tau,y) \\ &\quad - \nu_i(y)\nu_j(y)\frac{\partial\tilde{\varphi}}{\partial y_r}(\tau,y) + \nu_j(y)\nu_r(y)\frac{\partial\tilde{\varphi}}{\partial y_i}(\tau,y) \\ &= -\nu_r(y)M_{ij}[\varphi](\tau,y) \qquad \text{for all } (\tau,y) \in \partial_T\Omega. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{r=1}^{n}\Big\{\nu_i(x)\int_{-\infty}^{t}\int_{\partial\Omega}\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)M_{jr}[\varphi](\tau,y)\,d\sigma_y\,d\tau \\ &\qquad - \nu_j(x)\int_{-\infty}^{t}\int_{\partial\Omega}\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)M_{ir}[\varphi](\tau,y)\,d\sigma_y\,d\tau\Big\} \\ &\quad = \sum_{r=1}^{n}\{Q_r[\nu_i, M_{jr}[\varphi]](t,x) - Q_r[\nu_j, M_{ir}[\varphi]](t,x)\} + w[\partial_T\Omega, M_{ij}[\varphi]](t,x). \end{aligned} \tag{4.5}$$

The formulas (4.3), (4.4) and (4.5) imply

$$M_{ij}[w[\partial_T\Omega,\varphi]](t,x)=\sum_{r=1}^{n}\{Q_r[\nu_i,M_{jr}[\varphi]](t,x)-Q_r[\nu_j,M_{ir}[\varphi]](t,x)\} \\ +\nu_i(x)\tilde{Q}_t[\nu_j,\varphi](t,x)-\nu_j(x)\tilde{Q}_t[\nu_i,\varphi](t,x)+w[\partial_T\Omega,M_{ij}[\varphi]](t,x), \qquad (4.6)$$

and then (4.1) holds true for $\varphi\in C^{\frac{1+\beta}{2};1+\beta}(\partial_T\Omega)$.

Now let $\varphi\in C^{\frac{1}{2};1}(\partial_T\Omega)$. We consider only the case $T=+\infty$. Indeed, the case $T<+\infty$ can be treated similarly. We fix $t\in\,]-\infty,T[$, and we consider $\eta_1,\eta_2,\eta_3\in C_b^\infty(\mathbb{R})$ such that $\sum_{i=1}^3\eta_i=1$, $0\le\eta_i\le1$ for all $i=1,2,3$, $\eta_2(\tau)=1$ for all $\tau\in[t-1,t+1]$ and

$$\operatorname{supp}\eta_2\subseteq\,]t-2,t+2[,\quad \operatorname{supp}\eta_1\subseteq\,]-\infty,t-1[,\quad \operatorname{supp}\eta_3\subseteq\,]t+1,+\infty[.$$

Then we set $\varphi_i(\tau,x)=\varphi(\tau,x)\eta_i(\tau)$ for all $(\tau,x)\in\partial_T\Omega$ and for all $i=1,2,3$. Clearly,

$$\varphi(\tau,x)=\varphi_1(\tau,x)+\varphi_2(\tau,x)+\varphi_3(\tau,x)\quad\text{for all }(\tau,x)\in\partial_T\Omega.$$

We denote by $R_{ij}[\varphi]$ the right-hand side of (4.1). We note that by Theorem 3.3, Theorem 3.5 (ii) and by Theorem 4.1 (iii), $R_{ij}[\,\cdot\,]$ and $w[\partial_T\Omega,\cdot\,]$ are continuous from $C^{\frac{1}{2};1}(\partial_T\Omega)$ to $C_b^0(\partial_T\Omega)$. We now show that the weak M_{ij}-derivative of $w[\partial_T\Omega,\varphi_2]$ coincides with $R_{ij}[\varphi_2]$. Since φ_2 has compact support, by considering an extension of φ_2 of class $C^{\frac{1}{2};1}$ with compact support in $\mathbb{R}^{n+1}$ (see Ladyženskaja, Solonnikov and Ural'ceva [16, Chapter 1.1, pp. 9–10]), by considering a sequence of mollifiers of such an extension and by taking the restriction to $\partial_T\Omega$, we conclude that there exists a sequence $\{\varphi_{2l}\}_{l\in\mathbb{N}}$ in $C^{\frac{1+\alpha}{2},1+\alpha}(\partial_T\Omega)$ such that φ_{2l} converges to φ_2 in $C^{\frac{1}{2};1}(\partial_T\Omega)$. Moreover, we note that (4.1) holds for φ_{2l}. Then if $\mu\in C^1(\partial\Omega)$, we have

$$\begin{aligned}\int_{\partial\Omega}w[\partial_T\Omega,\varphi_2](t,y)M_{ij}[\mu](y)\,d\sigma_y&=\lim_{l\to\infty}\int_{\partial\Omega}w[\partial_T\Omega,\varphi_{2l}](t,y)M_{ij}[\mu](y)\,d\sigma_y\\&=-\lim_{l\to\infty}\int_{\partial\Omega}M_{ij}[w[\partial_T\Omega,\varphi_{2l}]](t,y)\mu(y)\,d\sigma_y\\&=-\lim_{l\to\infty}\int_{\partial\Omega}R_{ij}[\varphi_{2l}](t,y)\mu(y)\,d\sigma_y\\&=-\int_{\partial\Omega}R_{ij}[\varphi_2](t,y)\mu(y)\,d\sigma_y.\end{aligned}$$

Hence $R_{ij}[\varphi_2](t,\cdot)$ coincides with the weak M_{ij}-derivative of $w[\partial_T\Omega,\varphi_2](t,\cdot)$ for all $i,j\in\{1,\dots,n\}$ in $\partial\Omega$. Since both $R_{ij}[\varphi_2]$ and $w[\partial_T\Omega,\varphi_2]$ are continuous functions, it follows that $w[\partial_T\Omega,\varphi_2](t,\cdot)\in C^1(\partial\Omega)$ and that $M_{ij}[w[\partial_T\Omega,\varphi_2]](t,\cdot)=R_{ij}[\varphi_2](t,\cdot)$ classically in $\partial\Omega$.

Moreover, $M_{ij}[w[\partial_T\Omega,\varphi_1]](t,\cdot)=R_{ij}[\varphi_1](t,\cdot)$ in $\partial\Omega$. Indeed, $\varphi_1(\tau,\cdot)=0$ for all $\tau\in\,]t-1,+\infty[$. Thus the integral operators involved show no singularities, and then formulas (4.2)–(4.6) hold with φ replaced with φ_1 by classical differentiation theorems for integrals depending on a parameter.

Finally, since $\varphi_3(\tau,\cdot)=0$ for all $\tau\in\,]-\infty,t+1[$, the definition of $w[\partial_T\Omega,\cdot\,]$, Q_r and $\tilde{Q}_t$ implies

$$M_{ij}[w[\partial_T\Omega,\varphi_3]](t,\cdot)=R_{ij}[\varphi_3](t,\cdot)=0\quad\text{in }\partial\Omega.$$

Then we have $M_{ij}[w[\partial_T\Omega,\varphi]]=R_{ij}[\varphi]$ on $\partial_T\Omega$. □

Then we have the following:

Theorem 4.4. *Let $\alpha\in\,]0,1[$. Let $T\in\,]-\infty,+\infty]$, and let Ω be a bounded open subset of $\mathbb{R}^n$ of class C^2. Let $\varphi\in C^{1;2}(\partial_T\Omega)$. Then*

$$\frac{\partial}{\partial t}w[\partial_T\Omega,\varphi](t,x)=w[\partial_T\Omega,\partial_t\varphi](t,x)\quad\textit{for all }(t,x)\in\partial_T\Omega.$$

Proof. We consider $x \in \partial\Omega$ fixed, and we note that

$$w[\partial_T\Omega, \varphi](t,x) = \int_{-\infty}^{t} \int_{\partial\Omega} \frac{\partial}{\partial\nu(y)} \Phi_n(t-\tau, x-y)\varphi(\tau,y)\, d\sigma_y\, d\tau$$
$$= \int_{0}^{+\infty} \int_{\partial\Omega} \frac{\partial}{\partial\nu(y)} \Phi_n(\tau, x-y)\varphi(t-\tau,y)\, d\sigma_y\, d\tau \quad \text{for all } t \in \overline{]-\infty, T[}.$$

By the Hölder continuity of ν and by (2.3) there exists a constant $b_{\Omega,\alpha} > 0$ such that

$$\left|\frac{\partial}{\partial\nu(y)} \Phi_n(\tau, x-y)\partial_t\varphi(t-\tau,y)\right| \le \|\partial_t\varphi\|_{C_b^0(\partial_T\Omega)} b_{\Omega,\alpha}|x-y|^{1+\alpha}\tau^{-\frac{n}{2}-1}e^{-\frac{|x-y|^2}{4\tau}}$$

for all $t \in \overline{]-\infty, T[}$, $(\tau, y) \in]0, +\infty[\times \partial\Omega$ (see also [17, Lemma 5.1 (i)]). Moreover, there exists a constant $c'_{\Omega,n-1-\alpha} > 0$ such that

$$\int_{0}^{+\infty} \int_{\partial\Omega} |x-y|^{1+\alpha}\tau^{-\frac{n}{2}-1}e^{-\frac{|x-y|^2}{4\tau}}\, d\sigma_y\, d\tau = 4^{\frac{n}{2}} \int_{0}^{+\infty} u^{-\frac{n}{2}-1}e^{-\frac{1}{u}}\, du \int_{\partial\Omega} \frac{1}{|x-y|^{n-1-\alpha}}\, d\sigma_y \le 4^{\frac{n}{2}}\Gamma\Big(\frac{n}{2}\Big)c'_{\Omega,n-1-\alpha}$$

(cf., e.g., [17, Lemma 3.3 (i)]). Then the statement follows by classical differentiation theorems for integrals depending on a parameter. □

Now, invoking Theorems 3.10, 4.2, 4.3, 4.4, we are ready to prove the regularizing properties of the integral operator associated to the double layer heat potential.

Theorem 4.5. *Let $\alpha \in]0,1[$, $\beta \in]0,\alpha[$, $m \in \mathbb{N} \setminus \{0\}$, $T \in]-\infty, +\infty]$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Then the following statements hold.*
(i) *The operator from $C^{\frac{m}{2};m}(\partial_T\Omega)$ to $C^{\frac{m+\beta}{2};m+\beta}(\partial_T\Omega)$, which takes φ to $w[\partial_T\Omega, \varphi]$, is linear and continuous.*
(ii) *The operator from $C^{\frac{m+\beta}{2};m+\beta}(\partial_T\Omega)$ to $C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$, which takes φ to the function $w[\partial_T\Omega, \varphi]$, is linear and continuous.*

Proof. We first prove statement (i). We proceed by induction on m. The case $m = 1$ follows by the continuity of the pointwise product in Schauder spaces, by Remarks 2.6 and 2.7, by formula (4.1) of Theorem 4.3 for the tangential derivatives of the double layer potential and by Theorems 4.1 (iii), 4.2 (iv) and 3.10 (i), (ii).

We now prove that if statement (i) holds for all $m' \le m$ and $m \ge 1$, then it holds for $m+1$. It suffices to prove that the following three statements hold.
(j) $w[\partial_T\Omega, \cdot\,]$ is continuous from $C^{\frac{m+1}{2};m+1}(\partial_T\Omega)$ to $C_b^0(\partial_T\Omega)$.
(jj) $M_{ij}[w[\partial_T\Omega, \cdot\,]]$ is continuous from $C^{\frac{m+1}{2};m+1}(\partial_T\Omega)$ to $C^{\frac{m+\beta}{2};m+\beta}(\partial_T\Omega)$.
(jjj) $\frac{\partial}{\partial t}w[\partial_T\Omega, \cdot\,]$ is continuous from $C^{\frac{m+1}{2};m+1}(\partial_T\Omega)$ to $C^{\frac{m-1+\beta}{2};m-1+\beta}(\partial_T\Omega)$.
Statement (j) holds by the case $m = 1$ and by Remark 2.7. We now consider statement (jj). The continuity of the pointwise product in Schauder spaces, Remarks 2.6 and 2.7, Theorems 4.3, 3.10 (i), (ii) and the inductive assumption imply the validity of the statement (jj). Next we consider statement (jjj). Remarks 2.6 and 2.7, Theorem 4.4 and the inductive assumption imply the validity of statement (jjj). Accordingly, the proof of statement (i) is complete.

The proof of statement (ii) follows along the lines of the proof of statement (i) by replacing the use of Theorems 4.1 (iii), 4.2 (iv) and 3.10 (i), (ii) by that of Theorems 4.2 (ii), 4.2 (iii), 3.10 (iii), (iv), respectively. □

Finally, we consider another relevant integral operator associated to the normal derivative of the single layer heat potential, which is important in the analysis of boundary value problems for the heat equation. If $\varphi \in L^\infty(\partial_T\Omega)$, then we define

$$w_*[\partial_T\Omega, \varphi](t,x) \equiv \int_{-\infty}^{t} \int_{\partial\Omega} \frac{\partial}{\partial\nu(x)} \Phi_n(t-\tau, x-y)\varphi(\tau,y)\, d\sigma_y\, d\tau \quad \text{for all } (t,x) \in \partial_T\Omega.$$

Then we have the following.

Theorem 4.6. *Under the assumptions of Theorem 4.5, the following statements hold.*
(i) *The operator from* $C^{\frac{m-1}{2};m-1}(\partial_T\Omega)$ *to* $C^{\frac{m-1+\beta}{2};m-1+\beta}(\partial_T\Omega)$*, which takes* φ *to* $w_*[\partial_T\Omega,\varphi]$*, is linear and continuous.*
(ii) *The operator from* $C^{\frac{m-1+\beta}{2};m-1+\beta}(\partial_T\Omega)$ *to* $C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$*, which takes* φ *to* $w_*[\partial_T\Omega,\varphi]$*, is linear and continuous.*

Proof. First we note that

$$\begin{aligned} w_*[\partial_T\Omega,\varphi](t,x) &= \int_{-\infty}^{t}\int_{\partial\Omega} \frac{\partial}{\partial\nu(x)}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\,d\sigma_y\,d\tau \\ &= \sum_{i=1}^{n}\int_{-\infty}^{t}\int_{\partial\Omega} \nu_i(x)\frac{\partial}{\partial x_i}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\,d\sigma_y\,d\tau \\ &= \sum_{i=1}^{n}\int_{-\infty}^{t}\int_{\partial\Omega} (\nu_i(x)-\nu_i(y))\frac{\partial}{\partial x_i}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\,d\sigma_y\,d\tau \\ &\qquad - \sum_{i=1}^{n}\int_{-\infty}^{t}\int_{\partial\Omega} \nu_i(y)\frac{\partial}{\partial y_i}\Phi_n(t-\tau,x-y)\varphi(\tau,y)\,d\sigma_y\,d\tau \\ &= \sum_{i=1}^{n} Q_i[\nu_i,\varphi](t,x) - w[\partial_T\Omega,\varphi](t,x) \quad \text{for all } (t,x)\in\partial_T\Omega. \end{aligned} \tag{4.7}$$

Now we consider statement (i). Formula (4.7) and Theorems 3.10 (i), 4.1 (iii), 4.5 (i) imply the validity of statement (i).

The proof of statement (ii) follows along the lines of the proof of statement (i) by replacing the use of Theorems 3.10 (i), 4.1 (ii), 4.5 (i) by that of Theorems 3.10 (iii), 4.2 (ii), 4.5 (ii) □

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