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A BONNET-MYERS TYPE THEOREM FOR QUATERNIONIC CONTACT STRUCTURES

DAVIDE BARILARI[♭] AND STEFAN IVANOV[♯]

Abstract. We prove a Bonnet-Myers type theorem for quaternionic contact manifolds of dimension bigger than 7. If the manifold is complete with respect to the natural sub-Riemannian distance and satisfies a natural Ricci-type bound expressed in terms of derivatives up to the third order of the fundamental tensors, then the manifold is compact and we give a sharp bound on its sub-Riemannian diameter.

Contents

1. Introduction and main results

Bonnet-Myers theorem is classical among comparison theorems in Riemannian geometry [37]. It states that, if the Ricci curvature of a complete d-dimensional Riemannian manifold (M, g) is bounded below by $(d-1)\kappa > 0$, then the manifold M is compact and its diameter is at most $\pi/\sqrt{\kappa}$.

Several generalizations of this theorem, in variuos smooth settings (and even in the non-smooth one of metric measure spaces, see for instance [38]) have been recently investigated, introducing suitable notion of curvature or Ricci bound. Among these, different versions of Bonnet-Myers theorem have been obtained in the setting of sub-Riemannian geometry (cf. discussion in Section 1.4).

Recall that a sub-Riemannian structure $(\mathcal{D}, g)$ on a smooth, connected manifold M of dimension $d \geq 3$ is defined by a vector distribution $\mathcal{D}$ of constant rank $k \leq d$ and a smooth metric g assigned on $\mathcal{D}$. The distribution is required to satisfy the Hörmander condition, or to be bracket-generating, that means

$$\text{span}\{[X_{j_1}, [X_{j_2}, [\dots, [X_{j_{m-1}}, X_{j_m}]]]](x) \mid m \geq 1\} = T_xM, \qquad \forall x \in M, \tag{1}$$

for some (and then any) set $X_1, \dots, X_k \in \Gamma(\mathcal{D})$ of local generators for $\mathcal{D}$.

Given a sub-Riemannian structure on M, the *sub-Riemannian distance* is defined by:

$$d_{SR}(x, y) = \inf\{\ell(\gamma) \mid \gamma(0) = x,\, \gamma(T) = y,\, \gamma \text{ horizontal}\}.$$

[♭] Institut de Mathématiques de Jussieu-Paris Rive Gauche, UMR CNRS 7586, Université Paris-Diderot, Batiment Sophie Germain, Case 7012, 75205 Paris Cedex 13, France

[♯] University of Sofia, Faculty of Mathematics and Informatics, blvd. James Bourchier 5, 1164, Sofia, Bulgaria, & Institute of Mathematics and Informatics, Bulgarian Academy of Sciences

E-mail addresses: davide.barilari@imj-prg.fr , ivanovsp@fmi.uni-sofia.bg .

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where a Lipschitz continuous path $\gamma : [0, T] \to \mathbb{R}$ is *horizontal* if it satisfies $\dot\gamma(t) \in \mathcal{D}_{\gamma(t)}$ for almost every t, and in this case we set

$$\ell(\gamma) = \int_0^T \sqrt{g(\dot\gamma(t), \dot\gamma(t))}dt.$$

By the classical Chow-Rashevskii theorem (see for instance [6, Chapter 3]), the condition (1) implies that d_{SR} is finite and continuous on $M \times M$. We say that the sub-Riemannian manifold is complete if (M, d_{SR}) is complete as a metric space.

A sub-Riemannian Bonnet-Myers theorem states, under suitable curvature conditions, that the manifold M is compact and gives a bound on its sub-Riemannian diameter. For more details on sub-Riemannian geometry we refer to classical references such as [16, 36] and the more recent ones [6, 35, 39].

Remark 1. Notice that if the sub-Riemannian structure is defined as the restriction of a Riemannian metric g on M to a distribution $\mathcal{D}$, in general the sub-Riemannian diameter is bigger than the Riemannian one. Thus, even if one is able to control the Riemannian curvature of (M, g) and apply a classical Bonnet-Myers theorem, one can prove compactness of M, but has no a priori estimate on the sub-Riemannian diameter.

In this paper we focus on quaternionic contact structure. A quaternionic contact (qc) structure, introduced in [17], appears naturally as the conformal boundary at infinity of the quaternionic hyperbolic space. The qc structure gives a natural geometric setting for the quaternionic contact Yamabe problem, [23, 43, 31, 29]. A particular case of this problem amounts to find the extremals and the best constant in the L^2 Folland-Stein Sobolev-type embedding, [21] and [22], with a complete description of the extremals and the best constant on the quaternionic Heisenberg groups [29, 30, 27].

A quaternionic contact structure carry a natural sub-Riemannian structure with a codimension three distribution. Curvature conditions are expressed in terms on bounds on standard curvature tensors of quaternionic contact geometry. These conditions can be expressed only in terms of sub-Riemannian quantities (cf. Theorems 1 and 2) and are obtained through the computation of the sub-Riemannian coefficients of the generalized Jacobi equation, first introduced in [9, 44] and subsequently developed in [10, 11, 2].

1.1. **Quaternionic contact structure.**

A quaternionic contact manifold $(M, \mathbb{Q}, g)$ is a $(4n + 3)$-dimensional manifold M with a codimension-three distribution $\mathcal{D}$ equiped with $\mathrm{Sp}(n)\mathrm{Sp}(1)$ structure. Explicitly, the distribution $\mathcal{D}$ is locally described as the kernel of a 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in $\mathbb{R}^3$ together with a compatible Riemannian metric g and a rank-three bundle $\mathbb{Q}$ consisting of endomorphisms of $\mathcal{D}$ locally generated by three almost complex structures $I_1, I_2, I_3 : \mathcal{D} \to \mathcal{D}$ satisfying the identities of the imaginary unit quaternions. Namely, if $\{\alpha, \beta, \tau\}$ is any cyclic permutation of $\{1, 2, 3\}$ we have

$$I_\alpha I_\beta = -I_\beta I_\alpha = I_\tau, \quad I_\alpha^2 = I_\beta^2 = I_\tau^2 = I_\alpha I_\beta I_\tau = -\mathrm{id}_{|\mathcal{D}}. \tag{2}$$

Moreover I_1, I_2, I_3 are compatible with the metric g, in the following sense: for every $\alpha = 1, 2, 3$ and $X, Y \in \mathcal{D}$ we have

$$g(I_\alpha X, I_\alpha Y) = g(X, Y), \qquad 2g(I_\alpha X, Y) = d\eta_\alpha(X, Y).$$

From the sub-Riemannian view-point, these structures are *fat*, i.e. for any non zero section X of $\mathcal{D}$, TM is (locally) generated by $\mathcal{D}$ and $[X, \mathcal{D}]$. This is a direct consequence of the quaternionic relations of the almost complex structures. For completeness a proof is given in Section 2. The fat condition is open in the C^1 topology, however it gives some restriction on the rank k of the distribution (for example $\dim M \le 2k - 1$, [36, Prop. 5.6.3]).

Example 1 (Quaternionic Hopf fibration). A classical example of quaternionic contact structure is the quaternionic Hopf fibration

$$\mathbb{S}^3 \hookrightarrow \mathbb{S}^{4n+3} \xrightarrow{\pi} \mathbb{HP}^n, \qquad n \ge 1. \tag{3}$$

Here $\mathcal{D} = (\ker \pi_*)^\perp$ is the orthogonal complement of the kernel of the differential of the Hopf map π, and the sub-Riemannian metric is the restriction to $\mathcal{D}$ of the standard round metric on $\mathbb{S}^{4n+3}$. The sub-Riemannian distance on the quaternionic Hopf fibration can be computed explicitly and its diameter is π, as it is proved in [15]. This example is one of the simplest (non-Carnot) sub-Riemannian structures of corank greater than 1, and is included in the sub-class of 3-*Sasakian* structures.

Example 2 (Quaternionic Heisenberg group). An example of quaternionic contact structure that is not 3-sasakian is the quaternionic Heisenberg group. It is defined as

$$\mathbb{R}^{4n+3} = \mathbb{H}^n \oplus \mathrm{Im}(\mathbb{H})$$

endowed with the group law

$$(z, w) \cdot (z', w') = \left(z + z', w + w' + \frac{1}{2}\mathrm{Im}(z\bar{z}') \right).$$

If we take $\mathcal{D} = \mathbb{H}^n$ (which has dimension $4n$) with the standard Euclidean metric, it easy to see that it is bracket generating and defines a quaternionic contact structure.

1.2. **Biquard connection, torsion and curvature.** On a qc manifold of dimension $4n+3$ with $n \geq 2$ with a fixed metric g on the *horizontal distribution* $\mathcal{D}$ there exists a canonical connection, called *Biquard connection*, defined in [17]. Biquard shows that there exists a unique supplementary subspace V to $\mathcal{D}$ in TM and a unique connection ∇ with torsion T, such that:

(i) ∇ preserves the decomposition $H \oplus V$ and the $\mathrm{Sp}(n)\mathrm{Sp}(1)$ structure on $\mathcal{D}$, $\nabla g = 0$, $\nabla \sigma \in \Gamma(\mathbb{Q})$ for $\sigma \in \Gamma(\mathbb{Q})$, and its torsion on $\mathcal{D}$ is given by $T(X, Y) = -[X, Y]_{|V}$;

(ii) for $\xi \in V$, the endomorphism $T(\xi, .)_{|\mathcal{D}}$ of $\mathcal{D}$ lies in[1] $(\mathrm{sp}(n) \oplus \mathrm{sp}(1))^\perp \subset \mathrm{gl}(4n)$;

(iii) the connection on V is induced by the natural identification φ of V with the subspace $\mathrm{sp}(1)$ of the endomorphisms of $\mathcal{D}$, i.e., $\nabla\varphi = 0$.

When the dimension of M is at least eleven, [17] shows that the supplementary *vertical distribution* V is (locally) generated by three *Reeb vector fields* ξ_1, ξ_2, ξ_3 determined by the conditions

$$\eta_\alpha(\xi_\beta) = \delta_{\alpha\beta}, \qquad (\xi_\alpha \lrcorner d\eta_\alpha)_{|\mathcal{D}} = 0, \qquad (\xi_\alpha \lrcorner d\eta_\beta)_{|\mathcal{D}} = -(\xi_\beta \lrcorner d\eta_\alpha)_{|\mathcal{D}}, \tag{4}$$

where $\lrcorner$ denotes the interior multiplication: more explicitly $X \lrcorner \Phi = \Phi(X, \cdot)$ where X is a vector field and Φ is a differential 2-form.

Remark 2. In this paper we restrict our attention to quaternionic contact structure of dimension strictly bigger that seven. If the dimension of M is seven Duchemin shows in [20] that if we assume, in addition, the existence of Reeb vector fields as in (4), then the Biquard result holds. Henceforth, by a qc structure in dimension 7 we shall mean a qc structure satisfying (4). This implies the existence of the connection with properties (i), (ii) and (iii) above.

The fundamental 2-forms ω_α of the qc structure are defined by

$$2\omega_{\alpha|\mathcal{D}} = d\eta_{\alpha|\mathcal{D}}, \quad \xi \lrcorner \omega_\alpha = 0, \quad \xi \in V.$$

The torsion restricted to $\mathcal{D}$ has the form

$$T(X, Y) = -[X, Y]_{|V} = 2\sum_{\alpha=1}^{3} \omega_\alpha(X, Y)\xi_\alpha.$$

The properties of the Biquard connection are encoded in the torsion endomorphism $T(\xi, .)_{|\mathcal{D}}$. It is completely trace-free, $\mathrm{tr}(T(\xi, .)_{|\mathcal{D}}) = \mathrm{tr}T(\xi, .)_{|\mathcal{D}} \circ I_\alpha = 0$ and can be decomposed into symmetric and skew-symmetric parts, $T(\xi_\alpha, .)_{|\mathcal{D}} = T^0(\xi_\alpha, .)_{|\mathcal{D}} + I_\alpha u$, respectively

[1] the perpendicular is computed with respect to the inner product $\langle A \,|\, B \rangle = \sum_{i=1}^{4n} g(A(e_i), B(e_i))$, for $A, B \in \mathrm{End}(H)$.

where u is a traceless symmetric (1,1)-tensor on $\mathcal{D}$ which commutes with I_1, I_2, I_3, see [17]. When $n = 1$ the tensor u vanishes identically and the torsion is a symmetric tensor, $T_\xi = T^0_\xi$.

The two Sp(n)Sp(1)-invariant trace-free symmetric 2-tensors T^0, U on $\mathcal{D}$ defined in (33) having the properties (34) determine completely the symmetric and the skew-symmetric parts of torsion endomorphism, respectively [31] (cf. (35) and (36) in the Appendix.)

The *qc-Ricci tensor* Ric and the *normalized qc-scalar curvature* S of the Biquard connection are defined with the usual horizontal traces of the curvature of the Biquard connection (cf. (37) in the Appendix.)

A qc structure is said to be *qc-Einstein* if the horizontal qc-Ricci tensor is a scalar multiple of the metric. As shown in [31, 28] the qc-Einstein condition is equivalent to the vanishing of the torsion endomorphism of the Biquard connection. In this case S is constant and the vertical distribution is integrable. It is also worth recalling that the horizontal qc-Ricci tensors and the integrability of the vertical distribution can be expressed in terms of the torsion of the Biquard connection according to (38) in the Appendix (see [31], cf. also [29, 33, 32]) .

Any 3-Sasakian manifold has zero torsion endomorphism, and the converse is locally true if in addition the qc scalar curvature is a positive constant [31].

1.3. **Main results.** Our main result reads as follows, in terms of the qc Ricci tensor and the curvature tensor associated with the Biquard connection.

Theorem 1. *Let $(M, g, \mathbb{Q})$ be a $4n+3$-dimensional complete qc manifold with $n > 1$. Assume that there exists a constant $\kappa > 0$ such that*

$$\mathrm{Ric}(X, X) - \sum_{\alpha=1}^{3} R(X, I_\alpha X, I_\alpha X, X) \geq 4(n-1)\kappa, \qquad \forall\, X \in \mathcal{D}. \tag{5}$$

Then $(M, g, \mathbb{Q})$ is compact manifold with finite fundamental group, and its sub-Riemannian diameter is not greater than $\pi/\sqrt{\kappa}$.

Remark 3. The bound on the sub-Riemannian diameter given in Theorem 1 is sharp since the equality is attained for the quaternionic Hopf fibration, where κ can be chosen equal to 1 and the sub-Riemannian diameter is π (cf. Example 1 and Remark 5). Moreover, since $R(X, X, X, X) = 0$, the left hand side in (5) is indeed a trace on a $4(n-1)$-dimensional subspace of $\mathcal{D}$.

Theorem 1 can be also restated as follows, in terms of the horizontal part of torsion tensors and scalar curvature.

Theorem 2. *Let $(M, g, \mathbb{Q})$ be a $4n+3$-dimensional complete qc manifold with $n > 1$. Assume that there exists a constant $\kappa > 0$ such that*

$$2nT^0(X, X) + (4n-8)U(X, X) + 2(n-1)S \geq 4(n-1)\kappa, \qquad \forall\, X \in \mathcal{D}. \tag{6}$$

Then $(M, g, \mathbb{Q})$ is compact manifold with finite fundamental group, and its sub-Riemannian diameter not greater than $\pi/\sqrt{\kappa}$.

The proofs of Theorems 1 and 2 are given in Section 5. We stress that, thanks to the results in [31] and the proof of [34, Theorem 4.2.5], the condition (6) can be rewritten only in terms of the qc structure and its Lie derivatives.

Proposition 3. *Let $\{X_1, \ldots, X_{4n}\}$ be a local orthonormal basis for $\mathcal{D}$ and $\{\alpha, \beta, \tau\}$ be any cyclic permutation of $\{1, 2, 3\}$. We have the following relations:*

(i) *The symmetric part of the torsion endomorphism is determined entirely by the Lie derivative of the metric*

$$T^0(\xi, X, Y) = \frac{1}{2}\mathcal{L}_\xi g(X, Y), \qquad T^0(X, Y) = \frac{1}{2}\sum_{\alpha=1}^{3}(\mathcal{L}_{\xi_\alpha} g)(I_\alpha X, Y).$$

(ii) *The skew-symmetric part of the torsion described by U satisfies*

$$\begin{aligned} U(X,Y) =& \frac{1}{4} g((\mathcal{L}_{\xi_\beta} I_\alpha)X, I_\tau Y) - \frac{1}{4} g((\mathcal{L}_{\xi_\beta} I_\alpha) I_\tau X, Y) \\ &+ \frac{1}{2n} \sum_{i=1}^{4n} g((\mathcal{L}_{\xi_\beta} I_\alpha) I_\tau X_i, X_i) g(X,Y), \end{aligned}$$

(iii) *The normalized qc scalar curvature is written as*

$$S = d\eta_\alpha(\xi_\beta, \xi_\tau) - d\eta_\tau(\xi_\alpha, \xi_\beta) - d\eta_\beta(\xi_\tau, \xi_\alpha) - \frac{1}{2n} \sum_{i=1}^{4n} g((\mathcal{L}_{\xi_\beta} I_\alpha) I_\tau X_i, X_i).$$

Remark 4. We note that Theorems 1 and 2 generalize Bonnet-Myers results for the sub-class of qc manifold with integrable vertical space obtained in [24] simplifying considerably the Bonnet-Myers positivity condition and giving moreover explicit diameter bounds.

Remark 5 (3-Sasakian case). Assume that the qc manifold is 3-Sasakian. In this case, we have from [31, Corollary 4.13] and [31, Theorem 3.12] that

$$T^0 = U = 0, \qquad S = 2.$$

Therefore, (6) is satisfied with $\kappa = 1$ and we recover the universal diameter bound for 3-Sasakian manifold, established in [40]. The diameter bound is attained for the quaternionic Hopf fibration (cf. Example 1).

Notice that in [40] the authors use curvature tensors R^g associated with the Levi-Civita connection. Using the relation between R^g and the curvature tensor R associated with the Biquard connection (see [31, Corollary 4.13]) we have

$$\sum_{\alpha=1}^{3} R(X, I_\alpha X, I_\alpha X, X) = \sum_{\alpha=1}^{3} R^g(X, I_\alpha X, I_\alpha X, X) + 9 = 12,$$

where we apply the identity $\sum_{\alpha=1}^{3} R^g(X, I_\alpha X, I_\alpha X, X) = 3$, valid for 3-Sasakian manifolds, and proved in [42, Prop. 3.2].

We also state the following interesting corollary of Theorem 2, when $n = 2$.

Corollary 4. *Let $(M, g, \mathbb{Q})$ be a 11-dimensional complete qc manifold. Assume $T^0 = 0$ and $S \geq 2\kappa > 0$, then M is compact with sub-Riemannian diameter not greater than $\pi/\sqrt{\kappa}$.*

We note that qc manifolds with $T^0 = 0$ are characterized with the condition that the almost contact structure on the corresponding twistor space is normal, see [19].

1.4. **Relation with previous literature.** Other sub-Riemannian Bonnet-Myers type results are found in the literature, proved with different techniques and for different sub-Riemannian structures. The three dimensional contact case has been considered using second variation like formulas in [41] (for CR structures) and in [25]. In [12] and [13], a version of Bonnet-Myers has been proved using heat semigroup approach for Yang-Mills type structures with transverse symmetries and Riemannian foliations with totally geodesic leaves, respectively. Using Riccati comparison techniques for 3D contact [7], 3-Sasakian [40] and general contact sub-Riemannian structure [1]. See also [10] for a general approach to sub-Riemannian Bonnet-Myers theorem through curvature invariants.

A compactness result, obtained by applying Riemannian classical Bonnet-Myers theorem to a suitable Riemannian extension of the metric (cf. Remark 1) is obtained in [14] for contact manifolds and in [24] for quaternionic contact manifolds with integrable vertical space.

1.5. **Structure of the paper.** In Section 2 we recall some results about the sub-Riemannian Jacobi equation and the curvature invariants. In Sections 3 and 4 we carefully compute these invariants for quaternionic contact structures and express them with respect to standard tensors of quaternionic contact geometry. To perform these computations, we introduce a generalized Fermi frame along the geodesic. In Section 5 we use these computations to prove the Bonnet-Myers theorem. Appendix A resumes geometric properties of quaternionic contact structures.

2. Curvature of sub-Riemannian qc structures

In this section we resume the basic facts on sub-Riemannian geodesic flows and curvature needed to prove our results. For a more comprehensive presentation we refer the reader to [2, 10, 11].

2.1. **Quaternionic contact sub-Riemannian structures are fat.** A sub-Riemannian structure is said to be *fat* if for any non zero section X of $\mathcal{D}$, TM is (locally) generated by $\mathcal{D}$ and $[X,\mathcal{D}]$. This is equivalent to show that for every non zero horizontal vector $X \in \mathcal{D}$ the following map is surjective

$$\mathcal{L}_X : \mathcal{D} \to TM/\mathcal{D}, \qquad \mathcal{L}_X(Y) := [X,Y] \mod \mathcal{D}$$

Notice that the map $\mathcal{L}_X$ is tensorial, in the sense that for each $x \in M$ the value of $[X,Y](x)$ mod $\mathcal{D}_x$ depends only on $X(x)$ and $Y(x)$. Moreover $\dim T_xM/\mathcal{D}_x = 3$. The fat property follows from the following linear algebra observation.

Lemma 5. *The vectors* $\{\mathcal{L}_X(I_\alpha X)\}_{\alpha=1,2,3}$ *are linearly independent in* $T_xM/\mathcal{D}_x$.

Proof. Let us start by showing that, for every non zero horizontal vector $X \in \mathcal{D}$ the four vectors X, I_1X, I_2X, I_3X are mutually orthogonal. First notice that for every $\tau = 1,2,3$ and every horizontal vector $X \in \mathcal{D}$ one has

$$g(I_\tau X, X) = d\eta(X,X) = 0. \tag{7}$$

Moreover, if $\{\alpha\beta\tau\}$ is a cyclic permutation of $\{123\}$, thanks to (2) and (7), one has for $\alpha \neq \beta$

$$g(I_\alpha X, I_\beta X) = -g(X, I_\alpha I_\beta X) = -g(X, I_\tau X) = 0.$$

To prove that the sub-Riemannian structure is fat it is sufficient to show that the image through L_X of the vectors $I_\beta X$ for $\beta = 1,2,3$ is a basis of $T_xM/\mathcal{D}_x$, for every $x \in M$. This is equivalent to say that for every X the matrix $\Omega_{\alpha\beta} = \eta_\alpha(L_X(I_\beta X))$ is invertible, which follows from

$$\Omega_{\alpha\beta} = \eta_\alpha(\mathcal{L}_X(I_\beta X)) = -d\eta_\alpha(X, I_\beta X) = -2g(I_\alpha X, I_\beta X) = -2\delta_{\alpha\beta}g(X,X). \qquad \square$$

2.2. **Sub-Riemannian geodesic flow.** Sub-Riemannian *geodesics* are horizontal curves that are locally minimizers for the length (between curve with same endpoints). The *sub-Riemannian Hamiltonian* $H : T^*M \to \mathbb{R}$ is defined as

$$H(\lambda) := \frac{1}{2}\sum_{i=1}^{k} \langle \lambda, X_i\rangle^2, \qquad \lambda \in T^*M,$$

where $X_1, \dots, X_k$ is any local orthonormal frame for $\mathcal{D}$ and $\langle \lambda, v\rangle$ denotes the action of a covector $\lambda \in T_x^*M$ on a vector $v \in T_xM$, based at $x \in M$. Let σ be the canonical symplectic form on T^*M. The *Hamiltonian vector field* $\vec{H}$ is defined by the identity $\sigma(\cdot, \vec{H}) = dH$. Then the Hamilton equations are

$$\dot{\lambda}(t) = \vec{H}(\lambda(t)). \tag{8}$$

Solutions of (8) are called *extremals*, and one can prove that their projections $\gamma(t) := \pi(\lambda(t))$ on M are geodesics [6, Chapter 4]. The Hamiltonian H is constant along an extremal $\lambda(t)$ and we say that the extremal is *length-parametrized* if $H(\lambda(t)) = 1/2$.

Since the sub-Riemannian structure defined on the qc manifold is fat, any minimizer can be recovered uniquely in this way. This statement is not true in full generality, since there can exist minimizing trajectories might not satisfy the Hamiltonian equation (8). These trajectories are called *abnormal minimizers* and are related to the main open problems in sub-Riemannian geometry (see for instance [5] for a discussion).

2.3. **Jacobi equation revisited.** Given an extremal $\lambda(t)$ of the sub-Riemannian Hamiltonian flow and a vector field $V(t)$ along $\lambda(t)$ we define

$$\dot V(t) := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} e_*^{-\varepsilon\vec H} V(t+\varepsilon).$$

the Lie derivative of V in the direction of $\vec H$. A vector field $\mathcal{J}(t)$ along $\lambda(t)$ is a *sub-Riemannian Jacobi field* if

$$\dot{\mathcal{J}} = 0. \tag{9}$$

If M has dimension d, the set of solutions of (9) is a $2d$-dimensional vector space. The projections $\pi_*\mathcal{J}(t)$ are vector fields on the manifold M corresponding to one-parameter variations of $\gamma(t) = \pi(\lambda(t))$ through geodesics. In the Riemannian case, this coincides with the classical construction of Jacobi fields.

Next, let us write (9) using the symplectic structure σ of T^*M. Observe that on T^*M there is a natural notion of *vertical subspace* at $\lambda \in T^*M$, namely

$$\mathcal{V}_\lambda := \ker \pi_*|_\lambda = T_\lambda(T^*_{\pi(\lambda)}M) \subset T_\lambda(T^*M).$$

Then $\mathcal{V}$ is a smooth (Lagrangian) sub-bundle of $T(T^*M)$. If one considers the frame $E_i = \partial_{p_i}|_{\lambda(t)}$, and $F_j = \partial_{x_j}|_{\lambda(t)}$ induced by coordinates $(x_1, \dots, x_d)$ on M, then the vector field $\mathcal{J}(t)$ has components $(p(t), x(t)) \in \mathbb{R}^{2d}$, that means

$$\mathcal{J}(t) = \sum_{i=1}^d p_i(t)E_i(t) + x_i(t)F_i(t).$$

and the elements of the frame satisfy the equation

$$\frac{d}{dt}\begin{pmatrix} E \\ F \end{pmatrix} = \begin{pmatrix} \mathcal{A}(t) & -\mathcal{B}(t) \\ \mathcal{R}(t) & -\mathcal{A}(t)^* \end{pmatrix}\begin{pmatrix} E \\ F \end{pmatrix}, \tag{10}$$

for some smooth families of $d \times d$ matrices $\mathcal{A}(t), \mathcal{B}(t), \mathcal{R}(t)$, where $\mathcal{B}(t) = \mathcal{B}(t)^*$ and $\mathcal{R}(t) = \mathcal{R}(t)^*$. The structure of (10) follows from the fact that the frame is Darboux, namely

$$\sigma(E_i, E_j) = \sigma(F_i, F_j) = \sigma(E_i, F_j) - \delta_{ij} = 0, \qquad i, j = 1, \dots, d.$$

The idea is then to look for a suitable Darboux frame $\{E_i(t), F_i(t)\}_{i=1}^d$ along $\lambda(t)$ such that the equations above are in normal form.

2.4. **Curvature coefficients in quaternionic contact.** The normal form of the sub-Riemannian Jacobi equation (10) has been first studied by Agrachev-Zelenko in [3, 4] and subsequently completed by Zelenko-Li in [44]. In particular, there exist a normal form of (10) where the matrices $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are constant. Here we give an ad-hoc statement for quaternionic contact sub-Riemannian structures, following the notation and the presentation of [11].

Remark 6. It is convenient to split the set of indices $1, \dots, 4n+3$ into three subsets a, b, c with cardinality $|a| = |b| = 3$ and $|c| = 4n - 3$. The index a parametrizes the three-dimensional complement to the distribution, while b and c together parametrize the set of indices on the distribution.

This splitting is related to the fact that the Lie derivative $\mathcal{L}_X : \mathcal{D}_x \to T_xM/\mathcal{D}_x$ in the direction of a nontrivial horizontal vector $X \in \mathcal{D}_q$ induces a well defined, surjective linear map with 3-dimensional image (the "a" space) and a $4n-3$-dimensional kernel (the "c"

space). The orthogonal complement of the kernel within $\mathcal{D}_x$ is a 3-dimensional space (the "b" space).

Accordingly to this decomposition, any $(4n+3)\times(4n+3)$ matrix L is written in the block form

$$L = \begin{pmatrix} L_{aa} & L_{ab} & L_{ac} \\ L_{ba} & L_{bb} & L_{bc} \\ L_{ca} & L_{cb} & L_{cc} \end{pmatrix},$$

with similar notation for row or column vectors.

Theorem 6. *Let $\lambda(t)$ be a sub-Riemannian extremal of a qc sub-Riemannian structure. There exists a smooth moving frame along $\lambda(t)$*

$$E(t) = (E_a(t), E_b(t), E_c(t))^*, \qquad F(t) = (F_a(t), F_b(t), F_c(t))^*,$$

such that the following holds true for any t:

(i) $\mathrm{span}\{E_a(t), E_b(t), E_c(t)\} = \mathcal{V}_{\lambda(t)}$.

(ii) *It is a Darboux basis, namely*

$$\sigma(E_\mu, E_\nu) = \sigma(F_\mu, F_\nu) = \sigma(E_\mu, F_\nu) - \delta_{\mu\nu} = 0, \qquad \mu,\nu = a,b,c.$$

(iii) *The frame satisfies the* structural equations

$$\dot{E}_a = E_b, \qquad \dot{E}_b = -F_b, \qquad \dot{E}_c = -F_c,$$

$$\dot{F}_a = \sum_{\mu=a,b,c} \mathcal{R}_{a\mu}(t)E_\mu, \qquad \dot{F}_b = \sum_{\mu=a,b,c} \mathcal{R}_{b\mu}(t)E_\mu - F_a \qquad \dot{F}_c, = \sum_{\mu=a,b,c} \mathcal{R}_{c\mu}(t)E_\mu.$$

where the curvature matrix $\mathcal{R}(t) = \mathcal{R}(t)^*$ is

$$\mathcal{R}(t) = \begin{pmatrix} \mathcal{R}_{aa}(t) & \mathcal{R}_{ab}(t) & \mathcal{R}_{ac}(t) \\ \mathcal{R}_{ba}(t) & \mathcal{R}_{bb}(t) & \mathcal{R}_{bc}(t) \\ \mathcal{R}_{ca}(t) & \mathcal{R}_{cb}(t) & \mathcal{R}_{cc}(t) \end{pmatrix},$$

and satisfies the additional condition $\mathcal{R}_{ab}(t) = -\mathcal{R}_{ab}(t)^*$.

Remark 7. If we fix another frame $\{\widetilde{E}(t), \widetilde{F}(t)\}$ satisfying (i)-(iii) for some matrix $\widetilde{\mathcal{R}}(t)$, then there exists a constant $n\times n$ orthogonal matrix O that preserves the structural equations and such that

$$\widetilde{E}(t) = OE(t), \qquad \widetilde{F}(t) = OF(t), \qquad \widetilde{\mathcal{R}}(t) = O\mathcal{R}(t)O^*.$$

For more details about the uniqueness of this frame we refer the reader to [11].

2.5. **Ricci curvature and Bonnet-Myers theorem.** We can state now a corollary of the general results obtained in [10] (an analogue statement of the one mentioned here is [40, Thm. 5]) that gives a Bonnet-Myers type theorem that we will use to prove our results.

Theorem 7. *Let $(M, \mathcal{D}, g)$ be a complete qc sub-Riemannian manifold. Assume that there exists $\kappa > 0$ such that for any length-parametrized extremal $\lambda(t)$ one has*

$$\mathrm{tr}(\mathcal{R}_{cc}(t)) \geq 4(n-1)\kappa,$$

Then M is compact and its sub-Riemannian diameter is bounded by $\pi/\sqrt{\kappa}$. Moreover M has finite fundamental group.

In the following sections we will compute the quantity $\mathrm{tr}(\mathcal{R}_{cc}(t))$ for every sub-Riemannian extremal on a qc manifold and deduce the main theorems stated in the Introduction.

3. Structural equations for the coordinate frame

In what follows latin indices $i, j, k, \dots$ belong to $\{1, \dots, 4n\}$ and Greek ones $\alpha, \beta, \tau, \dots$ belong to $\{1, 2, 3\}$, corresponding to quaternions (following the same quaternionic indices notation of Appendix A).

We start by choosing a convenient local frame on M, associated with a given trajectory. Here $\{X_1, \dots, X_{4n}\}$ will denote a local orthonormal frame for the metric g on $\mathcal{D}$.

3.1. **Fermi frame.** Given a geodesic $\gamma(t) = \pi(\lambda(t))$, we define a convenient local frame on M which is an application of a standard result in differential geometry, called Fermi normal frame along a smooth curve.

Lemma 8. *Given a geodesic $\gamma(t)$, there exists a $\mathbb{Q}$-orthonormal frame, i.e., a horizontal frame X_i, $i \in \{1, \dots, 4n\}$, and vertical frame ξ_α, $\alpha = 1, 2, 3$ in a neighborhood of $\gamma(0)$, such that for all $\alpha, \beta \in \{1, 2, 3\}$ and $i, j \in \{1, \dots, 4n\}$,*

(i) *the frame is orthonormal for the Riemannian metric $g + \sum_\beta \eta_\beta^2$,*
(ii) $\nabla_{X_i} X_j|_{\gamma(t)} = \nabla_{\xi_\alpha} X_j|_{\gamma(t)} = \nabla_{X_i} \xi_\beta|_{\gamma(t)} = \nabla_{\xi_\alpha} \xi_\beta|_{\gamma(t)} = 0.$

In particular, for all $\alpha, \beta, \tau \in \{1, 2, 3\}$ and $i, j \in \{1, \dots, 4n\}$

$$((\nabla_{X_i} I_\alpha) X_j)|_{\gamma(t)} = ((\nabla_{X_i} I_\alpha) \xi_\beta)|_{\gamma(t)} = ((\nabla_{\xi_\beta} I_\alpha) X_j)|_{\gamma(t)} = ((\nabla_{\xi_\beta} I_\alpha) \xi_\tau)|_{\gamma(t)} = 0.$$

The proof of this Lemma is postponed to Appendix A.5.

3.2. **Commutator relations and Poisson brackets.** Fix $\{X_1, X_2, \dots, X_{4n}\}$ a horizontal frame and ξ_α for $\alpha = 1, 2, 3$ vertical frame and introduce the *momentum functions* $u_i, v_\alpha : T^*M \to \mathbb{R}$ defined by

$$u_i(\lambda) = \langle \lambda, X_i \rangle, \qquad i = 1, \dots, 4n,$$
$$v_\alpha(\lambda) = \langle \lambda, \xi_\alpha \rangle, \qquad \alpha = 1, 2, 3.$$

The momentum functions define coordinates (u, v) on each fiber of T^*M. In turn, they define local vector fields ∂_{v_α} and ∂_{u_i} on T^*M (satisfying $\pi_* \partial_{v_\alpha} = \pi_* \partial_{u_i} = 0$). Moreover, they define also the Hamiltonian vector fields $\vec{u}_i$ and $\vec{v}_\alpha$. The *Hamiltonian frame* associated with $\{\xi_\alpha, X_i\}$ is the local frame on T^*M around $\lambda(0)$ given by $\{\partial_{u_i}, \partial_{v_\alpha}, \vec{u}_i, \vec{v}_\alpha\}$.

The sub-Riemannian Hamiltonian and the corresponding Hamiltonian vector field are

$$H = \frac{1}{2} \sum_{i=1}^{4n} u_i u_i, \qquad \vec{H} = \sum_{i=1}^{4n} u_i \vec{u}_i.$$

We will use the short notation $\alpha\beta = \tau$, where $\alpha, \beta = 1, 2, 3$, for the quaternionic multiplication. The following 3×3 skew-symmetric matrix contains the vertical part of the covector:

$$V = V_{\alpha\beta} = v_{\alpha\beta} = v_\alpha v_\beta$$

with the convention $v_{\alpha^2} = -v_1 = 0$ which is the standard identification $\mathbb{R}^3 \backsimeq \mathfrak{so}(3)$.

Remark 8. For functions $f, g \in C^\infty(T^*M)$, the symbol $\{f, g\}$ denotes their Poisson bracket. The symbol $\dot{f}$ always denotes the Lie derivative in the direction of $\vec{H}$. We make systematic use of symplectic calculus (see for instance [8] for reference).

In what follows, we fix a geodesic $\gamma(t)$ with corresponding lift $\lambda(t)$ and a Fermi frame associated with it and given by Lemma 8. Repeated indices are implicitly summed over.

Lemma 9 (Commutators)**.** *We have the following identities*

(a') $g([X, Y], \xi_\alpha) = -d\eta_\alpha(X, Y) = -2g(I_\alpha X, Y)$
(b') $g([X, Y], Z) = g(\nabla_X Y, Z) - g(\nabla_Y X, Z)$
(c') $g([\xi_\alpha, X], \xi_\beta) = d\eta_\alpha(\xi_\beta, X) = -d\eta_\beta(\xi_\alpha, X) = -g(\nabla_X \xi_\alpha, \xi_\beta)$.
(d') $g([\xi_\alpha, X], Y) = -T(\xi_\alpha, X, Y) + g(\nabla_{\xi_\alpha} X, Y)$
(e') $g([\xi_\alpha, \xi_\beta], \xi_\gamma) = -d\eta_\gamma(\xi_\alpha, \xi_\beta) = Sg(\xi_{\alpha\beta}, \xi_\gamma) + g(\nabla_{\xi_\alpha} \xi_\beta, \xi_\gamma) - g(\nabla_{\xi_\beta} \xi_\alpha, \xi_\gamma)$
(f') $g([\xi_\alpha, \xi_\beta], X) = \rho_{\alpha\beta}(I_\beta X, \xi_\beta) = \rho_{\alpha\beta}(I_\alpha X, \xi_\alpha)$

Along the curve we have the following simplifications

(a) $g([X, Y], \xi_\alpha) = -d\eta_\alpha(X, Y) = -2g(I_\alpha X, Y)$
(b) $g([X, Y], Z) = 0$
(c) $g([\xi_\alpha, X], \xi_\beta) = 0$
(d) $g([\xi_\alpha, X], Y) = -T(\xi_\alpha, X, Y)$
(e) $g([\xi_\alpha, \xi_\beta], \xi_\gamma) = -d\eta_\gamma(\xi_\alpha, \xi_\beta) = Sg(\xi_{\alpha\beta}, \xi_\gamma)$

(f) $g([\xi_\alpha, \xi_\beta], X) = \rho_{\alpha\beta}(I_\beta X, \xi_\beta)$

Proof. It is obtained by a direct computation by combining the definition and the properties of the torsion of the Biquard connection (38) and using the choice of Fermi frame. □

As a consequence of the previous identities we compute the following Poisson brackets of momentum functions.

Lemma 10 (Poisson brackets)**.** *The momentum functions u_i, v_α have the following properties:*

(a') $\{v_\alpha, u_i\} = d\eta_\alpha(\xi_\tau, X_i)v_\tau + (-T(\xi_\alpha, X_i, X_k) + g(\nabla_{\xi_\alpha} X_i, X_k))\, u_k$,
(b') $\{v_\alpha, v_\beta\} = -d\eta_\tau(\xi_\alpha, \xi_\beta)v_\tau + \rho_{\alpha\beta}(I_\beta X_k, \xi_\beta)u_k$,
(c') $\{u_i, u_j\} = -2g(I_\tau X_i, X_j)v_\tau + g(X_k, [X_i, X_j])u_k$.

Moreover, when evaluated along the extremal $\lambda(t)$, one has

(a) $\{v_\alpha, u_i\} = -T(\xi_\alpha, X_i, \dot\gamma)$,
(b) $\{v_\alpha, v_\beta\} = -d\eta_\tau(\xi_\alpha, \xi_\beta)v_\tau + \rho_{\alpha\beta}(I_\beta\dot\gamma, \xi_\beta) = Sg(\xi_{\alpha\beta}, \xi_\tau)v_\tau + \rho_{\alpha\beta}(I_\beta\dot\gamma, \xi_\beta)$,
(c) $\{u_i, u_j\} = -2g(I_\tau X_i, X_j)v_\tau$.
(d) $\partial_{u_k}\{u_i, u_j\} = 0$,
(e) $\partial_{v_\alpha}\{u_i, u_j\} = -2g(I_\alpha X_i, X_j)$,

Proof. These formulas comes as a direct consequence of Lemma 9, thanks to the following observation: let X, Y be two smooth vector fields on M and $h_X(\lambda) = \langle \lambda, X(x)\rangle$ and $h_Y(\lambda) = \langle \lambda, Y(x)\rangle$ the associated Hamiltonians that are linear on fibers (here $x = \pi(\lambda)$). Then we have the identity $\{h_X, h_Y\}(\lambda) = \langle \lambda, [X, Y](x)\rangle =: h_{[X,Y]}$.

Let us now prove, as an example, formula (c'). We have, by definition

$$[X_i, X_j] = g(X_k, [X_i, X_j])X_k + g(\xi_\alpha, [X_i, X_j])\xi_\alpha.$$

Using then the above observation one has

$$\{u_i, u_j\} = v_\alpha g(\xi_\alpha, [X_i, X_j]) + u_k g(X_k, [X_i, X_j]).$$

and using then Lemma 9, one gets

$$\begin{aligned}\{u_i, u_j\} &= v_\alpha g(\xi_\alpha, [X_i, X_j]) + u_k g(X_k, [X_i, X_j]) \\ &= -v_\alpha d\eta_\alpha(X_i, X_j) + u_k g(X_k, [X_i, X_j]) = -2v_\alpha g(I_\alpha X_i, X_j) + u_k g(X_k, [X_i, X_j]).\end{aligned}$$

which proves (c'). Observe that the last term vanishes when evaluated along the extremal, thanks to the properties of Fermi frame. This proves (c). All other formulas follow analogously. □

Lemma 11 (Some arrows)**.** *We have the following expressions along the extremal*

(a) $\overrightarrow{\{u_i, u_j\}} = -2g(I_\alpha X_i, X_j)\vec v_\alpha - u_k X_\ell g([X_i, X_j], X_k)\partial_{u_\ell} - u_k\xi_\beta g([X_i, X_j], X_k)\partial_{v_\beta}$,
(b) $\overrightarrow{\{v_\alpha, u_i\}} = -T(\xi_\alpha, X_i, X_k)\vec u_k - K^\ell_{\alpha i}\partial_{u_\ell} - J^\beta_{\alpha i}\partial_{v_\beta}$,

where we set

$$\begin{aligned}K^\ell_{\alpha i} &:= v_\tau X_\ell d\eta_\alpha(\xi_\tau, X_i) - u_k X_\ell T(\xi_\alpha, X_i, X_k) + u_k X_\ell g(\nabla_{\xi_\alpha} X_i, X_k), \\ J^\beta_{\alpha i} &:= v_\tau \xi_\beta d\eta_\alpha(\xi_\tau, X_i) - u_k\xi_\beta T(\xi_\alpha, X_i, X_k) + u_k\xi_\beta g(\nabla_{\xi_\alpha} X_i, X_k).\end{aligned} \tag{11}$$

Proof. In this proof we make use of the following two facts. Let f be a smooth function on T^*M and let $\{\partial_{u_i}, \partial_{v_\alpha}, \vec u_i, \vec v_\alpha\}$ be the Hamiltonian frame, one has

$$\vec f = -X_i(f)\partial_{u_i} - \xi_\alpha(f)\partial_{v_\alpha} + \partial_{u_i}(f)\vec u_i + \partial_{v_\alpha}(f)\vec v_\alpha.$$

Moreover, if f, g are smooth functions on T^*M, we have $\overrightarrow{fg} = g\vec f + f\vec g$. To prove (a), one then computes

$$\begin{aligned}\overrightarrow{\{u_i, u_j\}} &= -2g(I_\alpha X_i, X_j)\vec v_\alpha + \vec u_k \cancel{g(X_k, [X_i, X_j])} - 2v_\alpha \overrightarrow{g(I_\alpha X_i, X_j)} + u_k\overrightarrow{g(X_k, [X_i, X_j])} \\ &= -2g(\phi_\alpha X_i, X_j)\vec v_\alpha + 2v_\alpha \cancel{X_\ell g(I_\alpha X_i, X_j)}\partial_{u_\ell} + 2v_\alpha \cancel{\xi_\tau g(I_\alpha X_i, X_j)}\partial_{v_\tau}\end{aligned}$$

$$- u_k X_\ell g(X_k,[X_i,X_j])\partial_{u_\ell} - u_k\xi_\tau g(X_k,[X_i,X_j])\partial_{v_\tau},$$

where the barred terms vanishes by Fermi frame. Similarly for (b) one gets

$$\begin{aligned}\overrightarrow{\{v_\alpha,u_i\}} &= \overrightarrow{d\eta_\alpha(\xi_\tau,X_i)v_\tau + (-T(\xi_\alpha,X_i,X_k)+g(\nabla_{\xi_\alpha}X_i,X_k))\,u_k}\\ &= \cancel{d\eta_\alpha(\xi_\tau,X_i)}\vec{v}_\tau + (-T(\xi_\alpha,X_i,X_k)+\cancel{g(\nabla_{\xi_\alpha}X_i,X_k)})\,\vec{u}_k\\ &\quad + \overrightarrow{d\eta_\alpha(\xi_\tau,X_i)}v_\tau + \left(\overrightarrow{-T(\xi_\alpha,X_i,X_k)+g(\nabla_{\xi_\alpha}X_i,X_k)}\right)u_k\\ &= -T(\xi_\alpha,X_i,X_k)\vec{u}_k - K^\ell_{\alpha i}\partial_{u_\ell} - J^\beta_{\alpha i}\partial_{v_\beta}\end{aligned}$$

again the barred terms vanishes by Fermi frame. The expression of the coefficients $K^\ell_{\alpha i}$ and $J^\beta_{\alpha i}$ are obtained from direct computations. □

Lemma 12. *Let $v_\alpha(t) = \langle\lambda(t),\xi_\alpha|_{\gamma(t)}\rangle$, for $\alpha = 1,2,3$. Then, along the geodesic, we have*

$$\nabla_{\dot\gamma}\dot\gamma = -2v_\alpha I_\alpha\dot\gamma. \tag{12}$$

Proof. Indeed $\gamma(t) = u_i(t)X_i|_{\gamma(t)}$, with $u_i(t) = \langle\lambda(t),X_i|_{\gamma(t)}\rangle$. Then, suppressing the explicit dependence on t one has

$$\begin{aligned}\nabla_{\dot\gamma}\dot\gamma &= \dot u_i X_i + u_i u_k \cancel{\nabla_{X_k}X_i} = \{H,u_i\}X_i\\ &= u_k\{u_k,u_i\}X_i = -2u_k v_\alpha g(I_\alpha X_k,X_i)X_i = -2v_\alpha g(I_\alpha\dot\gamma,X_i)X_i = -2v_\alpha I_\alpha\dot\gamma,\end{aligned}$$

where the barred term vanishes along the trajectory thanks to the properties of Fermi frame. □

3.3. **Fundamental computations.** The frame $\{\partial_u,\partial_v,\vec{u},\vec{v}\}$ is a basis of the tangent space to T^*M. We compute the differential equations of this frame along an extremal.

Lemma 13. *Along the extremal $\lambda(t)$, we have*

$$\begin{aligned}\dot\partial_v &= 2A\partial_u,\\ \dot\partial_u &= -\vec{u} + G\partial_v,\\ \dot{\vec{u}} &= 2C\vec{u} - 2A^*\vec{v} + B\partial_u + D\partial_v,\\ \dot{\vec{v}} &= L\vec{u} + M\partial_u + 2N\partial_v,\end{aligned}$$

where we defined the following matrices, computed along $\lambda(t)$:

$$\begin{aligned}A_{\beta i} &:= g(I_\beta\dot\gamma,X_i), && 3\times 4n \text{ matrix},\\ G_{i\alpha} &:= -T(\xi_\alpha,\dot\gamma,X_i) && 4n\times 3 \text{ matrix},\\ B_{i\ell} &:= -u_j u_k X_\ell g([X_j,X_i],X_k) = R(\dot\gamma,X_\ell,X_i,\dot\gamma)\\ C_{ij} &:= v_\alpha g(I_\alpha X_i,X_j), && 4n\times 4n \text{ skew-symmetric matrix},\\ D_{i\beta} &:= -u_j u_k\xi_\beta g([X_j,X_i],X_k) = R(\dot\gamma,\xi_\beta,X_i,\dot\gamma)\\ L_{\beta j} &:= -\frac{1}{2}\left(T^0(I_\beta X_j,\dot\gamma)+T^0(I_\beta\dot\gamma,X_j)\right)\\ M_{\alpha\ell} &:= K^\ell_{\alpha j}u_j, \quad M_{\alpha\ell} = -2v_\tau\rho_\zeta(X_\ell,\dot\gamma) + \frac{1}{2}(\nabla_{X_\ell}T^0)(I_\alpha\dot\gamma,\dot\gamma),\\ N_{\alpha\beta} &:= J^\beta_{\alpha j}u_j, \quad 2N_{\alpha\beta} = -2v_\tau\rho_\zeta(\xi_\beta,\dot\gamma) + \frac{1}{2}(\nabla_{\xi_\beta}T^0)(I_\alpha\dot\gamma,\dot\gamma).\end{aligned}$$

in the last two formulas $\{\alpha\tau\zeta\}$ is a cyclic permutation of $\{123\}$, or $\zeta = \alpha\tau$ (as product of quaternions).

Proof. By a direct computation we get (simplifications are due to Fermi frame properties)

$$\begin{aligned}\dot\partial_{v_\beta} &= [u_j\vec{u}_j,\partial_{v_\beta}] = -\partial_{v_\beta}(u_j)\vec{u}_j + u_j[\vec{u}_j,\partial_{v_\beta}] = u_j[\vec{u}_j,\partial_{v_\beta}](u_i)\partial_{u_i} + u_j[\vec{u}_j,\partial_{v_\beta}](v_\alpha)\partial_{v_\alpha}\\ &= -u_j\partial_{v_\beta}\{u_j,u_i\}\partial_{u_i} - u_j\cancel{\partial_{v_\beta}\{u_j,v_\alpha\}}\partial_{v_\alpha} = u_j g(2I_\beta X_j,X_i)\partial_{u_i} = 2g(I_\beta\dot\gamma,X_i)\partial_{u_i}.\end{aligned}$$

$$\dot{\partial}_{u_i} = [u_j\vec{u}_j, \partial_{u_i}] = -\partial_{u_i}(u_j)\vec{u}_j + u_j[\vec{u}_j, \partial_{u_i}] = -\vec{u}_i - u_j\partial_{u_i}\{u_j, u_\ell\}\partial_{u_\ell} - u_j\partial_{u_i}\{u_j, v_\alpha\}\partial_{v_\alpha}$$
$$= -\vec{u}_i - u_jT(\xi_\alpha, X_j, X_k)\partial_{u_i}(u_k)\partial_{v_\alpha} = -\vec{u}_i - T(\xi_\alpha, \dot{\gamma}, X_i)\partial_{v_\alpha}$$
$$\dot{\vec{u}}_i = [u_j\vec{u}_j, \vec{u}_i] = -\vec{u}_i(u_j)\vec{u}_j + u_j[\vec{u}_j, \vec{u}_i] = -\{u_i, u_j\}\vec{u}_j + u_j\overrightarrow{\{u_j, u_i\}}$$
$$= 2v_\alpha g(I_\alpha X_i, X_j)\vec{u}_j + 2g(I_\alpha X_i, \dot{\gamma})\vec{v}_\alpha + u_ju_kX_\ell g([X_i, X_j], X_k)\partial_{u_\ell} + u_ju_k\xi_\beta g([X_i, X_j], X_k)\partial_{v_\beta}.$$
$$\dot{\vec{v}}_\beta = [u_j\vec{u}_j, \vec{v}_\beta] = -\vec{v}_\beta(u_j)\vec{u}_j + u_j[\vec{u}_j, \vec{v}_\beta] = -\{v_\beta, u_j\}\vec{u}_j + u_j\overrightarrow{\{u_j, v_\beta\}}$$
$$= T(\xi_\beta, X_j, X_k)u_k\vec{u}_j + u_j\overrightarrow{\{u_j, v_\beta\}}$$
$$= T(\xi_\beta, X_j, X_k)u_k\vec{u}_j - u_j[-T(\xi_\alpha, X_j, X_k)\vec{u}_k - K^\ell_{\alpha j}\partial_{u_\ell} - J^\beta_{\alpha j}\partial_{v_\beta}]$$
$$= [T(\xi_\beta, X_j, \dot{\gamma}) + T(\xi_\beta, \dot{\gamma}, X_j)]\vec{u}_j + K^\ell_{\alpha j}u_j\partial_{u_\ell} + J^\beta_{\alpha j}u_j\partial_{v_\beta} \qquad \square$$

The previous proof is completed thanks to the next lemma.

Lemma 14. *In terms of Biquard curvature we have*

(a) $B_{i\ell} = -u_ju_kX_\ell g([X_j, X_i], X_k) = R(\dot{\gamma}, X_\ell, X_i, \dot{\gamma})$
(b) $D_{i\beta} = -u_ju_k\xi_\beta g([X_j, X_i], X_k) = R(\dot{\gamma}, \xi_\beta, X_i, \dot{\gamma})$
(c) $M_{\alpha\ell} = -2v_\tau\rho_\zeta(X_l, \dot{\gamma}) + \frac{1}{2}(\nabla_{X_\ell}T^0)(I_\alpha\dot{\gamma}, \dot{\gamma})$,
(d) $2N_{\alpha\beta} = -2v_\tau\rho_\zeta(\xi_\beta, \dot{\gamma}) + \frac{1}{2}(\nabla_{\xi_\beta}T^0)(I_\alpha\dot{\gamma}, \dot{\gamma})$,

where in the last two formulas $\{\alpha\tau\zeta\}$ *is a cyclic permutation of* $\{123\}$.

Proof of Lemma 14. We use classical tricks in curvature calculations

$$R(\dot{\gamma}, X_i, X_\ell, \dot{\gamma}) = u_ku_jg(\nabla_{X_k}\nabla_{X_i}X_\ell - \nabla_{X_i}\nabla_{X_k}X_\ell - \nabla_{[X_k, X_i]}X_\ell, X_j)$$
$$= u_ku_jX_kg(\nabla_{X_i}X_\ell, X_j) - u_ku_jg(\nabla_{X_i}X_\ell, \nabla_{X_k}X_j) - u_ku_jX_ig(\nabla_{X_k}X_\ell, X_j)$$
$$+ u_ku_jg(\nabla_{X_k}X_\ell, \nabla_{X_i}X_j) - u_ku_jg(\nabla_{[X_k, X_i]}X_\ell, X_j).$$

where the cancellation XX follows from properties of Fermi frame, while YY is due to the identity $u_kX_kg(\nabla_{X_i}X_\ell, X_i) = 0$ (notice that the last identity is the derivative in the direction of $\dot{\gamma}(t)$ of the following one $g(\nabla_{X_i}X_\ell, X_j)|_{\gamma(t)} = 0$). Thus using that the torsion among horizontal vector fields is vertical

$$R(\dot{\gamma}, X_i, X_\ell, \dot{\gamma}) = -u_ku_jX_ig(\nabla_{X_k}X_\ell, X_j)$$
$$= -u_ku_jX_ig(\nabla_{X_\ell}X_k, X_j) - u_ku_jX_ig([X_k, X_\ell], X_j).$$

On the other hand, the term $g(\nabla_{X_\ell}X_k, X_j)$ is skew-symmetric in k, j and we have a symmetric sum so the first term is zero and

$$R(\dot{\gamma}, X_i, X_\ell, \dot{\gamma}) = -u_ku_jX_ig([X_k, X_\ell], X_j).$$

Along the same path one can show that

$$R(\dot{\gamma}, \xi_\beta, X_\ell, \dot{\gamma}) = -u_ku_j\xi_\beta g([X_k, X_\ell], X_j).$$

Applying (41), (42), we have

$$R(\dot{\gamma}, \xi_\beta, X_\ell, \dot{\gamma}) = (\nabla_{\dot{\gamma}}U)(I_\beta X_\ell, \dot{\gamma}) - \frac{1}{4}(\nabla_{\dot{\gamma}}T^0)(I_\beta X_\ell, \dot{\gamma})$$
$$- \frac{1}{4}(\nabla_{\dot{\gamma}}T^0)(X_\ell, I_\beta\dot{\gamma}) + \frac{1}{2}(\nabla_{X_\ell}T^0)(I_\beta\dot{\gamma}, \dot{\gamma})$$
$$- 2g(I_\tau\dot{\gamma}, X_\ell)\rho_\zeta(I_\beta\dot{\gamma}, \xi_\beta) + 2g(I_\zeta\dot{\gamma}, X_\ell)\rho_\tau(I_\beta\dot{\gamma}, \xi_\beta),$$

where $\{\beta\tau\zeta\}$ is a cyclic permutation of $\{123\}$. Further, we have using (11)

$$M_{\alpha\ell} = K^\ell_{\alpha i}u_i = u_iv_\tau X_\ell d\eta_\alpha(\xi_\tau, X_i) - u_iu_kX_\ell T(\xi_\alpha, X_i, X_k) + u_iu_kX_\ell g(\nabla_{\xi_\alpha}X_i, X_k)$$
$$= -u_iv_\tau X_\ell g(\nabla_{X_i}\xi_\alpha, \xi_\tau) + \frac{1}{4}u_iu_k(\nabla_{X_\ell}T^0)[(I_\alpha X_i, X_k) + (I_\alpha X_k, X_i)]$$
$$= -2v_\tau\rho_\zeta(X_l, \dot{\gamma}) + \frac{1}{2}(\nabla_{X_\ell}T^0)(I_\alpha\dot{\gamma}, \dot{\gamma}),$$

where $\{\alpha\tau\zeta\}$ is a cyclic permutation of $\{123\}$, the skew-symmetric parts in i and k of the first line are cancelled and the first term of the second line is evaluated as follows

$$\begin{aligned} u_i X_\ell g(\nabla_{X_i}\xi_\alpha, \xi_\tau) &= u_i g(\nabla_{X_\ell}\nabla_{X_i}\xi_\alpha, \xi_\tau) + \cancel{u_i g(\nabla_{X_i}\xi_\alpha, \nabla_{X_\ell}\xi_\tau)} \\ = u_i g(\nabla_{X_i}\nabla_{X_\ell}\xi_\alpha, \xi_\tau) &+ u_i R(X_\ell, X_i, \xi_\alpha, \xi_\tau) + \cancel{u_i g(\nabla_{[X_\ell, X_i]}\xi_\alpha, \xi_\tau)} \\ &= 2\rho_\zeta(X_\ell, \dot\gamma) + \cancel{u_i X_i g(\nabla_{X_\ell}\xi_\alpha, \xi_\tau)} \end{aligned}$$

Similarly one gets the formula for $N_{\alpha\beta}$ □

4. Symplectic products and canonical frame

In this section, for n-tuples v, w of vectors, the symbol $\sigma(v, w)$ denotes the matrix

$$\sigma(v, w) := (\sigma(v_i, w_j))_{i,j=1,\dots,n},$$

whose entries are symplectic products. Notice that with this convention one has the identities

$$\sigma(v, w)^* = -\sigma(w, v), \qquad \sigma(Av, Bw) = A\sigma(v, w)B^*.$$

where, for n-tuple of vectors v and a matrix L, the juxtaposition Lv denotes the n-tuple of vectors obtained by matrix multiplication. When v, w are vector fields defined along an extremal $\lambda(t)$, the following Leibniz rule holds

$$\frac{d}{dt}\sigma(v, w) = \sigma(\dot v, w) + \sigma(v, \dot w).$$

4.1. **Derivatives of ∂_v and symplectic products of the coordinate frame.** By a direct computation one gets from Lemma 13 the following relations.

Lemma 15. *Along the extremal, we have*

$$\begin{aligned} \dot\partial_v &= 2A\partial_u, \\ \ddot\partial_v &= 2\dot A\partial_u + 2AG\partial_v - 2A\vec u, \\ \dddot\partial_v &= (4\dot A G + 2A\dot G - 2AD)\partial_v + (2\ddot A + 4AGA - 2AB)\partial_u + 4AA^*\vec v + (-4\dot A - 4AC)\vec u \\ &= (4\dot A G + 2A\dot G - 2AD)\partial_v + (2\ddot A + 4AGA - 2AB)\partial_u + 4\vec v - 4(3VA + v\dot\gamma^*)\vec u \end{aligned}$$

To prove the fourth equality we used (13) and (15) below. We then compute symplectic products of the elements of the basis.

Lemma 16. *The non-zero brackets between $\partial_{u_i}, \partial_{v_\alpha}, \vec u_i, \vec v_\alpha$ are*

(a) $\sigma(\partial_u, \vec u) = \mathbb{1}$,
(b) $\sigma(\partial_v, \vec v) = \mathbb{1}$,
(c) $\sigma(\vec u, \vec u) = \{u_i, u_j\} = -2C$,
(d) $\sigma(\vec v, \vec v) = \{v_\alpha, v_\beta\} = SV + \chi$,
(e) $\sigma(\vec u, \vec v) = \{u_i, v_\alpha\} = P$,

where, according to (38),

$$\chi_{\alpha\beta} = -\rho_{\alpha\beta}(I_\beta\dot\gamma, \xi_\beta) = \rho_{\alpha\beta}(\xi_\beta, I_\beta\dot\gamma) = -g([\xi_\alpha, \xi_\beta], \dot\gamma) = -\chi_{\beta\alpha},$$

and, thanks to (36),

$$P_{i\alpha} = \{u_i, v_\alpha\} = T(\xi_\alpha, X_i, \dot\gamma) = T(\xi_\alpha, \dot\gamma, X_i) + 2U(I_\alpha X_i, \dot\gamma) = -G_{i\alpha} + 2U(I_\alpha X_i, \dot\gamma).$$

Notice that, by bilinearity of σ, Lemma 16 permits to compute the symplectic product of any pair of vectors.

Lemma 17 (Several identities)**.** *We have the following identities*

$$AA^* = \mathbb{1}, \quad A\dot\gamma = 0, \quad Vv = 0, \quad AC = VA - v\dot\gamma^*, \quad V^2 = vv^* - ||v||^2\mathbb{1}, \tag{13}$$

$$\ddot\gamma = -2A^*v = 2C\dot\gamma, \tag{14}$$

$$\dot A = 2VA + 2v\dot\gamma^*, \quad \ddot A = 2\dot V A + 2\dot v\dot\gamma^* - 4\|v\|^2 A. \tag{15}$$

Proof. The identities (13) follow directly from the definitions while (14) is precisely (12) written in terms of A, C and v. For (15), working in the Fermi frame along γ, we have

$$\begin{aligned}\dot{A}_{\beta i} &= \{H, A_{\beta i}\} = u_j \vec{u_j}(A_{\beta i}) = -u_j \overrightarrow{A_{\beta i}}(u_j) \\ &= u_j X_l(g(I_\beta \dot\gamma, X_i))\partial_{u_l}(u_j) + u_j \xi_\tau (g(I_\beta \dot\gamma, X_i))\partial_{v_\tau}(u_j) \\ &= u_j X_j(g(I_\beta\dot\gamma, X_i)) = g(I_\beta \nabla_{\dot\gamma}\dot\gamma, X_i) = -2v_\alpha g(I_\beta I_\alpha \dot\gamma, X_i) = 2V_{\beta\alpha}A_{\alpha i} + 2v\dot\gamma^*\end{aligned}$$

where we used (12) to get the last equality. □

Corollary 18. *We also have*

$$\dot{A}A^* = 2V, \quad A\dot{A}^* = -2V, \quad ACA^* = V, \quad \dot{A} + AC = 3VA + v\dot\gamma^*, \quad V^3 = -||v||^2 V, \tag{16}$$

$$\ddot{A} = 2\dot{V}A + 2\dot{v}\dot\gamma^* - 4\|v\|^2 A, \quad A\ddot{A}^* = -2\dot{V} - 4\|v\|^2 \mathbb{1} \tag{17}$$

Proof. The identities (16) follow directly from (13) and (15). For the first one in (17), we take the derivative of (15) applying (13) and (14) to get that

$$\begin{aligned}\ddot{A} &= 2\dot{V}A + 2V\dot{A} + 2\dot{v}\dot\gamma^* + 2v\ddot\gamma^* = 2\dot{V}A + 2V(2VA + 2v\dot\gamma^*) + 2\dot{v}\dot\gamma^* - 4vv^*A \\ &= 2\dot{V}A + 4(vv^* - ||v||^2\mathbb{1})A + 2\dot{v}\dot\gamma^* - 4vv^*A = 2\dot{V}A + 2\dot{v}\dot\gamma^* - 4\|v\|^2 A.\end{aligned}$$

The second identity in (17) follows from the first one applying (13). □

Lemma 19 (Derivative of V)**.** *We have $\dot{V} \neq 0$. In particular*

$$\dot{v}_\tau = T(\xi_\tau, \dot\gamma, \dot\gamma). \tag{18}$$

In vector notation $\dot{v} = -G^*\dot\gamma$. In particular if $T^0 = 0$ then $\dot{v} = \dot{V} = 0$.

Proof. It easily follows by

$$\dot{v}_\tau = \{H, v_\tau\} = u_j\{u_j, v_\tau\} = u_j T(\xi_\tau, X_j, \dot\gamma) = T(\xi_\tau, \dot\gamma, \dot\gamma). \quad \square$$

4.2. **Computation of symplectic products.** Now we deduce symplectic products of derivatives of the vector ∂_v using Lemma 15, Lemma 16, Lemma 17 and Corollary 18.

Lemma 20. *We have*

$$\sigma(\partial_v, \partial_v) = 0, \quad \sigma(\partial_v, \dot\partial_v) = 0, \quad \sigma(\partial_v, \ddot\partial_v) = 0, \quad \sigma(\partial_v, \dddot\partial_v) = 4\mathbb{1}, \tag{19}$$

$$\sigma(\dot\partial_v, \dot\partial_v) = 0, \quad \sigma(\dot\partial_v, \ddot\partial_v) = -4\mathbb{1}, \quad \sigma(\dot\partial_v, \dddot\partial_v) = 24V, \quad \sigma(\ddot\partial_v, \ddot\partial_v) = -24V, \tag{20}$$

Proof. The identities (19) and the first two ones in (20) follow from Lemma 15, Lemma 16 and Lemma 17. We calculate $\sigma(\dot\partial_v, \dddot\partial_v) = 2A(-12VA - 4v\dot\gamma^*)^* = -24AA^*V^* - 8A\dot\gamma v^* = 24V$ which proves the third one in (20). Differentiating the second identity in (20), one gets $0 = \sigma(\ddot\partial_v, \ddot\partial_v) + \sigma(\dot\partial_v, \dddot\partial_v) = \sigma(\ddot\partial_v, \ddot\partial_v) + 24V$ which yields the third one in (20). □

4.3. **Canonical frame.** To compute $\mathcal{R}_{cc} = \sigma(\dot{F}_c, F_c)$, we need to compute the elements of the canonical basis up to F_c. The algorithm to recover them (following the general construction developed in [44]) starts from identifying E_a and then works as follows:

$$E_a \to E_b \to F_b \to E_c \to F_c \to \mathcal{R}_{cc}$$

The triplet E_a is determined by the following four conditions:

(i) $\pi_* E_a = 0$,
(ii) $\pi_* \dot{E}_a = 0$,
(iii) $\sigma(\ddot{E}_a, \dot{E}_a) = \mathbb{1}$,
(iv) $\sigma(\ddot{E}_a, \ddot{E}_a) = \mathbb{0}$.

Items (i) and (ii) imply that there exists $M \in \mathrm{GL}(3)$ such that $E_a = M\partial_v$. Condition (iii) implies that $M = \frac{1}{2}O$ with $O \in \mathrm{O}(3)$. Finally, (iv) implies that O satisfies the differential equation

$$\dot{O} = \frac{1}{16}O\sigma(\ddot{\partial}_{v_\alpha}, \ddot{\partial}_{v_\beta}) = -\frac{3}{2}OV. \tag{21}$$

Its solution is unique up to an orthogonal transformation (the initial condition, that we set $O(0) = \mathbb{1}$). Using the structural equations together with (21), we have

$$\begin{aligned} E_a &= \tfrac{1}{2}O\partial_v, \\ E_b &= \dot{E}_a = \tfrac{1}{2}O(-\frac{3}{2}V\partial_v + \dot{\partial}_v), \\ F_b &= -\dot{E}_b = -\tfrac{1}{2}O[(\frac{9}{4}V^2 - \frac{3}{2}\dot{V})\partial_v - 3V\dot{\partial}_v + \ddot{\partial}_v] \end{aligned} \tag{22}$$

Thus we can also compute

$$\dot{F}_b = -\tfrac{1}{2}O[((-\frac{27}{8}V^3 + \frac{27}{4}V\dot{V} - \frac{3}{2}\ddot{V})\partial_v + 3(\frac{9}{4}V^2 - \frac{3}{2}\dot{V})\dot{\partial}_v - \frac{9}{2}V\ddot{\partial}_v + \dddot{\partial}_v]. \tag{23}$$

The next step is to compute E_c. It is determined by the following conditions:

(i) $\pi_* E_c = 0$,
(ii) $\sigma(E_c, F_c) = \mathbb{1}$ and $\sigma(E_c, F_b) = \sigma(E_c, F_a) = \mathbb{0}$,
(iii) $\pi_* \ddot{E}_c = 0$.

For (i) we can write

$$E_c = Y\partial_u + W\partial_v, \quad F_c = -\dot{E}_c = -(\dot{Y} + 2WA)\partial_u - (YG + \dot{W})\partial_v + Y\vec{u}. \tag{24}$$

where Y is a $(4n-3)\times 4n$ matrix and W is a $(4n-3)\times 3$ matrix.

To compute $\sigma(E_c, F_c), \sigma(E_c, F_b), \sigma(E_c, \dot{F}_b)$ using (24), (22) and (23), we need to know

$$\sigma(\partial_u, \dot{\partial}_v),\ \sigma(\partial_u, \ddot{\partial}_v),\ \sigma(\partial_u, \dddot{\partial}_v),\ \sigma(\partial_v, \dot{\partial}_v),\ \sigma(\partial_v, \ddot{\partial}_v),\ \sigma(\partial_v, \dddot{\partial}_v).$$

The only non-zero terms using Lemma 15 are given by:

$$\sigma(\partial_u, \ddot{\partial}_v) = -2A^*, \quad \sigma(\partial_v, \dddot{\partial}_v) = 4\mathbb{1},$$
$$\sigma(\partial_u, \dddot{\partial}_v) = -4(\dot{A}^* + C^*A^*) = -4(3A^*V^* + \dot{\gamma}v^*).$$

For (ii), observing that $\sigma(E_c, F_a) = \mathbb{0}$ implies $\sigma(E_c, \dot{F}_b) = \mathbb{0}$, we get

$$YY^* = \mathbb{1}, \qquad YA^* = 0, \qquad W = Y(\dot{A}^* + C^*A^*) = Y(3A^*V^* + \dot{\gamma}v^*) = Y\dot{\gamma}v^*. \tag{25}$$

We obtain from (24) and (25)

$$\pi_* \ddot{E}_c = 2(\dot{Y} + WA + YC)\vec{u}.$$

Finally, using (iii) and the equality above, we get that Y must satisfy

$$\dot{Y} = -WA - YC = -Y(\dot{A}^*A + C^*A^* + C) = -Y(\dot{\gamma}v^*A + C). \tag{26}$$

Using Lemma 17, (25), (26) and (18), we have

$$\begin{aligned} \dot{W} &= \dot{Y}\dot{\gamma}v^* + Y\ddot{\gamma}v^* + Y\dot{\gamma}\dot{v}^* = -Y\dot{\gamma}v^*A\dot{\gamma}v^* - YC\dot{\gamma}v^* - 2YA^*vv^* + Y\dot{\gamma}T(\xi_\tau, \dot{\gamma}, \dot{\gamma}) \\ &= YA^*vv^* + Y\dot{\gamma}T(\xi_\tau, \dot{\gamma}, \dot{\gamma}) = Y\dot{\gamma}T(\xi_\tau, \dot{\gamma}.\dot{\gamma}) \end{aligned} \tag{27}$$

Observe that Y represents an orthogonal projection on $\mathcal{D} \cap \mathrm{span}\{I_\alpha\dot{\gamma}, I_\beta\dot{\gamma}, I_\tau\dot{\gamma}\}^\perp$. Then

$$Y^*Y = \mathbb{1} - A^*A.$$

Substitute (26) into the second equality of (24) to get

$$F_c = (YC - WA)\partial_u - (YG + \dot{W})\partial_v + Y\vec{u}. \tag{28}$$

We calculate from (28) using Lemma 13, (25), (26) and (27) that

$$\dot{F}_c = (\dot{Y}C + Y\dot{C} - \dot{W}A - W\dot{A})\partial_u - (\dot{Y}G + Y\dot{G} + \ddot{W})\partial_v + \dot{Y}\vec{u} \tag{29}$$

$$
\begin{aligned}
&+(YC-WA)(-\vec{u}+G\partial_v)-2(YG+\dot{W})A\partial_u+Y(2C\vec{u}-2A^*\vec{v}+B\partial_u+D\partial_v)\\
&=(\dot{Y}C+Y\dot{C}-\dot{W}A-W\dot{A}+YB-2YGA-2\dot{W}A))\partial_u-(\dot{Y}G+Y\dot{G}+\ddot{W}-YCG+WAG-YD))\partial_v\\
&\qquad+(\dot{Y}+WA-YC+2YC)\vec{u}-2YA^*\vec{v}\\
&=(\dot{Y}C+Y\dot{C}-\dot{W}A-W\dot{A}+YB-2YGA-2\dot{W}A))\partial_u-(\dot{Y}G+Y\dot{G}+\ddot{W}-YCG+WAG-YD))\partial_v,
\end{aligned}
$$

where the cancellations in the fourth line follow from (26) and (25).

Applying Lemma 16, Lemma 17, (25) and (26) we obtain from (28) and (29)

$$
\begin{aligned}
\mathcal{R}_{cc}=\sigma(\dot{F}_c,F_c)&=(\dot{Y}C+Y\dot{C}-\dot{W}A-W\dot{A}+YB-2YGA-2\dot{W}A)Y^*\\
&=\dot{Y}CY^*+Y\dot{C}Y^*-\dot{W}AY^*-W\dot{A}Y^*+YBY^*-\dot{W}AY^*\\
&=-Y(\dot{\gamma}v^*AC+CC)Y^*+YBY^*-W(2VA+2v\dot{\gamma}^*)Y^*+Y\dot{C}Y^*\\
&=-Y(\dot{\gamma}v^*AC+CC)Y^*+YBY^*-2Y\dot{\gamma}v^*v\dot{\gamma}^*Y^*+Y\dot{C}Y^*\\
&=-Y(\dot{\gamma}v^*AC+CC-B+2\|v\|^2\dot{\gamma}\dot{\gamma}^*-\dot{C})Y^*\\
&=-Y(\dot{\gamma}v^*(VA-v\dot{\gamma}^*)-\|v\|^2\mathbb{1}-B+2\|v\|^2\dot{\gamma}\dot{\gamma}^*-\dot{C})Y^*\\
&=Y[B+\dot{C}+\|v\|^2(1-\dot{\gamma}\dot{\gamma}^*)]Y^*,
\end{aligned}
\tag{30}
$$

where we used $C_{ij}C_{jk}=-v_\alpha v_\beta g(I_\alpha X_i,I_\beta X_k)=-||v||^2\mathbb{1}$.

Lemma 21. *(Derivative of C along $\gamma(t)$) We have*

$$\dot{C}_{ki}=-g(I_\tau X_k,X_i)T(\xi_\tau,\dot{\gamma},\dot{\gamma})$$

Proof. By a direct computation

$$
\begin{aligned}
\dot{C}_{ki}=\{H,C_{ki}\}&=-u_j\vec{C}_{ki}(u_j)=-u_jX_j(v_\alpha g(I_\alpha X_k,X_i)+u_j\partial_{v_\tau}(v_\alpha g(I_\alpha X_k,X_i)\vec{v_\tau}(u_j)\\
&=-v_\alpha[g(\nabla_{\dot{\gamma}}I_\alpha X_k,X_i)+g(I_\alpha X_k,\nabla_{\dot{\gamma}}X_i)]-u_jg(I_\tau X_k,X_i)T(\xi_\tau,X_j,\dot{\gamma})\\
&=-g(I_\tau X_k,X_i)T(\xi_\tau,\dot{\gamma},\dot{\gamma})
\end{aligned}
$$

□

Finally we derive the following formula for the trace of the matrix $\mathcal{R}_{cc}$.

Lemma 22. *We have*

$$\operatorname{tr}(\mathcal{R}_{cc})=\operatorname{Ric}(\dot{\gamma},\dot{\gamma})-\sum_{\alpha=1}^{3}R(\dot{\gamma},I_\alpha\dot{\gamma},I_\alpha\dot{\gamma},\dot{\gamma})+4(n-1)\|v\|^2.$$

Proof. Indeed, we obtain from (30)

$$
\begin{aligned}
\operatorname{tr}(\mathcal{R}_{cc})&=\operatorname{tr}((B+\dot{C}+\|v\|^2(\mathbb{1}-\dot{\gamma}\dot{\gamma}^*))Y^*Y)\\
&=\operatorname{tr}((B+\dot{C}+\|v\|^2(\mathbb{1}-\dot{\gamma}\dot{\gamma}^*))(\mathbb{1}-A^*A))\\
&=\sum_{i=1}^{4n}R(\dot{\gamma},X_i,X_i,\dot{\gamma})-\sum_{\alpha=1}^{3}R(\dot{\gamma},I_\alpha\dot{\gamma},I_\alpha\dot{\gamma},\dot{\gamma})+(4n-4)\|v\|^2\\
&=\operatorname{Ric}(\dot{\gamma},\dot{\gamma})-\sum_{\alpha=1}^{3}R(\dot{\gamma},I_\alpha\dot{\gamma},I_\alpha\dot{\gamma},\dot{\gamma})+4(n-1)\|v\|^2.
\end{aligned}
$$

where we used $\operatorname{tr}(\dot{C}\mathbb{1})=0$ and

$$
\begin{aligned}
\operatorname{tr}(\dot{C}A^*A)=\dot{C}_{ki}A_{i\beta}A_{\beta k}&=-T(\xi_\tau,\dot{\gamma},\dot{\gamma})g(I_\tau X_k,X_i)g(I_\beta\dot{\gamma},X_i)g(I_\beta\dot{\gamma},X_k)\\
&=T(\xi_\tau,\dot{\gamma},\dot{\gamma})g(I_\beta\dot{\gamma},I_\tau I_\beta\dot{\gamma})=0.
\end{aligned}
$$

□

5. Proof of Theorems 1 and 2

Theorem 1 is a direct consequence of Theorem 7 and Lemma 22. To prove Theorem 2 we use the following Lemma to rewrite the condition in Theorem 1.

Lemma 23. *We have the identity*

$$\mathrm{Ric}(X,X) - \sum_{\alpha=1}^{3} R(X, I_\alpha X, I_\alpha X, X) = 2nT^0(X,X) + (4n-8)U(X,X) + 2(n-1)S. \tag{31}$$

Proof. Fix a unit vector X and set $Y = Z = I_1X$, $V = X$ into (40). One obtains the following expression

$$\begin{aligned} 3R(X,I_1X,I_1X,X) + R(I_1X,X,I_1X,X) + R(I_2X,I_3X,I_1X,X) - R(I_3X,I_2X,I_1X,X) \\ = 2R(X,I_1X,I_1X,X) + 2R(I_2X,I_3X,I_1X,X) \\ = 4(T^0(X,X) + T^0(I_1X,I_1X)) + 8U(X,X) + 4S. \end{aligned}$$

Do the same for I_2, I_3 and sum the obtained equalities, we obtain applying the first Bianchi identity for the Biquard connection (39).

$$\begin{aligned} 2R(X,I_1X,I_1X,X) &+ 2R(I_2X,I_3X,I_1X,X) + 2R(X,I_2X,I_2X,X) \\ &+ 2R(I_3X,I_1X,I_2X,X) + 2R(X,I_3X,I_3X,X) + 2R(I_1X,I_2X,I_3X,X) \\ &= 2\sum_{\alpha=1}^{3} R(X,I_\alpha X,I_\alpha X,X) + 2b(I_1X,I_2X,I_3X,X) \\ &= 4\Big[3T^0(X,X) + \sum_{\alpha=1}^{3} T^0(I_\alpha X,I_\alpha X)\Big] + 24U(X,X) + 12S \\ &= 8T^0(X,X) + 24U(X,X) + 12S. \end{aligned}$$

From (39) we have $b(I_1X,I_2X,I_3X,X) = 2T^0(X,X) - 6U(X,X)$. Hence,

$$\begin{aligned} \sum_{\alpha=1}^{3} R(X,I_\alpha X,I_\alpha X,X) &= 6S + 4T^0(X,X) + 12U(X,X) - 2T^0(X,X) + 6U(X,X) \\ &= 2T^0(X,X) + 18U(X,X) + 6S. \end{aligned} \tag{32}$$

and the first identity of (38) combined with (32) gives (31). ☐

Now, Theorem 2 follows from Theorem 1 and Lemma 23.

Appendix A. Some technical facts and useful identities

Here we recall some properties of the torsion and curvature of the Biquard connection. See also [17, 31, 33, 28, 32] for a comprehensive exposition.

A.1. **Invariant decompositions.** Any endomorphism Ψ of $\mathcal{D}$ can be decomposed with respect to the quaternionic structure $(\mathbb{Q}, g)$ uniquely into four $\mathrm{Sp}(n)$-invariant parts $\Psi = \Psi^{+++} + \Psi^{+--} + \Psi^{-+-} + \Psi^{--+}$, where Ψ^{+++} commutes with all three I_i, Ψ^{+--} commutes with I_1 and anti-commutes with the others two, etc. The two $\mathrm{Sp}(n)\mathrm{Sp}(1)$-invariant components are given by

$$\Psi_{[3]} = \Psi^{+++}, \quad \Psi_{[-1]} = \Psi^{+--} + \Psi^{-+-} + \Psi^{--+}.$$

They are the projections on the eigenspaces of the Casimir operator $\Upsilon = I_1 \otimes I_1 + I_2 \otimes I_2 + I_3 \otimes I_3$, corresponding, respectively, to the eigenvalues 3 and -1, see [18]. Note here that each of the three 2-forms ω_s belongs to the $[-1]$-component and constitute a basis of the Lie algebra $\mathrm{sp}(1)$.

If $n = 1$ then the space of symmetric endomorphisms commuting with all I_α is 1-dimensional, i.e., the [3]-component of any symmetric endomorphism Ψ on $\mathcal{D}$ is proportional to the identity, $\Psi_{[3]} = -\frac{tr\Psi}{4}\mathrm{id}_{|H}$.

A.2. **The torsion tensor.** The torsion endomorphism $T_\xi = T(\xi,\cdot) : H \to H, \quad \xi \in V$ will be decomposed into its symmetric part T^0_ξ and skew-symmetric part $b_\xi, T_\xi = T^0_\xi + b_\xi$. Biquard showed in [17] that the torsion T_ξ is completely trace-free, $tr\, T_\xi = tr\, T_\xi \circ I_\alpha = 0$, its symmetric part has the properties

$$T^0_{\xi_\alpha} I_\alpha = -I_\alpha T^0_{\xi_\alpha}, \quad I_2(T^0_{\xi_2})^{+--} = I_1(T^0_{\xi_1})^{-+-},$$
$$I_3(T^0_{\xi_3})^{-+-} = I_2(T^0_{\xi_2})^{--+}, \quad I_1(T^0_{\xi_1})^{--+} = I_3(T^0_{\xi_3})^{+--}.$$

The skew-symmetric part can be represented as $b_{\xi_\alpha} = I_\alpha U$, where U is a traceless symmetric (1,1)-tensor on $\mathcal{D}$ which commutes with I_1, I_2, I_3. Therefore we have $T_{\xi_\alpha} = T^0_{\xi_\alpha} + I_\alpha U$. When $n = 1$ the tensor U vanishes identically, $U = 0$, and the torsion is a symmetric tensor, $T_\xi = T^0_\xi$. The two Sp(n)Sp(1)-invariant trace-free symmetric 2-tensors on $\mathcal{D}$

$$T^0(X,Y) = g((T^0_{\xi_1} I_1 + T^0_{\xi_2} I_2 + T^0_{\xi_3} I_3)X, Y) \quad \text{and} \quad U(X,Y) = g(uX,Y) \tag{33}$$

were introduced in [31] and enjoy the properties

$$\begin{aligned} T^0(X,Y) + T^0(I_1X, I_1Y) + T^0(I_2X, I_2Y) + T^0(I_3X, I_3Y) = 0, \\ U(X,Y) = U(I_1X, I_1Y) = U(I_2X, I_2Y) = U(I_3X, I_3Y). \end{aligned} \tag{34}$$

From [33, Proposition 2.3] we have

$$4T^0(\xi_\alpha, X, Y) = -T^0(I_\alpha X, Y) - T^0(X, I_\alpha Y), \tag{35}$$

hence, taking into account (35) it follows

$$T(\xi_\alpha, X, Y) = -\frac{1}{4}\Big[T^0(I_\alpha X, Y) + T^0(X, I_\alpha Y)\Big] + U(I_\alpha X, Y). \tag{36}$$

Any 3-Sasakian manifold has zero torsion endomorphism, and the converse is true if in addition the qc-Einstein curvature (see (37)) is a positive constant [31].

A.3. **Torsion and curvature.** Let $R = [\nabla, \nabla] - \nabla_{[\ ,\ ]}$ be the curvature tensor of ∇ and the dimension is $4n + 3$. We denote the curvature tensor of type (0,4) and the torsion tensor of type (0,3) by the same letter, $R(A,B,C,D) := g(R(A,B)C,D), \quad T(A,B,C) := g(T(A,B),C)$, $A,B,C,D \in \Gamma(TM)$. The *qc-Ricci tensor Ric*, *normalized qc-scalar curvature* S, *qc-Ricci forms* ρ_α of the Biquard connection are defined, respectively, by the following formulas

$$\begin{aligned} &\mathrm{Ric}(A,B) = \sum_{i=1}^{4n} R(X_i, A, B, X_i), \quad 4n\rho_\alpha(A,B) = \sum_{b=1}^{4n} R(A,B,X_i,I_\alpha X_i), \\ &8n(n+2)S = \sum_{i=1}^{4n} \mathrm{Ric}(X_i,X_i) = \sum_{i,j=1}^{4n} R(X_j,X_i,X_i,X_j), \end{aligned} \tag{37}$$

where $X_1, \dots, X_{4n}$ is an orthonormal basis of $\mathcal{D}$.

A qc structure is said to be qc-Einstein if the horizontal qc-Ricci tensor is a scalar multiple of the metric, $\mathrm{Ric}(X,Y) = 2(n+2)Sg(X,Y)$.

As shown in [31, 28] the qc-Einstein condition is equivalent to the vanishing of the torsion endomorphism of the Biquard connection. In this case S is constant and the vertical distribution is integrable. It is also worth recalling that the horizontal qc-Ricci tensors and the integrability of the vertical distribution can be expressed in terms of the torsion of the Biquard connection [31] (see also [29, 33, 32]). For example, we have

$$\begin{aligned} &\mathrm{Ric}(X,Y) = (2n+2)T^0(X,Y) + (4n+10)U(X,Y) + 2(n+2)Sg(X,Y), \\ &\rho_\alpha(X, I_\alpha Y) \ = \ -\frac{1}{2}\Big[T^0(X,Y) + T^0(I_\alpha X, I_\alpha Y)\Big] - 2U(X,Y) - Sg(X,Y), \\ &T(\xi_\alpha, \xi_\beta) = -S\xi_\tau - [\xi_\alpha, \xi_\beta]_{|H}, \qquad S = -g(T(\xi_1,\xi_2),\xi_3), \\ &g(T(\xi_\alpha,\xi_\beta),X) = -\rho_\tau(I_\alpha X, \xi_\alpha) = -\rho_\tau(I_\beta X, \xi_\beta) = -g([\xi_\alpha,\xi_\beta],X). \end{aligned} \tag{38}$$

Note that for $n = 1$ the above formulas hold with $U = 0$.

A.4. **Bianchi identity.** We shall also need the first Bianchi identity and the general formula for the curvature [31, 33, 34].

The first Bianchi identity for the Biquard connection reads

$$(39)\quad \begin{aligned} b(X,Y,Z,V) &= \sum_{(X,Y,Z)} R(X,Y,Z,V) = \sum_{(X,Y,Z)} \Big\{(\nabla_X T)(Y,Z,V) + T(T(X,Y),Z,V)\Big\} \\ &= \sum_{(X,Y,Z)} T(T(X,Y),Z,V) = 2\sum_{(X,Y,Z)}\sum_{\alpha=1}^{3} g(I_\alpha X,Y)T(\xi_\alpha,Z,V). \end{aligned}$$

where $\sum_{(X,Y,Z)}$ denotes the cyclic sum over $\{X,Y,Z\}$ and we used that $(\nabla_X T)(Y,Z,V) = 0$ for horizontal vectors.

We also have the identities, cf. [33, Theorem 3.1] or [34, Theorem 4.3.11],

$$(40)\quad \begin{aligned} &3R(X,Y,Z,V) - R(I_1X,I_1Y,Z,V) - R(I_2X,I_2Y,Z,V) - R(I_3X,I_3Y,Z,V) \\ &= 2\Big[g(Y,Z)T^0(X,V) + g(X,V)T^0(Z,Y) - g(Z,X)T^0(Y,V) - g(V,Y)T^0(Z,X)\Big] \\ &-2\sum_{\alpha=1}^{3}\Big[\omega_\alpha(Y,Z)T^0(X,I_\alpha V)+\omega_\alpha(X,V)T^0(Z,I_\alpha Y)-\omega_\alpha(Z,X)T^0(Y,I_\alpha V)-\omega_\alpha(V,Y)T^0(Z,I_\alpha X)\Big] \\ &+\sum_{\alpha=1}^{3}\Big[2\omega_\alpha(X,Y)\Big(T^0(Z,I_\alpha V)-T^0(I_\alpha Z,V)\Big)-8\omega_\alpha(Z,V)U(I_\alpha X,Y)-4S\omega_\alpha(X,Y)\omega_\alpha(Z,V)\Big]; \end{aligned}$$

$$(41)\quad \begin{aligned} R(\xi_\alpha,X,Y,Z) &= -(\nabla_X U)(I_\alpha Y,Z) + \omega_\beta(X,Y)\rho_\tau(I_\alpha Z,\xi_\alpha) \\ &- \omega_\tau(X,Y)\rho_\beta(I_\alpha Z,\xi_\alpha) - \frac{1}{4}\Big[(\nabla_Y T^0)(I_\alpha Z,X) + (\nabla_Y T^0)(Z,I_\alpha X)\Big] \\ &+ \frac{1}{4}\Big[(\nabla_Z T^0)(I_\alpha Y,X) + (\nabla_Z T^0)(Y,I_\alpha X)\Big] - \omega_\beta(X,Z)\rho_\tau(I_\alpha Y,\xi_\alpha) \\ &+ \omega_\tau(X,Z)\rho_\beta(I_\alpha Y,\xi_\alpha) - \omega_\beta(Y,Z)\rho_\tau(I_\alpha X,\xi_\alpha) + \omega_\tau(Y,Z)\rho_\beta(I_\alpha X,\xi_\alpha), \end{aligned}$$

where the Ricci two forms are given by

$$(42)\quad \begin{aligned} 6(2n+1)\rho_\alpha(\xi_\alpha,X) &= (2n+1)X(S) + \frac{1}{2}(\nabla_{e_a}T^0)[(e_a,X) - 3(I_\alpha e_a,I_\alpha X)] \\ &\qquad - 2(\nabla_{e_a}U)(e_a,X), \\ 6(2n+1)\rho_\alpha(\xi_\beta,I_\tau X) &= (2n-1)(2n+1)X(S) - \frac{4n+1}{2}(\nabla_{e_a}T^0)(e_a,X) \\ &\qquad - \frac{3}{2}(\nabla_{e_a}T^0)(I_\alpha e_a,I_\alpha X) - 4(n+1)(\nabla_{e_a}U)(e_a,X). \end{aligned}$$

A.5. **Fermi frame.** Here we prove the existence of Fermi frame. We recall the statement

Lemma 24. *Given a geodesic $\gamma(t)$, there exists a $\mathbb{Q}$-orthonormal frame, i.e., a horizontal frame X_i, $i \in \{1,\dots,4n\}$, and vertical frame ξ_α, $\alpha = 1,2,3$ in a neighborhood of $\gamma(0)$, such that for all $\alpha,\beta \in \{1,2,3\}$ and $i,j \in \{1,\dots,4n\}$,*

(i) *the frame is orthonormal for the Riemannian metric $g + \sum_\beta \eta_\beta^2$,*

(ii) $\nabla_{X_i}X_j|_{\gamma(t)} = \nabla_{\xi_\alpha}X_j|_{\gamma(t)} = \nabla_{X_i}\xi_\beta|_{\gamma(t)} = \nabla_{\xi_\alpha}\xi_\beta|_{\gamma(t)} = 0.$

In particular, for all $\alpha,\beta,\tau \in \{1,2,3\}$ and $i,j \in \{1,\dots,4n\}$

$$((\nabla_{X_i}I_\alpha)X_j)|_{\gamma(t)} = ((\nabla_{X_i}I_\alpha)\xi_\beta)|_{\gamma(t)} = ((\nabla_{\xi_\beta}I_\alpha)X_j)|_{\gamma(t)} = ((\nabla_{\xi_\beta}I_\alpha)\xi_\tau)|_{\gamma(t)} = 0.$$

Proof. Since ∇ preserves the splitting $H \oplus V$ we can apply the standard arguments for the existence of a Fermi normal frame along a smooth curve with respect to a metric connection (see e.g., [26]). We sketch the proof for completeness.

Let $\{\tilde{X}_1, \dots, \tilde{X}_{4n}, \tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3\}$ be a $\mathbb{Q}$-orthonormal basis around $p = \gamma(0)$ such that $\tilde{X}_{a|p} = X_a(p)$ and $\tilde{\xi}_{\alpha|p} = \xi_\alpha(p)$. We want to find a modified frame $X_a = o_a^b \tilde{X}_b$ and $\xi_\alpha = o_\alpha^\tau \tilde{\xi}_\tau$, which satisfies the normality conditions along the smooth geodesic $\gamma(t)$.

Let ϖ be the sp$(n)\oplus$sp(1)-valued connection 1-forms with respect to the frame $\tilde{X}_1, \dots, \tilde{X}_{4n}$, and $\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3$, namely

$$\nabla_D \tilde{X}_b = \varpi_b^c(D)\tilde{X}_c, \quad \nabla_D \tilde{\xi}_\alpha = \varpi_\alpha^\tau(D)\tilde{\xi}_\tau, \qquad D \in \Gamma(TM).$$

Consequently, we have

$$\nabla_D X_b = \omega_b^c(D) X_c = [D(o_b^a) + o_b^d \varpi_d^a(D)](o^{-1})_a^c X_c,$$
$$\nabla_D \xi_\beta = \omega_\beta^\tau(D)\xi_\tau = [D(o_\beta^\alpha) + o_\beta^\nu \varpi_\nu^\alpha(D)](o^{-1})_\alpha^\tau X_\tau$$

Since the Biquard connection preserves the splitting $H \oplus V$, the existence of a Fermi normal frame along $\gamma(t)$ is equivalent to the existence of a smooth solution to the system

$$[D(o_b^a) + o_b^d \varpi_d^a(D)]_{|\gamma(t)} = 0, \quad [D(o_\alpha^\beta) + o_\alpha^\tau \varpi_\tau^\beta(D)]_{|\gamma(t)} = 0.$$

A smooth solution to this system on a small neighborhood along $\gamma(t)$ exists, see e.g., [26, Theorem 3.1]. Clearly, the solution o_b^a belongs to Sp(n), $o_b^a \in$ Sp(n) since the connection 1-forms belong to the Lie algebra sp(n) and the solution $o_\alpha^\beta \in$ Sp(1) because the connection 1-forms are in the Lie algebra sp(1). □

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References

[1] A. Agrachev, D. Barilari, and L. Rizzi. Sub-Riemannian Curvature in Contact Geometry. *J. Geom. Anal.*, 27(1):366–408, 2017.

[2] A. Agrachev, D. Barilari, and L. Rizzi. Curvature: a variational approach. *Mem. Amer. Math. Soc.*, 256(1225):v+142, 2018.

[3] A. Agrachev and I. Zelenko. Geometry of Jacobi curves. I. *J. Dynam. Control Systems*, 8(1):93–140, 2002.

[4] A. Agrachev and I. Zelenko. Geometry of Jacobi curves. II. *J. Dynam. Control Systems*, 8(2):167–215, 2002.

[5] A. A. Agrachev. Some open problems. In *Geometric control theory and sub-Riemannian geometry*, volume 5 of *Springer INdAM Ser.*, pages 1–13. Springer, Cham, 2014.

[6] A. A. Agrachev, D. Barilari, and U. Boscain. A Comprehensive Introduction to sub-Riemannian Geometry, Cambridge University Press. 2019.

[7] A. A. Agrachev and P. W. Y. Lee. Generalized Ricci curvature bounds for three dimensional contact subriemannian manifolds. *Math. Ann.*, 360(1-2):209–253, 2014.

[8] A. A. Agrachev and Y. L. Sachkov. *Control theory from the geometric viewpoint*, volume 87 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2004. Control Theory and Optimization, II.

[9] A. A. Agrachev and I. Zelenko. Geometry of Jacobi curves. I. *J. Dynam. Control Systems*, 8(1):93–140, 2002.

[10] D. Barilari and L. Rizzi. Comparison theorems for conjugate points in sub-Riemannian geometry. *ESAIM Control Optim. Calc. Var.*, 22(2):439–472, 2016.

[11] D. Barilari and L. Rizzi. On Jacobi fields and a canonical connection in sub-Riemannian geometry. *Arch. Math. (Brno)*, 53(2):77–92, 2017.

[12] F. Baudoin and N. Garofalo. Curvature-dimension inequalities and Ricci lower bounds for sub-Riemannian manifolds with transverse symmetries. *Journal of the EMS (to appear)*, Jan. 2011.

[13] F. Baudoin, B. Kim, and J. Wang. Transverse Weitzenböck formulas and curvature dimension inequalities on Riemannian foliations with totally geodesic leaves. *Comm. Anal. Geom. (to appear)*, Aug. 2014.

[14] F. Baudoin and J. Wang. Curvature dimension inequalities and subelliptic heat kernel gradient bounds on contact manifolds. *Potential Anal.*, 40(2):163–193, 2014.

[15] F. Baudoin and J. Wang. The subelliptic heat kernels of the quaternionic Hopf fibration. *Potential Anal.*, 41(3):959–982, 2014.

[16] A. Bellaïche. The tangent space in sub-Riemannian geometry. In *Sub-Riemannian geometry*, volume 144 of *Progr. Math.*, pages 1–78. Birkhäuser, Basel, 1996.
[17] O. Biquard. Métriques d'Einstein asymptotiquement symétriques. *Astérisque*, (265):vi+109, 2000.
[18] M. M. Capria and S. M. Salamon. Yang-Mills fields on quaternionic spaces. *Nonlinearity*, 1(4):517–530, 1988.
[19] J. Davidov, S. Ivanov, and I. Minchev. The twistor space of a quaternionic contact manifold. *Quart. J. Math. (Oxford)*, 63(4):873–890, 2012.
[20] D. Duchemin. Quaternionic contact structures in dimension 7. *Ann. Inst. Fourier (Grenoble)*, 56(4):851–885, 2006.
[21] G. Folland. Subelliptic estimates and function spaces on nilpotent lie groups. *Ark. Math.*, 13(2):161–207, 1975.
[22] G. B. Folland and E. M. Stein. Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group. *Comm. Pure Appl. Math.*, 27:429–522, 1974.
[23] N. Garofalo and D. Vassilev. Symmetry properties of positive entire solutions of yamabe type equations on groups of heisenberg type. *Duke Math. J.*, 106(3):411–448, 201.
[24] R. K. Hladky. The topology of quaternionic contact manifolds. *Ann. Global Anal. Geom.*, 47(1):99–115, 2015.
[25] W. K. Hughen. *The sub-Riemannian geometry of three-manifolds*. ProQuest LLC, Ann Arbor, MI, 1995. Thesis (Ph.D.)–Duke University.
[26] B. Z. Iliev. *Handbook of normal frames and coordinates*, volume 42 of *Progress in Mathematical Physics*. Birkhäuser Verlag, Basel, 2006.
[27] S. Ivanov, I. Minchev, and D. Vassilev. The qc yamabe problem on 3-sasakian manifolds and the quaternionic heisenberg group. *arXiv:1504.03142*.
[28] S. Ivanov, I. Minchev, and D. Vassilev. Quaternionic contact einstein manifolds. *to appear in Math. Research Letters (2015); arXiv:1306.0474*.
[29] S. Ivanov, I. Minchev, and D. Vassilev. Extremals for the Sobolev inequality on the seven-dimensional quaternionic Heisenberg group and the quaternionic contact Yamabe problem. *J. Eur. Math. Soc. (JEMS)*, 12(4):1041–1067, 2010.
[30] S. Ivanov, I. Minchev, and D. Vassilev. The optimal constant in the l^2 folland-stein inequality on the quaternionic heisenberg group. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 11(3):635–662, 2012.
[31] S. Ivanov, I. Minchev, and D. Vassilev. Quaternionic contract Einstein structures and the quaternionic contact Yamabe problem. *Mem. Amer. Math. Soc.*, 231(1086):vi+82, 2014.
[32] S. Ivanov, A. Petkov, and D. Vassilev. The sharp lower bound of the first eigenvalue of the sub-Laplacian on a quaternionic contact manifold. *J. Geom. Anal.*, 24(2):756–778, 2014.
[33] S. Ivanov and D. Vassilev. Conformal quaternionic contact curvature and the local sphere theorem. *J. Math. Pures Appl. (9)*, 93(3):277–307, 2010.
[34] S. P. Ivanov and D. N. Vassilev. *Extremals for the Sobolev inequality and the quaternionic contact Yamabe problem*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011.
[35] F. Jean. *Control of nonholonomic systems: from sub-Riemannian geometry to motion planning*. Springer Briefs in Mathematics. Springer, Cham, 2014.
[36] R. Montgomery. *A tour of subriemannian geometries, their geodesics and applications*, volume 91 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.
[37] S. B. Myers. Riemannian manifolds with positive mean curvature. *Duke Math. J.*, 8:401–404, 1941.
[38] S.-i. Ohta. On the measure contraction property of metric measure spaces. *Comment. Math. Helv.*, 82(4):805–828, 2007.
[39] L. Rifford. *Sub-Riemannian geometry and optimal transport*. Springer Briefs in Mathematics. Springer, Cham, 2014.
[40] L. Rizzi and P. Silveira. Sub-riemannian ricci curvatures and universal diameter bounds for 3-sasakian manifolds. *Journal of the Institute of Mathematics of Jussieu*, pages 1–45, 2017.
[41] M. Rumin. Formes différentielles sur les variétés de contact. *J. Differential Geom.*, 39(2):281–330, 1994.
[42] S. Tanno. Killing vectors on contact Riemannian manifolds and fiberings related to the Hopf fibrations. *Tôhoku Math. J. (2)*, 23:313–333, 1971.
[43] W. Wang. The yamabe problem on quaternionic contact manifolds. *Ann. Mat. Pura Appl. (4)*, 186(2):359–380, 2007.
[44] I. Zelenko and C. Li. Differential geometry of curves in Lagrange Grassmannians with given Young diagram. *Differential Geom. Appl.*, 27(6):723–742, 2009.