Two different solution techniques for an optimal control problem with a stochastic switching time

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Abstract: In optimal control theory, strategic decision making requires the consideration of unforeseen disruptions that may arise within a predetermined time horizon. In this context, we introduce the concept of "stochastic switching time" as a random moment in time at which a sudden, irreversible alteration takes place in the system's dynamics or in the payoff function. To address optimal decision-making under such uncertain conditions, the literature presents two prominent methodologies: the "backward" approach and the "heterogeneous" approach. In this study, we offer an exposition and a comparative analysis of these two approaches. Finally, we present an illustrative example to show, in a detailed context, the advantages and disadvantages associated with these two solution strategies.

Key-Words: Optimal control, Regime shifts, Hamilton-Jacobi-Bellman equation, Pontryagin maximum principle.

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1 Introduction

In the context of dynamical systems, we call *stochastic switching time* an event which

- occurs at a random time τ;
- changes abruptly the nature of the system;
- splits the time horizon into two stages: a Stage 1 before the occurrence of τ, and a Stage 2 afterwards.

This framework finds many applications in various areas, such as epidemiology, [1], rational risk, [2], and renewable resources, [3], open source software, [4], to name a few. In the context of optimal control, we are interested to see how an optimal strategy adjusts to the change of the regime, ie how it changes going from Stage 1 to Stage 2, upon the occurrence of τ.

An optimal control problem is a dynamic optimisation problem in which the agent sets the value of the control variable $u(\cdot)$, for every time in a given programming interval $[0, T]$, choosing it from a given set U of feasible controls. Control enters state dynamics, influencing the evolution of the state variable $x(\cdot)$, whose initial value is given. Assuming the existence and uniqueness of the solution of such dynamics, the pair $(u(\cdot), x(\cdot))$ of control and the corresponding state trajectory is called a process, as done in [5]). For simplicity, in this paper, we assume that both the control variable and the state variable are unidimensional.

The planner's objective is to maximise a certain payoff, which is the sum of an intertemporal term and a salvage value. The first is the integral of a profit flow over time, which depends on the strategy and the corresponding state trajectory; the latter is a lump sum which depends on the final state $x(T)$:

$$\max_{u(t)\in U} \left[\int_0^T g\big(t, x(t), u(t)\big)dt + S\big(x(T)\big) \right]$$

subject to:

$$\begin{cases} \dot{x}(t) = f\big(t, x(t), u(t)\big) & \text{for } t \in [0, T] \\ x(0) = x_0 \end{cases}$$

where:

- $g(t, x, u)$ is the running payoff;
- $S(x)$ is the salvage value function;
- $f(t, x, u)$ is the state dynamics.

Two interesting applications of optimal control theory to economic growth models can be found in [6] and in [7]. There are two solution approaches to this problem: *Dynamic programming* and *Pontryagin's maximum principle*, see, [5], for the standard details of these two different techniques.

We are interested in integrating a stochastic switching time into the optimal control framework described in the previous paragraph. We model the stochastic switching time τ as an absolutely continuous random variable taking values in $[0, +\infty)$. Due to the finiteness of the time horizon, τ could occur during the programming interval $[0, T]$ (splitting it into a Stage 1 and a Stage 2, as in Figure 1) or after T (leaving the whole interval in Stage 1, as depicted in Figure 2).

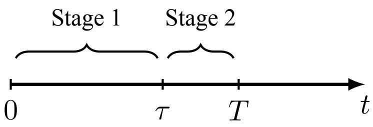

Figure 1: Case $\tau < T$

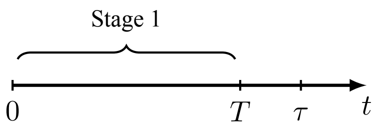

Figure 2: Case $\tau \geq T$

We describe the probability distribution of τ through a quantity called the *hazard rate* of τ at time t:

$$\lim_{h \to 0^+} \frac{\mathbb{P}(\tau \leq t + h \,|\, \tau > t)}{h} = \eta(t, x(t)) \tag{1}$$

where $\eta(t, x)$ is the hazard rate function. Actually, the hazard rate may be exogenous or endogenous, and in the latter case it may depend both on the state variable and the control variable. Nevertheless, in this paper, we assume it to be endogenous and dependent only on the state of the system.

The switch may have one or more simultaneous effects on the system, such as a change in the state dynamics, in the running payoff, in the salvage value function, and in the control set. Moreover, it may induce a jump discontinuity in the state trajectory such that

$$x(\tau^+) = \phi\big(\tau, x(\tau)\big).$$

See the following table for a schematic representation (and the respective notation) of the effects of the switch. Observe that if the state is assumed to be continuous in τ, then $\phi(\tau, x) = x$.

	Stage 1	**Switch**	**Stage 2**
Dynamics	$f_1(t, x, u)$		$f_2(\tau, t, x, u)$
Payoff	$g_1(t, x, u)$		$g_2(\tau, t, x, u)$
Salvage	$S_1(x)$		$S_2(\tau, x)$
Control	U_1		U_2
Jump		$\phi(\tau, x)$	

Because the planner does not know when τ will occur, it is necessary to plan a Stage 1 strategy for the entire time horizon, with associated process

$$(u_1(t),\ \ x_1(t)), \quad \text{for } t \in [0, T]$$

If the switching time occurs during the programming interval, the planner realises that τ has occurred and when. Therefore, in the Stage 2 interval $[\tau, T]$ they will be able to implement the optimal strategy for any specific occurrence of τ. The Stage 2 process hence depends on two variables: the realisation $s \in [0, T]$ of the switching time τ, and the time t in the Stage 2 interval $[s, T]$:

$$(u_2(s, t),\ \ x_2(s, t)), \quad \text{for } s \in [0, T],\ t \in [s, T]$$

Due to the stochasticity of τ, the planner can only aim at maximising the expectation of the total payoff, which takes different forms depending on the fact that τ occurs during the programming interval or afterwards. The switching time optimal control problem is:

$$\begin{aligned} \max_{\substack{u_1(t) \in U_1 \\ u_2(s,t) \in U_2}} \mathbb{E}\Bigg[& \mathbf{1}_{\{\tau < T\}} \Big\{ \int_0^\tau g_1(t, x_1(t), u_1(t)) dt \\ & + \int_\tau^T g_2(\tau, t, x_2(\tau, t), u_2(\tau, t)) dt \\ & + S_2(\tau, x_2(\tau, T)) \Big\} + \\ & \mathbf{1}_{\{\tau \geq T\}} \Big\{ \int_0^T g_1\big(t, x_1(t), u_1(t)\big) dt \\ & + S_1\big(x_1(T)\big) \Big\} \Bigg] \end{aligned} \tag{2}$$

subject to:

$$\begin{cases} \dot{x}_1(t) = f_1(t, x_1(t), u_1(t)), & t \in [0, T] \\ x_1(0) = x_0 \\ \dot{x}_2(s, t) = f_2(s, t, x_2(s, t), u_2(s, t)), & t \in [s, T] \\ x_2(s, s) = \phi\big(s, x_1(s)\big) \\ \text{Hazard rate of } \tau \text{ at time } t\text{: } \eta\big(t, x_1(t)\big) \end{cases}$$

We observe that the dynamics is deterministic and is parametrically fixed for every possible realisation s of the random variable τ. Therefore, the decision maker fixes both the control $u_1(\cdot)$ and the control $u_2(s, \cdot)$, which are parametric in s. With abuse of notation in the formulas above, we set: $\dot{x}_2(s, t) = \partial_t x_2(s, t)$. We will use the same notation in the rest of the paper.

It should be noted that the initial condition of the Stage 2 problem is a function of the Stage 1 variables at the switch: $x_2(s, s) = \phi\big(s, x_1(s)\big)$.

This implies that the Stage 2 problem cannot be solved independently, *unless* one applies dynamic programming, where an optimal control problem is solved for every possible initial value. We compute the expectation using the auxiliary Stage 1 state variable $z_1(t) := \mathbb{P}(\tau > t)$, which is the probability of still being in Stage 1 at time t. To view it as a state variable, we write its dynamics and initial value:

$$\begin{cases} \dot{z}_1(t) = -\eta\big(t, x_1(t)\big) z_1(t) \\ z_1(0) = 1 \end{cases}$$

where the dynamics is derived from the definition of hazard rate (1). Then, the probability density of τ at time t is:

$$f_\tau(t) = -\dot{z}_1(t) = \eta\big(t, x_1(t)\big) z_1(t). \tag{3}$$

Both expressions in (3) are used to compute the expectation in (2). After basic integral manipulations, the resulting objective is:

$$\begin{aligned} \max_{\substack{u_1(t)\in U_1 \\ u_2(s,t)\in U_2}} \Bigg[\int_0^T & z_1(t) \Big\{ g_1\big(t, x_1(t), u_1(t)\big) \\ & + \eta(t, x_1(t)) \Big[S_2\big(t, x_2(t,T)\big) \\ & + \int_t^T g_2(t, \theta, x_2(t,\theta), u_2(t,\theta)) d\theta \Big] \Big\} dt \\ & + z_1(T) S_1\big(x_1(T)\big) \Bigg] \end{aligned} \tag{4}$$

subject to:

$$\begin{cases} \dot{x}_1(t) = f_1\big(t, x_1(t), u_1(t)\big) \\ x_1(0) = x_0 \\ \dot{z}_1(t) = -\eta\big(t, x_1(t)\big) z_1(t) \\ z_1(0) = 1 \\ \dot{x}_2(s,t) = f_2\big(s, t, x_2(s,t), u_2(s,t)\big) \\ x_2(s,s) = \phi\big(s, x_1(s)\big) \end{cases}$$

where we updated the Stage 1 dynamics with the new variable z_1.

2 Solution approaches

There are two possible ways of solving the two-stage optimal control problem defined above: the *backward* approach and the *heterogeneous* one.

The backward approach is based on dynamic programming: it involves solving the Stage 2 problem for every possible occurrence s of the switch and for every possible initial state, and then plugging the Stage 2 value function into the Stage 1 problem, which is solved as a simple optimal control problem, assuming optimal behavior in Stage 2. The two stages are solved separately (in reverse order) at the cost of computing the Stage 2 value function for every possible value of x_2 at the switch, instead of just the value that it will take from the condition $x_2(s,s) = \phi\big(s, x_1(s)\big)$. For more details on this approach, see e.g., [5].

The heterogeneous approach is based on Pontryagin's Maximum Principle (see, [5]): one derives necessary conditions for the solution by calculating the co-state functions for both stages, as solutions of the corresponding adjoint system, and letting the optimal strategies satisfy the resulting maximality conditions. The two stages are necessarily solved together, because x_1 and u_1 enter the initial conditions of Stage 2, and (as we will see) the Stage 2 co-states enter the Stage 1 adjoint equations.

In what follows, we will determine the optimal control of the switching time problem, applying the two approaches and comparing their results.

2.1 Backward approach

First, let us solve the Stage 2 problem with dynamic programming. Since, in general, the Stage 2 data may depend on the realization s of the switching time τ, the Stage 2 value function V_2 will depend on s as well:

$$\begin{aligned} & V_2(s,t,\mathtt{x}) := \\ & \sup_{u(\theta)\in U_2} \left[\int_t^T g_2(s,\theta,x(\theta),u(\theta)) d\theta + S_2\big(s, x(T)\big) \right] \end{aligned} \tag{5}$$

subject to:

$$\begin{cases} \dot{x}(\theta) = f_2\big(s, \theta, x(\theta), u(\theta)\big) & \text{for } \theta \in [t,T] \\ x(t) = \mathtt{x} \end{cases}$$

If $V_2(s,\cdot,\cdot)$ is differentiable, then it is the solution of the corresponding system of HJB equation and terminal condition (parametrized by s):

$$\begin{cases} -\partial_t V_2(s,t,x) = & \max_{\mathtt{u}\in U_2} \big\{ g_2(s,t,x,\mathtt{u}) \\ & +\partial_x V_2(s,t,x) \cdot f_2(s,t,x,\mathtt{u}) \big\} \\ V_2(s,T,x) = & S_2(s,x) \end{cases} \tag{6}$$

feedback The optimal feedback strategy $\Phi_2(s,t,x)$ maximizes the RHS of the HJB equation in (6):

$$\begin{aligned} \Phi_2(s,t,x) \in \arg\max_{\mathtt{u}\in U_2} & \big\{ g_2(s,t,x,\mathtt{u}) \\ & + \partial_x V_2(s,t,x) \cdot f_2(s,t,x,\mathtt{u}) \big\}. \end{aligned}$$

Let (u_1, x_1) be a feasible Stage 1 process. Let the trajectory $t \mapsto x_2(s,t)$ satisfy the Cauchy problem

$$\begin{cases} \dot{x}_2(s,t) = f_2\big(s,t,x_2(s,t),\Phi_2(s,t,x_2(s,t))\big) \\ x_2(s,s) = \phi\big(s,x_1(s)\big), \end{cases}$$

defined for $t \in [s,T]$; then, the optimal control for Stage 2, given (u_1, x_1), is

$$u_2(s,t) = \Phi_2\big(s,t,x_2(s,t)\big).$$

By Bellman's Principle of Optimality, given (u_1, x_1) and assuming (u_2, x_2) as above, we can write:

$$\int_t^T g_2(s,\theta,x_2(s,\theta),u_2(s,\theta))d\theta + S_2\big(s,x_2(s,T)\big) = V_2\big(s,t,x_2(s,t)\big).$$

In particular, for $s = t$, we can substitute $x_2(t,t) = \phi\big(t,x_1(t)\big)$, yielding:

$$V_2\big(t,t,x_2(t,t)\big) = V_2\big(t,t,\phi(t,x_1(t))\big). \tag{7}$$

Assuming optimal behavior in Stage 2, in (4) we can substitute the Stage 2 payoff with (7), obtaining the following objective for Stage 1:

$$\max_{u_1(t)\in U_1} \Bigg[\int_0^T z_1(t)\Big\{ g_1\big(t,x_1(t),u_1(t)\big) + \eta\big(t,x_1(t)\big)V_2\big(t,t,\phi(t,x_1(t))\big)\Big\}dt + z_1(T)S_1\big(x_1(T)\big)\Bigg] \tag{8}$$

subject to:

$$\begin{cases} \dot{x}_1(t) = f_1\big(t,x_1(t),u_1(t)\big) \\ x_1(0) = x_0 \\ \dot{z}_1(t) = -\eta\big(t,x_1(t)\big)z_1(t) \\ z_1(0) = 1 \end{cases} \tag{9}$$

This is a simple optimal control problem, which can be solved using either *Dynamic programming* or *Pontryagin's maximum principle*. It is worth observing that the state variable $z_1(t)$ plays the role of a discount factor.

2.2 Heterogeneous approach

By [8], if the suitable regularity assumptions on the data hold and if $(u_1, x_1, z_1, u_2, x_2, z_2)$ constitute an optimal 2-stage process, then there exist the co-state functions $\lambda_x(t)$, $\lambda_z(t)$, and $\xi_x(s,t)$, $\xi_z(s,t)$ such that, once defined the "current" co-state functions

$$\lambda_x^c(t) := \lambda_x(t)/z_1(t), \quad \xi_x^c(s,t) := \xi_x(s,t)/z_2(s,t),$$

the following conditions hold:

Maximality condition for Stage 1

$$u_1(t) \in \arg\max_{\mathrm{u}\in U_1} \Big\{ g_1\big(t,x_1(t),\mathrm{u}\big) + \lambda_x^c(t)\cdot f_1\big(t,x_1(t),\mathrm{u}\big)\Big\}$$

Maximality condition for Stage 2

$$u_2(s,t) \in \arg\max_{\mathrm{u}\in U_2} \Big\{ g_2\big(s,t,x_2(s,t),\mathrm{u}\big) + \xi_x^c(s,t)\cdot f_2\big(s,t,x_2(s,t),\mathrm{u}\big)\Big\}$$

The co-state functions are solutions of the following adjoint system:

$$\begin{cases} -\dot{\lambda}_x^c(t) = & \partial_x g_1\big(t,x_1(t),u_1(t)\big) \\ & +\lambda_x^c(t)\partial_x f_1\big(t,x_1(t),u_1(t)\big) \\ & +[\xi_x^c(t,t)\partial_x\phi(t,x_1(t)) - \lambda_x^c(t)]\eta(t,x_1(t)) \\ & +[\xi_z(t,t) - \lambda_z(t)]\partial_x\eta(t,x_1(t)) \\ \lambda_x^c(T) = & \partial_x S_1\big(x_1(T)\big) \\ -\dot{\xi}_x^c(s,t) = & \partial_x g_2\big(s,t,x_2(s,t),u_2(s,t)\big) \\ & +\xi_x^c(s,t)\partial_x f_2\big(s,t,x_2(s,t),u_2(s,t)\big) \\ \xi_x^c(s,T) = & \partial_x S_2\big(s,x_2(s,T)\big) \\ -\dot{\lambda}_z(t) = & g_1\big(t,x_1(t),u_1(t)\big) + \\ & +\big[\xi_z(t,t) - \lambda_z(t)\big]\eta\big(t,x_1(t)\big) \\ \lambda_z(T) = & S_1\big(x_1(T)\big) \\ -\dot{\xi}_z(s,t) = & g_2\big(s,t,x_2(s,t),u_2(s,t)\big) \\ \xi_z(s,T) = & S_2\big(s,x_2(s,T)\big) \end{cases}$$

Observe that if $\eta \equiv 0$ (that is, without switching time) the Stage 1 current co-state λ_x^c coincides with the co-state of a single stage optimal control problem. On the other hand, when η does not vanish, the two additional terms:

$$+[\xi_x^c(t,t)\partial_x\phi(t,x_1(t)) - \lambda_x^c(t)]\eta(t,x_1(t))$$

and

$$+[\xi_z(t,t) - \lambda_z(t)]\partial_x\eta(t,x_1(t))$$

represent the anticipating effect on the Stage 1 shadow value of the state variable. Observe that also in the heterogeneous approach, variables $z_1(t)$ and $z_2(t)$ play the role of discount factors necessary to obtain the "current" co-state functions from the regular ones.

The heterogeneous approach can also be useful in addressing optimal control problems with age-structured dynamics. These models are well described in the literature, [5], and allow for the study of interesting practical problems. A very recent example of such an application can be found in [9]. Moreover, in [10], the heterogeneous approach is used to characterise the necessary conditions for an age-structured optimal control problem.

3 Numerical example

In the following numerical example, we compare the two techniques presented in the previous sections to solve a specific switching time optimal control problem. Let us set the data of the problem:

- $T = 1, \quad x_0 = 1$
- $f_1(t, x, u) = u,$
  $g_1(t, x, u) = -u,$
  $S_1(x) = 0, \quad U_1 = [0, 1]$
- $\eta(t, x) = x$
- $\phi(s, x) = x$
- $f_2(s, t, x, u) = 0,$
  $g_2(s, t, x, u) = \alpha - u, \alpha \in R$
  $S_2(s, x) = 0 \quad U_2 = [0, +\infty)$

3.1 Backward approach

The maximisation in (6) gives the Stage 2 optimal control $u_2^*(s, t) \equiv 0$, therefore the HJB system becomes

$$\begin{cases} -\partial_t V_2(s, t, x) = \alpha, \\ V_2(s, 1, x) = 0 \end{cases}$$

that solved gives the value function $V_2(s, t, x) = \alpha(1-t)$. Using this solution, we can write the optimal control problem defined in (8) and (9):

$$\max_{u_1 \in [0,1]} \int_0^1 z_1(t) \left\{-u_1(t) + \alpha x_1(t)(1 - t)\right\} dt$$

subject to

$$\begin{cases} \dot{x}_1(t) = u_1(t) \\ x_1(0) = 1 \\ \dot{z}_1(t) = -x_1(t)z_1(t) \\ z_1(0) = 1 \end{cases}$$

We now apply Pontryagin's Maximum Principle, [5] to this standard optimal control problem with associated Hamiltonian function

$$H = z_1 \left\{-u_1 + \alpha x_1(1 - t)\right\} + p_x u_1 - p_z x_1 z_1.$$

The optimal control in feedback form is

$$u_1^* = \mathbf{1}_{\{p_x - z_1\}},$$

while the co-state equations are

$$\begin{cases} \dot{x}_1(t) = \mathbf{1}_{\{p_x(t) - z_1(t)\}}(t) \\ x_1(0) = 1 \\ \dot{z}_1(t) = -x_1(t)z_1(t) \\ z_1(0) = 1 \\ \dot{p}_x(t) = \left(-\alpha(1 - t) + p_z(t)\right) z_1(t) \\ p_x(1) = 0 \\ \dot{p}_z(t) = \mathbf{1}_{\{p_x(t) - z_1(t)\}}(t) - (\alpha(1 - t) - p_z(t))x_1(t) \\ p_z(1) = 0 \end{cases} \tag{10}$$

This system of ODEs can be solved using only a numerical procedure.

3.2 Heterogeneus approach

Using the heterogeneous approach we can write the two Maximality conditions:
For Stage 1 we get

$$u_1^*(t) \in \arg \max_{u \in [0,1]} \left\{-u + \lambda_x^c(t) u\right\}$$

hence

$$u_1^*(t) = \mathbf{1}_{\{\lambda_x^c(t) - 1\}}(t),$$

while for Stage 2 we obtain

$$u_2^*(s, t) \in \arg \max_{u \in [0,+\infty)} \left\{\alpha - u\right\}$$

therefore $u_2^*(s, t) \equiv 0$.
The adjoint system is

$$\begin{cases} \dot{x}_1(t) = \mathbf{1}_{\{\lambda_x^c(t) - 1\}}(t) \\ x_1(0) = 1 \\ \dot{\lambda}_x^c(t) = (-\alpha(t - 1) + \lambda_z(t)) + \lambda_x^c(t)x_1(t) \\ \lambda_x^c(1) = 0 \\ \dot{\lambda}_z(t) = \mathbf{1}_{\{\lambda_x^c(t) - 1\}}(t) - (\alpha(t - 1) - \lambda_z(t))x_1(t) \\ \lambda_z(1) = 0 \end{cases} \tag{11}$$

and this problem can only be solved using a numerical procedure.
Even if these two approaches seem to give a different solution, it is simple to go from (10) to (11) observing that $z_1(t)$ is always strictly positive and

$$\frac{d}{dt} \frac{p_x(t)}{z_1(t)} = (-\alpha(1 - t) + p_z(t)) + \frac{p_x(t)}{z_1(t)} x_1(t)$$

which is exactly the co-state equation for $\lambda_x^c(t)$.

4 Conclusion

Comparing the two approaches to solve a stochastic switching time optimal control problem, we can underline the respective pros and cons.

The backward approach offers the advantage of deriving the optimal strategy in feedback form, which is very handy if the planner has access to the value of the state variable at all times. However, this comes at the cost of having to compute $V_2(s,t,x)$ for every (s,t,x), which is generally not an easy task. Moreover, the computation of a value function suffers from the curse of dimensionality.

The heterogeneous approach, on the contrary, allows one to derive the strategy only in the open-loop form. On the other hand, it allows for a compact and unified representation of the necessary conditions where the interaction between the two stages is made explicit.

Both approaches require a mathematical formulation that involves a complex notation. To facilitate comprehension, numerical examples can certainly be helpful. In Section 3, we provide a simple example that shows the equivalence of these two approaches in terms of ODEs. In future research, it would be useful to study additional examples that elucidate the details of this intricate relationship.

Finally, we notice that, even starting from a very simple example, the complexity of the system of ODEs obtained by the necessary conditions immediately requires a numerical procedure to find an explicit solution.

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