# GENERALIZED HEEGNER CYCLES ON MUMFORD CURVES 

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#### Abstract

We study generalised Heegner cycles, originally introduced by Bertolini-DarmonPrasanna for modular curves in BDP13], in the context of Mumford curves. The main result of this paper relates generalized Heegner cycles with the two variable anticyclotomic $p$-adic $L$-function attached to a Coleman family $f_{\infty}$ and an imaginary quadratic field $K$, constructed in BD07 and Sev14. While in BD07 and Sev14 only the restriction to the central critical line of this 2 variable $p$-adic $L$-function is considered, our generalised Heegner cycles allow us to study the restriction of this function to non-central critical lines. The main result expresses the derivative along the weight variable of this anticyclotomic $p$-adic $L$-function restricted to non necesserely central critical lines as a combination of the image of generalized Heegner cycles under a p-adic Abel-Jacobi map. In studying generalised Heegner cycles in the context of Mumford curves, we also obtain an extension of a result of Masdeu Mas12 for the (one variable) anticyclotomic $p$-adic $L$-function of a modular form $f$ and $K$ at non-central critical integers.


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## 1. Introduction

Generalized Heegner cycles have been introduced by Bertolini-Darmon-Prasanna in BDP13] with the aim of studying certain anticyclotomic $p$-adic $L$-functions of modular forms of level $\Gamma_{1}(N)$, where $p \nmid N$ is a prime number, twisted by Hecke characters of an imaginary quadratic field $K / \mathbb{Q}$ in which all primes dividing $N$ are split. These cycles are defined by means of the cohomology of the motive ( $\mathcal{E}^{n} \times E^{n}, \epsilon$ ), where $\mathcal{E}^{n}$ is a smooth compactification of the $n$-fold product of the universal elliptic curve $\mathcal{E} \rightarrow X_{1}(N), E$ is an auxiliary elliptic curve with CM by $\mathcal{O}_{K}$ and $\epsilon$ is a suitable projector in the ring of rational correspondences on $\mathcal{E}^{n} \times E^{n}$. The work BDP13 has been generalised by Brooks HB15 to the case when $X_{1}(N)$ is replaced by a Shimura curve, and therefore $\mathcal{E}$ and $E$ are replaced by a universal false elliptic curve $\mathcal{A}$ and a false elliptic curve $A$ with CM by $\mathcal{O}_{K}$; in HB15, $N$ is assumed to be prime to $p$ as in BDP13, and one allows factorisations of $N$ into coprime integers $N=N^{+} \cdot N^{-}$where all primes dividing $N^{+}$are split in $K$, all primes dividing $N^{-}$are inert in $K$, and $N^{-}$is the square-free product of an even number of distinct prime factors.

Along a different direction Masdeu in Mas12 has defined generalized Heegner cycles for Mumford curves; in this setting we fix a modular form $f$ of weight $k$ and level $\Gamma_{0}(N)$, an imaginary quadratic field $K$ and a factorisation $N=p \cdot N^{+} \cdot N^{-}$into coprime factors so that $p$ is a prime number, all primes dividing $N^{+}$are split in $K$, all primes dividing $p N^{-}$are inert
in $K$, and $N^{-}$is the square-free product of an odd number of distinct prime factors. The fiber at $p$ of Shimura curves attached to quaternion algebras $\mathcal{B}$ of discriminant $p N^{-}$can be described by means of Mumford curves, i.e. quotients $\Gamma \backslash \mathcal{H}_{p}$ of the $p$-adic upper half plane $\mathcal{H}_{p}=\mathbb{C}_{p}-\mathbb{Q}_{p}$ by an arithmetic group $\Gamma \subseteq B^{\times}$, where $B$ is the definite quaternion algebra of discriminant $N^{-}$obtained from $\mathcal{B}$ by interchanging the invariants $\infty$ and $p$. In this case, generalized Heegner cycles are constructed by means of the cohomology of $\left(\mathcal{A}^{\frac{n}{2}} \times E^{n}, \epsilon_{M}\right)$, where $\mathcal{A}$ is the universal false elliptic curve as in HB15, $E$ is a fixed elliptic curve with CM by $\mathcal{O}_{K}$ as in [BDP13, $n:=k-2$ and $\epsilon_{M}$ is a suitable projector on $\mathcal{A}^{\frac{n}{2}} \times E^{n}$. The main result of Mas12] expresses the derivative of the anticyclotomic $p$-adic $L$-function attached to $f$ and $K$ at integers $j$ in the critical strip $1 \leq j \leq k-1$ as linear combinations of the images of generalised Heegner cycles via the $p$-adic Abel-Jacobi map, evaluated at suitable differential forms. The main tools used in [Mas12] is the analysis by Iovita-Spiess [IS03] of the realisations (étale and de Rham) of the motive $\left(\mathcal{A}^{\frac{n}{2}}, \epsilon_{\mathcal{A}}\right)$.

This paper continues the work initiated by Mas12 in the context of Mumford curves, but instead of the motive $\left(\mathcal{A}^{\frac{n}{2}} \times E^{n}, \epsilon_{M}\right)$ considered in Mas12 we study the motive $\left(\mathcal{A}^{\frac{n}{2}} \times A^{\frac{n}{2}}, \epsilon\right)$ with $\mathcal{A}$ a universal false elliptic curve over a Shimura curve, $A$ a fixed false elliptic curve with CM by $\mathcal{O}_{K}$ and $\epsilon$ a projector in $\operatorname{Corr}_{X}\left(\mathcal{A}^{\frac{n}{2}} \times A^{\frac{n}{2}}\right)$. Here the setting is the same as in Mas12: we fix a modular form $f$ of level $\Gamma_{0}(N)$, an imaginary quadratic field $K$ and a factorisation $N=p \cdot N^{+} \cdot N^{-}$into coprime factors so that $p$ is a prime number, all primes dividing $N^{+}$ are split in $K$, all primes dividing $p N^{-}$are inert in $K$, and $N^{-}$is the square-free product of an odd number of distinct prime factors. It turns out that our motive seems to be more flexible than the motive considered in Mas12, and more natural because both the universal abelian variety and the fixed abelian variety with CM are false elliptic curves. In this context we define generalised Heegner cycles, and we study them using techniques from [IS03] and BDP13.

Our main results investigate the relation between generalised Heegner cycles and anticyclotomic $p$-adic $L$-functions, especially in the context of $p$-adic variation of modular forms. Fix an imaginary quadratic field $K$ and a modular form $f$ of weight $k_{0}$ and level $\Gamma_{0}(N)$, with $N=p N^{+} N^{-}$as above, having finite slope at $p$. Let $f_{\infty}$ be the Coleman family of modular forms passing through $f$. In the ordinary case with $k_{0}=2$, Bertolini and Darmon introduced in BD07] a $p$-adic $L$-function in the weight-variable $k$ interpolating special values at central critical points of anticyclotomic $L$-functions of newforms $f_{k}^{\sharp}$ whose $p$-stabilisations are the classical specialisations $f_{k}$ of $f_{\infty}$. In particular, this $p$-adic $L$-function is non-zero and vanishes at $k=2$. When $f$ corresponds to an elliptic curve, the main result of [BD07] expresses the first derivative along the weight variable $k$ of this anticyclotomic $p$-adic $L$-function valued at the point $k=2$ as linear combination of Heegner points. This results has been extended by Seveso Sev14 in the finite slope case and $k_{0} \geq 2$ by expressing the first derivative along the weight variable $k$ of this anticyclotomic $p$-adic $L$-function at $k=k_{0}$ as linear combination of Heegner cycles.

The $p$-adic $L$-functions studied in BD 07 and Sev 14 are restriction to the central critical line $s=k / 2$ of a $p$-adic $L$-function $\mathcal{L}_{p}(s, k)$ in two $p$-adic variables $s$ and $k$; in light of the results of Mas12, it is then natural to investigate the restriction $\mathcal{L}_{p}^{(j)}(k)=\mathcal{L}_{p}(k, k+j)$ of these $p$-adic $L$-functions along directions $s=k / 2+j$, with $-k / 2<j<k / 2$ an integer in the critical strip. In the spirit of [BD07] and [Sev14], for each $j$ such that $\mathcal{L}_{p}^{(j)}\left(k_{0}\right)=0$, we show that the derivative $\frac{d}{d k} \mathcal{L}_{p}^{(j)}(k)$ at $k=k_{0}$ can be expressed as linear combinations of our generalised Heegner cycles.

We now state our main result in a more precise form. Let $f \in S_{k_{0}}\left(\Gamma_{0}(N)\right)$ be a newform of weight $k_{0}$ and level $\Gamma_{0}(N), K$ an imaginary quadratic field, $N=p \cdot N^{+} \cdot N^{-}$a factorisation of $N$ into coprime integers such that $p$ is a prime, all prime factors dividing $N^{+}$(respectively,
$p N^{-}$) are split (respectively, inert) in $K$, and $N^{-}$is a square-free product of an odd number of primes. Let $\mathcal{B} / \mathbb{Q}$ be the indefinite quaternion algebra of discriminant $p N^{-}$, and $\Gamma \subseteq B^{\times}$ the arithmetic subgroup corresponding to the choice of an Eichler order of level $N^{+}$in the definite quaternion algebra $B / \mathbb{Q}$ of discriminant $N^{-}$. Let $X$ be the Shimura curve of level $\Gamma$. After choosing an auxiliary prime integer $M \geq 5$ prime to $N$ and a $\Gamma_{1}(M)$-level structure $\Gamma_{M} \subseteq \Gamma$, consider the Shimura curve $X_{M} \rightarrow X$ of level $\Gamma_{M}$ and the universal false elliptic curve $\mathcal{A} \rightarrow X_{M}$. Fix a false elliptic curve $A_{0}$ with CM by $\mathcal{O}_{K}$. For any isogeny $\varphi: A_{0} \rightarrow A$ we construct a generalised Heegner cycle $\Delta_{\varphi}$ in the Chow group $\mathrm{CH}^{n_{0}+1}(\mathcal{D})$ of the Chow motive $\mathcal{D}:=\left(\mathcal{A}^{\frac{n_{0}}{2}} \times A_{0}^{\frac{n_{0}}{2}}, \epsilon\right)$, where $n_{0}=k_{0}-2$. For any positive even integer $k$, let $M_{k}(\Gamma)$ be the $\mathbb{C}_{p}$-vector space of rigid analytic quaternionic modular forms of weight $k$ and level $\Gamma$; elements of $M_{k}(\Gamma)$ are functions from $\mathcal{H}_{p}=\mathbb{C}_{p}-\mathbb{Q}_{p}$ to $\mathbb{C}_{p}$ which transform under the action of $\Gamma$ by the automorphic factor of weight $k$. In particular, the Jacquet-Langlands correspondence allows us to see $f$ as an element of $M_{k_{0}}(\Gamma)$. Let $V_{n_{0}}$ denote the dual of the $\mathbb{C}_{p}$-vector space $\mathcal{P}_{n_{0}}$ of polynomials in one variable of degree at most $n_{0}$. We construct a $p$-adic Abel-Jacobi map

$$
\mathrm{AJ}_{p}: \mathrm{CH}^{n_{0}+1}(\mathcal{D}) \longrightarrow\left(M_{k_{0}}(\Gamma) \otimes V_{n_{0}}\right)^{\vee}
$$

where the target denotes the $\mathbb{C}_{p}$-linear dual of $M_{k_{0}}(\Gamma) \otimes V_{n_{0}}$. On the other hand, denote

$$
\mathcal{W}=\operatorname{Hom}_{\text {cont }}\left(\mathbb{Z}_{p}^{\times}, \mathbb{Q}_{p}^{\times}\right)
$$

the weight space, and view $\mathbb{Z} \subseteq \mathcal{W}$ by the map $k \mapsto\left[x \mapsto x^{k-2}\right]$. For any integer $j$ with $-k_{0} / 2<j<k_{0} / 2$, we construct a function $k \mapsto \mathcal{L}_{p}^{(j)}(k)$ defined in a sufficiently small connected neighborhood of $U$ of $k_{0} \in \mathcal{W}$. When $j \equiv 0(\bmod p+1), \mathcal{L}_{p}^{(j)}(k)$ coincides with the restriction to the line $s=k / 2+j$ of the two variable $p$-adic $L$-function of [BD07, Sev14]; thus in particular the value of this function at $j=0$ correspond to the one variable $p$-adic $L$ function studied in BD07, [Sev14. The notation used below to denote this function is more involved, but in the introduction we prefer to keep the notational complexity at minimum stating our main result, Theorem [1.1, in the case when the class number of $K$ is equal to 1 : see Definitions 6.6 and 6.8 for the complete notation, keeping in mind that if the class number of $K$ is 1 then the two functions in Definitions 6.6 and 6.8 are the same, and $\chi$ in loc. cit. is trivial. Thus, our main result, for which as remarked above we assume that the class number of $K$ is one to simplify the statement, is the following:

Theorem 1.1. For integers $-k_{0} / 2<j<k_{0} / 2$ with $j \equiv 0(\bmod p+1)$ we have

$$
\mathcal{L}_{p}^{(j)}\left(k_{0}\right)=0
$$

and there exists an isogeny $\varphi: A_{0} \rightarrow A$ and are elements $v_{\varphi}^{(j)}$ and $\bar{v}_{\varphi}^{(j)}$ in $V_{n_{0}}$ such that we have

$$
\left(\frac{d}{d k} \mathcal{L}_{p}^{(j)}(k)\right)_{\mid k=k_{0}}=c_{\varphi}\left(\operatorname{AJ}_{p}\left(\Delta_{\varphi}\right)\left(f \otimes v_{\varphi}^{(j)}\right)+\omega_{p} \operatorname{AJ}_{p}\left(\Delta_{\bar{\varphi}}\right)\left(f \otimes \bar{v}_{\varphi}^{(j)}\right)\right)
$$

In the theorem above, $c_{\varphi} \in \overline{\mathbb{Q}}_{p}^{\times}$is an explicit constant which only depends on $\varphi, \omega_{p} \in\{ \pm 1\}$ is the eigenvalue of the Atkin-Lehner involution acting on $f$, and if $\varphi: A_{0} \rightarrow A$ is an isogeny, we denote by $\bar{\varphi}: A_{0} \rightarrow \bar{A}$ the isogeny obtained by $\varphi$ composing with the generator of $\operatorname{Gal}(K / \mathbb{Q})$ (recall that $A$ is defined over $K$ by the theory of complex multiplication, under the assumption that $K$ has class number one). This result is a special case of Theorem 6.9 below, which also considers twists by certain anticyclotomic characters of $K$, and holds for arbitrary class number of $K$.

In studying our generalized Heegner cycles, we also obtain a second result similar in spirit to that of [Mas12], expressing the first derivative of the anticyclotomic $p$-adic $L$-function attached to $f$ and $K$ at integers $j$ in the critical strip $1 \leq j \leq k_{0}-1$ as linear combinations of our generalized Heegner cycles, valued at suitable differential forms; although the result is
similar in spirit to that of Mas12, it has a different shape, due to the different motives used, and furthermore generalises that of Mas12 to certain anticyclotomic characters. We state a simplified version (again for trivial characters and class number of $K$ equal to 1 ) of this result, referring to Theorem 6.13 and the comments following it for the notation.

Theorem 1.2. Let $L_{p}(f / K, s)$ be the anticyclotomic p-adic L-function attached to $f$ and $K$ and $-k_{0} / 2<j<k_{0} / 2$ an integer with $j \equiv 0(\bmod p+1)$. For each such $j$ we have $L_{p}\left(f / K, k_{0} / 2+j\right)=0$ and there exists an isogeny $\varphi: A_{0} \rightarrow A$ such that

$$
L_{p}^{\prime}(f / K, k / 2+j)=d_{\varphi} \cdot\left(\operatorname{AJ}_{p}\left(\Delta_{\varphi}\right)\left(f \otimes v_{\varphi}^{(j)}\right)-\omega_{p} \cdot \mathrm{AJ}_{p}\left(\Delta_{\bar{\varphi}}\right)\left(f \otimes \bar{v}_{\varphi}^{(j)}\right)\right)
$$

Here $d_{\varphi} \in \overline{\mathbb{Q}}_{p}^{\times}$is an explicit constant which only depends on $\varphi$. This result is a special case of Theorem 6.13 below, which, as for Theorem 1.1, also considers twists by certain anticyclotomic characters of $K$, and holds for arbitrary class number of $K$.

## 2. Shimura curves

In this section we collect come preliminaries on Shimura curves which will be needed in this paper. We fix an integer $N$ with a coprime factorization $N=p N^{+} N^{-}$such that $p \nmid N^{+} N^{-}$ is a prime number, and $N^{-}$is a square free product of an odd number of primes factors.
2.1. Shimura curves. Let $\mathcal{B}$ be the indefinite quaternion algebra over $\mathbb{Q}$ of discriminant $p N^{-}$. Fix a maximal order $\mathcal{R}^{\max }$ in $\mathcal{B}$ and an Eichler order $\mathcal{R}$ of level $N^{+}$contained in $\mathcal{R}^{\text {max }}$. The Shimura curve $X=X_{N^{+}, p N^{-}} / \mathbb{Q}$ is the coarse moduli scheme representing the functor which takes a $\mathbb{Q}$-scheme $S$ to isomorphism classes of abelian surfaces with quaternonic multiplication by $\mathcal{R}^{\max }$ and level $N^{+}$-structure, i.e. triples $(A, \iota, C)$ where
(1) $A$ is an abelian surface over a $\mathbb{Q}$-scheme $S$;
(2) $\iota: \mathcal{R}^{\max } \rightarrow \operatorname{End}_{S}(A)$ is an inclusion defining an $\mathcal{R}^{\text {max }}$-module structure on $A / S$;
(3) $C \subset A$ is a subsgroup scheme locally isomorphic to $\mathbb{Z} / N^{+} \mathbb{Z}$, stable and locally cyclic under the action of $\mathcal{R}$.
The scheme $X$ is a smooth, projective and geometrically connected curve over $\mathbb{Q}$. A triple $(A, \iota, C)$ is called a false elliptic curve with level $N^{+}$-structure, and the abelian surface $A$ is called a false elliptic curve. An isogeny $\varphi: A \rightarrow A^{\prime}$ of false elliptic curves is said to be a false isogeny if it commutes with the action of $\mathcal{R}^{\max }$.

The fiber at $p$ of $X$ is a Mumford curve, as we will review now. Let $\mathcal{H}_{p}$ denote the rigid analytic space over $\mathbb{Q}_{p}$ whose points over field extensions $L / \mathbb{Q}_{p}$ are given by $\mathcal{H}_{p}(L)=L-\mathbb{Q}_{p}$ (see for example [Dar04, Chapter 5] or DT08, Section 1], where the rigid analytic structure of $\mathcal{H}_{p}$ is also carefully described). Let $B / \mathbb{Q}$ be the definite quaternion algebra over $\mathbb{Q}$ of discriminant $N^{-}$and let $R$ be an Eichler $\mathbb{Z}\left[\frac{1}{p}\right]$-order of level $N^{+}$in $B$. By fixing an isomorphism $\iota_{p}: B_{p} \rightarrow \mathrm{M}_{2}\left(\mathbb{Q}_{p}\right)$ the group $\Gamma$ of elements of reduced norm 1 in $R$ can be identified with a discrete subgroup of $\mathrm{SL}_{2}\left(\mathbb{Q}_{p}\right)$. We let $\mathrm{SL}_{2}\left(\mathbb{Q}_{p}\right)$ act on $\mathcal{H}_{p}(L)$, for each field extension $L / \mathbb{Q}_{p}$, by fractional linear transformations $z \mapsto \frac{a z+b}{c z+d}$ for $\left(\begin{array}{cc}a & b \\ c & d\end{array}\right) \in \mathrm{SL}_{2}\left(\mathbb{Q}_{p}\right)$ and $z \in \mathcal{H}_{p}(L)$. We may then form the quotient $X_{\Gamma, \mathbb{Q}_{p}}=\Gamma \backslash \mathcal{H}_{p}$ and for any field extension $F / \mathbb{Q}_{p}$, its base change $X_{\Gamma, F}=X_{\Gamma, \mathbb{Q}_{p}} \otimes_{\mathbb{Q}_{p}} F$, which, when $F$ contains $\mathbb{Q}_{p^{2}}$, is a Mumford curve defined over $F$ (in general, it is a twist of a Mumford curve by a quadratic character). The Cerednik-Drinfeld theorem states the existence of an isomorphism

$$
\begin{equation*}
X_{\mathbb{Q}_{p^{2}}} \simeq X_{\Gamma, \mathbb{Q}_{p^{2}}} \tag{1}
\end{equation*}
$$

of algebraic curves defined over $\mathbb{Q}_{p^{2}}$. See [JL85, Section 4], BD96, Theorem 1.3] or [BC91, Chapitre III] for details. We put $X_{\Gamma}=X_{\Gamma, \hat{\mathbb{Q}}_{p}^{u n r}}$ to simplify the notation, where $\hat{\mathbb{Q}}_{p}^{\text {unr }}$ is the completion of the maximal unramified extension $\mathbb{Q}_{p}^{\text {unr }}$ of $\mathbb{Q}_{p}$.
2.2. An auxiliary fine moduli problem. Fix $M \geq 3$ an integer relatively prime to $N$. Let $X_{M}$ be the fine moduli scheme representing abelian surfaces with quaternionic multiplication by $\mathcal{R}^{\max }$, level $N^{+}$-structure and full level $M$-structure over $\mathbb{Q}$-schemes, i.e. quadruples $(A, \iota, C, \bar{\nu})$ where
(1) $(A, \iota, C)$ is an abelian surface with quaternionic multiplication by $\mathcal{R}^{\max }$ and level $N^{+}$-structure over a $\mathbb{Q}$-scheme $S$;
(2) $\bar{\nu}:\left(\mathcal{R}^{\max } / M \mathcal{R}^{\max }\right)_{S} \rightarrow A[M]$ is a $\mathcal{R}^{\max }$-equivariant isomorphism from the constant group scheme $\left(\mathcal{R}^{\max } / M \mathcal{R}^{\max }\right)$ to the group scheme of $M$-division points of $A$.
Quadruplets $(A, \iota, C, \bar{\nu})$ are called false elliptic curves with level $\left(N^{+}, \nu\right)$-structure. The scheme $X_{M}$ is a smooth projective curve over $\mathbb{Q}$ which is not geometrically connected. The morphism $X_{M} \rightarrow X$ given by forgetting the level $M$-structure is a Galois covering with Galois group isomorphic to $G_{M} /\{ \pm 1\}$, where

$$
G_{M}:=\left(\mathcal{R}^{\max } / M \mathcal{R}^{\max }\right)^{\times}
$$

We denote $\mathcal{A} \rightarrow X_{M}$ the universal abelian surface.
Over $\hat{\mathbb{Q}}_{p}^{\mathrm{unr}}$, the curve $X_{M}$ decomposes as a disjoint union of Mumford curves

$$
\begin{equation*}
X_{M, \hat{\mathbb{Q}}_{p}^{\mathrm{unr}}}=X_{M} \otimes_{\mathbb{Q}} \hat{\mathbb{Q}}_{p}^{\mathrm{unr}} \simeq \coprod_{(\mathbb{Z} / M \mathbb{Z})^{\times}} \Gamma_{M} \backslash \mathcal{H}_{p}^{\mathrm{unr}} \tag{2}
\end{equation*}
$$

for a suitable congruence subgroup $\Gamma_{M} \subset \mathrm{SL}_{2}\left(\mathbb{Q}_{p}\right)$, where we write $\mathcal{H}_{p}^{\text {unr }}=\mathcal{H}_{p} \otimes_{\mathbb{Q}_{p}} \hat{\mathbb{Q}}_{p}^{\text {unr }}$ (this isomorphism can be realised over any extension of $\mathbb{Q}_{p^{2}}$ containing all $\varphi(M)$-roots of unity, where $\left.\varphi(M)=\sharp(\mathbb{Z} / M \mathbb{Z})^{\times}\right)$. See [IS03, Section 5] for more details.
2.3. Modular forms. We introduce in this subsection two definitions of modular forms on quaternion algebras.

Definition 2.1. Let $F$ be a field of characteristic zero and $k \geq 2$ an even integer. A $F$-valued modular form of weight $k$ on $X$ is a global section of the sheaf $\left(\Omega_{X_{F} / F}^{1}\right)^{\otimes \frac{k}{2}}$. We denote the space of these modular forms by $M_{k}(X, F)$.

The Jacquet-Langlands correspondence implies the existence of an isomorphism of $K$-vector spaces

$$
M_{k}(X, F) \simeq S_{k}\left(\Gamma_{0}(N), F\right)^{p N^{-}-\text {new }}
$$

where the right hand side denotes the $F$-vector space of $F$-valued cusp forms of weight $k$ and level $\Gamma_{0}(N)$ which are new at the primes dividing $p N^{-}$. This isomorphism is compatible with the action of the Hecke operators and Atkin-Lehner involutions, defined on both sides. For details, see BD96, Theorem 1.2].

Definition 2.2. Let $F$ be a field of characteristic zero and $k \geq 2$ an even integer. A $p$-adic modular form of weight $k$ for $\Gamma$ defined over $F$ is a rigid analytic function $f$ on $\mathcal{H}_{p}$ defined over $F$ satisfying the rule

$$
f(\gamma z)=(c z+d)^{k} f(z) \quad \text { for all } \gamma=\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right) \in \Gamma \text { and } z \in \mathcal{H}_{p}\left(\mathbb{C}_{p}\right)=\mathbb{C}_{p}-\mathbb{Q}_{p}
$$

The space of these $p$-adic modular forms will be denoted by $M_{k}(\Gamma, F)$ and for $F=\hat{\mathbb{Q}}_{p}^{u n r}$ we set $M_{k}(\Gamma)=M_{k}\left(\Gamma, \hat{\mathbb{Q}}_{p}^{\mathrm{unr}}\right)$.

Using the Cerednik-Drinfeld isomorphism (11), one easily shows that the map $f \mapsto f(z) d z^{\otimes \frac{k}{2}}$ establishes and isomorphism between $M_{k}(\Gamma, F)$ and $M_{k}(X, F)$ for all fields $F$ containing $\mathbb{Q}_{p^{2}}$.

## 3. The generalised Kuga-Sato motive

Let $N=p N^{+} N^{-}$be fixed as in 92 Let $k \geq 4$ be an even integer and put $n=k-2$ and $m=n / 2$. Fix a quadratic imaginary field $K$ satisfying the following assumption: all primes dividing $N^{+}$(respectively, $p N^{-}$) are split (respectively, inert) in $K$.
3.1. Definition. We begin by recalling some generalities on Chow motives, following mainly [IS03, §5]. Let $F$ be a field of characteristic zero, and $S$ a smooth quasi-projective connected variety over $F$. We denote $\mathcal{M}(S)$ the category of effective relative Chow motives over $S$ with respect to graded correspondences ([DM91, §1.3], [Sch94, Sec. 1]). We will only use motives of the form $(X, \epsilon)=(X, \epsilon, 0)$ where $X$ is a smooth projective $S$-scheme and $\epsilon \in \operatorname{Corr}_{S}^{0}(X, X)$ is a projector (i.e. $\epsilon \circ \epsilon=\epsilon$ ) in the ring of correspondences on $X$ of degree 0 (see [Sch94, §1.2, §1.3] for details). If $S=\operatorname{Spec}(F)$, write $\mathcal{M}(F)=\mathcal{M}(\operatorname{Spec}(F))$. Denote $R_{p}: \mathcal{M}(S) \rightarrow D^{b}\left(S, \mathbb{Q}_{p}\right)$ the $p$-adic realisation functor to the bounded derived category $D^{b}\left(S, \mathbb{Q}_{p}\right)$ of $\mathbb{Q}_{p}$-sheaves over $S$ ([DM91, §1.8]), thus for $\mathcal{M}=(X, \epsilon) \in \mathcal{M}(S), R_{p}(\mathcal{M})$ denotes the $p$-adic realization of $\mathcal{M}$ as a motive over $S$. We can also consider $\mathcal{M}$ as a Chow motive over $F$ by applying the canonical functor $\mathcal{M}(S) \rightarrow \mathcal{M}(F)$, and if $\mathcal{M}=(X, \epsilon)$ for an abelian scheme $\pi: X \rightarrow S$, the $p$-adic realization $H_{p}(\mathcal{M})$ of $\mathcal{M}$ as a motive over $F$ is given by

$$
H_{p}^{r}(\mathcal{M})=H^{r}\left(\bar{S}, R_{p}(\mathcal{M})\right)=\epsilon_{*} \cdot \bigoplus_{i+j=r} H^{i}\left(\bar{S}, R^{j} \pi_{*} \mathbb{Q}_{p}\right)
$$

where $\bar{S}=S \otimes_{F} \bar{F}$ (see Bes95, Proposition 5.9] for the argument).
We denote

$$
\begin{equation*}
c \ell_{\mathcal{M}}^{(i)}: \mathrm{CH}^{i}(\mathcal{M}) \longrightarrow H_{p}^{2 i}(\mathcal{M}) \tag{3}
\end{equation*}
$$

the cycle class map $([\boxed{I S 03},(40)])$, whose kernel is denoted $\mathrm{CH}(\mathcal{M})_{0}$ (this map will not be used until Section 4, but we prefer to introduce it here to collect all notations concerning Chow motives; the same applies to (4) and (5) below).

Let $F$ be an unramified field extension of $\mathbb{Q}_{p}$. For a semistable representation of $G_{F}=$ $\operatorname{Gal}(\bar{F} / F)$, let $D_{\text {st }, F}$ denote the semistable Dieudonné functor over $F$ (see [IS03, §2]); so if $V$ is a semistable representation of $G_{F}$, then $D_{\mathrm{st}, F}(V)$ is a filtered Frobenius monodromy module over $F$ (see [IS03, §2]); the category of such objects is denoted $\mathrm{MF}_{F}^{\phi, N}$, and for an object $D$ in this category we denote $F^{\bullet}(D)$ its filtration. For an object $D$ in $\mathrm{MF}_{F}^{\phi, N}$, define

$$
\begin{equation*}
\Gamma(D)=\operatorname{Hom}_{\mathrm{MF}_{F}^{\phi, N}}(F, D)=\operatorname{Ext}_{\mathrm{MF}_{F}^{\phi, N}}^{0}(F, D)=F^{0}(D) \cap D^{\phi=\mathrm{id}, N=0} \tag{4}
\end{equation*}
$$

Here $\operatorname{Hom}_{\mathrm{MF}_{F}^{\phi, N}}(\cdot, \cdot)$ denotes homomorphisms in the category $\mathrm{MF}_{F}^{\phi, N}, \phi$ is the Frobenius morphism, id is the identity morphism, and $N$ is the monodromy operator of the object $D$. In particular, if the $p$-adic realization $H_{p}(\mathcal{M})$ of $\mathcal{M}$ is semistable, then the cycle class map $c \ell_{\mathcal{M}}^{(i)}$ takes the form ([IS03, (47)])

$$
\begin{equation*}
c \ell_{\mathcal{M}}^{(i)}: \mathrm{CH}^{i}(\mathcal{M}) \longrightarrow \Gamma\left(D_{\text {st }, F}\left(H_{p}^{2 i}(\mathcal{M})(F)\right)\right) \tag{5}
\end{equation*}
$$

Let $A_{0}$ be a fixed abelian surface with quaternionic multiplication and full level- $M$ structure, defined over $H$ (the Hilbert class field of $K$ ) and with complex multiplication by $\mathcal{O}_{K}$; the action of $\mathcal{O}_{K}$ is required to commute with the quaternionic action, and this implies that $A_{0}$ is isogenous to $E \times E$ for an elliptic curve $E$ with CM by $\mathcal{O}_{K}$. Fix a field $F \supset H$ and consider the $(2 n+1)$-dimensional variety $Y_{m}$ over $F$ given by

$$
Y_{m}:=\mathcal{A}^{m} \times A_{0}^{m}
$$

Here $\mathcal{A}^{m}$ is the $m$-fold fiber product of $\mathcal{A}$ over $X_{M}$. The variety $Y_{m}$ is equipped with a proper morphism $\pi: Y_{m} \rightarrow X_{M}$ with $2 n$-dimensional fibers: the fibers above points $x$ of $X_{M}$ are products of the form $A_{x}^{m} \times A_{0}^{m}$, where $A_{x}$ is the fiber of $\mathcal{A} \rightarrow X_{M}$ at $x$.

Denote $\epsilon_{\mathcal{A}}$ the projector in [IS03, Appendix 10.1]; this is an idempotent in the ring of correspondences $\operatorname{Corr}_{X_{M}}\left(\mathcal{A}^{m}, \mathcal{A}^{m}\right)$. The projector $\epsilon_{\mathcal{A}}$ defines then a projector $\epsilon_{A_{0}}$. One can then define the motive

$$
\mathcal{D}_{M}:=\left(Y_{m}, \epsilon_{M}\right)
$$

defined over $F$, where $\epsilon_{M}=\left(\epsilon_{\mathcal{A}}, \epsilon_{A_{0}}\right)$. In the previous notation, $\mathcal{D}_{M} \in \mathcal{M}\left(X_{M}\right)$.
We now descent $\mathcal{D}_{M}$ to a motive over the Shimura curve $X$. Observe that the group

$$
G_{M}:=\left(\mathcal{R}^{\max } / M \mathcal{R}^{\max }\right)^{\times} \simeq \mathrm{GL}_{2}(\mathbb{Z} / M \mathbb{Z})
$$

acts as $X$-automorphism on $X_{M}$ and $\mathcal{A}^{m}$. It follows that the element $p_{G}:=\frac{1}{\left|G_{M}\right|} \sum_{g \in G_{M}} g$ can be seen as a projector in $\operatorname{Corr}_{X}\left(Y_{m}, Y_{m}\right)$, which acts trivially on $A_{0}^{m}$. Since it commutes with $\epsilon_{M}$ (viewed as projector in $\operatorname{Corr}_{X}\left(Y_{m}, Y_{m}\right)$ ), their product $\epsilon=p_{G} \cdot \epsilon_{M}$ is a projector, and we can define a new motive $\mathcal{D}$ over $X$, the generalised Kuga-Sato motive, as

$$
\mathcal{D}:=\left(Y_{m}, \epsilon\right) .
$$

In the previous notation, $\mathcal{D} \in \mathcal{M}(X)$. We also denote $\mathcal{M}_{M}=\left(\mathcal{A}^{m}, \epsilon_{\mathcal{A}}\right)$ the motive in $\mathcal{M}\left(X_{M}\right)$ considered in [IS03], and we write $\mathcal{M}_{A_{0}}=\left(A_{0}^{m}, \epsilon_{A_{0}}\right)$, also in $\mathcal{M}\left(X_{M}\right)$; then $\mathcal{D}_{M}=\mathcal{M}_{M} \otimes \mathcal{M}_{A_{0}}$. Moreover, if we write $\mathcal{M}=\left(\mathcal{A}^{m}, p_{G} \cdot \epsilon_{\mathcal{A}}\right)$ then we have

$$
\begin{equation*}
\mathcal{D}=\mathcal{M} \otimes \mathcal{M}_{A_{0}} \tag{6}
\end{equation*}
$$

viewing $\mathcal{M}_{A_{0}}$ as a motive over $X$ (recall that the tensor product on the category of Chow motives is induced by the fiber product DM91, page 203]). Finally, note that $H_{p}^{2 n+1}(\mathcal{D})$ is equipped with a structure of $G_{F}=\operatorname{Gal}(\bar{F} / F)$-representation.
3.2. The étale realization. We consider now the sheaf $\mathbb{L}_{n}$ over $X_{M}$ introduced in IS03, Section 5], which is defined as follows. First, define $\mathbb{L}_{2}$ as the intersection of the kernels of the maps $b-\operatorname{Nr}(b): R^{2} \pi_{*} \mathbb{Q}_{p} \rightarrow R^{2} \pi_{*} \mathbb{Q}_{p}$, as $b$ varies in $\mathcal{B}$, where Nr denote the reduced norm map; next, for any integer $n>2$, consider the non-degenerate pairing $R^{2} \pi_{*} \mathbb{Q}_{p} \otimes R^{2} \pi_{*} \mathbb{Q}_{p} \rightarrow \mathbb{Q}_{p}(-2)$ given by cup product and the Laplace operator $\Delta_{m}: \operatorname{Sym}^{m}\left(\mathbb{L}_{2}\right) \rightarrow\left(\operatorname{Sym}^{m-2}\left(\mathbb{L}_{2}\right)\right)(-2)$ associated with this pairing, and define $\mathbb{L}_{n}$ to be the kernel of $\Delta_{m}$.

Let $x_{A_{0}}$ be the closed point of $X_{M}$ corresponding to the abelian surface $A_{0}$ and $\bar{x}_{A_{0}}=$ $x_{A_{0}} \otimes_{F} \overline{\mathbb{Q}}$.

Proposition 3.1. The $p$-adic realization $H_{p}(\mathcal{D})$ of $\mathcal{D}$ is different from zero in degree $2 n+1$ only, and we have

$$
H_{p}^{2 n+1}(\mathcal{D})=H^{1}\left(\bar{X}_{M}, \mathbb{L}_{n}\right)^{G_{M}} \otimes\left(\mathbb{L}_{n}\right)_{\bar{x}_{A_{0}}} .
$$

Proof. The $p$-adic realization $R_{p}\left(\mathcal{M}_{M}\right)$ of the motive $\mathcal{M}_{M}$ over $X_{M}$ is $\mathbb{L}_{n}[-n]$ ([IS03, (71)]); by [IS03, Lemma 10.1] the $p$-adic realization $H_{p}(\mathcal{M})$ of $\mathcal{M}$ is concentrated in degree $n+1$ and we have

$$
H_{p}^{n+1}(\mathcal{M}) \simeq H^{1}\left(\bar{X}_{M}, \mathbb{L}_{n}\right)^{G_{M}}
$$

On the other hand, the $p$-adic realization $R_{p}\left(\mathcal{M}_{A_{0}}\right)$ of the motive $\mathcal{M}_{A_{0}}$ over $X_{M}$ is the fiber at $x_{A_{0}}$ of $\left.R_{p}\left(\mathcal{M}_{M}\right)=\mathbb{L}_{n}[-n](\underline{\underline{I S 03}},(71)]\right)$; therefore, $H_{p}\left(\mathcal{M}_{A_{0}}\right)=H^{*}\left(\bar{X}_{M},\left(\mathbb{L}_{n}[-n]\right)_{x_{A_{0}}}\right)$. Since $H^{i}\left(\bar{X}_{M},\left(\mathbb{L}_{n}\right)_{x_{A_{0}}}\right)=0$ for $i \neq 0$, we see that $H_{p}^{n}\left(\mathcal{M}_{A_{0}}\right)=\left(\mathbb{L}_{n}\right) \bar{x}_{A_{0}}$ and $H_{p}^{i}\left(\mathcal{M}_{A_{0}}\right)=0$ for $i \neq n$. The Kunneth formula ([DM91, §1.8]) implies the result.

Remark 3.2. Considered as a $G_{\widehat{\mathbb{Q}}_{p}^{u n r}}$-representation $H_{p}(\mathcal{D})$ is semistable since the category of semistable representations is an abelian tensor category.
3.3. The de Rham realisation. Let $V_{n}:=\mathcal{P}_{n}^{\vee}$ be the dual of the vector space of polynomials of degree $\leq n$ with coefficients in $\mathbb{Q}_{p}$ equipped with the left $\mathrm{GL}_{2}$-action given by

$$
(A \cdot R)(P(X))=R(P(X) \cdot A)
$$

for all $A=\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)$, where the right action of $A$ on a polynomial $P(X) \in \mathcal{P}_{n}$ is via the formula $P(X) \cdot A=(c X+d)^{n} P\left(\frac{a X+b}{c X+d}\right)$. The $\mathbb{Q}_{p}$-vector space $V_{n}$ is equipped with a symmetric bilinear form

$$
\begin{equation*}
\langle,\rangle_{V_{n}}: V_{n} \otimes V_{n} \longrightarrow \operatorname{det}^{\otimes n} \tag{7}
\end{equation*}
$$

in $\operatorname{Rep}_{\mathbb{Q}_{p}}\left(\mathrm{GL}_{2}\right)$, the category of $\mathbb{Q}_{p}$-representations of $\mathrm{GL}_{2}$, defined as follows. First, we define $\langle,\rangle_{V_{2}}$ for $n=2$. Let $\operatorname{ad}^{0}=\left\{U \in \mathrm{M}_{2} \mid \operatorname{trace}(U)=0\right\}$, where trace $: \mathrm{M}_{2}\left(\mathbb{Q}_{p}\right) \rightarrow \mathbb{Q}_{p}$ is the trace map; $\mathrm{ad}^{0}$ is equipped with a right $\mathrm{GL}_{2}$-action by $U \cdot A=\bar{A} \cdot U \cdot A$ for $U \in \operatorname{ad}^{0}$ and $A \in \mathrm{GL}_{2}$, where for $A=\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)$, we put $\bar{A}=\left(\begin{array}{cc}d & -b \\ -c & a\end{array}\right)$. The map ad ${ }^{0} \rightarrow \mathcal{P}_{2}$ which takes $U$ to

$$
P_{U}(X)=\operatorname{trace}\left(U \cdot\left(\begin{array}{cc}
X & -X^{2}  \tag{8}\\
1 & -X
\end{array}\right)\right)
$$

is an isomorphism of right $\mathrm{GL}_{2}$-modules. For $P_{U_{1}}, P_{U_{2}} \in \mathcal{P}_{2}$, we define a pairing on $\mathcal{P}_{2}$ by $\left\langle P_{U_{1}}, P_{U_{2}}\right\rangle_{\mathcal{P}_{2}}=\left\langle U_{1}, U_{2}\right\rangle_{V_{2}}=-\operatorname{trace}\left(U_{1} \cdot \bar{U}_{2}\right)$. This defines the pairing on $V_{2}$ by duality. More generally, we define a pairing $\langle,\rangle_{V_{n}}$ on $\operatorname{Sym}^{n / 2}\left(\operatorname{ad}^{0}\right)$ by the formula

$$
\left\langle u_{1} \cdots u_{n / 2}, v_{1} \cdots v_{n / 2}\right\rangle_{V_{n}}=\frac{1}{(n / 2)!} \sum_{\sigma \in \Sigma_{n / 2}}\left\langle u_{1}, v_{\sigma(1)}\right\rangle_{V_{2}} \cdot\left\langle u_{n}, v_{\sigma(n / 2)}\right\rangle_{V_{2}}
$$

The map $\operatorname{Sym}^{n / 2}\left(\operatorname{ad}^{0}\right) \rightarrow \mathcal{P}_{n}$ induced by $U \mapsto P_{U}(X)$ gives by duality a map $V_{n} \rightarrow \operatorname{Sym}^{n / 2}\left(\operatorname{ad}^{0}\right)$, and we obtain a pairing on $V_{n}$, which we also denote $\langle,\rangle_{V_{n}}$, from that on $\operatorname{Sym}^{n / 2}\left(\operatorname{ad}^{0}\right)$.

We consider the $\mathrm{GL}_{2} \times \mathrm{GL}_{2}$-representation $V_{n}\{m\}=V_{n} \odot \operatorname{det}^{\otimes m}$ (see [IS03, page 345] for the notation) and let $\mathcal{V}_{n}=\mathcal{E}\left(V_{n}\{m\}\right)$ denote the filtered $F$-isocrystal on $\hat{\mathcal{H}}_{p}^{\text {unr }}$ associated with $V_{n}\{m\}$; here $\hat{\mathcal{H}}_{p}$ is the formal model of $\mathcal{H}_{p}$ over $\mathbb{Z}_{p}$, and $\hat{\mathcal{H}}_{p}^{\text {unr }}$ its base change to $\hat{\mathbb{Z}}_{p}^{\text {unr }}$, the valuation ring of $\hat{\mathbb{Q}}_{p}^{\mathrm{unr}}$. See [IS03, page 346] and [Mas12, page 1024] for more details on this definition. Define the sheaf of $\mathcal{O}_{X_{\Gamma}}$-modules $\mathcal{V}_{n, n}=\mathcal{V}_{n} \otimes\left(\mathcal{V}_{n}\right)_{z_{A_{0}}}$ where $z_{A_{0}}$ is a point in $\mathcal{H}_{p}\left(\mathbb{Q}_{p^{2}}\right)$ such that $\Gamma z_{A_{0}}$ corresponds to the abelian surface $A_{0}$. Then

$$
\begin{equation*}
H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n, n}\right)=H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n}\right) \otimes_{\hat{\mathbb{Q}}_{p}^{\mathrm{unr}}}\left(\mathcal{V}_{n}\right)_{z_{A_{0}}} \tag{9}
\end{equation*}
$$

The vector space $H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n, n}\right)$ has a stucture of filtered Frobenius monodromy module in $\mathrm{MF}_{\mathbb{\mathbb { Q }}_{p}^{\text {unr }}}^{\phi, N}$.

Proposition 3.3. $D_{\text {st, } \hat{\mathbb{Q}}_{p}^{\text {unr }}}\left(H_{p}^{2 n+1}(\mathcal{D})\right) \simeq H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n, n}\right)$ as filtered Frobenius monodromy modules in $\mathrm{MF}_{\mathbb{\mathbb { Q }}_{p}^{\text {unr }}}^{\phi, N}$.

Proof. By [IS03, Theorem 5.9] we have

$$
D_{\mathrm{st}, \hat{\mathbb{Q}}_{p}^{\mathrm{unr}}}\left(H_{p}^{n+1}(\mathcal{M})\right) \simeq D_{\text {st }, \hat{\mathbb{Q}}_{p}^{\text {unr }}}\left(H^{1}\left(\bar{X}_{M}, \mathbb{L}_{n}\right)^{G_{M}}\right) \simeq H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n}\right)
$$

and, by [IS03, Remark 5.14] we have $D_{\text {st, } \hat{\mathbb{Q}}_{p}^{\text {unr }}}\left(\left(\mathbb{L}_{n}\right)_{\bar{x}_{A_{0}}}\right) \simeq\left(\mathcal{V}_{n}\right)_{z_{A_{0}}}$. The result follows from Proposition (3.1, equation (9) and the compatibility of the functor $D_{\text {st }}$ with tensor products (see [BC, pg. 145]).

We now describe of $D_{\text {st, } \hat{\mathbb{Q}}_{p}^{u n r}}\left(H_{p}^{2 n+1}(\mathcal{D})\right)$ as filtered Frobenius monodromy module.

We begin with the filtration. For $i=0, \ldots, n$ and $z \in \mathcal{H}_{p}\left(\hat{\mathbb{Q}}_{p}^{\text {unr }}\right)$, define $\partial^{i} \in\left(\mathcal{V}_{n}\right)_{z} \simeq V_{n}$ by

$$
\partial^{i}(P(X))=\left(\frac{d^{i}}{d X^{i}} P(X)\right)_{X=z}
$$

for $P(X) \in \mathcal{P}_{n}$. Let $\mathbb{Q}_{p} \cdot \partial^{i}$ be the $\mathbb{Q}_{p}$-subspace of $V_{n}$ generated by $\partial^{i}$. The $i$-th step of the filtration of $V_{n}$ is given by

$$
F^{i}\left(V_{n}\right)=\left\{\begin{array}{l}
V_{n} \text { if } i \leq 0 \\
\sum_{j=0}^{n-i} \mathbb{Q}_{p} \cdot \partial^{j} \text { for } 0 \leq i \leq n \\
0 \text { if } i \geq n+1
\end{array}\right.
$$

The $i$-th step of the filtration of $H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n}\right)$ is

$$
F^{i}\left(H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n}\right)\right)=\left\{\begin{array}{l}
H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n}\right) \text { if } i \leq 0  \tag{10}\\
M_{k}(\Gamma) \text { if } 1 \leq i \leq n+1 \\
0 \text { is } i \geq n+2
\end{array}\right.
$$

See [IS03, Proposition 6.1] for proofs. In particular, the isomorphism $M_{k}(\Gamma) \rightarrow F^{n+1}\left(H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n}\right)\right)$ is given by

$$
\begin{equation*}
f \longmapsto \omega_{f}:=f(z) \partial^{0} \otimes d z \tag{11}
\end{equation*}
$$

From (10) and Proposition 3.3 we see that the $(n+1)$-step of the filtration of $D_{\text {st, } \hat{\mathbb{Q}}_{p}^{\text {unr }}}\left(H_{p}^{2 n+1}(\mathcal{D})\right)$ is

$$
\begin{equation*}
F^{n+1}\left(D_{\text {st, }, \hat{\mathbb{Q}}_{p}^{\text {unr }}}\left(H_{p}^{2 n+1}(\mathcal{D})\right)\right)=M_{k}(\Gamma) \otimes\left(\mathcal{V}_{n}\right)_{z_{A_{0}}} \tag{12}
\end{equation*}
$$

We also need an explicit description of the monodromy operator on $D_{\text {st, } \hat{\mathbb{Q}}_{p}^{\text {unr }}}\left(H_{p}^{2 n+1}(\mathcal{D})\right)$. We first describe the monodromy operator on $H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n}\right)$. Let $\mathcal{T}$ denote the Bruhat-Tits tree of $\mathrm{PGL}_{2}\left(\mathbb{Q}_{p}\right)$, and denote $\overrightarrow{\mathcal{E}}$ and $\mathcal{V}$ the set of oriented edges and vertices of $\mathcal{T}$, respectively. If $e=\left(v_{1}, v_{2}\right) \in \overrightarrow{\mathcal{E}}$, we denote by $\bar{e}$ the oriented edge $\left(v_{2}, v_{1}\right)$. Let $C^{0}\left(\left(V_{n}\right)_{\hat{\mathbb{Q}}_{p}^{\text {unr }}}\right)$ be the set of maps $\mathcal{V} \rightarrow\left(V_{n}\right)_{\widehat{\mathbb{Q}}_{p}^{\text {unr }}}$ and $C^{1}\left(\left(V_{n}\right)_{\hat{\mathbb{Q}}_{p}^{u n r}}\right)$ be the set of maps $\overrightarrow{\mathcal{E}} \rightarrow\left(V_{n}\right)_{\widehat{\mathbb{Q}}_{p}^{\text {unr }}}$ such that $f(\bar{e})=$ $-f(e)$ for all $e \in \overrightarrow{\mathcal{E}}$, where $\left(V_{n}\right)_{\widehat{\mathbb{Q}}_{p}^{\text {unr }}}=V_{n} \otimes_{\mathbb{Q}_{p}} \hat{\mathbb{Q}}_{p}^{\text {unr }}$. The group $\Gamma$ acts on $f \in C^{i}\left(\left(V_{n}\right)_{\hat{\mathbb{Q}}_{p}^{\text {unr }}}\right)$ by $\gamma(f)=\gamma \circ f \circ \gamma^{-1}$. Let

$$
\epsilon: C^{1}\left(\left(V_{n}\right)_{\hat{\mathbb{Q}}_{p}^{\text {unr }}}\right)^{\Gamma} \longrightarrow H^{1}\left(\Gamma,\left(V_{n}\right)_{\hat{\mathbb{Q}}_{p}^{\text {unr }}}\right)
$$

be the connecting homomorphism arising from the short exact sequence

$$
\left.0 \longrightarrow\left(V_{n}\right)_{\widehat{\mathbb{Q}}_{p}^{\text {unr }}} \longrightarrow C^{0}\left(\left(V_{n}\right)_{\widehat{\mathbb{Q}}_{p}^{\text {unr }}}\right) \xrightarrow{\delta} C^{1}\left(\left(V_{n}\right)_{\widehat{\mathbb{Q}}_{p}^{\text {unr }}}\right)\right) \longrightarrow 0
$$

where $\delta$ is the homomorphism defined by $\delta(f)(e)=f\left(v_{1}\right)-f\left(v_{2}\right)$ for $e=\left(v_{1}, v_{2}\right)$. The map $\epsilon$ induces the following isomorphism that we also denote by $\epsilon$

$$
\epsilon: C^{1}\left(\left(V_{n}\right)_{\widehat{\mathbb{Q}}_{p}^{\text {unr }}}\right)^{\Gamma} / C^{0}\left(\left(V_{n}\right)_{\hat{\mathbb{Q}}_{p}^{\text {unr }}}\right)^{\Gamma} \longrightarrow H^{1}\left(\Gamma,\left(V_{n}\right)_{\widehat{\mathbb{Q}}_{p}^{\text {unr }}}\right) .
$$

Let $A_{e} \subset \mathcal{H}_{p}$ be the oriented annulus in $\mathcal{H}_{p}$ corresponding to $e$ and $U_{v} \subset \mathcal{H}_{p}$ be the affinoid corresponding to $v \in \mathcal{V}$, which are obtained as inverse images of the reduction map (see [IS03, page 342$]$ ). Recall that $H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n}\right)$ can be identified with the $\hat{\mathbb{Q}}_{p}^{\text {unr }}$-vector space of $V_{n}$-valued, $\Gamma$-invariant differential forms of the second kind on $\mathcal{H}_{p}$ modulo exact forms ([IS03, page 348]). Let $\omega$ be a $V_{n}$-valued $\Gamma$-invariant differential of the second kind on $\mathcal{H}_{p}$. We define $I(\omega)$ to be the map which assigns to an oriented edge $e \in \overrightarrow{\mathcal{E}}$ the value $I(\omega)(e)=\operatorname{Res}_{e}(\omega)$, where $\operatorname{Res}_{e}$
denotes the annular residue along $A_{e}$. If $\omega$ is exact, $I(\omega)=0$. Thus $I$ gives a well-defined map

$$
\begin{equation*}
I: H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n}\right) \longrightarrow C^{1}\left(\left(V_{n}\right)_{\hat{\mathbb{Q}}_{p}^{\mathrm{unr}}}\right)^{\Gamma} \tag{13}
\end{equation*}
$$

The $\hat{\mathbb{Q}}_{p}^{\text {unr }}$-vector space $\bigoplus_{v \in \mathcal{V}} H_{\mathrm{dR}}^{0}\left(U_{v}, \mathcal{V}_{n}\right)$ can be identified with $C^{0}\left(\left(V_{n}\right)_{\widehat{\mathbb{Q}}_{p}^{\text {unr }}}\right)$, and the subspace of $\bigoplus_{e \in \overrightarrow{\mathcal{E}}} H_{\mathrm{dR}}^{0}\left(A_{e}, \mathcal{V}_{n}\right)$ consisting of elements $\left\{f_{e}\right\}_{e \in \overrightarrow{\mathcal{E}}}$ such that $f_{\bar{e}}=-f_{e}$ can be identified with $C^{1}\left(\left(V_{n}\right)_{\widehat{\mathbb{Q}}_{p}^{\text {unr }}}\right)$. Since the set $\left\{U_{v}\right\}_{v \in \mathcal{V}}$ is an admissible covering of $\mathcal{H}_{p}$, the MayerVietoris sequence yields an embedding

$$
C^{1}\left(\left(V_{n}\right)_{\widehat{\mathbb{Q}}_{p}^{\text {unr }}}\right)^{\Gamma} / C^{0}\left(\left(V_{n}\right)_{\widehat{\mathbb{Q}}_{p}^{\text {unr }}}\right)^{\Gamma} \longleftrightarrow H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n}\right)
$$

Precomposing with $\epsilon$, we obtain an embedding

$$
\begin{equation*}
\iota: H^{1}\left(\Gamma,\left(V_{n}\right)_{\hat{\mathbb{Q}}_{p}^{\mathrm{unr}}}\right) \longleftrightarrow H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n}\right) \tag{14}
\end{equation*}
$$

This map admits a natural left inverse

$$
\begin{equation*}
P: H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n}\right) \longrightarrow H^{1}\left(\Gamma,\left(V_{n}\right)_{\widehat{\mathbb{Q}}_{p}^{\text {unr }}}\right) \tag{15}
\end{equation*}
$$

which takes $\omega$ to the class of the cocycle $\gamma \mapsto \gamma\left(F_{\omega}\right)-F_{\omega}$. Here $F_{\omega}$ is a primitive of $\omega$ in the sense of Coleman, i.e. $d F_{\omega}=\omega$ (see Col82, Lemma 4.4]).

Define now the monodromy operator $N_{n}$ on $H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n}\right)$ as the composite $\iota \circ(-\epsilon) \circ I$. On the other hand, the monodromy operator $N_{\left(\mathcal{V}_{n}\right)_{z_{A_{0}}}}$ on the filtered $(\phi, N)$-module $\left(\mathcal{V}_{n}\right)_{z_{A_{0}}}$ is trivial. Therefore, since $D_{\text {st, } \hat{\mathbb{Q}}_{p}^{u n r}}\left(H_{p}^{2 n+1}(\mathcal{D})\right)$ is isomorphic to $H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n}\right) \otimes\left(\mathcal{V}_{n}\right)_{z_{A_{0}}}$ in the category of filtered Frobenius monodromy modules, its monodromy operator is given by

$$
\begin{equation*}
N=\operatorname{id}_{n} \otimes N_{\left(\mathcal{V}_{n}\right)_{z_{A_{0}}}}+N_{n} \otimes \operatorname{id}_{\left(\mathcal{V}_{n}\right)_{z_{A_{0}}}}=N_{n} \otimes \operatorname{id}_{\left(\mathcal{V}_{n}\right)_{z_{A_{0}}}} \tag{16}
\end{equation*}
$$

where id. denote identity operators.
We now describe the Frobenius operator on $D_{\text {st, } \hat{\mathbb{Q}}_{p}^{\text {unr }}}\left(H_{p}^{2 n+1}(\mathcal{D})\right)$. First, $H^{1}\left(\Gamma,\left(V_{n}\right)_{\hat{\mathbb{Q}}_{p}^{\text {unr }}}\right)$ has a Frobenius endomorphism induced by the map $p^{\frac{n}{2}} \otimes \sigma$ on $\left(V_{n}\right)_{\hat{\mathbb{Q}}_{p}^{u n r}}=V_{n} \otimes_{\mathbb{Q}_{p}} \hat{\mathbb{Q}}_{p}^{\text {unr }}$, where $\sigma$ denotes the absolute Frobenius automorphism on $\hat{\mathbb{Q}}_{p}^{u n r}$. As defined in [IS03, Section 4], $\Phi_{n}$ is the unique operator on $H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n}\right)$ satisying $N_{n} \Phi_{n}=p \Phi_{n} N_{n}$ and which is compatible (with respect to $\iota$ and $P$ ) with the Frobenius on $H^{1}\left(\Gamma,\left(V_{n}\right)_{\mathbb{Q}_{p}^{\text {unr }}}\right)$. On the other hand, the Frobenius on the filtered $(\phi, N)$-module $\left(\mathcal{V}_{n}\right)_{z_{A_{0}}}$ is given by $\Phi_{\left(\mathcal{V}_{n}\right)_{z_{A_{0}}}}=p^{\frac{n}{2}} \otimes \sigma$ acting on the underlying vector space $V_{n} \otimes_{\mathbb{Q}_{p}} \hat{\mathbb{Q}}_{p}^{u n r}$. The Frobenius operator on $D_{\text {st }, \hat{\mathbb{Q}}_{p}^{\text {unr }}}\left(H_{p}^{2 n+1}(\mathcal{D})\right)$ is given by

$$
\Phi=\Phi_{n} \otimes \Phi_{\left(\mathcal{V}_{n}\right)_{z_{A_{0}}}}
$$

Note that $N$ and $\Phi$ satisfy the relation $N \Phi=p \Phi N$.
Recall that $H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n}\right)$ is equipped with a non-degenerate pairing

$$
\langle,\rangle_{\mathcal{V}_{n}}: H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n}\right) \otimes H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n}\right) \longrightarrow \hat{\mathbb{Q}}_{p}^{\mathrm{unr}}[n+1]
$$

in $\mathrm{MF}_{\widehat{\mathbb{Q}}_{p}^{\text {unr }}}^{\phi, N}$, which is induced from $\langle,\rangle_{V_{n}}$; see [IS03, §5], especially [IS03, Remark 5.12], for definitions and details. Let

$$
\begin{equation*}
\langle,\rangle_{\mathcal{V}_{n, n}}: H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n, n}\right) \otimes H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n, n}\right) \longrightarrow \hat{\mathbb{Q}}_{p}^{\mathrm{unr}}[n+1] \otimes \operatorname{det}^{\otimes n} \tag{17}
\end{equation*}
$$

be the induced symmetric non-degenerate pairing defined by $\langle,\rangle_{\mathcal{V}_{n, n}}=\langle,\rangle_{\mathcal{V}_{n}} \otimes\langle,\rangle_{V_{n}}$ (where we also use the isomorphisms $\left(\mathcal{V}_{n}\right)_{z_{A_{0}}} \simeq V_{n}$ to define a pairing on $\left(\mathcal{V}_{n}\right)_{z_{A_{0}}}$ via that on $\left.V_{n}\right)$. If
we denote $V^{\vee}$ the $F$-linear dual of a $F$-vector space $V$, from (12) and the non-degeneracy of $\langle,\rangle_{\mathcal{V}_{n, n}}$ we obtain an isomorphism of $\hat{\mathbb{Q}}_{p}^{\text {urr }}$-vector spaces:
(18)

$$
\frac{D_{\mathrm{st}, \hat{\mathbb{Q}}_{p}^{\text {unr }}}\left(H_{p}^{2 n+1}(\mathcal{D})\right)}{F^{n+1}\left(D_{\mathrm{st}, \hat{\mathbb{Q}}_{p}^{\text {un }}}\left(H_{p}^{2 n+1}(\mathcal{D})\right)\right)} \simeq\left(F^{n+1}\left(D_{\mathrm{st}, \hat{\mathbb{Q}}_{p}^{\text {unr }}}\left(H_{p}^{2 n+1}(\mathcal{D})\right)\right)^{\vee} \simeq\left(M_{k}(\Gamma) \otimes\left(\mathcal{V}_{n}\right)_{z_{A_{0}}}\right)^{\vee}\right.
$$

## 4. $p$-adic Abel-Jacobi maps

Let the notation be as in Section 3: $N=p N^{+} N^{-}$is a factorisation of the integer $N \geq 1$ into coprime integers $p, N^{+}, N^{-}$with $p \nmid N^{+} N^{-}$a prime number, and $N^{-}$be a square-free product of an odd number of factors; $k \geq 4$ is an even integer and put $n=k-2$ and $m=n / 2$; $K$ is a quadratic imaginary field such that all primes dividing $N^{+}$(respectively, $p N^{-}$) are split (respectively, inert) in $K$.
4.1. Definition of the Abel-Jacobi map. Let [ $\Delta$ ] be the class of a null-homologous cycle $\Delta$ of codimension $n+1$ in $\mathrm{CH}^{n+1}(\mathcal{D})(F)$, where $F \subseteq \overline{\mathbb{Q}}$ is a field containing the Hilbert class field of $K$; here

$$
\mathrm{CH}^{n+1}(\mathcal{D})(F)=\epsilon \cdot \mathrm{CH}^{n+1}\left(Y_{m}\right)(F)
$$

and null-homologous means that $[\Delta]$ belongs to $\mathrm{CH}_{0}^{n+1}(\mathcal{D})$, the kernel of the cycle class map $c \ell_{\mathcal{D}}^{(n+1)}$ in (33). Let $\operatorname{Ext}_{G_{F}}^{1}(\cdot, \cdot)$ be the first Ext functor in the category of continuous $G_{F^{-}}$ representations. For a $G_{F}$-representation $M$, let $M(i)$ denote its $i$-th Tate twist. One may associate to $[\Delta]$ the isomorphism class in

$$
\operatorname{Ext}_{G_{F}}^{1}\left(\mathbb{Q}_{p}, \epsilon_{*} \cdot H_{\mathrm{et}}^{2 n+1}\left(\bar{Y}_{m}, \mathbb{Q}_{p}(n+1)\right)\right)=H^{1}\left(F, H_{p}^{2 n+1}(\mathcal{D})(n+1)\right)
$$

of the extension

$$
\begin{equation*}
0 \longrightarrow \epsilon_{*} \cdot H_{\mathrm{et}}^{2 n+1}\left(\bar{Y}_{m}, \mathbb{Q}_{p}(n+1)\right) \longrightarrow E \longrightarrow \mathbb{Q}_{p} \longrightarrow 0 \tag{19}
\end{equation*}
$$

given by the pull-back of the exact sequence (which comes from the Gysin exact sequence Mil80, Remark 5.4(b)])
(20) $0 \longrightarrow \epsilon_{*} \cdot H_{\mathrm{et}}^{2 n+1}\left(\bar{Y}_{m}, \mathbb{Q}_{p}(n+1)\right) \longrightarrow \epsilon_{*} \cdot H_{\mathrm{et}}^{2 n+1}\left(\bar{U}, \mathbb{Q}_{p}(n+1)\right) \longrightarrow \epsilon_{*} \cdot H_{\bar{\Delta}}^{2 n+2}\left(\bar{Y}_{m}, \mathbb{Q}_{p}(n+1)\right) \longrightarrow 0$
(where $\left.U=Y_{m}-\Delta, \bar{U}=U \otimes_{F} \bar{F}, \bar{\Delta}=\Delta \otimes_{F} \bar{F}\right)$ via the map $\mathbb{Q}_{p} \rightarrow \epsilon_{*} \cdot H \frac{2 n+2}{\bar{\Delta}}\left(\bar{Y}_{m}, \mathbb{Q}_{p}(n+1)\right.$ ) sending 1 to the cycle class of $\Delta$; see [Jan90, Remark 9.1] for the definition of the Abel-Jacobi map, and use Proposition 3.1 to obtain the above recipe (see also a similar argument using projectors as in (BDP13, §3.3]). This association defines a map, called p-adic étale Abel-Jacobi map

$$
\begin{equation*}
c \ell_{\mathcal{D}, 0}^{(n+1)}: \mathrm{CH}_{0}^{n+1}(\mathcal{D})(F) \longrightarrow \operatorname{Ext}_{G_{F}}^{1}\left(\mathbb{Q}_{p}, H_{p}^{2 n+1}(\mathcal{D})(n+1)\right)=H^{1}\left(F, H_{p}^{2 n+1}(\mathcal{D})(n+1)\right) . \tag{21}
\end{equation*}
$$

4.2. Semistability. We now use $p$-adic Hodge theory to describe the restriction of $\mathrm{AJ}_{p}$ to $\mathrm{CH}^{n+1}(\mathcal{D})\left(F_{v}\right)$, where $v$ is the place of $F$ above $p$ induced by the inclusion $F \subseteq \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_{p}$, which for simplicity we assume to be unramified over $p$; here $F_{v}$ is the completion of $F$ at $v$, which we also assume to contain $\mathbb{Q}_{p^{2}}$. The motive $\mathcal{D}$ is then defined over $F_{v}$, because the prime $p$, being inert in $K$, splits completely in its Hilbert class field $H$. Consider the base change of $Y_{m}$ to $F_{v}$ that we also denote by $Y_{m}$ by a slight abuse of notation, and the Abel-Jacobi map

$$
c \ell_{\mathcal{D}, 0}^{(n+1)}: \mathrm{CH}_{0}^{n+1}(\mathcal{D})\left(F_{v}\right) \longrightarrow \operatorname{Ext}_{G_{F_{v}}}^{1}\left(\mathbb{Q}_{p}, H_{p}^{2 n+1}(\mathcal{D})(n+1)\right)=H^{1}\left(F_{v}, H_{p}^{2 n+1}(\mathcal{D})(n+1)\right)
$$

obtained by restriction. For a $G_{F_{v}}=\operatorname{Gal}\left(\bar{F}_{v} / F_{v}\right)$-representation $V$, let $H_{\mathrm{st}}^{1}\left(G_{F_{v}}, V\right)$ be the semistable Bloch-Kato Selmer group ([BK90, §3], or [IS03, page 361]). By a result of Nekovář
[Nek00, Theorem 3.6] (see also [IS03, Lemma 7.1] and the remarks following it), we know that the image of $\mathrm{AJ}_{p}$ is contained in $H_{\mathrm{st}}^{1}\left(F_{v}, H_{p}^{2 n+1}(\mathcal{D})(n+1)\right)$. We have

$$
H_{\mathrm{st}}^{1}\left(F_{v}, H_{p}^{2 n+1}(\mathcal{D})(n+1)\right) \simeq \operatorname{Ext}_{\operatorname{Rep}_{\mathrm{st}}\left(G_{F_{v}}\right)}^{1}\left(F_{v}(n+1), H_{p}^{2 n+1}(\mathcal{D})\right)
$$

where $\operatorname{Rep}_{\mathrm{st}}\left(G_{F_{v}}\right)$ denotes the category of semistable $p$-adic representations of $G_{F_{v}}$, and $\operatorname{Ext}_{\operatorname{Rep}_{\mathrm{st}}\left(G_{F_{v}}\right)}^{1}(\cdot, \cdot)$ is the first Ext functor in this category. The functor $D_{\mathrm{st}, F_{v}}$ gives an isomorphism

$$
\operatorname{Ext}_{\operatorname{Rep}_{\mathrm{st}}\left(G_{F_{v}}\right)}^{1}\left(F_{v}(n+1), H_{p}^{2 n+1}(\mathcal{D})\right) \simeq \operatorname{Ext}_{\mathrm{MF}_{F_{v}}^{\phi, N}}^{1}\left(F_{v}[n+1], D_{\mathrm{st}, F_{v}}\left(H_{p}^{2 n+1}(\mathcal{D})\right)\right)
$$

where now $\operatorname{Ext}_{\mathrm{MF}_{F_{v}}^{\phi, N}}^{1}(\cdot, \cdot)$ denotes the first Ext functor in the category $\mathrm{MF}_{F_{v}}^{\phi, N}([\overline{\mathrm{IS} 03},(44)])$, and for an object $M$ in this category, $M[i]$ is its $i$-th fold twist described in [IS03, §2]. By [IS03, Lemma 2.1],

$$
\operatorname{Ext}_{\mathrm{MF}_{F_{v}}^{\phi, N}}^{1}\left(F_{v}[n+1], D_{\mathrm{st}, F_{v}}\left(H_{p}^{2 n+1}(\mathcal{D})\right)\right) \simeq \frac{D_{\mathrm{st}, F_{v}}\left(H_{p}^{2 n+1}(\mathcal{D})\right)}{F^{n+1}\left(D_{\mathrm{st}, F_{v}}\left(H_{p}^{2 n+1}(\mathcal{D})\right)\right)}
$$

Therefore we conclude that

$$
H_{\mathrm{st}}^{1}\left(F_{v}, H_{p}^{2 n+1}(\mathcal{D})(n+1)\right) \simeq \frac{D_{\mathrm{st}, F_{v}}\left(H_{p}^{2 n+1}(\mathcal{D})\right)}{F^{n+1}\left(D_{\mathrm{st}, F_{v}}\left(H_{p}^{2 n+1}(\mathcal{D})\right)\right)}
$$

Finally, using the canonical map $D_{\text {st }, F_{v}}\left(H_{p}^{2 n+1}(\mathcal{D})\right) \hookrightarrow D_{\text {st, }, \hat{\mathbb{Q}}_{p}^{u n r}}\left(H_{p}^{2 n+1}(\mathcal{D})\right)$ (which respects the filtrations on both sides) and (18), we obtain from $c \ell_{\mathcal{D}, 0}^{(n+1)}$ a map $\mathrm{AJ}_{p}$ still called $p$-adic Abel-Jacobi map,

$$
\begin{equation*}
\mathrm{AJ}_{p}: \mathrm{CH}^{n+1}(\mathcal{D})\left(F_{v}\right) \longrightarrow\left(M_{k}(\Gamma) \otimes\left(\mathcal{V}_{n}\right)_{z_{A_{0}}}\right)^{\vee} \tag{22}
\end{equation*}
$$

4.3. The de Rham realization. We now introduce, following [IS03, a more concrete description of the map (22). Fix a point $x_{A} \in X_{M}(F)$ (as above, $F \subseteq \overline{\mathbb{Q}}$ ) which reduces to a non-singular point in the special fiber of $X_{M}$, and let $A^{m} \times A_{0}^{m}$ be the fiber of $\mathcal{A}^{m} \times A_{0}^{m} \rightarrow X_{M}$ at $x_{A}$. Define

$$
H^{1}\left(\bar{X}_{M}, \mathbb{L}_{n, n}\right)=H^{1}\left(\bar{X}_{M}, \mathbb{L}_{n}\right) \otimes\left(\mathbb{L}_{n}\right)_{\bar{x}_{A_{0}}}
$$

Let $\bar{x}_{A}=x_{A} \otimes_{F} \bar{F}, U_{x_{A}}=X_{M}-\left\{x_{A}\right\}$ and $\bar{U}_{x_{A}}=U_{x_{A}} \otimes_{F} \bar{F}$. The Gysin sequence gives rise to an exact sequence

$$
0 \longrightarrow H^{1}\left(\bar{X}_{M}, \mathbb{L}_{n, n}\right)(n+1) \longrightarrow H^{1}\left(\bar{U}_{x_{A}}, \mathbb{L}_{n, n}\right)(n+1) \longrightarrow\left(\left(\mathbb{L}_{n}\right)_{\bar{x}_{A}} \otimes\left(\mathbb{L}_{n}\right)_{\bar{x}_{A_{0}}}\right)(n) \longrightarrow 0
$$

whose surjectivity follows from the analogous exact sequence in [IS03, (51)] tensoring with the constant sheaf $\left(\mathbb{L}_{n}\right)_{\bar{x}_{A_{0}}}$. Applying the projector $\left(p_{G}\right)_{*}$ we obtain an exact sequence

$$
\begin{equation*}
0 \longrightarrow H_{p}(\mathcal{D})(n+1) \longrightarrow E \longrightarrow\left(\left(\mathbb{L}_{n}\right)_{\bar{x}_{A}} \otimes\left(\mathbb{L}_{n}\right)_{\bar{x}_{A_{0}}}\right)(n) \longrightarrow 0 \tag{23}
\end{equation*}
$$

Suppose $F \subseteq \hat{\mathbb{Q}}_{p}^{\text {unr }}$. Let $z_{A}$ and $z_{A_{0}}$ be the points in $\mathcal{H}_{p}\left(\hat{\mathbb{Q}}_{p}^{\text {unr }}\right)$ lying over $x_{A}$ and $x_{A_{0}}$, respectively (using (2)). Define $U_{z_{A}}=X_{\Gamma}-\left\{z_{A}\right\}$ and put

$$
\begin{equation*}
H_{\mathrm{dR}}^{1}\left(U_{z_{A}}, \mathcal{V}_{n, n}\right)=H_{\mathrm{dR}}^{1}\left(U_{z_{A}}, \mathcal{V}_{n}\right) \otimes\left(\mathcal{V}_{n}\right)_{z_{A_{0}}} \tag{24}
\end{equation*}
$$

Let $\operatorname{Res}_{z}: H_{\mathrm{dR}}^{1}\left(U, \mathcal{V}_{n}\right) \rightarrow\left(\mathcal{V}_{n}\right)_{z}$ be the residue map at a point $z \in X_{\Gamma}\left(\hat{\mathbb{Q}}_{p}^{\mathrm{unr}}\right)$. The Gysin sequence of [IS03, Theorem 5.13] gives rise, after tensoring with $\left(\mathcal{V}_{n}\right)_{z_{A_{0}}}$ and using (9), (24), to an exact sequence in $\mathrm{MF}_{\widehat{\mathbb{Q}}_{p}^{\phi \text { ur }}}^{\phi, N}$ :

$$
\begin{equation*}
0 \longrightarrow H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n, n}\right)[-(n+1)] \longrightarrow H_{\mathrm{dR}}^{1}\left(U_{z_{A}}, \mathcal{V}_{n, n}\right)[-(n+1)] \xrightarrow{\operatorname{Res}_{z_{A}}}\left(\left(\mathcal{V}_{n}\right)_{z_{A}} \otimes\left(\mathcal{V}_{n}\right)_{z_{A_{0}}}\right)[-n] \longrightarrow 0 \tag{25}
\end{equation*}
$$

This exact sequence is obtained by applying $D_{\text {st, } \hat{\mathbb{Q}}_{p}^{u n r}}$ to (23).

Remark 4.1. The shift in (25) is due to the definition of Tate twists adopted in [IS03, page 337]; see [FO, §7.1.3] or [BC, §8.3] for a different convention.

We have the cycle class map

$$
\begin{aligned}
c \ell=c \ell_{\left(A^{m} \times A_{0}^{m}, \epsilon_{M}\right)}^{(n)}: \mathrm{CH}^{n}\left(\left(A^{m} \times A_{0}^{m}, \epsilon_{M}\right)\right) \longrightarrow & \Gamma\left(D_{\mathrm{st}, \hat{\mathbb{Q}}_{p}^{\mathrm{unr}}}\left(H_{p}^{2 n}\left(\left(A^{m} \times A_{0}^{m}, \epsilon_{M}\right)(n)\right)\right)\right) \\
& \simeq \Gamma\left(D_{\mathrm{st}, \hat{\mathbb{Q}}_{p}^{\mathrm{unr}}}\left(H^{2 n}\left(\bar{X}_{M},\left(\mathbb{L}_{n}\right)_{x_{A}} \otimes\left(\mathbb{L}_{n}\right)_{x_{A_{0}}}\right)(n)\right)\right) \\
& \left.\simeq \Gamma\left(D_{\mathrm{st}, \hat{\mathbb{Q}}_{p}^{\mathrm{unr}}}\left(\left(\left(\mathbb{L}_{n}\right)_{\bar{x}_{A}} \otimes\left(\mathbb{L}_{n}\right)_{\bar{x}_{A_{0}}}\right)(n)\right)\right)\right) \\
& \simeq \Gamma\left(\left(\left(\mathcal{V}_{n}\right)_{z_{A}} \otimes\left(\mathcal{V}_{n}\right)_{z_{A_{0}}}\right)[-n]\right) .
\end{aligned}
$$

Next, from (25) we obtain a connecting homomorphism in the sequence of Ext groups

$$
\begin{aligned}
\Gamma\left(\left(\left(\mathcal{V}_{n}\right)_{z_{A}} \otimes\left(\mathcal{V}_{n}\right)_{z_{A_{0}}}\right)[-n]\right) \xrightarrow{\partial} & \operatorname{Ext}_{\mathrm{MF}_{\mathbb{Q}_{p}^{\text {unr }}}^{\phi, N}}^{1}\left(\hat{\mathbb{Q}}_{p}^{\mathrm{unr}}, H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n, n}\right)[-(n+1)]\right) \\
& \simeq \operatorname{Ext}_{\mathrm{MF}_{\widehat{\mathbb{Q}}_{p}^{\text {unr }}}^{1}}^{1}\left(\hat{\mathbb{Q}}_{p}^{\mathrm{unr}}[n+1], H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n, n}\right)\right) \\
& \simeq\left(M_{k}(\Gamma) \otimes\left(\mathcal{V}_{n}\right)_{z_{A_{0}}}\right)^{\vee}
\end{aligned}
$$

where the last isomorphism comes, as before, from (18) and [IS03, Lemma 2.1]. On the other hand, we have a canonical map

$$
i: \mathrm{CH}^{n}\left(\left(A^{m} \times A_{0}^{m}, \epsilon_{M}\right)\right) \longrightarrow \mathrm{CH}^{n+1}(\mathcal{D})
$$

The definition of the Abel-Jacobi map ([Jan90, §9]) shows that the following diagram is commutative:


Suppose that $\Delta$ is supported in the fiber of $\mathcal{D}$ above $x_{A} \in X_{M}(F)$, then $\mathrm{AJ}_{p}(\Delta)$ is the extension class determined by the following diagram (in which the right square is cartesian)

where the vertical left map sends $1 \longmapsto c \ell(\Delta)[n+1]$.

## 5. Generalized Heegner cycles

5.1. Definition of the cycles. We fix a field $F$ containing the Hilbert class field $H$ of $K$. Recall the fixed abelian surface $A_{0}$ with QM and complex multiplication by $\mathcal{O}_{K}$. Consider the set of pairs $(\varphi, A)$, where $A$ is an abelian surface with QM and $\varphi: A_{0} \rightarrow A$ is a false isogeny (defined over $\bar{K}$ ) of false elliptic curves, of degree prime to $N^{+} M$, i.e. whose kernel intersects the level structures of $A_{0}$ trivially. Let $x_{A}$ be the point on $X_{M}$ corresponding to $A$ with level structure given by composing $\varphi$ with the level structure of $A_{0}$. We associate to any pair $(\varphi, A)$ a codimension $n+1$ cycle $\Upsilon_{\varphi}$ on $Y_{m}$ by defining

$$
\Upsilon_{\varphi}:=\left(\Gamma_{\varphi}\right)^{m} \subset\left(A_{0} \times A\right)^{m} \simeq A^{m} \times A_{0}^{m} \subset \mathcal{A}^{m} \times A_{0}^{m}
$$

where $\Gamma_{\varphi} \subset A_{0} \times A$ is the graph of $\varphi$ and the inclusion $A^{m} \times A_{0}^{m} \subset \mathcal{A}^{m} \times A_{0}^{m}$ is $\mathrm{id}_{A_{0}}^{m}$ on the second component. We then set

$$
\Delta_{\varphi}:=\epsilon \Upsilon_{\varphi} .
$$

The cycle $\Delta_{\varphi}$ of $\mathcal{D}$ is supported on the fiber above $x_{A}$ and has codimension $n+1$ in $\mathcal{A}^{m} \times A_{0}^{m}$, thus $\Delta_{\varphi} \in \mathrm{CH}^{n+1}(\mathcal{D})$. Since the cycle class map sends $\Delta_{\varphi}$ to the $p$-adic realization $H_{p}^{2 n+2}(\mathcal{D})$ and $H_{p}^{2 n+2}(\mathcal{D})=0$, the cycle $\Delta_{\varphi}$ is homologous to zero.
5.2. The image of $\Delta_{\varphi}$ under the $p$-adic Abel-Jacobi map. For any $D \in \mathrm{MF}_{\mathbb{Q}_{p}^{\text {unr }}}^{\phi, N}$, write $D=\oplus_{\lambda} D_{\lambda}$ for its slope decomposition, where $\lambda \in \mathbb{Q}$ ([IS03, (2)]). Recall the monodromy operator $N$ introduced in (16).
Lemma 5.1. $N$ induces an isomorphism $H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n, n}\right)_{n+1} \simeq H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n, n}\right)_{n}$.
Proof. Since the monodromy operator $N$ and the Frobenius $\Phi$ on $H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n, n}\right)$ satisfy the relation $N \Phi=p \Phi N$, we have $N\left(H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n, n}\right)_{n+1}\right) \subseteq H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n, n}\right)_{n}$. Since $\left(\mathcal{V}_{n}\right)_{z_{A_{0}}}$ is isotypical of slope $n / 2$, we have

$$
H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n, n}\right)_{n+1}=H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n}\right)_{\frac{n}{2}+1} \otimes\left(\mathcal{V}_{n}\right)_{z_{A_{0}}}
$$

and

$$
H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n, n}\right)_{n}=H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n}\right)_{\frac{n}{2}} \otimes\left(\mathcal{V}_{n}\right)_{z_{A_{0}}}
$$

By 【S03, we know that $N_{n}: H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n}\right)_{\frac{n}{2}+1} \rightarrow H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n}\right)_{\frac{n}{2}}$ is an isomorphism, thus the restriction of $N$ to $H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n, n}\right)_{n+1}$ is an isomorphism by the definition of the monodromy operator $N$ given in (16).

Fix $f \in M_{k}(\Gamma)$ and $v \in\left(\mathcal{V}_{n}\right)_{z_{A_{0}}}$. Thanks to Lemma [5.1, we can apply [IS03, Lemma 2.1] (see also Mas12, Lemma 3.3]) to compute $\operatorname{AJ}_{p}\left(\Delta_{\varphi}\right)(f \otimes v)$. With the notation as in (27), and following loc. cit, choose $\alpha \in H_{\mathrm{dR}}^{1}\left(U_{z_{A}}, \mathcal{V}_{n, n}\right)_{n+1}$ such that

$$
\operatorname{Res}_{z_{A}}(\alpha)=c \ell_{A}\left(\Delta_{\varphi}\right)
$$

and $N(\alpha)=0$. Choose $\beta$ in $H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n, n}\right)$ such that

$$
j_{*}(\beta) \equiv \alpha \bmod F^{n+1}\left(H_{\mathrm{dR}}^{1}\left(U_{z_{A}}, \mathcal{V}_{n, n}\right)\right) .
$$

Then the image of the extension $c l_{\mathcal{D}, 0}^{(n+1)}\left(\Delta_{\varphi}\right)$ in

$$
H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n, n}\right) / F^{n+1}\left(H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n, n}\right)\right) \simeq\left(M_{k}(\Gamma) \otimes\left(\mathcal{V}_{n}\right)_{z_{A_{0}}}\right)^{\vee}
$$

is the class of $\beta$ (which we denote by the same symbol $\beta$ ) in this quotient. Let $\omega_{f}$ be the class in $F^{n+1}\left(H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n}\right)\right)$ corresponding to $f \in M_{k}(\Gamma)$ under the isomorphism (10). Recall the pairing $\langle,\rangle_{\mathcal{V}_{n, n}}$ defined in (17). Then by definition

$$
\begin{equation*}
\operatorname{AJ}_{p}\left(\Delta_{\varphi}\right)(f \otimes v)=\left\langle\omega_{f} \otimes v, \beta\right\rangle_{\mathcal{V}_{n, n}} . \tag{28}
\end{equation*}
$$

From the proof of [IS03, Theorem 6.4] we know that $H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n}\right)$ decomposes as the direct sum of $H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n}\right)_{\frac{n}{2}}$ and $F^{\frac{n}{2}+1}\left(H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n}\right)\right)$. Since

$$
F^{\frac{n}{2}+1}\left(H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n}\right)\right)=F^{n+1}\left(H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n}\right)\right)
$$

and $F^{n+1}\left(\left(\mathcal{V}_{n}\right)_{z_{A_{0}}}\right)=0$, using the previous decomposition, and the fact that, as above, $\left(\mathcal{V}_{n}\right)_{z_{A_{0}}}$ is isotypical of slope $n / 2$, we obtain a decomposition

$$
H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n, n}\right) \simeq H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n, n}\right)_{n} \oplus F^{n+1}\left(H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n, n}\right)\right)
$$

We may therefore assume that the element $\beta$ considered above belongs to $H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n, n}\right)_{n}$. Moreover, again from the proof of [IS03, Theorem 6.4] we know that

$$
\begin{equation*}
\operatorname{ker}\left(N_{n}\right)=\iota\left(H^{1}\left(\Gamma,\left(V_{n}\right)_{\mathbb{Q}_{p}^{\text {unr }}}\right)\right)=H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n}\right)_{\frac{n}{2}} \tag{29}
\end{equation*}
$$

where $\iota$ is the map considered in (14). To simplify the notation we put

$$
H^{1}\left(\Gamma, V_{n, n}\right)=H^{1}\left(\Gamma,\left(V_{n}\right)_{\hat{\mathbb{Q}}_{p}^{\mathrm{unr}}}\right) \otimes\left(\mathcal{V}_{n}\right)_{z_{A_{0}}}
$$

We now extend $\iota$ to a map, still denoted by the same symbol,

$$
\iota=\iota \otimes \operatorname{id}_{\left(\mathcal{V}_{n}\right)_{z_{A}}}: H^{1}\left(\Gamma, V_{n, n}\right) \longleftrightarrow H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n, n}\right)
$$

and (29) shows that there exists an isomorphisms

$$
\operatorname{ker}(N)=\iota\left(H^{1}\left(\Gamma, V_{n, n}\right)\right)=H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n, n}\right)_{n}
$$

Therefore we may assume $\beta=\iota(c)$ for some $c \in H^{1}\left(\Gamma, V_{n, n}\right)$.
We now introduce still an other pairing $\langle,\rangle_{\Gamma}$. Let $C_{\text {har }}\left(V_{n}\right)^{\Gamma}$ denote the $\mathbb{Q}_{p}$-vector space of $\Gamma$-invariant $V_{n}$-valued harmonic cocycles (see for example [DT08, Definition 2.2.9]). We denote

$$
\langle,\rangle_{\Gamma}^{\prime}: C_{\mathrm{har}}\left(V_{n}\right)^{\Gamma} \otimes H^{1}\left(\Gamma, V_{n}\right) \longrightarrow \mathbb{Q}_{p}
$$

the pairing introduced in [IS03, (75)]. To simplify the notation, we set

$$
C_{\mathrm{har}}\left(V_{n, n}\right)^{\Gamma}=C_{\mathrm{har}}\left(V_{n}\right)^{\Gamma} \otimes\left(\mathcal{V}_{n}\right)_{z_{A_{0}}}
$$

We then define the pairing

$$
\langle,\rangle_{\Gamma}: C_{\mathrm{har}}\left(V_{n, n}\right)^{\Gamma} \otimes H^{1}\left(\Gamma, V_{n, n}\right) \longrightarrow \mathbb{Q}_{p}
$$

by $\langle,\rangle_{\Gamma}=\langle,\rangle_{\Gamma}^{\prime} \otimes\langle,\rangle_{V_{n}}$ (where as above we identify $\left(\mathcal{V}_{n}\right)_{z_{A_{0}}}$ and $V_{n}$ ). Recall the map $I$ is defined in (13).

Lemma 5.2. $\left\langle\omega_{f} \otimes v, \beta\right\rangle_{\mathcal{V}_{n, n}}=-\left\langle I\left(\omega_{f}\right) \otimes v, c\right\rangle_{\Gamma}$.
Proof. Write $\beta=\sum_{i} \beta_{i} \otimes v_{i}$ and $c=\sum_{j} c_{j} \otimes w_{j}$. The assumption $\iota(c)=\beta$ shows that $i=j$, $v_{i}=w_{i}$ and $\iota\left(c_{i}\right)=\beta_{i}$ where here $\iota$ is the map in (14). By [IS03, Theorem 10.2] we know that for each $i$ we have $\left\langle\omega_{f}, \beta_{i}\right\rangle_{\mathcal{V}_{n}}=-\left\langle I\left(\omega_{f}\right), c_{i}\right\rangle_{\Gamma}^{\prime}$. The definitions of $\langle,\rangle_{\mathcal{V}_{n, n}}$ and $\langle,\rangle_{\Gamma}$ imply the result.

Recall the open set $U_{z_{A}}=X_{\Gamma}-\left\{z_{A}\right\}$. Write $\alpha-j_{*}(\beta)=\sum_{i} \gamma_{i} \otimes v_{i}$. For each $i$, let $\chi_{i}$ be a $\Gamma$-invariant $V_{n}$-valued meromorphic differential form on $\mathcal{H}_{p}$ which is holomorphic outside $\pi^{-1}\left(U_{z_{A}}\right)$, with a simple pole at $z_{A}$, and whose class $\left[\chi_{i}\right]$ in $F^{\frac{n}{2}+1}\left(H_{\mathrm{dR}}^{1}\left(U_{z_{A}}, \mathcal{V}_{n}\right)\right)$ represents $\gamma_{i}$. Then the class of $\chi=\sum_{i} \chi_{i} \otimes v_{i}$ represents $\alpha-j_{*}(\beta)$.

Having identified $H_{\mathrm{dR}}^{1}\left(X_{\Gamma}, \mathcal{V}_{n}\right)$ with the $\hat{\mathbb{Q}}_{p}^{u n r}$-vector space of $\Gamma$-invariant $V_{n}$-valued differential forms of the second kind on $\mathcal{H}_{p}$ modulo exact forms, denote $F_{\omega_{f}} \in H^{0}\left(X_{\Gamma}, \mathcal{V}_{n}\right)$ the Coleman primitive of $\omega_{f}$ (dS89, §2.3]). Having fixed $n$, we write $\langle,\rangle_{z, z_{A_{0}}}$ for the restriction of $\langle,\rangle_{\mathcal{V}_{n, n}}$ to the stalk of $\mathcal{V}_{n, n}$ at $z$. Then $\langle,\rangle_{z, z_{A_{0}}}$ is a pairing on $\left(\mathcal{V}_{n}\right)_{z} \otimes\left(\mathcal{V}_{n}\right)_{z_{A_{0}}}$.

Lemma 5.3. $-\left\langle I\left(\omega_{f}\right) \otimes v, c\right\rangle_{\Gamma}=\left\langle F_{\omega_{f}}\left(z_{A}\right) \otimes v, \operatorname{Res}_{z_{A}}(\chi)\right\rangle_{z_{A}, z_{A_{0}}}$.
Proof. As in the proof of Lemma 5.2 write $c=\sum_{j} c_{j} \otimes w_{j}$. By definition,

$$
\left\langle I\left(\omega_{f}\right) \otimes v, c\right\rangle_{\Gamma}=\sum_{j}\left\langle I\left(\omega_{f}\right), c_{j}\right\rangle_{\Gamma}^{\prime} \cdot\left\langle v, w_{j}\right\rangle_{V_{n}}
$$

By [IS03, Corollary 10.7],

$$
\left\langle I\left(\omega_{f}\right), c_{j}\right\rangle_{\Gamma}^{\prime}=\left\langle F_{\omega_{f}}\left(z_{A}\right), \operatorname{Res}_{z_{A}}\left(\chi_{j}\right)\right\rangle_{V_{n}}
$$

where in the last pairing we identify $\left(\mathcal{V}_{n}\right)_{z_{A}}$ with $V_{n}$. The result follows now from the definition of the pairing $\langle,\rangle_{\mathcal{V}_{n, n}}$ in (17).

For a smooth projective variety $X$ defined over $F$, denote

$$
\cup: H_{\mathrm{dR}}^{p}(X) \otimes H_{\mathrm{dR}}^{q}(X) \longrightarrow H_{\mathrm{dR}}^{p+q}(X)
$$

the cup product pairing on the de Rham cohomology of $X$. If $d$ is the dimension of $X$, we also denote $\eta_{X}: H^{2 d}(X) \rightarrow F$ the trace isomorphism.

Let $A_{z}$ be the fiber at $z$ of $\mathcal{A} \rightarrow X_{M}$. The projector $\epsilon$ defines a projector $\epsilon_{z}$ on $A_{z}^{m}$ and we have (Bes95, Theorem 5.8 (iii)])

$$
\begin{equation*}
\left(\epsilon_{z}\right)_{*} H_{\mathrm{dR}}^{n}\left(A_{z}^{m}\right) \simeq\left(\mathcal{V}_{n}\right)_{z} . \tag{30}
\end{equation*}
$$

We also have a canonical map

$$
\begin{equation*}
\left(\epsilon_{z_{A}}\right)_{*} H_{\mathrm{dR}}^{n}\left(A^{m}\right) \otimes\left(\epsilon_{z_{A_{0}}}\right)_{*} H_{\mathrm{dR}}^{n}\left(A_{0}^{m}\right) \longleftrightarrow H_{\mathrm{dR}}^{n}\left(A^{m}\right) \otimes H_{\mathrm{dR}}^{n}\left(A_{0}^{m}\right) \longleftrightarrow H_{\mathrm{dR}}^{2 n}\left(A^{m} \times A_{0}^{m}\right) \tag{31}
\end{equation*}
$$

arising from the Kunneth decomposition; explicitly, this is the map which takes $\alpha \otimes \beta$ to $p_{A}^{*}(\alpha) \cup p_{A_{0}}^{*}(\beta)$, where $p_{A}: A^{m} \times A_{0}^{m} \rightarrow A^{m}$ and $p_{A_{0}}: A^{m} \times A_{0}^{m} \rightarrow A_{0}^{m}$ are the two projections. Composing (30) with (31) we obtain a map

$$
\Theta:\left(\mathcal{V}_{n}\right)_{z_{A}} \otimes\left(\mathcal{V}_{n}\right)_{z_{A_{0}}} \hookrightarrow H_{\mathrm{dR}}^{2 n}\left(A^{m} \times A_{0}^{m}\right) .
$$

Recall that, given a false isogeny $\varphi: A_{0} \rightarrow A$, we have pull-back an push-forward maps $\varphi^{*}: H_{\mathrm{dR}}^{i}(A) \rightarrow H_{\mathrm{dR}}^{i}\left(A_{0}\right)$ and $\varphi_{*}: H_{\mathrm{dR}}^{i}\left(A_{0}\right) \rightarrow H_{\mathrm{dR}}^{i}(A)$. Applying the projectors $\epsilon_{z_{A_{0}}}$ and $\epsilon_{z_{A}}$ and using (30) we thus obtain maps $\varphi^{*}:\left(\mathcal{V}_{n}\right)_{z_{A}} \rightarrow\left(\mathcal{V}_{n}\right)_{z_{A_{0}}}$ and $\varphi_{*}:\left(\mathcal{V}_{n}\right)_{z_{A_{0}}} \rightarrow\left(\mathcal{V}_{n}\right)_{z_{A}}$.
Lemma 5.4. Fix $v_{A} \otimes v_{A_{0}} \in\left(\mathcal{V}_{n}\right)_{z_{A}} \otimes\left(\mathcal{V}_{n}\right)_{z_{A_{0}}}$ and an isogeny $\varphi: A_{0} \rightarrow A$. Then

$$
\left\langle v_{A} \otimes v_{A_{0}}, c l\left(\Delta_{\varphi}\right)\right\rangle_{z_{A}, z_{A_{0}}}=\left\langle v_{A}, \varphi_{*}\left(v_{A_{0}}\right)\right\rangle_{z_{A}} .
$$

Proof. For each $z$, the pairing $\langle,\rangle_{z}$ on $\left(\mathcal{V}_{n}\right)_{z}$ is induced by the pairing on $V_{n}$ and the isomorphism $\left(\mathcal{V}_{n}\right)_{z} \simeq V_{n}$ corresponds under the above map to the cup product pairing $\cup$ on the de Rham cohomology of $A_{z}^{m}$ (see [IS03, Remark 5.12]). Let $\varrho=\left(\varphi^{m}, \mathrm{id}^{m}\right): A_{0}^{m} \rightarrow A^{m} \times A_{0}^{m}$. Then we have $\varrho\left(A_{0}^{m}\right)=\Upsilon_{\varphi}$ and $\varrho_{*}\left(1_{A_{0}^{m}}\right)=c \ell_{A^{m} \times A_{0}^{m}}\left(\Upsilon_{\varphi}\right)$, where $1_{A_{0}^{m}} \in H_{\mathrm{dR}}^{0}\left(A_{0}^{m}\right)$ is the identity element. Thus

$$
\begin{aligned}
\left\langle v_{A} \otimes v_{A_{0}}, c \ell\left(\Delta_{\varphi}\right)\right\rangle_{z_{A}, z_{A_{0}}} & =\eta_{A^{m} \times A_{0}^{m}}\left(\Theta\left(v_{A} \otimes v_{A_{0}}\right) \cup\left(c c_{A^{m} \times A_{0}^{m}}\left(\Upsilon_{\varphi}\right)\right)\right) \\
& =\eta_{A^{m} \times A_{0}^{m}}\left(\Theta\left(v_{A} \otimes v_{A_{0}}\right) \cup \varrho_{*}\left(1_{A_{0}^{m}}\right)\right) \\
& =\eta_{A^{m} \times A_{0}^{m}}\left(\left(p_{A}^{*}\left(v_{A}\right) \cup p_{A_{0}}^{*}\left(v_{A_{0}}\right)\right) \cup \varrho_{*}\left(1_{A_{0}^{m}}\right)\right) .
\end{aligned}
$$

It turns out that

$$
\begin{aligned}
\eta_{\left(A \times A_{0}\right)^{m}}\left(\left(p_{A}^{*}\left(v_{A}\right) \cup p_{A_{0}}^{*}\left(v_{A_{0}}\right)\right) \cup \varrho_{*}\left(1_{A_{0}^{m}}\right)\right) & =\eta_{A_{0}^{m}}\left(\varrho^{*}\left(p_{A}^{*}\left(v_{A}\right) \cup p_{A_{0}}^{*}\left(v_{A_{0}}\right)\right) \cup 1_{A_{0}^{m}}\right) \\
& =\eta_{A_{0}^{m}}\left(\varphi^{*}\left(v_{A}\right) \cup v_{A_{0}}\right)
\end{aligned}
$$

Therefore

$$
\left\langle v_{A} \otimes v_{A_{0}}, c l\left(\Delta_{\varphi}\right)\right\rangle_{z_{A}, z_{A_{0}}}=\eta_{A_{0}^{m}}\left(\varphi^{*}\left(v_{A}\right) \cup v_{A_{0}}\right)=\eta_{A^{m}}\left(v_{A} \cup \varphi_{*}\left(v_{A_{0}}\right)\right) .
$$

Now the term on the right of the last displayed equation coincides with $\left\langle v_{A}, \varphi_{*}\left(v_{A_{0}}\right)\right\rangle_{z_{A}}$, and the result follows.

Theorem 5.5. Let $\varphi: A_{0} \rightarrow A$ and $v \in\left(\mathcal{V}_{n}\right)_{z_{A_{0}}}$. Then

$$
\operatorname{AJ}_{p}\left(\Delta_{\varphi}\right)(f \otimes v)=\left\langle F_{\omega_{f}}\left(z_{A}\right), \varphi_{*}(v)\right\rangle_{z_{A}} .
$$

Proof. Recall that $\operatorname{Res}_{z_{A}}(\chi)=\operatorname{Res}_{z_{A}}(\alpha)=c \ell\left(\Delta_{\varphi}\right)$, where the first equality follows because $\operatorname{Res}_{z_{A}}\left(j_{*}(\beta)\right)=0$. Combining this with (28), Lemma 5.2 and Lemma 5.3) we obtain

$$
\operatorname{AJ}_{p}\left(\Delta_{\varphi}\right)(f \otimes v)=\left\langle F_{\omega_{f}}\left(z_{A}\right) \otimes v, c \ell\left(\Delta_{\varphi}\right)\right\rangle_{z_{A}, z_{A_{0}}} .
$$

The result follows then from Lemma 5.4 .

Corollary 5.6. Let $\varphi: A_{0} \rightarrow A, \varphi^{\vee}: A \rightarrow A_{0}$ the dual isogeny, and $v \in\left(\mathcal{V}_{n}\right)_{z_{A}}$. Denote $\operatorname{deg}(\varphi)$ the degree of $\varphi$. Then

$$
\operatorname{AJ}_{p}\left(\Delta_{\varphi}\right)\left(f \otimes \varphi_{*}^{\vee}(v)\right)=\operatorname{deg}(\varphi) \cdot\left\langle F_{\omega_{f}}\left(z_{A}\right), v\right\rangle_{z_{A}}
$$

Proof. Let $\operatorname{deg}(\varphi)$ denote multiplication by $\operatorname{deg}(\varphi)$ map on $A$ and $A_{0}$. The result follows from Theorem 5.5 observing that $\operatorname{deg}(\varphi)_{*}=\left(\varphi \circ \varphi^{\vee}\right)_{*}=\varphi_{*} \circ \varphi_{*}^{\vee}$.

## 6. Anticyclotomic $p$-Adic $L$-FUnCtions

This section contains the main result of this paper, in which we connect our generalised Heegner cycles to certain semidefinite integrals and anticyclotomic $p$-adic $L$-functions extensively studied in the literature, especially in [BD96, BD98, BD07, BDIS02, [IS03], Sev14]. The setting is as before: $N=p N^{+} N^{-}$is a factorisation of the integer $N \geq 1$ into coprime integers $p, N^{+}, N^{-}$with $p \nmid N^{+} N^{-}$a prime number, and $N^{-}$be a square-free product of an odd number of factors; $K$ is a quadratic imaginary field such that all primes dividing $N^{+}$ (respectively, $p N^{-}$) are split (respectively, inert) in $K$. We also fix an integer $k_{0} \geq 4$, and a modular form $f$ of level $\Gamma_{0}(N)$ and weight $k_{0}$. We put $n_{0}=k_{0}-2$ and $m_{0}=n_{0} / 2$.
6.1. Measure valued modular forms. We begin by recalling some results from BD07] and [Sev14], to which the reader is referred to for details. Let $\mathcal{D}\left(\mathbb{Z}_{p}^{\times}\right)$the $\mathbb{Q}_{p}$-algebra of locally analytic distributions on $\mathbb{Z}_{p}^{\times}$. For each $\mathbb{Z}_{p}$-lattice $L \subseteq \mathbb{Q}_{p}^{2}$, denote $L^{\prime}$ the subset of $L$ consisting of primitive vectors (if $L=\mathbb{Z}_{p} v_{1} \oplus \mathbb{Z}_{p} v_{2}$, then $L^{\prime}$ consists of those $v=a v_{1}+b v_{2}$ such that at least one of $a$ and $b$ is not divisible by $p$. For each lattice $L$, denote by $\mathcal{D}\left(L^{\prime}\right)$ the $\mathbb{Q}_{p}$-vector space of locally analytic distributions on $L^{\prime}$, i.e. $\mathcal{D}\left(L^{\prime}\right)=\operatorname{Hom}_{\mathbb{Q}_{p} \text {-cont }}\left(\mathcal{A}\left(L^{\prime}\right), \mathbb{Q}_{p}\right)$, where $\mathcal{A}\left(L^{\prime}\right)$ is the $\mathbb{Q}_{p}$-vector space of $\mathbb{Q}_{p}$-valued locally analytic functions on $L^{\prime}$. Since $L^{\prime}$ is $\mathbb{Z}_{p}^{\times}$-stable, there is a natural $\mathcal{D}\left(\mathbb{Z}_{p}^{\times}\right)$-module structure on $\mathcal{D}\left(L^{\prime}\right)$, defined by the formula

$$
\int_{L^{\prime}} F(x, y) d(r \mu)(x, y):=\int_{\mathbb{Z}_{p}^{\times}}\left(\int_{L^{\prime}} F(t x, t y) d \mu(x, y)\right) d r(t)
$$

Let $A(U)$ be the $\mathbb{Q}_{p}$-affinoid algebra of an open affinoid disk $U \subset \mathcal{W}$, where

$$
\mathcal{W}:=\operatorname{Hom}_{\operatorname{cont}}\left(\mathbb{Z}_{p}^{\times}, \mathbb{Q}_{p}^{\times}\right)
$$

We view $\mathbb{Z} \subseteq \mathcal{W}$ via the map which takes $k$ to the homomorphism $x \mapsto x^{k-2}$. The $\mathbb{Q}_{p}$-affinoid algebra $A(U)$ has a $\mathcal{D}\left(\mathbb{Z}_{p}^{\times}\right)$-module structure given by the map $\mathcal{D}\left(\mathbb{Z}_{p}^{\times}\right) \rightarrow A(U)$ defined by $r \mapsto\left[\kappa \mapsto \int_{\mathbb{Z}_{p}^{\times}} \kappa(t) d r(t)\right]$. Let

$$
\mathcal{D}\left(L^{\prime}, U\right):=A(U) \hat{\otimes}_{\mathcal{D}\left(\mathbb{Z}_{p}^{\times}\right)} \mathcal{D}\left(L^{\prime}\right)
$$

Let $B$ be the definite quaternion algebra over $\mathbb{Q}$ with discriminant $N^{-}$, and let $R$ be a fixed Eichler $\mathbb{Z}[1 / p]$-order of level $N^{+}$in $B$. Fix an Eichler $\mathbb{Z}$-order $\underline{R}$ of $B$ of level $N^{+}$in such a way that $\underline{R}[1 / p]=R$, and let $\mathcal{O}_{B}$ be a maximal $\mathbb{Z}$-order of $B$ containing $\underline{R}$. We will write $\underline{\hat{R}}$ for the adelisation $\underline{R} \otimes \hat{\mathbb{Z}}$ of $\underline{R}$. For each prime number $\ell \nmid N^{-}$fix a $\mathbb{Q}_{\ell^{\ell}}$-algebra isomorphisms $\iota_{\ell}: B \otimes \mathbb{Q}_{\ell} \xrightarrow{\sim} \mathrm{M}_{2}\left(\mathbb{Q}_{l}\right)$ sending $\mathcal{O}_{B} \otimes \mathbb{Z}_{\ell}$ isomorphically onto $\mathrm{M}_{2}\left(\mathbb{Z}_{\ell}\right)$. Write $\hat{\mathbb{Q}}$ for the ring of finite adéles of $\mathbb{Q}$ and $\hat{B}$ for $B \otimes \hat{\mathbb{Q}}$. Define the level structures $\Sigma=\Sigma\left(N^{+} p, N^{-}\right)=\prod_{\ell} \Sigma_{\ell}$ for

$$
\Sigma_{\ell}= \begin{cases}\left(\mathcal{O}_{B} \otimes \mathbb{Z}_{\ell}\right)^{\times} & \text {if } \ell \nmid N^{+} p \\ \iota_{\ell}^{-1}\left(\Gamma_{0}\left(N^{+} p \mathbb{Z}_{\ell}\right)\right) & \text { if } \ell \mid N^{+} p\end{cases}
$$

where $\Gamma_{0}\left(N^{+} p \mathbb{Z}_{\ell}\right)$ denotes the subgroup of $\mathrm{GL}_{2}\left(\mathbb{Z}_{\ell}\right)$ consisting of matrices which are upper triangular modulo $N^{+} p$. Write $\Sigma_{\infty}$ to denote the open compact subroup obtained from the group $\Sigma$ by replacing the local condition at $p$ with the local condition $\iota_{p}\left(\Sigma_{\infty, p}\right)=\mathrm{GL}_{2}\left(\mathbb{Z}_{p}\right)$. Let $S$ be any commutative ring, and $A$ be any $S$-module with an $S$-linear left action of the
semigroup $\mathrm{M}_{2}\left(\mathbb{Z}_{p}\right)$ of matrices with entries in $\mathbb{Z}_{p}$ and non-zero determinant. We define the $S$-module $S(\Sigma, A)$ as the space of $A$-valued automorphic forms on $B^{\times}$of level $\Sigma$, i.e.

$$
S(\Sigma, A)=\left\{\phi: \hat{B}^{\times} \rightarrow A: \phi(g b \sigma)=\iota_{p}\left(\sigma_{p}^{-1}\right) \phi(b)\right\},
$$

where $g \in B^{\times}$(embedded diagonally in $\hat{B}^{\times}$), $b \in \hat{B}^{\times}$and $\sigma \in \Sigma$. Observe that, by the strong approximation theorem for $B, \hat{B}^{\times}=B^{\times} B_{p}^{\times} \Sigma$ and a modular form $\phi$ in $S(\Sigma, A)$ can be viewed as a function on $R^{\times} \backslash B_{p}^{\times} / \iota_{p}^{-1}\left(\Gamma_{0}\left(p \mathbb{Z}_{p}\right)\right)$ or, equivalently, as a function on $\mathrm{GL}_{2}\left(\mathbb{Q}_{p}\right)$ satisfying $\phi(\gamma b \sigma)=\sigma^{-1} \phi(b)$, for all $\gamma \in \iota_{p}\left(R^{\times}\right), b \in \mathrm{GL}_{2}\left(\mathbb{Q}_{p}\right)$ and $\sigma \in \Gamma_{0}\left(p \mathbb{Z}_{p}\right)$.

For any integer $n \geq 0$, we still use the symbol $\mathcal{P}_{n}$ for the $\mathbb{Q}_{p}$-vector space of homogeneous polynomials in two variables of degree $n$, and the same for the dual space $V_{n}$. If $k=n-2$, the space $S\left(\Sigma, V_{n}\right)$ is referred to as the space of weight $k$ automorphic forms on $B$ of level $\Sigma$, and it is denoted by $S_{k}(\Sigma)$. Fix $U \subseteq \mathcal{W}$ a neighborhood of $k_{0}$. Set $L_{*}=\mathbb{Z}_{p}^{2}$. For every integer $k \geq 2$ in $U$, there exists a specialization map

$$
\rho_{k}: S\left(\Sigma_{\infty}, \mathcal{D}\left(L_{*}^{\prime}, U\right)\right) \longrightarrow S_{k}(\Sigma)
$$

defined by

$$
\left(\rho_{k}(\Phi)(g)\right)(P):=\int_{\mathbb{Z}_{p}^{\times} \times p \mathbb{Z}_{p}} P(x, y) d \Phi(g),
$$

for all $g \in \mathrm{GL}_{2}\left(\mathbb{Q}_{p}\right)$ and $P \in \mathcal{P}_{n}$, where $n=k-2$.
Let $\varphi_{f} \in S_{k_{0}}\left(\Sigma\left(N^{+} p, N^{-}\right)\right)$be the modular form corresponding to $f$ via the JacquetLanglands correspondence, normalised as in [Sev14, §3.2]. By [Sev14, Theorem 3.7] (see also [LV12]) there exists a connected neighborhood $U \subseteq \mathcal{W}$ of $k_{0}$ and

$$
\begin{equation*}
\Phi \in S\left(\Sigma_{\infty}, \mathcal{D}\left(L_{*}^{\prime}, U\right)\right) \tag{32}
\end{equation*}
$$

such that $\rho_{k_{0}}(\Phi)=\varphi_{f}$.
6.2. Semidefinite integrals and generalised Heegner cycles. Choose the branch of the $p$-adic logarithm $\log _{f}$ as in Sev14, §5.2]. Recall the element $\Phi$ in (32). Out of $\Phi$, one constructs as explained in Sev14 Proposition 3.5], a collection of measure $\left\{\mu_{L}\right\}_{L}$ with $\mu_{L} \in \mathcal{D}\left(L^{\prime}, U\right)$ indexed by lattices $L$ of $\mathbb{Q}_{p}^{2}$.

For the next definition of semidefinite integral, which can be found in Sev14, Section 5.2], we use the following notation: for any point $z \in \mathcal{H}_{p}\left(\mathbb{Q}_{p^{2}}\right)$ whose reduction to the special fiber is non-singular, we denote $L_{z}$ the lattice associated with the reduction of $z$ and $\left|L_{z}\right|$ its $p$-adic size; see [Sev14, page 115], to which the reader is referred to for details.
Definition 6.1. The semidefinite integral is the function

$$
\begin{equation*}
(z, Q) \longmapsto \int^{z} Q \omega_{f}:=\frac{1}{\left|L_{z}\right|^{m_{0}}} \frac{d}{d k}\left(\int_{L_{z}^{\prime}} Q(x, y)\langle x-z y\rangle^{k-k_{0}} d \mu_{L_{z}}(x, y)\right)_{\left.\right|_{k=k_{0}}} \tag{33}
\end{equation*}
$$

defined for $Q \in \mathcal{P}_{n_{0}}$ and $z \in \mathcal{H}_{p}\left(\mathbb{Q}_{p^{2}}\right)$ whose reduction to the special fiber is non-singular.
We now connect semidefinite integrals and generalised Heegner cycles. For each $Q \in \mathcal{P}_{n_{0}}$, denote $Q^{\vee}$ the element in $V_{n_{0}}$ defined by $Q^{\vee}(P)=\langle Q, P\rangle_{\mathcal{P}_{n_{0}}}$ for $P \in \mathcal{P}_{n_{0}}$. For a fixed $z \in \mathcal{H}_{p}\left(\mathbb{Q}_{p}^{\text {unr }}\right)$ define the following element of $V_{n_{0}}$ :

$$
Q \longmapsto\left\langle F_{\omega_{f}}(z), Q^{\vee}\right\rangle_{V_{n_{0}}}
$$

where we identify as above $\left(\mathcal{V}_{n_{0}}\right)_{z}$ with $V_{n_{0}}$; recall that $F_{\omega_{f}}$ is the Coleman primitive of $\omega_{f}$.
Lemma 6.2. One has
(1) $\left\langle F_{\omega_{f}}(\gamma(z)), Q^{\vee}\right\rangle_{V_{n_{0}}}=\left\langle F_{\omega_{f}}(z),(Q \cdot \gamma)^{\vee}\right\rangle_{V_{n_{0}}}$, for every $\gamma \in \Gamma$;
(2) $\left\langle F_{\omega_{f}}\left(z_{2}\right), Q^{\vee}\right\rangle_{V_{n_{0}}}-\left\langle F_{\omega_{f}}\left(z_{1}\right), Q^{\vee}\right\rangle_{V_{n_{0}}}=\int_{z_{1}}^{z_{2}} f(z) Q(z) d z$.

Proof. The second statement is a consequence of (11) and the definition of Coleman primitive, since

$$
d\left\langle F_{\omega_{f}}(z), Q^{\vee}\right\rangle=f(z)\left\langle\partial^{0}, Q^{\vee}\right\rangle d z=f(z) Q(z) d z
$$

We need to prove (1). Since $f$ has level $\Gamma$, its Coleman primitive $F_{\omega_{f}}$ is $\Gamma$-invariant, i.e. $\gamma F_{\omega_{f}}=F_{\omega_{f}}$ for every $\gamma \in \Gamma$, where $\left(\gamma F_{\omega_{f}}\right)(z):=\gamma F_{\omega_{f}}\left(\gamma^{-1} z\right)$ (note that the action on the right hand side is the one on $\left.V_{n}\right)$. This means that $F_{\omega_{f}}(\gamma(z))=\gamma F_{\omega_{f}}(z)$ for every $\gamma \in \Gamma$. Recall that $\left\langle A v_{1}, v_{2}\right\rangle_{V_{n}}=\left\langle v_{1}, \bar{A} v_{2}\right\rangle_{V_{n}}$; thus, for every $\gamma \in \Gamma$ we have

$$
\left\langle\gamma F_{\omega_{f}}(z), Q^{\vee}\right\rangle_{V_{n_{0}}}=\left\langle F_{\omega_{f}}(z), \gamma^{-1} Q^{\vee}\right\rangle_{V_{n_{0}}}=\left\langle F_{\omega_{f}}(z),(Q \cdot \gamma)^{\vee}\right\rangle_{V_{n_{0}}}
$$

which proves (1).
Theorem 6.3. Let $\varphi: A_{0} \rightarrow A$ be an isogeny and $Q^{\vee}=v$ for some $v \in\left(\mathcal{V}_{n_{0}}\right)_{z_{A}}$. Then

$$
\operatorname{deg}(\varphi) \cdot \int^{z_{A}} Q \omega_{f}=\operatorname{AJ}_{p}\left(\Delta_{\varphi}\right)\left(f \otimes \varphi_{*}^{\vee}(v)\right)
$$

Proof. By [Sev14, Lemma 5.6], there is a unique function $(z, Q) \mapsto F(z, Q)$ for $z \in \mathcal{H}\left(\mathbb{Q}_{p^{2}}\right)$, $Q \in \mathcal{P}_{n_{0}}$ satisfying the following properties:
(1) $F(\gamma(z), Q)=F(z, Q \cdot \gamma)$,
(2) $F\left(z_{1}, Q\right)-F\left(z_{2}, Q\right)=\int_{z_{2}}^{z_{1}} f(z) Q(z) d z$,
for all $z, z_{1}, z_{2}$ and all $Q$. By Lemma 6.2 we have

$$
\begin{equation*}
\int^{z_{A}} Q \omega_{f}=\left\langle F_{\omega_{f}}\left(z_{A}\right), Q^{\vee}\right\rangle_{V_{n_{0}}} \tag{34}
\end{equation*}
$$

The result follows then from Corollary 5.6.
6.3. Heegner points, optimal embeddings and false isogenies. A Heegner point (of conductor 1) on the Shimura curve $X=X_{N^{+}, p N^{-}}$is a point on $X$ corresponding to an abelian surface $A$ with quaternionic multiplication and level $N^{+}$structure, such that the ring of endomorphisms of $A$ (over an algebraic closure of $\mathbb{Q}$ ) which commute with the quaternionic action and respect the level $N^{+}$structure is isomorphic to $\mathcal{O}_{K}$. The theory of complex multiplication implies that they are all defined over the Hilbert class field $H$ of $K$. We denote Heeg $\left(\mathcal{O}_{K}\right)$ denotes the set of Heegner points of conductor 1 on $X$.

We now recall Shimura reciprocity law, referring to [HB15, §2.5] for details. Fix an ideal $\mathfrak{a} \subseteq \mathcal{O}_{K}$ and an Heegner point $z$. We have then an embedding $\iota_{z}: K \hookrightarrow \mathcal{B}$, and since the class number of the indefinite quaternion algebra $\mathcal{B}$ is equal to 1 , there is $\alpha \in \mathcal{B}$ such that $\iota_{z}(\mathfrak{a}) \mathcal{R}_{\max }=\alpha \mathcal{R}_{\max }$. Right multiplication by $\alpha$ gives a false isogeny $\varphi_{\alpha}: A_{z} \rightarrow A_{\alpha(z)}$, where for any point $x \in X$ we let $A_{x}$ denote the false elliptic curve corresponding to $x$. If $\left(\mathfrak{a}, N^{+} M\right)=1$ then this is a false isogeny of degree prime to $N^{+} M$. Since $\alpha(z)$ only depends on $\mathfrak{a}$ and not on the choice of $\alpha$, we may write $\alpha(z)=\mathfrak{a} \star z, A_{\mathfrak{a} \star z}=A_{\alpha(z)}$ and $\varphi_{\mathfrak{a}}=\varphi_{\alpha}$. If we denote $\sigma_{\mathfrak{a}}$ the element in $\operatorname{Gal}(H / K)$ corresponding to $\mathfrak{a}$ via the arithmetically normalized Artin reciprocity map, Shimura reciprocity law shows that $\sigma_{\mathfrak{a}}(z)=\mathfrak{a} \star z$. Moreover, if we denote $\mathcal{W}$ the group of Atkin-Lehner involutions acting on $X$, the action of $\mathcal{W} \times \operatorname{Gal}(H / K)$ on the set $\operatorname{Heeg}\left(\mathcal{O}_{K}\right)$ is simply transitive (see BD07, §2.3] or [IS03, page 366]). Fixed a point $z_{0}$ corresponding to the false elliptic curve $A_{0}$, the correspondences $\mathfrak{a} \mapsto \mathfrak{a} \star z_{0}$ and $\mathfrak{a} \mapsto \varphi_{\mathfrak{a}}: A_{0}=A_{z_{0}} \rightarrow A_{\mathfrak{a} \star z_{0}}$ set up a bijection

$$
\begin{equation*}
\operatorname{Heeg}\left(\mathcal{O}_{K}\right) \longleftrightarrow \operatorname{Isog}\left(A_{0}\right) \tag{35}
\end{equation*}
$$

where $\operatorname{Isog}\left(A_{0}\right)$ denotes the set of false isogenies $\varphi: A_{0} \rightarrow A$ of degree prime to $N^{+} M$.
An embedding of $\mathbb{Q}$-algebras $\Psi: K \rightarrow B$ is called optimal of level $N^{+}$if $\Psi^{-1}(R)=\mathcal{O}_{K}[1 / p]$. The group $\Gamma$ acts by conjugation on the set of optimal embeddings. Let $\operatorname{Emb}\left(\mathcal{O}_{K}\right)$ be the set
of $\Gamma$-conjugacy classes of optimal embeddings, which is non-empty under our assumption (see [BD96, Lemma 2.1]). By [BD98, Theorem 5.3] there exists a bijection

$$
\begin{equation*}
\operatorname{Heeg}\left(\mathcal{O}_{K}\right) \longleftrightarrow \operatorname{Emb}\left(\mathcal{O}_{K}\right) \tag{36}
\end{equation*}
$$

We briefly describe how this bijection is obtained. Let $z_{A}$ be an Heegner point corresponding to the abelian surface $A$. Let $\operatorname{End}(A)$ denote the endomorphism rings of $A$ and $\operatorname{End}(\bar{A})$ the endomorphism rings of the reduction $\bar{A}$ of the abelian varietiy $A$ modulo $p$. Define $\operatorname{End}^{0}(A)=\operatorname{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\operatorname{End}^{0}(\bar{A})=\operatorname{End}(\bar{A}) \otimes_{\mathbb{Z}} \mathbb{Q}$ and let $\operatorname{End}_{\mathcal{B}}^{0}(A)$ and $\operatorname{End}_{\mathcal{B}}^{0}(\bar{A})$ denote the endomorphisms which commute with the action of the quaternion algebra $\mathcal{B}$. We then have $\operatorname{End}_{\mathcal{B}}^{0}(A) \simeq K$ and $\operatorname{End}_{\mathcal{B}}^{0}(\bar{A}) \simeq B$, and the map $\Psi$ associated with $z_{A}$ as in (36) is the reduction of endomorphisms:

$$
\Psi_{A}: K=\operatorname{End}_{\mathcal{B}}^{0}(A) \longrightarrow \operatorname{End}_{\mathcal{B}}^{0}(\bar{A})=B
$$

On the other hand, let $\Psi: K \longrightarrow B$ be an optimal embedding of level $N^{+}$. It determines a local embedding $\Psi: K_{p} \longrightarrow B_{p}$ which we denote in the same way by an abuse of notation. The local embedding $\Psi$ defines an action of $K_{p}^{\times}$on $\mathcal{H}_{p}\left(K_{p}\right)$ which has two fixed points, $z_{\Psi}$ and $\bar{z}_{\Psi}$. The Heegner point associated to $\Psi$ by (36) is the point on $X$ corresponding via the Cerednik-Drinfeld uniformization to the class modulo $\Gamma$ of $z_{\Psi}$. Abusing notation, in the following we will use the symbol $z_{\Psi}$ to denote both the fixed point in $\mathcal{H}_{p}\left(K_{p}\right)$ and its class in $\Gamma \backslash \mathcal{H}_{p}\left(K_{p}\right)=X\left(K_{p}\right)$.

In light of the previous paragraphs, given $\varphi: A_{0} \rightarrow A \in \operatorname{Isog}\left(A_{0}\right)$, we denote $z_{\varphi}$ the Heegner points corresponding to $\varphi$ by (35) and $\Psi_{\varphi}$ the optimal embedding corresponding to $z_{\varphi}$ by (36). For $\varphi$ the identity map, we denote $z_{\varphi}$ by $z_{0}$ and $\Psi_{\varphi}$ by $\Psi_{0}$. Moreover, if we start with an optimal embedding $\Psi$, we denote $z_{\Psi}$ the Heegner point corresponding to $\Psi$ by (36) and $\varphi_{\Psi}$ the false isogeny corresponding to $z_{\Psi}$ by (35). Finally, if we start with an Heegner point $z$, we denote $\Psi_{z}$ the optimal embedding corresponding to $z$ via (36) and $\varphi_{z}: A_{0} \rightarrow A_{z}$ the false isogeny corresponding to $z$ via (35). We also introduce a convention for the Galois action: for any $\sigma=\sigma_{\mathfrak{a}} \in \operatorname{Gal}(H / K)$, we denote $z_{A}^{\sigma}=\mathfrak{a} \star z_{A}, A^{\sigma}=A_{\sigma\left(z_{A}\right)}$ and $\Psi^{\sigma}=\Psi_{z_{A}}$.

Denote $z \mapsto \bar{z}$ the action of the non-trivial automorphism $c \in \operatorname{Gal}\left(\mathbb{Q}_{p^{2}} / \mathbb{Q}_{p}\right)$ on $\mathcal{H}_{p}\left(\mathbb{Q}_{p^{2}}\right)$. For each optimal embedding $\Psi$, denote

$$
P_{\Psi}(x, y)=c x^{2}+(d-a) x y-b y^{2}=A_{\Psi}\left(x-z_{\Psi} y\right)\left(x-\bar{z}_{\Psi} y\right)
$$

the polynomial associated to $\Psi$, where $\iota_{p}(\Psi(\sqrt{D}))=\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)$ and $D$ is the discriminant of $K$ (cf. as in BD07, (84)]). Define the the polynomials

$$
Q_{\Psi}^{(j)}(x, y)=\left(x-z_{\Psi} y\right)^{m_{0}+j}\left(x-\bar{z}_{\Psi} y\right)^{m_{0}-j}
$$

for any positive integer $k$ and any integer $j=-n_{0} / 2, \ldots, n_{0} / 2$. Put $v_{\Psi}^{(j)}=\left(Q_{\Psi}^{(j)}\right)^{\vee}$ and define

$$
v_{\varphi}^{(j)}=\varphi_{*}^{\vee}\left(v_{\Psi \varphi}^{(j)}\right)
$$

Proposition 6.4. Let $\varphi: A_{0} \rightarrow A$ be a false isogeny. Then

$$
\operatorname{deg}(\varphi) \cdot \int^{z_{\varphi}} Q_{\Psi_{\varphi}}^{(j)} \omega_{f}=\operatorname{AJ}_{p}\left(\Delta_{\varphi}\right)\left(f \otimes v_{\varphi}^{(j)}\right)
$$

Proof. Since $\varphi_{*}\left(v_{\varphi}^{(j)}\right)=\operatorname{deg}(\varphi) \cdot v_{\Psi \varphi}^{(j)}$, the proposition follows from Theorem 6.3,
Let $\varphi: A_{0} \rightarrow A$ be a false isogeny. The abelian variety $A$ is defined over $H$, and therefore it is also defined over $\mathbb{Q}_{p^{2}}$, because $p$ is inert in $K$ and therefore splits completely in $H$. Let $\bar{A}$ denote the abelian variety obtained by applying to $A$ the non-trivial automorphism $c$ of $\operatorname{Gal}\left(\mathbb{Q}_{p^{2}} / \mathbb{Q}_{p}\right)$, and still denote $c: A \rightarrow \bar{A}$ the map induced by $c$. If $\varphi: A_{0} \rightarrow A$ is a false isogeny, then we denote $\bar{\varphi}=c \circ \varphi: A_{0} \rightarrow \bar{A}$ the isogeny obtained by composition $\varphi$ with $c: A \rightarrow \bar{A}$. Let $W_{p}: X \rightarrow X$ denote the Atkin-Lehner involution at $p$. If we denote $w_{p}$ any
element of $\mathcal{R}^{\times}$such that the $p$-adic valuation of its norm is equal to 1 , which we fix from now on, then we have $W_{p}(z)=w_{p}(z)$. We have (see e.g. [BD98, Theorem 4.7])

$$
\begin{equation*}
z_{\bar{A}}=w_{p}\left(\bar{z}_{A}\right) \tag{37}
\end{equation*}
$$

For the next result, define

$$
\bar{v}_{\varphi}^{(j)}=\bar{\varphi}_{*}^{\vee}\left(\left(Q_{\Psi_{\varphi}}^{(j)} \mid w_{p}\right)^{\vee}\right)
$$

Proposition 6.5. Let $\varphi: A_{0} \rightarrow A$ be an isogeny. Then

$$
\operatorname{deg}(\varphi) \cdot \int^{\bar{z}_{\varphi}} Q_{\Psi_{\varphi}}^{(j)} \omega_{f}=\omega_{p} \cdot \operatorname{AJ}_{p}\left(\Delta_{\bar{\varphi}}\right)\left(f \otimes \bar{v}_{\varphi}^{(j)}\right)
$$

where $\omega_{p} \in\{ \pm 1\}$ is the eigenvalue of the Atkin-Lehner involution at $p$ acting on $f$.
Proof. By (37), and the fact that $W_{p}$ is an involution, we have

$$
\int^{\bar{z}_{\varphi}} Q_{\Psi \varphi}^{(j)} \omega_{f}=\int^{w_{p}\left(z_{\bar{\varphi}}\right)} Q_{\Psi}^{(j)} \omega_{f}
$$

Since $W_{p}$ acts on $F_{\omega_{f}}$ as multiplication by $\omega_{p} \in\{ \pm 1\}$, one easily checks (using the same calculations as in Lemma 6.2) that

$$
\int^{w_{p}\left(z_{\bar{\varphi}}\right)} Q_{\Psi_{\varphi}}^{(j)} \omega_{f}=\omega_{p} \cdot \int^{z_{\bar{\varphi}}}\left(Q_{\Psi_{\varphi}}^{(j)} \mid w_{p}\right) \omega_{f}
$$

The result follows then from Theorem 6.3.
6.4. Two variables anticyclotomic $p$-adic $L$-functions. For each optimal embedding $\Psi$, we consider the lattice $L_{\Psi}=L_{z_{\Psi}}$; recall that this lattice is characterised up to homothety by the condition that $L_{\Psi}$ is stable by the action of $\Psi\left(K_{p}^{\times}\right)$, where $K_{p}=K \otimes_{\mathbb{Q}} \mathbb{Q}_{p} \simeq \mathbb{Q}_{p^{2}}$ ( cf. BD07, §3.2]). Recall that the function $(x, y) \mapsto \operatorname{ord}_{p}\left(P_{\Psi}(x, y)\right)$ is constant on $L_{\Psi}^{\prime}$, and its constant value is equal to $\operatorname{ord}_{p}\left(\left|L_{\Psi}\right|\right)$ (see BD07, Lemma 3.7]). Therefore, by eventually translating $\left(\Psi, L_{\Psi}\right)$ by an appropriate element of $R^{\times}$in such a way that $\left|L_{\Psi}\right|=1$, we have $\left\langle P_{\Psi}(x, y)\right\rangle=P_{\Psi}(x, y)$ for all $(x, y) \in L_{\Psi}^{\prime}$. Moreover, note that

$$
Q_{\Psi}^{(j)}(x, y)=\frac{P_{\Psi}(x, y)^{m_{0}}}{A_{\Psi}^{m_{0}}}\left(\frac{x-z_{\Psi} y}{x-\bar{z}_{\Psi} y}\right)^{j}
$$

If $j \equiv 0(\bmod p+1)$, then we have

$$
\left(\frac{x-z_{\Psi} y}{x-\bar{z}_{\Psi} y}\right)^{j}=\left\langle\frac{x-z_{\Psi} y}{x-\bar{z}_{\Psi} y}\right\rangle^{j}
$$

for all $(x, y) \in L_{\Psi}^{\prime}$. In fact, the $p$-adic valuation of $x-z_{\Psi} y$ and $x-\bar{z}_{\Psi} y$ are equal and, if $x-z_{\Psi} y=\zeta\left\langle x-z_{\Psi} y\right\rangle$ then $x-\bar{z}_{\Psi} y=\bar{\zeta}\left\langle x-\bar{z}_{\Psi} y\right\rangle$, where $\zeta$ is a $\left(p^{2}-1\right)$-th root of unity. Since $\frac{\zeta}{\zeta}=\frac{\zeta}{\zeta^{p}}=\zeta^{p^{2}-p}$, if $j \equiv 0(\bmod p+1)$ then $\zeta^{j\left(p^{2}-p\right)}=1$.

Definition 6.6. The partial two-variable anticyclotomic p-adic L-function associated to $\Phi$ and $[\Psi] \in \operatorname{Emb}\left(\mathcal{O}_{K}\right)$ is the function defined for $(k, s) \in U \times \mathbb{Z}_{p}$ as

$$
\mathcal{L}_{p}(\Phi / K, \Psi, k, s)=\frac{A_{\Psi}^{\frac{k-k_{0}}{2}}}{\left|L_{\Psi}\right|^{m_{0}}} \int_{L_{\Psi}^{\prime}} P_{\Psi}^{m_{0}}(x, y)\left\langle x-z_{\Psi} y\right\rangle^{s-k_{0} / 2}\left\langle x-\bar{z}_{\Psi} y\right\rangle^{k-s-k_{0} / 2} d \mu_{L_{\Psi}}
$$

The restriction of $\mathcal{L}_{p}(\Phi / K, \Psi, k, s)$ to the line $s=k / 2+j$, for $-n / 2 \leq j \leq n / 2$ an integer, is then the function

$$
\mathcal{L}_{p}^{(j)}(\Phi / K, \Psi, k)=\frac{A_{\Psi}^{\frac{k-k_{0}}{2}}}{\left|L_{\Psi}\right|^{m_{0}}} \int_{L_{\Psi}^{\prime}} P_{\Psi}^{m_{0}}(x, y)\left\langle\frac{x-z_{\Psi} y}{x-\bar{z}_{\Psi} y}\right\rangle^{j}\left\langle x-z_{\Psi} y\right\rangle^{\frac{k-k_{0}}{2}}\left\langle x-\bar{z}_{\Psi} y\right\rangle^{\frac{k-k_{0}}{2}} d \mu_{L_{\Psi}}
$$

Proposition 6.7. Let $\varphi: A_{0} \rightarrow A$ be a false isogeny. Suppose that $j \equiv 0(\bmod p+1)$. Then we have $\mathcal{L}_{p}^{(j)}\left(\Phi / K, \Psi_{\varphi}, k_{0}\right)=0$ and

$$
\frac{d}{d k}\left(\mathcal{L}_{p}^{(j)}\left(\Phi / K, \Psi_{\varphi}, k\right)\right)_{\left.\right|_{k=k_{0}}}=\frac{A_{\Psi_{\varphi}}^{m_{0}}}{2 \operatorname{deg}(\varphi)}\left(\operatorname{AJ}_{p}\left(\Delta_{\varphi}\right)\left(f \otimes v_{\varphi}^{(j)}\right)+\omega_{p} \cdot \operatorname{AJ}_{p}\left(\Delta_{\bar{\varphi}}\right)\left(f \otimes \bar{v}_{\varphi}^{(j)}\right)\right) .
$$

Proof. The congruence conditions imposed to $j$ combined with the observations before Definition 6.6 imply that

$$
\mathcal{L}_{p}^{(j)}\left(\Phi / K, \Psi_{\varphi}, k\right)=\frac{A_{\Psi_{\varphi}}^{\frac{k-k_{0}}{2}}}{\left|L_{\Psi_{\varphi}}\right|^{m_{0}}} \int_{L_{\Psi_{\varphi}}^{\prime}} A_{\Psi_{\varphi}}^{m_{0}} Q_{\Psi_{\varphi}}^{(j)}(x, y)\left\langle x-z_{\varphi} y\right\rangle^{\frac{k-k_{0}}{2}}\left\langle x-\bar{z}_{\varphi} y\right\rangle^{\frac{k-k_{0}}{2}} d \mu_{L_{\Psi_{\varphi}}} .
$$

The value at $k_{0}$ is then

$$
\mathcal{L}_{p}^{(j)}\left(\Phi / K, \Psi_{\varphi}, k_{0}\right)=\frac{A_{\Psi_{\varphi}}^{m_{0}}}{\left|L_{\Psi_{\varphi}}\right|^{m_{0}}} \int_{L_{\Psi_{\varphi}}^{\prime}} Q_{\Psi_{\varphi}}^{(j)}(x, y) d \mu_{L_{\Psi_{\varphi}}}
$$

which is equal to 0 by [Sev14, Propositions 3.8 and 6.2]. By [Sev14, Proposition 3.1], for any $Q \in \mathcal{P}_{n_{0}}$, any lattice $L$ and any $z_{1}, z_{2} \in \mathcal{H}_{p}\left(\mathbb{Q}_{p^{2}}\right)$ we have

$$
\frac{d}{d k}\left(\int_{L^{\prime}} Q(x, y)\left\langle x-z_{1} y\right\rangle^{\frac{k-k_{0}}{2}}\left\langle x-z_{2} y\right\rangle^{\frac{k-k_{0}}{2}} d \mu_{L}\right)_{\mid k=k_{0}}
$$

is the sum

$$
\frac{1}{2} \frac{d}{d k}\left(\int_{L^{\prime}} Q(x, y)\left\langle x-z_{1} y\right\rangle^{k-k_{0}} d \mu_{L}\right)_{\mid k=k_{0}}+\frac{1}{2} \frac{d}{d k}\left(\int_{L^{\prime}} Q(x, y)\left\langle x-z_{2} y\right\rangle^{k-k_{0}} d \mu_{L}\right)_{\mid k=k_{0}}
$$

If we take $L=L_{z_{\varphi}}=L_{\Psi_{\varphi}}, z_{1}=z_{\varphi}$ and $z_{2}=\bar{z}_{\varphi}$, the first summand in the above formula is $\frac{1}{2}\left|L_{\Psi_{\varphi}}\right|^{m_{0}} \int^{z_{\varphi}} Q \omega_{f}$, while the second summand is

$$
\frac{1}{2} \frac{d}{d k}\left(\int_{L_{z_{\varphi}}} Q(x, y)\left\langle x-\bar{z}_{\varphi} y\right\rangle^{k-k_{0}} d \mu_{L_{z_{\varphi}}}\right)_{\mid k=k_{0}} .
$$

We now observe that $L_{z_{\varphi}}=L_{\bar{z}_{\varphi}}$ and therefore the second summand is $\frac{1}{2}\left|L \Psi_{\varphi}\right|^{m_{0}} \int^{\bar{z}_{\varphi}} Q \omega_{f}$ : this is because, as recalled above, lattice $L_{\Psi}$ attached to an optimal embedding $\Psi: K \rightarrow B$ is characterised up to homothety by the condition that $L_{\Psi}$ is stable by the action of $\Psi\left(\mathbb{Q}_{p^{2}}\right)$. The result then follows from Proposition 6.4 and Proposition 6.5

Let $K_{\infty}$ be the maximal anticyclotomic extension of $K$ which is unramified outside $p$. Write $\tilde{G}$ for $\operatorname{Gal}\left(K_{\infty} / K\right)$ and $\Delta$ for $\operatorname{Gal}(H / K)$. As recalled above, the group $\mathcal{W} \times \Delta$ acts freely and transitively on $\operatorname{Emb}\left(\mathcal{O}_{K}\right)$, and, by the Shimura Reciprocity Law, this action corresponds to the natural action of $\mathcal{W} \times \Delta$ on the set of Heegner points under the bijection (36). Denote by $\Xi$ the set of $\Delta$-orbits in $\operatorname{Emb}\left(\mathcal{O}_{K}\right)$ and fix $\xi \in \Xi$. If $\Delta=\left\{\bar{\delta}_{1}, \ldots, \bar{\delta}_{h}\right\}$ then $\Psi_{i}=\Psi_{0}^{\bar{\delta}_{i}^{-1}}$ are representatives for the elements of $\Xi$, for a fixed $\Psi_{0} \in \operatorname{Emb}\left(\mathcal{O}_{K}\right)$. Let $\chi: \tilde{G} \rightarrow \overline{\mathbb{Q}}_{p}^{\times}$be a character factoring through $\Delta$. The optimal embeddings $\Psi_{i}$ correspond to Heegner points $z_{i}=z_{0}^{\bar{\delta}_{i}^{-1}}$, and these come from isogenies $\varphi_{i}=\varphi_{0}^{\bar{\delta}_{i}^{-1}}: A_{0} \rightarrow A_{i}=A_{z_{i}}$.
Definition 6.8. The two-variable anticyclotomic p-adic L-function associated to $\Phi$ and the character $\chi$ is the function defined for $(k, s) \in U \times \mathbb{Z}_{p}$ as

$$
\mathcal{L}_{p}(\Phi / K, k, s, \chi)=\sum_{i=1}^{h} \chi\left(\delta_{i}\right) \cdot \mathcal{L}_{p}\left(\Phi / K, \Psi_{i}, k, s\right),
$$

where $\delta_{i} \in \tilde{G}$ is a lift of $\bar{\delta}_{i} \in \Delta$.

The restriction of $\mathcal{L}_{p}(\Phi / K, \Psi, k, s)$ to the line $s=k / 2+j$, for $-n / 2 \leq j \leq n / 2$ an integer, is then the function

$$
\mathcal{L}_{p}^{(j)}(\Phi / K, k, \chi)=\sum_{i=1}^{h} \chi\left(\delta_{i}\right) \cdot \mathcal{L}_{p}^{(j)}\left(\Phi / K, \Psi_{i}, k\right)
$$

Theorem 6.9. Suppose that $j \equiv 0(\bmod p+1)$. Then we have $\mathcal{L}_{p}^{(j)}(\Phi / K, k, \chi)=0$ and

$$
\frac{d}{d k}\left(\mathcal{L}_{p}^{(j)}(\Phi / K, k, \chi)\right)_{\mid k=k_{0}}=\sum_{i=1}^{h} \frac{A_{\Psi_{i}}^{m_{0}} \cdot \chi\left(\delta_{i}\right)}{2 \cdot \operatorname{deg}\left(\varphi_{i}\right)}\left(\operatorname{AJ}_{p}\left(\Delta_{\varphi_{i}}\right)\left(f \otimes v_{\varphi_{i}}^{(j)}\right)+\omega_{p} \operatorname{AJ}_{p}\left(\Delta_{\bar{\varphi}_{i}}\right)\left(f \otimes \bar{v}_{\varphi_{i}}^{(j)}\right)\right)
$$

Proof. The result follows from Proposition 6.7 and the definitions.
Remark 6.10. The function $\mathcal{L}_{p}(\Phi / K, k, \chi)=\mathcal{L}_{p}^{(0)}(\Phi / K, k, \chi)$ is a square-root $p$-adic $L$-function, in the sense that the value of

$$
L_{p}(\Phi / K, k, \chi)=\mathcal{L}_{p}(\Phi / K, k, \chi) \cdot \mathcal{L}_{p}\left(\Phi / K, k, \chi^{-1}\right)
$$

at integers $k \geq 2, k \equiv k_{0}(\bmod p-1), k \neq k_{0}$, satisfies an interpolation formula of the following shape:

$$
L_{p}(\Phi / K, k, \chi) \doteq L_{K}^{\mathrm{alg}}\left(f_{k}^{\sharp}, \chi, k / 2\right) .
$$

In the formula above we adopt the following notation. First, for each even integer $k$ as above, let $f_{k}$ be the classical modular form of $\Gamma_{0}(N)$ and weight $k$ which correspond under the Jacquet-Langlands correspondence to the specialization $\rho_{k}(\Phi) \in S_{k}(\Sigma)$ of $\Phi$ in weight $k$, it is well defined up to scalars; a version for families of the Jacquet-Langlands correspondence allows us to see these forms as classical specialisations of a Coleman family $f_{\infty}$ of modular forms. Denote $f_{k}^{\sharp}$ the newform of level $N / p$ whose $p$-stabilisation is $f_{k}$ if $k \neq k_{0}$ or $f$ is old at $p$, and $f_{k_{0}}^{\sharp}=f$ otherwise; $L_{K}^{\text {alg }}\left(f_{k}^{\sharp}, \chi, k / 2\right)$ denote the algebraic part of the value at $s=k / 2$ of the complex $L$-function $L_{K}\left(f_{k}^{\sharp}, \chi, s\right)$, which is obtained by dividing $L_{K}\left(f_{k}^{\sharp}, \chi, k / 2\right)$ by a suitable complex period; the symbol $\doteq$ means that the equality is up to explicit algebraic factors. See [Sev14, Theorem 9.1] for details. It is a very interesting task to investigate similar interpolation properties of $\mathcal{L}_{p}^{(j)}(\Phi / K, k, \chi)$ : the natural question is if $\mathcal{L}_{p}^{(j)}(\Phi / K, k, \chi)$ is related to $L_{K}^{\text {alg }}\left(f_{k}, \chi, k / 2+j\right)$ in a way similar to what happens in the case $j=0$.
6.5. One variable anticyclotomic $p$-adic $L$-functions. In this section we use the results collected in the previous sections to give an extension of the results in Mas12 on the first derivative of the 1 -variable anticyclotomic $p$-adic $L$-function.

Denote by $L_{p}(f / K, \Psi, \star, s)$ the partial anticyclotomic $p$-adic $L$-function of $f$ and $K$ attached to the pair $(\Psi, \star)$, where $\Psi$ is an optimal embedding as in 6 6.3 and $\star \in \mathbb{P}^{1}\left(\mathbb{Q}_{p}\right)$ a base point ([BDIS02); this is a function of the $p$-adic variable $s \in \mathbb{Z}_{p}$ defined by

$$
L_{p}(f / K, \Psi, \star, s)=\int_{G}\langle\alpha\rangle^{s-\frac{k_{0}}{2}} d \mu_{f, \Psi, \star}(\alpha),
$$

where $\langle\alpha\rangle^{t}=\exp \left(t \log _{f}(\langle\alpha\rangle)\right)$ for all $t \in \mathbb{Z}_{p}$ and $\mu_{f, \Psi, \star}$ is the local analytic distribution on $G=K_{p, 1}^{\times}$, the compact subgroup of $K_{p}^{\times}$of elements of norm 1, defined in BDIS02, Section 2.4].

Proposition 6.11. Let $\varphi: A_{0} \rightarrow A$ be a false isogeny. For integer $-n_{0} / 2 \leq j \leq n_{0} / 2$ with $j \equiv 0(\bmod p+1)$ we have $L_{p}\left(f / K, \Psi_{\varphi}, \infty, k_{0} / 2+j\right)=0$ and

$$
L_{p}^{\prime}\left(f / K, \Psi_{\varphi}, \infty, s\right)_{\left\lvert\, s=\frac{k_{0}}{2}+j\right.}=\frac{A_{\Psi_{\varphi}}^{m_{0}}}{\operatorname{deg}(\varphi)}\left(\operatorname{AJ}_{p}\left(\Delta_{\varphi}\right)\left(f \otimes v_{\varphi}^{(j)}\right)-\omega_{p} \cdot \operatorname{AJ}_{p}\left(\Delta_{\bar{\varphi}}\right)\left(f \otimes \bar{v}_{\varphi}^{(j)}\right)\right) .
$$

Proof. We sketch the proof, following closely (Mas12, Theorem 5.3] (but see Remark 6.12). Thanks to the congruence conditions imposed to $j$, have

$$
L_{p}\left(f / K, \Psi_{\varphi}, \infty, k_{0} / 2+j\right)=\int_{G} \alpha^{j} d \mu_{f, \Psi_{\varphi}, \infty}(\alpha)
$$

where now $\alpha^{j}$ is the usual $j$-fold product of $\alpha$ by itself, and therefore the above integral vanishes thanks to Mas12, Lemma 5.1]. For the value of the derivative, we begin by observing that, thanks to the congruence conditions imposed to $j$, we have $\langle\alpha\rangle^{j}=\alpha^{j}$, and therefore

$$
L_{p}^{\prime}\left(f / K, \Psi_{\varphi}, \infty, s\right)_{\left\lvert\, s=\frac{k_{0}}{2}+j\right.}=\int_{G} \log _{f}(\langle\alpha\rangle)\langle\alpha\rangle^{j} d \mu_{f, \Psi_{\varphi}, \infty}(\alpha)
$$

Let now $\mu_{f}$ the measure on $\mathbb{P}^{1}\left(\mathbb{Q}_{p}\right)$ attached to $f$ in Tei90, Proposition 9] using the harmonic cocycle attached to $f$. Then we have

$$
\begin{aligned}
\int_{G} \log _{f}(\langle\alpha\rangle)\langle\alpha\rangle^{j} d \mu_{f, \Psi_{\varphi}, \infty}(\alpha) & =\int_{\mathbb{P}^{1}\left(\mathbb{Q}_{p}\right)} \log _{f}\left(\frac{x-z_{\varphi}}{x-\bar{z}_{\varphi}}\right) \cdot\left(\frac{x-z_{\varphi}}{x-\bar{z}_{\varphi}}\right)^{j} P_{\Psi_{\varphi}}^{m_{0}}(x) d \mu_{f}(x) \\
& =\int_{\mathbb{P}^{1}\left(\mathbb{Q}_{p}\right)}\left(\int_{\bar{z}_{\varphi}}^{z_{\varphi}} \frac{d z}{z-x}\right) \cdot\left(\frac{x-z_{\varphi}}{x-\bar{z}_{\varphi}}\right)^{j} P_{\Psi_{\varphi}}^{m_{0}}(x) d \mu_{f}(x) \\
& =\int_{\bar{z}_{\varphi}}^{z_{\varphi}}\left(\int_{\mathbb{P}^{1}\left(\mathbb{Q}_{p}\right)} \frac{1}{z-x} \cdot\left(\frac{x-z_{\varphi}}{x-\bar{z}_{\varphi}}\right)^{j} P_{\Psi_{\varphi}}^{m_{0}}(x) d \mu_{f}(x)\right) d z \\
& =\int_{\bar{z}_{\varphi}}^{z_{\varphi}}\left(\int_{\mathbb{P}^{1}\left(\mathbb{Q}_{p}\right)} \frac{d \mu_{f}(x)}{z-x}\right) \cdot\left(\frac{z-z_{\varphi}}{z-\bar{z}_{\varphi}}\right)^{j} P_{\Psi_{\varphi}}^{m_{0}}(z) d z \\
& =\int_{\bar{z}_{\varphi}}^{z_{\varphi}} f(z)\left(\frac{z-z_{\varphi}}{z-\bar{z}_{\varphi}}\right)^{j} P_{\Psi_{\varphi}}^{m_{0}}(z) d z
\end{aligned}
$$

where the first equality follows from the definition of the $p$-adic $L$-function in [BDIS02, §2.4], the second equality follows from the definition of Coleman integral, the third follows from the fact that we can reverse the order of integration by applying the reasoning in the proof of Theorem 4 of Tei90, the fourth from the fact that

$$
\int_{\mathbb{P}^{1}\left(\mathbb{Q}_{p}\right)} \frac{1}{z-x} \cdot\left(\frac{x-z_{\Psi}}{x-\bar{z}_{\Psi}}\right)^{j} P_{\Psi}^{m_{0}}(x) d \mu_{f}(x)=\int_{\mathbb{P}^{1}\left(\mathbb{Q}_{p}\right)} \frac{1}{z-x} \cdot\left(\frac{z-z_{\Psi}}{z-\bar{z}_{\Psi}}\right)^{j} P_{\Psi}^{m_{0}}(z) d \mu_{f}(x)
$$

since the two functions inside the integral differ by a polynomial of degree at most $n_{0}$ in $x$, and the last equality follows from Teitelbaum's $p$-adic Poisson inversion formula (we refer to the proof of Mas12, Theorem 5.3] and BDIS02, Theorem 3.5] for details). Combining the above equations we find:

$$
\begin{aligned}
L_{p}^{\prime}\left(f / K, \Psi_{\varphi}, \infty, s\right)_{\left\lvert\, s=\frac{k_{0}}{2}+j\right.} & =\int_{\bar{z}_{\varphi}}^{z_{\varphi}} f(z)\left(\frac{z-z_{\varphi}}{z-\bar{z}_{\varphi}}\right)^{j} P_{\Psi_{\varphi}}^{m_{0}}(z) d z \\
& =A_{\Psi_{\varphi}}^{m_{0}} \int_{\bar{z}_{\varphi}}^{z_{\varphi}} f(z)\left(z-z_{\varphi}\right)^{m_{0}+j}\left(z-\bar{z}_{\varphi}\right)^{m_{0}-j} d z \\
& =A_{\Psi_{\varphi}}^{m_{0}} \int_{\bar{z}_{\varphi}}^{z_{\varphi}} f(z) Q_{\Psi_{\varphi}}^{(j)} d z \\
& =A_{\Psi_{\varphi}}^{m_{0}}\left(\int^{z_{\varphi}} Q_{\Psi_{\varphi}}^{(j)} \omega_{f}-\int^{\bar{z}_{\varphi}} Q_{\Psi_{\varphi}}^{(j)} \omega_{f}\right)
\end{aligned}
$$

The result follows then from Propositions 6.4 and Proposition 6.5,

Remark 6.12. It seem to the authors that [Mas12, Theorem 5.3] only works under the congruence condition, $j \equiv 0(\bmod p+1)$. In the general case we have the equality

$$
L_{p}\left(f / K, \Psi_{\varphi}, \infty, k_{0} / 2+j\right)=\int_{G}\langle\alpha\rangle^{j} d \mu_{f, \Psi_{\varphi}, \infty}(\alpha)
$$

where now the function $\alpha \mapsto\langle\alpha\rangle^{j}$ is locally analytic, and is a polynomial only under the congruence conditions on $j$ considered above. Therefore, if $j$ does not satisfy the congruence conditions $j \equiv 0(\bmod p+1)$ then one can not directly apply [Mas12, Lemma 5.1] to conclude that the value of the $p$-adic $L$-function at $k_{0} / 2+j$ vanishes.

Recall that we denoted by $K_{\infty}$ the maximal anticyclotomic extension of $K$ which is unramified outside $p$, by $\tilde{G}$ the Galois group $\operatorname{Gal}\left(K_{\infty} / K\right)$ and by $\Delta$ the Galois group $\operatorname{Gal}(H / K)$. Class field theory implies that the group $G$ can be identified with $\operatorname{Gal}\left(K_{\infty} / H\right)$. Let $\operatorname{Emb}_{0}\left(\mathcal{O}_{K}\right)$ be the set of $\Gamma$-conjugacy classes of pairs $(\Psi, \star)$ where $\Psi$ is an optimal embedding and $\star \in \mathbb{P}_{1}\left(\mathbb{Q}_{p}\right)$ a base point. The action of $\mathcal{W} \times \Delta$ on $\operatorname{Emb}\left(\mathcal{O}_{K}\right)$ lifts to a simply transitive action of $\mathcal{W} \times \tilde{G}$ on $\operatorname{Emb}_{0}\left(\mathcal{O}_{K}\right)$ such that $G$ acts trivially on $\operatorname{Emb}\left(\mathcal{O}_{K}\right)$. Using this action the distribution $\mu_{f, \Psi, \star}$ on $G$ can be canonically extended to a distribution on $\tilde{G}$ denoted $\mu_{f, K, \xi}$ where $\xi=(\Psi, \star) \in \operatorname{Emb}_{0}\left(\mathcal{O}_{K}\right)$ (see [BDIS02, Section 2.5]). This distribution depends on the choice of $(\Psi, \star)$ only up to translation by an element of $\tilde{G}$, and up to multiplication by $-\omega_{p}= \pm 1$, the negative of the sign of the Atkin-Lehner involution $W_{p}$ acting on $f$ (see [BDIS02, Lemma 2.15]).

Let $\left\{\delta_{1}, \ldots, \delta_{h}\right\}$ be a set of representatives of the elements of $\Delta$ in $\tilde{G}$, and write

$$
\left(\Psi_{i}, \star_{i}\right):=\delta_{i}(\Psi, \star) .
$$

Let $\chi: \tilde{G} \rightarrow \overline{\mathbb{Q}}_{p}^{\times}$be a continuous character of finite order. We can define the anti-cyclotomic $p$-adic L-function attached to $f$ and $K$ twisted by $\chi$ as

$$
L_{p}(f / K, \xi, \chi, s)=\int_{\tilde{G}} \chi(\alpha)\langle\alpha\rangle^{s-\frac{k_{0}}{2}} d \mu_{f, K, \xi}(\alpha) .
$$

If $\chi$ factors through $\Delta, L_{p}(f / K, \xi, \chi, s)$ can be written as a twisted sum of partial $L$-functions

$$
L_{p}(f / K, \xi, \chi, s)=\sum_{i=1}^{h} \chi\left(\delta_{i}\right) L_{p}\left(f / K, \Psi_{i}, \star_{i}, s\right) .
$$

Since $\mathcal{W} \times \tilde{G}$ acts simply transitively on $\operatorname{Emb}_{0}\left(\mathcal{O}_{K}\right)$, for every pair $\left(\Psi_{i}, \star_{i}\right)$ in the previous sum, there exists a unique $\alpha_{i} \in \mathcal{W} \times G \subseteq \mathcal{W} \times \tilde{G}$ such that $\left(\Psi_{i}, \star_{i}\right)=\alpha_{i}\left(\Psi_{i}, \infty\right)$. If we assume that $\alpha_{i} \in G=K_{p, 1}^{\times}$, then we have $L_{p}\left(f / K, \Psi_{i}, \star_{i}, s\right)=\left(\alpha_{i}\right)^{s-\frac{k_{0}}{2}} L_{p}\left(f / K, \Psi_{i}, \infty, s\right)$. We can always do this since the $\Psi_{i}$ 's are in the same $\mathcal{W}$-orbit and, for $w \in \mathcal{W}$

$$
L_{p}(f / K, w \xi, \chi, s)= \pm L_{p}(f / K, \xi, \chi, s)
$$

Thus, up to sign, we can express the first derivative of the anticyclotomic $p$-adic $L$-function as an explicit combination of values of the Abel-Jacobi images of the cycles $\Delta_{\varphi_{i}}$. Here $\varphi_{i}$ denotes the isogeny $A_{0} \rightarrow A_{\Psi_{i}}$ associated to $\Psi_{i}$.

Theorem 6.13. Let $\chi: \tilde{G} \rightarrow \overline{\mathbb{Q}}_{p}^{\times}$be a character factoring through $\Delta$. Then for every integer $j$ such that $-n_{0} / 2 \leq j \leq n_{0} / 2$ and $j \equiv 0(\bmod p+1)$, we have

$$
L_{p}^{\prime}\left(f / K, \xi, \chi, k_{0} / 2+j\right)=\sum_{i=1}^{h} \chi\left(\delta_{i}\right) \alpha_{i}^{j} \frac{A_{\Psi_{\varphi_{i}}}^{m_{0}}}{\operatorname{deg}\left(\varphi_{i}\right)}\left(\mathrm{AJ}_{p}\left(\Delta_{\varphi_{i}}\right)\left(f \otimes v_{\varphi_{i}}^{(j)}\right)-\omega_{p} \cdot \mathrm{AJ}_{p}\left(\Delta_{\bar{\varphi}_{i}}\right)\left(f \otimes \bar{v}_{\varphi_{i}}^{(j)}\right)\right)
$$

Proof. This follows directly from the definitions and Proposition 6.11,

Remark 6.14. The interpolation properties satisfied by the $p$-adic $L$-function $L_{p}(f / K, \xi, \chi, s)$ and the value of the complex $L$-function $L_{K}(f, \chi, s)$ at the central critical point $s=k_{0} / 2$ are well-known and carefully discussed in BDIS02, to which the reader is referred to for details. In particular, in our setting both the $p$-adic $L$-function and the complex $L$-function vanish at $s=k_{0} / 2$. It is an interesting task to investigate similar interpolation properties satisfied by the $p$-adic $L$-function $L_{p}(f / K, \xi, \chi, s)$ and the complex $L$-function $L_{K}\left(f_{k}, \chi, s\right)$ at integers $s=k_{0} / 2+j$ with $n_{0} / 2 \leq j \leq n_{0} / 2$.

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