Discrete and Continuous Dynamical Systems
Vol. 45, No. 5, May 2025, pp. 1366-1398
doi:10.3934/dcds.2024132

ON THE CONTINUUM LIMIT OF THE FOLLOW-THE-LEADER MODEL AND ITS STABILITY

Fabio Ancona[1], Mohamed Bentaibi[1] and Francesco Rossi[*2]

[1]Dipartimento di Matematica "Tullio Levi-Civita", Università di Padova, Italy

[2]Dipartimento di Culture del Progetto, Università Iuav di Venezia, Italy

(Communicated by Alberto Bressan)

Abstract. We consider the Follow-the-Leader (FtL) model and study which properties of the initial positioning of the vehicles ensure its convergence to the classical Lighthill-Whitham-Richards (LWR) model for traffic flow. Robustness properties of both FtL and LWR models with respect to the initial discretization schemes are investigated. Some numerical simulations are also discussed.

1. **Introduction.** Vehicular traffic on a one-lane road can be described at two fundamentally different levels: microscopic and macroscopic. The first one is based on the individual modeling of each vehicle, whose dynamics is governed by the distance to the nearest vehicle in front. This is the so-called Follow-the-Leader (FtL) model [3, 12], which consists of a system of ordinary differential equations. The macroscopic level, relying on a continuum assumption (better justified in the context of heavy traffic), describes the traffic flow in terms of an averaged density that evolves according to a partial differential equation. Assuming that the number of vehicles is conserved, we get the classical Lighthill-Whitham-Richards (LWR) model [17, 19], an hyperbolic conservation law in which the averaged velocity is an explicit function of the density.

The analysis of convergence of the microscopic FtL model toward the macroscopic nonlinear conservation law LWR, as the number of vehicles tends to infinity and their length tends to 0, has been recently investigated by several authors (see [1, 7, 9, 10, 13, 14, 20] and references therein). The question can be summarized as follows; see also Figure 1:

- Take an initial macroscopic description, i.e., a probability density $\bar{\rho}$;
- Discretize it in a suitable manner, finding a microscopic description with $N+1$ vehicles with initial positions
$$\bar{x}_0^N < \bar{x}_1^N < \cdots < \bar{x}_{N-1}^N < \bar{x}_N^N;$$
- Let the microscopic model evolve in time via the FtL;
- Compare it with the solution of LWR starting from $\bar{\rho}$.

2020 *Mathematics Subject Classification.* Primary: 65M75; Secondary: 34D20, 35L65.

Key words and phrases. Lighthill-Whitham-Richards model, Follow-the-Leader model, micro-to-macro convergence, stability of dynamical systems, road traffic.

F. Ancona and F. Rossi are members of G.N.A.M.P.A. (I.N.d.A.M.).

*Corresponding author: Francesco Rossi.

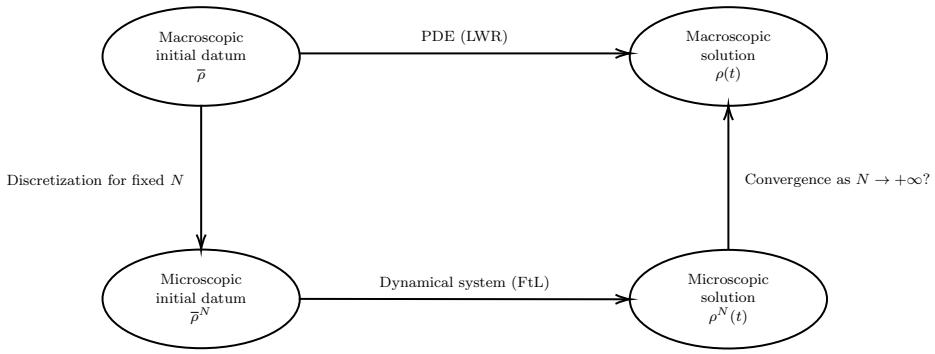


FIGURE 1. Problem statement

The first rigorous proof of convergence of this large particle limit was established in [10], in which a very natural but specific discretization was proposed. The theory is based on a form of L^1 convergence of a suitable miscroscopic-like density to $\bar{\rho}$. Our article, relying on this first result, provides a more general answer, by showing that other discretization schemes can be chosen. Roughly speaking, we show that some form of weak convergence of the microscopic-like density is sufficient.

We now formally describe the framework of our contribution. Consider an initial probability density $\bar{\rho}$ with compact support that satisfies $\|\bar{\rho}\|_{L^\infty} \leq \rho_{max} := 1$. Fix $N \in \mathbb{N}$ and choose

$$\overline{x}_{min} := \bar{x}_0^N < \bar{x}_1^N < \cdots < \bar{x}_{N-1}^N < \bar{x}_N^N =: \overline{x}_{max},$$

that can be interpreted as the initial positions of $N+1$ ordered vehicles with mass $l := \frac{\|\bar{\rho}\|_{L^1}}{N}$. Remark that we do not require $\bar{x}_i^N$ to belong to the support of $\bar{\rho}$, in general; this is a key difference from previous contributions, in which the values $\bar{x}_i^N$ are completely determined by $\bar{\rho}$ (see Section 1.1 for details). Let the x_i^N evolve according to the ordinary differential equation (ODE)

$$\dot{x}_i^N = v\left(\frac{l}{x_{i+1}^N - x_i^N}\right), \qquad i = 0, \dots, N-1, \tag{1}$$

where the velocity function $v = v(\rho)$ satisfies the standing assumptions:

$$\begin{gathered} v \in \mathrm{Lip}([0, \rho_{\max}]) \quad \text{with Lipschitz constant } L, \\ v(\rho_{\max}) = 0, \qquad v'(\rho) \leq c < 0 \quad \text{for a.e. } \rho. \end{gathered} \tag{V1}$$

In some cases, an additional assumption will be required:

$$\text{the map } [0, +\infty) \ni \rho \mapsto \rho\, v'(\rho) \in [0, +\infty) \text{ is non-increasing.} \tag{V2}$$

Together with (V1), it implies the strict concavity of the map

$$\rho \mapsto f(\rho) := \rho\, v(\rho)\,. \tag{2}$$

To close the system of ODEs (1), we prescribe the velocity of the first (leading) vehicle as the maximum possible velocity:

$$\dot{x}_N^N = v_{\max} := v(0)\,. \tag{3}$$

One can view the quantity $l/(x_{i+1}^N - x_i^N)$ in (1) as a discrete density and consider as zero the value of the discrete density on the right of x_N^N, since there is no other vehicle ahead of it. Next, letting $x_i^N(t)$, $i = 0, \dots, N$ and denoting the corresponding

solutions of (1),(3) with initial positions $\bar{x}_i^N$, one can define the discretized Eulerian density as

$$\rho^{E,N}(t,x) := \sum_{j=0}^{N-1} \frac{l}{x_{j+1}^N(t) - x_j^N(t)} \chi_{[x_j^N(t), x_{j+1}^N(t))}(x) \qquad x \in \mathbb{R}, \tag{4}$$

where χ_A is the indicator function of a set A. Then, it is shown in [10] that, for a precise discretization scheme (that we recall in (15) below), one has convergence in $L^1_{\rm loc}([0,+\infty)\times\mathbb{R};[0,1])$ of $\{\rho^{E,N}(t,x)\}_{N\in\mathbb{N}}$ to the weak entropy solution $\rho(t,x)$ of the Cauchy problem for the LWR model

$$\begin{cases} \rho_t + f(\rho)_x = 0, & t>0, \quad x\in\mathbb{R}, \\ \rho(0,x) = \bar{\rho}(x), & x\in\mathbb{R}, \end{cases} \tag{5}$$

with the flux $f(\rho)$ as in (2). Notice that, by construction, here the initial discretized density $\{\rho^{E,N}(0)\}_{N\in\mathbb{N}}$ converges in $L^1(\mathbb{R})$ to the initial density $\bar{\rho}$. A similar result was obtained in [14, 15] for traffic density uniformly away from vacuum, assuming the L^1 convergence of the inverse Lagrangian discrete density (see Section 1.1).

As explained above, in this paper we address the following questions:

- which properties of the initial positioning of the vehicles and of the convergence of the discretized initial data ensure the convergence of the microscopic density $\rho^{E,N}$ to the macroscopic one ρ as $N\to\infty$?
- which kind of stability is enjoyed by these discretization schemes?

An answer to these questions sheds light on the range of applicability, accuracy, and robustness (with respect to errors, gaps in data collection, and oscillations) of the many particle limits in the context of traffic flow. Moreover, from the modeling point of view, the analysis of the discrete-to-continuum limit provides a theoretical background to reconstruct the traffic state of a region through data collected from stationary detectors and GPS devices. On the other hand, these results can be applied to validate the adoption of the macroscopic LWR model in cases where the use of microscopic dynamics is better justified than the macroscopic one.

Our results are all formulated for initial discretization schemes that have uniformly bounded support. Namely, we shall require that the initial positions $x_i^N(0) = \bar{x}_i^N$ of all vehicles are contained in a fixed bounded set.

Definition 1.1 (Condition of uniformly bounded initial support)**.** We say that $\{x_j^N(t)\}_{j=0}^N$ satisfies the condition of uniformly bounded initial support if there exists a bounded set K such that there holds

$$x_i^N(0) \in K \qquad \forall\, i=0,\dots,N, \quad \forall\, N\in\mathbb{N}. \tag{6}$$

For sake of simplicity, we shall assume throughout the paper that $\|\bar{\rho}\|_{L^1}=1$, but all results remain valid without this assumption. The first main result of this paper basically shows that we can replace the requirement of L^1 convergence of the initial discretization (present both in [10] and in [14, 15]) with weak convergence.

Theorem 1.2. *Assume that the velocity map v satisfies (V1). Let $\bar{\rho}\in L^\infty(\mathbb{R};[0,1])$ be with compact support and such that $\|\bar{\rho}\|_{L^1(\mathbb{R})}=1$. Let $\{x_j^N(t)\}_{j=0}^N$ be solutions of the FtL system (1), (3) that moreover satisfy the condition of uniformly bounded initial support (6). Consider the corresponding Eulerian discrete density $\rho^{E,N}\in L^\infty([0,+\infty)\times\mathbb{R};[0,1])$ defined by (4). Assume that*

$$\rho^{E,N}(0) \rightharpoonup \bar{\rho} \qquad \textit{weak}^* \ \textit{in} \ L^\infty(\mathbb{R}), \tag{7}$$

and that one of the two following conditions hold:

(H1) $\bar\rho \in BV(\mathbb{R})$ *and there exists* $C>0$ *such that* $\mathrm{TV}(\rho^{E,N}(0);\mathbb{R}) < C$ *for all* N*, i.e., such that*

$$\left(\frac{1}{x_1^N(0)-x_0^N(0)}+\frac{1}{x_N^N(0)-x_{N-1}^N(0)}+\sum_{j=0}^{N-2}\left|\frac{1}{x_{j+2}^N(0)-x_{j+1}^N(0)}-\frac{1}{x_{j+1}^N(0)-x_j^N(0)}\right|\right) < N\,C, \tag{8}$$

for all N*;*

(H2) the velocity function v *satisfies* (V2)*.*

Then, the sequence $\{\rho^{E,N}\}_{N\in\mathbb{N}}$ *converges in* $L^1_{\mathrm{loc}}([0,+\infty)\times\mathbb{R};[0,1])$ *to the weak entropy solution* ρ *of the Cauchy problem* (5)*.*

Remark 1.3. The proof of convergence, up to a subsequence, of the sequence of Eulerian discrete density $\{\rho^{E,N}\}_{N\in\mathbb{N}}$ is based on an estimate of the L^1 Cauchy property for a subsequence of $\{\rho^{E,N}\}_{N\in\mathbb{N}}$ in terms of the L^1 Cauchy property of the cumulative distribution associated to $\rho^{E,N}$. Then, one can conclude that such a subsequence converges, relying only on the convergence of the cumulative and pseudoinverse functions associated to $\rho^{E,N}$, and on the 1-Wasserstein convergence of $\{\rho^{E,N}\}_{N\in\mathbb{N}}$ that was established in [10]. Finally, relying on (7), one deduces that the limit point ρ of the converging subsequence of $\{\rho^{E,N}\}_{N\in\mathbb{N}}$ is uniquely determined since it must be the unique weak entropy solution of the Cauchy problem (5). As a consequence, the whole sequence $\{\rho^{E,N}\}_{N\in\mathbb{N}}$ converges to ρ. This proof is simpler than the one presented in [10, Theorem 3], where the authors achieve the L^1-compactness of $\{\rho^{E,N}\}_{N\in\mathbb{N}}$ taking advantage also of the Wasserstein equicontinuity of $\rho^{E,N}(t)$, which allows to apply a generalization of the Aubin-Lions lemma.

The second main contribution of this paper is a stability result with respect to the 1-Wasserstein distance W_1. It is a microscopic stability result for the evolution of two different initial discretization schemes, which in turn yields a stability result with respect to the L^1 norm that is uniform in time. Such a result is rather surprising in view of the instability of the FtL dynamics.

Theorem 1.4 (Discrete Eulerian stability theorem)**.** *Assume that the velocity map* v *satifies* (V1)*. Let* $\{x_j^N(t)\}_{j=0}^N$,$\{\tilde x_j^N(t)\}_{j=0}^N$ *be solutions of the FtL system* (1), (3)*, that moreover satisfy the condition of uniformly bounded initial support* (6)*. Consider the corresponding Eulerian discrete densities* $\rho^{E,N}$, $\tilde\rho^{E,N}\in L^\infty(([0,+\infty)\times\mathbb{R});[0,1])$ *defined by* (4)*. Then, for all* $T>0$*, and for all* $N\in\mathbb{N}$*, there holds*

$$\begin{aligned}&\sup_{t\in[0,T]} W_1(\rho^{E,N}(t),\tilde\rho^{E,N}(t))\\ \le& W_1(\rho^{E,N}(0),\tilde\rho^{E,N}(0)) + +2LT\sum_{j=0}^{N-1}|x_{j+1}(0)-x_j(0)-(\tilde x_{j+1}(0)-\tilde x_j(0))|,\end{aligned} \tag{9}$$

where L *is the Lipschitz constant of* v*. Moreover, if there holds* $x_N^N(0)=\tilde x_N^N(0)$ *for all* $N\in\mathbb{N}$*, and*

$$\lim_{N\to+\infty}\sum_{j=0}^{N-1}|x_{j+1}(0)-x_j(0)-(\tilde x_{j+1}(0)-\tilde x_j(0))|=0, \tag{10}$$

then the following two properties are satisfied:

(i) if there exists $C > 0$ *such that* $\mathrm{TV}\left(\rho^{E,N}(0);\mathbb{R}\right), \mathrm{TV}\left(\tilde{\rho}^{E,N}(0);\mathbb{R}\right) < C$ *for all* N*, then for all* $T > 0$*, there holds*

$$\lim_{N\to+\infty} \sup_{t\in[0,T]} \left\|\rho^{E,N}(t) - \tilde{\rho}^{E,N}(t)\right\|_{L^1(\mathbb{R})} = 0; \tag{11}$$

(ii) if the velocity v *satisfies* (V2)*, then for all* $T > 0$*, there holds*

$$\lim_{k\to+\infty} \sup_{t\in[1/k,\,T]} \left\|\rho^{E,N_k}(t) - \tilde{\rho}^{E,N_k}(t)\right\|_{L^1(\mathbb{R})} = 0, \tag{12}$$

for some subsequences $\{\rho^{E,N_k}\}_k$, $\{\tilde{\rho}^{E,N_k}\}_k$.

Remark 1.5. If $x_N^N(0) = \tilde{x}_N^N(0)$ for all N and there holds (10), then Proposition 5.1 below ensures

$$\lim_{N\to+\infty} W_1(\rho^{E,N}(0), \tilde{\rho}^{E,N}(0)) = 0\,.$$

This in turn implies that $\rho^{E,N}(0) - \tilde{\rho}^{E,N}(0) \rightharpoonup 0$. Thus, letting $\bar{\rho}, \tilde{\rho}$ denote the weak* limit of $\{\rho^{E,N}(0)\}_N$, $\{\tilde{\rho}^{E,N}(0)\}_N$, respectively, we have $\bar{\rho} = \tilde{\rho}$. Hence, applying Theorem 1.2, we deduce that both sequences $\left\{\rho^{E,N}\right\}_{N\in\mathbb{N}}$, $\left\{\rho^{\tilde{E},N}\right\}_{N\in\mathbb{N}}$ converge in $L^1_{\mathrm{loc}}([0,+\infty)\times\mathbb{R})$ to the weak entropy solution of the Cauchy problem (5), which implies

$$\lim_{N\to+\infty} \left\|\rho^{E,N}(t) - \tilde{\rho}^{E,N}(t)\right\|_{L^1(\mathbb{R})} = 0 \qquad \text{for a.e. } t > 0\,. \tag{13}$$

The main new property provided by Theorem 1.4 is the fact that, thanks to the stability estimate (9), the convergence in (13) is actually uniform in time.

Notice also that property (i) of Theorem 1.4 implies that, if $x_N^N(0) = \tilde{x}_N^N(0)$ for all N, and if we have a uniform bound on the total variation of $\rho^{E,N}(0)$, $\tilde{\rho}^{E,N}(0)$, then the assumption (10) in particular yields the L^1 convergence $\rho^{E,N}(0) - \tilde{\rho}^{E,N}(0) \to 0$.

One final contribution of our work shows that a crucial question is still open. In Proposition 4.8, we give an example of a discretization scheme that does not fulfill the assumption (H1) of Theorem 1.2. Therefore, we cannot apply our result for such a scheme in the case of fluxes $f(\rho)$ that are not concave. However, it is surprising to remark that the numerical simulations presented in Remark 4.9 seem to suggest that the Eulerian discrete density defined with such a scheme exhibits essentially the same behavior of the one produced by those for which Theorem 1.2 can be applied, ensuring convergence to solutions of LWR. This is an interesting phenomenon that shows that, in the case of non-concave fluxes, the relation between the convergence of $\rho^{E,N}(0)$ to $\bar{\rho}$ and of $\rho^{E,N}(t)$ to the solution ρ of (5) has not yet been properly understood and needs further investigation.

The paper is organized as follows. In Section 1.1, we compare our main results with the contributions of [10] and [14, 15]. In Section 2, we recall the definition of the Follow-the-Leader dynamics and provide a stability result for it. In Section 3, we define the Eulerian and Lagrangian discrete densities, and their cumulative functions with the corresponding pseudo-inverses and discuss their properties and interpretations. In Section 4, we prove the first main result of the article, i.e., Theorem 1.2. We also discuss in this section an atomization scheme different from the ones in [10, 14, 15], which leads to an Eulerian discrete density that converges to the solution of the LWR model when the velocity v satisfies the additional assumption (V2). A numerical simulation indicating that this is not the case for velocities v that do not satisfy the assumption (V2) is also discussed in this section. Finally, in Section 5, we establish the main stability result, i.e., Theorem 1.4.

1.1. **Comparison with the literature.** In this section, we compare our results with the most relevant contributions in the field.

The main reference here is clearly [10]. The main result there is Theorem 3, that provides the same convergence result under the following explicit discretization scheme: given $\bar\rho$ with compact support, define

$$x_0^N := \min(\mathrm{supp}(\bar\rho)) \tag{14}$$

and recursively

$$x_j^N(0) := \sup\left\{ x \in \mathbb{R} : \int_{x_{j-1}^N(0)}^{x} \bar\rho(y)dy < \frac{1}{N} \right\}, \qquad j = 1, ..., N. \tag{15}$$

This amounts to splitting the subgraph of $\bar\rho$ into N adjoining intervals of mass $l = 1/N$ and to choose x_i^N to be the extremes of these intervals. Note that $x_0^N = \inf(\mathrm{supp}(\bar\rho))$ and $x_N^N = \sup(\mathrm{supp}(\bar\rho))$, for any N, i.e., that all discretizations share the initial and final points. Moreover, we will show in Proposition 4.6 that this discretization ensures L^1-convergence of the Eulerian discrete density $\rho^{E,N}(0,x)$ to $\bar\rho$.

In our contribution, instead, we only require weak convergence of the Eulerian discrete density $\rho^{E,N}(0,x)$ to $\bar\rho$. In particular, we accept discretizations in which the $x_i^N(0)$ do not belong to the support of the initial datum $\bar\rho$: as an example, it may well happen that $x_0^N(0) \neq \inf(\mathrm{supp}(\bar\rho))$ and $x_N^N(0) \neq \sup(\mathrm{supp}(\bar\rho))$. Since weak convergence does not provide information on the position of the initial and final point of the discretization scheme, we are forced to add the condition of uniformly bounded initial support (6).

A very similar result is obtained for dense traffic regions (i.e., away from the vacuum) in [14, Theorem 2.5], [15, Theorem 4.1], where instead it is assumed the following: the L^1 convergence of the inverse Lagrangian discrete density $y^{L,N}(0)$ (see Definition 3.3 below) as $N \to \infty$; that $y^{L,N}(0)$ has uniformly (in N) bounded total variation; and that the discrete density $\rho^{E,N}(0)$ is uniformly bounded away from zero. Moreover, in the same non-vacuum setting, [15, Lemma 3.1] provides an L^1 stability estimate for different discretization schemes. In our contribution, in Theorem 1.2 we are essentially providing a result showing that weak convergence implies strong convergence, even for initial data possibly containing vacuum regions. In Theorem 1.4, we provide the stability of two different discretization schemes $\rho^{E,N}$ and $\tilde\rho^{E,N}$, by exploiting the fact that weak convergence combined with a control of the total variation implies strong convergence. Weak convergence here is ensured by condition (10), which is based on the discretization scheme only. Such an hypothesis is assumed for instance in [14, (2.11)] in the case of initial data $\bar\rho \in BV(\mathbb{R})$ away from vacuum.

Finally, in [18, Theorem 3.6], the authors provide a Cauchy property and the rate of convergence of a Eulerian microscopic density for non-local conservation laws. Also in this case, the result holds with a specific discretization scheme of the initial data $\bar\rho \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ that satisfies $\bar\rho > 0$ and $\int_{\mathbb{R}} |x|\bar\rho(x)dx < \infty$. This is given in the form of a microscopic stability between $\rho^{E,N}$ and $\rho^{E,M}$, for $M, N \in \mathbb{N}$ large enough. The main idea is that the Eulerian microscopic density is a quasi-entropy solution of the conservation law. The generalization of our results to non-local conservation laws seems interesting, since they are somehow more naturally connected to microscopic dynamics, e.g., via the mean-field limit. This is a future research topic that we aim to address.

2. **The Follow-the-Leader model.** In this section, we introduce the Follow-the-Leader (FtL) model and study its behavior. This is a classical model for road traffic on a one-lane road with no overtaking, see e.g., [5, 12]. The goal here is to investigate its stability properties with respect to the initial data. We first define the dynamics of the positions of vehicles $x_j(t)$, then consider the associated discrete density $\rho_j(t)$, and finally introduce the inverse discrete density $y_j(t)$. For each of these quantities, we analize the dynamics and some useful properties.

We start by considering $N+1$ vehicles, of length l, with initial positions

$$\bar{x}_0^N < \cdots < \bar{x}_N^N \tag{16}$$

satisfying

$$\bar{x}_{i+1}^N - \bar{x}_i^N \geq l, \qquad \text{with} \quad l \coloneqq \frac{1}{N}. \tag{17}$$

This standard condition ensures non-overlapping of vehicles.

We now define the FtL dynamics.

Definition 2.1. The FtL dynamics is

$$\begin{cases} \dot{x}_N^N = v_{max}, \\ \dot{x}_j^N = v\left(\dfrac{l}{x_{j+1}^N - x_j^N}\right), & \text{for } j = 0, ..., N-1, \\ x_j^N(0) = \bar{x}_j^N, & \text{for } j = 0, ..., N, \end{cases} \tag{18}$$

where the initial positions $\bar{x}_j^N$ satisfy the condition (17).

The FtL model describes the evolution of each vehicle x_j^N, which adapts its speed based on the distance to the vehicle immediately in front x_{j+1}^N. We now introduce the corresponding definition of discrete density and of its dynamics.

Definition 2.2. Given $\{x_j^N(t)\}_{j=0}^N$ a solution of (18), define the discrete density as

$$\rho_j^N(t) \coloneqq \frac{l}{x_{j+1}^N(t) - x_j^N(t)} \qquad j = 0, ..., N-1. \tag{19}$$

Because of (18), the discrete density satisfies the dynamics

$$\begin{cases} \dot{\rho}_{N-1}^N = -N(\rho_{N-1}^N)^2 \left(v_{\max} - v(\rho_{N-1}^N)\right), \\ \dot{\rho}_j^N = N(\rho_j^N)^2 \left(v(\rho_j^N) - v(\rho_{j+1}^N)\right), & \text{for } j = 0, ..., N-2, \\ \rho_j^N(0) = \bar{\rho}_j^N, & \text{for } j = 0, ..., N-1, \end{cases} \tag{20}$$

where the initial data is

$$\bar{\rho}_j^N \coloneqq \frac{l}{\bar{x}_{j+1}^N - \bar{x}_j^N}, \qquad \text{for } j = 0, ..., N-1.$$

We finally consider the inverse discrete density introduced in [14].

Definition 2.3. Given $\{x_j^N(t)\}_{j=0}^N$ a solution of (18), define the inverse discrete density as

$$y_j^N(t) \coloneqq \frac{x_{j+1}^N(t) - x_j^N(t)}{l} = \frac{1}{\rho_j^N(t)} \qquad j = 0, ..., N-1. \tag{21}$$

Because of (18), the inverse discrete density satisfies the dynamics

$$\begin{cases} \dot{y}^N_{N-1} = N\left(v_{\max} - V(y^N_{N-1})\right), & \\ \dot{y}^N_j = N\left(V(y^N_{j+1}) - V(y^N_j)\right), & \text{for } j = 0, ..., N-2 \\ y^N_j(0) = \bar{y}^N_j := \dfrac{\bar{x}^N_{j+1}(t) - \bar{x}^N_j(t)}{l}, & \text{for } j = 0, ..., N-1, \end{cases} \tag{22}$$

where the velocity of the inverse discrete density is defined by

$$V(y) := v\left(\frac{1}{y}\right).$$

Here, the first equation of (22) prescribes that the inverse discrete density of the leading vehicle evolves with the maximum velocity

$$V(y^N_N) = v(0) = v_{\max}, \tag{23}$$

which could be viewed as setting "$y^N_N = +\infty$", corresponding to an empty road in front of the leader x^N_N. As a consequence of (V1), the velocity of the inverse discrete density satisfies the conditions

$$V \in \mathrm{Lip}([1, +\infty)) \text{ with Lipschitz constant } L, \qquad V(1) = 0, \qquad V \text{ is increasing.}$$

Remark 2.4 (Discrete minimum/maximum principle)**.** The solution of the FtL model (18) and the corresponding discrete density (20) satisfy a discrete minimum/maximum principle. This is the microscopic version of the well-known maximum principle that applies to solutions of (5); see, for example, [8, Theorem 6.2.7]. Indeed, the following estimates hold:

$$\begin{aligned} &\min_{j=0,...,N-1}(x^N_{j+1}(t) - x^N_j(t)) \;\geq \min_{j=0,...,N-1}(\bar{x}^N_{j+1} - \bar{x}^N_j) \;\geq l; \\ &\max_{j=0,...,N-1}(x^N_{j+1}(t) - x^N_j(t)) \leq \bar{x}^N_N - \bar{x}^N_0 + t\, v_{\max}. \end{aligned} \tag{24}$$

Thus, by virtue of (17)-(19), we also deduce

$$\max_{j=0,...N-1} \rho^N_j(t) \leq \max_{j=0,...N-1} \bar{\rho}^N_j \leq 1. \tag{25}$$

A proof of (24) can be found in [10, Lemma 1]. Similarly, the solution of the discrete inverse density (22) satisfies a discrete minimum principle due to (24). Indeed, it holds

$$\min_{j=0,...N-1} y^N_j(t) \geq \min_{j=0,...N-1} \bar{y}^N_j \geq 1.$$

In the same spirit of [14, Lemma 2.3], we now prove a stability estimate for two different solutions of (20).

Proposition 2.5. *Consider two solutions* $\{x^N_j(t)\}^N_{j=0}$, $\{\tilde{x}^N_j(t)\}^N_{j=0}$ *of* (18)*, with initial positions* $\{\bar{x}^N_j\}^N_{j=0}$, $\{\tilde{\bar{x}}^N_j\}^N_{j=0}$, *respectively. Let* $\{\rho^N_j(t)\}^{N-1}_{j=0}$, $\{\tilde{\rho}^N_j(t)\}^{N-1}_{j=0}$ *be the corresponding discrete densities defined by* (19)*, and let* $\{y^N_j(t)\}^{N-1}_{j=0}$, $\{\tilde{y}^N_j(t)\}^{N-1}_{j=0}$ *be the corresponding inverse discrete densities defined by* (21)*. Then, there holds*

$$\sum_{j=0}^{N-1} |\rho^N_j(t) - \tilde{\rho}^N_j(t)| \leq \sum_{j=0}^{N-1} |y^N_j(0) - \tilde{y}^N_j(0)| \qquad \forall\, t \geq 0. \tag{26}$$

Proof. Throughout the proof, we drop the superscript N for simplicity of notation. We consider two solutions of (22) parametrized by two different variables t and τ, and use the Kruzkov's doubling of variables method to provide the contraction estimate for the inverse densities. We finally rely on the maximum principle for the discrete densities to conclude. With this aim, we define

$$V_j(t) \coloneqq V(y_j(t)), \qquad \tilde{V}_j(\tau) \coloneqq V(\tilde{y}_j(\tau)).$$

We then notice that, for $j = 0, ..., N-2$, it holds

$$\begin{aligned}
\frac{d}{dt}|y_j(t) - \tilde{y}_j(\tau)| &= N \operatorname{sign}(y_j(t) - \tilde{y}_j(\tau))(V_{j+1}(t) - V_j(t)),\\
\frac{d}{d\tau}|y_j(t) - \tilde{y}_j(\tau)| &= N \operatorname{sign}(y_j(t) - \tilde{y}_j(\tau))(\tilde{V}_j(\tau) - \tilde{V}_{j+1}(\tau)).
\end{aligned}$$

Therefore, we deduce that, for $j = 0, ..., N-2$, we have

$$\begin{aligned}
&\left(\frac{d}{dt} + \frac{d}{d\tau}\right)|y_j(t) - \tilde{y}_j(\tau)|\\
&\quad = N \operatorname{sign}(y_j(t) - \tilde{y}_j(\tau))[V_{j+1}(t) - V_j(t) - \tilde{V}_{j+1}(\tau) + \tilde{V}_j(\tau)]\\
&\quad = N \Big[- \operatorname{sign}(y_j(t) - \tilde{y}_j(\tau))(V_j(t) - \tilde{V}_j(\tau))\\
&\qquad + \operatorname{sign}(y_{j+1}(t) - \tilde{y}_{j+1}(\tau))(V_{j+1}(t) - \tilde{V}_{j+1}(\tau))\\
&\qquad\quad + (V_{j+1}(t) - \tilde{V}_{j+1}(\tau))[\operatorname{sign}(y_j(t) - \tilde{y}_j(\tau)) - \operatorname{sign}(y_{j+1}(t) - \tilde{y}_{j+1}(\tau))\Big]\\
&\quad \le N \Big[- \operatorname{sign}(y_j(t) - \tilde{y}_j(\tau))(V_j(t) - \tilde{V}_j(\tau))\\
&\qquad + \operatorname{sign}(y_{j+1}(t) - \tilde{y}_{j+1}(\tau))(V_{j+1}(t) - \tilde{V}_{j+1}(\tau))\Big]. \qquad (27)
\end{aligned}$$

The last inequality can be recovered as follows:

- If

$$y_j(t) \ge \tilde{y}_j(\tau) \text{ and } y_{j+1}(t) \le \tilde{y}_{j+1}(\tau), \qquad (28)$$

then one has

$$V_{j+1}(t) - \tilde{V}_{j+1}(\tau) \le 0, \qquad \operatorname{sign}(y_j(t) - \tilde{y}_j(\tau)) - \operatorname{sign}(y_{j+1}(t) - \tilde{y}_{j+1}(\tau)) \ge 0.$$

- If

$$y_j(t) \le \tilde{y}_j(\tau) \text{ and } y_{j+1}(t) \ge \tilde{y}_{j+1}(\tau), \qquad (29)$$

then one has

$$V_{j+1}(t) - \tilde{V}_{j+1}(\tau) \ge 0, \qquad \operatorname{sign}(y_j(t) - \tilde{y}_j(\tau)) - \operatorname{sign}(y_{j+1}(t) - \tilde{y}_{j+1}(\tau)) \le 0.$$

- Otherwise, if neither (28) nor (29) are satisfied, then one has

$$\operatorname{sign}(y_j(t) - \tilde{y}_j(\tau)) - \operatorname{sign}(y_{j+1}(t) - \tilde{y}_{j+1}(\tau)) = 0.$$

Summing up the inequalities in (27) and by the monotonicity of V, we find

$$\begin{aligned}
\sum_{j=0}^{N-2} \left(\frac{d}{dt} + \frac{d}{d\tau}\right)|y_j(t) - \tilde{y}_j(\tau)| &\le N \operatorname{sign}(y_{N-1}(t) - \tilde{y}_{N-1}(\tau))[V_{N-1}(t) - \tilde{V}_{N-1}(\tau)]\\
&\quad - N \operatorname{sign}(y_0(t) - \tilde{y}_0(\tau))[V_0(t) - \tilde{V}_0(\tau)]\\
&\le N \operatorname{sign}(y_{N-1}(t) - \tilde{y}_{N-1}(\tau))[V_{N-1}(t) - \tilde{V}_{N-1}(\tau)].
\end{aligned}$$

On the other hand, for $j = N-1$, it holds

$$\begin{aligned}\left(\frac{d}{dt}+\frac{d}{d\tau}\right)&|y_{N-1}(t)-\tilde{y}_{N-1}(\tau)|\\ &= N\,\mathrm{sign}(y_{N-1}(t)-\tilde{y}_{N-1}(\tau))[v_{\max}-V_{N-1}(t)-v_{\max}+\tilde{V}_{N-1}(\tau)]\\ &= N\,\mathrm{sign}(y_{N-1}(t)-\tilde{y}_{N-1}(\tau))[\tilde{V}_{N-1}(\tau)-V_{N-1}(t)].\end{aligned}$$

Therefore, we conclude that

$$\sum_{j=0}^{N-1}\left(\frac{d}{dt}+\frac{d}{d\tau}\right)|y_j(t)-\tilde{y}_j(\tau)|\le 0. \tag{30}$$

Relying on (30), we can complete the proof with the same arguments of the proof of [14, Lemma 2.3]. Namely, multiplying (30) by a non-negative test function $\phi(t,\tau)$ with $\phi\in C_0^\infty((0,\infty)\times(0,\infty))$, and then integrating by parts, one obtains

$$\int_0^\infty\int_0^\infty(\phi_t+\phi_\tau)\sum_{j=0}^{N-1}|y_j(t)-\tilde{y}_j(\tau)|dtd\tau\ge 0. \tag{31}$$

Next, choose

$$\phi(t,\tau)=\psi\left(\frac{t+\tau}{2}\right)\eta_\epsilon(t-\tau),$$

where $\psi\in C_0^\infty((0,\infty)\times(0,\infty))$ is a non-negative function, and η_ϵ is a standard mollifier converging to the Dirac delta at the origin as $\epsilon\to 0$. Then, plugging this test function in (31) and sending $\epsilon\to 0$, we get

$$\int_0^\infty\psi'(t)\sum_{j=0}^{N-1}|y_j(t)-\tilde{y}_j(t)|dt\ge 0 \tag{32}$$

Now, taking ψ in (32) to be a smooth approximation of the characteristic function of the interval $(t_1,t_2)\subset(0,t)$ we get

$$\sum_{j=0}^{N-1}|y_j(t_2)-\tilde{y}_j(t_2)|\le\sum_{j=1}^{N-1}|y_j(t_1)-\tilde{y}_j(t_1)|. \tag{33}$$

Then, letting $t_1\to 0$ and $t_2\to t$ in (33), we obtain

$$\sum_{j=0}^{N-1}|y_j(t)-\tilde{y}_j(t)|\le\sum_{j=1}^{N-1}|y_j(0)-\tilde{y}_j(0)|. \tag{34}$$

Finally, by using (34) and the maximum principle (25), we find

$$\begin{aligned}&\sum_{j=0}^{N-1}|\rho_j(t)-\tilde{\rho}_j(t)|=\sum_{j=0}^{N-1}\rho_j(t)\tilde{\rho}_j(t)\,|y_j(t)-\tilde{y}_j(t)|\le\\ &\sum_{j=0}^{N-1}|y_j(t)-\tilde{y}_j(t)|\le\sum_{j=0}^{N-1}|y_j(0)-\tilde{y}_j(0)|,\end{aligned}$$

thus establishing (26). □

Remark 2.6. In [14, 15], the authors established the contractive estimate (34) assuming a uniform bound on the inverse discrete density $y_j^N(0)$ and on the total variation of the corresponding inverse Eulerian discrete density $y^{E,N}$ (see Definition 3.2 below). The estimates in [14] were obtained in two settings:

- either they assume to have infinitely many equally spaced vehicles in front of the leading one located at x_N^N, with a distance M/N between two consecutive ones, for some constant $M > 1$,
- or they assume that the location of the vehicles is periodic in an interval $[a, b]$, so that the distance between the vehicle located in x_N^N and the one located at x_1^N is $(b - x_N^N) + (x_1^N - a)$.

This corresponds to defining the inverse discrete density related to the leading vehicle as

$$y_N^N = \begin{cases} M & \text{in non-periodic case,} \\ N(b - x_N + x_1 - a) & \text{in periodic case.} \end{cases}$$

In the non-periodic setting, this definition leads to prescribing the velocity

$$V(y_N^N) = v\left(\frac{1}{M}\right)$$

for the inverse discrete density in front of the leader. Here, instead, we obtain the contractive estimate (34) by observing that $\dot{x}_N^N = v_{max}$ in (18) implies the first equation in (22) with V given by (23). Therefore, Proposition 2.5 provides an extension of [14, Lemma 2.3], in the non-periodic setting, to the case "$M = +\infty$" corresponding to an empty road ahead of the leader, and removing any boundedness assumption on $y_j^N(0)$ and on the total variation of $y^{E,N}$.

Finally, we recall the discrete Oleinik-type condition proved in [10, Corollary 1 of Lemma 6] in the case of the particular discretization scheme considered therein, which remains valid for a general discretization scheme. Such one-sided estimate yields the uniform bounds on the total variation of the discrete densities stated in Proposition 3.4-(ii) below.

Lemma 2.7 (Discrete Oleinik-type condition)**.** *Consider a solution* $\{x_j^N(t)\}_{j=0}^N$ *of* (18)*, and let* $\{\rho_j^N(t)\}_{j=0}^{N-1}$ *be the corresponding discrete density defined by* (19)*. Assume that* v *satisfies* (V1) *and* (V2)*. Then, for any* $j = 0, \dots, N-2$*, there holds*

$$\frac{v(\rho_{j+1}^N(t, x_{j+1}^N(t))) - v(\rho_j^N(t, x_j^N(t)))}{x_{j+1}^N(t) - x_j^N(t)} \leq \frac{1}{t} \qquad \forall\, t \geq 0.$$

Proof. The proof in [10, Corollary 1 of Lemma 6] is completely independent of the initial profile $\{x_j^N(0)\}_{j=0}^N$ and on the initial discretization $\{\rho_j^N(0)\}_{j=0}^N$. It depends solely on the dynamics given by (20). □

3. Eulerian and Lagrangian densities.

In this section, we present several different densities that approximate the solution of (5). They have a simple structure, being either piecewise constant or a combination of Dirac deltas. This section is mainly based on the analysis developed in [10].

We first introduce the Eulerian discrete density, which can be understood as a discrete approximation of the solution of the LWR model (5), based on the dynamics of the FtL (18).

Definition 3.1. Given $\{x_j^N(t)\}_{j=0}^N$ solution of (18), define the Eulerian discrete density as

$$\rho^{E,N}(t,x) := \sum_{j=0}^{N-1} \rho_j^N(t)\chi_{[x_i^N(t),x_{i+1}^N(t))}(x), \tag{35}$$

where the ρ_j^N are defined by (19).

Notice that the Eulerian discrete density can be seen as a quasi-entropy solution to (5), as discussed in [18]. We now define the inverse Eulerian discrete density and the (Dirac) empirical measure.

Definition 3.2. Given $\{x_j^N(t)\}_{j=0}^N$ solution of (18), define the inverse Eulerian discrete density as

$$y^{E,N}(t,x) := \sum_{j=0}^{N-1} y_j^N(t)\chi_{[x_i^N(t),x_{i+1}^N(t))}(x), \qquad x \in \mathbb{R},$$

and the (Dirac) empirical measure as

$$\rho^{D,N}(t,x) := \frac{1}{N}\sum_{j=0}^{N-1} \delta_{x_j(t)}(x), \qquad x \in \mathbb{R}, \tag{36}$$

where the y_j^N are defined by (21), and δ_x denotes the Dirac delta at point x.

We finally define the Lagrangian discrete density and the inverse Lagrangian density. The latter can be understood as a piecewise constant approximation of the solution of the Lagrangian version of the LWR model; see [14]. We recall that $l = 1/N$.

Definition 3.3. Given $\{x_j^N(t)\}_{j=0}^N$ solution of (18), define the Lagrangian discrete density as

$$\rho^{L,N}(t,z) := \sum_{j=0}^{N-1} \rho_j^N(t)\chi_{[jl,(j+1)l)}(z), \qquad z \in [0,1], \tag{37}$$

and the inverse Lagrangian discrete density as

$$y^{L,N}(t,z) := \sum_{j=0}^{N-1} y_j^N(t)\chi_{[jl,(j+1)l)}(z), \qquad z \in [0,1]. \tag{38}$$

The coordinate $z \in [0,1]$ can be seen as a Lagrangian mass coordinate. As pointed out in [14], the integer part of $\frac{z}{l}$ measures how many vehicles are located to the left of z.

Notice that, while the L^1 norm of the Eulerian discrete density $\rho^{E,N}$ represents the total mass of vehicles, the L^1 norm of the inverse Lagrangian discrete density $y^{L,N}$ provides the measure of their support. Indeed, given $\{x_j^N(t)\}_{j=0}^N$ solution of (18), it holds

$$\left\|y^{L,N}(t)\right\|_{L^1([0,1])} = \sum_{j=0}^{N-1} y_j^N(t)\cdot l = \sum_{j=0}^{N-1} x_{j+1}^N(t) - x_j^N(t) = x_N^N(t) - x_0^N(t). \tag{39}$$

Therefore, if $\{x_j^N(t)\}_{j=0}^N$ satisfy the condition of uniformly bounded initial support (6), relying on the discrete maximum principle (24), we deduce that the corresponding inverse Lagrangian discrete density $y^{L,N}(t)$ has a bound in $L^1([0,1])$ that is uniform with respect to N, for all $t>0$.

The discrete Eulerian and Lagrangian densities enjoy a bounded variation (BV) contraction property in the case of initial data with bounded variation, and uniform BV estimates for initial data with velocity satisfying assumption (V2). These results are established in [10, Propositions 5-6] (see also [9, Propositions 1-3]), and collected in the next proposition.

Proposition 3.4. *Assume that v satisfies (V1). Let $\{x_j^N(t)\}_{j=0}^N$ be a solution of (18). Consider the corresponding Eulerian discrete density $\rho^{E,N} \in L^\infty([0,+\infty)\times\mathbb{R};[0,1])$ defined by (35) and the Lagrangian discrete density $\rho^{L,N}\in L^\infty([0,+\infty)\times[0,1])$ defined by (37). Then, the following holds:*

(i) if $\bar\rho \in BV(\mathbb{R})$, then

$$\mathrm{TV}\left(\rho^{E,N}(t);\mathbb{R}\right) = \mathrm{TV}\left(\rho^{L,N}(t);\mathbb{R}\right) \le \mathrm{TV}\left(\rho^{E,N}(0);\mathbb{R}\right) \qquad \forall\, t\ge 0, \quad \forall\, N\in\mathbb{N};$$

(ii) if the velocity function v satisfies (V2) and $\bar\rho\in L^\infty(\mathbb{R})$, then, for any $\delta>0$, there exists a constant $C>0$, depending on δ, such that

$$\sup_{t\ge\delta}\mathrm{TV}\left(\rho^{E,N}(t);\mathbb{R}\right)\le C, \qquad \sup_{t\ge\delta}\mathrm{TV}\left(\rho^{L,N}(t);\mathbb{R}\right)\le C, \qquad \forall\, N\in\mathbb{N}.$$

Proof. We prove statement (i) by proving that

$$\frac{d}{dt}\mathrm{TV}\left(\rho^{E,N}(t)\right)\le 0, \qquad \forall\, t>0, \quad \forall\, N\in\mathbb{N}\,.$$

It is sufficient to apply the exact same computations of the proof of [10, Proposition 5], which are independent on the particular initial discretization scheme $\{x_j^N(0)\}_{j=0}^N$.

We prove statement (ii) by following the exact same computations as in [10, Proposition 6], that are also independent on the particular initial discretization scheme $\{x_j^N(0)\}_{j=0}^N$. By relying on Lemma 2.7, we find that, for any $N\in\mathbb{N}$ and for all $t\ge\delta$, it holds

$$\begin{aligned}\mathrm{TV}\left(v\left(\rho^{E,N}(t)\right);\mathbb{R}\right) = \mathrm{TV}\left(v\left(\rho^{L,N}(t)\right);\mathbb{R}\right) &\le \left[3v_{\max}+2\frac{x_N^N(0)-x_0^N(0)}{\delta}\right]\\ &\le \left[3v_{\max}+2\frac{\mathrm{meas}(K)}{\delta}\right].\end{aligned}$$

In the last inequality, K is the bounded set such that (6) holds. Since v is invertible and $(v^{-1})'$ is bounded because of condition (V1), statement (ii) follows. □

3.1. **Cumulative and pseudo-inverse functions.** We now define the cumulative distribution of a function and the corresponding pseudo-inverse function.

Definition 3.5. Consider the space of probability densities

$$\mathcal{P}_c(\mathbb{R}) := \{\rho \text{ Radon measure on } \mathbb{R} \text{ with compact support with } \rho(\mathbb{R})=1\}.$$

Given $\rho\in\mathcal{P}_c(\mathbb{R})$, define the cumulative distribution $F_\rho:\mathbb{R}\to[0,1]$:

$$F_\rho(x) := \rho((-\infty,x]), \qquad x\in\mathbb{R}, \tag{40}$$

and its associated pseudo-inverse $X_\rho:[0,1]\to\mathbb{R}$ as

$$X_\rho(z) := \inf\{x\in\mathbb{R} \,:\, F_\rho(x)\ge z\}, \qquad z\in[0,1].$$

Observe that F_ρ is non-decreasing and right-continuous.

We recall that the one-dimensional Wasserstein distance can be defined using the cumulative or the pseudo-inverse functions, see e.g., [22].

Definition 3.6. The one-dimensional 1-Wasserstein distance is

$$W_1(\rho,\tilde\rho) \coloneqq \|F_\rho - F_{\tilde\rho}\|_{L^1(\mathbb{R})} = \|X_\rho - X_{\tilde\rho}\|_{L^1([0,1])}\,. \tag{41}$$

Recall that the discrete density $\rho^{E,N}$ is a probability measure in $\mathcal{P}_c(\mathbb{R})$. We can apply Definition 3.5 to $\rho^{E,N}$ and find that its cumulative distribution takes the form:

$$\begin{aligned} F_{\rho^{E,N}}(t,x) &= \int_{-\infty}^{x} \rho^{E,N}(t,y)dy \\ &= \sum_{j=0}^{N-1} \left[jl + \rho_j^N(t)(x - x_j(t))\right] \chi_{[x_j(t),x_{j+1}(t))}(x) + \chi_{[x_N(t),+\infty)}(x). \end{aligned} \tag{42}$$

Notice that the cumulative distribution $F_{\rho^{E,N}}$ is 1-Lipschitz in the x-variable. The corresponding pseudo-inverse takes the form:

$$X_{\rho^{E,N}}(t,z) = \sum_{j=0}^{N-1} \left[x_j^N(t) + \frac{z-jl}{\rho_j^N(t)}\right] \chi_{[jl,(j+1)l)}(z) + \left[x_N^N(t)\right] \chi_{\{1\}}(z), \quad z\in[0,1]. \tag{43}$$

The pseudo-inverse $X_{\rho^{E,N}}$ satisfies

$$\rho^{L,N}(t,z) = \rho^{E,N}(t, X_{\rho^{E,N}}(t,z)), \quad y^{L,N}(t,z) = y^{E,N}(t, X_{\rho^{E,N}}(t,z))$$

for all $t\geq 0,\ z\in[0,1]$. The cumulative function $F_{\rho^{E,N}}$ then satisfies

$$\rho^{L,N}(t, F_{\rho^{E,N}}(t,x)) = \rho^{E,N}(t,x), \quad y^{L,N}(t, F_{\rho^{E,N}}(t,x)) = y^{E,N}(t,x) \tag{44}$$

for all $t\geq 0,\ z\in[0,1]$. Relying on (42) and (44), one deduces that

$$\left\|\rho^{L,N}(t)\right\|_{L^1([0,1])} = \int_{\mathbb{R}} \rho^{E,N}(t,x)\,\frac{d}{dx}F_{\rho^{E,N}}(t,x)\,dx = \left\|\rho^{E,N}(t)\right\|^2_{L^2(\mathbb{R})}\,.$$

Similarly, if we apply Definition 3.5 to the (Dirac) empirical measure $\rho^{D,N}$, the cumulative distribution takes the form

$$F_{\rho^{D,N}}(t,x) = \sum_{j=0}^{N-1} [(j+1)l]\,\chi_{[x_j(t),x_{j+1}(t))}(x) + \chi_{[x_N(t),+\infty)}(x). \tag{45}$$

The corresponding pseudo-inverse takes the form:

$$X_{\rho^{D,N}}(t,z) = \sum_{j=0}^{N-1} \left[x_j^N(t)\right] \chi_{[jl,(j+1)l)}(z) + \left[x_N^N(t)\right] \chi_{\{1\}}(z). \tag{46}$$

3.2. **Evolution of the supports.** In this short section, we provide a first rough estimate on the support of all the functions defined above. The starting point is condition (6), which ensures the existence of a uniformly bounded initial support for the x_i.

Proposition 3.7. *Let $\{x_j^N(t)\}_{j=0}^N$ satisfy the condition of uniformly bounded initial support (6) for some set $K\subset\mathbb{R}$. Let v_{max} be given by (3). For each $T\geq 0$, define*

$$K_T := K + [0, Tv_{max}] = \{x+z \text{ such that } x\in K, z\in[0,Tv_{max}]\}. \tag{47}$$

It then holds

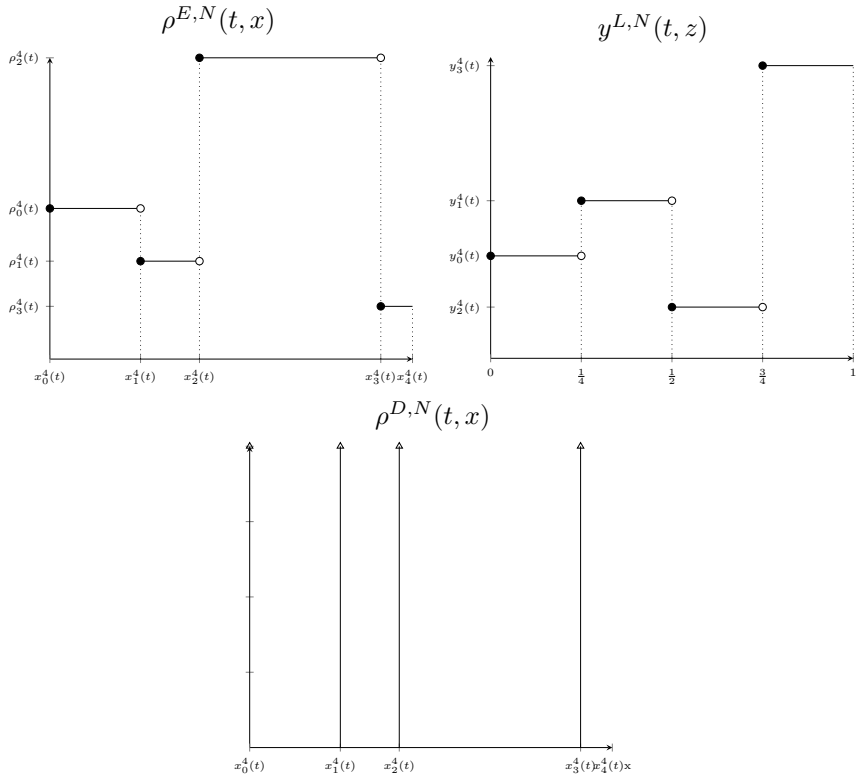


FIGURE 2. The Eulerian discrete density, the inverse Lagrangian discrete density and the (Dirac) empirical measure profiles ($N = 4$).

(i) $x_i^N(t) \in K_T$ *for all* $N \in \mathbb{N}$, $i = 0, \dots, N$, $t \in [0,T]$;
(ii) $\mathrm{supp}(\rho^{E,N}(t,\cdot)), \mathrm{supp}(y^{E,N}(t,\cdot)), \mathrm{supp}(\rho^{D,N}(t,\cdot)) \subset K_T$ *for all* $t \in [0,T]$;
(iii) $F_{\rho^{E,N}}(t,x) = F_{\rho^{D,N}}(t,x) = 0$ *for all* $x < \inf(K) = \inf(K_T)$ *and*
$F_{\rho^{E,N}}(t,x) = F_{\rho^{D,N}}(t,x) = 1$ *for all* $x > \max(K_T) = \max(K) + Tv_{max}$.

Proof. Statement (i) is a direct consequence of the fact that $\dot{x}_i \in [0, v_{\max}]$ in (1), (3). Statements (ii)-(iii) are then direct consequences of the definitions. □

3.3. **Convergence results for the cumulative and pseudo-inverse functions.** We now recall some results about the limits of $X_{\rho^{E,N}}$, $X_{\rho^{D,N}}$, $F_{\rho^{E,N}}$ and $F_{\rho^{D,N}}$, first given in [10]. The proofs are valid for any initial data $\{x_j^N(0)\}_{j=0}^N$ of system (18) that satisfies the condition of uniformly bounded initial support (6).

Proposition 3.8. *Let* $\{x_j^N(t)\}_{j=0}^N$ *be a solution of* (18) *that satisfies the condition of uniformly bounded initial support* (6). *Consider the corresponding Eulerian discrete density* $\rho^{E,N} \in L^\infty([0,+\infty) \times \mathbb{R}; [0,1])$ *defined by* (35) *and the (Dirac) empirical measure* $\rho^{D,N} \in L^\infty([0,+\infty); W_1(\mathcal{P}_c(\mathbb{R})))$ *defined by* (36). *Let* $F_{\rho^{E,N}}$, $X_{\rho^{E,N}}$, $F_{\rho^{D,N}}$, $X_{\rho^{D,N}}$ *be the corresponding cumulative distributions and pseudo-inverses defined by* (42), (43), (45), *and* (46), *respectively. Then, the following holds:*

(i) *there exists a non-decreasing function* $X \in L^\infty([0,+\infty)\times[0,1])$ *such that, up to a subsequence, both* $\{X_{\rho^{E,N}}\}_N$ *and* $\{X_{\rho^{D,N}}\}_N$ *converge to* X *in* $L^1_{loc}([0,+\infty)\times[0,1])$*;*

(ii) *define the map* $F:[0,+\infty)\times\mathbb{R}\to[0,1]$ *as*

$$F(t,x) := \mathrm{meas}\{z\in[0,1] : X(t,z)\le x\}, \qquad t\ge 0,\ x\in\mathbb{R}, \tag{48}$$

where $X(t,z)$ *is given by statement (i). Then, up to a subsequence, both* $\{F_{\rho^{E,N}}\}_N$ *and* $\{F_{\rho^{D,N}}\}_N$ *converge to* F *in* $L^1_{\mathrm{loc}}([0,+\infty)\times\mathbb{R})$.

Proof. See [10, Propositions 1-2, Lemma 4] with $L=1$, and $R=1$ (due to the maximum principle (24)), using their notation. □

Remark 3.9. Notice that, differently from the results in [10], Proposition 3.8 here only states the convergence of $\{X_{\rho^{E,N}}\}_{N\in\mathbb{N}}$, $\{X_{\rho^{D,N}}\}_{N\in\mathbb{N}}$ and $\{F_{\rho^{E,N}}\}_{N\in\mathbb{N}}$, $\{F_{\rho^{D,N}}\}_{N\in\mathbb{N}}$ up to a subsequence, which is obtained relying on Helly's compactness theorem. In [10], the authors conclude that the whole sequences $\{X_{\rho^{E,N}}\}_{N\in\mathbb{N}}$, $\{X_{\rho^{D,N}}\}_{N\in\mathbb{N}}$ converge, exploiting the fact that their atomization scheme for the FtL model guarantees that $X_{\rho^{D,N+1}}(t,z)\le X_{\rho^{D,N}}(t,z)$ for all $t\ge 0$ and $z\in[0,1]$. In turn, by the definition of the Wasserstein distance (41), the convergence of the whole sequences $\{X_{\rho^{E,N}}\}_{N\in\mathbb{N}}$ and $\{X_{\rho^{D,N}}\}_{N\in\mathbb{N}}$ yields the convergence of $\{F_{\rho^{E,N}}\}_{N\in\mathbb{N}}$ and $\{F_{\rho^{D,N}}\}_{N\in\mathbb{N}}$.

We now provide a refinement of Proposition 3.8-(ii).

Proposition 3.10. *Consider two sequences* $\{F_{\rho^{E,N}}\}_N$, $\{F_{\rho^{D,N}}\}_N$ *of cumulative distributions associated to the Eulerian discrete density* $\rho^{E,N}$ *and to the (Dirac) empirical measure* $\rho^{D,N}$*, respectively, that converge to a function* F *defined by* (48)*, which is Lipschitz continuous with respect to* x*. For any* $t\ge 0$*, let* $\rho(t)$ *be the distributional derivative of* $x\mapsto F(t,x)$*. Then, the following holds:*

(i) $\rho(t)\in\mathcal{P}_c(\mathbb{R})$ *for all* $t\ge 0$,

(ii) $0\le\rho(t)\le 1$ *for almost every* $t\ge 0$ *and* $x\in\mathbb{R}$,

(iii) $\{\rho^{E,N}\}_N$ *and* $\{\rho^{D,N}\}_N$ *converge to* ρ *in* $L^1_{\mathrm{loc}}([0,+\infty);W_1(\mathcal{P}_c(\mathbb{R})))$.

Proof. See [10, Proposition 3] with $L=1$, and $R=1$ (due to the maximum principle (24)), using their notation. □

Remark 3.11. Given a map $F:[0,+\infty)\times\mathbb{R}\to[0,1]$, denote with $\rho(t)$ the distributional derivative of $x\mapsto F(t,x)$, and assume that $\rho(t)\in\mathcal{P}_c(\mathbb{R})$ for all $t\ge 0$. Then, if we consider the cumulative distribution $F_{\rho(t)}$ as defined in (40), one has

$$F_{\rho(t)}(x) = F(t,x) \quad \text{for a.e. } x\in\mathbb{R}.$$

Lemma 3.12. *Let* $\{x_j^N(t)\}_{j=0}^N$ *be a solution of* (18)*, and consider the Lagrangian discrete density* $\rho^{L,N}\in L^\infty([0,+\infty)\times[0,1])$ *defined by* (37)*. Then, there exists* $\rho^L\in L^\infty([0,T]\times[0,1])$ *such that, up to a subsequence,* $\{\rho^{L,N}\}_{N\in\mathbb{N}}$ *converges to* ρ^L *weakly-* in* $L^\infty([0,+\infty)\times[0,1])$.

Proof. See [10, Lemma 5] with $L=1$, and $R=1$ (due to the maximum principle (24)), using their notation. □

4. **Proof of Theorem 1.2.** In this section, we prove the first main result of this article, i.e., Theorem 1.2. With this goal, we first recall standard tools to study the Cauchy problem (5): the definition of weak solution and classical results of existence and uniqueness of entropy solutions. Then, after proving a technical lemma, we present the proof of Theorem 1.2.

Given the Cauchy problem (5), we recall the definition of weak and entropy weak solution.

Definition 4.1. A function $\rho \in L^\infty([0,+\infty)\times\mathbb{R})$ is a weak solution to (5) if it holds

$$\int_{\mathbb{R}}\int_{\mathbb{R}_+} [\rho(t,x)\varphi_t(t,x) + (\rho(t,x)v(\rho(t,x)))\varphi_x(t,x)]\,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}} \bar\rho(x)\varphi(0,x)\mathrm{d}x = 0$$

for all $\varphi \in C_c^\infty([0,+\infty)\times\mathbb{R})$.

Definition 4.2. A function $\rho \in L^\infty([0,+\infty)\times\mathbb{R})$ is a Kružkov's entropy solution to (5) if it satisfies the entropy inequality

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}_+} [|\rho(t,x)-k|\varphi_t(t,x) + \mathrm{sign}(\rho(t,x)-k)[f(\rho(t,x))-f(k)]\varphi_x(t,x)]dtdx \\ + \int_{\mathbb{R}} |\bar\rho(x)-k|\varphi(0,x)dx \geq 0\end{aligned} \tag{49}$$

for all $\varphi \in C_c^\infty([0,+\infty)\times\mathbb{R})$ with φ non-negative, and for all constants $k \in \mathbb{R}$.

We now present two well-known results about the existence and uniqueness of the weak entropy solution to the Cauchy problem (5).

Theorem 4.3 (Uniqueness of Kružkov's solution, [16])**.** *Assume that the flux $f(\rho)$ is locally Lipschitz. For any given initial data $\bar\rho \in L^\infty$ with compact support, there exists a unique Kružkov's entropy solution $\rho \in L^\infty([0,+\infty)\times\mathbb{R})$ to (5).*

Theorem 4.4 (Chen and Rascle's entropy solution, [6])**.** *Assume that the flux is genuinely nonlinear almost everywhere, i.e., there exists no nontrivial interval on which the flux $f(\rho)$ is affine. For a given initial data $\bar\rho \in L^\infty$ with compact support, there exists a unique weak solution $\rho \in L^\infty([0,+\infty)\times\mathbb{R})$ of (5) in the sense of Definition 4.1 that satisfies the entropy inequality*

$$\int_{\mathbb{R}}\int_{\mathbb{R}_+} [|\rho(t,x)-k|\varphi_t(t,x) + \mathrm{sign}(\rho(t,x)-k)[f(\rho(t,x))-f(k)]\varphi_x(t,x)]\,dtdx \geq 0 \tag{50}$$

for all $\varphi \in C_c^\infty((0,+\infty)\times\mathbb{R})$ with φ non-negative and for all constants $k \in \mathbb{R}$. Moreover, ρ is the unique Kružkov's entropy solution to (5).

In Theorem 4.4, we see that, if the flux is genuinely nonlinear almost everywhere, the uniqueness of entropy solution is preserved for a relaxed notion of entropy solution, which does not require the entropy inequality (49) to be satisfied at $t = 0$. This is due to the fact that the nonlinearity of the flux ensures the existence of a strong trace at $t = 0$ of a weak solution to (5) in the sense of Definition 4.1.

We now present the following lemma, which is used in the proof of Theorem 1.2.

Lemma 4.5. *Consider a function $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ which is $1-$Lipschitz. It holds*

$$\|f\|_{L^\infty(\mathbb{R})} \leq \sqrt{\|f\|_{L^1(\mathbb{R})}}.$$

Proof. Since $|f|$ is 1-Lipschitz, for every $\bar{x} \in \mathbb{R}$, it holds

$$|f(x)| \geq \max\{|f(\bar{x})| - |x - \bar{x}|, 0\} \quad \forall\, x \in \mathbb{R}.$$

By integrating in space, it holds

$$\|f\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} |f(x)| dx \geq \int_{\mathbb{R}} \max\{|f(\bar{x})| - |x - \bar{x}|, 0\} dx = |f(\bar{x})|^2.$$

Take now $\bar{x}_n$ such that $\lim_{n\to+\infty} |f(\bar{x}_n)| = \|f\|_{L^\infty(\mathbb{R})}$. By passing to the limit, we have

$$\|f\|_{L^1(\mathbb{R})} \geq \lim_{n\to+\infty} |f(\bar{x}_n)|^2 = \|f\|^2_{L^\infty(\mathbb{R})},$$

and thus the result follows. □

We are now ready to provide the proof of the first main result of this article.

Proof of Theorem 1.2. Consider the Eulerian discrete density $\rho^{E,N} \in L^\infty([0,+\infty) \times \mathbb{R}; [0,1])$ defined by (4). To ease notation, we set

$$F^N(t) := F_{\rho^{E,N}(t)}, \tag{51}$$

from now on, where $F_{\rho^{E,N}}$ denotes the cumulative distribution of $\rho^{E,N}$ given by (42). Let $F(t,x)$ be the function defined by (48), which is equal to the cumulative distribution $F_{\rho(t)}(x)$ of its x-distributional derivative $\rho(t)$ (see Remark 3.11).

The proof is based on two steps.

1. In this step, we prove that $\{\rho^{E,N}\}_{N\in\mathbb{N}}$, up to a subsequence, is a Cauchy sequence in $L^1_{\mathrm{loc}}([0,+\infty) \times \mathbb{R})$, under either assumption (H1) or (H2) in Theorem 1.2. Thus, a subsequence of $\{\rho^{E,N}\}_{N\in\mathbb{N}}$ converges in $L^1_{\mathrm{loc}}([0,+\infty) \times \mathbb{R})$ to some limit function $\rho \in L^1_{\mathrm{loc}}([0,+\infty) \times \mathbb{R})$.

Recall by Propositions 3.8-3.10 that, up to a subsequence, and for every $T > 0$, it holds

$$\lim_{N\to+\infty} \int_0^T W_1(\rho^{E,N}(t), \rho(t)) dt = \lim_{N\to+\infty} \int_0^T \left\| F^N(t) - F(t) \right\|_{L^1(\mathbb{R})} dt = 0. \tag{52}$$

Since F^N, F^M are monotone non-decreasing and $1-$Lipschitz in the x variable, then also the function $F^N - F^M$ is $1-$Lipschitz in the x variable. Therefore, by Lemma 4.5, it holds

$$\left\| F^N(t) - F^M(t) \right\|_{L^\infty(\mathbb{R})} \leq \sqrt{\|F^N(t) - F^M(t)\|_{L^1(\mathbb{R})}} \quad \forall\, N, M \in \mathbb{N}, \quad \forall\, t > 0. \tag{53}$$

It moreover holds $\mathrm{supp}(F^N(t) - F^M(t)) \subset K_T$, $\mathrm{supp}(\rho^{E,N}(t) - \rho^{E,M}(t) \subset K_T$, for all $t \in [0,T]$, as a consequence of Proposition 3.7. Integrating by parts, we find

$$\begin{aligned}
&\int_{\mathbb{R}} (\rho^{E,N}(t,x) - \rho^{E,M}(t,x))^2 dx \\
&= \int_{\mathbb{R}} \frac{d}{dx} \left(F^N(t,x) - F^M(t,x) \right) \left(\rho^{E,N}(t,x) - \rho^{E,M}(t,x) \right) dx \\
&= - \int_{\mathbb{R}} \left(F^N(t,x) - F^M(t,x) \right) \frac{d}{dx} \left(\rho^{E,N}(t,x) - \rho^{E,M}(t,x) \right) dx \\
&\leq \left\| F^N(t) - F^M(t) \right\|_{L^\infty(\mathbb{R})} \mathrm{TV} \left(\rho^{E,N}(t) - \rho^{E,M}(t); \mathbb{R} \right) \\
&\leq \left\| F^N(t) - F^M(t) \right\|_{L^\infty(\mathbb{R})} \left[\mathrm{TV} \left(\rho^{E,N}(t); \mathbb{R} \right) + \mathrm{TV} \left(\rho^{E,M}(t); \mathbb{R} \right) \right].
\end{aligned}$$

By Hölder inequality and by using (53), we thus get that for all $N, M \in \mathbb{N}$, and for all $t > 0$, it holds

$$\begin{aligned}
&\left\|\rho^{E,N}(t) - \rho^{E,M}(t)\right\|^2_{L^1(\mathbb{R})} \leq \operatorname{meas}(K_T) \left\|\rho^{E,N}(t) - \rho^{E,M}(t)\right\|^2_{L^2(\mathbb{R})} \\
&\leq \operatorname{meas}(K_T) \left\|F^N(t) - F^M(t)\right\|_{L^\infty(\mathbb{R})} \left[\operatorname{TV}\left(\rho^{E,N}(t);\mathbb{R}\right) + \operatorname{TV}\left(\rho^{E,M}(t);\mathbb{R}\right)\right] \\
&\leq \operatorname{meas}(K_T) \sqrt{\left\|F^N(t) - F^M(t)\right\|_{L^1(\mathbb{R})}} \left[\operatorname{TV}\left(\rho^{E,N}(t);\mathbb{R}\right) + \operatorname{TV}\left(\rho^{E,M}(t);\mathbb{R}\right)\right].
\end{aligned} \tag{54}$$

The further treatment of this inequality is now addressed by considering separately the two cases of assumption (H1) and (H2) in Theorem 1.2.

Case (H1). We assume that (8) holds. Because of the BV contractivity property enjoyed by $\rho^{E,N}$ and $\rho^{E,M}$ (see Proposition 3.4-(i)) and relying on the hypothesis on the total variation of $\rho^{E,N}(0)$ and $\rho^{E,M}(0)$, it holds

$$\operatorname{TV}\left(\rho^{E,N}(t);\mathbb{R}\right) + \operatorname{TV}\left(\rho^{E,M}(t);\mathbb{R}\right) \leq \operatorname{TV}\left(\rho^{E,N}(0);\mathbb{R}\right) + \operatorname{TV}\left(\rho^{E,M}(0);\mathbb{R}\right) \leq 2C.$$

Thus, we deduce from (54) that, for all $N, M \in \mathbb{N}$, and for all $t > 0$, it holds

$$\left\|\rho^{E,N}(t) - \rho^{E,M}(t)\right\|^2_{L^1(\mathbb{R})} \leq 2C \operatorname{meas}(K_T) \sqrt{\left\|F^N(t) - F^M(t)\right\|_{L^1(\mathbb{R})}}. \tag{55}$$

Notice that, by Hölder's inequality, we have

$$\begin{aligned}
&\int_0^T \sqrt{\left\|F^N(t) - F^M(t)\right\|_{L^1(\mathbb{R})}} dt \\
\leq\ & \|1\|_{L^2([0,T])} \left\| \sqrt{\left\|F^N(t) - F^M(t)\right\|_{L^1(\mathbb{R})}} \right\|_{L^2([0,T])} \\
=\ & \sqrt{T} \sqrt{\int_0^T \left\|F^N(t) - F^M(t)\right\|_{L^1(\mathbb{R})} dt}.
\end{aligned} \tag{56}$$

Then, integrating (55) in the time interval $[0,T]$ and using (56), we find that for all $N, M \in \mathbb{N}$, it holds

$$\begin{aligned}
&\int_0^T \left\|\rho^{E,N}(t) - \rho^{E,M}(t)\right\|^2_{L^1(\mathbb{R})} dt \\
&\leq 2C \operatorname{meas}(K_T) \sqrt{T} \sqrt{\int_0^T \left\|F^N(t) - F^M(t)\right\|_{L^1(\mathbb{R})} dt}.
\end{aligned} \tag{57}$$

Finally, by Hölder's inequality, we derive from (57) that

$$\begin{aligned}
&\int_0^T \left\|\rho^{E,N}(t) - \rho^{E,M}(t)\right\|_{L^1(\mathbb{R})} dt \leq \sqrt{T} \sqrt{\int_0^T \left\|\rho^{E,N}(t) - \rho^{E,M}(t)\right\|^2_{L^1(\mathbb{R})} dt} \\
&\leq \sqrt{2C \operatorname{meas}(K_T)} \cdot T^{\frac{3}{4}} \left(\int_0^T \left\|F^N(t) - F^M(t)\right\|_{L^1(\mathbb{R})} dt \right)^{\frac{1}{4}}.
\end{aligned}$$

Therefore, in Case (H1), the convergence result (52) implies that, for every $T > 0$, the sequence $\left\{\rho^{E,N}\right\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^1([0,T] \times \mathbb{R})$.

Case (H2). We assume that (V2) holds. By Proposition 3.4-(ii) and Proposition 3.7, for any fixed $T, \delta > 0$, there exists a constant $C_{\delta,T} > 0$ such that, for all $N, M \in \mathbb{N}$, it holds

$$\sup_{t\in[\delta,T]} \left[\operatorname{TV}\left(\rho^{E,N}(t);\mathbb{R}\right) + \operatorname{TV}\left(\rho^{E,M}(t);\mathbb{R}\right)\right] \leq C_{\delta,T}.$$

and $\mathrm{supp}(\rho^{E,N}(t)), \mathrm{supp}(\rho^{E,M}(t)) \subset K_T$ with K_T compact, given by (47).

With the same analysis in (54), (55), it thus follows that, for all $N, M \in \mathbb{N}$, and for all $t \in [\delta, T]$, it holds

$$\begin{aligned}
&\left\|\rho^{E,N}(t) - \rho^{E,M}(t)\right\|^2_{L^1(\mathbb{R})} \leq \\
&\mathrm{meas}(K_T) \sup_{t\in[\delta,T]} \left[\mathrm{TV}\left(\rho^{E,N}(t);\mathbb{R}\right) + \mathrm{TV}\left(\rho^{E,M}(t);\mathbb{R}\right)\right] \sqrt{\|F^N(t) - F^M(t)\|_{L^1(\mathbb{R})}} \\
&\leq 2C_{\delta,T}\,\mathrm{meas}(K_T)\sqrt{\|F^N(t) - F^M(t)\|_{L^1(\mathbb{R})}}. \qquad (58)
\end{aligned}$$

Integrating in the time interval $[\delta, T]$ and using the Hölder's inequality as in the previous step, we then find that, for all $N, M \in \mathbb{N}$, it holds

$$\begin{aligned}
&\int_\delta^T \left\|\rho^{E,N}(t) - \rho^{E,M}(t)\right\|^2_{L^1(\mathbb{R})}\,dt \leq \\
&2C_{\delta,T}\,\mathrm{meas}(K_T)\sqrt{T}\sqrt{\int_\delta^T \|F^N(t) - F^M(t)\|_{L^1(\mathbb{R})}\,dt},
\end{aligned}$$

and

$$\begin{aligned}
&\int_\delta^T \left\|\rho^{E,N}(t) - \rho^{E,M}(t)\right\|_{L^1(\mathbb{R})}\,dt \leq \\
&\sqrt{2C_{\delta,T}\,\mathrm{meas}(K_T)} \cdot \left(T^3 \int_\delta^T \left\|F^N(t) - F^M(t)\right\|_{L^1(\mathbb{R})}\,dt\right)^{\frac{1}{4}}. \qquad (59)
\end{aligned}$$

Observe now that, for any fixed $\epsilon > 0$, setting $\delta_\epsilon := \epsilon/(2\,\mathrm{meas}(K_T))$, we have

$$\int_0^{\delta_\epsilon} \left\|\rho^{E,N}(t) - \rho^{E,M}(t)\right\|_{L^1(\mathbb{R})}\,dt \leq \frac{\epsilon}{2}, \qquad \forall N, M \in \mathbb{N}\,. \qquad (60)$$

On the other hand, the convergence result (52), together with (59), implies that there exists $N(\epsilon) > 0$ such that

$$\int_{\delta_\epsilon}^T \left\|\rho^{E,N}(t) - \rho^{E,M}(t)\right\|_{L^1(\mathbb{R})}\,dt \leq \frac{\epsilon}{2}, \qquad \forall N, M \geq N(\epsilon)\,. \qquad (61)$$

Therefore, combining (60)-(61), we find that, also in case (H2), for every $T > 0$, the sequence $\left\{\rho^{E,N}\right\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^1([0,T]\times\mathbb{R})$. Then, we conclude as in case (H1).

2. In this step, we show that the function ρ determined in the previous step is the (unique) weak entropy solution of the Cauchy problem (5), and that actually the whole sequence $\left\{\rho^{E,N}\right\}_{N\in\mathbb{N}}$ converges in $L^1_{\mathrm{loc}}([0,+\infty)\times\mathbb{R})$ to ρ.

Recalling that $\left\{\rho^{E,N}(0)\right\}_{N\in\mathbb{N}}$ weakly converges to $\bar\rho$ by hypothesis (7), and following the same procedure as in Step 1-Case 1 of the proof of [9, Theorem 2], we deduce that ρ is a weak solution to (5) in the sense of Definition 4.1. Furthermore, it also holds that ρ satisfies the entropy inequality (50) by applying the exact same computations as done in the part (vi) of the proof of [10, Theorem 3]. In turn, this implies that ρ is a weak entropy solution of the Cauchy problem (5), thanks to Theorem 4.4. By merging Step 1 and Step 2, we conclude that, up to a subsequence, $\left\{\rho^{E,N}\right\}_{N\in\mathbb{N}}$ converges in $L^1_{\mathrm{loc}}([0,+\infty)\times\mathbb{R})$ to the unique weak entropy solution of (5). Since, with the same arguments, we can show that any subsequence of $\left\{\rho^{E,N}\right\}_{N\in\mathbb{N}}$ admits a subsubsequence converging to the unique weak entropy solution of (5), it follows that the whole sequence $\left\{\rho^{E,N}\right\}_{N\in\mathbb{N}}$ converges to ρ. □

If $\bar\rho \in BV(\mathbb{R})$ satisfies the assumptions of Theorem 1.2, relying on the analysis performed in Step 1 of the above proof, one can derive the convergence rate for the initial Eulerian discrete density $\rho^{E,N}(0)$ associated to the atomization scheme introduced in [10, (19a)-(19b)]. We recall it here and prove some relevant properties.

Proposition 4.6. *Let $\bar\rho \in \mathcal{P}_c(\mathbb{R})$, with $\|\bar\rho\|_{L^\infty(\mathbb{R})} \le 1$. Assume that v satisfies (V1). Define $x_i^N(0)$ by (14)-(15). Let $\rho^{E,N}(0)$ be the corresponding Eulerian discrete density at time $t = 0$, defined as in (35). Then, the following properties hold:*

(i)

$$\rho^{E,N}(0) \to \bar\rho \quad \text{in} \quad L^1(\mathbb{R}). \tag{62}$$

(ii) If $\bar\rho \in BV(\mathbb{R})$, then there holds

$$\left\|\rho^{E,N}(0) - \bar\rho\right\|_{L^1(\mathbb{R})} \le \frac{C}{N^{1/4}} \qquad \forall\, N \in \mathbb{N} \setminus \{0\}\,, \tag{63}$$

for some constant $C > 0$ depending on $\mathrm{TV}(\bar\rho;\mathbb{R})$ and on the measure of the support of $\bar\rho$.

Proof. We prove (i). Observe that, by definitions (14), (15), one has

$$x_N^N(0) = \bar x_{\max}. \tag{64}$$

For any $x \in (\bar x_{\min}, \bar x_{\max})$, let

$$I^N(x) := [x_{j_N}^N(0), x_{j_N+1}^N(0)),$$

be the interval containing x for some $j_N \in \{0,\dots,N-1\}$, and set

$$I(x) := \cap_N I^N(x).$$

Then, we can decompose $(\bar x_{\min}, \bar x_{\max})$ as the disjoint union of the sets

$$\mathcal{I}_1 := \{x \in (\bar x_{\min}, \bar x_{\max}) \mid I(x) = \{x\}\}, \quad \mathcal{I}_2 := \{x \in (\bar x_{\min}, \bar x_{\max}) \mid \{x\} \subsetneq I(x)\}\,. \tag{65}$$

Notice that

$$\lim_{N\to\infty} \mathrm{meas}(I^N(x)) = 0 \qquad \forall\, x \in \mathcal{I}_1. \tag{66}$$

Moreover, observe that, by definitions (17), (19), (35), (15), we have

$$\rho^{E,N}(0,x) - \bar\rho(x) = \frac{1}{\mathrm{meas}(I^N(x))} \int_{I^N(x)} (\bar\rho(y) - \bar\rho(x))\, dy\,, \qquad \forall\, x \in (\bar x_{min}, \bar x_{max}). \tag{67}$$

Therefore, since $\bar\rho \in L^1(\mathbb{R})$, by the Lebesgue differentiation theorem (e.g., see [11, § 3.4]) we deduce from (66), (67) that there holds

$$\lim_{N\to\infty} \rho^{E,N}(0,x) = \bar\rho(x) \qquad \text{for a.e.} \quad x \in \mathcal{I}_1\,. \tag{68}$$

On the other hand, by definition (65), the set $\mathcal{I}_2$ is the union of intervals $J = [x_J, x_J']$ with the property that

$$[x_J, x_J'] \subset [x_{j_N}^N(0), x_{j_N+1}^N(0)), \qquad \forall\, N, \tag{69}$$

for some sequence of indices $j_N \in \{0,\dots,N-1\}$. Hence, by definition, we derive

$$\rho^{E,N}(0,x) = \frac{1/N}{x_{j_N+1}^N(0) - x_{j_N}^N(0)} \le \frac{1/N}{x_J' - x_J} \qquad \forall\, x \in [x_J, x_J'], \qquad \forall\, N \in \mathbb{N}\,,$$

which yields

$$\lim_{N\to\infty} \rho^{E,N}(0,x) = 0 \qquad \forall\, x \in [x_J, x_J']. \tag{70}$$

Next, observe that by definition (15) and because of (69), we have

$$\int_{x_J}^{x_J'} \bar\rho(x)\,dx \le \int_{x_{j_N}^N(0)}^{x_{j_N+1}^N(0)} \bar\rho(x)\,dx = \frac{1}{N} \qquad \forall N,$$

which implies

$$\int_{x_J}^{x_J'} \bar\rho(x)\,dx = 0,$$

and thus we find

$$\bar\rho(x) = 0 \qquad \text{for a.e.} \quad x \in [x_J, x_J']\,. \tag{71}$$

Since $\mathcal{I}_2$ is the union of intervals of the form $[x_J, x_J']$, we deduce from (70)-(71) that

$$\lim_{N\to\infty} \rho^{E,N}(0,x) = \bar\rho(x) \qquad \text{for a.e.} \quad x \in \mathcal{I}_2\,. \tag{72}$$

Then, by the dominated convergence theorem, we derive from (68), (72) that (62) is verified.

We now prove (ii). By the proof of [10, Proposition 4], it holds

$$W_1(\rho^{E,N}(0), \bar\rho) \le \frac{2(\bar x_{\max} - \bar x_{\min})}{N} \qquad \forall\ N \in \mathbb{N}\setminus\{0\}. \tag{73}$$

By using the notation in (51), this implies that $\{F^N(0)\}_{N\in\mathbb{N}}$ converges to $F_{\bar\rho}$ in $L^1(\mathbb{R})$, as $N\to\infty$. Moreover, by [10, Proposition 5] we have $\mathrm{TV}\,(\rho^{E,N}(0);\mathbb{R}) \le \mathrm{TV}\,(\bar\rho;\mathbb{R})$ for all N. On the other hand, relying on (62) and taking the limit as $M\to\infty$ in the inequality (54) at $t=0$, with $K=[\bar x_{\min}, \bar x_{\max}]$, we get that, for all N, there holds

$$\begin{aligned} &\left\|\rho^{E,N}(0) - \bar\rho\right\|_{L^1(\mathbb{R})} \\ &= \left\|\rho^{E,N}(0) - \bar\rho\right\|_{L^1(K)} \le \sqrt{2C_1(\bar x_{\max} - \bar x_{\min})}\left\|F^N(0) - F_{\bar\rho}\right\|_{L^1(\mathbb{R})}^{\frac14} \\ &= \sqrt{2C_1(\bar x_{\max} - \bar x_{\min})}\left(W_1(\rho^{E,N}(0),\bar\rho)\right)^{\frac14}, \end{aligned} \tag{74}$$

where $C_1 = \mathrm{TV}\,(\bar\rho;\mathbb{R})$. Thus, combining (73)-(74), we deduce (63). □

Remark 4.7. Under the same assumptions of Theorem 1.2, we can also deduce that the sequence of empirical measures $\{\rho^{D,N}\}_{N\in\mathbb{N}}$ defined in (36) converges in $L^1_{\mathrm{loc}}([0,+\infty];W_1)$ to the unique weak entropy solution ρ of (5). Indeed, fix $T>0$ and notice that

$$\int_0^T W_1(\rho^{D,N}(t),\rho(t))dt \le \int_0^T W_1(\rho^{D,N}(t),\rho^{E,N}(t))dt + \int_0^T W_1(\rho^{E,N}(t),\rho(t))dt. \tag{75}$$

Moreover, recalling (41), (43) and (46), it holds

$$\begin{aligned} W_1(\rho^{D,N}(t),\rho^{E,N}(t)) &= \int_0^1 |X^{E,N}(z) - X^{D,N}(z)|dz = \sum_{j=0}^{N-1} y_j^N \int_{jl}^{(j+1)l} [z-jl]dz \\ &= \frac{l}{2}\sum_{j=0}^{N-1} x_{j+1}^N(t) - x_j^N(t) = \frac{l}{2}\left(x_N^N(t) - x_0^N(t)\right). \end{aligned}$$

Therefore, we have

$$W_1(\rho^{D,N}(t),\rho^{E,N}(t)) = \int_0^T\int_0^1 |X^{E,N}(t,z) - X^{D,N}(t,z)|dz \le$$

$$\frac{T(x_N^N(0)-x_0^N(0)+v_{\max}T)}{2N}.$$

Thanks to the condition of uniformly bounded initial support (6), it holds

$$\lim_{N\to+\infty}\int_0^T W_1(\rho^{D,N}(t),\rho^{E,N}(t))dt=0. \tag{76}$$

Observe now that, invoking Proposition 3.7, recalling (41)-(42), and using Poincaré's inequality, we have

$$W_1(\rho^{E,N}(t),\rho(t))=\left\|F^N(t)-F(t)\right\|_{L^1(\mathbb{R})}\le C_1\left\|\rho^{E,N}(t)-\rho(t)\right\|_{L^1(\mathbb{R})}\quad\forall\, t\in[0,T],$$

for some constant $C_1>0$. Then, integrating on $[0,T]$, we derive

$$\int_0^T W_1(\rho^{E,N}(t),\rho(t))dt\le C_1\left\|\rho^{E,N}-\rho\right\|_{L^1([0,T]\times\mathbb{R})}. \tag{77}$$

By merging (75), (76), (77) and Theorem 1.2, it holds

$$\lim_{N\to+\infty}\int_0^T W_1(\rho^{D,N}(t),\rho(t))dt=0\qquad\forall\, T>0.$$

The next Proposition shows how to define an atomization scheme $\{\tilde{x}_j^N(0)\}_{j=0}^N$ different from the one in [10], whose corresponding initial Eulerian discrete density satisfies the assumption (7) of Theorem 1.2. Thus, if the velocity function satisfies (V2), according with Theorem 1.2, the scheme $\{\tilde{x}_j^N(0)\}_{j=0}^N$ leads to an Eulerian discrete density $\tilde{\rho}^{E,N}(t,x)$ that still converges in L^1_{loc} to the weak entropy solution of the Cauchy problem (5). On the other hand, we also show that the initial Eulerian discrete density associated with $\{\tilde{x}_j^N(0)\}_{j=0}^N$ does not satisfy assumption (H1) of Theorem 1.2. Hence, in this case, one would not expect that $\tilde{\rho}^{E,N}(t,x)$ converges to the weak entropy solution of the Cauchy problem (5). However, we will discuss in Remark 4.9 some numerical simulations that seem to suggest that such convergence holds for $\tilde{\rho}^{E,N}(t,x)$ too.

Proposition 4.8. *Let $\bar{\rho}\in\mathcal{P}_c(\mathbb{R})$, with $\|\bar{\rho}\|_{L^\infty(\mathbb{R})}\le 1/2$. Assume that v satisfies (V1). Define $\{x_j^N(0)\}_{j=0}^N$ as in (14)-(15). Also, define $\{\tilde{x}_j^N(0)\}_{j=0}^N$ as follows*

$$\tilde{x}_j^N(0)=\begin{cases}x_j^N(0) & \text{if } j \text{ is even or } j=N,\\ \dfrac{x_{j+1}^N(0)+x_j^N(0)}{2} & \text{if } j<N \text{ is odd.}\end{cases} \tag{78}$$

Let $\tilde{\rho}^{E,N}(0)$ be the corresponding Eulerian discrete density at time $t=0$, defined as in (35). Then, the following properties hold:

(i) $\tilde{\rho}^{E,N}(0)\rightharpoonup\bar{\rho}$ weak* in $L^\infty(\mathbb{R})$;
(ii) $\tilde{\rho}^{E,N_k}(0)\twoheadrightarrow\bar{\rho}$ in $L^1(\mathbb{R})$, for every subsequence $\{\tilde{\rho}^{E,N_k}(0)\}_k$;
(iii) The sequence $TV(\tilde{\rho}^{E,N}(0))$ is unbounded.

Proof. Denote with $\lfloor a\rfloor$ the integer part of a and define $N':=\lfloor\frac{N}{2}\rfloor-1$.

We first prove (i). By (15), (78), and because of definitions (17), (19), (35), we have

$$\int_{x_{2j}^N(0)}^{x_{2j+2}^N(0)}\tilde{\rho}^{E,N}(0,x)\,dx=\int_{x_{2j}^N(0)}^{x_{2j+2}^N(0)}\bar{\rho}(x)\,dx=\frac{2}{N}\qquad\forall\, j=0,\dots,N'. \tag{79}$$

In the same way, if N is odd, and thus $2\lfloor \frac{N}{2} \rfloor = N-1$, we find

$$\int_{x^N_{N-1}(0)}^{x^N_N(0)} \tilde\rho^{E,N}(0,x)\,dx = \int_{x^N_{N-1}(0)}^{x^N_N(0)} \bar\rho(x)\,dx = \frac{1}{N}.$$

Consider a test function $\varphi \in C^\infty_c(\mathbb{R})$, and set

$$a^N_j := \min_{x\in[x^N_{2j}(0),x^N_{2j+2}(0)]} \varphi(x), \quad b^N_j := \max_{x\in[x^N_{2j}(0),x^N_{2j+2}(0)]} \varphi(x) \qquad \forall\, j=0,\dots,N'.$$

By monotonicity of the integral (i.e., $\varphi \le \psi \Rightarrow \int \varphi\rho\,dx \le \int \psi\rho\,dx$ for $\rho \ge 0$), it follows from (79) that

$$\int_{x^N_{2j}(0)}^{x^N_{2j+2}(0)} \varphi(x)\bar\rho(x)\,dx \in \frac{2}{N}[a^N_j, b^N_j], \quad \int_{x^N_{2j}(0)}^{x^N_{2j+2}(0)} \varphi(x)\tilde\rho^{E,N}(0,x)\,dx \in \frac{2}{N}[a^N_j, b^N_j]. \tag{80}$$

Moreover, letting L be the Lipschitz costant of φ, it holds

$$b^N_j - a^N_j \le L(x^N_{2j+2}(0) - x^N_{2j}(0)).$$

As a consequence, we derive from (80) that

$$\left| \int_{x^N_{2j}(0)}^{x^N_{2j+2}(0)} \varphi(x)(\bar\rho(x) - \tilde\rho^{E,N}(0,x))\,dx \right| \le \frac{2\,L(x^N_{2j+2}(0) - x^N_{2j}(0))}{N} \qquad \forall\, j=0,\dots,N'.$$

For N even, this implies

$$\begin{aligned} \left| \int_{\mathbb{R}} \varphi(x)(\bar\rho(x) - \tilde\rho^{E,N}(0,x))\,dx \right| &\le \frac{2\,L}{N} \sum_{j=0}^{\frac{N}{2}-1} (x^N_{2j+2}(0) - x^N_{2j}(0)) \\ &= \frac{2\,L\,(x^N_N(0) - x^N_0(0))}{N}. \end{aligned} \tag{81}$$

For N odd, one needs to consider also the additional estimate

$$\left| \int_{x^N_{N-1}(0)}^{x^N_N(0)} \varphi(x)(\bar\rho(x) - \tilde\rho^{E,N}(0,x))\,dx \right| \le \frac{L(x^N_N(0) - x^N_{N-1}(0))}{N}$$

that anyway leads to (81). In both cases, recalling (14), (64), we deduce from (81) that

$$\left| \int_{\mathbb{R}} \varphi(x)(\bar\rho(x) - \tilde\rho^{E,N}(0,x))\,dx \right| \le \frac{2\,L\,(\bar x_{\max} - \bar x_{\min})}{N},$$

which proves (i).

We now prove (ii). In view of Proposition 4.6-(i), in order to establish (ii), it will be sufficient to show that

$$\tilde\rho^{E,N_k}(0) - \rho^{E,N_k}(0) \rightharpoonup 0 \quad \text{in} \quad L^1(\mathbb{R}), \tag{82}$$

for every subsequence $\{\tilde\rho^{E,N_k}(0)\}_k$. To this end, denote with $\rho^{E,N}(0)$ the Eulerian discrete densities at time $t=0$, defined as in (35) in connection with the scheme $\{x^N_j(0)\}_{j=0}^N$ in (14)-(15). By (17), (19), and (78), we have

$$\tilde x^N_{2j+2}(0) - \tilde x^N_{2j+1}(0) = x^N_{2j+2}(0) - \frac{x^N_{2j+2}(0) - x^N_{2j+1}(0)}{2} = \frac{x^N_{2j+2}(0) - x^N_{2j+1}(0)}{2},$$

and

$$\rho^{E,N}(0,x) = \frac{1/N}{x^N_{2j+2}(0) - x^N_{2j+1}(0)}, \qquad \tilde\rho^{E,N}(0,x) = \frac{2/N}{x^N_{2j+2}(0) - x^N_{2j+1}(0)},$$

for all $x \in [\tilde{x}^N_{2j+1}(0),\, \tilde{x}^N_{2j+2}(0)]$, and for all $j = 0, \dots, N'$. As a consequence we find

$$\int_{\tilde{x}^N_{2j+1}(0)}^{\tilde{x}^N_{2j+2}(0)} \left| \tilde{\rho}^{E,N}(0,x) - \rho^{E,N}(0,x) \right| dx = \frac{1}{2N}, \qquad \forall\, j = 0, \dots, N',$$

which yields

$$\begin{aligned} \|\tilde{\rho}^{E,N}(0) - \rho^{E,N}(0)\|_{L^1(\mathbb{R})} &\geq \sum_{j=0}^{N'} \int_{\tilde{x}^N_{2j+1}(0)}^{x^N_{2j+2}(0)} \left| \tilde{\rho}^{E,N}(0,x) - \rho^{E,N}(0,x) \right| dx \\ &= \frac{1}{2N}(N'+1) \geq \frac{N-1}{4N} > \frac{1}{8} \qquad \forall\, N > 2\,. \end{aligned}$$

This implies (82), thus completing the proof of (ii).

We finally prove (iii). By contradiction, assume that there exists a subsequence (that we do not relabel) such that $TV(\tilde{\rho}^{E,N}(0))$ is uniformly bounded. Since $\tilde{\rho}^{E,N}(0)$ is uniformly bounded in $L^\infty(\mathbb{R})$, by Helly's compactness theorem there exists a further subsequence (that we do not relabel) that converges in $L^1(\mathbb{R})$ to some function $\tilde{\rho}$. Statement (i) ensures that $\bar{\rho} = \tilde{\rho}$, that is a contradiction with (ii). This proves (iii). □

Remark 4.9. Given $\bar{\rho} \in \mathcal{P}_c(\mathbb{R})$, consider the atomization scheme $\{\tilde{x}^N_j(0)\}_{j=0}^N$ defined in (78), and let $\{\tilde{x}^N_j(t)\}_{j=0}^N$ be the solution of (18), with traffic velocity given by

$$v(\rho) = \begin{cases} \exp\left(\dfrac{\rho}{\rho - 1}\right) & \text{if } \ \rho < 1, \\ 0 & \text{if } \ \rho = 1, \end{cases} \tag{83}$$

according with the Bonzani and Mussone's model [2]. It is clear that the associated flux is not strictly concave, see Figure 3.

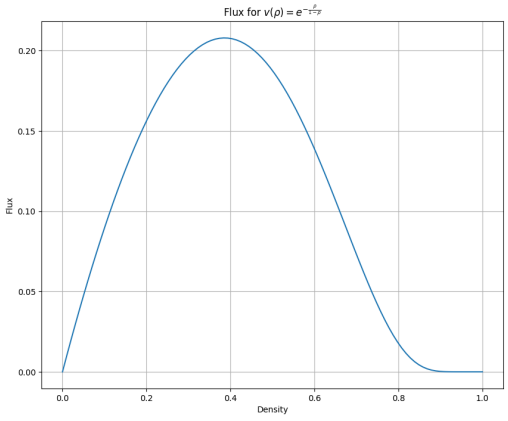


FIGURE 3. Flux associated with (83).

More precisely, it holds $\rho\, v'(\rho) = -\dfrac{\rho\, v(\rho)}{(1-\rho)^2}$, which is a decreasing function in the interval $[0,\, \frac{\sqrt{5}-1}{2}]$, and an increasing function in the interval $[\frac{\sqrt{5}-1}{2},\, 1]$. Therefore, the velocity $v(\rho)$ in (83) does not satisfy the assumption (V2).

On the other hand, we have shown in Proposition 4.8-(iii) that, letting $\tilde{\rho}^{E,N}(0)$ be the Eulerian discrete densities at time $t = 0$ corresponding to $\{\tilde{x}_j^N(0)\}_{j=0}^N$, one has that $TV(\tilde{\rho}^{E,N}(0))$ is unbounded. Hence, neither condition (H1) nor (H2) of Theorem 1.2 are satisfied. Thus, although $\tilde{\rho}^{E,N}(0)$ satisfies the assumption (7) of Theorem 1.2 (as shown in Proposition 4.8-(i)), one cannot expect the sequence $\{\tilde{\rho}^{E,N}\}_{N\in\mathbb{N}}$ to converge to the weak entropy solution ρ of the Cauchy problem (5), in general.

To investigate this behavior, we discuss here the numerical simulations corresponding to the discretization scheme $\{\tilde{x}_j^N(t)\}_{j=0}^N$, with velocity $v(\rho)$ in (83), and initial datum

$$\bar{\rho}(x) = \frac{1}{2}\chi_{[\frac{1}{2},\frac{5}{2}]}(x)\,.$$

The parameters for the numerical simulation are chosen as follows. We set the space step-size $\Delta x = 0.01$, time step-size $\Delta t = 0.001$, and the time period $[0, 3]$. Since the system becomes stiffer as N increases, we choose an implicit method to numerically solve the system. In particular, we use an implicit method based on backward-differentiation formulas (BDF) of automatically-varying order (from 1 to 5), already implemented in the Python library "scipy" as "BDF". The general framework of such algorithm is described in [4] and the Python implementation follows a quasi-constant step size as explained in [21].

In Figure 4, we show snapshots of the evolution of the two different profiles $\rho^{E,5}(t)$ and $\tilde{\rho}^{E,5}(t)$. It is remarkable that for some initial short period of time, the fluxes experimented by both profiles are visibly distinct; see Figure 4a. Yet, after some further time, both profiles start exhibiting an extremely similar behavior (e.g., starting at $t = 1$ in Figure 4b). Indeed, an approximation of a rarefaction wave can be seen on the right and a shock on the left. They then interact, thus causing the decrease of the total variation at the interaction point. This corresponds to the behavior at the macroscopic level. Afterward, both profiles remain in the same configuration with decreasing density with respect to x, i.e., $\rho^{E,N}(x_1) \le \rho^{E,N}(x_2)$ with $x_2 \le x_1$. This property is indeed preserved forward in time by the FtL.

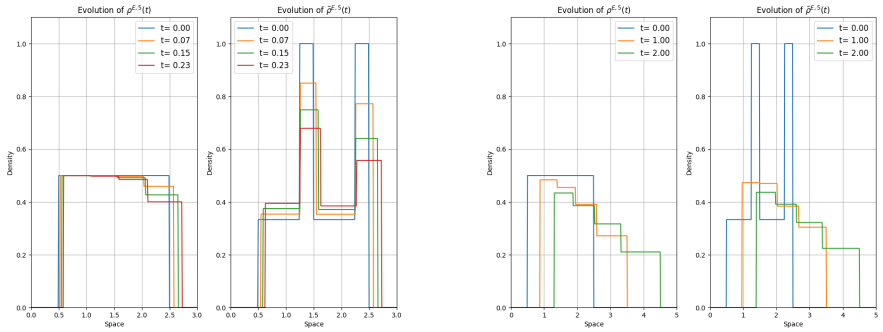
In Figure 5, we show the evolution in time of the L^1 distance (in the space variable) between $\rho^{E,N}(t)$ and $\rho^{\tilde{E},N}(t)$, for different values of N. The striking phenomenon is that the L^1 distance is a decreasing function of time, with a very strong decay as N increases. This suggests that the following result might hold for this initial datum: for every $t > 0$ and for arbitrary $\epsilon > 0$, there exists $N > 0$ such that

$$\int_0^t \left\|\rho^{E,N}(t) - \tilde{\rho}^{E,N}(t)\right\|_{L^1(\mathbb{R})} \le \epsilon.$$

In particular, it seems that the discrepancy between the approximating solutions is arbitrarily small for arbitrarily small time. In other terms, Theorem 1.2 might be non-sharp, and a more general result of convergence might be available.

5. **Proof of Theorem 1.4.** In this section, we prove Theorem 1.4, which provides a stability result for two different Eulerian discrete densities, in both the Wasserstein and the L^1 norm. Here, we compare two solutions $\{x_j^N(t)\}_{j=0}^N$, $\{\tilde{x}_j^N(t)\}_{j=0}^N$ of the FtL model (18) and the corresponding Eulerian discrete densities $\rho^{E,N}$, $\tilde{\rho}^{E,N}$ defined by (35).

We first state three propositions, which lead to the proof of the main theorem.



(A) Evolution in the time period $[0, 0.23]$ (B) Evolution in the time period $[0, 2]$

FIGURE 4. Snapshots of the dynamics of $\rho^{E,5}(t)$ and $\tilde{\rho}^{E,5}(t)$.

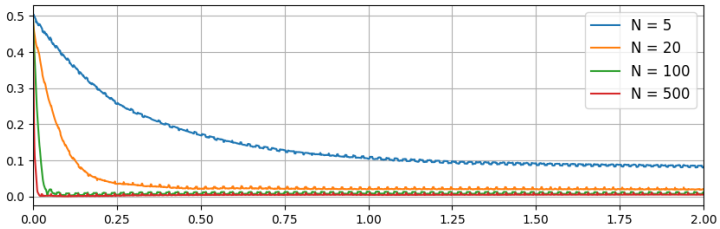


FIGURE 5. Evolution of $\|\rho^{E,N}(t) - \tilde{\rho}^{E,N}(t)\|_{L^1(\mathbb{R})}$ for $N = 5, 20, 100, 500$.

Proposition 5.1. *Given two sequences* $\{x_j^N\}_{j=0}^N, \{\tilde{x}_j^N\}_{j=0}^N$, *satisfying condition* (17), *assume that* $x_N^N = \tilde{x}_N^N$. *Consider the corresponding Eulerian densities* $\rho^{E,N}$, $\tilde{\rho}^{E,N} \in L^\infty(\mathbb{R})$ *defined by* (35). *Then, it holds*

$$W_1(\rho^{E,N}, \tilde{\rho}^{E,N}) \leq 2 \sum_{j=0}^{N-1} |x_{j+1} - x_j - (\tilde{x}_{j+1} - \tilde{x}_j)| \tag{84}$$

Proof. For any fixed $j = 0, ..., N-1$, and for every $z \in [jl, (j+1)l)$, recalling the definition of y_j^N in (21), we have

$$\begin{aligned}
&\left|x_j^N - \tilde{x}_j^N + (z - jl)\left(y_j^N - \tilde{y}_j^N\right)\right| = \\
&\left|x_j^N - \tilde{x}_j^N + \frac{z - jl}{l}\left(x_{j+1}^N - \tilde{x}_{j+1}^N - (x_j^N - \tilde{x}_j^N)\right)\right| \\
&\leq \left|x_j^N - \tilde{x}_j^N\right| + \left|x_{j+1}^N - \tilde{x}_{j+1}^N - (x_j^N - \tilde{x}_j^N)\right| \\
&\leq 2\left|x_j^N - \tilde{x}_j^N - (x_{j+1}^N - \tilde{x}_{j+1}^N)\right| + \left|x_{j+1}^N - \tilde{x}_{j+1}^N\right| \\
&\leq 2\left(\sum_{k=j}^{N-1} \left|x_k^N - \tilde{x}_k^N - (x_{k+1}^N - \tilde{x}_{k+1}^N)\right|\right) + \left|x_N^N - \tilde{x}_N^N\right|,
\end{aligned}$$

where in the last inequality we repeatedly make use of the triangular inequality

$$\left|x_k^N - \tilde{x}_k^N\right| \le \left|x_k^N - \tilde{x}_k^N - (x_{k+1}^N - \tilde{x}_{k+1}^N)\right| + \left|x_{k+1}^N - \tilde{x}_{k+1}^N\right| \quad k = j+1, \dots, N-1.$$

Therefore, by summing in j, and since $x_N^N = \tilde{x}_N^N$, it holds

$$\begin{aligned}
&\sum_{j=0}^{N-1} \left|x_j^N - \tilde{x}_j^N + (z - jl)\left(y_j^N - \tilde{y}_j^N\right)\right| \le \\
&2\left(\sum_{j=0}^{N-1}\sum_{k=j}^{N-1} \left|x_k^N - \tilde{x}_k^N - (x_{k+1}^N - \tilde{x}_{k+1}^N)\right|\right) + N\left|x_N^N - \tilde{x}_N^N\right| \\
&\le 2N\left(\sum_{j=0}^{N-1} \left|x_j^N - \tilde{x}_j^N - (x_{j+1}^N - \tilde{x}_{j+1}^N)\right|\right) + 0 = 2\sum_{j=0}^{N-1}\left|y_j^N - \tilde{y}_j^N\right|. \quad (85)
\end{aligned}$$

Then, relying on (85) and recalling (43), we find

$$\begin{aligned}
&\int_0^1 |X_{\rho^{E,N}}(z) - X_{\tilde{\rho}^{E,N}}(z)|dz \\
&= \int_0^1 \sum_{j=0}^{N-1} \left|x_j^N - \tilde{x}_j^N + (z - jl)\left(y_j^N - \tilde{y}_j^N\right)\right| \chi_{[jl,(j+1)l)}(z)dz \\
&\le \int_0^1 2\sum_{j=0}^{N-1} \left|y_j^N - \tilde{y}_j^N\right| \chi_{[jl,(j+1)l)}(z)dz = 2\left\|y^{L,N} - \tilde{y}^{L,N}\right\|_{L^1([0,1])}.
\end{aligned}$$

By (39), (41), this proves (84). □

Proposition 5.2. *Assume that the velocity map v satifies (V1). Let $\{x_j^N(t)\}_{j=0}^N$, $\{\tilde{x}_j^N(t)\}_{j=0}^N$ be solutions of (18) that satisfy the condition of uniformly bounded initial support (6). Consider the corresponding Eulerian discrete densities $\rho^{E,N}$, $\tilde{\rho}^{E,N} \in L^\infty([0,+\infty)\times\mathbb{R})$ defined by (35). Then, for any fixed $T > 0$ and for all N, it holds*

$$\begin{aligned}
&\sup_{t\in[0,T]} W_1(\rho^{E,N}(t), \tilde{\rho}^{E,N}(t)) \\
&\le W_1(\rho^{E,N}(0), \tilde{\rho}^{E,N}(0)) + +2LT\sum_{j=0}^{N-1} |x_{j+1}(0) - x_j(0) - (\tilde{x}_{j+1}(0) - \tilde{x}_j(0))|,
\end{aligned} \quad (86)$$

where L is the Lipschitz constant of v.

Proof. **1.** In this step, we show that, for $z \in [1-l, 1)$, it holds

$$\begin{aligned}
&\left|x_{N-1}^N(t) - \tilde{x}_{N-1}^N(t) + (z - 1 + l)\left(\frac{1}{\rho_{N-1}^N(t)} - \frac{1}{\tilde{\rho}_{N-1}^N(t)}\right)\right| \le \\
&\qquad \le \left|x_{N-1}^N(0) - \tilde{x}_{N-1}^N(0) + (z - 1 + l)\left(\frac{1}{\rho_{N-1}^N(0)} - \frac{1}{\tilde{\rho}_{N-1}^N(0)}\right)\right| + \\
&\qquad\quad + L\int_0^t \left|\rho_{N-1}^N(s) - \tilde{\rho}_{N-1}^N(s)\right| ds,
\end{aligned} \quad (87)$$

and, for all $j = 0, \dots, N-2$, and $z \in [jl, (j+1)l)$, it holds

$$\begin{aligned}\left|x_j^N(t)-\tilde{x}_j^N(t)+(z-jl)\left(\frac{1}{\rho_j^N(t)}-\frac{1}{\tilde{\rho}_j^N(t)}\right)\right| \leq \\ \leq \left|x_j^N(0)-\tilde{x}_j^N(0)+(z-jl)\left(\frac{1}{\rho_j^N(0)}-\frac{1}{\tilde{\rho}_j^N(0)}\right)\right| + \\ + L\int_0^t \Big(\left|\rho_j^N(s)-\tilde{\rho}_j^N(s)\right| + \left|\rho_{j+1}^N(s)-\tilde{\rho}_{j+1}^N(s)\right| \Big) ds.\end{aligned} \tag{88}$$

To this end, first notice that (18) ensures

$$x_N^N(t)-\tilde{x}_N^N(t) = x_N^N(0)-\tilde{x}_N^N(0), \tag{89}$$

$$x_j^N(t)-\tilde{x}_j^N(t) = x_j^N(0)-\tilde{x}_j^N(0)+\int_0^t v(\rho_j^N(t))-v(\tilde{\rho}_j^N(t))dt, \qquad j=0,\dots,N-1. \tag{90}$$

Moreover, observe that

$$0 \leq 1-\frac{z-1+l}{l} \leq 1 \qquad \forall\, z\in[1-l,1],$$

and that, recalling (19), we have the identity

$$\begin{aligned}&(x_{N-1}^N(t)-\tilde{x}_{N-1}^N(t))\left(1-\frac{z-1+l}{l}\right)+\frac{z-1+l}{l}(x_N^N(t)-\tilde{x}_N^N(t)) = \\ &= x_{N-1}^N(t)-\tilde{x}_{N-1}^N(t)+(z-1+l)\left(\frac{1}{\rho_{N-1}^N(t)}-\frac{1}{\tilde{\rho}_{N-1}^N(t)}\right).\end{aligned} \tag{91}$$

Then, relying on (89)-(91), and using the Lipschitz continuity of the velocity v, we derive that, for $z\in[1-l,1)$, it holds

$$\begin{aligned}&\left|x_{N-1}^N(t)-\tilde{x}_{N-1}^N(t)+(z-1+l)\left(\frac{1}{\rho_{N-1}^N(t)}-\frac{1}{\tilde{\rho}_{N-1}^N(t)}\right)\right| \\ &= \left|(x_{N-1}^N(t)-\tilde{x}_{N-1}^N(t))\left(1-\frac{z-1+l}{l}\right)+\frac{z-1+l}{l}(x_N^N(t)-\tilde{x}_N^N(t))\right| \\ &= \left|\left(x_{N-1}^N(0)-\tilde{x}_{N-1}^N(0)+\int_0^t (v(\rho_{N-1}^N(s))-v(\tilde{\rho}_{N-1}^N(s)))ds\right)\left(1-\frac{z-1+l}{l}\right)\right. \\ &\quad \left.+\ \frac{z-1+l}{l}\left(x_N^N(0)-\tilde{x}_N^N(0)\right)\right| \\ &\leq \left|(x_{N-1}^N(0)-\tilde{x}_{N-1}^N(0))\left(1-\frac{z-1+l}{l}\right)+\frac{z-1+l}{l}(x_N^N(0)-\tilde{x}_N^N(0))\right| \\ &\quad +\left(1-\frac{z-1+l}{l}\right)\int_0^t |v(\rho_{N-1}^N(s))-v(\tilde{\rho}_{N-1}^N(s))|ds \\ &\leq \left|x_{N-1}^N(0)-\tilde{x}_{N-1}^N(0)+(z-1+l)\left(\frac{1}{\rho_{N-1}^N(0)}-\frac{1}{\tilde{\rho}_{N-1}^N(0)}\right)\right| \\ &\quad + L\int_0^t \left|\rho_{N-1}^N(s)-\tilde{\rho}_{N-1}^N(s)\right| ds,\end{aligned}$$

which proves (87).

Next, observe that, for $j = 0, ..., N-2$, one has

$$1 - \frac{z - jl}{l} \leq 1 \qquad \text{and} \qquad \frac{z - jl}{l} \leq 1 \qquad \forall\, z \in [jl, (j+1)l], \tag{92}$$

and the identity

$$\begin{aligned} &(x_j^N(t) - \tilde{x}_j^N)(t)\left(1 - \frac{z - jl}{l}\right) + \frac{z - jl}{l}(x_{j+1}^N(t) - \tilde{x}_{j+1}^N(t)) \\ &= x_j^N(t) - \tilde{x}_j^N(t) + (z - jl)\left(\frac{1}{\rho_j^N(t)} - \frac{1}{\tilde{\rho}_j^N(t)}\right). \end{aligned} \tag{93}$$

Then, relying on (90), (92), (93), with similar computations as above, we find that, for $j = 0, ..., N-2$, and for $z \in [jl, (j+1)l)$, it holds

$$\begin{aligned} &\left| x_j^N(t) - \tilde{x}_j^N(t) + (z - jl)\left(\frac{1}{\rho_j^N(t)} - \frac{1}{\tilde{\rho}_j^N(t)}\right)\right| \\ &= \left| (x_j^N(0) - \tilde{x}_j^N(0))\left(1 - \frac{z - jl}{l}\right) + \frac{z - jl}{l}(x_{j+1}^N(0) - \tilde{x}_{j+1}^N(0)) + \right. \\ &\left(1 - \frac{z - jl}{l}\right)\int_0^t \Big(v(\rho_j^N(s)) - v(\tilde{\rho}_j^N(s))\Big)ds + \\ &\left. \frac{z - jl}{l}\int_0^t \Big(v(\rho_{j+1}^N(s)) - v(\tilde{\rho}_{j+1}^N(s))\Big)ds \right| \\ &\leq \left| (x_j^N(0) - \tilde{x}_j^N(0))\left(1 - \frac{z - jl}{l}\right) + \frac{z - jl}{l}(x_{j+1}^N(0) - \tilde{x}_{j+1}^N(0))\right| + \\ &\qquad + L\int_0^t \Big(\left|\rho_j^N(s) - \tilde{\rho}_j^N(s)\right| + \left|\rho_{j+1}^N(s) - \tilde{\rho}_{j+1}^N(s)\right|\Big)ds \\ &\leq \left| x_j^N(0) - \tilde{x}_j^N(0) + (z - jl)\left(\frac{1}{\rho_j^N(0)} - \frac{1}{\tilde{\rho}_j^N(0)}\right)\right| + \\ &\qquad + L\int_0^t \Big(\left|\rho_j^N(s) - \tilde{\rho}_j^N(s)\right| + \left|\rho_{j+1}^N(s) - \tilde{\rho}_{j+1}^N(s)\right|\Big)ds, \end{aligned}$$

which proves (88).

2. By definition of the pseudo-inverse given in (43), using the bounds (87), (88) and recalling (17), we get

$$\begin{aligned} &\int_0^1 |X_{\rho^{E,N}(t)}(z) - X_{\tilde{\rho}^{E,N}(t)}(z)|dz \\ &= \int_0^1 \sum_{j=0}^{N-1} \left| x_j^N(t) - \tilde{x}_j^N(t) + (z - jl)\left(\frac{1}{\rho_j^N(t)} - \frac{1}{\tilde{\rho}_j^N(t)}\right)\right| \chi_{[jl,(j+1)l)}(z)dz \\ &\leq \int_0^1 \sum_{j=0}^{N-1} \left| x_j^N(0) - \tilde{x}_j^N(0) + (z - jl)\left(\frac{1}{\rho_j^N(0)} - \frac{1}{\tilde{\rho}_j^N(0)}\right)\right| \chi_{[jl,(j+1)l)}(z)dz \\ &\qquad + 2L\int_0^1 \int_0^t \sum_{j=0}^{N-1} \left|\rho_j^N(s) - \tilde{\rho}_j^N(s)\right| \chi_{[jl,(j+1)l)}(z)dsdz \end{aligned}$$

$$= \int_0^1 \sum_{j=0}^{N-1} \left| x_j^N(0) - \tilde{x}_j^N(0) + (z - jl) \left(\frac{1}{\rho_j^N(0)} - \frac{1}{\tilde{\rho}_j^N(0)} \right) \right| \chi_{[jl,(j+1)l)}(z) dz$$
$$+ \frac{2L}{N} \int_0^t \sum_{j=0}^{N-1} \left| \rho_j^N(s) - \tilde{\rho}_j^N(s) \right| ds.$$

Then, applying Proposition 2.5, we deduce

$$\begin{aligned} &\int_0^1 |X_{\rho^{E,N}(t)}(z) - X_{\tilde{\rho}^{E,N}(t)}(z)| dz \leq \\ &\left\| X_{\rho^{E,N}}(0) - X_{\tilde{\rho}^{E,N}}(0) \right\|_{L^1([0,1])} + \frac{2Lt}{N} \sum_{j=0}^{N-1} \left| y_j^N(0) - \tilde{y}_j^N(0) \right|. \end{aligned} \tag{94}$$

Notice that, by (17), (38), we have

$$\begin{aligned} \left\| y^{L,N}(0) - \tilde{y}^{L,N}(0) \right\|_{L^1([0,1])} &= \sum_{j=0}^{N-1} |y_j^N(0) - \tilde{y}_j^N(0)| \int_0^1 \chi_{[jl,(j+1)l)}(z) dz \\ &= \frac{1}{N} \sum_{j=0}^{N-1} |y_j^N(0) - \tilde{y}_j^N(0)|. \end{aligned} \tag{95}$$

Therefore, from (94)-(95), we obtain

$$\begin{aligned} &\left\| X_{\rho^{E,N}}(t) - X_{\tilde{\rho}^{E,N}}(t) \right\|_{L^1([0,1])} \leq \\ &\left\| X_{\rho^{E,N}}(0) - X_{\tilde{\rho}^{E,N}}(0) \right\|_{L^1([0,1])} + \quad + 2Lt \left\| y^{L,N}(0) - \tilde{y}^{L,N}(0) \right\|_{L^1([0,1])} \end{aligned} \tag{96}$$

for all $t > 0$. By Definition 3.6, we can restate (96) in terms of the Wasserstein distance as

$$W_1(\rho^{E,N}(t), \tilde{\rho}^{E,N}(t)) \leq W_1(\rho^{E,N}(0), \tilde{\rho}^{E,N}(0)) + 2Lt \left\| y^{L,N}(0) - \tilde{y}^{L,N}(0) \right\|_{L^1([0,1])} \tag{97}$$

for all $t > 0$. Taking the supremum in (97) over the time interval $[0, T]$, and recalling (39), we recover the inequality (86). □

Proposition 5.3. *Under the same assumptions of Proposition 5.2, for any $T > 0$, the following holds:*

(i) if there exists $C > 0$ such that $\mathrm{TV}\left(\rho^{E,N}(0); \mathbb{R}\right), \mathrm{TV}\left(\tilde{\rho}^{E,N}(0); \mathbb{R}\right) < C$ for all N, then there exists $C_T > 0$ such that for all N and $t \in [0, T]$ there holds

$$\left\| \rho^{E,N}(t) - \tilde{\rho}^{E,N}(t) \right\|_{L^1(\mathbb{R})}^2 \leq 2\, C_T \sqrt{\sup_{t \in [0,T]} \left\| F_{\rho^{E,N}}(t) - F_{\tilde{\rho}^{E,N}}(t) \right\|_{L^1(\mathbb{R})}};$$

(ii) if the velocity v satisfies (V2), then for any $\delta > 0$, there exists $C_\delta, C_{\delta,T} > 0$ such that for all N it holds

$$\mathrm{TV}\left(\rho^{E,N}(t); \mathbb{R}\right) < C_\delta, \qquad \mathrm{TV}\left(\tilde{\rho}^{E,N}(t); \mathbb{R}\right) < C_\delta \qquad \forall\ t \geq \delta,$$

and for all $t \in [\delta, T]$ it holds

$$\left\| \rho^{E,N}(t) - \tilde{\rho}^{E,N}(t) \right\|_{L^1(\mathbb{R})}^2 \leq 2\, C_{\delta,T} \sqrt{\sup_{\tau \in [0,T]} \left\| F_{\rho^{E,N}}(\tau) - F_{\tilde{\rho}^{E,N}}(\tau) \right\|_{L^1(\mathbb{R})}}.$$

Proof. The proofs can be obtained with precisely the same arguments of Step 1 of the proof of Theorem 1.2, replacing $\rho^{E,M}, F^M$, with $\tilde{\rho}^{E,N}, \tilde{F}^N$, respectively; see (55), (58). □

We are now ready to establish the proof of the second main result of this paper.

Proof of Theorem 1.4. The proof is based on the concatenation of the above propositions. Namely, notice that the estimate (9) is an immediate consequence of Proposition 5.2. Also, observe that, by Definition 3.6, we have

$$\left\| F_{\rho^{E,N}}(t) - F_{\tilde{\rho}^{E,N}}(t) \right\|_{L^1(\mathbb{R})} = W_1(\rho^{E,N}(t), \tilde{\rho}^{E,N}(t)).$$

Applying Proposition 5.1 together with Proposition 5.3, and relying on (10), we derive the uniform limits (11) and

$$\lim_{N\to+\infty} \sup_{t\in[\delta,\,T]} \left\| \rho^{E,N}(t) - \tilde{\rho}^{E,N}(t) \right\|_{L^1(\mathbb{R})} = 0, \qquad \forall\ \delta > 0. \tag{98}$$

Then, we recover the uniform limit (12) from the limit in (98) with $\delta = 1/k$, for some subsequences $\{\rho^{E,N_k}\}_k$, $\{\tilde{\rho}^{E,N_k}\}_k$ constructed by a diagonal procedure. This completes the proof of the theorem. □

Acknowledgments. The first author is supported by the PRIN Project 20204NT8-W4 (PI Stefano Bianchini), and by the PRIN Project PNRR P2022XJ9SX of the European Union – Next Generation EU (PI Roberta Bianchini).

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Received March 2024; 1st revision July 2024; 2nd revision September 2024; early access October 2024.