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Philosophiae Doctoral Course in PHYSICS

# Supergravity and the Swampland

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*To my sister, my mother and my father*
*To my grandparents*
*To my grandaunt Silvia*
*To my estimable teachers*
*To my close friends*

# Contents

# Abstract

As we know, our Universe is governed by few fundamental forces: the electromagnetic interaction, the weak and the strong interactions, the Higgs-mediated interaction and the gravitational interaction. The first four forces are governed by a quantum field theory called the Standard Model (and extensions of it), whereas the gravitational interaction is described by General Relativity, which is a classical (non-quantum) theory. If we are convinced that gravity is a fundamental force of our Universe, then, for instance by consistency with all the other known fundamental interactions, it has to be quantic and the way to quantize it has to be found.

Within such an attempt, the Swampland Program has been recently developed. If compared to the attempts that have been pursued in the last decades, it represents a completely new paradigm for the study of the quantum properties of gravity: instead of trying to infer the phenomenology by taking advantage of a definition of a quantum theory of gravity, the Swampland Program would like to establish some general principles that are independent of the details of the theory at high energies, i.e. in the ultraviolet (UV) regime, in order to constrain the phenomenology itself. More specifically, the aim of the Sewampland Proposal is to identify the properties that a consistent theory of quantum gravity should have, more and more strictly circumscribing the landscape of the effective theories that are completable in a theory of quantum gravity from the swampland of those effective constructions that, despite seeming consistent, are not compatible with the quantum structure of gravity. The distinction between the Landscape and the Swampland is made by conjectural statements that are motivated for example by String Theory (which is by now the only quantum theory of gravity that we have at our disposal and with which we can make calculations and predictions), by black hole physics or by holography.

One of the better posed conjectures of the Swampland Program is the Weak Gravity Conjecture (WGC), which is roughly the statement that gravity always acts as the weakest force. In the present work we are going to discuss the WGC both as a property of a theory in the Landscape, which is interesting to investigate *per se*, and as a tool to address the out-standing and cosmologically relevant problem of realizing de Sitter

vacua in String Theory or, coherently with the spirit of the Swampland Program, in four-dimensional supergravity theories.

Within such a framework, animated by the presence of scalar fields in Nature (as the Higgs field or possibly the inflaton field), we are going to study the intriguing generalization of the WGC to forces mediated by light scalar fields. In particular, analyzing the BPS black hole solutions in extended supergravity theories, we will describe two interesting relations involving first and second derivatives of combinations of the central charges. One relation is a new identity that solely relies on the geometric properties of the scalar manifolds of extended supergravities, and the other relation is a generalization of a scalar weak gravity conjecture recently proposed by E. Palti and uses properties of the underlying black hole solution. We will also provide for the first time an explicit covariant construction of the BPS squared action for such solutions.

After that, we will prove that, while respecting the magnetic WGC, a charged gravitino can not have parametrically small or vanishing Lagrangian mass in de Sitter vacua of extended Supergravity. This allows to place large classes of de Sitter solutions of gauged Supergravity, interestingly including all known stable solutions of the N=2 theory, in the Swampland.

In doing so, we will start attacking the long-standing problem of realizing de Sitter space within String Theory trying to uncover such difficulty already at the level of four-dimensional Supergravity.

Inspired by the conclusions that we have just presented, we are going to deal with the intensively discussed Kachru–Kallosh–Linde–Trivedi (KKLT) model, one of the few proposed constructions of de Sitter vacua in a string theory framework. We will actually challenge this scenario by showing that anti-brane uplifting procedures may suffer from a tachyonic instability towards goldstino condensation.
Having in mind that the embedding of the KKLT-type uplift within Supergravity includes the coupling to a nilpotent superfield, one of the cleanest ways to make its alleged pathologies evident is to possibly bring them out in the low energy 4D N=1 supergravity description. Since (within a stringy setup) anti-branes induce spontaneous supersymmetry breaking and a goldstinio sector consequently appears on their world-volume, we will focus on the Volkov–Akulov (VA) model, which is the minimal supersymmetric theory that describes the low energy dynamics of a goldstino. Confining ourselves to rigid supersymmetry (as a first step), after recasting the VA model in terms of constrained superfields, we will show via the exact renormalization group (ERG) technique combined with a supersymmetric rendition of the local potential approximation the emergence of composite states of the goldstino. We will also provide their effective low energy charac-

terization by means of a Kähler potential and a superpotential. This in turn allows us to reveal an inherent non-perturbative tachyonic instability of the pure VA theory.

Willing to give firmer physical substance to the goldstino condensation phenomenon, we are finally going to discuss the standard component-form 4D Volkov–Akulov action in the presence of N non-linear supersymmetries. This is an interesting ground to explore, because, as our analysis will clarify, a large number N of non-linearly realized supersymmetries corresponds to an actual *large N* limit for which the vacuum structure of the tree-level dual bosonic theory is controlled by the classical behaviour. Within this framework, we will find that the effective scalar potential, written in terms of two composite real scalar fields, exhibits at least two stationary points, one representing the original supersymmetry breaking configuration and the other one corresponding to goldstino condensation, where supersymmetry seems to be restored in the deep IR. This result clearly supports the ERG analysis of the goldstino condensation phenomenon from a different perspective and might indicate a way to study the reasons and the structure (e.g. its end-point) of the instability highlighted there.

# A brief introduction to the general framework

Our Universe is governed by few fundamental forces: the electromagnetic interaction, the weak interaction, the strong interaction, the Higgs-mediated interaction and the gravitational interaction.

The first four forces are described by a quantum field theory called the Standard Model (SM) and extensions of it.
A quantum field theory (QFT) Lagrangian is determined by the invariance group $\mathcal{G}$ of the theory itself; by the spectrum of spin-0, spin-$\frac{1}{2}$ and spin-1 particles composing the matter content of the model with their representations with respect to $\mathcal{G}$ and by the interactions respecting the symmetry $\mathcal{G}$. In particular [1], the Standard Model is based on the symmetry group

$$\mathcal{G} = \mathrm{SU}(3)_{\mathrm{color}} \times \mathrm{SU}(2)_{\mathrm{weak}} \times \mathrm{U}(1)_{\mathrm{hypercharge}},$$

where the factor $\mathrm{SU}(3)_{\mathrm{color}}$ refers to the strong interaction and the sector $\mathrm{SU}(2)_{\mathrm{weak}} \times \mathrm{U}(1)_{\mathrm{hypercharge}}$ takes into account the unified weak and electromagnetic interactions. The matter content of the theory is composed by the Higgs scalar field; by three similarly organized generations of fermionic particles (quarks and leptons), which are

$$q_1 = \begin{pmatrix} u \\ d \end{pmatrix}, \quad l_1 = \begin{pmatrix} e \\ \nu_e \end{pmatrix}; \quad q_2 = \begin{pmatrix} c \\ s \end{pmatrix}, \quad l_2 = \begin{pmatrix} \mu \\ \nu_\mu \end{pmatrix}; \quad q_3 = \begin{pmatrix} t \\ b \end{pmatrix}, \quad l_3 = \begin{pmatrix} \tau \\ \nu_\tau \end{pmatrix},$$

and by the gluons, the $W^\pm$ and the $Z^0$ bosons and the photon. By means of the Higgs mechanism (together with the Yukawa couplings), the Higgs field lets (almost all) these matter particles to get a mass. The SM is also requested to be a renormalizable theory: its Lagrangian density $\mathcal{L}_{\mathrm{SM}}$, whose schematic structure is

$$\mathcal{L}_{\mathrm{SM}} = \sum_i c_i O_i \tag{i}$$

(for the operators $\{O_i\}_i$ with associated coefficients $\{c_i\}_i$), has to be such that the mass dimensions of $O_i$ and $c_i$ (for any $i$) have to satisfy

$$[O_i] \leq 4 \quad \text{and} \quad [c_i] \geq 0. \tag{ii}$$

Once all these ingredients are taken under consideration, the Lagrangian (or the action) of the Standard Model is entirely determined and its study can be pushed forward[1].

The gravitational interaction is described by General Relativity (GR), instead. The crucial idea on which General Relativity is based is that gravity is a manifestation of the spacetime geometry. As a consequence, it does not influence the motion of a body in the same way as all the other interactions do: a body that feels the gravitational interaction moves freely in a deformed background spacetime. This brilliant intuition that A. Einstein was able to make evident substantiates in the so called Equivalence Principle.
The geometric nature of gravity and its relation with the other constituents of our Universe is then expressed by the action

$$S_{\rm GR} = \int d^4x \sqrt{-g} \left[ \frac{M_P^2}{2} R + \mathcal{L}_{\text{rest of the world}} \right] , \tag{iii}$$

where $M_P$ is the (reduced) Planck mas; $g_{\mu\nu}$ is the spacetime metric; $R$ is the corresponding Ricci scalar and $\mathcal{L}_{\text{rest of the world}}$ is the Lagrangian density grouping the contributions from all the components of the Universe but the gravitational field. By the use of the Variational Principle applied to $S_{\rm GR}$ the famous Einstein's equations can be deduced.
General Relativity is amazingly well tested by experiments [2–5]. For instance, Eat–Wash torsion balance experiment is able to test the Weak Equivalence Principle with a precision of $\mathcal{O}(10^{13})$. Some experiments involving an appropriately modified version of Dirac equation (to take into account of possible violations of local Lorentz invariance) and based on measures of nuclear energy levels allow to confirm the validity of the Einstein Equivalence Principle with $\mathcal{O}(10^{29})$ of precision; some measurements of the gravitational red-shift effect performed with gravitational clocks test the Einstein Equivalence Principle with a precision of $\mathcal{O}(10^4)$. By studying the system Earth-Moon in the gravitational field of the Sun the Laser Lunar Ranging Experiment has managed to confirm the Strong Equivalence Principle with a precision level of $\mathcal{O}(10^{13})$.

General Relativity is a classical (in the sense of non-quantum) theory. However, being convinced that gravity has to be a fundamental force governing our Universe, for consistency with the other fundamental interactions, it has to be quantic too and the way to quantize it has to be found.

Nowadays, String Theory is the only consistent framework we can refer to and we can make calculations and formulate predictions with in the attempt of getting some information on the construction of a theory of quantum gravity.
One of the main ideas on which String Theory is based is that the fundamental entities

[1] Once, in light of (i) and (ii), such a study is performed, one bumps into a problem soon: the masses of the neutrinos can not be accounted for. Extensions of the SM beyond (ii) have therefore to be necessarily considered.

composing our Universe are extended (rather than point-like) objects. When supersymmetry (which is basically a symmetry that exchanges bosonic and fermionic degrees of freedom [6]) enters the game, in order to preserve Lorentz invariance, String Theory becomes consistent in ten (or eleven) spacetime dimensions [7]. Since we are sensitive to four dimensions only, there emerge six (or seven) extra dimensions. They are associated to a manifold (denoted as internal manifold) and have to be "small" enough so that they do not apparently affect any experimental result. This is in substance the idea lying behind the compactification procedure. Because of the enormously rich variety of internal manifolds allowed by String Theory to which the extra dimensions can be referred and the influence that the properties of the internal manifold actually exert on the four observable dimensions, an infinite number of potential four-dimensional universes is obtained. Once fluxes are turned on in the compactification, this number reduces but is still incredibly large ($\mathcal{O}(10^{500})$ according to [8, 9])[2].

Animated by the belief that String Theory should be predictive regardless of the previous estimate, the question of how our Universe can be recognized and selected in this "jungle" of potential universes spontaneously arises. Thinking that there is a way by which Nature has made such a choice and being interested in uncovering this mechanism, C. Vafa has introduced the distinction between the so called Landscape and Swampland in String Theory [11].

The string Landscape can be defined as the set of those effective QFTs that admit a high-energy completion in String Theory. Still, because of the richness of choices for the geometry of the internal space, studying the Landscape by means of the compactification technique is hard. One could then be led to the construction of consistent-looking four-dimensional theories and to the deduction of the relevant four-dimensional physics to which they give rise without caring about their possible origin from a compactification procedure. From this perspective String Theory would become useless. However, the majority of seemingly consistent four-dimensional theories can not be deduced as descendants of String Theory. All those effective QFTs that appear consistent but are not completable in String Theory at high energies are said to belong to the string Swampland (see Figure 1). The concepts of string Landscape and Swampland can be extrapolated to Quantum Gravity (QG). In this respect the quantum gravitational Landscape is made by all consistent-looking effective QFTs that descend from Quantum Gravity and the quantum gravitational Swampland is composed by all those seemingly consistent effective QFTs that do not admit a completion in Quantum Gravity. Since the proper characteristics of the quantum theory of gravity relevant for our Universe are not known and

[2]The reader should be aware of the fact that this estimate is under debate, as e.g. [10] points out.

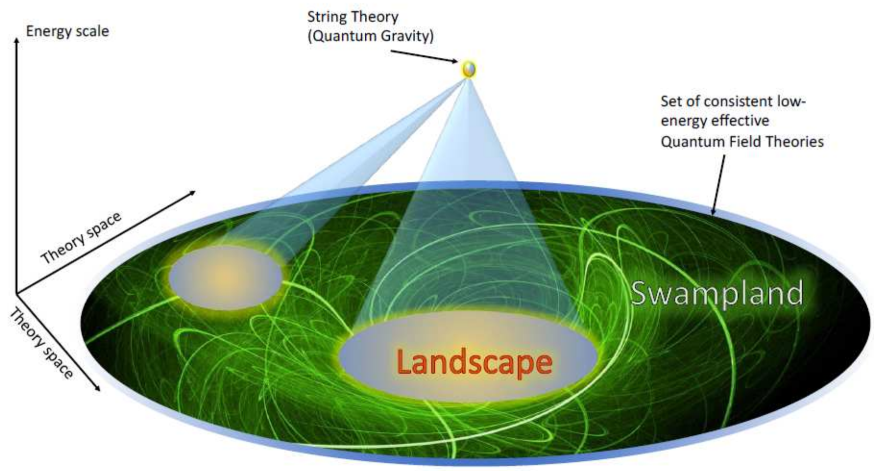


Figure 1 [13]: The figure schematically shows the set of apparently self-consistent effective QFTs. The subset that can arise from String Theory is called the string Landscape; all the other theories are said to belong to the string Swampland.

String Theory is not necessarily such a theory, the QG Landscape and Swampland and the string Landscape and Swampland do not *a priori* coincide[3]. We do not know how to circumscribe the QG Landscape from the QG Swampland or, either said, we do not know what are the (additional) properties that define a theory consistently accounting for the quantization of gravity and are absent (instead) when gravity does not play any role.

The attempt of getting (at least) some line-guide principle to uncover the characteristics of a consistent theory of quantum gravity and the absence of an evident alternative way of proceeding lead to identify, as far as practical purposes only are concerned, the QG Landscape and Swampland with the string Landscape and Swampland. In this framework evidence for some criteria distilling out the Landscape from the Swampland can be gained. These criteria are formulated as conjectural statements and are motivated for instance by examples coming from String Theory (as it could be easily guessed) and by arguments arising from black hole physics or the holographic principle. (For an extensive review see e.g. [13]).

Among the various Swampland conjectures the present work deals in particular with the so called Weak Gravity Conjecture (WGC). It can be phrased as the claim that gravity acts as the weakest force in any circumstance.

In its best known and understood version the WGC can be stated as follows [13, 14]:

[3]Some very recent developments within the Swampland Program, in particular related to the Cobordism Conjecture [12], seem to suggest that String Theory is actually the unique theory of quantum gravity that can exist. This perspective, which is by now posed on a mostly speculative ground, needs to be further investigated.

*Consider a theory, coupled to gravity, with a $U(1)$ gauge symmetry (whose gauge coupling is $g$) in four spacetime dimensions:*

$$S = \int \mathrm{d}^4x \sqrt{-g} \left[ \frac{M_P^2}{2} R - \frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + \dots \right]. \tag{iv}$$

*Electric WGC. There exists a particle in the theory with mass $m$ and charge $q$ satisfying the inequality*

$$m \leq \sqrt{2} g q M_P. \tag{v}$$

*Magnetic WGC. The cut-off scale $\Lambda$ of the effective theory is bounded from above approximately by the gauge coupling as*

$$\Lambda \lesssim g M_P. \tag{vi}$$

The first inequality guarantees that the gravitational interaction between two identical particles of mass $m$ and charge $q$ set at a mutual distance $r$ is beaten in strength by the electromagnetic force that is acting between the two, namely

$$\frac{m^2}{r^2} \lesssim \frac{q^2}{r^2} \tag{vii}$$

(in appropriate units), coherently expressing, as mentioned above, the weakness of gravity with respect (for instance) to the electromagnetic interaction.
Let us try to investigate more deeply the motivation for (v) exploiting a black hole physics argument.
Consider a black hole with mass $M$ and charge $Q$ under a U(1) gauge symmetry (in four dimensions). The black hole ($M$,$Q$) is meant to be the solution of the Einstein's equations expressed by

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + f(r)^{-1}\,\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \tag{viii}$$

with

$$f(r) = 1 - \frac{2M_{\mathrm{BH}}}{r} + \frac{2g^2Q^2}{r^2}, \tag{ix}$$

where $M_{\mathrm{BH}}$ is $M_{\mathrm{BH}} = G_N M$ ($G_N$ being the Newton's constant); $g$ is the gauge coupling constant and coordinates ($t$,$r$,$\theta$,$\phi$) adapted to an observer at infinity are used.
Since $f(r)$ is quadratic, there are two horizons located at

$$r_\pm = M_{\mathrm{BH}} \pm \sqrt{M_{\mathrm{BH}}^2 - 2g^2Q^2}. \tag{x}$$

To make the previous solution a black hole, the extremality bound

$$M_{\mathrm{BH}}^2 \geq 2g^2Q^2 \tag{xi}$$

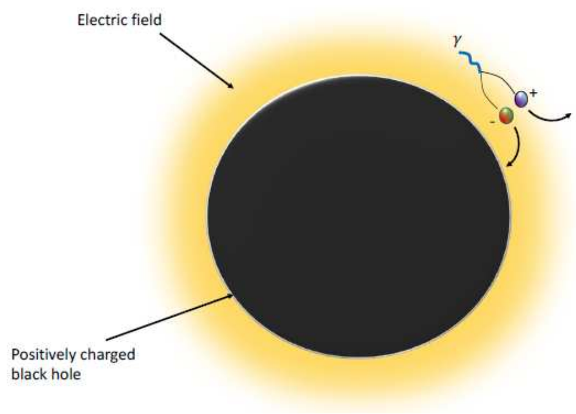


Figure 2 [13]: The figure shows a black hole's evaporation and discharge processes. A pair of charged particle and antiparticle are produced in the electric field outside the black hole; for instance, the antiparticle crosses the black hole's horizon and the particle escapes.

has to be satisfied. When this bound is saturated, the black hole is said to be *extremal*, meaning that it has the minimal mass to admit a horizon, once its charge $Q$ has been fixed. A violation of the extremality bound leads to a naked singularity; but naked singularities are thought not to be there according to the Cosmic Censorship [15]. Let us suppose that the black hole ($M$,$Q$) is extremal; indeed, $M = Q$ in appropriate units (where the reduced Planck mass $M_P$ has been set to 1). This black hole can loose mass thanks to Hawking radiation and discharge through an analogous process made possible by the field around the horizon that the black hole's charge induces [16, 17]. The two main discharging processes are the thermal one, occurring when the black hole's Hawking temperature is greater than the mass of the particles in which the black hole is discharging, and the Schwinger pair production process which is relevant for extremal or near-extremal black holes. While evaporating, the black hole emits particles with mass and charge ($m_i$,$q_i$), as it is illustratively depicted in Figure 2. For the black hole to remain a black hole while evaporating, step by step in the emission process the mass of the black hole should be greater or equal to its charge. Moreover, the decay of the charged black hole is constrained by energy and charge conservation ($M \geq \sum_i m_i$ and $Q = \sum_i q_i$) such that

$$\frac{M}{Q} \geq \frac{\sum_i m_i}{Q} = \frac{1}{Q}\sum_i \frac{m_i}{q_i} q_i \geq \frac{m}{q}\Big|_{\text{min}}. \tag{xii}$$

As a consequence of the relation (xii), we can argue for the existence of at least a particle whose charge-to-mass ratio is greater or equal than that of the black hole. By then exploiting the extremality condition (hence $M = Q$) we constrain further these particles to be such that gravity acts as the weakest force on them (since $m \leq q$).

The weakness of gravity with respect to the other interactions is really the physical principle behind the (electric) WGC.
To further motivate the conjecture as a Swampland criterion, let us try to understand what happens if we set the electromagnetic force to be weaker than the gravitational one for the particle(s) with the largest charge-to-mass ratio in the theory.
Attracting rather than repelling, such two WGC particles would form a bound state. Because of energy and charge conservation the energy of the bound state would be smaller than $2m$ and its charge would be exactly $2q$. Having a charge-to-mass ratio larger than the charge-to-mass ratio of the particle(s) with the largest charge-to-mass ratio in the theory, the bound state just formed could not discharge emitting particles: it would be stable. By adding more and more particles, since they attract each other, it would be possible to produce stable bound states with arbitrary charge. These $(m,q)$ particle bound states can be weakly coupled and are stable due to their charge. Even though the comprehension of what goes wrong with them microscopically is still an open question, it is sensible not to expect the existence of such bound states.
These observations provide sensible evidence for the electric WGC.

This being settled, we would now like to give an argument in favour of the magnetic WGC. To this aim let us come back to black holes and try to distinguish the cases in which they are charged under a global or gauged U(1) symmetry.
While in the presence of a U(1) global symmetry we can in principle create an infinite number of black hole states with an arbitrary (thus not specifiable) global charge and the same finite mass, for a U(1) gauge symmetry the number of states below a given energy scale is finite due to the extremality bound, which implies that any charge increase corresponds to a mass increase for an otherwise naked singularity to be shielded. Once a mass scale $\Lambda$ has been fixed, the number of possible black holes $\mathrm{N_{BH}}$ is [13]

$$\mathrm{N_{BH}} = \frac{\Lambda}{g M_P}. \tag{xiii}$$

This relation gives interesting constraints in the limit $g \to 0$. In fact, at least theoretically, it is possible to measure the black hole's charge thanks to the flux of the gauge field. However, when $g$ is made smaller and smaller, $\mathrm{N_{BH}}$ diverges and it becomes impossible to determine such a charge, because there is no more flux emanating from the black hole. In other words, when the gauge coupling of a gauge symmetry is sent to zero, the circumstance where a global symmetry is in the game is retrieved. If we agree with the argument against global symmetries in QG, i.e. with the No Global Symmetry Conjecture [13], we have to accept the black hole argument against the vanishing gauge coupling limit of a gauge symmetry[4]. This naturally inspires the elaboration of a statement expressing

[4]Let us notice that making this argument quantitative is rather difficult. The infinite amount of time

how Quantum Gravity opposes to the continuous flow (in the coupling space) towards the forbidden global symmetry limit and thus gives a motivation for the magnetic WGC.

Besides the previous supportive arguments to the WGC, it is actually an open question whether the black hole discharging process can be considered a good condition to chart the Swampland and so if charged black holes must be able to decay or not. In this respect, showing that stable charged black holes at a given energy scale (which can be, for instance, the scale that the magnetic WGC fixes) carry an intrinsic inconsistency would amount to a proof of the electric WGC.

When dealing with charged black holes, it is interesting to note that they may have a self-instability and therefore no charged particle is requested in order for them to decay. In other words, it is possible that a charged black hole discharges in smaller charged black holes. For instance, a charged black hole ($M$,$Q$) with horizon area $A$ can bifurcate in two charged black holes ($M_1$,$Q_1$) and ($M_2$,$Q_2$) with horizon areas $A_1$ and $A_2$ (respectively) if

$$M_1 = M_2 = \frac{M}{2}; \quad Q_1 = Q_2 = \frac{Q}{2} \tag{xiv}$$

and (consequently) $A = A_1 + A_2$, saturating the constraints $A \geq A_1 + A_2$, $M \geq M_1 + M_2$ and $Q = Q_1 + Q_2$, and representing (in this particular case) a decay without the emission of gravitational waves. Such a cascade of charged black hole bifurcations can be followed down towards the Planck scale and problems such as those related to black hole entropy bounds arise [13].

Considering Einstein–Maxwell theory and including some other massive structure, the low energy effective theory receives corrections from higher-derivative terms, which come out of the integration on the massive structure itself. In this context, there are examples in which extremal black hole solutions do not saturate any more the inequality (xi) (or (v)). The possible success of the efforts in showing that the higher-derivative terms increase the charge-to-mass ratio of black holes would amount to prove a formulation of the WGC where the state is a black hole and that can not be valid indeed in the regime in which the state is a particle. In some cases, the structure of the higher-derivative terms has been found to be coherent with the idea that the charge-to-mass ratio of extremal black holes is raised above one. This has been recently shown by using arguments of scattering amplitudes' positivity in [18], where also a $S$-matrix proof of (a weak version of) the WGC has been provided. Anyway, further work on this line of research is still needed.

These last observations (and [18] too) also suggest some subtleties of the WGC, which

requested to measure precisely the black hole's charge (the sphere measuring the flux is at infinity) is an obstacle to the desired quantification. For small gauge couplings the uncertainty on the black hole's charge becomes larger and larger and so the Bekenstein–Hawking entropy may be violated.

we would like to briefly present before moving on. One criticism has to do with the fact that we have formulated the WGC using the word *particle* meaning a state whose mass is below the Planck scale. However, the conjecture may be referred also to states which are much heavier than $M_P$; they can be regarded as extended objects such as black holes[5]. Another criticism of the WGC is related to the gauge coupling $g$, which the action (iv) does not fix. The normalization of $g$ can be given by choosing the gauge field normalization to have canonical coupling to matter currents. (We will comment on this ambiguity in Chapter 2). A further subtlety of the WGC deals with the unclear meaning of the cut-off scale $\Lambda$ in (vi). By taking advantage of the intrinsically relational character of the Swampland web of conjectures, an interpretation of $\Lambda$ can be gained through the Swampland Distance Conjecture [13]: there, $\Lambda$ represents the mass scale of an infinite tower of states.

Having discussed the Weak Gravity Conjecture on some extent, which is one of the main ingredients and tools of the present work, let us pick up the threads of our general introduction to describe the other relevant pillar of this thesis.

As we mentioned above, one of the aims of the Swampland Program is to identify the vacuum where our Universe sits among all those configurations that Quantum Gravity (or String Theory) plausibly allows.

At the beginning of $20^{\text{th}}$ century the first steps for the construction of a scientific theory describing the Universe and its properties were accomplished. Since there were no well-structured empirical basis to found these theories upon, some leading principles were adopted.

Having in mind that it is possible to reduce the degrees of freedom of a system by exploiting symmetries, the Cosmological Principle was formulated to model the Universe, its kinematics and its dynamics. It states that

*Any comoving observer observes the Universe around itself at fixed (cosmic) time (in its reference frame) to be isotropic and homogeneous on average.*

An observer is said to be *comoving* if it moves integrally with the source of the geometry of the Universe. Practically, a comoving observer is one that measures the Cosmic Microwave Background (CMB) to be isotropic at per million level (and up to the intrinsic anisotropies)[6]. The *cosmic time* is the proper time of comoving observers. The proper-

[5]This supports a weak version of the WGC: rather than referring to the lightest object in the theory, one could claim that the conjecture is satisfied by those states with the smallest mass-to-charge ratio. More evidence favouring the weak version of the WGC rather than the strong one can be provided [14,18].

[6]Around the end of the '60s and the beginning of the '70s a dipole anisotropy of CMB (whose mean temperature is 2.725 K) was measured: CMB is "hotter" along a direction and "colder" in the opposite

ties of average *isotropy* and *homogeneity* are referred to the mass-energy distribution on great scales, when observing the Universe with small spatial resolution. The hypothesis of isotropy is confirmed (at an appropriate precision level) by experiments revealing the CMB or the abundance of elements such as Helium or measuring the isotropy in the statistic properties in the scattering of galaxies. On the contrary, because of our limited ability in the direct exploration of the Universe, the hypothesis of homogeneity can not be tested experimentally on large scales and has to be assumed. To understand the hypothesis of homogeneity in the part of the Universe to which we could have access, a principle of General Relativity can be used: it claims that isotropy around any (comoving) observer at fixed time is equivalent to homogeneity. The Cosmological Principle is an abstract statement that is not actually realistic, but it is really helpful in writing down the equations governing the dynamics of the Universe itself.

The Universe is composed by a four-dimensional spacetime with a maximally symmetric three-dimensional space. Spatial rotations and translations surviving as invariance properties, the cosmological spacetime symmetry group has six generators. With respect to Minkowski spacetime, because the Universe is expanding and there is a privileged reference frame (that of comoving observers) due to the presence of cosmic matter and energy, time translation invariance and Lorentz boost invariance are lost as symmetries. Coherently with the Cosmological Principle the geometric properties of the Universe are described thanks to the so called Robertson–Walker metric that can be expressed as

$$ds^2 = c^2dt^2 - a(t)^2 \left[ \frac{dr^2}{1-kr^2} + r^2 d\Omega_2^2 \right], \tag{xv}$$

with $d\Omega_2^2 = d\theta^2 + \sin^2\theta\, d\phi^2$, $(t,r,\theta,\phi)$ being the coordinates adapted to a comoving observer. The coordinate $r$ is adimensional and $k$ is an adimensional constant that can take three values: $-1$, $0$ or $+1$. They correspond to the three equivalence classes of (would be) geometries of the Universe: $k = -1$ stands for the infinite set of open and negative curvature spaces; $k = 0$ denotes the case of a spatially flat universe and $k = +1$ groups the infinite class of close and positive curvature spaces. The factor $a(t)$ (which has the dimensions of a length) allows to describe the expansion or the contraction of the Universe and is named scale factor.

After having chosen (xv) as spacetime metric, Einstein's equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G_N}{c^4} T_{\mu\nu} \tag{xvi}$$

one at per mill level. The Earth is not a comoving reference frame with respect to the average mass-energy distribution of the Universe; and even taking into account the motion of the Earth around the Sun, of the Sun with respect to the center of mass of our Galaxy and of the Milky Way with respect to the Local Group of Galaxies, a residual dipole anisotropy persists: it can be interpreted as the result of the Doppler effect due to the velocity of the Local Group relative to an observer moving with CMB. This velocity is estimated to be 600 Km/s.

(where $g_{\mu\nu}$ is the metric, $R_{\mu\nu}$ is the Ricci tensor, $R$ is the Ricci scalar and $T_{\mu\nu}$ is strength energy tensor) result in the so called Friedman's equations

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G_N}{3}\rho - \frac{kc^2}{a^2}, \quad \frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}\left(\rho + 3\frac{P}{c^2}\right), \quad \dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) \tag{xvii}$$

(denoting with the dot the derivative with respect to the cosmic time)[7].
In (xvii) $\rho$ and $P$ are the energy density and the isotropic pressure of the constituents of substance of the Universe. They can be modelled as perfect fluids characterized by the equation of state

$$P = w\rho c^2, \tag{xviii}$$

$w$ being a constant depending on the constituent.

By evaluating the first of the equations in (xvii) ignoring the spatial curvature term, a critical energy density

$$\rho_c(t) = \frac{3H(t)^2}{8\pi G_N}, \tag{xix}$$

where $H(t) = \frac{\dot{a}}{a}$ is the Hubble parameter, can be defined. Together with $\rho_c$ the measurable quantity

$$\Omega(t) = \frac{\rho(t)}{\rho_c(t)} \tag{xx}$$

is introduced: $\Omega(t)$ is the density parameter at cosmic time $t$. The Planck Mission managed to estimate the deviation from 1 of the total density parameter of the Universe "today" (at $t_0$): it is

$$\Omega_{\text{tot}}(t_0) - 1 = -0.001 \pm 0.002. \tag{xxi}$$

When Friedman's equations were written down for the first time, scientists thought that the Universe was made of ordinary matter. Then, after the surprising observation and analysis of the rotational curves of spiral galaxies (at the beginning of the '70s), the existence of another constituent, called dark matter (DM), was proposed (and confirmed later on by solid evidence coming, for example, from the study of nucleosynthesis processes and the formation of clusters of galaxies).
By consistency between theory and experiments, $\Omega_{\text{matter}}(t_0)$ can be fixed to be $\Omega_{\text{matter}}(t_0) \sim 0.05$ [8] and $\Omega_{\text{DM}}(t_0)$ can be set to be $\Omega_{\text{DM}}(t_0) \sim 0.25$. If one accounts for these components of substance only, the trustful experimental result (xxi) can not be reproduced. The inclusion of radiation (CMB) and massive neutrinos which contribute with $\Omega_{\text{radiation}}(t_0) \sim 10^{-5}$ and $\Omega_{\text{neutrinos}}(t_0) \sim 10^{-4}$ (respectively) to the evaluation of the energetic budget

[7] The scale factor $a$ is different from 0 at any time after the Big Bang, if the Big Bang occurred.

[8] In order for the abundance of elements (such as $He^4$, $Li^3$, $H^3$ or $H^2$) in the Universe to be as observations state, the theory of nucleosynthesis imposes that $0.011 < \Omega_{\text{matter}}(t_0)\, h^2 < 0.025$, where $h$ is a constant giving $H(t_0)$ as $H(t_0) = 100\, h$ (km/s)/Mpc.

of the Universe does not solve the problem.
The analysis of the anisotropies of CMB or the study of the spectra of cosmic standard candles as Supernovae of Type IA seem then to suggest the existence of another constituent of the Universe: it is named dark energy (DE).

As with dark matter, we do not know what dark energy really is. A way to interpret dark energy was unwillingly given by Einstein.
At the beginning of the 20$^{\text{th}}$ century the scientific community was debating on the staticity of the Universe: the majority of scientists (and Einstein too) thought that the Universe was static and only a few were convinced that the Universe had to be dynamic.
If the Universe is composed by matter (as it was originally believed), a static universe can not be regarded as a solution of Einstein's equations. This can be easily seen by requiring $P = 0$ (for matter) and $\dot{a} = \ddot{a} = 0$ in (xvii). Having noticed that and afraid of the fact the static universe could not be a solution to his equations, Einstein decided to modify them. He proposed

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = \frac{8\pi G_N}{c^4} T_{\mu\nu}, \tag{xxii}$$

where $\Lambda$ is the so called cosmological constant. Einstein introduced the cosmological constant as a modification of the Universe spacetime geometry. By moving $\Lambda g_{\mu\nu}$ to the right hand side of (xxii), the cosmological constant term can be intended (*a posteriori*) as an ingredient participating to the definition of the content of substance of the Universe. In this respect, the original strength energy tensor $T_{\mu\nu} = \text{diag}(\rho c^2, -P, -P, -P)$ has to be substituted with $\tilde{T}_{\mu\nu} = \text{diag}(\tilde{\rho} c^2, -\tilde{P}, -\tilde{P}, -\tilde{P})$ [9], where

$$\tilde{\rho} = \rho + \frac{\Lambda c^2}{8\pi G_N} \quad \text{and} \quad \tilde{P} = P + \frac{\Lambda c^4}{8\pi G_N}. \tag{xxiii}$$

If one repeats the calculation that has led to (xvii) from Einstein's equations with $T_{\mu\nu}$ for the modified strength energy tensor $\tilde{T}_{\mu\nu}$, the Friedman's equations

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G_N}{3}\tilde{\rho} - \frac{kc^2}{a^2}, \quad \frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}\left(\tilde{\rho} + 3\frac{\tilde{P}}{c^2}\right), \quad \dot{\tilde{\rho}} = -3\frac{\dot{a}}{a}\left(\tilde{\rho} + \frac{\tilde{P}}{c^2}\right) \tag{xxiv}$$

are obtained. They are the analogue of (xvii) but with the replacements $\rho \to \tilde{\rho}$ and $P \to \tilde{P}$. Convinced that a static universe should exist, Einstein required $P = 0$ (for matter) and $\dot{\rho} = \dot{a} = \ddot{a} = 0$ and found the desired static solution corresponding to a closed universe with the cosmological constant given in terms of the scale factor as $\Lambda = \frac{1}{a^2}$. However, Friedman noticed soon that this solution was unstable and Einstein claimed that the introduction of the cosmological constant was the greatest mistake of his life.

[9]This, in the convention $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$.

As already mentioned, even though Einstein's idea of the cosmological constant was wrong, the cosmological constant can be regarded as a constituent of the Universe. More precisely, dark energy can be described as a cosmological constant participating to the energy budget of the Universe today with $\Omega_{\mathrm{DE}}(t_0) \sim 0.70$ (as the study of the CMB anisotropies or the spectra of Supernovae of Type IA or the analysis of how galaxies group together suggest, leaving behind the Hubble tension problem).
More precisely, the recent cosmological observation of the CMB and the experimental data relative to the spectra of Supernovae of Type IA allow to conclude that our Universe is entering a phase of accelerated expansion [19–23][10]. Since an ordinary matter or dark matter distributions give rise to an attractive gravitational field, in order to have

$$\ddot{a} > 0 \tag{xxv}$$

the second Friedman's equation requires an exotic substance, whose isotropic pressure is (sufficiently) negative

$$P < -\frac{1}{3}\rho c^2. \tag{xxvi}$$

Dark energy in the form of the cosmological constant plays this role: in fact, it satisfies

$$P_{\mathrm{DE}} = w_{\mathrm{DE}}\rho_{\mathrm{DE}}c^2 = -\rho_{\mathrm{DE}}c^2 \tag{xxvii}$$

(as it can be deduced from (xxiii)).

In the presence of the cosmological constant only and so imposing $P = \rho = 0$, the relevant Friedman's equations (xxiv) become

$$\frac{\dot{a}^2}{a^2} = \frac{\Lambda c^2}{3} - \frac{kc^2}{a^2} \quad \text{and} \quad \frac{\ddot{a}}{a} = -\frac{\Lambda c^2}{3}. \tag{xxviii}$$

If, for simplicity, the spatial curvature term is ignored, one obtains

$$a(t) = e^{Ht}, \tag{xxix}$$

where

$$H = \left(\frac{\Lambda c^2}{3}\right)^{\frac{1}{2}} = \text{constant}. \tag{xxx}$$

This is the de Sitter solution to Einstein's equations when the cosmological constant dominates and the spatial curvature is negligible.

[10] To have successful nucleosynthesis in the radiation-dominated era and an appropriate ambience for the formation of cosmic structures during the matter-dominated epoch, the present acceleration of the Universe has started during a recent past.

When DE is regarded as vacuum energy[11], a great problem emerges.
As precised above, experimental observations set the DE energy density to be roughly

$$\rho_{\text{DE}}^{\text{exp}} \sim 0.7\rho_0^c \sim 0.7\big(3 \times 10^{-3}\text{eV}\big)^4 \sim 10^{-11}\text{eV}^4, \tag{xxxi}$$

where $\rho_0^c$ is the critical energy density today. As this estimate suggests, $\rho_{\text{DE}}$ is tiny with respect to the typical energy scales of Particle Physics (ignoring neutrino mass scales). Then, by introducing a cut-off scale at $M_P$ for instance (so that quantum gravity effects modifying the behaviour of the theory in the UV can be ignored), the theoretical expectation for the DE energy density is

$$\rho_{\text{DE}}^{\text{th}} \sim M_P^4 \sim \big(10^{19}\text{GeV}\big)^4 = 10^{112}\text{eV}^4. \tag{xxxii}$$

It can be easily seen that $\rho_{\text{DE}}^{\text{th}}$ is more less 123 orders of magnitude greater $\rho_{\text{DE}}^{\text{exp}}$. This huge discrepancy between what theoretical predictions and experimental results suggest is known as the Cosmological Constant Problem [21, 26, 27][12]. It can be understood both as a fine-tuning problem and, on top of that, as a UV-sensitivity issue: even accepting a fine-tuning of the cosmological constant to make it as small as it should be, its value remains incredibly sensitive to changes of the parameters of the UV model. Let us furthermore note that the Cosmological Constant Problem remains (in a milder form) also when exploring the possibility that DE is not due to vacuum energy.

As already observed, String Theory predicts (once fluxes are turned on) $\mathcal{O}(10^{500})$ vacua [8, 9]. In order to face the apparently lack of predictive value by which String Theory seems to be characterized, three approaches are viable.
One possibility is to not caring that there are $\mathcal{O}(10^{500})$ vacua: independently on how we have reached it, we are in a vacuum and we can simply try to describe and understand what happens in its own vicinity. Another approach consists in thinking that there is actually a mechanism that operates a selection among the $\mathcal{O}(10^{500})$ universes to which String Theory gives rise: by studying the Swampland Program we can endeavour to gain comprehension on how such a mechanism works and on how our Universe has been selected. The third possibility founds itself on the observation that not all the stringy $\mathcal{O}(10^{500})$ vacua are compatible with "life" in the form that we know: there are some conditions that have to be satisfied in order for observers to exist and only a few universes

[11] It is worth noting that, (also) because dark energy can not be observed directly, its actual composition is still unknown. Despite of being considered as a cosmological constant, there are other possible DE candidates [24, 25]. They are all characterized by negative pressure and are able to drive the accelerated expansion of the Universe.

[12] When using dimensional regularization instead of the cut-off regularization prescription or even accounting for supersymmetry and its breaking in the real world, such a discrepancy is reduced to be between 50 and 60 orders of magnitude: the Cosmological Constant Problem persists (e.g. see [28]).

respect such constraints. This approach is based on the so called Anthropic Principle according to which, among the “jungle” of possible vacua originating from String Theory, the only ones we should care of are those that we can in principle inhabit [26,27]. It is fair to further specify that the physical attitude inspiring the Anthropic Principle is that, if there is a mechanism to populate all vacua in the landscape of possible vacuum configurations and our universe corresponds to a sufficiently stable vacuum of such a landscape, then, upon evolution, we could inhabit only those configurations that are compatible with “life” in the form that we know. Along these lines, there would be an anthropic selection rather than a dynamical selection of our universe among all the existing vacua in the Landscape.

An attempt to deal with the Cosmological Constant Problem consists in making reference to the Anthropic Principle. The Landscape selected by the Anthropic Principle provides a very large (but discrete) number of vacua where the cosmological constant can take a value that is as small as anthropic arguments tell us that it should be. More precisely, one way to think of the idea behind the anthropic selection solution to the Cosmological Constant Problem is that a scalar field “sitting” on the profile of a scalar potential (induced, for instance, by a compactification of the underlying higher dimensional theory) gives rise to an expanding universe with a certain value of the cosmological constant; because of its fluctuations, the scalar field may be subjected to a phase transition that brings it to a new local minimum configuration (where the value of the potential is less than it was previously): this determines other subuniverses (vacuum bubbles) that are characterized by a different value of the cosmological constant (and so on). By waiting sufficiently long, the majority of the regions of the Universe (at least those with which we are in causal contact) are characterized by the present value of the cosmological constant. In this framework a de Sitter vacuum of String Theory is meant to be a vacuum that is a local minimum of an appropriate scalar potential whose value at the minimum itself is positive. The present acceleration epoch that our Universe is undergoing may be due to a positive cosmological constant.

It seems very difficult to construct de Sitter vacua in String Theory and this may be due to the fact that the starting theory is supersymmetric whereas de Sitter space is not or because de Sitter vacua require the stabilization of all the moduli in the theory but there are no well-understood mechanisms to do so [13]. The attitude in facing these difficulties might be to consider them just as technical problems or as a substantial obstruction to the construction of de Sitter vacua in String Theory. In the spirit of the first attitude, before the technical difficulties would be fully overcome, some (but few) proposals on how de Sitter vacua could be constructed within Sting Theory, as the KKLT construc-

tion [29, 30], have been formulated and are under further investigation. If (instead) the second circumstance realized, de Sitter vacua would fall in the Swampland.
The possibility that String Theory does not admit de Sitter vacua seems to be in contrast with the experimental results that show that the Universe is entering a late-time acceleration phase. However, this is not necessarily the case. As inflation, which was a primordial phase of accelerated expansion through which our Universe has passed, is likely led by a scalar field rolling down a potential [31][13], it is reasonable to think that such a mechanism may allow to describe also the expansion of the Universe "today". This scenario is known as quintessence or dynamical dark energy (DDE) [32, 33]. Then, by adopting the perspective of DDE models to explain the present cosmological acceleration epoch, de Sitter vacua may fall in the Swampland and no contradiction with cosmological observations can arise. Having in mind a DDE scenario and recovering the anthropic selection solution to the Cosmological Constant Problem (which is not lost, if de Sitter vacua are in the Swampland), we can then think that String Theory may allow for a landscape of potentials which have flat enough regions to lead to the accelerated expansion of the Universe and that anthropic arguments can limit the magnitude of the potential in those regions.

The idea that String Theory does not allow for de Sitter vacua has recently gained impetus thanks to a proposal for a constraint that potentials that are in the Landscape have to obey. Animated by examples coming from String Theory, the de Sitter Conjecture (dSC) states that [35]

*The scalar potential of a theory coupled to gravity must satisfy a bound on its derivative with respect to the scalar fields*

$$|\nabla V| \geq \frac{C}{M_P} V, \tag{xxxiii}$$

*where* $|\nabla V|$ *is the norm of the vector of derivatives of* $V$ *with respect to the scalar fields in the theory and* $C$ *is a constant of* $\mathcal{O}(1)$.

Even though the conjecture does not fix the value of the constant $C$, the experimental data concerned with the present acceleration of the Universe pose $C < 0.6$ [36]. However, the de Sitter Conjecture in the form that we have just proposed is incoherent with the Standard Model. As [37] shows, the top of the Higgs potential would violate (xxxiii):

[13]The New Inflation model proposed by A. Guth was characterized by the so called "graceful exit" problem according to which the phase transition to the true vacuum was never complete in a sizeable part of the actual volume of the Universe [34]. To get out of this puzzle A. D. Linde introduces an inflationary model where a scalar field slowly rolls down its potential: this ensures that there is sufficient time available for the phase transition throughout the actual volume of the Universe.

$\frac{|\nabla V|}{V} \sim \frac{10^{-55}}{M_P}$. In order to avoid the possible counter-examples coming from the Standard Model and extensions of it [38] a refinement of (xxxiii) has to be found. The refined de Sitter Conjecture is [35]:

*The scalar potential of a theory coupled to quantum gravity satisfy either*

$$|\nabla V| \geq \frac{C}{M_P} V \tag{xxxiv}$$

*or*

$$\min\left(\nabla_i \nabla_j V\right) \leq -\frac{C'}{M_P^2} V, \tag{xxxv}$$

*where $C$ and $C'$ are positive constants of $\mathcal{O}(1)$ and $\min\left(\nabla_i \nabla_j V\right)$ is the minimum eigenvalue of the Hessian of $V$ (in an orthonormal frame).*

Regardless of the violation of (xxxiii), the top of the Higgs potential satisfies the refined de Sitter Conjecture and (xxxv). In particular: $\frac{\min(\nabla_i \nabla_j V)}{V} \sim -\frac{10^{35}}{M_P^2}$ .
Similarly, for QCD axions and QCD phase transitions there could be violations of the original de Sitter Conjecture [39, 40], but, as before, they are prevented thanks to the refined de Sitter Conjecture.

It is also worth noting that for an axion-like particle, whose potential has as leading contribution $V \sim -\cos\frac{\phi}{f}$ with $\frac{\min(\nabla_i \nabla_j V)}{V} \leq -\frac{1}{f^2}$, the refined de Sitter Conjecture is satisfied whenever $f \leq M_P$. This result is coherent with what the WGC for axions prescribes [13, 41, 42].

Having presented the de Sitter Conjecture and some motivations for it, let us further comment on the cosmological implications of the conjecture itself.
The observation that our Universe is entering a phase of late-time acceleration suggests that the scalar potential of the Universe should have a positive value, $V > 0$. The de Sitter Conjecture implies that it can not be at a minimum (where $|\nabla V| = 0$). So, the Universe is rolling down a potential slowly enough that the potential energy dominates over the kinetic one and the accelerated expansion can take place. The dynamical dark energy scenario in comparison with the cosmological constant setup is illustrated in Figure 3. A prediction of the quintessence models is that the equation of state of DE has to vary with time. If DE is described as a fluid with

$$P_{\mathrm{DE}} = w_{\mathrm{DE}} \rho_{\mathrm{DE}} c^2, \tag{xxxvi}$$

for a scalar field rolling down a potential

$$w_{\mathrm{DE}} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}. \tag{xxxvii}$$

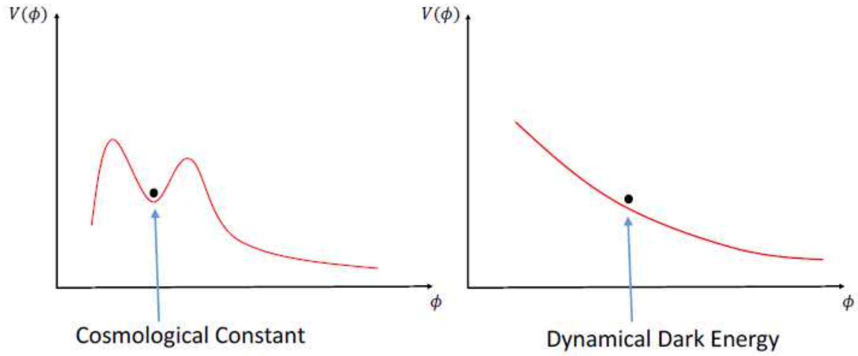


Figure 3 [13]: The figure shows the scalar potential of the Universe along a particular scalar direction denoted by $\phi$. The current state of the Universe is indicated with a black dot. The possibility on the left hand side corresponds to a cosmological constant driving the present accelerated expansion; it violates the de Sitter Conjecture. The potential on the right hand side represents a DDE scenario where the accelerated expansion is driven by a rolling scalar field; it is compatible with the de Sitter Conjecture.

The cosmological constant scenario takes place in the limit $w_{\mathrm{DE}} \to -1$. Current observations of the DE equation of state parameter $w_{\mathrm{DE}}$ bound its possible deviation from a cosmological constant. These bounds constrain the constant $C$ in the de Sitter Conjecture to be $C < 0.6$ [36, 43]. The de Sitter Conjecture interacts also with inflationary models, because the parameter $C$ is strictly related to the slow-roll parameter during inflation. The constraints from inflation (and from the non observation of tensor modes, in particular) set $C < 0.09$ [43]. This is somehow in tension with the conjecture, but this depends on how sharply the condition $C \sim \mathcal{O}(1)$ is interpreted.

The de Sitter Conjecture implies that the Universe "today" must correspond to a scalar field that is rolling (down) to larger and larger expectation values. It is possible that this would lead to an effective negative cosmological constant causing a phase transition in the Universe or to an expectation value of the scalar field that is so large that the light states of the Swampland Distance Conjecture [13] start to affect the Universe with a possible consequent phase transition.
Besides of being connected with the distance criteria, another important aspect of the de Sitter conjecture is that it has the purpose of linking microscopic and quantum aspects of de Sitter space with properties of scalar potentials of effective field theories arising from String Theory. Even though String Theory may not admit de Sitter vacua, this forms a framework where the attempt of constructing de Sitter vacua in String Theory may find a basis of development. Within this line of research the KKLT proposal sets itself.

Before coming into business, because of its relevance for later discussions, let us briefly summarize the logic behind the KKLT construction by following the original paper [29] (up to its subsequent investigations and improved understanding of which the reader should be anyway aware through e.g. [44–47])[14]. As we will see in the subsequent chapters, the question whether the KKLT scenario leads to true de Sitter vacua of String Theory remains nowadays subjective and controversial.

Let us consider F-theory compactified on an elliptic Calabi–Yau fourfold X, which is constructed via a base manifold of the fibration, M, and an elliptic fiber [48]. The base manifold M encodes the type IIB geometry data, while the variation of the complex structure $\tau$ of the elliptic fiber describes the profile of the type IIB axio-dilaton. In such a model, a tadpole condition

$$\frac{\chi(X)}{24} = \mathrm{N_{D3}} + \frac{1}{2\mathrm{k}_{10}^2\mathrm{T}_3}\int_{\mathrm{M}} \mathrm{H}_3 \wedge \mathrm{F}_3 \tag{xxxviii}$$

has to be satisfied. $T_3$ is the tension of a D3-brane, $\mathrm{N_{D3}}$ is the net number of (D3-$\overline{\mathrm{D3}}$) branes and $H_3$ and $F_3$ are the three-form fluxes in the type IIB theory that arise in the Neveu–Schwarz-Neveu–Schwarz and Ramond-Ramond sectors, respectively.
In the presence of fluxes a superpotential for the Calabi–Yau moduli of the form

$$W = \int_M G_3 \wedge \Omega_3 \tag{xxxix}$$

is generated, where $G_3 = F_3 - \tau H_3$ and $\Omega_3$ is the holomorphic three-form [49]. In combination with the Kähler potential

$$K = -3\log[-i(\rho - \overline{\rho})] - \ln[-i(\tau - \overline{\tau})] - \ln\left[-i\int_M \Omega_3 \wedge \overline{\Omega}_3\right] \tag{xl}$$

($\rho$ being the single volume modulus of the present construction), one gets the scalar potential

$$V = e^K\left(g^{a\bar{b}} D_a W \overline{D_b W} - 3W\overline{W}\right), \tag{xli}$$

where the indices $a$ and $b$ run over all moduli fields. However, since $\rho$ is not part of (xxxix), one is left with

$$V = e^K g^{i\bar{j}} D_i W \overline{D_j W}, \tag{xlii}$$

$i$ and $j$ running over all moduli fields except from $\rho$. In general, the complex structure moduli of the F-theory fourfold (i.e. the complex structure moduli, the dilaton and the moduli of D7-branes in the type IIB language) are completely fixed. The authors of [29]

[14]For simplicity, we will set $M_P = 1$.

assume that an appropriate choice of flux quanta allows to fix the complex structure moduli at the typical scale $m \simeq \frac{\alpha'}{R^3}$ (where $R$ is the radius of the manifold) and then focus on the effective theory of the volume modulus $\rho$ [15].
If the no-scale model (xlii) is corrected by modifying the superpotential (xxxix) through an exponential superpotential term for $\rho$ at large volume [50, 51][16]

$$W = \cdots + Ae^{ia\rho}, \tag{xliii}$$

the volume modulus can be stabilized in a supersymmetry preserving AdS vacuum. In order to make this statement explicit the authors consider the tree-level Kähler potential

$$K = -3\ln[-i(\rho - \overline{\rho})] \tag{xliv}$$

and the superpotential

$$W = W_0 + Ae^{ia\rho}, \tag{xlv}$$

imagining the complex structure moduli to be fixed in their VEVs and assuming the tadpole condition to be satisfied by turning on only flux (with no additional D3-branes). A supersymmetric vacuum is identified by solving the condition $D_\rho W = 0$. If one parameterizes $\rho$ as $\rho = \frac{b}{\sqrt{2}} + i\sigma$ and then sets the axion $b$ to 0 for simplicity, the critical configuration

$$W_0 = -Ae^{-a\sigma_*}\left(1 + \frac{2}{3}a\sigma_*\right) \quad \text{with} \quad V_* = -\frac{a^2A^2e^{-2a\sigma_*}}{6\sigma_*} \tag{xlvi}$$

(where $A$, $a$ and $W_0 \in \mathbb{R}$ and $W_0 < 0$) is found, thus stabilizing the volume modulus $\rho$. The supergravity approximation is valid when, retaining possible to arrange for $|W_0| \ll 1$, $\sigma \gg 1$ (for $a\sigma > 1$ and $a > 1$). After having realized such a supersymmetric AdS vacuum configuration the authors of [29] propose to turn on a bit more flux so that the tadpole condition can only be satisfied by introducing an $\overline{\text{D3}}$-brane. The $\overline{\text{D3}}$-brane gives an additional contribution to the scalar potential of the form

$$\delta V = \frac{D}{(\text{Im}\rho)^3}, \tag{xlvii}$$

where $D$ is a tunable parameter, depending (for instance) on the warp factor of the warped compactification geometry underlying the model [52]. The potential

$$V = \frac{aAe^{-a\sigma}}{2\sigma^2}\left(\frac{1}{3}\sigma aAe^{-a\sigma} + W_0 + Ae^{-a\sigma}\right) + \frac{D}{\sigma^3} \tag{xlviii}$$

[15] This is self-consistent if the final mass of $\rho$ is small with respect to $m$.

[16] In principle there are also corrections to the Kähler potential. These could be ignored in the original KKLT proposal *a posteriori* (and so we do here for presentation purposes) because the volume modulus is stabilized at values which are parametrically large compared to the string scale.

interestingly admits a de Sitter minimum for a suitable range of values of $D$: the addition of a $\overline{\text{D3}}$-brane to the stringy setup, or the introduction of $\delta V$ in the scalar potential of the effective theory for the volume modulus, guarantees the uplifting of a supersymmetric AdS minimum to a (metastable) de Sitter critical configuration, which may be cosmologically relevant.
The KKLT model that we have just schematically discussed represents one of the few toy-constructions of de Sitter vacua within String Theory, but, as we already alluded to and we will see, its history and destiny seem quite controversial.

Having presented the general framework where the present thesis work sets itself and the problems and the concepts we would like to deal with, let us proceed us follows.

In Chapter 1, taking as motivation the existence of scalar fields in Nature (such as the Higgs field or possibly the inflaton field), we will discuss the application of the WGC to forces mediated by light scalar fields (e.g. [53–58]). We will analyze the BPS black hole equations in extended supergravity theories and consequently find two interesting relations involving first and second derivatives of combinations of the central charges. One relation is a new identity that solely relies on the geometric properties of the scalar manifolds of extended supergravities, and the other relation is a generalization of a scalar weak gravity conjecture proposed by E. Palti [53] and uses properties of the underlying black hole solution. We will also provide for the first time an explicit covariant construction of the BPS squared action for such solutions [59].

After that, in Chapter 2, we will focus on the WGC in its magnetic formulation and prove that, while it is respected, a charged gravitino can not have parametrically small or vanishing Lagrangian mass in de Sitter vacua of extended (two-derivative) Supergravity [60]. This allows us to place large classes of de Sitter solutions of gauged Supergravity, interestingly including all known stable solutions of the N=2 theory, in the Swampland.

In doing so, we will start attacking the long-standing problem of realizing de Sitter space within String Theory trying, within the spirit of the Swampland Program, to uncover such difficulty already at the level of four-dimensional Supergravity.

Inspired by the conclusions that we got in [60], we will deal with the intensively discussed Kachru–Kallosh–Linde–Trivedi model [29], one of the few proposed constructions of de Sitter vacua in a string theory framework. In Chapter 3 we will actually challenge this scenario by showing that anti-brane uplifting procedures may suffer from a tachyonic instability towards goldstino condensation [61].
Having in mind that the embedding of the KKLT-type uplift within Supergravity includes the coupling to a nilpotent superfield [62, 63], one of the cleanest ways to make its alleged pathologies [64–68] evident is to possibly bring them out in the low energy 4D N=1

supergravity description. Since (within a stringy setup) anti-branes induce spontaneous supersymmetry breaking and a goldstinio sector consequently appears on their world-volume [69, 70], we will focus on the Volkov–Akulov (VA) model, which is the minimal supersymmetric theory that describes the low energy dynamics of a goldstino. Confining ourselves to rigid supersymmetry (as a first step), after recasting the VA model in terms of constrained superfields, we will show via the exact renormalization group (ERG) technique combined with a supersymmetric rendition of the local potential approximation the emergence of composite states of the goldstino. We will also provide their effective low energy characterization by means of a Kähler potential and a superpotential. This in turn allows to reveal an inherent non-perturbative tachyonic instability of the pure VA theory.

Taking inspiration from [71], willing to give firmer physical substance to the goldstino condensation phenomenon, in Chapter 4 we will discuss the standard component-form 4D Volkov–Akulov action in the presence of N non-linear supersymmetries [72, 73]. This is an interesting ground to explore, because, as our analysis will show, a large number N of non-linearly realized supersymmetries corresponds to an actual *large N* limit for which the vacuum structure of the tree-level dual bosonic theory is controlled by the classical behaviour. Within this framework, we will find that the effective scalar potential, written in terms of two composite real scalar fields, exhibits at least two stationary points, one representing the original supersymmetry breaking configuration and the other one corresponding to goldstino condensation, where supersymmetry is restored in the deep IR. This result clearly supports the conclusions of [61] from a different perspective and indicates a path for the study of the reasons and structure (e.g. its end-point) of the instability highlighted there.

# Chapter 1

# The electric WGC and BPS black holes

As we have just discussed in the introductory chapter, the analysis of the necessary conditions for a generic effective theory to be compatible with the existence of an underlying quantum theory of gravity has recently led to the formulation of a number of conjectures that allow to distinguish the good models belonging to the Landscape from those that are in the Swampland [11].

The Weak Gravity Conjecture (WGC) [14], which for a U(1) boson coupled to gravity states that there must always exist a charged particle with mass $m$ and charge $q$ such that $m \leq g\, q\, M_P$, is one of the first conjectural statements that were put to the forefront. As we have already mentioned, there is by now strong evidence that such conjecture is correct [13] and it has been generalized in various directions.

An interesting generalization of the WGC is its application to forces mediated by light scalar fields (e.g. [53–58, 74–78]).
The first formulation of a version of the WGC applied to scalar fields is due to E. Palti [53], who considered particles whose masses $m$ depend on some light scalar $\phi$ by means of tri-linear couplings $\partial_\phi m$. In this case the conjecture states that $(\partial_\phi m)^2 \geq m^2/M_P^2$, so that the force mediated by $\phi$ is stronger than the gravitational force. While this applies only to the WGC scalars whose mass is a function of $\phi$, it still can give constraints on effective theories, which may even be too strong with respect to expectations [53].
This idea has been pushed even further by E. Gonzalo and L. Ibáñez in [75], where a strong version of the scalar WGC has been proposed. The idea is that scalar self-interactions should always be stronger than gravity, for any scalar in the theory. This was summarized by the inequality

$$2(V''')^2 - V''V'''' \geq \frac{(V'')^2}{M_P^2}, \tag{1.i}$$

where the primes denote derivatives of the scalar potential $V$ with respect to the scalar in exam. This conjecture is much stronger than Palti's proposal, because it applies to any scalar, including massive mediators, and results in very strong constraints on effective theory models containing scalars. While equation (1.i) has nice implications and seems compatible with the Swampland Distance Conjecture [79, 80], it mixes ingredients that are clearly long-range with others that are related to short-range interactions (like the quartic couplings). Its derivation from first principles, even in simple situations, is therefore challenging.

A different bound involving cubic and quartic interactions has been suggested in a footnote of [53], where it was noted that, in the context of N=2 supergravity theories, the masses of supersymmetric black holes have to fulfill an interesting relation, which follows from special geometry, which is the geometry underlying the scalar $\sigma$-model.

In N=2 Supergravity the central charge satisfies the algebraic identity [53, 81]

$$g^{i\bar{\jmath}} D_i \overline{D}_{\bar{\jmath}} |Z|^2 = n_V \, |Z|^2 + g^{i\bar{\jmath}} D_i Z \overline{D}_{\bar{\jmath}} \overline{Z}, \tag{1.ii}$$

where $n_V$ is the number of vector multiplets. This identity is rather easily derived from the application of some special geometry identities [82], namely

$$\overline{D}_{\bar{\jmath}} Z = 0, \quad D_i \overline{D}_{\bar{\jmath}} \overline{Z} = g_{i\bar{\jmath}} \, \overline{Z}. \tag{1.iii}$$

Based on the relation (1.ii), in a footnote of [53] there is a proposal for a scalar WGC constraint of the form

$$n \, m^2 + g^{ij} \partial_i m \partial_j m \leq \frac{1}{2} \, g^{ij} D_i \partial_j (m^2), \tag{1.iv}$$

where $n$ is the number of scalar fields coupling to the WGC state. This is also a relation between mass, three-point and four-point couplings of the WGC states to scalar fields, but very different from (1.i).

In this chapter we would like to give a stronger basis to a scalar WGC relation like (1.iv) by analyzing what happens for N>2 theories, where the central charge matrix has N(N − 1)/2 entries and the supersymmetric black hole mass is equal to the largest of its eigenvalues

$$M_{\rm ADM} = |Z_1| > |Z_2| > \ldots > |Z_{\rm N/2}|, \tag{1.v}$$

where $Z_1, \ldots, Z_{\rm N/2}$ are the eigenvalues of the central charge antisymmetric matrix $Z_{AB}$, written in normal form [83].

Before generalizing (1.ii), one should note that, if we want to interpret it as a bound on the black hole mass, we should rewrite it fully in terms of $M_{\rm ADM} = |Z|$. In this case the relation above can be expressed as

$$D_i \overline{D}^i (|Z|^2) = 4 \, \partial_i |Z| \overline{\partial}^i |Z| + n_V |Z|^2. \tag{1.vi}$$

It is interesting to note the factor in front of the first derivative terms: it is going to be crucial in the correct identification of the generalization of such identity.

Along the upcoming chapter we will prove two distinct relations. The first is a purely algebraic identity, valid for any number of supersymmetries and reducing to (1.ii) for N=2:

$$D_a \overline{D}^a \left( Z_{AB} Z^{AB} \right) = D_a Z_{AB} \overline{D}^a Z^{AB} + n \, Z_{AB} Z^{AB}, \tag{1.vii}$$

where

$$n = n_V + \frac{(\mathrm{N}-2)(\mathrm{N}-3)}{2} \tag{1.viii}$$

(where we use flat complex indices for the scalar derivatives). The intriguing aspect is that the number $n$ corresponds precisely to half of the rank of the Hessian matrix of the black hole potential at “fixed scalars”, therefore giving credit to the fact that in the relation between the mass and the three and four-point couplings only *active scalars* should appear, where by active we mean scalars that support the black hole solutions and are not moduli. As briefly mentioned above, this relation is not suitable to be interpreted as a form of scalar WGC because the various derived quantities in (1.vii) can not be identified with the (square of the) ADM mass (1.v). We will therefore analyze more in detail the black hole solutions for N>2 and find that there is also a general differential relation on the ADM mass of such black holes, which uses some insights from the black hole solution. This is going to be the generalization of (1.vi) to an arbitrary number of supersymmetries and coincides with (1.vii) for N=2. This second relation is

$$P^a{}_b D_a \overline{D}^b W^2 = 4 \, D_a W \overline{D}^a W + n \, W^2, \tag{1.ix}$$

where

$$W = \sqrt{\frac{1}{2} Z^{AB} P_A{}^C P_B{}^D Z_{CD}} \tag{1.x}$$

is the superpotential that can be identified with the ADM mass for BPS black holes in extended theories; $P^a{}_b$ is a projector on the space of active complex scalars and $P^A{}_B$ is a projector on the R-symmetry vector space to the bidimensional eigenspace related to the largest central charge value, according to (1.v).
While deriving this last identity, we will also work out a fully covariant formulation of the BPS equations and of the BPS squaring of the reduced action on the black hole solution. Since this has a general value for analyzing BPS black hole solutions in extended supergravities, we will explicitly provide this construction for N=3 and N=4 theories.

We will in the end comment on the physics of (1.ix) and discuss its compatibility and relation to the Swampland Distance Conjecture.

## 1.1 Preliminaries

When considering N>2 supergravity theories one should note and use the fact that the scalar $\sigma$-model is described by a homogeneous manifold $G/H$ of restricted type, because $H$ must contain the R-symmetry group U(N) (SU(8) for N=8). Also, duality invariance in four dimensions imposes that $G \subset \mathrm{Sp}(2\mathrm{n_V}, \mathbb{R})$, where $n_V$ is the total number of vector fields in the theory. These facts allow us to perform a rather general analysis by considering the general structure of homogeneous manifolds and declining the various formulas to specific N when necessary. For the sake of self-consistency of this work, we recall here some preliminary relations already presented in [84–86], whose conventions we mostly follow.

In order to parameterize the scalar manifold we choose a coset representative $L$ in a basis that makes duality relations manifest. We therefore take $L \in \mathrm{USp}(\mathrm{n_V}, \mathrm{n_V})$, i.e. satisfying $L^\dagger \eta L = \eta = \mathrm{diag}\{\mathbb{1}_{n_V}, -\mathbb{1}_{n_V}\}$ and $L^T \Omega L = \Omega$, where $\Omega = \begin{pmatrix} 0 & \mathbb{1}_{n_V} \\ -\mathbb{1}_{n_V} & 0 \end{pmatrix}$. A generic parameterization, which will be useful in the following, is

$$L = \frac{1}{\sqrt{2}} \begin{pmatrix} f + i\,h & f^* + i\,h^* \\ f - i\,h & f^* - i\,h^* \end{pmatrix}, \tag{1.1.1}$$

where

$$f^T h = h^T f, \tag{1.1.2}$$

$$i(f^T h^* - h^T f^*) = -\mathbb{1}. \tag{1.1.3}$$

Maurer–Cartan equations define the generic structure of the coset by producing its vielbeins and connection as

$$\mathcal{W} = L^{-1} dL = \begin{pmatrix} \omega & P^* \\ P & \omega^* \end{pmatrix}, \tag{1.1.4}$$

which leads to the definitions

$$\omega = i(f^\dagger dh - h^\dagger df), \tag{1.1.5}$$

$$P = i(h^T df - f^T dh), \tag{1.1.6}$$

and to the relations

$$d\omega + \omega \wedge \omega = P \wedge P^*, \tag{1.1.7}$$

$$DP = dP + \omega^* \wedge P + P \wedge \omega = 0. \tag{1.1.8}$$

We can make everything explicit by introducing flat indices on the coset manifold $G/H$. Since $H$ = (S)U(N) $\times H'$, we can write flat indices using a multi-index structure, combining U(N) indices $A, B = 1, \ldots, N$ and $H'$ indices $I, J = 1, \ldots, n_h$, where $n_h$ is the dimension of the fundamental representation of $H'$. More in detail, we split the real symplectic vector representation[1] as $V^M = (V^\Lambda, V_\Lambda)$, $\Lambda = 1, \dots, n_V$, and use the transformation properties of $L$ under the right action of $H$ to split the same vector in terms of a twofold complex tensor representation of (S)U(N) and $H'$. This means that the generic coset representative can be split accordingly, so that

$$\begin{aligned} f &= (f^\Lambda{}_{AB}, f^\Lambda{}_I), \\ h &= (h_{\Lambda AB}, h_{\Lambda I}), \end{aligned} \tag{1.1.9}$$

and

$$\begin{aligned} f^* &= (f^{\Lambda AB}, f^{\Lambda I}), \\ h^* &= (h_\Lambda{}^{AB}, h_\Lambda{}^I). \end{aligned} \tag{1.1.10}$$

By using this decomposition we find

$$P_{ABI} = P_{IAB} = i(h_{\Lambda AB} df^\Lambda{}_I - f^\Lambda{}_{AB} dh_{\Lambda I}), \tag{1.1.11}$$

$$P_{IJ} = i(h_{\Lambda I} df^\Lambda{}_J - f^\Lambda{}_I dh_{\Lambda J}), \tag{1.1.12}$$

$$P_{ABCD} = i(h_{\Lambda AB} df^\Lambda{}_{CD} - f^\Lambda{}_{AB} dh_{\Lambda CD}) \tag{1.1.13}$$

and $P^{IAB} = (P_{IAB})^*$, $P^{IJ} = (P_{IJ})^*$ and $P^{ABCD} = (P_{ABCD})^*$. Clearly such 1-forms correspond to vielbeins of $G/H$ in different ways according to the number of supersymmetries N.

For N=3, the scalar manifold is $G/H = \mathrm{SU}(3, n_V)/\left[\mathrm{SU}(3) \times \mathrm{SU}(n_V) \times \mathrm{U}(1)\right]$, which has dimension $3\,n_V$. This means that the flat vielbein indices lie in the $(3, n_V)$ representation of $H$ and hence $P_{ABCD} = P_{IJ} = 0$.
For N=4 the scalar manifold is $G/H = \mathrm{SU}(1,1)/\mathrm{U}(1) \times \mathrm{SO}(6,n_V)/[\mathrm{SO}(6) \times \mathrm{SO}(n_V)]$ and therefore the vielbein splits in two, $P_p$ being the complex vielbein of the first factor and $P_{IAB}$ in the $(6, n_V)$ representation of $\mathrm{SU}(4) \times \mathrm{SO}(n_V)$ the complex vielbein of the second factor. This implies $P_{ABCD} = \epsilon_{ABCD} P_p$, $P_{IJ} = \delta_{IJ} \overline{P}_{\bar{p}}$. Moreover one should note that there is a complex self-duality condition on the vielbeins so that

$$P_{IAB} = \frac{1}{2} \delta_{IJ}\, \epsilon_{ABCD} P^{JCD} = (P^{IAB})^*. \tag{1.1.14}$$

For N=5, 6 and 8 there are no vector multiplets and the scalar manifolds are SU(1,5)/U(5), SO*(12)/U(6) and $E_{7(7)}$/SU(8), respectively of dimension 10, 30 and 70. The vielbeins lie in the **5**, **15** and **35** representations of U(5), U(6) and SU(8) and are therefore always

[1]The real embedding $G \subset \mathrm{Sp}(2\mathrm{n_V}, \mathbb{R})$ is appropriate for the explicit action of the duality group on the vector field strengths, while the complex embedding in $\mathrm{USp}(n_V, n_V)$ is useful to write down the fermion transformation laws.

described by the complex $P_{ABCD}$. However, the vector fields are in the **10**, **15**+**1** and **28** dimensional representations of their respective R-symmetry groups. This means that in the N=6 case there is a vector field that behaves as a matter vector field, being a singlet of the R-symmetry group. We therefore have $P_{IJ} = 0$ and $P_{IAB} = 0$ for N=5,8, while for N=6 we also have $P_{\cdot AB} = \frac{1}{4!}\epsilon_{ABCDEF}P^{CDEF}$, where the $\cdot$ stands for the U(6) singlet. Finally, in the N=8 case we also have a complex self-duality condition of the form

$$P_{ABCD} = \frac{1}{4!}\epsilon_{ABCDEFGH}P^{EFGH}. \tag{1.1.15}$$

From the relation $dL = L\mathcal{W}$ we can now obtain general relations for the covariant derivatives of the coset representatives:

$$\begin{aligned} Df^{\Lambda}{}_{AB} &= f^{\Lambda I}P_{IAB} + \frac{1}{2}f^{\Lambda CD}P_{CDAB}, \\ Df^{\Lambda}{}_{I} &= \frac{1}{2}f^{\Lambda CD}P_{ICD} + f^{\Lambda J}P_{JI}, \end{aligned} \tag{1.1.16}$$

where we also used that $f^* = (f^{\Lambda AB}, f^{\Lambda I})$.

In the following we are interested in relations that involve derivatives of the central charges of N-extended supergravities, for N>2. Central charges are introduced as a symplectic product of a charge vector $Q = (p^{\Lambda}, q_{\Lambda})$ and the section vector $\mathcal{V} = (f^{\Lambda}, h_{\Lambda})$. We therefore see that we have two types of charges

$$Z_{AB} = p^{\Lambda}h_{\Lambda AB} - q_{\Lambda}f^{\Lambda}{}_{AB}, \tag{1.1.17}$$

$$Z_{I} = p^{\Lambda}h_{\Lambda I} - q_{\Lambda}f^{\Lambda}{}_{I}. \tag{1.1.18}$$

The first set $Z_{AB}$ defines the actual central charges associated to the N(N−1)/2 graviphotons in the theory, while $Z_I$ are the matter charges, related to the possible additional vector multiplets (with the exception of the N=6 theory, as mentioned above). It is then straightforward to obtain relations between these charges by taking their covariant derivatives, using (1.1.16), (1.1.17) and (1.1.18):

$$DZ_{AB} = Z^{I}P_{IAB} + \frac{1}{2}Z^{CD}P_{CDAB}, \tag{1.1.19}$$

$$DZ_{I} = Z^{J}P_{JI} + \frac{1}{2}Z^{CD}P_{ICD}. \tag{1.1.20}$$

In order to compute (second) derivatives of the central charges and of the ADM mass, we need the explicit expression of the derivatives we can obtain from (1.1.19) when projecting on the scalar $\sigma$-model vielbeins. The exercise is straightforward and we report here the outcome for the different values of N:

$$\begin{aligned} \mathrm{N}=3: \quad & D = \frac{1}{2}P_{IAB}D^{IAB} + \frac{1}{2}P^{IAB}D_{IAB}, \\ & D^{ICD}Z_{AB} = 2\,\delta^{CD}_{AB}Z^{I}, \qquad D_{ICD}Z_{AB} = 0, \\ & D^{ICD}Z_{J} = \delta^{I}_{J}\,Z^{CD}, \qquad D_{ICD}Z_{J} = 0. \end{aligned} \tag{1.1.21}$$

$$\text{N} = 4: \quad \begin{aligned} & D = \frac{1}{4} P_{IAB} D^{IAB} + \frac{1}{4} P^{IAB} D_{IAB} && + P_p D_p + \overline{P}_{\bar{p}} D_{\bar{p}}, \\ & D_{ICD} Z_{AB} = \epsilon_{ABCD}\, \delta_{IJ}\, Z^J, && D^{ICD} Z_{AB} = 2\, \delta^{CD}_{AB} Z^I, \\ & D_{JAB} Z_I = \frac{1}{2}\, \delta_{IJ}\, \epsilon_{ABCD}\, Z^{CD}, && D^{JAB} Z_I = \delta^J_I Z^{AB}, \\ & D_p Z_{AB} = \frac{1}{2} \epsilon_{ABCD} Z^{CD}, && D_{\bar{p}} Z_{AB} = 0, \\ & D_p Z_I = 0, && D_{\bar{p}} Z_I = \delta_{IJ} Z^J. \end{aligned} \tag{1.1.22}$$

$$\text{N} = 5: \quad \begin{aligned} & D = \frac{1}{4!} P_{ABCD} D^{ABCD} + \frac{1}{4!} P^{ABCD} D_{ABCD}, \\ & D^{ABCD} Z_{EF} = 12\, \delta^{[AB}_{EF} Z^{CD]}, \qquad D_{ABCD} Z_{EF} = 0. \end{aligned} \tag{1.1.23}$$

$$\text{N} = 6: \quad \begin{aligned} & D = \frac{1}{4!} P_{ABCD} D^{ABCD} + \frac{1}{4!} P^{ABCD} D_{ABCD}, \\ & D_{ABCD} Z_{EF} = \epsilon_{ABCDEF} \bar{Z}, \qquad D^{ABCD} Z_{EF} = \frac{4!}{2} \delta^{[AB}_{EF} Z^{CD]}, \\ & D_{ABCD} Z = \frac{1}{2} \epsilon_{ABCDEF} Z^{EF}, \quad D^{ABCD} Z = 0. \end{aligned} \tag{1.1.24}$$

$$\text{N} = 8: \quad \begin{aligned} & D = \frac{1}{2} \frac{1}{4!} P_{ABCD} D^{ABCD} + \frac{1}{2} \frac{1}{4!} P^{ABCD} D_{ABCD}, \\ & D_{ABCD} Z_{EF} = \frac{1}{2}\, \epsilon_{ABCDEFGH} Z^{GH}. \quad D^{ABCD} Z_{EF} = 12\, \delta^{[AB}_{EF} Z^{CD]}. \end{aligned} \tag{1.1.25}$$

## 1.2 The identity

In this section we provide the details of the derivation of the general algebraic identity (1.vii). The formula encompasses the specific forms that we obtained for similar calculations done for different numbers of supersymmetries. We therefore perform our calculations by using the derivative relations on the central charges obtained in the previous section, declined for specific N in (1.1.21)–(1.1.25), and applying them to the square of the central charges $Z_{AB} Z^{AB}$, which is an $H$-invariant tensor.

**N=3 identity.** The computation of the second derivative of the sum of the squares of the central charges can be easily obtained by applying the rules described in (1.1.21) and leads directly to the desired result:

$$\frac{1}{2} D^{ICD} D_{ICD} (Z_{AB} Z^{AB}) = \frac{1}{2} D^{ICD} Z_{AB} D_{ICD} Z^{AB} + n_V\, Z_{AB} Z^{AB}. \tag{1.2.1}$$

**N=4 identity.** In the N=4 case one has to be more careful because there are two factors in the $\sigma$-model and there is a duality constraint between $P_{IAB}$ and $P^{IAB}$. This is also reflected in the numerical factors needed to obtain the correct result:

$$\begin{aligned} &\frac{1}{4}D^{ICD}D_{ICD}\left(Z_{AB}Z^{AB}\right)+D_pD_{\bar{p}}\left(Z_{AB}Z^{AB}\right)= \\ =&\frac{1}{4}D^{ICD}Z_{AB}D_{ICD}Z^{AB}+\frac{1}{4}D_{ICD}Z_{AB}D^{ICD}Z^{AB}+D_pZ_{AB}D_{\bar{p}}Z^{AB}+ \\ &+(1+n_V)Z_{AB}Z^{AB}, \end{aligned} \tag{1.2.2}$$

where we identify $1=(\mathrm{N}-2)(\mathrm{N}-3)/2$.

**N=5 identity.** In this case the identity follows again straightforwardly from the application of (1.1.23):

$$\frac{1}{4!}D_{CDEF}D^{CDEF}\left(Z_{AB}Z^{AB}\right)=\frac{1}{4!}D_{CDEF}Z^{AB}D^{CDEF}Z_{AB}+3\,Z^{AB}Z_{AB}, \tag{1.2.3}$$

where we identify $3=(\mathrm{N}-2)(\mathrm{N}-3)/2$.

**N=6 identity.** While the final relation in the N=6 case has the same structure as the previous ones, the derivation is a bit more delicate, because there is a vector in the gravity multiplet that is a singlet of the R-symmetry group and therefore its central charge behaves as a matter charge. Anyway, by repeatedly using (1.1.24) one obtains

$$\frac{1}{4!}D_{CDEF}D^{CDEF}\left(Z_{AB}Z^{AB}\right)=\frac{1}{4!}D_{CDEF}Z^{AB}D^{CDEF}Z_{AB}+7\,Z^{AB}Z_{AB}, \tag{1.2.4}$$

where we identify $7=1+(\mathrm{N}-2)(\mathrm{N}-3)/2$ and the extra unity corresponds to the vector that acts as a matter multiplet.

**N=8 identity.** The only delicate point is once more the duality relation between the vielbeins. This is the reason for the different coefficient in the formula with respect to the N=6 case. Using (1.1.25) we obtain

$$\begin{aligned} &\frac{1}{2}\frac{1}{4!}D_{CDEF}D^{CDEF}\left(Z_{AB}Z^{AB}\right)= \\ =&\frac{1}{2}\frac{1}{4!}D_{CDEF}Z^{AB}D^{CDEF}Z_{AB}+\frac{1}{2}\frac{1}{4!}D_{CDEF}Z_{AB}D^{CDEF}Z^{AB}+15\,Z^{AB}Z_{AB}, \end{aligned} \tag{1.2.5}$$

where we identify $15=(\mathrm{N}-2)(\mathrm{N}-3)/2$.

**General Form.** Altogether we can summarize all these identities in a single formula, where we use a single-index complex notation for the scalar fields:

$$D_a\overline{D}^a\left(Z_{AB}Z^{AB}\right)=D_aZ_{AB}\overline{D}^aZ^{AB}+n\,Z_{AB}Z^{AB}, \tag{1.2.6}$$

where

$$n = n_V + \frac{(\mathrm{N}-2)(\mathrm{N}-3)}{2}. \tag{1.2.7}$$

As previously noted, the number $n$ corresponds to half of the rank of the Hessian matrix of the black hole potentials at fixed scalars, but our derivation was fully general and did not make use of the black hole solution at any stage. It is indeed an identity that follows by purely algebraic relations imposed by the geometry of the scalar $\sigma$-model.

## 1.3 BPS black holes in N=3 Supergravity

The identity derived in the previous section has general validity and reduces to the N=2 identity noted in [53] to argue that there may be a scalar WGC constraining cubic and quartic interactions. However, for N>2 the combination $Z_{AB}Z^{AB}$ can not be identified directly with the ADM mass and the first-derivative terms do not act on duality-invariant quantities, but directly on the central charges, hence giving expressions that depend on the basis.

For this reason we now analyze in detail the BPS rewriting of the reduced action of N-extended Supergravity and propose a new relation that generalizes (1.vi) for arbitrary N.

The general metric ansatz for an extremal, asymptotically flat black hole solution [87] depends on a unique unknown function:

$$ds^2 = -e^{2U(r)}dt^2 + e^{-2U(r)}dr^2 + r^2 d\Omega_2^2, \tag{1.3.1}$$

where $d\Omega_2^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ is the line-element of a two-sphere and $U$ is the warp factor, which depends only on the radial variable to respect spherical symmetry. The vector and scalar fields also satisfy the same spherical symmetry requirement, with electric and magnetic charges located at $r = 0$. We can therefore reduce the four-dimensional supergravity action to a one-dimensional action depending only on the $r$ variable, denoting derivatives with respect to $r$ by a prime.

In the case of N=3 Supergravity [88], the reduced Lagrangian is

$$\mathcal{L} = \frac{1}{2} P'_{IAB} P'^{IAB} + (U')^2 + e^{2U} V_{\mathrm{BH}}, \tag{1.3.2}$$

where [84]

$$V_{\mathrm{BH}} = \frac{1}{2} Z_{AB}Z^{AB} + Z_I Z^I. \tag{1.3.3}$$

The BPS equations [84] follow from requiring the vanishing of the supersymmetry transformation of the fermions on this background:

$$\varepsilon'_A - \frac{i}{2} e^U \gamma^0 Z_{AB} \varepsilon^B = 0, \tag{1.3.4}$$

$$U'\varepsilon_A - ie^U\gamma^0 Z_{AB}\varepsilon^B = 0, \tag{1.3.5}$$

$$Z_{AB}\varepsilon_C\, \epsilon^{ABC} = 0, \tag{1.3.6}$$

$$P'_{IAB}\varepsilon_C\, \epsilon^{ABC} = 0, \tag{1.3.7}$$

$$P'_{IAB}\varepsilon^B = ie^U\, Z_I\gamma^0\varepsilon_A. \tag{1.3.8}$$

The interpretation of these equations is that the first one fixes the radial dependence of the supersymmetry spinor, the second one gives the flow of the warp factor, the third one projects away one component of the spinor, the fourth one constrains the number of scalars flowing and finally the last one gives the flow equations of the scalar fields. Essentially, we have first a reduction from N=3 to N=2 because of (1.3.6) and then we recover the same type of equations as for the N=2 case, with the addition of a constraint on the active scalars. To see this in detail, we define the normalized vector

$$V_A \equiv \left(2Z_{EF}Z^{EF}\right)^{-1/2}\, \epsilon_{ABC}Z^{BC}, \tag{1.3.9}$$

which is going to give the direction orthogonal to the preserved supersymmetry, according to (1.3.6), and we use it to define the projector to its orthogonal subspace:

$$P^A{}_B = \delta^A_B - V^A V_B. \tag{1.3.10}$$

The correct set of BPS equations follows now as gradient flows activated by the superpotential

$$W = \sqrt{\frac{1}{2}\, Z_{CD}P^C{}_A P^D{}_B Z^{AB}}, \tag{1.3.11}$$

which coincides with the ADM mass of the solution. We emphasize this definition of the superpotential, because it is going to be the expression that will be generalized to arbitrary N.
The first thing to note is that in this special instance (N=3) the superpotential reduces to

$$W = \sqrt{\frac{1}{2}Z_{AB}Z^{AB}}, \tag{1.3.12}$$

because the central charge is automatically orthogonal to the $V$ vector:

$$Z^{AC}V_C \sim \epsilon_{CDE}Z^{AC}Z^{DE} = \epsilon_{CDE}Z^{A[C}Z^{DE]} = \epsilon_{CDE}Z^{[AC}Z^{DE]} = 0. \tag{1.3.13}$$

In order to derive bosonic flow equations, we then have to impose two projectors on the Killing spinors to reduce supersymmetry to N=1 along the solution. One projection follows straightforwardly from (1.3.6), and the other one can be read from the (1.3.5) equation and is needed to relate the action of the $\gamma^0$ matrix on the spinor with the action of the central charge matrix:

$$i\,\gamma^0\varepsilon^A = \frac{1}{W}\,Z^{AB}\varepsilon_B, \tag{1.3.14}$$

$$V^A\varepsilon_A = 0 \quad \Leftrightarrow \quad P_A^B\varepsilon_B = \varepsilon_A. \tag{1.3.15}$$

Consistency of these projection operations is easy to check. For instance,

$$(\gamma^0)^2\varepsilon_A = i\,\frac{Z_{AB}}{W}\gamma^0\varepsilon^B = \frac{1}{W^2}Z^{BC}Z_{AB}\varepsilon_C = \frac{1}{W^2}\left(W\epsilon^{BCD}V_D\,W\,\epsilon_{ABE}V^E\right)\varepsilon_C = -P^C{}_A\varepsilon_C, \tag{1.3.16}$$

which correctly produces $(\gamma^0)^2\varepsilon_A = -\varepsilon_A$ once (1.3.15) is employed. We see that, in addition to the equation fixing the Killing spinor, the 1/3 BPS black hole solution is determined by the following two BPS equations:

$$U' \;=\; -\,e^U W, \tag{1.3.17}$$

$$P'_{IAB} \;=\; -2\,e^U D_{IAB}W, \tag{1.3.18}$$

where the derivative of the superpotential can be obtained by applying (1.1.21):

$$D_{IAB}W = \frac{1}{2W}Z_I Z_{AB}. \tag{1.3.19}$$

The flow equations (1.3.17) and (1.3.18) have been derived previously in [89, 90], where also the correct superpotential (1.3.12) has been identified, though using a different approach.

Note that the explicit expression of $D_{IAB}W$ implies right away that only $2n_V$ scalars flow rather than $3n_V$, because

$$V^A P'_{IAB} \sim V^A D_{IAB}W \sim Z_I V^A Z_{AB} = 0. \tag{1.3.20}$$

Once the flow equations have been fixed we can provide the identification of the superpotential with the ADM mass by the BPS rewriting of the Lagrangian (1.3.2). The first thing to note is that, using (1.1.21), the black hole potential can be rewritten as a squared expression in terms of the superpotential

$$V_{\rm BH} = 4\left(\frac{1}{2}D_{IAB}W D^{IAB}W\right) + W^2, \tag{1.3.21}$$

which mimics what happens in N=2 in terms of the absolute value of the central charge. The action then vanishes on the BPS solutions, up to a boundary term, which is identified with the ADM mass:

$$\mathcal{L} = \left[U' + e^U W\right]^2 + \frac{1}{2}\left(P'_{IAB} + 2\,e^U D_{IAB}W\right)\left(P'^{IAB} + 2\,e^U D^{IAB}W\right) - \left[2\,e^U W\right]'. \tag{1.3.22}$$

### 1.3.1 ADM mass constraint

Once identified $W$ with the ADM mass, we can prove that it satisfies the relation

$$\frac{1}{2}P^C{}_A P^D{}_B \, D_{ICD} D^{IAB} \left(W^2\right) = 4\left(\frac{1}{2}\, D_{IAB} W \, D^{IAB} W\right) + n_V W^2. \tag{1.3.23}$$

As explained in the introductory section to this chapter, it is crucial to project the second derivatives of the superpotential on the set of scalars active on the black hole solution, otherwise additional terms appear on the right hand side of the equation. The reason for this has to do with the fact that even if the only derivatives of the superpotential different from zero are along the directions of the running scalars, the second derivative may contain non-zero contributions from orthogonal directions because of the connection terms. While this projection may seem *ad hoc*, we stress that this is precisely what we should expect if we want to interpret such relation as a scalar WGC constraint. Only the scalar mediating the interaction between the black holes should be taken into account.

The derivation is rather easy once one applies the derivatives correctly and uses their properties:

$$\begin{aligned}
\frac{1}{2}P^C{}_A P^D{}_B \, D_{ICD} D^{IAB} \left(W^2\right) &= \frac{1}{4}\, P^C{}_A P^D{}_B \, D_{ICD} D^{IAB} \left(Z_{EF} Z^{EF}\right) && (1.3.24)\\
&= \frac{1}{4}\, P^C{}_A P^D{}_B \, D_{ICD} \left(Z^{EF} D^{IAB} Z_{EF}\right) && (1.3.25)\\
&= \frac{1}{2}\, P^C{}_A P^D{}_B \, D_{ICD} \left(Z^{AB} Z^I\right) && (1.3.26)\\
&= \frac{1}{2}\, P^C{}_A P^D{}_B \, \left(n_V Z_{CD} Z^{AB} + Z_I Z^I \, 2\, \delta^{AB}_{CD}\right) && (1.3.27)\\
&= n_V W^2 + 2 Z_I Z^I. && (1.3.28)
\end{aligned}$$

From (1.3.24) to (1.3.27) we just use the definition of $W$ and the derivative relations (1.1.21). The last equality uses once more the definition of $W$ and the fact that $P^A{}_A = 2$. Finally we recover (1.3.23) by using (1.3.19).

## 1.4 BPS black holes in N=4 Supergravity

In the case of N=4 Supergravity [91,92], the scalar manifold is factorized and we need to introduce two different types of complex vielbeins, $P_p$ and $P_{IAB}$. They are in one to one correspondence to the first and second factor in

$$\mathcal{M}_{\text{scalar}} = \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(6, n_V)}{\mathrm{SO}(6) \times \mathrm{SO}(n_V)}. \tag{1.4.1}$$

Using the same ansatz for the metric, scalars and vector fields as in the N=2 and N=3 cases, we can write the reduced one-dimensional Lagrangian as

$$\mathcal{L} = \frac{1}{4}\, P'_{IAB}P'^{IAB} + P_p P_{\bar{p}} + (U')^2 + e^{2U} V_{\rm BH}, \tag{1.4.2}$$

where once more [84]

$$V_{\rm BH} = \frac{1}{2}\, Z_{AB}Z^{AB} + Z_I Z^I. \tag{1.4.3}$$

Note that in this case the kinetic term of the vector multiplet scalars has an additional 1/2 factor to take into account the redundancy in the representation with the $P_{IAB}$ vielbeins, which indeed satisfy a complex self-duality constraint.

The BPS equations for such theory are

$$\varepsilon'_A - \frac{i}{2}\, e^U\, \gamma^0 Z_{AB}\varepsilon^B = 0, \tag{1.4.4}$$

$$U'\varepsilon_A - i\, e^U\, \gamma^0 Z_{AB}\varepsilon^B = 0, \tag{1.4.5}$$

$$P'_p\varepsilon^A = -\frac{i}{2}\, e^U\, \epsilon^{ABCD} Z_{BC}\gamma^0\varepsilon_D, \tag{1.4.6}$$

$$P'_{IAB}\varepsilon^B = i\, e^U\, Z_I\gamma^0\varepsilon_A, \tag{1.4.7}$$

and the resulting configurations should preserve 1/4 of the original supersymmetry. As in the previous case we would like to obtain such configurations by means of two projectors, one that reduces supersymmetries by half and projects on the subspace determined by the highest eigenvalue of the central charge and another one that further reduces supersymmetry by half, relating the projections on the SU(4) indices and on the spinor indices. The N=4 central charge can be skew-diagonalized, so that the squared matrix $M^A{}_B = Z^{AC}Z_{BC}$ has two distinct eigenvalues $e_1$ and $e_2$ with multiplicity 2. If we assume that $e_1 > e_2 \geq 0$, the ADM mass of the black hole should be identified with $\sqrt{e_1}$ [84]. We therefore want to construct the BPS flow equations as gradient flow equations deriving from a superpotential that coincides with this eigenvalue. In order to do so, we employ the same technique we employed in the N=3 case and construct a projector $P_-^A{}_B$ that projects along the $e_1$ eigenspace and define the superpotential as in (1.3.11):

$$W = \sqrt{\frac{1}{2}\, P_-^C{}_A P_-^D{}_B Z_{CD}Z^{AB}}. \tag{1.4.8}$$

The projectors can be easily constructed following Schwinger's procedure as

$$P_1^A{}_B = \frac{Z^{AC}Z_{BC} - e_2\,\delta^A_B}{e_1 - e_2}, \quad P_2^A{}_B = \frac{Z^{AC}Z_{BC} - e_1\,\delta^A_B}{e_2 - e_1}. \tag{1.4.9}$$

In order to have a covariant expression in terms of the central charges, we note that we can write the following combinations:

$$e_1 + e_2 \;=\; \frac{1}{2}\, Z^{AB}Z_{AB}, \tag{1.4.10}$$

$$(e_1 - e_2)^4 \quad = \quad \det\mathrm{A}, \tag{1.4.11}$$

where

$$A^A{}_B = 2Z^{AC}Z_{CB} + \frac{1}{2}\,\delta^A_B\, Z_{EF}Z^{EF}. \tag{1.4.12}$$

Hence, after some simple algebra, we see that the projections to the two distinct eigenspaces can be rewritten in terms of

$$P^A_{\pm\,B} = \frac{1}{2}\left(\delta^A_B \pm \Pi^A{}_B\right), \tag{1.4.13}$$

where

$$\Pi^A{}_B = \frac{A^A{}_B}{(\det A)^{1/4}} \tag{1.4.14}$$

and

$$P_- = P_1, \quad P_+ = P_2. \tag{1.4.15}$$

Note that $\Pi^2 = \mathbb{1}_4$, as expected for a projector and therefore we also have the identities

$$A^2 = \sqrt{\det\, A}\,\mathbb{1}_4 = \left[Z^{AB}Z_{BC}Z^{CD}Z_{DA} - \frac{1}{4}\,\left(Z_{EF}Z^{EF}\right)^2\right]\mathbb{1}_4. \tag{1.4.16}$$

It is also interesting to note that in this case the projector on the central charge satisfies

$$P^A_{-\,C}P^B_{-\,D}Z^{CD} = P^A_{-\,C}Z^{CB} = -P^B_{-\,C}Z^{CA}, \tag{1.4.17}$$

as follows from the algebraic identities

$$\Pi^A{}_C\Pi^B{}_D Z^{CD} = Z^{AB}, \tag{1.4.18}$$

$$\Pi^A{}_C Z^{CB} = -\Pi^B{}_C Z^{CA}. \tag{1.4.19}$$

Using this notation, the superpotential can also be expressed as

$$W = \frac{1}{2}\sqrt{Z_{AB}Z^{AB} + 2\,(\det\mathrm{A})^{1/4}}, \tag{1.4.20}$$

which can be better handled to compute its derivatives.

Before dealing with the BPS equations we give here the outcome of the application of the covariant derivatives on the superpotential, which can be computed directly by using (1.1.22) on (1.4.20):

$$D_pW \quad = \quad \frac{1}{4W}\,\mathrm{Pf}\,\bar{Z}, \tag{1.4.21}$$

$$D_{IAB}W \quad = \quad \frac{1}{2W}\,Z_I Z_{AC}P^C_{-\,B} + \frac{1}{4W}\,\delta_{IJ}\,Z^J\,\epsilon_{ABCD}Z^{CF}P^D_{-\,F}, \tag{1.4.22}$$

where we introduced the shorthand notation

$$\mathrm{Pf}\,Z = \frac{1}{4}\,\epsilon^{ABCD}Z_{AB}Z_{CD} \tag{1.4.23}$$

for the Pfaffian of the matrix $Z$.

The BPS flow equations can be obtained from (1.4.4)–(1.4.7) by employing the projectors

$$P_{+}{}^{A}{}_{B}\varepsilon^{B} = 0, \tag{1.4.24}$$

$$i\gamma^{0}\varepsilon^{A} = \frac{1}{W}Z^{AB}\varepsilon_{B}. \tag{1.4.25}$$

The first projector halves the supersymmetries leaving only the spinors in the eigenspace of the maximum eigenvalue of $Z$, while the second further reduces by half the supersymmetries relating different spinor components between them. We can check consistency of the two projections noting that the first implies

$$A^{A}{}_{B}\varepsilon^{B} = -(\det A)^{1/4}\varepsilon^{A} = \left(-2W^{2} + \frac{1}{2}\,Z_{CD}Z^{CD}\right)\varepsilon^{A} \tag{1.4.26}$$

and therefore

$$Z^{AB}Z_{BC}\varepsilon^{C} = -W^{2}\,\varepsilon^{A}, \tag{1.4.27}$$

while

$$(\gamma^{0})^{2}\varepsilon^{A} = -\frac{i}{W}\,Z^{AB}\gamma^{0}\varepsilon_{B} = \frac{1}{W^{2}}Z^{AB}Z_{BC}\varepsilon^{C} = -\varepsilon^{A}, \tag{1.4.28}$$

by using the first projection.
Once we use the projectors in the BPS equations we get

$$U' = -e^{U}\,W, \tag{1.4.29}$$

$$P'_{p} = -2\,e^{U}\,D_{\bar{p}}W, \tag{1.4.30}$$

$$P'_{IAB} = -2\,e^{U}\,D_{IAB}W. \tag{1.4.31}$$

These flow equations (1.4.29)–(1.4.31) have also been discussed in [89, 90], together with the superpotential (1.4.20), though for the case where only the gravity multiplet is present.
Note that out of the $6n_V$ scalars in $P_{IAB}$, only $2n_V$ flow, because

$$P_{-}{}^{C}{}_{A}P_{+}{}^{D}{}_{B}\,D_{ICD}W = 0, \tag{1.4.32}$$

which gives $4n_V$ conditions. This follows from (1.4.22), noting that the first term is fully projected on the $P_-$ subspace and the second is fully projected on the $P_+$ subspace and $P_{-}{}^{A}{}_{B}P_{+}{}^{B}{}_{C} = 0$.

The BPS squaring of the action follows by recognizing that

$$4\left(D_{p}WD_{\bar{p}}W + \frac{1}{4}\,D_{IAB}WD^{IAB}W\right) = \frac{1}{4W^{2}}|\mathrm{Pf}Z|^{2} + Z_{I}Z^{I} \tag{1.4.33}$$

and

$$W^2 + \frac{1}{4W^2}|\mathrm{Pf}Z|^2 = \frac{1}{2}Z_{AB}Z^{AB}, \tag{1.4.34}$$

so that

$$V_{\mathrm{BH}} = 4\left(D_pW D_{\bar{p}}W + \frac{1}{4}\, D_{IAB}W D^{IAB}W\right) + W^2. \tag{1.4.35}$$

Plugging this into the Lagrangian we eventually obtain

$$\begin{aligned}\mathcal{L} = &\ \left(U' + e^U\, W\right)^2 + |P'_p + 2e^U\, D_{\bar{p}}W|^2 + \\ &+\frac{1}{4}\left(P'_{IAB} + 2\, e^U\, D_{IAB}W\right)\left(P'^{IAB} + 2\, e^U\, D^{IAB}W\right) - (2e^U W)',\end{aligned} \tag{1.4.36}$$

so that again we identify $W$ with the ADM mass.

### 1.4.1 ADM mass constraint

The superpotential satisfies an interesting relation, which is the N=4 instance of the general expression (1.vi):

$$\begin{aligned}&D_pD_{\bar{p}}(W^2) + \frac{1}{4}\left(P_{-}^{A}{}_{C}P_{-}^{B}{}_{D} + P_{+}^{A}{}_{C}P_{+}^{B}{}_{D}\right) D_{IAB}D^{ICD}(W^2) = \\ &= 4\left(\frac{1}{4}\, D_{IAB}WD^{IAB}W\right) + (n_V+1)W^2.\end{aligned} \tag{1.4.37}$$

Also in this case it is crucial to project on the subspace of flowing complex scalars, given by the ++ and −− combinations of the projectors.
Before beginning with the actual derivation, we note two identities:

$$\epsilon^{ABCD}\Pi^{[E}_{A}\Pi^{F]}_{B} = \epsilon^{ABEF}\Pi^{[C}_{A}\Pi^{D]}_{B}, \tag{1.4.38}$$

$$\epsilon^{ABCD}P_{+}^{[E}{}_{A}P_{+}^{F]}{}_{B}P_{-}^{G}{}_{D} = \epsilon^{EFBD}P_{-}^{C}{}_{B}P_{-}^{G}{}_{D}. \tag{1.4.39}$$

We then compute

$$\begin{aligned}&\frac{1}{4}\left(P_{-}^{A}{}_{C}P_{-}^{B}{}_{D} + P_{+}^{A}{}_{C}P_{+}^{B}{}_{D}\right) D_{IAB}D^{ICD}(W^2) = \\ &= \frac{1}{4}\left(P_{-}^{A}{}_{C}P_{-}^{B}{}_{D} + P_{+}^{A}{}_{C}P_{+}^{B}{}_{D}\right) D_{IAB}\left(Z^IZ^{CE}P_{-}^{D}{}_{E} + \frac{1}{2}\,\delta^{IJ}\; Z_J\,\epsilon^{CDEF}Z_{EG}P_{-}^{G}{}_{F}\right) = \\ &= \frac{1}{4}\left(P_{-}^{A}{}_{C}P_{-}^{B}{}_{D} + P_{+}^{A}{}_{C}P_{+}^{B}{}_{D}\right)\left(n_VZ_{AB}Z^{CE}P_{-}^{D}{}_{E} + 2\delta^{CE}_{AB}\,Z_IZ^IP_{-}^{D}{}_{E} - \frac{1}{2}Z^IZ^{CE}D_{IAB}\Pi^{D}{}_{E}\right. \\ &+\frac{1}{4}n_V\,\epsilon_{ABPQ}Z^{PQ}\epsilon^{CDEF}Z_{EG}P_{-}^{G}{}_{F} + \frac{1}{2}\,Z_IZ^I\epsilon_{ABEG}\epsilon^{CDEF}P_{-}^{G}{}_{F} + \\ &\left.-\frac{1}{4}\delta^{IJ}\; Z_J\,\epsilon^{CDEF}Z_{EG}D_{IAB}\Pi^{G}{}_{F}\right).\end{aligned} \tag{1.4.40}$$

Using projector identities like $P_-^2 = P_-$, $P_+P_- = 0$, $\epsilon^{ABCD}P_-^{[E}{}_A P_-^{F]}{}_B P_-^G{}_D = 0$ and $\Pi^A{}_B D_{IEF}\Pi^B{}_C = 0$, we see that

$$\begin{aligned}
&\frac{1}{4}\left(P_-^A{}_C P_-^B{}_D + P_+^A{}_C P_+^B{}_D\right) D_{IAB}D^{ICD}(W^2) = \\
&= n_V W^2 + Z_I Z^I - \frac{1}{4} Z_I Z^I \left(D_{IAB}\Pi^B{}_C - \Pi^E{}_A\Pi^B{}_C D_{IEF}\Pi^F{}_B\right) Z^{AC} + \\
&-\frac{1}{8}\delta^{IJ} Z_J\, \epsilon^{ABCD} Z_{CG}\left(D_{IAB}\Pi^G{}_D + \Pi^E{}_A\Pi^F{}_B\, D_{IEF}\Pi^G{}_D\right).
\end{aligned} \tag{1.4.41}$$

Now, recalling that $Z^{AC} = Z^{EF}\Pi^A{}_E\Pi^C{}_F$, the third term vanishes, and we can see that also the last one vanishes upon using the identities above:

$$\begin{aligned}
&-\frac{1}{8}\,\delta^{IJ}\, Z_J\, \epsilon^{ABCD} Z_{CG}\left(D_{IAB}\Pi^G{}_D + \Pi^E{}_A\Pi^F{}_B\, D_{IEF}\Pi^G{}_D\right) = \\
&= -\frac{1}{4} Z_J Z_{CG} D^{JCD}\Pi^G{}_D - \frac{1}{8}\,\delta^{IJ} Z_J\, \epsilon^{ABCD} Z_{CG}\Pi^E{}_A\Pi^F{}_B D_{IEF}\Pi^G{}_D = \\
&= -\frac{1}{4}\, Z_J\left(Z_{AG} - Z_{EF}\Pi^E{}_A\Pi^F{}_G\right) D^{JAB}\Pi^G{}_B = 0.
\end{aligned} \tag{1.4.42}$$

Finally, we use (1.4.35) in (1.4.41) to recover (1.4.37).

## 1.5 Comments

In the previous sections of this chapter we have built evidence that for asymptotically flat BPS black holes in four dimensions we have a differential constraint on their ADM mass of the form

$$P^a{}_b D_a \overline{D}^b (M^2) = 4 D_a M \overline{D}^a M + n\, M^2, \tag{1.5.1}$$

where derivatives are taken only with respect to the running complex scalars. Starting from this result, we can now use the WGC to obtain a general constraint on the scalar-dependent masses of the various fields. For a generic charged black hole in the presence of scalar fields we have that

$$M^2 + \Sigma^2 - Q_\infty^2 \geq 0, \tag{1.5.2}$$

where $M$ is the ADM mass of the black hole, $\Sigma$ represents the scalar charges and $Q_\infty$ are the U(1) charges at infinity. Our relation can also be written as

$$D^2 M^2 = n\, M^2 + \Sigma^2 = (n-1)M^2 + (M^2 + \Sigma^2), \tag{1.5.3}$$

where $\Sigma = 2DW = 2DM$. Using the black hole relation (1.5.2) we therefore find

$$M^2 + \Sigma^2 - Q_\infty^2 = n\, M^2 + \Sigma^2 - D^2 M^2 \geq 0, \tag{1.5.4}$$

which implies that the particle needed to discharge the black hole should satisfy the opposite inequality

$$D^2 m^2(\phi) \geq n\, m(\phi)^2 + 4(Dm(\phi))^2. \tag{1.5.5}$$

This is a rather strong constraint on the possible moduli dependence of the masses of particles in effective theories.

While we would like to take such a relation and use it as a novel scalar WGC, we should first inspect it more closely to better understand its requirements and limits. First of all we would like to point out that it is difficult to extract a simple universal behaviour of the masses as a function of the scalar fields. Take for instance conjugate BPS configurations in the N=2 STU model

$$K = -\log\left[i(s-\bar{s})(t-\bar{t})(u-\bar{u})\right], \tag{1.5.6}$$

$$Z_1 = e^{K/2}\left(p^0 stu - q_1 s - q_2 t - q_3 u\right), \tag{1.5.7}$$

$$Z_2 = e^{K/2}\left(-q^0 stu + p_1 tu + p_2 su + p_3 st\right). \tag{1.5.8}$$

Using a real parameterization

$$s = \frac{\sigma}{M} + i\, e^{-\sqrt{2}\phi_s/M}, \quad t = \frac{\tau}{M} + i\, e^{-\sqrt{2}\phi_t/M}, \quad u = \frac{\nu}{M} + i\, e^{-\sqrt{2}\phi_u/M}, \tag{1.5.9}$$

we see that only the $\phi_{s,t,u}$ scalars flow along the black hole solution and the ADM mass $M_{\mathrm{ADM}} = |Z|$ has a very simple and yet different dependence on them, namely

$$\begin{aligned} M_{\mathrm{ADM}} \sim \; & -p^0 e^{-(\phi_s+\phi_t+\phi_u)/(\sqrt{2}M)} + q_1 e^{(-\phi_s+\phi_t+\phi_u)/(\sqrt{2}M)} + \\ & + q_2 e^{(\phi_s-\phi_t+\phi_u)/(\sqrt{2}M)} + q_3 e^{(\phi_s+\phi_t-\phi_u)/(\sqrt{2}M)} \end{aligned} \tag{1.5.10}$$

for the $Z_1$ charge, and

$$\begin{aligned} M_{\mathrm{ADM}} \sim \; & -q_0 e^{(\phi_s+\phi_t+\phi_u)/(\sqrt{2}M)} + p^1 e^{(\phi_s-\phi_t-\phi_u)/(\sqrt{2}M)} + \\ & + p^2 e^{(-\phi_s+\phi_t-\phi_u)/(\sqrt{2}M)} + p^3 e^{(-\phi_s-\phi_t+\phi_u)/(\sqrt{2}M)} \end{aligned} \tag{1.5.11}$$

for the $Z_2$ charge. We can interpret the resulting behaviour as the outcome of the sum over different states, whose masses either exponentially vanish or blow-up in the $\phi_{s,t,u} \to \pm\infty$ limit towards the boundary of the moduli space. This is the expected behaviour to be compatible with the Swampland Distance Conjecture and also with the microscopic interpretation of the black hole charges with D-branes wrapping cycles of the internal manifold (in this case D0, D2, D4 and D6-branes on 0, 2, 4 and 6-cycles of $\mathbb{T}^6/(\mathbb{Z}_2\times\mathbb{Z}_2)$). Another aspect that emerges from this analysis is that it is crucial in the relation to have a second covariant derivative spanning over all active *complex* scalars. In the N=2 example that we have just presented, $\sigma = 0 = \tau = \nu$ along the whole solution [93], but the identity is fulfilled only if the terms $g^{\sigma\sigma}\partial_\sigma^2|Z|$ and $g^{\sigma\sigma}\gamma^{\phi_s}_{\sigma\sigma}\partial_{\phi_s}|Z|$ are taken into account (and their analogous terms for the $t$ and $u$ fields). Without considering these terms one would not obtain a differential equation on $M_{\mathrm{ADM}}$ resulting in the expected behaviour in

$\phi_{s,t,u}$. This clearly hampers the possibility of a straightforward generalization to theories where the moduli fields do not come in complex form.
The last point that is quite peculiar of this relation is that its validity rests on the *sum* over all complex scalars contributing to the BPS configuration. This means that we are not able at this stage to extract a strong form of the inequality, to be valid for *each* scalar, like the one proposed in [75].

While the formula that we proposed for the differential relation on the ADM mass of a BPS black hole has been written in a form that is independent of the number of supersymmetries, we should stress that we completed the proof only for N=2, N=3 and N=4. We do not foresee obstacles to a further extension to N=5, N=6 or N=8, and in fact in [89] one can find the identification of the superpotential with the appropriate eigenvalue of the central charge matrix. However, computations become technically much more involved because the projectors needed have a rather complicated expression in terms of traces and determinants of combinations of the central charges. This is anyway an obvious possible extension of the results reported here.
Another possible extension is the analysis of the extremal non-BPS configurations in extended Supegravity, along the lines of what done in [94] for the N=2 case. The N=2 case has been already been discussed in [53], but for N>2 one can imagine that different possible superpotentials appear, according to the branch of non-BPS extremal solutions (see [95] for an overview of the possibilities). Some instances of such superpotentials have been discussed in [90] and it would be interesting to see if they all satisfy the same differential constraint.

# Chapter 2

# The magnetic WGC and dS space in Supergravity

In Chapter 1 we have focused on the Weak Gravity Conjecture in its electric formulation and on its possible improvements to forces mediated by light scalar fields, thus discussing or deriving some restrictions on the properties of scalar potentials when gravity is required to remain the weakest force [53, 59, 75]. In this chapter we will deal instead with the magnetic version of the WGC, which posits that there is a prematurely low ultraviolet (UV) cut-off that depends on the gauge coupling. Inspired by [96–98], we would like to apply the magnetic WGC to Supergravity to question the realization of de Sitter space within String Theory [10], or in Quantum Gravity more broadly [99]. This may furthermore allow us to make contact with the de Sitter Conjecture and other conjectural statements that explicitly forbid stable [100–103] or long-lived [104–106] de Sitter solutions in a UV-completable EFT.

More precisely, we will further build on the results of [98] using N=2 matter-coupled gauged Supergravity as our main framework [107, 108]. We will present for the first time a general proof that parametrically light or massless charged gravitini[1] at a de Sitter critical point result in a violation of the magnetic WGC in N=2 Supergravity. We will illustrate this result with several N=2 models that have de Sitter critical points, both stable and unstable, or even have flat directions corresponding to moduli. We will thus be able to exclude a majority of these de Sitter solutions, including some that fail to be excluded even by the de Sitter Swampland criteria.

[1] We stress that the mass is not a good quantum number to describe physical states in de Sitter and Anti-de Sitter spacetimes, because $p^2$ is not a good Casimir of the corresponding symmetry algebras. Moreover, gravitini are never truly massless in de Sitter, even when they have vanishing Lagrangian mass, in the sense that they always propagate four degrees of freedom instead of two. When we will refer to “massless gravitini”, we will mean gravitini with vanishing Lagrangian mass, whereas by “light gravitini” we will mean a vanishing or parametrically small Lagrangian mass compared to the Hubble scale. We will omit the word “Lagrangian” to avoid clutter and we hope that the reader will not be confused by this omission.

We will also discuss examples that evade exclusion by the WGC by having either uncharged gravitini or by breaking all gauge symmetry at the critical point.

After examining the N=2 models, we will provide a parallel proof that de Sitter critical points with light charged gravitini are similarly excluded in N=8 Supergravity. Our findings strongly indicate that an analogous result for other extended supergravities with 8>N>2 should hold.

As a cross-validation of our findings, we will see that our results are strongly consonant with the "festina lente" (FL) bound [109–111], which places a lower bound on the masses of all charged particles in a de Sitter background. Conversely, our findings can be interpreted as a highly non-trivial check on the consistency of the FL bound, by simply applying it on the N=2 gravitini and demanding gravity to be the weakest force.
As an aside, let us note that complementary arguments that deal with the lowering of the EFT cut-off in the limit of light gravitini, including considerations based on the WGC, appear in [112] and are further established in [113][2].

## 2.1 General considerations

In this section we will present our main arguments for why (quasi) de Sitter backgrounds with charged light gravitini belong to the Swampland. First we will recall how the magnetic Weak Gravity Conjecture for a U(1) places a restriction on the energy density of a theory and then we will further argue for an extension of this restriction to the case of non-Abelian gauge symmetry. After that we will present a general proof that de Sitter critical points in N=2 gauged Supergravity with charged massless gravitini belong to the Swampland and we will show that it also applies to the case of parametrically light masses. We will close the section by relating our results to other Swampland conjectures.

### 2.1.1 Magnetic WGC and de Sitter

In order to review the implications that the Weak Gravity Conjecture can have on de Sitter backgrounds let us first recall that the magnetic WGC postulates that for any U(1) gauge symmetry there is a quantum gravity-induced UV cut-off [118]. The value of that cut-off $\Lambda_{\text{UV}}$ for an EFT is bounded from above by the formula

$$\Lambda_{\text{UV}} < g_{\text{U(1)}}\, q_{\text{el.}}\, M_P \ , \quad \text{for every charged object}\,, \tag{2.1.1}$$

[2]For a different line of investigation concerning massless gravitini based on post-inflationary cosmological considerations, see [114–117].

where $g_{\mathrm{U}(1)}$ is the U(1) gauge coupling and $q_{\mathrm{el.}}$ is the charge with respect to that U(1). From now on we will call $g_{\mathrm{U}(1)}q_{\mathrm{el.}}$ the physical coupling of an object

$$q_{\mathrm{phys.}} = g_{\mathrm{U}(1)}q_{\mathrm{el.}}. \tag{2.1.2}$$

Clearly, the object with the lowest physical coupling sets the strongest restriction on the allowed UV cut-off, and for uncharged objects (2.1.1) does not apply. The way the UV cut-off manifests in the EFT is not known *a priori* unless one also knows the UV completion of the theory. For example, it can be due to towers of massive states that have a mass controlled by $\Lambda_{\mathrm{UV}}$. For us here this UV cut-off will be simply used as a device to signal when higher order corrections to the effective theory become important. If one wants to safely ignore such corrections, then one should work at energies parametrically lower than $\Lambda_{\mathrm{UV}}$.

On a de Sitter background there is one simple condition that should be satisfied such that higher order gravitational corrections do not immediately threaten the EFT. For a background with Hubble constant $H$ this condition is

$$H \ll \Lambda_{\mathrm{UV}}\,. \tag{2.1.3}$$

An extended discussion justifying this condition can be found in [98]. If (2.1.3) does not hold, then the two-derivative gravitational theory may be subject to strong quantum corrections and, as a result, it is not a trustworthy EFT.

### Warm-up: Gauged R-symmetry in N=1

A simple illustration of the restrictions placed by the magnetic WGC on supergravity theories is the following observation, already presented in [119]. If we consider the Freedman model [120], then the Lagrangian contains only gravitation with a positive cosmological constant, a U(1) gauge field ($v_\mu$) that gauges the R-symmetry, and a massless, but charged, gravitino ($\psi_\mu$). In the unitary gauge this is

$$\begin{aligned} e^{-1}\mathcal{L} = &-\frac{1}{2}R + \frac{1}{2}\epsilon^{\kappa\lambda\mu\nu}\left(\overline{\psi}_\kappa\overline{\sigma}_\lambda D_\mu\psi_\nu - \psi_\kappa\sigma_\lambda D_\mu\overline{\psi}_\nu\right) + \\ &- \frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + iq\,\epsilon^{\kappa\lambda\mu\nu}\overline{\psi}_\kappa\overline{\sigma}_\lambda\psi_\mu v_\nu - 4g^2q^2, \end{aligned} \tag{2.1.4}$$

where $D_\mu$ here is the spacetime Lorentz covariant derivative, which includes the gravitino-dependent spin-connection. This de Sitter space is characterized by a Hubble scale that is of the same order as the gravitino charge multiplied by the gauge coupling. As a result, one can argue that the Hubble scale of such a simple model already hits the magnetic WGC cut-off and thus is faced with a Dine–Seiberg problem [121]. Interestingly, one could place this model in the Swampland equally well just by applying the FL conjecture [111].

### 2.1.2 WGC and non-Abelian gauge groups

The magnetic WGC is formulated for U(1) gauge symmetries, and a natural question is whether a similar expression for the UV cut-off exists involving the gauge couplings of non-Abelian groups[3]. In theories with charged scalar fields one should expect this to be the case, because the gauge group itself can be broken or enhanced depending on the expectation values of the fields. This is especially true in gauged extended supergravities, where the gauging of a non-Abelian group forces the theory to contain the would-be Goldstones of the non-Abelian gauge symmetries.

The simplest case to consider would be a theory where a certain vacuum preserves a non-Abelian gauge symmetry but also has a charged modulus such that giving it an expectation value breaks the non-Abelian symmetry to a U(1). For any value of this modulus we have a vacuum of the theory where one can clearly apply the WGC using the gauge coupling of this U(1). We can then take the limit as the expectation value approaches the original "central" vacuum where the full non-Abelian symmetry is restored. In this limit the U(1) gauge coupling will approach the non-Abelian coupling of the central vacuum.

In this sequence of U(1) gauge theories we can determine the WGC cut-off in the usual manner and then, by continuity, we conclude that the cut-off of the vacuum with non-Abelian symmetry must be the limit of the cut-offs of the broken phase. Since this cut-off is determined from the U(1) gauge coupling, which in turn approaches the non-Abelian coupling, we conclude that the non-Abelian coupling can also be used to determine the UV cut-off for the original non-Abelian theory.

If we interpret the WGC cut-off as, for example, coming from the mass scale of some UV states that are not captured by the effective theory, these masses would not change drastically between nearby points in moduli space. We expect this to be the case, regardless of any gauge symmetry enhancement or breaking that may also be taking place. Thus, if the WGC is a good criterion for determining the cut-off of a U(1) gauge theory, it should work when we approach a point where this symmetry is enhanced. The same holds for any other UV origin of the cut-off.

A more subtle situation happens when the direction in field space that breaks the non-Abelian symmetry but preserves a U(1) is not a modulus. In this case, although we can still consider a sequence of points in moduli space that approach the central vacuum, they will not dynamically "stay in place", so to speak.

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[3] As far as the upcoming discussion is concerned, we remain conservative and we apply the WGC directly to a factored out U(1) gauge group. However, one could also apply the criterion to the Cartans of non-Abelian groups; the adoption of this perspective would make our arguments stronger.

In that case, the next best scenario is if the U(1) preserving direction also has the gradient of the potential tangent to it. Then, there exists a U(1) preserving classical trajectory and the situation is similar to the case when it is a modulus. Indeed, if we set conditions at $t = 0$ where this field has a sufficiently small but non-zero expectation value and vanishing kinetic energy, the subsequent trajectory will be along the U(1)-preserving direction. In this circumstance we can expand the action around that path and this expansion will still have a massless U(1) gauge field manifestly present, with the various field excitations charged under it. The magnetic WGC can then be applied as usual to determine a (possibly time-dependent) UV cut-off for the effective theory defined by the expansion around such a non-stationary classical path.

If we can carry out this procedure of defining effective theories around U(1) preserving non-stationary backgrounds, we can consider a sequence of such effective theories defined around paths with $t = 0$ conditions closer and closer to the central vacuum. Once again we expect the cut-offs for this sequence of effective theories to approach the cut-off of the non-Abelian theory, while the U(1) gauge coupling will approach the non-Abelian coupling, leading to the non-Abelian WGC.

In general theories, of course, we do not always have the ability to break the non-Abelian group while preserving a U(1) subgroup.

The above arguments are particularly relevant to N=2 Supergravity, where the vector multiplet scalars transform in the adjoint of the non-Abelian gauge group and can thus be used to break it to a U(1). In Section 2.3 we will see examples of both of the above scenarios, where a SU(2) gauge group in the central vacuum gets broken down to a U(1) either by a modulus or a tachyonic scalar that allows for U(1)-preserving classical trajectories. In both of those examples we will be able to exclude the central vacuum with non-Abelian gauge group by considering the WGC for the neighboring U(1) preserving points.

A final possible *caveat* is that, although a U(1) preserving direction in field space might exist, there might be no U(1) preserving classical trajectories, if the gradient of the potential is not aligned with the U(1) preserving direction. In this case it is not clear how to apply the WGC. We have not encountered such examples in our investigations.

### 2.1.3 N=2 with charged light gravitini

In this subsection we would like to provide a simple proof of the fact that in N=2 gauged Supergravity de Sitter critical points with charged massless gravitini are incompatible with the consistency requirements of the Weak Gravity Conjecture.

For the sake of clarity in the presentation we will directly give the argument in the

following, using only the N=2 ingredients that are directly relevant. The interested reader can find a summary of N=2 gauged Supergravity and all relevant references in Appendix 2.A.
Once we will establish that massless charged gravitini are in the Swampland, we will further show that, if they have a parametrically small mass, then the same results still apply.

In detail, we need three ingredients: the kinetic terms of the vectors in order to identify the gauge couplings; the gravitini-gauge vectors minimal couplings to identify the charge, and the value of the vacuum energy when the gravitino mass is vanishing. Since it is not restrictive, we assume that the gauging is purely electric.

The kinetic terms of the gauge vectors $A^\Lambda_\mu$ have the form

$$e^{-1}\mathcal{L}_{\text{kin.}} = \frac{1}{4}\mathcal{I}_{\Lambda\Sigma}\, F^\Lambda_{\mu\nu}F^{\mu\nu\,\Sigma}\,, \tag{2.1.5}$$

where $\mathcal{I}_{\Lambda\Sigma}$ is a negative definite scalar dependent matrix and $F^\Lambda_{\mu\nu} = 2\partial_{[\mu}A^\Lambda_{\nu]} + f^\Lambda{}_{\Sigma\Gamma}A^\Sigma_\mu A^\Gamma_\nu$. Once we define vielbeins and inverse vielbeins for the matrix $\mathcal{I}$ as follows

$$-\mathcal{I}_{\Lambda\Sigma} = \delta_{AB}\,\mathcal{E}^A_\Lambda \mathcal{E}^B_\Sigma\ , \quad \mathcal{E}^A_\Lambda \mathcal{E}^\Lambda_B = \delta^A_B, \tag{2.1.6}$$

we get the kinetic terms for the canonical vectors $v^A = \mathcal{E}^A_\Lambda A^\Lambda$

$$e^{-1}\mathcal{L}_{\text{kin.}} = -\frac{1}{4}\delta_{AB}F^A_{\mu\nu}F^{\mu\nu\,B}. \tag{2.1.7}$$

Within these $v^A_\mu$ vectors there is the massless U(1) gauge field we are interested in.

This being done, we wish to identify the physical charge of the gravitini under this U(1). To this end we focus on the minimal coupling between the gravitini and the U(1) vector. The relevant term takes the form

$$e^{-1}\mathcal{L}_{\text{kin.},\,3/2} = -\overline{\psi}^i_\mu\gamma^{\mu\nu\rho}D_\nu(\omega)\psi_{i\rho} - \frac{i}{2}\overline{\psi}^i_\mu\gamma^{\mu\nu\rho}v^A_\nu\left(\delta^j_i P^0_A + \sigma^x{}_i{}^j P^x_A\right)\psi_{j\rho}, \tag{2.1.8}$$

after having defined

$$P^0_\Lambda \mathcal{E}^\Lambda_B = P^0_B\ , \quad P^x_\Lambda \mathcal{E}^\Lambda_B = P^x_B. \tag{2.1.9}$$

Note that in (2.1.8) we have also included the kinetic term of the gravitino to stress that it is already canonically normalized.
In choosing the vielbein basis we have enough freedom to ensure that our U(1) gauge field, $u_\mu$, is along one specific basis element

$$u_\mu = v^{A=1}_\mu, \tag{2.1.10}$$

so that the corresponding minimal coupling to the gravitino is

$$e^{-1}\mathcal{L}_{\text{kin.},\,3/2} = -\overline{\psi}^i_\mu \gamma^{\mu\nu\rho} D_\nu(\omega)\psi_{i\rho} - i\overline{\psi}^i_\mu \gamma^{\mu\nu\rho} u_\nu Q_i{}^j \psi_{j\rho}, \tag{2.1.11}$$

where we have introduced the Hermitian matrix

$$2\,Q_i{}^j = \delta_i^j P_1^0 + \sigma^x{}_i{}^j P_1^x . \tag{2.1.12}$$

Since the two-by-two matrix $Q$ is Hermitian we can diagonalize it by a unitary transformation $U$, which we can also use to rotate the gravitini, namely

$$Q \to UQU^\dagger, \quad \psi \to U\psi, \quad \overline{\psi} \to \overline{\psi}U^\dagger, \tag{2.1.13}$$

so that the minimal coupling has the form

$$e^{-1}\mathcal{L}_{\text{kin.},\,3/2} = -\overline{\psi}^i_\mu \gamma^{\mu\nu\rho} D_\nu(\omega)\psi_{i\rho} - i\overline{\psi}^1_\mu \gamma^{\mu\nu\rho} u_\nu q_1 \psi_{1\rho} - i\overline{\psi}^2_\mu \gamma^{\mu\nu\rho} u_\nu q_2 \psi_{2\rho}. \tag{2.1.14}$$

Let us note that $q_1$ and $q_2$ are the physical couplings (i.e. gauge coupling $\times$ integer charge) between the canonical gauge bosons and the gravitini: they are the quantities that enter the WGC. As a result, the magnetic WGC for the U(1) under which the gravitini are charged states that

$$\Lambda_{\text{UV}} < q_1 \quad \wedge \quad \Lambda_{\text{UV}} < q_2, \tag{2.1.15}$$

where we remind the reader that we are working in Planck units.

We can now turn to the scalar potential. As an anticipation, we will see that under the assumption that the charged gravitini masses vanish the vacuum energy hits the WGC cut-off. The N=2 scalar potential with vanishing gravitini masses takes the form

$$\mathcal{V} = -\frac{1}{2}\mathcal{I}^{-1|\Lambda\Sigma}\left[P^0_\Lambda P^0_\Sigma + P^x_\Lambda P^x_\Sigma\right] + 4h_{uv}\, k^u_\Lambda k^v_\Sigma\, \bar{L}^\Lambda L^\Sigma \tag{2.1.16}$$

and consequently satisfies

$$\mathcal{V} \geq \frac{1}{2}\delta^{AB}\left[P^0_A P^0_B + P^x_A P^x_B\right]. \tag{2.1.17}$$

Then, we further have

$$\delta^{AB}\left[P^0_A P^0_B + P^x_A P^x_B\right] = \frac{1}{2}\delta^{AB}\left[\delta_i^j P^0_A + \sigma^x{}_i{}^j P^x_A\right]\left[\delta_j^i P^0_B + \sigma^y{}_j{}^i P^y_B\right] \geq \frac{1}{2}\left[\delta_i^j P^0_1 + \sigma^x{}_i{}^j P^x_1\right]^2. \tag{2.1.18}$$

Once we make use of the $Q$ matrix (2.1.12) and perform the rotation (2.1.13), we obtain

$$\mathcal{V} \geq \text{Tr}\left[UQU^\dagger UQU^\dagger\right] = \text{Tr}\left[QQ\right] = q_1^2 + q_2^2\,. \tag{2.1.19}$$

We thus conclude that

$$\mathcal{V} \geq q_1^2 \quad \wedge \quad \mathcal{V} \geq q_2^2 \quad \Rightarrow \quad \mathcal{V} \geq \Lambda_{\mathrm{UV}}^2, \tag{2.1.20}$$

which translates into

$$H \geq \frac{\Lambda_{\mathrm{UV}}}{\sqrt{3}} : \tag{2.1.21}$$

this means that the tree-level de Sitter critical points will receive large quantum corrections and can not be trusted, or, either said, it is a manifestation of the Dine–Seiberg problem [121], which challenges the consistency of such de Sitter vacua. It is important to keep in mind that we are always talking about charges of the gravitino under massless gauge fields, such that the Weak Gravity Conjecture can be directly applied. When, instead, a gauge symmetry is broken, even though a (covariantly) conserved current does still exist, one can not unambiguously define the charge any more, at least in Minkowski. As promised slightly above, we can extend our conclusions to the case of very light gravitini, in particular when they are parametrically lighter than the Hubble scale. Indeed, a gravitino mass matrix has the form

$$S_{ij} = i P_\Lambda^x L^\Lambda (\sigma_x)_i{}^k \epsilon_{jk}, \tag{2.1.22}$$

and only influences the supergravity scalar potential by the supersymmetry requirement that implies the inclusion of a new term of the form

$$\mathcal{V}_S = -4 \bar{L}^\Lambda L^\Sigma P_\Lambda^x P_\Sigma^x. \tag{2.1.23}$$

Having gravitino masses parametrically small compared to the Hubble scale means

$$\sqrt{\bar{L}^\Lambda L^\Sigma P_\Lambda^x P_\Sigma^x} \;\ll\; H : \tag{2.1.24}$$

this implies that the dominant contribution still comes from the term (2.1.20) and, therefore, the Hubble scale still hits the cut-off. We see that de Sitter backgrounds in N=2 Supergravity with charged light gravitini are faced with a Dine–Seiberg problem.
In the upcoming sections we will give explicit examples that show how the magnetic WGC restricts such vacua.

Let us observe that, if the gravitini are uncharged, the situation is different. In such a setup we would have $P_\Lambda^0 = 0 = P_\Lambda^x$ and the scalar potential would thus take the form

$$\mathcal{V} = 4 h_{uv}\, k_\Lambda^u k_\Sigma^v\, \bar{L}^\Lambda L^\Sigma \geq 0\,. \tag{2.1.25}$$

If we have an isometry with non-vanishing Killing vectors, this can lead to positive vacuum energy while maintaining vanishing gravitini mass. However, if such a background contains only spectator massless U(1) gauge fields, then in any case the WGC can not be directly applied and we can not conclude if it is in the Swampland or not. We will present an example where this happens in Subsection 2.4.2.

If one does not consider gauged supergravities, then the WGC is even less restrictive, at least at first sight. For example, de Sitter vacua with an underlying non-linear realization of N=2 would not require charged gravitini or any gauging at all [122, 123]. Such models can evade the restrictions that we just discussed, but this does not mean they can arise from String Theory, or, even if they do, they may still lead to short-lived vacua [124]. In addition, one may find complementary restrictions on such theories from EFT arguments as discussed in [125]. There are also examples where the N=2 de Sitter is supported by condensates of gravitini bi-linears [126], which hints that the vacuum does lie within a strongly coupled regime.

### 2.1.4 Main result and related conjectures

Our results have common ground with other conjectures and Swampland bounds. It is thus instructive to state clearly what we have found here and discuss what is the relation to the existing Swampland bounds.

Our results here can be expressed in the following way:

$$\text{Quasi de Sitter with} \quad m_{3/2} \ll H \quad \wedge \quad q_{3/2} \neq 0 \quad \text{has a Dine–Seiberg problem.} \tag{2.1.26}$$

Indeed, when the conditions described in (2.1.26) are met we find that the EFT has a very low cut-off and so the two-derivative truncation is inherently inconsistent. One can thus state that such EFTs belong to the Swampland.
We have already presented a general proof in a gauged N=2 framework in Subsection 2.1.3, and we will also give a proof for the N=8 case in Section 2.5. We will further illustrate this result in the various examples of the following sections.
Let us stress that the bound (2.1.26) follows from the magnetic WGC and it was already noticed in [98] for gauged N=2 without hypermultiplets. There it was rephrased as a conjecture, stating that de Sitter vacua with degenerate gravitino mass matrix belong to the Swampland. Our results here thus yield further credence to such a bound, also in the presence of hypermultiplets.

There is a non-trivial convergence between our results and the “festina lente” (FL) bound [109, 111], which roughly states that $m^2 \gtrsim qgH$ has to hold for every charged particle in the spectrum.
There are three instances where we can draw compatible conclusions.

Firstly, if we apply the FL bound on the gravitino, we can bring it exactly to the form:

$$m_{3/2} \ll H \quad \wedge \quad q_{3/2} \neq 0 \quad \Longrightarrow \quad \text{in the Swampland.} \tag{2.1.27}$$

We see that this exactly matches our main conclusion. Our results can be considered solid independent evidence that the gravitino abides by the FL bound. Conversely, if we had assumed the FL bound, then (2.1.26) would emerge simply as a particular instance of it.
Secondly, in [111] it is further argued that the Hubble scale is bounded from above by the magnetic WGC, which can be recast in a form that is relevant to us, that is:

$$H \gg q_{\text{phys.}} M_P \quad \Longrightarrow \quad \text{in the Swampland.} \tag{2.1.28}$$

Again this condition is at the core of the conclusions that we are drawing here and is already discussed in [98].
Thirdly, according to [111] the FL bound also gives restrictions on non-Abelian gauge theories, implying that they should either confine or spontaneously break at a scale above the Hubble scale, that is:

$$\text{de Sitter with perturbative non-Abelian gauging} \quad \Longrightarrow \quad \text{in the Swampland.} \tag{2.1.29}$$

This result again nicely aligns with our earlier discussion on non-Abelian gaugings, where we have used the WGC to argue that de Sitter N=2 vacua with perturbative non-Abelian groups and massless gravitini are in the Swampland.

Clearly our work also makes partial contact with the dSC/TCC conjectures [10, 100–105] which indicate that de Sitter space either does not exist as a solution within a theory of quantum gravity, or is inherently unstable. Our results, however, differs in that it is based solely on the magnetic Weak Gravity Conjecture without reference to the shape of the potential around the critical point. In particular, this leads to the elimination of certain de Sitter solutions that would be otherwise acceptable by the refined de Sitter Conjecture [101, 102], i.e. de Sitter points with steep tachyons.

Our analysis also makes contact with recent work [112, 113] claiming that the massless gravitini limit would correspond to a parametrically low cut-off due to towers of light states entering the EFT. Our conclusions here and the earlier work [98] are in agreement with these conjectures as the de Sitter points with vanishing gravitino mass are proposed to be in the Swampland precisely because of a very low cut-off, and thus, clearly, so is the limit when approaching such points.

## 2.2 Stable de Sitter vacua with massless gravitini

### 2.2.1 SO(2,1) × U(1) with one hypermultiplet

The first illustrative model with vanishing gravitino masses comes from the gauging of a SO(2,1) × U(1) group in a supergravity model with three vector multiplets and one hypermultiplet.

The scalar manifolds are

$$\mathcal{M}_{\rm SK} = \left[\frac{\rm SU(1,1)}{\rm U(1)}\right]^3, \quad \mathcal{M}_{\rm QK} = \frac{\rm SU(2,1)}{\rm SU(2)\times U(1)}. \tag{2.2.1}$$

For the Special-Kähler geometry we use as a starting point the symplectic frame where the prepotential is

$$F(X) = \sqrt{[(X^0)^2 + (X^1)^2]\,[(X^2)^2 + (X^3)^2]}, \tag{2.2.2}$$

which was shown in [127] to give a description in terms of the Calabi–Vesentini coordinates $z^I = \{S, y_0, y_1\}$ by means of the symplectic sections

$$Z = \begin{pmatrix} \frac{1}{2}(1+y_0^2+y_1^2) \\ \frac{i}{2}(1-y_0^2-y_1^2) \\ Sy_0 \\ Sy_1 \\ \frac{1}{2}S(1+y_0^2+y_1^2) \\ \frac{i}{2}S(1-y_0^2-y_1^2) \\ y_0 \\ y_1 \end{pmatrix}. \tag{2.2.3}$$

The gauging that we perform is not electric in this frame and therefore we introduce the symplectic rotation

$$S_{\rm Sp(8.\mathbb{R})} = \begin{pmatrix} \mathbb{1}_2 & & & & & \\ & 0 & 0 & & 1 & 0 \\ & 0 & -\sin\phi & & 0 & \cos\phi \\ & & & \mathbb{1}_2 & & \\ & -1 & 0 & & 0 & 0 \\ & 0 & -\cos\phi & & 0 & -\sin\phi \end{pmatrix} \tag{2.2.4}$$

acting on the symplectic section according to (2.A.2). The resulting holomorphic sections are

$$Z = \begin{pmatrix} \frac{1}{2}(1+y_0^2+y_1^2) \\ \frac{i}{2}(1-y_0^2-y_1^2) \\ y_0 \\ y_1(\cos\phi - S\ \sin\phi) \\ \frac{1}{2}S(1+y_0^2+y_1^2) \\ \frac{i}{2}S(1-y_0^2-y_1^2) \\ -Sy_0 \\ -y_1(S\ \cos\phi+\sin\phi) \end{pmatrix}, \tag{2.2.5}$$

which fix the Kähler potential as

$$e^{-\mathcal{K}} = -\mathrm{Im}S\big(1-2|y_0|^2-2|y_1|^2+|y_0^2+y_1^2|^2\big), \tag{2.2.6}$$

and the rest of the geometry according to the formulae in Appendix 2.A.
In this parameterization there is an obvious SO(2,1) symmetry acting on the first three sections, generated by the Killing vectors

$$\kappa_0^I = \begin{pmatrix} 0 \\ -\frac{i}{2}(1+y_0^2-y_1^2) \\ -iy_0y_1 \end{pmatrix}, \quad \kappa_1^I = \begin{pmatrix} 0 \\ \frac{1}{2}(1-y_0^2-y_1^2) \\ -y_0y_1 \end{pmatrix}, \quad \kappa_2^I = \begin{pmatrix} 0 \\ iy_0 \\ iy_1 \end{pmatrix}, \tag{2.2.7}$$

which we choose to gauge with the first three vectors (the graviphoton and two of the other vectors in the vector multiplets), hence fixing

$$k_\Lambda^I = e_0\left(\kappa_0^I\ ,\kappa_1^I\ ,\kappa_2^I\ ,\ 0\right)\ , \tag{2.2.8}$$

where we also introduced explicitly the SO(2,1) coupling $e_0$, which is going to be crucial in the following analysis.

The Quaternionic-Kähler geometry is that of the universal hypermultiplet (see for instance [128]), parametrized by the scalar fields $q^u = \{\rho,\sigma,\theta,\tau\}$, with metric

$$ds^2 = h_{uv}dq^udq^v = \frac{d\rho^2}{2\rho^2} + \frac{1}{2\rho^2}(d\sigma - 2\tau d\theta + 2\theta d\tau)^2 + \frac{2}{\rho}(d\theta^2 + d\tau^2). \tag{2.2.9}$$

In this sector we decide to gauge a compact U(1) symmetry generated by the Killing vector

$$\kappa_H^u = \begin{pmatrix} 4\rho\tau \\ 2\theta + 2\sigma\tau + 2\rho\theta + 2\theta(\theta^2+\tau^2) \\ 4\theta\tau - \sigma \\ 1-\rho-3\theta^2+\tau^2 \end{pmatrix}. \tag{2.2.10}$$

The gauging is performed using the last vector field available, hence fixing

$$k^u_\Lambda = e_1 \big(0\ ,\ 0\ ,\ 0\ , \kappa^u_H\big), \tag{2.2.11}$$

where, once again, we made explicit the coupling $e_1$.
Since the scalar potential (2.A.45) is determined not only by the Killing vectors of the hypermultiplets but also by their prepotentials, we also give here the explicit form of the prepotential associated to the isometry $\kappa^u_H$:

$$P^x_3 = e_1 \begin{pmatrix} -\frac{2}{\sqrt{\rho}}(1+\rho-3\theta^2+\tau^2) \\ \frac{2}{\rho}(\sigma-4\theta\tau) \\ \frac{2}{\rho}(\theta+3\rho\theta-\theta^3+\sigma\tau-\theta\tau^2) \end{pmatrix}. \tag{2.2.12}$$

Once one puts together the various pieces to the scalar potential, one can see that it has a critical point at

$$S = \cot\phi - \frac{i}{4}\left|\frac{e_0}{e_1\sin\phi}\right|, \quad \rho = 1, \quad y_0 = y_1 = \sigma = \theta = \tau = 0, \tag{2.2.13}$$

where

$$\mathcal{V} = 4\ |e_0\ e_1\ \sin\phi|. \tag{2.2.14}$$

This implies that the Hubble scale at this critical point is

$$H = \sqrt{\frac{4}{3}|e_0 e_1 \sin\phi|}. \tag{2.2.15}$$

Moreover, the gravitini mass matrix is identically vanishing at this critical point. The U(1) Killing vector also vanishes at this critical point, indicating that the U(1) symmetry is preserved and thus the WGC can be applied.

To explicitly check the consistency of such vacua against the Weak Gravity Conjecture, we first compute the gauge couplings at the critical point, which follow from

$$\mathcal{I}^{-1|\Lambda\Sigma} = -\frac{1}{2}\sin\phi \begin{pmatrix} 4e_1/e_0 & 0 & 0 & 0 \\ 0 & 4e_1/e_0 & 0 & 0 \\ 0 & 0 & 4e_1/e_0 & 0 \\ 0 & 0 & 0 & e_0/e_1 \end{pmatrix}. \tag{2.2.16}$$

We also note that the gravitino is only charged under the U(1) symmetry, with charge $q_{3/2} = 2e_1$, so that the magnetic WGC cut-off is

$$\Lambda_{\rm UV} = g_{\rm U(1)} q_{3/2} = \sqrt{2|e_0 e_1 \sin\phi|}\ : \tag{2.2.17}$$

the Hubble scale is of the order of the cut-off dictated by the magnetic WGC, and consequently there is a Dine–Seiberg problem.

The mass spectrum of the scalar fluctuations around the critical point includes two zero-modes, corresponding to the Goldstone modes eaten by the two broken non-compact SO(2,1) isometries. The rest of the spectrum is positive definite

$$m^2_{(\text{multiplicity})} = \big(0_{(2)}\ ,\ 1/4_{(4)}\ ,\ 1_{(2)}\ ,\ 2_{(2)}\big) \times \mathcal{V}, \tag{2.2.18}$$

so this critical point is also in violation of the de Sitter criterion.

For completeness, we note that there is also another unbroken U(1) isometry coming from the compact generator of SO(2,1) gauged on the vectors. The gravitino charge under that U(1) is given by $P_2^0$, which at the critical point simply evaluates to

$$q_{3/2} = \frac{1}{2} P_2^0 = e_0\,. \tag{2.2.19}$$

Multiplying by the appropriate component of $\mathcal{I}$ we obtain the same cut-off as before, $\Lambda_{\text{UV}} = \sqrt{2|e_0 e_1 \sin\,\phi|}$, which again points to the Dine–Seiberg problem that we have just highlighted.

### 2.2.2 SO(2,1) × U(1)$^3$ with two hypermultiplets

The model presented in this subsection has again massless gravitini at its critical point. A version of this model without hypermultiplets can be found in [129] and is already discussed from the WGC perspective in [98] and eliminated.
The modification that we consider here contains two hypermultiplets and is the first time that a model with two hypermultiplets and a fully stable de Sitter vacuum has been constructed. However, as we will see, it still suffers from a Dine–Seiberg problem that is signaled by the WGC.

The matter content of the model is given by five vector multiplets and two hypermultiplets, with scalar geometry

$$\mathcal{M}_{\text{SK}} = \frac{\text{SU}(1,1)}{\text{U}(1)} \times \frac{\text{SO}(2,4)}{\text{SO}(2)\times\text{SO}(4)}, \quad \mathcal{M}_{\text{QK}} = \frac{\text{SO}(4,2)}{\text{SO}(4)\times\text{SO}(2)}\,. \tag{2.2.20}$$

#### Vector multiplets

The geometry of the vector multiplet sector is described in a similar way as in the example of the previous subsection, starting from the prepotential

$$F(X) = \sqrt{[(X^0)^2 + (X^1)^2]\,\big(X^{\tilde{a}} X^{\tilde{b}} \delta_{\tilde{a}\tilde{b}}\big)}, \tag{2.2.21}$$

where $\tilde{a}, \tilde{b} = 2,3,4,5$. As in the case above, we can describe our gauging in the electric frame by introducing the Calabi–Vesentini coordinates [127, 129]

$$z^I = \{S\ ,\ y^a\}, \quad \text{where} \quad y^a = \{y^0, y^x\}, \quad x = 1,2,3\,, \tag{2.2.22}$$

and by performing an appropriate symplectic rotation analogous to (2.2.4).
The resulting holomorphic sections in the new frame are

$$Z = \begin{pmatrix} X^\Lambda \\ F_\Sigma \end{pmatrix}, \tag{2.2.23}$$

where

$$X^\Lambda(S, y^a) = \begin{pmatrix} \frac{1}{2}(1 + y^a y^a) \\ \frac{i}{2}(1 - y^a y^a) \\ y^0 \\ y^x(\cos\phi - S\sin\phi) \end{pmatrix}, \tag{2.2.24}$$

and

$$F_\Lambda(S, y^a) = \begin{pmatrix} \frac{1}{2}S(1 + y^a y^a) \\ \frac{i}{2}S(1 - y^a y^a) \\ -Sy^0 \\ -y^x(S\cos\phi + \sin\phi) \end{pmatrix}. \tag{2.2.25}$$

The geometry of this sector follows from these sections according to the formulae in Appendix 2.A. The Kähler potential is the sum of two factors

$$\mathcal{K} = -\log[i(\overline{X}^\Lambda F_\Lambda - \overline{F}_\Sigma X^\Sigma)] = \mathcal{K}_1 + \mathcal{K}_2, \tag{2.2.26}$$

where

$$\mathcal{K}_1 = -\log[i(S - \overline{S})], \tag{2.2.27}$$

$$\mathcal{K}_2 = -\log\left[\frac{1}{2}(1 - 2y^a\overline{y}^a + y^a y^a \overline{y}^a \overline{y}^a)\right]. \tag{2.2.28}$$

The metric for the scalar fields is factorized as

$$g_{I\overline{J}} = \begin{pmatrix} g_{S\overline{S}} & 0 \\ 0 & g_{a\bar{b}} \end{pmatrix}, \tag{2.2.29}$$

with

$$g_{S\overline{S}} = \frac{1}{(2\,\mathrm{Im}S)^2}, \quad g_{a\bar{b}} = \frac{\partial}{\partial y^a}\frac{\partial}{\partial \overline{y}^b}\mathcal{K}_2. \tag{2.2.30}$$

As before, also in this model we gauge the SO(2,1) symmetry in the vector sector that rotates the first three sections. This is generated by the Killing vectors

$$\kappa_0^I = \left(0, -\frac{i}{2}\left[1 + y_0^2 - \sum_x (y_x)^2\right], -iy_0y_1, -iy_0y_2, -iy_0y_3\right), \tag{2.2.31}$$

$$\kappa_1^I = \left(0, \frac{1}{2}\left[1 - y_0^2 + \sum_x (y_x)^2\right], -y_0y_1, -y_0y_2, -y_0y_3\right), \tag{2.2.32}$$

$$\kappa_2^I = (0, iy_0, iy_1, iy_2, iy_3)\,. \tag{2.2.33}$$

We gauge these isometries with the graviphoton and the first two vectors in the vector multiplet sector. The Killing vectors then are

$$k_\Lambda^I = e_0\left(\kappa_0^I\ , \kappa_1^I\ , \kappa_2^I\ ,\ 0\ ,\ 0\ ,\ 0\right), \tag{2.2.34}$$

where we made once more explicit the coupling $e_0$. From the above ingredients we can compute

$$\mathcal{V}_{D_1} = g_{I\overline{J}}\, k_\Lambda^I k_\Sigma^{\overline{J}}\, \overline{L}^\Lambda L^\Sigma, \tag{2.2.35}$$

which only depends on the vector multiplets and will therefore remain the same regardless of our choice of hypermultiplets or their gauging.

### Gauging $\mathbf{U(1)^3}$ on two hypermultiplets

We will now include two hypermultiplets in the model that we have studied in the previous subsection. The hyper-manifold $\mathcal{M}_{\rm QK}$ given above is a coset space; we can exploit this fact to explicitly provide the details of its construction in Appendix 2.B. We only report here its metric

$$\begin{aligned} ds^2 &= h_{uv}dq^u dq^v = \\ &= \frac{1}{q_1^2}\left[dq_1^2 + q_5^2 dq_4^2 + (dq_2 + \sqrt{2}q_7 dq_4)^2 + (dq_3 + \sqrt{2}q_8 dq_4)^2\right] + \\ &\quad + \frac{1}{72 q_1^2 q_5^2}\left[6\sqrt{2}dq_6 - 12 q_7 dq_2 - 12 q_8 dq_3 + 2\sqrt{2}q_4(q_7 dq_7 + q_8 dq_8) - 5\sqrt{2}(q_7^2 + q_8^2)dq_4)\right]^2 \\ &\quad + \frac{1}{q_5^2}\left(dq_5^2 + dq_7^2 + dq_8^2\right). \end{aligned} \tag{2.2.36}$$

Given that the isometries of $\mathcal{M}_{\rm QK}$ are a subset of those of $\mathcal{M}_{\rm SK}$, one could gauge an SO(2,1) × SO(3) gauge group using at the same time their action on the vector scalars and on the hypers. This is what was done in [129] to find one of the first examples of marginally stable de Sitter vacua. These models, however, do not lead to scalars with all masses positive and a simple analysis of their vacuum structure also shows that they are in tension with the WGC. We therefore decided to follow a different path and gauge three Abelian commuting isometries in the hypermultiplet geometry while leaving the SO(2,1) action confined to the vector multiplet sector. To summarize, our gauging is

$$\mathcal{G}_{\rm gauge} = {\rm SO}(2,1)_{\text{only on vectors}} \times \big({\rm U}(1)\times{\rm U}(1)\times{\rm U}(1)\big)_{\text{only on hypers}}. \tag{2.2.37}$$

This gauging is specified by the Killing vectors (2.2.34), together with the Killing vectors specifying the isometries that we want to gauge on the hypermultiplet sector

$$U(1)^3 : \, k^u_\Lambda = \big(0 \;,\; 0 \;,\; 0 \;,\; e_4 \, k^u_{T_{\underline{12}}} \;,\; e_5 \, k^u_{T_{\underline{34}}} \;,\; e_6 \, k^u_{T_{\underline{56}}}\big), \tag{2.2.38}$$

where the explicit expression of the $k^u_{T_{\underline{ab}}}$ can be found in Appendix 2.B. The gauging is electric in the frame given by (2.2.23). Notice that we could have made use of additional symplectic rotation parameters $\phi_i$ for each U(1) and indeed that would lead to different expressions for the masses. However, in all cases where the masses are positive, the properties of the vacuum do not significantly depend on the values of the angles and so we decided to take them to be all of the same value $\phi$ for simplicity.
In the end these ingredients contribute to the scalar potential that relates to the hypers and has the form

$$\mathcal{V}_{D_2} = 4 \, h_{uv} \, k^u_\Lambda k^v_\Sigma \overline{L}^\Lambda L^\Sigma. \tag{2.2.39}$$

When hypers are introduced, the would-be FI terms are field-dependent and are given by the appropriate prepotentials $P^x_\Lambda(q^u)$, which are determined by the isometries gauged on the hypers as reviewed in Appendix 2.A: we have

$$\mathcal{V}_F = \Big(U^{\Lambda\Sigma} - 3\overline{L}^\Lambda L^\Sigma\Big) \, P^x_\Lambda P^x_\Sigma \,. \tag{2.2.40}$$

The total potential is thus

$$\mathcal{V} = \mathcal{V}_{D_1} + \mathcal{V}_{D_2} + \mathcal{V}_F. \tag{2.2.41}$$

One can then verify that there is a central critical point at

$$q^1 = q^5 = 1 \;, \quad q^2 = q^3 = q^4 = q^6 = q^7 = q^8 = 0, \tag{2.2.42}$$

and

$$y^0 = y^x = 0 \;, \quad S = \cot\phi - i \left| \frac{e_0}{\sqrt{e_4^2 + e_5^2} \, \sin\phi} \right| . \tag{2.2.43}$$

It is interesting to note that at this critical point most prepotentials vanish

$$P^x_0 = P^x_1 = P^x_2 = P^x_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} , \quad P^x_3 = \begin{pmatrix} -e_4 \\ 0 \\ 0 \end{pmatrix} , \quad P^x_4 = \begin{pmatrix} -e_5 \\ 0 \\ 0 \end{pmatrix} , \tag{2.2.44}$$

as well as all Killing vectors of the compact U(1) isometries gauged in the hyper-sector:

$$k^u_\Lambda = 0. \tag{2.2.45}$$

This is in accordance with the fact that we have a residual $\mathrm{U}(1)^4$ gauge symmetry on the vacuum.
The value of the scalar potential is

$$\mathcal{V} = \sqrt{e_4^2 + e_5^2}\, |e_0 \sin\phi|, \tag{2.2.46}$$

and the canonically normalized mass eigenvalues are given by

$$m^2_{(\text{multiplicity})} = \left(0_{(2)}\ ,\ 1_{(6)}\ ,\ 2_{(2)}\ ,\ \frac{e_4^2}{e_4^2+e_5^2}{}_{(4)}\ ,\ \frac{e_5^2}{e_4^2+e_5^2}{}_{(4)}\right) \times \mathcal{V}, \tag{2.2.47}$$

which include two Goldstone modes, while all the other masses are positive definite.
We therefore see that we have a fully stabilized de Sitter critical point with both vector and hyper-multiplets. This is the first instance where a model with these properties is constructed.
We also see that the gravitini remain massless

$$S_{ij} = i P^x_\Lambda L^\Lambda (\sigma_x)_i{}^k \epsilon_{jk} = 0 \tag{2.2.48}$$

and hence we expect this model to fail to give a proper effective theory, according to the WGC.

In order to check this, we need the kinetic terms of the vectors, and in particular of the ones that are performing the U(1) gaugings. At the critical point the relevant sector of the Lagrangian becomes

$$e^{-1}\mathcal{L}_{\text{kin.vec.}} = -\frac{1}{4}\left|\frac{e_0}{\sqrt{e_4^2+e_5^2}\,\sin\phi}\right| \sum_{\Lambda=0}^{2} F^2(A^\Lambda) - \frac{1}{4}\frac{\sqrt{e_4^2+e_5^2}}{|e_0\sin\phi|}\sum_{\Lambda=3}^{5} F^2(A^\Lambda) \tag{2.2.49}$$

and we see that there is a rather intricate dependence of the gauge couplings on the charges $e_0$, $e_4$ and $e_5$.
The simplest way to check compatibility with the magnetic WGC is the following. We first notice that we have a spontaneous breaking of the SO(2,1) to a U(1) and the Goldstone modes associated to this symmetry breaking are the real and imaginary parts of $y_0$. This is seen from the fact that on the vacuum we have

$$k_0^I = e_0\left(0, -\frac{i}{2}, 0, 0, 0\right), \tag{2.2.50}$$

$$k_1^I = e_0\left(0, \frac{1}{2}, 0, 0, 0\right), \tag{2.2.51}$$

$$k_2^I = e_0\,(0, 0, 0, 0, 0)\,, \tag{2.2.52}$$

$$k_3^I = k_4^I = k_5^I = 0. \tag{2.2.53}$$

This means that the U(1) that survives the Higgsing is the one corresponding to $k_2^I$, which is just a standard U(1) acting on the $y^x$'s as follows

$$\mathrm{U}(1)_{\mathrm{residual}} \;:\; \mathrm{y}^{\mathrm{x}} \to \mathrm{e}^{\mathrm{i}\alpha\,\mathrm{e}_0}\,\mathrm{y}^{\mathrm{x}}. \tag{2.2.54}$$

As a result, we can identify the physical minimal coupling between the $y^x$'s and the residual massless U(1) gauge vector as

$$q_{\mathrm{phys.}} = e_0 \times \sqrt{\sqrt{e_4^2+e_5^2}\,\left|\frac{\sin\phi}{e_0}\right|}. \tag{2.2.55}$$

For any charged field the magnetic WGC tells us that in Planck units

$$\Lambda_{\mathrm{UV}} < q_{\mathrm{phys.}}, \tag{2.2.56}$$

while inspecting (2.2.46) and (2.2.55) we have that

$$H \sim \Lambda_{\mathrm{UV}}, \tag{2.2.57}$$

which is the signal that such vacua are faced with a Dine–Seiberg problem.
One can reach the same conclusion by identifying the gravitino charge under the residual U(1) from $P_2^0/2$ which gives $q_{3/2} = e_0$ so that the gravitino has once again physical coupling (2.2.55).

It is interesting to note that the model that we have just presented can also be obtained from a reduction from the SO(4,4) gauged N=8 Supergravity of [130]. In particular, if we set $e_5 = e_6 = 0$ in the model above and keep only the $e_4$ we get that the mass eigenvalues are given by

$$m^2_{(\text{multiplicity}\,+\,\text{Goldstones})} = \left(0_{(4+2)}, 1_{(10)}, 2_{(2)}\right) \times \mathcal{V}, \tag{2.2.58}$$

which match the truncated spectrum of the central vacuum in [130]. This can be understood from the fact that the scalar manifold (2.2.20) can be obtained as a N=2 truncation of the N=8 scalar manifold $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$, following [131], and that the SO(4,4) gauging produces an action on the scalar fields which is factorized in the same way as that of our truncated model.

### Gauging the SO(3)

The same model with scalar manifold (2.2.20), has been used in [129], but with a SO(3) gauging rather than a $\mathrm{U}(1)^3$. Also, both the SO(2,1) and the SO(3) factors have been gauged with a diagonal action on the vector and hyper-multiplets. The resulting scalar potential has a critical point where the hypers have non-negative masses. For completeness, we would like to show that these models are still faced with a Dine–Seiberg problem. Since all details of the model can be found in [129], we will only report here the details relevant for our discussion. Let us recall that $e_0$ corresponds to the coupling of the SO(2,1) factor (and $r_0 = 0, 1$ is the coefficient that signals the presence of a simultaneous action on the hyperscalars) and $e_1$ is the coupling of the SO(3) factor (and $r_1 = 0, 1$ again signals the action on the hyperscalars).

These models have a meta-stable vacuum with vacuum energy

$$\mathcal{V} = \sqrt{3(1+2r_0^2)}\,|e_0\,e_1\,r_1\,\sin\phi| > 0. \tag{2.2.59}$$

On this point the SO(2,1) gauge group is broken to a residual U(1), whose gauge vector has a kinetic term of the form

$$e^{-1}\mathcal{L}_{\text{residual U}(1)} = -\frac{1}{4}\left|\frac{e_0\,\sqrt{(1+2r_0^2)}}{\sqrt{3}\,e_1\,r_1\,\sin\phi}\right| F_{\mu\nu}F^{\mu\nu}\,. \tag{2.2.60}$$

Under the surviving U(1) the scalars of the vector multiplet are still charged with charge $e_0$, that is

$$\mathrm{U}(1)_{\text{residual}}\ :\delta \mathrm{y}^{\mathrm{x}} = \mathrm{i}\alpha\,\mathrm{e}_0\,\mathrm{y}^{\mathrm{x}}\,. \tag{2.2.61}$$

As a result the physical charge of the $y^x$ scalars under the residual U(1) is

$$q_{\text{phys.}} = \sqrt{\left|\frac{\sqrt{3}\,e_1\,r_1\,\sin\phi}{e_0\,\sqrt{(1+2r_0^2)}}\right|} \times e_0\,, \tag{2.2.62}$$

which sets the upper bound on the WGC cut-off. We conclude again that

$$H \sim \Lambda_{\text{UV}}\,, \tag{2.2.63}$$

both for $r_0 = 0$ and for $r_0 = 1$. Note that here we used the charge of the $y^x$ scalars under the U(1) to find the WGC cut-off, but we could have equally well used the gravitino charge.

Note that in all of the examples presented in this section, the central vacuum has vanishing gravitino mass. The contribution to the gravitino mass from the SO(2,1) gauging vanishes because the corresponding prepotentials vanish, while the contributions associated to the rest of the gauge group vanish due to the vanishing of the corresponding section components. Thus, all these examples serve to illustrate the result that critical points with charged massless gravitini violate the WGC. Of course, these examples also violate the de Sitter criterion directly, by virtue of their scalar mass spectra being positive (semi-)definite.

## 2.3 Multiple unstable dS vacua with various gravitini masses

We now turn to a different set of examples, where the de Sitter critical points are unstable and could survive the de Sitter Conjecture. We will show that whenever the gravitini masses are vanishing, we still can place these models in the Swampland. Moreover, among these examples we will find a case where there is a modulus such that the gravitini masses vary with its expectation value. This is a very instructive example because it shows explicitly how we can violate the WGC in a dynamic way, exhibiting that not only vanishing gravitini masses are dangerous but also very light ones. We will also give another example that we think is instructive, because it makes explicit the discussion of Section 2.1.2, having a central vacuum with non-Abelian gauge symmetry and massless gravitini for which the argument in Section 2.1.2 allows to point out a Dine–Seiberg problem.

### 2.3.1 Scalar manifolds

Both models that we consider in the following contain three vector multiplets and two hypermultiplets, parameterizing the scalar manifold

$$\mathcal{M}_{\rm SK} = \frac{\rm SU(1,3)}{\rm SU(3)\times U(1)}, \quad \mathcal{M}_{\rm QK} = \frac{\rm SO(4,2)}{\rm SO(4)\times SO(2)}. \tag{2.3.1}$$

The Special-Kähler manifold describes a vector multiplet geometry with minimal couplings and follows from the prepotential

$$F(X) = -\frac{i}{4} X^\Lambda X^\Sigma \, \eta_{\Lambda\Sigma}, \tag{2.3.2}$$

with $\eta = \mathrm{diag}\{1,-1,-1,-1\}$. The associated symplectic frame is described by the holomorphic sections

$$Z = \begin{pmatrix} 1 \\ z^I \\ -\frac{i}{2} \\ \frac{i}{2} z^I \end{pmatrix}, \quad I = 1,2,3, \tag{2.3.3}$$

where the $z^I$ are the three complex vector multiplet scalars. The Kähler potential is given by

$$\mathcal{K} = -\log\left[1 - z^I \bar{z}^I\right], \tag{2.3.4}$$

which makes explicit the SU(2) isometry that rotates the three scalars. The Killing vectors for these isometries are

$$\kappa_1^I = \begin{pmatrix} 0 \\ z_3 \\ -z_2 \end{pmatrix}, \quad \kappa_2^I = \begin{pmatrix} -z_3 \\ 0 \\ z_1 \end{pmatrix}, \quad \kappa_3^I = \begin{pmatrix} z_2 \\ -z_1 \\ 0 \end{pmatrix}. \tag{2.3.5}$$

The hypermultiplet scalar manifold is the same as in the previous section and the details of its parameterization are given in Appendix 2.B.

### 2.3.2 Gauging

The two models that we are going to analyze have a gauge group that is the direct product of a SO(3) factor and an Abelian compact or non-compact one-dimensional group.

The common SO(3) factor is taken to act simultaneously on the vector multiplet scalars as well as on the hyperscalars. The action on the vector scalar fields is identified with the isometries generated by the Killing vectors (2.3.5) and is gauged by the vector fields in the vector multiplets

$$k_\Lambda^I = e_1 \left(0 \ , \ \kappa_1^I \ , \ \kappa_2^I \ , \ \kappa_3^I\right). \tag{2.3.6}$$

The same SO(3) gauge group acts on the hyperscalars as specified by the generators $T_{\underline{12}}, T_{\underline{13}}, T_{\underline{23}}$ of the $\mathfrak{so}(4,2)$ algebra (see equation (2.B.1) in Appendix 2.B).
In addition, we either take a compact Abelian factor gauged by the graviphoton and acting on the hyperscalars as specified by $T_{\underline{56}}$, or a non-compact Abelian factor, always gauged by the graviphoton and acting on the hyperscalars as specified by $T_{\underline{46}}$. Overall, on the hypermultiplets we have the identifications

$$\mathrm{SO}(3) \times \mathrm{U}(1): \ \mathrm{k}_\Lambda^\mathrm{u} = \left(\mathrm{e}_0 \mathrm{k}^\mathrm{u}_{\mathrm{T}_{\underline{56}}} \ , \ \mathrm{e}_1 \mathrm{k}^\mathrm{u}_{\mathrm{T}_{\underline{12}}} \ , \ \mathrm{e}_1 \mathrm{k}^\mathrm{u}_{\mathrm{T}_{\underline{13}}} \ , \ \mathrm{e}_1 \mathrm{k}^\mathrm{u}_{\mathrm{T}_{\underline{23}}}\right), \tag{2.3.7}$$

$$\mathrm{SO}(3) \times \mathrm{O}(1,1): \ \mathrm{k}_\Lambda^\mathrm{u} = \left(\mathrm{e}_0 \mathrm{k}^\mathrm{u}_{\mathrm{T}_{\underline{46}}} \ , \ \mathrm{e}_1 \mathrm{k}^\mathrm{u}_{\mathrm{T}_{\underline{12}}} \ , \ \mathrm{e}_1 \mathrm{k}^\mathrm{u}_{\mathrm{T}_{\underline{13}}} \ , \ \mathrm{e}_1 \mathrm{k}^\mathrm{u}_{\mathrm{T}_{\underline{23}}}\right), \tag{2.3.8}$$

i.e. the SU(2) acting on the vector multiplets is identified with the SO(3) of the hyper manifold, while the U(1) or the O(1,1) symmetry is gauged by the graviphoton.
From the Killing vectors and the metric we can also compute the prepotentials $P^x$ using (2.A.38).

Both models have a critical configuration at the SO(3) invariant point

$$q_1 = q_5 = 1, \quad q_2 = q_3 = q_4 = q_6 = q_7 = q_8 = 0, \quad z^I = 0, \tag{2.3.9}$$

where the SO(3) Killing vectors vanish. The U(1) Killing vector also vanishes at this point, hence showing that the whole gauge group survives, while the O(1,1) Killing vector takes the form

$$k_{\underline{46}}^u = \delta_8^u, \tag{2.3.10}$$

thus signalling its breaking at the critical point. The corresponding prepotentials at the same point are

$$P_0^x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad P_1^x = \begin{pmatrix} e_1 \\ 0 \\ 0 \end{pmatrix}, \quad P_2^x = \begin{pmatrix} 0 \\ e_1 \\ 0 \end{pmatrix}, \quad P_3^x = \begin{pmatrix} 0 \\ 0 \\ e_1 \end{pmatrix}, \tag{2.3.11}$$

with $P_0^x$ vanishing for both the U(1) and the O(1,1) generators. As anticipated in the introductory comments to this section, both models have interesting features for our analysis, which we are now going to examine.

### 2.3.3 SO(3) × U(1)

The central critical point in this model has energy

$$\mathcal{V} = 3\, e_1^2, \tag{2.3.12}$$

while the eigenvalues of the scalar mass matrix are

$$m^2_{(\text{multiplicity})} = \left( -\frac{2}{3}_{(6)} \,,\, \frac{4}{3} r^2_{(2)} \,,\, \frac{4}{3}(r^2+1)_{(6)} \right) \times \mathcal{V}, \tag{2.3.13}$$

where $r = e_0/e_1$ is the ratio of the U(1) and SO(3) couplings.
The gravitino mass matrix vanishes at this critical point and hence we could be within the assumptions of our general proof of sSection 2.1.3. However, the gravitino charges under the four gauge bosons are

$$q_A = \left( \pm 0 \,,\, \pm\frac{1}{\sqrt{2}} e_1 \,,\, \pm\frac{1}{\sqrt{2}} e_1 \,,\, \pm\frac{1}{\sqrt{2}} e_1 \right), \tag{2.3.14}$$

so that the gravitini are not charged under the U(1), but only with respect to the SO(3) gauge group. The charges listed above have been computed by taking into account the normalization of the vector kinetic terms, given by the values of the gauge kinetic functions at the critical point, namely $\mathcal{I}_{\Lambda\Sigma} = -\frac{1}{2}\,\delta_{\Lambda\Sigma}$.

Since the gravitini are not charged under the U(1) gauge group, and the SO(3) factor is unbroken, we can not confidently apply the WGC in its usual form, using its gauge coupling. We must therefore resort to the argument presented in Section 2.1.2 where we look at nearby configurations that break the SO(3) symmetry down to a U(1). This allows us to use the SO(3) coupling in the magnetic WGC, giving a cut-off $\Lambda_{\text{UV}} = \frac{e_1}{\sqrt{2}}$, and the Hubble scale exceeds this.
In particular, we assume a small perturbation of the SO(3) point of the form

$$z_2 = i\,\epsilon, \quad z_1 = z_3 = 0. \tag{2.3.15}$$

This point is not a critical point of the theory of course, but it is still a legitimate configuration in our field space. Since Im$z_2$ gets a vacuum expectation value, the central SO(3) gauge group is Higgsed and only a U(1) remains under which fields are still charged with charge $e_1$. In addition, the total energy density, which is dominated by the vacuum energy, is still given approximately by $\rho \simeq 3\, e_1^2 + \epsilon\, \frac{dV}{d\mathrm{Im} z_2}$. As a result, for small $\epsilon$ we have

$$\Lambda_{\mathrm{WGC}}\Big|_{\epsilon\sim 0} \sim e_1 \sim H, \tag{2.3.16}$$

which can be extrapolated to the central vacuum as the limit $\epsilon \to 0$. We conclude that the central critical point is also threatened by the WGC cut-off.

It is worth pointing out that for non-zero Im$z_2$, the derivative of the potential also points in that direction. This allows for classical trajectories which preserve the U(1) symmetry throughout their entire duration. This puts us in the second scenario discussed in Section 2.1.2, where we do not have a modulus, but do still have a classical path. An example, where Im$z_2$ is a true modulus will be however presented in the next subsection.

Before moving on, we would like to observe that the same model possesses a second critical point, where the SO(3) gauge group is fully broken. This vacuum can be found by letting, for example, Re$z_1$ and Im$z_2$ vary. The new critical point appears at

$$\mathrm{Re}\, z_1 = \frac{1}{2}\,, \quad \mathrm{Im}\, z_2 = \frac{1}{2}, \tag{2.3.17}$$

and has energy $\mathcal{V} = 2\, e_1^2$. The normalized scalar mass spectrum is

$$m^2_{(\mathrm{mult.})} = \big(0_{(3)}, -1_{(2)}, 8_{(1)}, 2+4r^2-2r_{(2)}, \beta^2+\beta_{(2)}, \beta^2-\beta_{(2)}, 2+2r+4r^2_{(2)}\big) \times \mathcal{V}, \tag{2.3.18}$$

with $r = e_0/e_1$ describing the ratio of the charges and $\beta = 4r+1$. Once again this critical point respects the de Sitter criterion and also our criterion fails because the gravitino mass matrix at this saddle point is

$$S_{ij} = \begin{pmatrix} \sqrt{2}\, e_1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.3.19}$$

so that one gravitino acquires a mass of order the Hubble scale, while the other remains massless. We can also see that the gravitini are still uncharged under the residual U(1), because the gauge kinetic functions at this point are given by

$$-\frac{1}{2}\mathcal{I}^{-1} = \begin{pmatrix} 3 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.3.20}$$

and physical gravitino charges are then expressed by the eigenvalues of

$$(q_A)_i{}^j = \frac{1}{2}\mathcal{E}^\Lambda_A P^x_\Lambda (\sigma^x)_i{}^j, \tag{2.3.21}$$

where

$$\mathcal{E}^\Lambda{}_A = \begin{pmatrix} \sqrt{2} & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix}, \tag{2.3.22}$$

in such a way that $\mathcal{I}^{-1|\Lambda\Sigma} = \mathcal{E}^\Lambda{}_A \mathcal{E}^\Sigma{}_B \delta^{AB}$. After a straightforward calculation we find that

$$q_A = \left( \pm 0\ ,\ \pm e_1\ ,\ \pm e_1\ ,\ \pm \frac{1}{\sqrt{2}} e_1 \right). \tag{2.3.23}$$

One might hope to apply the WGC despite a complete breaking of the SO(3) gauge symmetry, if some of the gauge fields have masses below the Hubble scale and therefore still effectively mediate long-range forces within a Hubble patch. However, in this model this is not the case. The masses of the gauge bosons can be determined from the eigenvalues of

$$m^2_{AB} = \frac{1}{2}\big(\mathcal{E}^\Lambda{}_A k^\alpha_\Lambda g_{\alpha\bar\beta} k^{\bar\beta}_\Sigma \mathcal{E}^\Sigma{}_B + \text{h.c.}\big), \tag{2.3.24}$$

which are $(0,\ 8e_1^2,\ 4e_1^2,\ 4e_1^2)$. The zero mass corresponds to the unbroken U(1) under which the gravitino is uncharged. The remaining masses are clearly of order the Hubble scale and thus do not mediate long-range forces.

### 2.3.4 SO(3) × O(1,1)

A very instructive model is the one described by (2.3.1), but with a SO(3) × O(1,1) gauge group.

This model also has a critical point at (2.3.9) with a vacuum energy

$$\mathcal{V} = 2\, e_0^2 + 3\, e_1^2, \tag{2.3.25}$$

and scalar mass spectrum

$$\begin{aligned} m^2_{(\text{multiplicity})} = \Big(&0_{(1)},\ 2(e_0^2 - e_1^2)_{(3)},\ 4e_0^2{}_{(1)},\ 4e_1^2{}_{(2)}, x_1{}_{(1)},\ x_2{}_{(1)},\ x_3{}_{(1)}, \\ & e_1^2 + \sqrt{4e_0^4 - 4e_0^2 e_1^2 + 9e_1^4}_{(2)},\ e_1^2 - \sqrt{4e_0^4 - 4e_0^2 e_1^2 + 9e_1^4}_{(2)}\Big) \times \mathcal{V}, \end{aligned} \tag{2.3.26}$$

where $x_{1,2,3}$ are solutions of the cubic equation

$$x^3 + 6e_1 x^2 + \left(4e_0^2 e_1^2 - 4e_0^4\right) x - \left(16e_0^4 e_1^2 - 16e_0^2 e_1^4 + 32e_1^6\right) = 0. \tag{2.3.27}$$

This mass spectrum pushes the limits of the de Sitter criterion, but not parametrically so. Again, the normalization of the vector kinetic terms, given by the values of the gauge kinetic functions at the critical point, is trivial, because $\mathcal{I}_{\Lambda\Sigma} = -\frac{1}{2}\,\delta_{\Lambda\Sigma}$.

As in the previous model, the gravitino masses vanish at the central point, but, once again, the unbroken SO(3) prevents a straightforward application of the WGC and one must therefore resort to the argument presented in Section 2.1.2 where we look at nearby configurations that break the SO(3) symmetry down to a U(1). In particular, for the special choice $e_0 = e_1$, we find entire lines of critical points that pass from the center and are parametrized by any of the three imaginary components. Along each of these lines, except the central point, the SO(3) is broken to a U(1) and therefore we can confidently invoke the WGC.

For the rest of this subsection we set $e_0 = e_1$ and for concreteness we take $\mathrm{Im}z_2 = z$ as the modulus, with all other scalars remaining fixed at zero. The scalar mass spectrum along this line is given by

$$m^2_{(\text{multiplicity})} = \Bigg(0_{(4)}, -2/5_{(1)}, 4/5_{(3)}, \frac{4}{5-5z^2}_{(1)}, x_{1\ (1)}, x_{2\ (1)}, x_{3\ (1)}\\ \frac{1+z^2+\sqrt{9-14z+9z^2}}{5-5z^2}_{(1)}, \frac{1+z^2-\sqrt{9-14z+9z^2}}{5-5z^2}_{(1)}\Bigg) \times \mathcal{V}, \tag{2.3.28}$$

where $x_{1,2,3}$ are now solutions of

$$\begin{aligned} &x^3 + \sqrt{1-z^2}(6-2z^2)x^2 + (16z^2 - 32z^4 + 16z^6)x \\ &+ \sqrt{1-z^2}(-32 + 128z^2 - 192z^4 + 128z^6 - 32z^8) = 0. \end{aligned} \tag{2.3.29}$$

The behaviour of these normalized masses is shown in Figure 2.1. Notice that, in addition to the three Goldstone modes of the SU(2)×O(1,1) breaking to U(1), there is an additional massless scalar field, $z = \mathrm{Im}z_2$.

We now have a residual U(1) gauge group with respect to which the gravitini are charged. This is specified by the physical gravitino charges

$$q_A = \left(\pm 0\ ,\ \pm\, e_1\ ,\ \pm\, e_1\sqrt{\frac{1+z^2}{1-z^2}}\ ,\ \pm\, e_1\right), \tag{2.3.30}$$

where the normalizations follow from the gauge couplings at the line of critical points

$$-\frac{1}{2}\mathcal{I}^{-1} = \begin{pmatrix} \frac{1+z^2}{1-z^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1+z^2}{1-z^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.3.31}$$

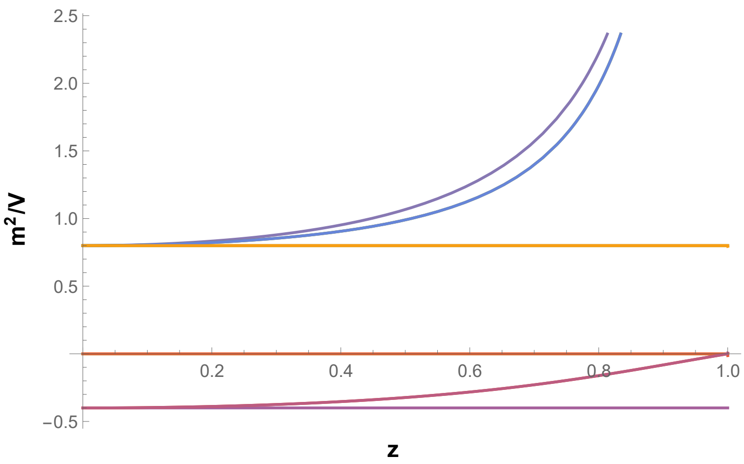


Figure 2.1: The figure illustrates the ratio $m^2/\mathcal{V}$ of all the scalar fields in the SU(2)×O(1,1) model, plotted as a function of the modulus $z = \mathrm{Im}z_2$. The central vacuum $z = 0$ preserves the full SU(2) gauge group. Tachyons with $m^2/\mathcal{V} = -2/5$ are present for all values of $z$.

The third eigenvalue of (2.3.30) is the coupling of the unbroken U(1) subgroup of the SO(3) symmetry and thus is the one that can be used for the WGC. This implies that the WGC cut-off is

$$\Lambda_{\mathrm{UV}} = e_1 \sqrt{\frac{1+z^2}{1-z^2}}. \tag{2.3.32}$$

On the other hand the gravitino mass matrix is

$$S_{ij} = \begin{pmatrix} e_1 \frac{z}{\sqrt{1-z^2}} & 0 \\ 0 & e_1 \frac{z}{\sqrt{1-z^2}} \end{pmatrix} \tag{2.3.33}$$

and therefore, as we move away from the central point, both gravitini become massive, while the gauge coupling increases, thus increasing the magnetic WGC cut-off.

As we mentioned before, this is very instructive for various reasons. First of all, we see that the vacuum at $z = 0$ has a symmetry enhancement, so we could not apply directly the WGC. However, this vacuum is now the limiting point of a series of critical points with a residual U(1) gauge group for which we can apply the WGC and, having charged gravitini, our argument. In fact, all the critical points close to the central one are not part of a good EFT, because the vacuum energy is larger than the cut-off scale ($\Lambda_{\mathrm{UV}}/H < 1$). Then, as the gravitini mass approaches the Hubble scale, the Hubble parameter itself becomes smaller than the cut-off energy, saving the EFT approximation. This can be seen explicitly in Figure 2.2, where both the ratios of the cut-off over the Hubble scale and gravitini mass over Hubble parameter are plotted, for the whole range of validity of the scalar vacuum expectation value $z$.

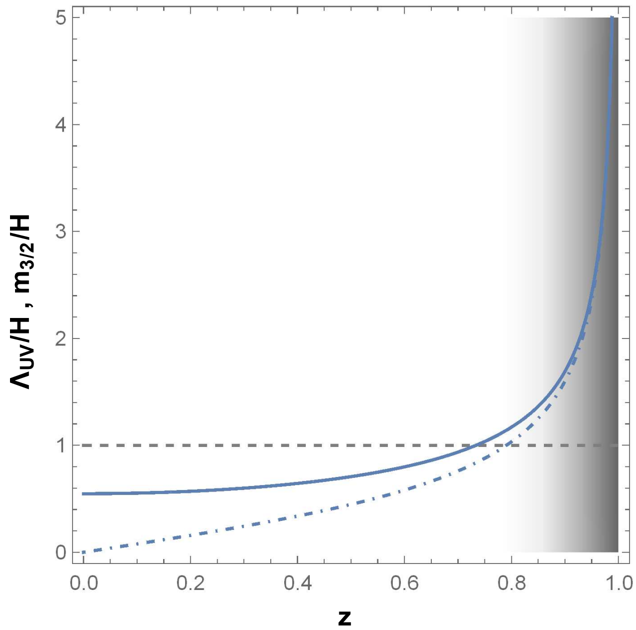


Figure 2.2: The figure shows the ratios $\Lambda_{\rm UV}/H$ (solid) and $m_{3/2}/H$ (dot-dashed) as a function of the modulus $z = {\rm Im} z_2$ in the SU(2)×O(1,1) model. For small values of $z$ the gravitino mass vanishes and the Hubble scale is above the cut-off, so the theory has a Dine–Seiberg problem. The shaded gray region denotes where the effective theory is increasingly well controlled; in the dark gray part $H \ll \Lambda_{\rm UV}$. The gravitino mass is always below the cut-off, approaching it as $z$ approaches the boundary of moduli space.

## 2.4 Unstable vacua with no massless U(1) couplings

We conclude our discussion of N=2 models by giving some simple examples for which our criteria do not apply, just to clarify the existence of situations that avoid our assumptions.

### 2.4.1 Massive gravitini

The first simple example is based on a method to construct stable de Sitter critical points in N=2 that was presented in [132]. Such critical configurations always have non-vanishing gravitini mass and, therefore, our general argument does not apply. In addition, as the only gauging involved is a Higgsed U(1), the WGC can not be applied to eliminate such vacua.

Taking however into account that in [132] only a general strategy is described but no explicit instance is given, we believe that it is useful to see a concrete example. We thus skip the general properties presented in [132] and focus on a model that contains a single hypermultiplet with a Quaternionic-Kähler geometry that is not homogeneous. This is a special case of the general metric of [132] for a Quaternionic-Kähler manifold with one isometry.

The specific example that we would like to study has a metric given by

$$ds^2 = \frac{1}{2\rho^2}\left[f d\rho^2 + f e^h (d\theta^2 + d\tau^2) + f^{-1}(d\sigma + \Theta)^2\right], \tag{2.4.1}$$

where the scalar fields are $q^u = \{\rho, \theta, \tau, \sigma\}$ and

$$h = \log(a\rho + b), \quad f = \frac{a\rho + 2b}{a\rho + b}, \quad \Theta = \frac{a}{2}\left(\theta d\tau - \tau d\theta\right), \tag{2.4.2}$$

for $a, b$ real parameters. This metric has an obvious shift symmetry along $\sigma$. After the gauging with the graviphoton of such isometry,

$$k^u = e_0\,(0, 0, 0, 1), \tag{2.4.3}$$

we get a scalar potential of the form

$$\mathcal{V} = \frac{e_0^2}{\rho^2}\left(\frac{a\rho + b}{a\rho + 2b} - \frac{3}{4}\right), \tag{2.4.4}$$

which exhibits a critical point at

$$\rho = \frac{(1 + \sqrt{5})b}{a} \tag{2.4.5}$$

with positive vacuum energy

$$\mathcal{V} = \frac{(5\sqrt{5} - 11)a^2 e_0^2}{32 b^2}. \tag{2.4.6}$$

The scalar $\rho$ is tachyonic, while the scalars $\theta$ and $\tau$ are flat directions. In particular, the canonically normalized mass of $\rho$ is

$$m_\rho^2 = -(5 + \sqrt{5}) \times \mathcal{V}\,. \tag{2.4.7}$$

As a result, the refined de Sitter Conjecture [101, 102] is not violated. We can also compute the gravitini masses: they are

$$m_{3/2}^2 = \frac{e_0^2 a^2}{[2(1 + \sqrt{5})b]^2}. \tag{2.4.8}$$

We see that this critical point is not threatened by a low cut-off because there is no U(1) to invoke the WGC, and at the same time both gravitini are massive.

One can work out different examples that include additional vector multiplets, but they essentially have the same property as far as our work is concerned.
It is further worth noting that in the procedure of [132] for constructing fully stable de Sitter vacua, there is always a spectator U(1) related to the graviphoton because the shift symmetry of the hypermultiplet is gauged by a vector belonging to a physical vector multiplet.

### 2.4.2 Massless uncharged gravitini

Another simple instance where our argument in Section 2.1 does not apply is when the gravitini are both massless and uncharged.
A model of this type is obtained, for instance, from the models of Subsection 2.3.4 by taking $e_1 = 0$, i.e. removing the SO(3) gauging. The central vacuum is still present, with energy $\mathcal{V} = 2e_0^2$, but there is no longer any physical U(1) charge to determine a WGC cut-off. Indeed, the vacuum energy only comes from the charges of the hypermultiplets under the broken symmetries, in this case under O(1,1).
Note that there are three more vectors in the theory on top of the graviphoton, but they are merely spectators and one can truncate them without changing the properties of the example.

We see that the fact that we can not exclude the existence of this de Sitter vacuum due to the massless but uncharged gravitini aligns nicely with the FL bound [109–111] which does not prohibit massless uncharged fields either.
Furthermore, models that contain only hypers and where the gauging is from the graviphoton have been proven to be tachyonic [133]. In particular, as shown in [133], such models always contain an $\mathcal{O}(1)$ (in Hubble units) mass tachyon and so do not violate the de Sitter criterion [100–102].

## 2.5 Maximal Supergravity with light gravitini

While all the discussions and examples provided so far are related to N=2 gauged Supergravity, we strongly believe that our arguments are fairly general and should apply to any gauged extended supergravity theory. In this section we show how the proof of Section 2.1 can be applied to the case of maximal Supergravity. We believe that similar results could be obtained for any N $\geq$ 2.

Let us first recall some crucial aspects of N=8 gauged Supergravity [134]. Maximal Supergravity contains a single gravity multiplet, whose fields are the graviton, $g_{\mu\nu}$, eight gravitini, $\psi_\mu^i$ $(i = 1, \ldots, 8)$, twenty-eight vector fields, $A_\mu^\Lambda$ (conventionally $\Lambda = 0, \ldots, 27$), fifty-six spin 1/2 dilatini, $\chi_{ijk} = \chi_{[ijk]}$, and seventy real scalar fields, $\varphi^u$ $(u = 1, \ldots, 70)$. The scalar fields describe a non-linear $\sigma$-model given by a homogeneous manifold

$$\mathcal{M}_{\text{scalar}} = \frac{\mathrm{E}_{7(7)}}{\mathrm{SU}(8)}. \tag{2.5.1}$$

The vector fields and their duals transform in the **56**-dimensional fundamental representation of $\mathrm{E}_{7(7)}$, which is a symplectic representation, defining an embedding of $\mathrm{E}_{7(7)}$ in $\mathrm{Sp}(56, \mathbb{R})$, i.e. $V^M = \{V^\Lambda, V_\Lambda\}$. The coset representative is customarily described by complex 56-dimensional vectors, $L_M{}^{ij} = -L_M{}^{ji}$, and their complex conjugates, $L_{M\,ij}$, which together build a matrix

$$L_M{}^{\underline{N}} = \left(L_M{}^{ij}, L_{M\,kl}\right). \tag{2.5.2}$$

This matrix transforms under rigid $E_{7(7)}$ transformations from the left and under local SU(8) transformations from the right. We also note the following properties of $L_M{}^{\underline{N}}$, which follow from their definition,

$$\begin{aligned} L_M{}^{ij} L_{N\,ij} - L_N{}^{ij} L_{M\,ij} &= i\,\Omega_{MN}, \\ \Omega^{MN} L_M{}^{ij} L_{N\,kl} &= i\,\delta^{ij}_{kl}, \\ \Omega^{MN} L_M{}^{ij} L^{kl}_N &= 0, \end{aligned} \tag{2.5.3}$$

where $\Omega$ is the symplectic invariant matrix.

The gauging procedure fixes the gauge generators $X_M$ from the $E_{7(7)}$ ones $t_\alpha$ specifying the embedding tensor $\Theta$

$$X_{MN}{}^P = \Theta_M{}^\alpha (t_\alpha)_N{}^P. \tag{2.5.4}$$

Of course, consistent gaugings have restrictions on the allowed form of $\Theta$ and consequently of $X$, as discussed in [134].

We are interested in the vector kinetic terms, in the Lagrangian sector describing the kinetic and mass terms for the gravitini, and in the scalar potential.
The vector kinetic terms have the usual form

$$e^{-1}\mathcal{L}_{\text{kin.}} = \frac{1}{4}\,\mathcal{I}_{\Lambda\Sigma}\,F^\Lambda_{\mu\nu}F^{\mu\nu\Sigma}, \tag{2.5.5}$$

although now the vector kinetic term can be expressed in terms of the coset representatives as

$$\mathcal{I}^{-1|\Lambda\Sigma} = -2\,L^\Lambda{}_{ij}\,L^{\Sigma\,ij}. \tag{2.5.6}$$

The relevant sector for the gravitini is

$$-\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\left(\bar{\psi}_\mu{}^i\gamma_\nu D_\rho(\omega,\mathcal{Q})\psi_{\sigma\,i} + \text{h.c.}\right) + g\,e\left(\frac{1}{2}\sqrt{2}\,A_{1\,ij}\,\bar{\psi}^{\,i}_\mu\gamma^{\mu\nu}\psi^j_\nu + \text{h.c.}\right), \tag{2.5.7}$$

where the covariant derivative acts as

$$D_\rho(\omega,\mathcal{Q})\psi_{\sigma\,i} = D_\rho(\omega)\psi_{\sigma\,i} + \frac{1}{2}\mathcal{Q}_{\rho i}{}^j\psi_{\sigma\,j}, \tag{2.5.8}$$

and the gauged SU(8) connection $\mathcal{Q}_\mu$ contains the gauging charges $\mathcal{Q}_{Mi}{}^j$ in the form

$$\mathcal{Q}_{\mu\,i}{}^j = \frac{2}{3}\mathrm{i}(L_{\Lambda\,ik}\,\partial_\mu L^{\Lambda\,jk} - L^\Lambda{}_{ik}\,\partial_\mu L_\Lambda{}^{jk}) - g\,A_\mu{}^M\,\mathcal{Q}_{M\,i}{}^j. \tag{2.5.9}$$

The explicit expression of the gauging charges can be obtained from the following identities:

$$\begin{aligned} \mathcal{Q}_{M\,ij}{}^{kl} &= \delta_{[i}{}^{[k}\,\mathcal{Q}_{M\,j]}{}^{l]} = i\,\Omega^{NP}\,L_{N\,ij}X_{MP}{}^{Q}\,L_{Q}{}^{kl}\,, \\ \mathcal{P}_{M\,ijkl} &= \frac{1}{24}\,\varepsilon_{ijklmnpq}\,\mathcal{P}_{M}^{mnpq} = i\,\Omega^{NP}\,L_{N\,ij}X_{MP}{}^{Q}\,L_{Qkl}, \end{aligned} \tag{2.5.10}$$

where $\mathcal{Q}_{M}{}^{i}{}_{j} = -\mathcal{Q}_{Mj}{}^{i}$ and $\mathcal{Q}_{Mi}{}^{i} = 0$, which means that $\mathcal{Q}_{Mi}{}^{j}$ is taken to be anti-Hermitian $\mathcal{Q}_{M}^{\dagger} = -\mathcal{Q}_{M}$. The Lagrangian mass $A_{1\,ij}$ is defined by the gauging procedure, together with the tensor $A_{2i}{}^{jkl}$, which will fix the scalar potential, as

$$\mathrm{i}\,\Omega^{MN}\,\mathcal{Q}_{M\,i}{}^{j}\,L_{N}{}^{kl} = -\,A_{2i}{}^{jkl} - 2\,A_{1}{}^{j[k}\,\delta^{l]}{}_{i}. \tag{2.5.11}$$

Finally, the scalar potential can be written by using the various structures that we have introduced so far as

$$\begin{aligned} \mathcal{V} &= g^{2}\Big\{\frac{1}{24}\,\big|A_{2i}{}^{jkl}\big|^{2} - \frac{3}{4}\,\big|A_{1}^{ij}\big|^{2}\Big\} = \\ &= \frac{1}{336}\,g^{2}\,\mathcal{M}^{MN}\,\big\{8\,\mathcal{P}_{M}{}^{ijkl}\mathcal{P}_{Nijkl} + 9\,\mathcal{Q}_{Mi}{}^{j}\,\mathcal{Q}_{Nj}{}^{i}\big\}\,, \end{aligned} \tag{2.5.12}$$

where

$$\mathcal{M}_{MN} \equiv L_{M}{}^{ij}\,L_{N\,ij} + L_{M\,ij}\,L_{N}{}^{ij}\,, \quad \mathcal{M}^{MN} = \Omega^{MP}\Omega^{NQ}\mathcal{M}_{PQ}, \tag{2.5.13}$$

and one notes the relations

$$\mathcal{M}^{MN}\mathcal{P}_{M}{}^{ijkl}\mathcal{P}_{Nijkl} = 4\,|A_{2l}{}^{ijk}|^{2}\,, \tag{2.5.14}$$

$$\mathcal{M}^{MN}\mathcal{Q}_{Mi}{}^{j}\,\mathcal{Q}_{Nj}{}^{i} = -2\,|A_{2l}{}^{ijk}|^{2} - 28\,|A_{1}{}^{ij}|^{2}. \tag{2.5.15}$$

Now that all necessary ingredients have been put forward, we can build our argument along the lines of Section 2.1.

First of all we assume that all gravitini are massless and hence that we have a de Sitter critical point where $A_{1}^{ij} = 0$. This implies that at the critical point the relation

$$\mathcal{M}^{MN}\mathcal{P}_{M}{}^{ijkl}\mathcal{P}_{Nijkl} = -2\mathcal{M}^{MN}\mathcal{Q}_{Mi}{}^{j}\,\mathcal{Q}_{Nj}{}^{i} \tag{2.5.16}$$

holds and, in turn, that the potential can be written as

$$\mathcal{V} = -\frac{1}{48}\,g^{2}\,\mathcal{M}^{MN}\mathcal{Q}_{Mi}{}^{j}\,\mathcal{Q}_{Nj}{}^{i} > 0. \tag{2.5.17}$$

If we move to an electric symplectic frame, we can further simplify this to

$$\mathcal{V} = -\frac{1}{48}\,g^{2}\,\mathcal{M}^{\Lambda\Sigma}\mathcal{Q}_{\Lambda i}{}^{j}\,\mathcal{Q}_{\Sigma j}{}^{i} > 0 \tag{2.5.18}$$

and using (2.5.6), (2.5.13) and the coset relations (2.5.3) we find $\mathcal{M}^{\Lambda\Sigma} = -\mathcal{I}^{\Lambda\Sigma}$

$$\mathcal{V} = \frac{1}{48}\, g^2\, \mathcal{I}^{\Lambda\Sigma} \mathcal{Q}_{\Lambda i}{}^{j}\, \mathcal{Q}_{\Sigma j}{}^{i} = -\frac{1}{48}\, g^2\, \mathcal{I}^{\Lambda\Sigma} \mathrm{Tr}(\mathcal{Q}^\dagger_\Lambda \mathcal{Q}_\Sigma) > 0, \tag{2.5.19}$$

where we recall that $\mathcal{I}$ is negative definite and $\mathcal{Q}$ is anti-Hermitian.

At this point the argument follows along the same lines as in Section 2.1. We define a set of vielbeins to put the kinetic terms of the vectors in canonical form

$$-\mathcal{I}_{\Lambda\Sigma} = \delta_{AB}\, \mathcal{E}^A_\Lambda \mathcal{E}^B_\Sigma\ , \quad \mathcal{E}^A_\Lambda \mathcal{E}^\Lambda_B = \delta^A_B, \tag{2.5.20}$$

and use the same vielbeins to identify the physical, now Hermitian, charges of the gravitini

$$Q_A = \frac{i}{2}\, g\, \mathcal{E}^\Lambda_A\, \mathcal{Q}_\Lambda, \tag{2.5.21}$$

so that (2.5.8) becomes

$$D_\rho \psi_{\sigma i} = D_\rho(\omega)\psi_{\sigma i} + \ldots + i A^A_\mu Q_A{}_i{}^j \psi_{\sigma j}, \tag{2.5.22}$$

and the scalar potential at the critical point is

$$\mathcal{V} = \frac{1}{12}\, \delta^{AB}\, \mathrm{Tr}\left(Q_A Q_B\right). \tag{2.5.23}$$

We therefore see that if there is a U(1) surviving at the critical point under which the gravitini are charged, the scalar potential is larger than the sum of the squares of the physical charges of the gravitini.

Clearly the only effect of switching-on a parametrically small gravitino mass is to slightly alter the vacuum energy. However, as long as such contribution is parametrically smaller than the Hubble scale it does not alter the fact that the vacuum energy hits the cut-off. As a result we conclude once again that (quasi) de Sitter with light charged gravitini belongs to the Swampland.

So far we do not have many examples of de Sitter vacua in maximal gauged Supergravity and all the examples that we have do not have Abelian factors in the residual gauge symmetry. However, we can once more employ the argument about the WGC for critical points with a non-Abelian gauge symmetry made in Section 2.1, because in maximal Supergravity every time that we gauge a non-Abelian symmetry of the scalar manifold, we will have scalars that potentially break this symmetry by acquiring a VEV. This would tell us that the SO(4) $\times$ SO(4) vacuum of the SO(4,4) gauging in [130] is not a consistent effective theory, because it has massless gravitini, charged under the residual gauge group. While this was expected for the vacuum coming from the regular gauging, which is a consistent truncation of type II compactifications on a hyperboloid, and therefore do not have a mass gap with the Kaluza–Klein states, it is certainly interesting for the deformed models, where the SO(4,4) gauge group was embedded in a new, rotated way inside $E_{7(7)}$. We also confirm that the second de Sitter vacuum found in [130], when the deformation parameter is non-vanishing, has non-vanishing gravitino masses and therefore could still survive the WGC constraints, while having a parametrically small tachyon[4].

## 2.6 Comments

In this chapter we have argued that de Sitter critical points in extended (two-derivative) Supergravity violate the magnetic WGC when they have charged light gravitini.
We have presented a general proof of this claim in N=2 and N=8 gauged Supergravity.
We have further illustrated this claim with several N=2 models with hypermultiplets, whose scalar potentials admit de Sitter critical points, both stable and unstable.
We have also presented examples of critical points that escape the WGC by having either massive gravitini or no U(1) gauge symmetry at the critical point, and a model where a modulus interpolates between respecting and violating the WGC.

It is interesting to observe that many of the unstable critical points that we ruled out respect the de Sitter criterion.
Our results are also especially consonant with the "festina lente" bound, which forbids charged particles that are too light in a de Sitter background.
In addition, our findings resonate with the arguments that rule out continuous non-compact gauge groups as a consequence of the Completeness Hypothesis [136], and they are similar in spirit to other works pointing towards a lowering of the UV cut-off of effective theories in the limit of vanishing gravitino mass (see e.g. [112]).

## 2.A Basic Ingredients of 4D N=2 gauged Supergravity

In this appendix we give a very short description of some identities in Special-Kähler geometry, Quaternionic-Kähler geometry and in the gauging procedure of N=2 Supergravity. It is by no means a comprehensive summary of gauged N=2 supergravity theories, but it contains the ingredients required to reproduce the calculations of this chapter. For the derivation of what follows we refer the reader to the original works that we used [82, 137–140] and to the review [107].

We should start by stressing that, although a full N=2 duality covariant supergravity action has not been built so far, decisive steps have been taken in this direction. As

[4]See also [135] for further solutions of type II on non-compact group manifolds.

shown in [141], whenever one introduces magnetic gaugings, tensor multiplets have to be introduced. In the case of Supergravity coupled to vector multiplets, one has therefore to improve couplings to vector-tensor multiplets [142, 143] (and its extension to non-trivial FI terms in [144]). For the general matter-coupled case, an outline of the general procedure by using the embedding tensor formalism can be found in [145] and general Lagrangians for N=2 conformal supergravity theories with arbitrary gaugings have been presented in [146]. Our formulae are straightforward applications of the results contained in the above references.

### 2.A.1 Vector multiplets

The geometry described by the scalar fields appearing in N=2 vector multiplets coupled to Supergravity is called Special-Kähler geometry. A Special-Kähler manifold is parameterized by complex coordinates $z^I$, $I = 1, \dots, n_V$. Since this is the geometry of the vector-multiplet sector, electric-magnetic duality plays a role in constraining the manifold: this is made manifest by describing the geometry by means of holomorphic sections

$$Z^M = \begin{pmatrix} X^\Lambda(z) \\ F_\Lambda(z) \end{pmatrix}, \quad \Lambda = 0, I, \tag{2.A.1}$$

where the additional sections with index 0 have been added to take into account the graviphoton and its dual, which do not have corresponding scalars in their multiplet. When a prepotential $F(X)$ exists, these sections can also be though of as projective coordinates and $F_\Lambda = \partial_\Lambda F(X)$. However, special geometry can be defined in the absence of such a prepotential and, unless specified otherwise, we do not assume that the sections are chosen in such a specific frame.

Let us note that two different patches of the manifold are related by

$$Z'(z) = e^{-f(z)} S Z(z), \tag{2.A.2}$$

where $S$ is a constant symplectic matrix and $f$ is a holomorphic function of the coordinates, generating the Kähler transformations of the Kähler potential. Defining the symplectic product

$$\langle A, B\rangle = A^T \Omega B = A^\Lambda B_\Lambda - B^\Lambda A_\Lambda, \tag{2.A.3}$$

the Kähler potential is then

$$K = -\log\left[-i\langle Z, \bar{Z}\rangle\right] \tag{2.A.4}$$

and from (2.A.2), i.e. changing patches, we get the usual Kähler transformation

$$K'(z, \bar{z}) \to K(z, \bar{z}) + f(z) + \bar{f}(\bar{z}). \tag{2.A.5}$$

On the Hodge bundle over the manifold one can also define covariantly-holomorphic sections

$$V^M = e^{\frac{K}{2}} Z^M \tag{2.A.6}$$

such that the whole geometric structure gets encoded in the following algebraic and differential constraints:

$$\langle V, \overline{V} \rangle = i; \tag{2.A.7}$$

$$U_I = D_I V = (f_I^\Lambda, h_{I\Lambda}); \tag{2.A.8}$$

$$D_I U_J = i\, \hat{C}_{IJK}\, g^{K\bar{K}} \overline{U}_{\bar{K}}; \tag{2.A.9}$$

$$D_I \overline{U}_{\bar{J}} = g_{I\bar{J}}\, \overline{V}; \tag{2.A.10}$$

$$D_I \overline{V} = 0, \tag{2.A.11}$$

where now $D_I$ is the covariant derivative with respect to the usual Levi–Civita connection and the Kähler connection $\partial_I K$. This means that under a Kähler transformation (2.A.5), a generic field $\chi^I$ with charge $p$, namely transforming as $\chi^I \to e^{-\frac{p}{2}f + \frac{\bar{p}}{2}\bar{f}} \chi^I$, has covariant derivative

$$D_I \chi^J = \partial_I \chi^J + \Gamma^I_{JK} \chi^K + \frac{p}{2}\, \partial_J K\, \chi^I, \tag{2.A.12}$$

and analogously for $\overline{D}_{\bar{J}}$, with $p \to \bar{p}$. According to the standard conventions, which we follow all along this chapter, $p = -\bar{p} = 1$ for the weight of $V$. Note also that

$$g_{I\bar{J}} = i\, \langle U_I, \overline{U}_{\bar{J}} \rangle. \tag{2.A.13}$$

One more ingredient that is needed is the matrix defining the non-minimal couplings of the vector multiplets

$$\mathcal{N}_{\Lambda\Sigma} = \mathcal{R}_{\Lambda\Sigma} + i\, \mathcal{I}_{\Lambda\Sigma} = \left( M_\Lambda, \overline{h}_{\bar{I}} \right) \left( L^\Sigma, \overline{f}^\Sigma_{\bar{I}} \right)^{-1}, \tag{2.A.14}$$

which results in the kinetic Lagrangian for the vector multiplets

$$\mathcal{L}_{\text{kin.}} = \frac{1}{4}\, e\, \mathcal{I}_{\Lambda\Sigma}\, F^\Lambda_{\mu\nu} F^{\Sigma\mu\nu} + \frac{1}{8} \mathcal{R}_{\Lambda\Sigma}\, \epsilon^{\mu\nu\rho\sigma}\, F^\Lambda_{\mu\nu} F^\Sigma_{\rho\sigma}, \tag{2.A.15}$$

which means that $\mathcal{I}$ is negative definite.

The scalar potential following from the gauging procedure has two main contributions. The first contribution, $\mathcal{V}_F$, comes from the N=2 Fayet–Iliopoulos (FI) terms, which are the relics of the possible coupling to hypermultiplets. If we consider full symplectic invariance, the FI terms are given in terms of the triplet of FI charge vectors $Q^{Mx} = \left( P^{\Lambda x}, P^x_\Lambda \right)$, with $x = 1, 2, 3$:

$$\mathcal{V}_F = g^{I\bar{J}} \langle Q^x, U_I \rangle \langle Q^x, \overline{U}_{\bar{J}} \rangle - 3 \langle Q^x, V \rangle \langle Q^x, \overline{V} \rangle. \tag{2.A.16}$$

The second contribution is the D-term $\mathcal{V}_D$ generated by the proper gauging of the isometries of the Special-Kähler scalar manifold. Again, trying to be general and maintaining symplectic invariance, for Special-Kähler manifolds the isometries can be derived by looking at their linear action on the sections. In fact, all isometries must preserve (2.A.2) and therefore

$$\delta_P Z^M = (T_P)_N{}^M Z^N - f_P(z) Z^M, \tag{2.A.17}$$

where $T_P$ is a symplectic matrix (the generator of $S$) satisfying

$$T_\Lambda^T \Omega + \Omega T_\Lambda = 0, \tag{2.A.18}$$

and $f_N(z)$ are compensating holomorphic functions, which are going to be related to how the Kähler potential transforms under such isometries. Using full Sp($2n_V+2$,$\mathbb{R}$) indices:

$$T_{M[N}{}^Q \Omega_{P]Q} = 0. \tag{2.A.19}$$

Consistency of the gauging also requires

$$T_{(MN}{}^Q \Omega_{P)Q} = 0. \tag{2.A.20}$$

Note that now the position of the index transforming with $S$ is fixed, so that indices $M, N, \ldots$ are lowered and raised with the symplectic matrix. Upper indices transform with $S$ and lower indices transform with $S^{-1} = -\Omega S^T \Omega$, so that $V^M W_M = V^M \Omega_{MN} W^N$ is symplectic invariant:

$$V^{M\prime} W'_M = V^{M\prime} \Omega_{MN} W^{N\prime} = V^P S^M{}_P \Omega_{MN} S^N{}_Q W^Q = V^M \Omega_{MN} W^N = V^M W_M. \tag{2.A.21}$$

The non-linear action on the coordinates can be obtained by means of holomorphic Killing vectors, which can be related to the linear action above in frames where the prepotential exists. In this case, the Killing vectors follow by introducing *normal coordinates* $z^I \equiv X^I/X^0$:

$$\begin{aligned} \delta_M z^I &= \frac{\delta_M X^I}{X^0} - \frac{X^I}{X^0}\frac{\delta_M X^0}{X^0} = \\ &= \frac{(T_M Z)^I}{X^0} - \frac{X^I}{X^0}\frac{(T_M Z)^0}{X^0} \equiv k^I_M(z). \end{aligned} \tag{2.A.22}$$

At the infinitesimal level

$$\delta_M V^N = -T_{MP}{}^N V^P, \tag{2.A.23}$$

$$\delta_M W_N = T_{MN}{}^P W_P. \tag{2.A.24}$$

Under an isometry the Kähler potential transforms as

$$\delta_M K = -e^K\, i\,(\delta_M Z^T \Omega \overline{Z} + Z^T \Omega\, \delta_M \overline{Z}) = f_M + \overline{f}_M. \tag{2.A.25}$$

As it is customary in Supergravity, the gauging procedure is enforced by the introduction of prepotentials (or moment maps) for the gauged isometries. In this context, the prepotential definition is

$$P^0_M = -ik^i_M \partial_i K + i\, f_M , \tag{2.A.26}$$

which, in the frame where a prepotential exists, becomes

$$P^0_M = e^K\, \overline{Z}^T \Omega T_M Z = e^K\, T_{MN}{}^Q \Omega_{QP} Z^N \overline{Z}^P \,. \tag{2.A.27}$$

Prepotentials satisfy the constraint

$$Z^M(z) P^0_M(z, \bar{z}) = 0, \tag{2.A.28}$$

which also implies

$$Z^M(z) k^I_M(z) = 0. \tag{2.A.29}$$

The relations between the prepotentials and the Killing vectors also imply that

$$\bar{Z}^M(\bar{z}) k^I_M(z) = i\, g^{I\bar{J}} \overline{U}^M_{\bar{J}}\, P^0_M \,. \tag{2.A.30}$$

After the gauging, the resulting scalar potential is therefore

$$\mathcal{V}_{D_1} = \overline{V}^M k^I_M V^N \bar{k}^{\bar{J}}_N g_{I\bar{J}} = g^{I\bar{J}} U^M_I \overline{U}^N_{\bar{J}}\, P^0_M P^0_N . \tag{2.A.31}$$

### 2.A.2 Hypermultiplets

Hyper-scalars, $q^u (u = 1, ..., 4n_H)$, span a Quaternionic-Kähler manifold, namely a $4n_H$-dimensional real manifold endowed with an invertible metric $h_{uv}$ and a triplet of complex structures $(J^x)_u{}^v$, $x = 1, 2, 3$, satisfying the quaternionic algebra

$$J^x J^y = -\delta^{xy} \mathbb{1} + \epsilon^{xyz} J^z , \tag{2.A.32}$$

and with respect to which the metric is Hermitian

$$(J^x)_u{}^w (J^x)_v{}^t h_{wt} = h_{uv}. \tag{2.A.33}$$

From the complex structures one can introduce a triplet of 2-forms $K^x = h_{uw}(J^x)_v{}^w\, dq^u \wedge dq^v$, which are proportional to the curvatures of a SU(2) bundle with connections $\omega^x$, i.e.

$$R^x = d\omega^x + \frac{1}{2}\, \epsilon^{xyz} \omega^y \wedge \omega^z = -K^x. \tag{2.A.34}$$

This implies that the quaternionic structures are preserved by the SU(2) connection:

$$\nabla K^x = dK^x + \epsilon^{xyz} \omega^y \wedge K^z = 0. \tag{2.A.35}$$

This same structure also implies that for each isometry of the manifold, $\delta q^u = \epsilon^M k^u_M$, we can introduce a triplet of moment maps by

$$2\, R^x_{uv}\, k^v_M = \partial_u P^x_M + \epsilon^{xyz} \omega^y_u P^z_M. \tag{2.A.36}$$

and satisfy the consistency condition

$$R^x_{uv}\, k^u_M k^v_N + \frac{1}{2}\, \epsilon^{xyz} P^y_M P^z_N = \frac{1}{2}\, f_{MN}{}^P P^z_P, \tag{2.A.37}$$

required by gauge invariance of the N=2 action, where $f_{MN}{}^P$ are the structure constants of the gauge algebra.
Using the properties of the SU(2) curvatures, one can also find

$$2n_H\, P^x_M = -(R^x)_u{}^v \nabla_v k^u_M. \tag{2.A.38}$$

In the absence of hypermultiplets we can still introduce constant $P^x_M$, which correspond to the FI-terms of the previous subsection.

The gauging of a non-Abelian gauge group introduces a new D-term potential

$$\mathcal{V}_{D_2} = 4\, \overline{V}^M k^u_M V^N \bar{k}^v_N h_{uv} \tag{2.A.39}$$

and the F-term potential gets improved from the U(1) charges of the previous section to the full prepotentials $P^{Mx} = \left(P^{\Lambda x}, P^x_\Lambda\right)$:

$$\mathcal{V}_F = g^{I\bar{J}} \langle P^x, U_I \rangle \langle P^x, \overline{U}_{\bar{J}} \rangle - 3 \langle P^x, V \rangle \langle P^x, \overline{V} \rangle. \tag{2.A.40}$$

### 2.A.3 Potential, gravitino mass and charge

Summarizing, the scalar potential of a generic N=2 matter-coupled gauged supergravity theory can be written as the sum of three pieces

$$\mathcal{V} = \mathcal{V}_{D_1} + \mathcal{V}_{D_2} + \mathcal{V}_F, \tag{2.A.41}$$

$$\mathcal{V}_{D_1} = \overline{V}^M k^I_M V^N \bar{k}^{\bar{J}}_N g_{I\bar{J}} = g^{I\bar{J}} U^M_I \overline{U}^N_{\bar{J}}\, P^0_M P^0_N, \tag{2.A.42}$$

$$\mathcal{V}_{D_2} = 4\, \overline{V}^M k^u_M V^N \bar{k}^v_N h_{uv}, \tag{2.A.43}$$

$$\mathcal{V}_F = g^{I\bar{J}} \langle P^x, U_I \rangle \langle P^x, \overline{U}_{\bar{J}} \rangle - 3 \langle P^x, V \rangle \langle P^x, \overline{V} \rangle. \tag{2.A.44}$$

A general result of consistent gaugings is that we can always rotate the symplectic frame from which we start in the description of the Lagrangian so that the couplings and the potentials result from purely electric gaugings [108]. This means that, once we introduce new sections as in (2.A.2) with an appropriate symplectic matrix $S$, we can write the scalar potential above as

$$\mathcal{V} = \mathcal{V}_{D_1} + \mathcal{V}_{D_2} + \mathcal{V}_F, \tag{2.A.45}$$

$$\mathcal{V}_{D_1} = \overline{L}^\Lambda k^I_\Lambda \, L^\Sigma \bar{k}^{\bar{J}}_\Sigma \, g_{I\bar{J}} = U^{\Lambda\Sigma} P^0_\Lambda P^0_\Sigma, \tag{2.A.46}$$

$$\mathcal{V}_{D_2} = 4\, \overline{L}^\Lambda k^u_\Lambda \, L^\Sigma \bar{k}^v_\Sigma \, h_{uv}, \tag{2.A.47}$$

$$\mathcal{V}_F = g^{I\bar{J}} f^\Lambda_I \overline{f}^\Sigma_{\bar{J}} P^x_\Lambda P^x_\Sigma - 3\, L^\Lambda \overline{L}^\Sigma P^x_\Lambda P^x_\Sigma = \left( U^{\Lambda\Sigma} - 3 L^\Lambda \overline{L}^\Sigma \right) P^x_\Lambda P^x_\Sigma, \tag{2.A.48}$$

where we note the useful identity

$$U^{\Lambda\Sigma} = g^{I\bar{J}} f^\Lambda_I f^\Sigma_{\bar{J}} = -\frac{1}{2} \mathcal{I}^{-1|\Lambda\Sigma} - \bar{L}^\Lambda L^\Sigma. \tag{2.A.49}$$

(Clearly, in all these expressions the $L^\Lambda$ refer to the new frame $V'$).

From the full Lagrangian [107, 138, 140] we can also extract two ingredients that are central in our analysis, the gravitino mass matrix

$$S_{ij} = \langle P^x, V \rangle \, i \, (\sigma_x)_i{}^k \epsilon_{jk}, \tag{2.A.50}$$

and the physical charges of the gravitini, which, in the electric frame, are the eigenvalues of

$$(q_A)_i{}^j = \frac{1}{2} \mathcal{E}^\Lambda_A \, P^x_\Lambda (\sigma^x)_i{}^j, \tag{2.A.51}$$

where $\mathcal{E}^\Lambda_A \mathcal{E}^\Sigma_B \, \delta^{AB} = \mathcal{I}^{-1|\Lambda\Sigma}$. These are the charges to be used when determining the magnetic WGC cut-off.

## 2.B SO(4,2)/SO(4)×SO(2) coset space hyper-geometry

In this appendix we describe the Quaternionic–Kähler geometry of the coset space $\frac{\mathrm{SO}(4,2)}{\mathrm{SO}(4)\times\mathrm{SO}(2)}$, parametrized by the scalars $q^u$, $u = 1, ..., 8$.

We start from the SO(4,2) generators

$$(T_{\underline{ab}})_{\underline{c}}{}^{\underline{d}} = \eta_{\underline{c}[\underline{a}} \delta^{\underline{d}}_{\underline{b}]}, \tag{2.B.1}$$

where $\underline{a} = 1, \ldots 6$ is in the fundamental of so$(4, 2)$ and $\eta_{\underline{ab}} = \mathrm{diag}\{1, 1, 1, 1, -1, -1\}$.

We use $C_1 = T_{\underline{15}}$ and $C_2 = T_{\underline{36}}$ as the non-compact Cartan generators and introduce the following set of positive roots with respect to $C_1$ and $C_2$

$$\begin{aligned} &E^{(1,1)}_0 = \frac{1}{\sqrt{2}} \left( T_{\underline{12}} + T_{\underline{25}} - T_{\underline{16}} + T_{\underline{56}} \right), \quad E^{(1,-1)}_0 = \frac{1}{\sqrt{2}} \left( T_{\underline{12}} + T_{\underline{25}} + T_{\underline{16}} - T_{\underline{56}} \right), \\ &E^{(1,0)}_a = T_{\underline{13}} + T_{\underline{35}}, \quad E^{(1,0)}_b = T_{\underline{14}} + T_{\underline{45}}, \quad E^{(0,1)}_a = T_{\underline{23}} + T_{\underline{36}}, \quad E^{(0,1)}_b = T_{\underline{24}} + T_{\underline{46}}, \end{aligned} \tag{2.B.2}$$

where the superscripts denote the weights under the non-compact Cartans and the $a, b$ subscript distinguishes between generators of different weight under the remaining compact Cartan. Together with $C_1$ and $C_2$, these six generators form a basis for the tangent space of the coset space, which we collectively denote as

$$G_a = \big(C_1\ ,\ E_a^{(1,0)}\ ,\ E_b^{(1,0)}\ ,\ E_0^{(1,-1)}\ ,\ C_2\ ,\ E_0^{(1,1)}\ ,\ E_a^{(0,1)}\ ,\ E_b^{(0,1)}\big). \tag{2.B.3}$$

We then write the coset representative as

$$\begin{aligned}\mathbb{L} = \exp\Bigg[\Bigg(&q_6 - \frac{q_2 q_7}{\sqrt{2}} - \frac{q_3 q_8}{\sqrt{2}}\Bigg) E_0^{(1,1)} + \Bigg(q_2 + \frac{q_7 q_4}{\sqrt{2}}\Bigg) E_a^{(1,0)} + \Bigg(q_3 + \frac{q_8 q_4}{\sqrt{2}}\Bigg) E_b^{(1,0)} + \\ &+ q_7 E_a^{(0,1)} + q_8 E_b^{(0,1)} + q_4 E_0^{(1,-1)}\Bigg] \exp\left[\log(q_1) C_1\right] \exp\left[\log(q_5) C_2\right],\end{aligned} \tag{2.B.4}$$

from which we can read off the vielbeins $e^a{}_m$ through

$$G_a e^a{}_u dq^u = \mathbb{L}^{-1} d\mathbb{L}\,. \tag{2.B.5}$$

The resulting vielbein is

$$e^a{}_u = \begin{pmatrix} \frac{1}{\sqrt{2}q_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}q_1} & 0 & \frac{q_7}{q_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}q_1} & \frac{q_8}{q_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{q_5}{\sqrt{2}q_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}q_5} & 0 & 0 & 0 \\ 0 & -\frac{q_7}{q_1 q_5} & -\frac{q_8}{q_1 q_5} & -\frac{5(q_7^2+q_8^2)}{6\sqrt{2}q_1 q_5} & 0 & \frac{1}{\sqrt{2}q_1 q_5} & \frac{q_4 q_7}{3\sqrt{2}q_1 q_5} & \frac{q_4 q_8}{3\sqrt{2}q_1 q_5} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}q_5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}q_5} \end{pmatrix} \tag{2.B.6}$$

and the metric is then

$$\begin{aligned} ds^2 &= h_{uv} dq^u dq^v = \delta_{ab}\, e^a{}_u e^b{}_v dq^u dq^v \\ &= \frac{1}{q_1^2}\left[dq_1^2 + q_5^2 dq_4^2 + (dq_2 + \sqrt{2}q_7 dq_4)^2 + (dq_3 + \sqrt{2}q_8 dq_4)^2\right] \\ &\quad + \frac{1}{72 q_1^2 q_5^2}\left[6\sqrt{2}dq_6 - 12 q_7 dq_2 - 12 q_8 dq_3 + 2\sqrt{2}q_4(q_7 dq_7 + q_8 dq_8) - 5\sqrt{2}(q_7^2 + q_8^2)dq_4\right]^2 \\ &\quad + \frac{1}{q_5^2}\big(dq_5^2 + dq_7^2 + dq_8^2\big). \end{aligned} \tag{2.B.7}$$

The homogeneous nature of the scalar manifold allows us to find the Killing vectors of the SO(4,2) isometries by the action of the generators on the coset representative (see, for instance, [108]):

$$k^u_{T_{\mathrm{SO}(4,2)}} \partial_u \mathbb{L} = T_{\mathrm{SO}(4,2)} \mathbb{L} - \mathbb{L}\, w^H\, T_{H=\mathrm{SO}(4)\times\mathrm{SO}(2)}, \tag{2.B.8}$$

where the last term is the H-compensator and cancels the part of the transformation that moves along the coset. For each generator this is a set of fifteen equations ($\mathbb{L}$ has fifteen independent components) in fifteen unknowns (eight components for the Killing vector $k^u$ and seven coefficients for the compensator $w^H$). Finding the solution is straightforward, but the resulting expressions are very elaborate and we do not present them here.

Finally, we give here the quaternionic structures

$$J^1 = T_{\underline{12}} + T_{\underline{34}}\ , \quad J^2 = -T_{\underline{13}} + T_{\underline{24}}\ , \quad J^3 = T_{\underline{23}} + T_{\underline{14}}\,, \tag{2.B.9}$$

which correspond to a normal SU(2) subgroup of the SO(4). The action of the generators on the vielbeins defined in (2.B.6) can be deduced from their commutators with the corresponding generators such that

$$[J^x, e^a G_a] = (J^x)^a{}_b e^b G_a \tag{2.B.10}$$

and take the form

$$J^1 = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$J^2 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \tag{2.B.11}$$

$$J^3 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}.$$

# Chapter 3

# An instability towards goldstino condensation

Animated by the conclusions of the previous discussion, in the upcoming chapter we would like to push forward the cosmologically relevant investigation of the challenges to find de Sitter vacua in String Theory while adopting, within the spirit of the Swampland Program, an effective four-dimensional perspective.

As we have explained in the introduction, one of the central ingredients in typical string theory de Sitter constructions is the use of anti-D3-branes to uplift an AdS vacuum to a de Sitter critical configuration [29, 147–151]. Even though the end result of such uplift is often challenged [10], and the existence of tachyons or loss of criticality is implied [100–103], there is no conclusive indication that the four-dimensional effective field theory suffers from the alleged instabilities. (Recent selected possible issues from a ten-dimensional perspective are further reported in the articles [152–159]). If one assumes that there is a consistent low energy effective field theory that can incorporate the anti-D3-brane uplift, then one expects to be able to embed it within 4D N=1 Supergravity with the use of a nilpotent chiral superfield that breaks supersymmetry and provides the uplift [160–165]. Indeed, the embedding of the KKLT-type uplift within Supergravity coupled to a nilpotent superfield was discussed in [166, 167], and the underlying non-linear supersymmetry of the anti-brane uplift was made manifest. More generally, the appearance of a goldstino sector on anti-Dp-brane worldvolumes was also investigated in [168–171].

If the aforementioned simple de Sitter constructions are unstable, one of the cleanest ways to see this would be to uncover this instability in the low energy 4D N=1 supergravity description. To support this approach, let us observe that there do exist hints that the Volkov–Akulov (VA) model coupled to pure 4D N=1 Supergravity [172, 173] is inherently unstable. This is seen by considering a real scalar representing a gravitino condensate

$\langle \psi_m \psi^m \rangle \sim \sigma(x)$, and studying the effective theory à la Nambu–Jona-Lasinio [174, 175] (also similarly to composite Higgs models [176]) for the four-gravitini interaction. Such an analysis has been performed in the early supergravity bibliography in [177, 178] to yield the effective scalar potential, but the actual existence of a condensate was later questioned [179, 180]. More recently, the condensation was revisited and put on firmer footing in [181–183]. Furthermore, both the Fierz ambiguity and the wavefunction renormalization of the condensate were addressed [184]. A comprehensive and analytic review of all this work can be found in [185]. The crucial result in these articles pertains to the behaviour of the effective scalar potential for the condensate $\sigma(x)$, which turns out to be *tachyonic* around the original central vacuum. This, therefore, indicates that an uplift from an AdS vacuum to a de Sitter one with a pure non-linear realization may be inherently unstable. The drawback of these articles, however, is that a manifestly supersymmetric analysis is missing and a precise and controlled form of the effective theory for the condensates is elusive. This also means that, if one wanted to extend the analysis to a matter coupled Supergravity, for example to the Kähler modulus of KKLT, the full procedure would have to be performed from scratch.

In this chapter we wiil report a first step towards filling this gap by providing a manifestly supersymmetric description of the composite states.
One could suspect that the effect that manifests itself in a supergravity framework as gravitino condensates may already exist in the rigid limit in the form of goldstino condensates. Indeed, such behaviour of the Volkov–Akulov fermion can be justified by the fact that the theory contains four-Fermi interactions, just like the fermions in the Nambu–Jona-Lasinio model. In a typical Nambu–Jona-Lasinio setup the way in which the effective theory for the composite states is uncovered is by recasting the four-Fermi interactions $\left(\overline{\Psi}\Psi\right)^2$ as $-\sigma^2 + \sigma\overline{\Psi}\Psi$, with $\sigma$ taken to be auxiliary at some UV scale, and then following the flow of the theory to the IR. This generates a kinetic term for the scalar $\sigma$ and also gives rise to new contributions to its effective potential, which leads to the formation of a new vacuum where the condensation takes place. However, for a single goldstino, a large N expansion (N being the number of fermion species) that helps to control the diagrams in the Nambu–Jona-Lasinio model is not available; as a consequence, a perturbative loop-diagram analysis is not tractable and a direct non-perturbative analysis is required: the Nambu–Jona-Lasinio model can be discussed for a single fermion species if a non-perturbative renormalization group flow is utilized [186].

The procedure that we are going to adopt in the subsequent sections will be then the following. We will first recast the full Volkov–Akulov model in terms of two unconstrained superfields, $X$ and $T$, where the latter is a Lagrange multiplier that is integrated out to impose the nilpotency condition $X^2 = 0$ on the former [187–189]. Once the nilpotency is imposed, we recover the typical Volkov–Akulov model. To uncover the low energy description of the theory, and possibly the existence of light composite states, we have to track the flow of the theory towards the IR. We will do this by following the flow determined by the *exact renormalization group* (ERG) equations within a *supersymmetric* rendition of the local potential approximation (which we will denote as SLPA) [190–196]. In particular, we will perform this analysis preserving supersymmetry off-shell and we will keep track of the full flow of the Kähler potential. Our final result will verify the emergence of composite states, which fall into standard chiral multiplets with *linearly realized* supersymmetry, with an effective low energy description where the Volkov–Akulov Kähler potential $K_{\text{VA}} = X\overline{X}$ is replaced by

$$K_{\text{composite states}} = Z_X \, X\overline{X} + Z_T \, T\overline{T} + \text{higher-order terms}\,, \tag{3.i}$$

where the $Z$'s indicate the wavefunction renormalizations, and where the superpotential remains the same as in the original Volkov–Akulov model,

$$W = fX + \frac{1}{2}TX^2. \tag{3.ii}$$

A study of the potential of the resulting supersymmetric theory will show that tachyons are generated near the origin of the $(X, T)$ field space, signaling an inherent non-perturbative instability of the pure Volkov–Akulov model. The compositeness of $T$ is manifested by the vanishing $Z_T$ at the UV point of the flow[1].

A further step would be to perform a similar procedure in Supergravity using again an ERG flow [201, 202]. An important obstacle to performing this analysis in Supergravity is the requirement of a supersymmetric regulator. However, for small fields, one can trust the supersymmetric analysis: the supergravity effects should only enter as we probe larger distances in field space, where the $1/M_P$ effects will become important. Therefore, the same low energy effective description for the Volkov–Akulov composite states can be justified to hold also in Supergravity, but only near the field space origin. We will use this approximation to show that the Volkov–Akulov model, now coupled to 4D N=1 Supergravity as in [163], again suffers from tachyonic instabilities; this verifies the earlier results regarding tachyons due to gravitino condensation [178, 184]. The same holds also for the KKLT supergravity embedding, when we introduce a single Kähler modulus [166]. Our results will actually indicate that, if a 4D N=1 supergravity theory flows in the IR

[1]Our approach can be described as replacing a Lagrangian where supersymmetry is non-linearly realized by an equivalent low energy model where supersymmetry has, instead, a linear realization. We could therefore say that we have an emergence of supersymmetry in the IR. Although the concept of "emergent supersymmetry" has previously appeared in [199, 200], one should investigate whether our findings could be embedded in that framework.

to the model of [163], then it will suffer from the same instability (as it happens for the KKLT 4D N=1 embedding of [166]).
The only way to avoid the instability associated with the formation of these composite states would be to always have some additional light states (possibly of non-perturbative origin) surviving in the IR and either disrupt the procedure that we described or alter the flow.

## 3.1 Composite supersymmetry from the VA model

### 3.1.1 The setup

The four-dimensional Volkov–Akulov model [172] can be described in terms of a constrained superfield $X$ that satisfies [187, 188]

$$X^2 = 0, \quad X| = \frac{G^2}{2F^X}, \tag{3.1.1}$$

with the Lagrangian, defined in terms of a Kähler potential $K$ and a superpotential $W$,

$$\mathcal{L} = \int d^4\theta\, K + \left(\int d^2\theta\, W + \text{c.c.}\right) = \int d^4\theta |X|^2 + \left(\int d^2\theta f X + \text{c.c.}\right). \tag{3.1.2}$$

This procedure is further described in [189] and the component form of the Lagrangian is

$$\mathcal{L} = -f^2 + iG\sigma^m \partial_m \overline{G} - \frac{1}{4f^2}\overline{G}^2 \partial^2 G^2 - \frac{1}{16f^6} G^2 \overline{G}^2 \partial^2 G^2 \partial^2 \overline{G}^2 \tag{3.1.3}$$

(using the mostly-minus convention for the spacetime metric, $\eta_{mn} = \text{diag}(+1, -1, -1, -1)$). The presence of the non-linear self-interaction terms opens the possibility of composite states of two or more goldstini, e.g. $G^2/f$. We wish to investigate the possible description of such states in the low energy theory[2].

First, we bring the theory into a form where supersymmetry is linearly realized, and use it as the form of the theory at the UV point. To this end we introduce a Lagrange multiplier multiplet $T$ (associated with no term in the Kähler potential) by changing the superpotential to

$$W = fX + \frac{1}{2}TX^2. \tag{3.1.4}$$

By varying $T$ we recover the superspace condition (3.1.1). Note that the Volkov–Akulov model naturally comes with a UV scale given by the supersymmetry breaking scale, which

[2] Note that, classically, the various forms of the Volkov–Akulov model have been shown to be equivalent [203, 204], and always reduce to (3.1.3) in the component form.

is controlled by $f$; it is therefore natural, though not mandatory, to match the two descriptions at that energy scale.

At this point, let us prove that (3.1.4) is the most general form of the superpotential that one can write in the UV point and that $K = |X|^2$ is the most general Kähler potential.

We firstly observe that, by definition, the Kähler potential has no dependence on $T$ in the UV point. We then assume that the superpotential takes the form $W = fX + \frac{1}{2}TX^2 + P(X)$, where $P(X) = \sum_{n\geq 0} P_n X^n = P_0 + P_1 X + P_2 X^2 + P_3 X^3 + \ldots$ is some arbitrary analytic function (as it can always taken to be, since we are dealing with a superpotential), and we further note that we can not have terms like $1/X^p$ with $p > 0$, because they will become ill-defined once the nilpotency condition is imposed. Now, $P_0$ clearly drops out due to the $\int d^2\theta$ and $P_1$ can be absorbed into $f$. So, we remain with $fX + \frac{1}{2}TX^2 + X^2\tilde{P}(X)$, where $\tilde{P}(X) = P_2 + P_3 X + P_4 X^2 + \ldots$, which is still granted to be an analytic function of $X$. If we simply shift $T$ to $T - 2\tilde{P}(X)$, we are left with our original superpotential (3.1.4). This shift is consistent precisely because the function $\tilde{P}(X)$ is analytic. Moreover, there are no $T$-dependent terms in the Kähler potential, so this shift does not generate new terms.

Analogously, let us consider the Kähler potential $K = |X|^2 + \sum_{m,n\geq 0} M_{nm} X^n \overline{X}^m$ assuming by *fiat* that it can be expanded in a power series (as part of our general assumptions). We can directly see that $M_{00}$, $M_{01}$ and $M_{10}$ drop out due to the $\int d^4\theta$, whereas $M_{11}$ is absorbed by redefining the $|X|^2$ term. Then, we are left with $K = |X|^2 + X^2\tilde{M}(X,\overline{X}) + \overline{X}^2\left(\tilde{M}(X,\overline{X})\right)^*$, where $\tilde{M}(X,\overline{X}) = \sum_{m,n\geq 0} \tilde{M}_{nm} X^n \overline{X}^m$. Now, once simply shifting $T$ to $T + \frac{1}{2}\overline{D}^2\tilde{M}(X,\overline{X})$, we are left with our original Kähler potential.

We are thus working with the most general Kähler potential and superpotential at the UV point.

The second step of our procedure is to lower the energy scale at which we probe the theory via a renormalization group (RG) flow. This makes the superfield $T$ acquire a kinetic term: the interpretation of this is that the new standard chiral superfields $X$ and $T$ now describe composite states of the original fermion goldstino. This means that the partition function of the system has the form

$$\mathcal{Z} = \frac{1}{N_0}\int \mathcal{D}[T]\mathcal{D}[X]\, e^{iS[T,X]}. \tag{3.1.5}$$

Notice that, once we start integrating out individual components of the superfields, up to the overall normalization, the partition function reduces to $\int \mathcal{D}[G]\, e^{iS[G]}$, where $S[G]$ comes from (3.1.3). We therefore get back the partition function for the Volkov–Akulov model.

We will study the RG flow by using the exact renormalization group method discussed in [190, 191, 196]. To follow this method it is typical to introduce all possible terms in the Kähler potential and the superpotential that are consistent with the symmetries that are expected to remain preserved, such that the flow can be properly described. Some terms may stay trivially zero throughout the flow depending on the initial conditions and the subsequent flow. Since such a procedure actually requires to introduce *infinite* terms, in order to make it tractable the local potential approximation (LPA) has been devised (see e.g. [192–195]). The idea standing behind the LPA is that it may be advantageous to apply approximations directly at the level of a differential equation rather than applying approximations at the level of the solution. In this way the LPA simply works by truncating all the contributions of derivative interactions to the ERG equations, which are expected to be irrelevant to the IR dynamics, anyway. For a supersymmetric theory, however, since the supersymmetry transformations contain derivatives, different orders of derivative interactions mix: a crude LPA would then explicitly violate supersymmetry. Here, we then use a type of local potential approximation that allows us to preserve supersymmetry manifestly, which we simply call, as already mentioned in the introductory remarks to this chapter, the *supersymmetric* local potential approximation (SLPA). It is easier to describe this approximation directly in the superspace language[3]. First of all, supersymmetric theories with chiral superfields $\mathcal{X}^i$ are in any case described by two types of superspace integrals, which are $\int d^4\theta$ and $\int d^2\theta$. If we think of the functions that we can insert in those integrals as having a (superspace) derivative expansion, then we can have

$$\int d^4\theta \left( K(\mathcal{X}^i, \overline{\mathcal{X}}^j) + \mathcal{O}\left(\partial_m, D_\alpha, \overline{D}_{\dot{\beta}}\right) \right) + \left( \int d^2\theta\, W(\mathcal{X}^i) + \text{c.c.} \right), \tag{3.1.6}$$

where the first term corresponds to the Kähler potential and the second one to the superpotential. Our SLPA can be now simply defined by stating that we do not keep track of superspace higher derivative terms. This means that any term in (3.1.6) that is not a a part of $K$ or $W$ are always ignored. Similarly, since we are going to work in component form, we are not going to keep track of any terms that do not correspond to a Kähler potential or a superpotential. We are going to ignore these terms when they are generated during the flow and we are not going to take into account their backreaction into the ERG equations.

Even though these types of approximations are in standard use in the ERG literature,

[3] A superspace account of the RG flow has been presented in [205], but for our purposes we will ultimately work with the component form. Other examples of truncating the flow equations based on a superderivative expansion include [206, 207].

it is important to understand and be conscious of their limitations regarding the physical conclusions that one can draw. We will discuss these issues carefully in the next section. For now, we proceed within this approximation and derive the SLPA RG flow of our model. We will, however, note immediately that in our chosen approach, even though we will not be able to see the anomalous dimensions of the fields, we will have a certain handle on the wavefunction renormalizations because of the presence of other fields besides the physical scalars.

Even within the LPA, one still makes an expansion in the fields to derive the RG flow, which means that we still need to make an expansion in terms of superfields in the Kähler potential. In our case, we will make an educated guess and keep only the terms that provide a self-consistent flow. Indeed, the gratifying result of applying the SLPA is that we get a solution to the ERG in a closed form that includes only a handful of new terms in the Kähler potential. As we will see, it suffices to work with the Kähler potential

$$K = \alpha|X|^2 + \beta|T|^2 + g\,|T|^2|X|^2 + \frac{1}{4}q\,|X|^4 \tag{3.1.7}$$

and the superpotential

$$W = fX + \frac{1}{2}TX^2, \tag{3.1.8}$$

where we have both dimensionful and dimensionless couplings: $[\alpha] = 0$, $[\beta] = 0$, $[g] = -2$ and $[q] = -2$. This theory is defined with a UV cut-off $\Lambda$. We want to study its properties as we integrate out the high energy modes down to a lower scale $\mu$, with

$$\mu \leq \Lambda. \tag{3.1.9}$$

This is often called the renormalization scale and it is related to the energy scale at which we probe the theory. Then, the renormalization time is defined as

$$t = \log\frac{\Lambda}{\mu}, \tag{3.1.10}$$

and the flow to the IR is described by $\mu \downarrow$ and $t \uparrow$. To match with the Volkov–Akulov action at the UV point, that is when $\mu = \Lambda$, we want to find the RG flow of the couplings with the boundary conditions

$$\alpha\Big|_{\mu=\Lambda} = 1\,, \quad \beta\Big|_{\mu=\Lambda} = 0\,, \quad g\Big|_{\mu=\Lambda} = 0\,, \quad q\Big|_{\mu=\Lambda} = 0. \tag{3.1.11}$$

Indeed, we notice that $\beta(\mu = \Lambda) = 0$ and $g(\mu = \Lambda) = 0$ are required in order for the superfield $T$ to act as a Lagrange multiplier at the UV scale and $q(\mu = \Lambda) = 0$ because, as we have showed above, $K = |X|^2$ is the most general expression of the UV Kähler potential that we can have.

Following [190, 191, 196], we re-organize the action into propagator and interaction parts. We do this in a supersymmetric manner by appropriately splitting the Kähler potential. In particular, we have

$$K_{\text{prop.}} = c^{-1}|X|^2 + c^{-1}|T|^2, \tag{3.1.12}$$

where $c$ is a scale-dependent regularization function, which we will discuss momentarily, with $[c] = 0$, and

$$K_{\text{int.}} = (\alpha - 1)|X|^2 + (\beta - 1)|T|^2 + g|T|^2|X|^2 + \frac{1}{4}q|X|^4. \tag{3.1.13}$$

Note that we leave the background field dependence in the interaction part of the Kähler potential. Similarly, the superpotential naturally only contributes to the interactions:

$$W_{\text{int.}} = W. \tag{3.1.14}$$

Moreover, since no background-independent mass term exists neither for $X$ nor for $T$ in the UV, and it will not be generated during the flow[4], we do not include a mass for these fields in the propagator piece (3.1.12).

Returning to the propagator regularization function $c$, working in Euclidean momentum space, we can write

$$c = \sum_{n=0}^{+\infty} c_n \hat{p}^{2n} \quad \text{with} \quad \hat{p} = \mu^{-1} p, \tag{3.1.15}$$

as long as some basic asymptotic properties are satisfied [196], where now we have $|\hat{p}| \leq 1$. For our calculations we do not need to work with an explicit form for $c$, but for the benefit of the reader we can give as a simple example the expression $c(p, \mu) = (1 - \hat{p}^2)\,\Theta\,(1 - \hat{p}^2)$, which is discussed in [196–198]. Such a regulator is manifestly supersymmetric, because the component field propagator terms are collectively described by the superspace integral $\int d^4\theta\, (c(-\partial^2))^{-1}\,(|X|^2 + |T|^2)$ and the superspace derivatives (the $D's$) commute with any combination of spacetime derivatives.

In addition, since we are only interested in the vacuum (in)stability of the theory, we do not include any source terms in our analysis or any derivative interactions. As discussed in [191], this allows, together with the use of the $c$ function (3.1.15) in the propagator piece, to simplify the calculations. We importantly stress that the choice of regulator does

[4]This is true at least perturbatively, and can also be checked explicitly within our SLPA analysis.

not ultimately affect any physical results in the IR. We further define the renormalization time derivative of the regulator $c$ as

$$\dot{c} \equiv \frac{\partial}{\partial t} c = -\mu \partial_\mu c = p\partial_p c = \hat{p}\partial_{\hat{p}} c = 2\hat{p}^2 c_1 + \mathcal{O}(\hat{p}^4). \tag{3.1.16}$$

In our procedure we are going to heavily rely on supersymmetry and, because we work directly in component form, this means that, within the SLPA, we will only evaluate the flow of the coefficients of the auxiliary field potential that are related to the Kähler potential. In other words, we will only keep track of the terms of the form

$$\mathcal{L} = g_{i\bar{j}} F^i \overline{F}^j + \dots, \tag{3.1.17}$$

and from these terms we will deduce the full flow of the Kähler potential. A similar procedure is used in [208] for the evaluation of the one-loop Kähler potential. For the study of the flow we will be using the Euclidean conventions of [209], where the Lorentzian conventions correspond to $\eta_{mn} = \mathrm{diag}(+1, -1, -1, -1)$, which also matches with the conventions of [190].

We now go to components and work with the momentum space fields. We define

$$\hat{X}(\hat{p}) = \frac{1}{\sqrt{2}}(\phi + i\chi), \quad \hat{T}(\hat{p}) = \frac{1}{\sqrt{2}}(\tau + i\sigma), \tag{3.1.18}$$

and

$$\hat{F}^X(\hat{p}) = F_1 + iF_2, \quad \hat{F}^T(\hat{p}) = B_1 + iB_2, \tag{3.1.19}$$

which have vanishing mass dimensions $[\phi] = 0$, $[F_1] = 0$, etc. We further follow [190] and obtain the propagator part of the action[5]

$$\begin{aligned} L_{\text{prop.}} = &\int \frac{d^4\hat{p}}{(2\pi)^4} \left[ -\frac{1}{2}\hat{p}^2 c^{-1}(\phi^2 + \chi^2 + \tau^2 + \sigma^2) \right] + \\ &+ \int \frac{d^4\hat{p}}{(2\pi)^4} \left[ c^{-1}(F_1^2 + F_2^2 + B_1^2 + B_2^2) \right] + \\ &+ \text{fermion propagator terms}, \end{aligned} \tag{3.1.20}$$

where we have $\phi^2 = \phi(\hat{p})\phi(-\hat{p})$, $F_1^2 = F_1(\hat{p})F_1(-\hat{p})$, etc. We also define dimensionless couplings via

$$\gamma = \mu^2 g(\mu) \quad \text{and} \quad \zeta = \mu^2 q(\mu), \tag{3.1.21}$$

so that $[\gamma] = 0$ and $[\zeta] = 0$. For the interacting part of the component field action we have

$$\begin{aligned} L_{\text{int.}} = &\frac{1}{2} B_1(\phi^2 - \chi^2) + F_1(\phi\tau - \chi\sigma) + (\alpha - 1)F_1^2 + (\beta - 1)B_1^2 + \\ &+ \frac{\gamma}{2}\left[ F_1^2(\tau^2 + \sigma^2) + B_1^2(\phi^2 + \chi^2) \right] + \frac{\zeta}{2} F_1^2(\phi^2 + \chi^2) + \dots, \end{aligned} \tag{3.1.22}$$

[5] To get to Euclidean momentum space, we first Wick-rotate and then we go to momentum space.

where the dots stand for many other interaction terms and many terms including fermions. Here $L_{\text{int.}}$ is only a formal compact expression and it really means that we should treat all terms in the expansion in the form

$$L_{\text{int.}} = \int \frac{d^4\hat{p}_1 \dots d^4\hat{p}_n}{(2\pi)^{4n-4}} \; \hat{Y}_{A_1 \dots A_n}(t) \; \hat{\Psi}_{A_1}(\hat{p}_1) \dots \hat{\Psi}_{A_n}(\hat{p}_n) \; \delta\left(\sum_{i=1}^{n} \hat{p}_i\right), \tag{3.1.23}$$

with $[\hat{Y}_{A_1 \dots A_n}(t)] = 0$ and $[\hat{\Psi}_{A_i}(\hat{p}_i)] = 0$ (for any $A_i$). A sample of illustrative terms gives

$$\begin{aligned} L_{\text{int.}} = & \int \frac{d^4\hat{p}_1 d^4\hat{p}_2}{(2\pi)^4} \; (\alpha(t) - 1) \; F_1(\hat{p}_1) F_1(\hat{p}_2) \; \delta(\hat{p}_1 + \hat{p}_2) \; + \\ & + \int \frac{d^4\hat{p}_1 d^4\hat{p}_2 d^4\hat{p}_3 d^4\hat{p}_4}{(2\pi)^{12}} \frac{\gamma(t)}{2} F_1(\hat{p}_1) F_1(\hat{p}_2) \tau(\hat{p}_3) \tau(\hat{p}_4) \delta(\hat{p}_1 + \hat{p}_2 + \hat{p}_3 + \hat{p}_4) \\ & + \dots . \end{aligned} \tag{3.1.24}$$

For completeness, let us mention that the full Euclidean partition function is

$$\mathcal{Z} = \int \mathcal{D}[X] \mathcal{D}[T] \, e^{L_{\text{prop.}} + L_{\text{int.}}}, \tag{3.1.25}$$

just as in [190]. As already mentioned, the symbol $L$ clearly refers to an action, but we keep the notation of [190], using the symbol $L$ instead of $S$. Note also, once again, that all fields, momenta and couplings are dimensionless.

We can now use the ERG equation from [190] to obtain

$$\begin{aligned} \dot{L}_{\text{int.}} = & - \int d^4\hat{p} \; \frac{(2\pi)^4}{2} \; \hat{p}^{-2} \dot{c}(\hat{p}) \sum_{\varphi^a = (\phi, \chi, \tau, \sigma)} \left( \frac{\partial L_{\text{int.}}}{\partial \varphi^a} \frac{\partial L_{\text{int.}}}{\partial \varphi^a} + \frac{\partial^2 L_{\text{int.}}}{\partial \varphi^a \partial \varphi^a} \right) + \\ & + \frac{1}{4} \int d^4\hat{p} \; (2\pi)^4 \; \dot{c}(\hat{p}) \sum_{h^a = (F_1, F_2, B_1, B_2)} \left( \frac{\partial L_{\text{int.}}}{\partial h^a} \frac{\partial L_{\text{int.}}}{\partial h^a} + \frac{\partial^2 L_{\text{int.}}}{\partial h^a \partial h^a} \right) + \\ & + \text{fermion propagator terms.} \end{aligned} \tag{3.1.26}$$

We have independent sums in the ERG equation (3.1.26) because all our scalars have diagonal propagator terms. Notice that in a slight abuse of notation, we denote by partial derivatives what should really be understood as variational derivatives, with a momentum matching $\delta$-function, i.e., say,

$$\frac{\partial \varphi^a(p)}{\partial \varphi^b(k)} = \delta^a_b \delta^{(4)}(p - k). \tag{3.1.27}$$

The lack of a $(2\pi)^4$ factor on the $\delta$-function above is due to the fact that it is already included explicitly in the expression (3.1.26). This choice of notation and normalization, which will be used throughout the rest of this section, corresponds to the one in [190].

At this stage we have to insert the interacting action (3.1.24) into the ERG equation and equate term by term to deduce the flow equations. We stress that one only needs to look at the terms related to the auxiliary field potential, namely

$$F_1^2\,, \quad \tau^2 F_1^2\,, \quad \text{etc..} \tag{3.1.28}$$

To this end the third line in (3.1.26) does not play any role as one can check by considering the fermionic contribution to the ERG (see e.g. [191]) and the fermion couplings to the auxiliary fields, which are only linear in the auxiliary fields (see e.g. [6]). Therefore, we will focus on the first two lines of (3.1.26) and we will only use the fermions as a cross-check.

One can easily prove that the superpotential does not receive any corrections within the SLPA by checking that no terms linear in the auxiliary fields are generated. This can be seen faster by writing (3.1.26) in a form where the scalars and the auxiliary fields are recast to be complex by a simple chain rule.

### 3.1.2 The RG flow within the supersymmetric local potential approximation

Our aim is to find the flow equations for the couplings $\alpha$, $\beta$, $\zeta$ and $\gamma$.

Let us first look at the equation governing the flow of $\zeta$. This means that on the left hand side of (3.1.26) we want to focus on the term

$$\frac{1}{2}(\dot{\zeta} + 2\zeta)F_1^2\phi^2 = \int \frac{\prod_{i=1}^4 d^4\hat{p}_i}{(2\pi)^{12}} \frac{1}{2}(\dot{\zeta} + 2\zeta)F_1(\hat{p}_1)F_1(\hat{p}_2)\phi(\hat{p}_3)\phi(\hat{p}_4)\delta\left(\sum_{i=1}^4 \hat{p}_i\right). \tag{3.1.29}$$

On the right hand side we have the term

$$-\int d^4\hat{k}\frac{(2\pi)^4}{2}\hat{k}^{-2}\dot{c}(\hat{k})\frac{\partial(F_1\phi\tau)}{\partial\tau(\hat{k})}\frac{\partial(F_1\phi\tau)}{\partial\tau(-\hat{k})}\,, \tag{3.1.30}$$

and a variety of seemingly relevant terms related to the auxiliary field propagators as, for instance, the term

$$+\int d^4\hat{k}(2\pi)^4\dot{c}(\hat{k})\left(\frac{\partial\left[(\alpha-1)F_1^2\right]}{\partial F_1(\hat{k})}\frac{\partial\left(\frac{\zeta}{2}F_1^2\phi^2\right)}{\partial F_1(-\hat{k})}\right). \tag{3.1.31}$$

As we will now see, only the term (3.1.30) actually contributes to this part of the flow, whereas (3.1.31) contributes to derivative interactions. Indeed, up to an overall coefficient, (3.1.31) gives

$$(\alpha-1)\zeta\int d^4\hat{p}_1 d^4\hat{p}_2 d^4\hat{p}_3 d^4\hat{p}_4\,\dot{c}(\hat{p}_1)\,F_1(\hat{p}_1)F_1(\hat{p}_2)\phi(\hat{p}_3)\phi(\hat{p}_4)\,\delta\left(\sum_i \hat{p}_i\right), \tag{3.1.32}$$

which means that this term contributes only to derivative terms: in fact, from (3.1.16) we have

$$\int d^4\hat{p}_1\, \dot{c}(\hat{p}_1)F_1(\hat{p}_1) \sim \int d^4\hat{p}_1 \left(2\hat{p}_1^2 c_1 + \mathcal{O}(\hat{p}_1^4)\right) F_1(\hat{p}_1)\,. \tag{3.1.33}$$

In a similar way, we can see that the only relevant part of (3.1.30) is given by the $c_1$ part of the expansion

$$\frac{1}{2}\hat{k}^{-2}\dot{c}(\hat{k}) = c_1 + \mathcal{O}(\hat{k}^2)\,. \tag{3.1.34}$$

We then evaluate

$$\begin{aligned}\frac{\partial(F_1\phi\tau)}{\partial\tau(\hat{k})} &= \int \frac{d^4\hat{p}_1 d^4\hat{p}_2 d^4\hat{p}_3}{(2\pi)^8} F_1(\hat{p}_1)\phi(\hat{p}_2)\delta(\hat{p}_3-\hat{k})\delta(\hat{p}_1+\hat{p}_2+\hat{p}_3) = \\ &= \int \frac{d^4\hat{p}_1 d^4\hat{p}_2}{(2\pi)^8} F_1(\hat{p}_1)\phi(\hat{p}_2)\delta(\hat{p}_1+\hat{p}_2+\hat{k})\,,\end{aligned} \tag{3.1.35}$$

and

$$\frac{\partial(F_1\phi\tau)}{\partial\tau(-\hat{k})} = \int \frac{d^4\hat{p}_3 d^4\hat{p}_4}{(2\pi)^8} F_1(\hat{p}_3)\phi(\hat{p}_4)\delta(\hat{p}_3+\hat{p}_4-\hat{k})\,. \tag{3.1.36}$$

Finally, (3.1.30) becomes

$$\begin{aligned}-c_1(2\pi)^4 \int d^4\hat{k} \frac{\partial(F_1\phi\tau)}{\partial\tau(\hat{k})}\frac{\partial(F_1\phi\tau)}{\partial\tau(-\hat{k})} &= \\ = -c_1 \int \frac{\prod_{i=1}^4 d^4\hat{p}_i}{(2\pi)^{12}} F_1(\hat{p}_1)\phi(\hat{p}_2)F_1(\hat{p}_3)\phi(\hat{p}_4)\delta\left(\sum_{i=1}^4 \hat{p}_i\right)&.\end{aligned} \tag{3.1.37}$$

This means that in the compact notation (3.1.23) the relevant part of (3.1.26) takes the form $\frac{1}{2}(\dot{\zeta}+2\zeta)F_1^2\phi^2 = -c_1 F_1^2\phi^2$, which delivers

$$\dot{\zeta} = -2\zeta - 2c_1\,, \tag{3.1.38}$$

where the $-2\zeta$ is due to the fact that $\zeta$ originates from a dimensionful coupling. Similarly, for the coupling $\gamma$ we are bound to get

$$\dot{\gamma} = -2\gamma - 2c_1\,. \tag{3.1.39}$$

In this way we see that the tree level interactions from the superpotential contribute to the flow of the higher order terms in the Kähler potential.

We now analyze the equations that govern the flow of $\beta$. The relevant term of left hand side of (3.1.26) is

$$\dot{\beta}B_1^2 = \int \frac{d^4\hat{p}_1 d^4\hat{p}_2}{(2\pi)^4} \dot{\beta}B_1(\hat{p}_1)B_1(\hat{p}_2)\delta(\hat{p}_1+\hat{p}_2)\,, \tag{3.1.40}$$

and on the right hand side we have

$$-\int d^4\hat{k}\frac{(2\pi)^4}{2}\hat{k}^{-2}\dot{c}(\hat{k})\sum_{\varphi^a=(\phi,\chi,\tau,\sigma)}\frac{\partial^2\left[\frac{\gamma}{2}B_1^2(\phi^2+\chi^2)\right]}{\partial\varphi^a(\hat{k})\partial\varphi^a(-\hat{k})}, \tag{3.1.41}$$

and a variety of seemingly relevant terms related to the auxiliary field propagators, as the term

$$+\int d^4\hat{k}(2\pi)^4\dot{c}(\hat{k})\sum_{h^a=(F_1,F_2,B_1,B_2)}\left(\frac{\partial\left[(\beta-1)B_1^2\right]}{\partial h^a(\hat{k})}\frac{\partial\left[(\beta-1)B_1^2\right]}{\partial h^a(-\hat{k})}\right). \tag{3.1.42}$$

As we can see right away, the term (3.1.42) and other similar terms from the second line of (3.1.26) do not enter this part of the flow and can be safely ignored. Indeed, focusing on (3.1.42) we find

$$\begin{aligned}
&\int d^4\hat{k}(2\pi)^4\dot{c}(\hat{k})\sum_{h^a=(F_1,F_2,B_1,B_2)}\left(\frac{\partial\left[(\beta-1)B_1^2\right]}{\partial h^a(\hat{k})}\frac{\partial\left[(\beta-1)B_1^2\right]}{\partial h^a(-\hat{k})}\right)=\\
&=4(\beta-1)^2\int d^4\hat{k}(2\pi)^4\dot{c}(\hat{k})\left(\int\frac{d^4\hat{p}_1d^4\hat{p}_2}{(2\pi)^4}\;B_1(\hat{p}_1)\delta(\hat{p}_2-\hat{k})\delta(\hat{p}_1+\hat{p}_2)\right)\times\\
&\qquad\times\left(\int\frac{d^4\hat{p}_3d^4\hat{p}_4}{(2\pi)^4}\;B_1(\hat{p}_3)\delta(\hat{p}_4+\hat{k})\delta(\hat{p}_3+\hat{p}_4)\right)=\\
&=4(\beta-1)^2\int\frac{d^4\hat{p}_1d^4\hat{p}_2d^4\hat{p}_3d^4\hat{p}_4}{(2\pi)^4}\dot{c}(\hat{p}_2)B_1(\hat{p}_1)\delta(\hat{p}_1+\hat{p}_2)B_1(\hat{p}_3)\delta(\hat{p}_4+\hat{p}_2)\delta(\hat{p}_3+\hat{p}_4)=\\
&=4(\beta-1)^2\int\frac{d^4\hat{p}_1d^4\hat{p}_2}{(2\pi)^4}\dot{c}(\hat{p}_2)B_1(\hat{p}_1)B_1(\hat{p}_2)\delta(\hat{p}_1+\hat{p}_2)\,.
\end{aligned} \tag{3.1.43}$$

From (3.1.16) we see that here effectively $\dot{c}(\hat{p}_2)=2c_1\hat{p}_2^2+\mathcal{O}(\hat{p}_2^4)$, which means that (3.1.42) does not contribute to the flow of $\beta$; it contributes, instead, to the higher order derivative terms. Therefore, the flow of $\beta$ is controlled by (3.1.41).

We have two terms in (3.1.41), but their contribution is the same: we will work out one of the two terms and double the result. We have

$$\begin{aligned}
&-\int d^4\hat{k}\frac{(2\pi)^4}{2}\hat{k}^{-2}\dot{c}(\hat{k})\frac{\gamma}{2}\frac{\partial^2\left(B_1^2\phi^2\right)}{\partial\phi(\hat{k})\partial\phi(-\hat{k})}=\\
&=-\frac{(2\pi)^4\gamma}{2}\int d^4\hat{k}\hat{k}^{-2}\dot{c}(\hat{k})\int\frac{\prod_{i=1}^4 d^4\hat{p}_i}{(2\pi)^{12}}B_1(\hat{p}_1)B_1(\hat{p}_2)\delta(\hat{p}_3-\hat{k})\delta(\hat{p}_4+\hat{k})\delta\left(\sum_{i=1}^4\hat{p}_i\right)=\\
&=-\frac{\gamma}{2}\int\frac{d^4\hat{p}_1d^4\hat{p}_2d^4\hat{p}_3d^4\hat{p}_4}{(2\pi)^8}\;\hat{p}_3^{-2}\dot{c}(\hat{p}_3)B_1(\hat{p}_1)B_1(\hat{p}_2)\delta(\hat{p}_3+\hat{p}_4)\delta(\hat{p}_1+\hat{p}_2+\hat{p}_3+\hat{p}_4)=\\
&=-\frac{\gamma}{2}\int\frac{d^4\hat{p}_1d^4\hat{p}_2d^4\hat{p}_3}{(2\pi)^8}\;\hat{p}_3^{-2}\dot{c}(\hat{p}_3)B_1(\hat{p}_1)B_1(\hat{p}_2)\delta(\hat{p}_1+\hat{p}_2)\,.
\end{aligned} \tag{3.1.44}$$

Now, we postulate that the momentum integral over $\hat{p}_3$ takes a value $N$, which is regulator-dependent, and is given by

$$N=\frac{1}{2}\int\frac{d^4\hat{p}_3}{(2\pi)^4}\hat{p}_3^{-2}\dot{c}(\hat{p}_3)\,. \tag{3.1.45}$$

The exact value of $N$ can thus be evaluated only once we have a specific regularization scheme at hand. We conclude that

$$\begin{aligned} -\int d^4\hat{k}\frac{(2\pi)^4}{2}\hat{k}^{-2}\dot{c}(\hat{k})\frac{\gamma}{2}\frac{\partial^2\left(B_1^2\phi^2\right)}{\partial\phi(\hat{k})\partial\phi(-\hat{k})} &= \\ = -N\gamma\int\frac{d^4\hat{p}_1 d^4\hat{p}_2}{(2\pi)^4}B_1(\hat{p}_1)B_1(\hat{p}_2)\delta(\hat{p}_1+\hat{p}_2)\,, \end{aligned} \tag{3.1.46}$$

and, once we sum over both scalars and referring to the compact notation of (3.1.23), we find that the relevant part of (3.1.26) takes the form $\dot{\beta}B_1^2 = 2\times(-N\gamma)B_1^2$, which delivers

$$\dot{\beta} = -2N\gamma. \tag{3.1.47}$$

A similar analysis for $F_1^2$ is bound to give the flow equation for the coupling $\alpha$, which is

$$\dot{\alpha} = -2N(\gamma+\zeta). \tag{3.1.48}$$

This completes the analysis of (3.1.26) and the reader can check that no other terms are required for the self-consistent flow of the auxiliary field scalar potential. This means that the Kähler potential will not need higher order terms and that our flow is exact.

We directly solve the flow equations (3.1.38), (3.1.39), (3.1.47) and (3.1.48), together with the boundary conditions

$$\alpha\Big|_{t=0} = 1, \quad \beta\Big|_{t=0} = 0, \quad \gamma\Big|_{t=0} = 0, \quad \zeta\Big|_{t=0} = 0 \tag{3.1.49}$$

to find

$$\zeta = -c_1\left(1-e^{-2t}\right), \quad \gamma = -c_1\left(1-e^{-2t}\right) \tag{3.1.50}$$

and

$$\alpha = 1 - 2c_1N + 4c_1N\left(t+\frac{1}{2}e^{-2t}\right), \quad \beta = -c_1N + 2c_1N\left(t+\frac{1}{2}e^{-2t}\right). \tag{3.1.51}$$

We conclude that the quantum effects make the Lagrange multiplier superfield $T$ become propagating and a non-ghost kinetic term requires $Nc_1 > 0$, which is in accordance with the typical properties (3.1.52). Indeed, it is typical to have a regularization scheme where [196]

$$c_1 < 0, \quad N < 0, \quad \text{so that} \quad Nc_1 > 0. \tag{3.1.52}$$

These conditions are satisfied for the probe regulator $c(\hat{p}) = (1-\hat{p}^2)\,\Theta\,(1-\hat{p}^2)$, for example, which gives $c_1 = -1$ and $N = -\frac{1}{32\pi^2}$ [197, 198]. Note that the condition $Nc_1 > 0$ also ensures that the multiplet $X$ doesn't become of ghost type in the IR. One can also work with different regulators $c(\hat{p})$, and, as we have already mentioned, find similar results. For instance, one can use an analytic function of the form $e^{-\hat{p}^2-4\hat{p}^4}$, which would give again $c_1 = -1$ and $N \simeq -10^{-3}$. An analytic regulator, however, will always have some contribution from the high UV modes because its support goes up to infinity.

We now shift to canonically normalized dimensionless superfields by redefining them as follows:

$$X \to \mu\, X/\sqrt{\alpha}, \quad T \to \mu\, T/\sqrt{\beta}\,, \tag{3.1.53}$$

$\mu$ being the scale compared to which we measure energies and lengths. In addition, for the action and the superspace integrals we have

$$\int d^4x\,(...) \;\to\; \int d^4x\,\mu^{-4}\,(...), \quad \int d^2\theta \;\to\; \mu \int d^2\theta, \tag{3.1.54}$$

where the new $x$ and $\theta$ are dimensionless. This means that, when the redefined fields take a VEV, e.g. $\langle T\rangle = 0.1$, the original field had a VEV of the form $0.1 \times \mu/\sqrt{\beta}$. We thus obtain a Lagrangian where all fields are dimensionless and all couplings are dressed with $\mu$, i.e. $g$ is always going to appear in the combination $\mu^2 g$ (and analogously for $q$). In the end, after we redefine the superfields $X$ and $T$ to be dimensionless and canonical, we have

$$\begin{aligned} K_{\text{norm.}} =& |X|^2 + |T|^2 + \frac{1}{4}\,\frac{-c_1(1-e^{-2t})}{\left[1-2c_1N+4c_1N(t+\frac{1}{2}e^{-2t})\right]^2}|X|^4 + \\ &+ \frac{-c_1(1-e^{-2t})}{\left[1-2c_1N+4c_1N(t+\frac{1}{2}e^{-2t})\right]\left[-c_1N+2c_1N(t+\frac{1}{2}e^{-2t})\right]}|X|^2|T|^2 \end{aligned} \tag{3.1.55}$$

and

$$\begin{aligned} W_{\text{norm.}} =& \frac{e^{2t}\xi_{\text{UV}}}{\left[1-2c_1N+4c_1N(t+\frac{1}{2}e^{-2t})\right]^{1/2}}X + \\ &+ \frac{1}{2}\,\frac{1}{\left[1-2c_1N+4c_1N(t+\frac{1}{2}e^{-2t})\right]\left[-c_1N+2c_1N(t+\frac{1}{2}e^{-2t})\right]^{1/2}}X^2T, \end{aligned} \tag{3.1.56}$$

where we have defined

$$f = \Lambda^2 \xi_{\text{UV}}. \tag{3.1.57}$$

Then, if we assume that the supersymmetry breaking scale of the Volkov–Akulov model serves also as the UV scale where the nilpotency of $X$ is imposed, we would have

$$f\Big|_{\mu=\Lambda} = \Lambda^2 \;\text{ so that }\; \xi_{\text{UV}} = 1. \tag{3.1.58}$$

If, instead, we consider the Volkov–Akulov model to be a low energy effective description of some supersymmetric model with supersymmetry breaking scale $\sqrt{f}$, then we might wish to impose the boundary conditions (3.1.11) at some lower scale $\Lambda$. This is captured by choosing another value for $\xi_{\rm UV}$, i.e. we would have

$$f\Big|_{\mu=\Lambda} > \Lambda^2 \text{ so that } \xi_{\rm UV} > 1. \tag{3.1.59}$$

We should note immediately that the qualitative results regarding the vacuum stability will not depend on $\xi_{\rm UV}$ as long as $\xi_{\rm UV} \geq 1$. In fact, the tachyonic behaviour that we will shortly demonstrate will only become more extreme as $\xi_{\rm UV}$ increases. For this reason, we will use $\xi_{\rm UV} = 1$ in our numerical examples, as the maximally benign option.

We should also observe that, here, we consider the pure Volkov–Akulov model, which we might imagine as the low energy limit of a supersymmetry breaking model where all other degrees of freedom are sufficiently massive and can be integrated out. More generally, it could be possible that some light degrees of freedom remain in the effective theory below $\Lambda$, and could have non-trivial couplings to the nilpotent superfield $X$. In this case, one would have to include the effects of these couplings on the RG flow. However, it is worth noting that the superfield $T$ starts out as a Lagrange multiplier that does not couple to any other degrees of freedom except $X$. This means that at least for small $t$ the presence of additional light degrees of freedom in the EFT would not be able to greatly affect the evolution of $\beta$ and $\gamma$. Thus, we expect the qualitative features of the results described in the next section to remain valid even in the context of more general models.

Finally, as a non-trivial cross-check of our results, we can study the flow of the coupling $q$ with the use of the fermionic terms that are related to the relevant part of the Kähler potential, which are given by

$$\begin{aligned} L_{\rm int.} \ni \int \frac{\prod_{i=1}^4 d^4 p_i}{(2\pi)^{12}} \Big[ & iq\overline{X}(-p_1)\overline{G}_{\dot\alpha}(-p_2)\overline{\sigma}^{m\dot\alpha\alpha}(p_3+p_4)_m G_\alpha(p_3) X(p_4) \times \\ & \times \delta(p_1+p_2-p_3-p_4)\Big], \end{aligned} \tag{3.1.60}$$

after Wick rotation, but still in the dimensionful notation. The term in (3.1.60) is influenced by the fermionic propagator and one needs the ERG equation for such fields. We have

$$\dot L_{\rm int.} = -i \int d^4\hat k (2\pi)^4 \hat k^{-2} \dot c(\hat k)\, \overline{\sigma}^{m\dot\alpha\alpha}\hat k_m \frac{\partial L_{\rm int.}}{\partial \overline{\lambda}^{\dot\alpha}(-\hat k)} \frac{\partial L_{\rm int.}}{\partial \lambda^\alpha(\hat k)} + \dots, \tag{3.1.61}$$

where $\lambda$ is the fermion component of the superfield $T$, which is explicitly defined as

$$\lambda_\alpha = \frac{1}{\sqrt{2}} D_\alpha T|. \tag{3.1.62}$$

In order to find the flow of $q$ we need the superpotential part that enters $L_{\rm int.}$, namely, in terms of dimensionless fields,

$$L_{\rm int.} = -X\lambda^\alpha G_\alpha + \overline{X}\overline{\lambda}^{\dot\alpha}\overline{G}_{\dot\alpha} + \dots. \tag{3.1.63}$$

The right hand side of the ERG equation is then

$$\begin{aligned}
&-i\int d^4\hat{k}(2\pi)^4\hat{k}^{-2}\dot{c}(\hat{k})\,\overline{\sigma}^{m\dot{\alpha}\alpha}\hat{k}_m\,\frac{\partial L_{\text{int.}}}{\partial\overline{\lambda}^{\dot{\alpha}}(-\hat{k})}\frac{\partial L_{\text{int.}}}{\partial\lambda^{\alpha}(\hat{k})}=\\
&=i\int\frac{d^4\hat{p}_1d^4\hat{p}_2d^4\hat{p}_3d^4\hat{p}_4}{(2\pi)^{12}}\int d^4\hat{k}\;\frac{\dot{c}(\hat{k})}{\hat{k}^2}\;\hat{k}^{\dot{\alpha}\alpha}\left(\overline{X}(-\hat{p}_1)\overline{G}_{\dot{\alpha}}(-\hat{p}_2)\right)\left(X(\hat{p}_3)G_{\alpha}(\hat{p}_4)\right)\delta_{1,2,\hat{k}}\delta_{3,4,\hat{k}}\\
&=\;-i\int\frac{\prod_{i=1}^4 d^4\hat{p}_i}{(2\pi)^{12}}\,\frac{\dot{c}(\hat{p}_3+\hat{p}_4)}{(\hat{p}_3+\hat{p}_4)^2}\;(\hat{p}_3+\hat{p}_4)^{\dot{\alpha}\alpha}\left(\overline{X}(-\hat{p}_1)\overline{G}_{\dot{\alpha}}(-\hat{p}_2)\right)\left(X(\hat{p}_3)G_{\alpha}(\hat{p}_4)\right)\delta_{1,2,-3,-4}\\
&=\;-2c_1\int\frac{d^4\hat{p}_1d^4\hat{p}_2d^4\hat{p}_3d^4\hat{p}_4}{(2\pi)^{12}}\,\overline{\sigma}^{m\dot{\alpha}\alpha}i(\hat{p}_3+\hat{p}_4)_m\;\left(\overline{X}(-\hat{p}_1)\overline{G}_{\dot{\alpha}}(-\hat{p}_2)\right)\left(X(\hat{p}_3)G_{\alpha}(\hat{p}_4)\right)\delta_{1,2,-3,-4},
\end{aligned}\tag{3.1.64}$$

where we have abbreviated $\hat{l}^{\dot{\alpha}\alpha}=\overline{\sigma}^{m\dot{\alpha}\alpha}\hat{l}_m$ in the middle lines. The left hand side of (3.1.61), always in terms of dimensionless fields and couplings, is then

$$\begin{aligned}
\dot{L}_{\text{int.}}=(\dot{\zeta}+2\zeta)\int\frac{\prod_{i=1}^4 d^4\hat{p}_i}{(2\pi)^{12}}\Big[&\overline{X}(-\hat{p}_1)\overline{G}_{\dot{\alpha}}(-\hat{p}_2)\overline{\sigma}^{m\dot{\alpha}\alpha}i(\hat{p}_3+\hat{p}_4)_m G_{\alpha}(\hat{p}_3)X(\hat{p}_4)\times\\
&\times\delta_{1,2,-3,-4}\Big],
\end{aligned}\tag{3.1.65}$$

so that we can deduce the equation

$$\dot{\zeta}=-2\zeta-2c_1,\tag{3.1.66}$$

which is in exact agreement with the result coming from the auxiliary field potential. Further cross-checks for the flow can be done. It is actually straightforward to show that all the component terms in $\int d^4\theta|X|^2$ share the same flow, and similarly for $\int d^4\theta|T|^2$. Let us note in passing that it would be interesting to perform a similar analysis for a Volkov–Akulov model in lower dimensions where different flow equations could apply (see e.g. [210]).

### 3.1.3 What are the composite states?

Here we wish to determine the nature of the composite states described by $X$ and $T$.

The scalar $X$ is clearly a multi-linear composite state of goldstini, which is controlled by $G^2/2F^X$, which contains a goldstino bilinear,

$$\langle X\rangle\;\sim\;\left\langle\frac{G^2}{f}\right\rangle+\ldots,\tag{3.1.67}$$

taking into account that $F^X$ has an on-shell expansion in terms of the goldstino.
The scalar $T$ is also multi-linear in the goldstini in the on-shell Volkov–Akulov theory. To see this let us first take the superspace equations of motion for $X$, before imposing the nilpotency condition via $T$, which give

$$-\frac{1}{4}\overline{D}^2\overline{X}=-f-TX.\tag{3.1.68}$$

On this superspace equation one can then impose the condition $X^2 = 0$, because it is derived from the independent variation of $T$. From (3.1.68) we see that, once we project to the lowest components, we have $TX = -f + \overline{D}^2\overline{X}|/4 = -f - \overline{F}^X$ and, using the on-shell value of $F^X$, we find

$$T \times G^2 = G^2 \left(1 - \frac{1}{4f^4}\overline{G}^2\partial^2 G^2\right)\left(\frac{1}{2f^2}\partial^2\overline{G}^2 + \frac{3}{8f^6}\overline{G}^2\partial^2 G^2\partial^2\overline{G}^2\right). \tag{3.1.69}$$

This equation shows that the on-shell value of $T$ in the Volkov–Akulov model can be determined in terms of goldstino multi-linears, up to a $G^2$ ambiguity, and it has the form

$$\langle T \rangle \sim \left\langle \frac{\partial^2\overline{G}^2}{f^2}\right\rangle + \ldots . \tag{3.1.70}$$

Clearly, both (3.1.67) and (3.1.70) can be recast in other forms due to the on-shell properties of $G$ and the inherent ambiguity of $T$. As it was noticed in [203], this ambiguity comes from the fact that the superpotential (3.1.8) remains invariant under the shift $T \to T + \mathcal{W}X$, for any holomorphic $\mathcal{W}$. Finally, the fermion component of the superfield $T$, which is $\lambda$, can be found by simply applying a supersymmetry transformation on (3.1.70).

## 3.2 Consequences for the pure Volkov–Akulov model

### 3.2.1 Critical point stability analysis

To analyze the properties of the model at a lower energy scale, we refer to (3.1.55) and (3.1.56) and use the regulator $c(\hat{p}^2) = (1 - \hat{p}^2)\Theta(1 - \hat{p}^2)$, which gives $c_1 = -1$ and $N = -\frac{1}{32\pi^2}$. We stress once again that we can make this choice without loss of generality as far as the lower energy dynamics are concerned. The Kähler potential and the superpotential become

$$\begin{aligned} K \equiv& |X|^2 + |T|^2 + \tilde{\zeta}|X|^4 + \tilde{\gamma}|X|^2|T|^2 = \\ =& |X|^2 + |T|^2 + \frac{1}{4}\frac{1 - e^{-2t}}{\left[1 - \frac{1}{16\pi^2} + \frac{1}{8\pi^2}(t + \frac{1}{2}e^{-2t})\right]^2}|X|^4 + \\ &+ \frac{1 - e^{-2t}}{\left[1 - \frac{1}{16\pi^2} + \frac{1}{8\pi^2}(t + \frac{1}{2}e^{-2t})\right]\left[-\frac{1}{32\pi^2} + \frac{1}{16\pi^2}(t + \frac{1}{2}e^{-2t})\right]}|X|^2|T|^2 \end{aligned} \tag{3.2.1}$$

and

$$\begin{aligned} W \equiv& \tilde{f}X + \tilde{g}X^2T = \\ =& \frac{e^{2t}\xi_{\rm UV}}{\left[1 - \frac{1}{16\pi^2} + \frac{1}{8\pi^2}(t + \frac{1}{2}e^{-2t})\right]^{1/2}}X + \\ &+ \frac{1}{2}\frac{1}{\left[1 - \frac{1}{16\pi^2} + \frac{1}{8\pi^2}(t + \frac{1}{2}e^{-2t})\right]\left[-\frac{1}{32\pi^2} + \frac{1}{16\pi^2}(t + \frac{1}{2}e^{-2t})\right]^{1/2}}X^2T, \end{aligned} \tag{3.2.2}$$

where the couplings $\tilde{\zeta} = \frac{\zeta}{4\alpha^2}$, $\tilde{\gamma} = \frac{\gamma}{\alpha\beta}$, $\tilde{f} = \frac{e^{2t}\xi_{\rm UV}}{\sqrt{\alpha}}$ and $\tilde{g} = \frac{1}{2\alpha\sqrt{\beta}}$ are obtained by canonically normalizing the fields, i.e. dividing by appropriate powers of the wavefunction renormalization.

The scalar potential is defined as

$$V = g^{i\bar{j}}\partial_i W \partial_{\bar{j}} \overline{W}, \tag{3.2.3}$$

where the indices $i, j, ... = 1, 2$ run over the complex scalar fields $X$ and $T$ and $g^{i\bar{j}}$ is the inverse of the scalar field space metric $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$.

Once $X$ and $T$ (and their complex conjugates) are expressed in terms of real scalar fields via

$$X = \frac{1}{\sqrt{2}}(\phi + i\chi), \quad T = \frac{1}{\sqrt{2}}(\tau + i\sigma), \tag{3.2.4}$$

it can be easily shown that the scalar potential has a critical point at

$$\phi\Big|_* = 0, \quad \chi\Big|_* = 0, \quad \tau\Big|_* = 0, \quad \sigma\Big|_* = 0, \tag{3.2.5}$$

where its value is

$$V\Big|_* = \frac{e^{4t}\xi_{\rm UV}^2}{1 - \frac{1}{16\pi^2} + \frac{1}{8\pi^2}(t + \frac{1}{2}e^{-2t})}. \tag{3.2.6}$$

This is a positive energy configuration, where supersymmetry is spontaneously broken. The (in)stability of such critical point is deduced from the scalar mass matrix associated with $V$, evaluated on the configuration itself. The (multiplicity two) eigenvalues are

$$m_\pm^2 = -\tilde{f}^2 \left[ \left(\tilde{\gamma} + 4\tilde{\zeta}\right) \pm \sqrt{\frac{16\,\tilde{g}^2}{\tilde{f}^2} + \left(\tilde{\gamma} - 4\tilde{\zeta}\right)^2} \,\right]. \tag{3.2.7}$$

As it can be clearly seen, at least one of the eigenvalues (3.2.7) is negative: the scalar field space origin is always tachyonic. We can further observe that, as the RG time flows, the term $\left(\tilde{g}/\tilde{f}\right)^2$ decreases, rendering all the eigenvalues negative for sufficiently large $t$ ($t \gtrsim 0.35$ for $\xi_{\rm UV} = 1$), provided $\tilde{\gamma}$ and $\tilde{\zeta}$ are positive, which is the case. Choosing $\xi_{\rm UV}$ larger will simply increase $\tilde{f}$ and make the tachyonic behaviour more extreme.

The investigation of the existence of other critical points and the possible consequent discussion of their relevance can be simplified by exploiting the following observation. The Kähler potential and the superpotential have a R-symmetry under which the superfields $X$ and $T$ have opposite non-vanishing R-charges. This means that the scalar potential will have a R-symmetry, even though the latter may or may not be preserved by all the other (e.g. higher derivative) interactions. Therefore, once we leave the central configuration, one of the scalar fields is bound to behave like a R-axion Goldstone mode, at least as far as the scalar potential is concerned. Then, by definition, such a mode will be massless and it will have a shift symmetry *on* the critical points that are away from the central one. Therefore, we can set it to vanish without loss of generality. We can consistently choose $\sigma$, which is bound to contribute to the R-axion as long as $\langle\tau\rangle \neq 0$, to be

$$\sigma = 0. \tag{3.2.8}$$

With this choice, we see that in order for the gradient of $V$ to be able to vanish, $\chi$ has to be set to zero too. We can thus restrict to the $\phi$ and $\tau$ directions to search for other possible critical points. As long as

$$t > \frac{1}{2}\log\frac{1+48\pi^2+\sqrt{1+224\pi^2+2304\pi^4}}{64\pi^2} \sim 0.20 \quad (\text{for } \xi_{\text{UV}}=1), \tag{3.2.9}$$

a positive energy critical point, which is also tachyonic, can be found. Moreover, for larger $t$, the characteristic field values for this critical configuration are $\phi^3 \sim \tilde{f}/\tilde{g}$ and $\tau^3 \sim \tilde{f}^2/\tilde{g}^2$. Since $\tilde{f}$ grows exponentially with $t$, coherently with the small field approximation with which we are working, these critical points become untrustable very soon (already for $t \sim 1$) along the RG flow.

To give a flavour of the behaviour of the scalar potential $V$ along the non-axionic directions $\phi$ and $\tau$, we include in Figure 3.1 the stream plots of the (opposite of the) potential gradient at $t = 0.1$ and $t = 1$, for $\xi_{\text{UV}} = 1$, where one can observe the appearance of the tachyons.

The formation of a tachyonic instability and the resulting goldstino condensation might seem somewhat unexpected, since the original fermion self-interaction terms in the Volkov–Akulov action are not strong. However, weak coupling does not necessarily imply the absence of important non-perturbative effects that have qualitative consequences. Indeed, our non-perturbative analysis of the composite state dynamics does not yield any constraints on the coupling and the tachyonic instability persists even for weak goldstino self-interaction. It would, of course, be interesting to have a complementary interpretation of these effects in terms of the purely fermionic formulation of the Volkov–Akulov model.

### 3.2.2 Why is there a tachyon in the central critical point?

We will now argue that the existence of the central tachyon that we have just encountered is unavoidable for a consistent RG flow.

First, let us note that the superpotential at the UV point (identified by an energy scale, say, $\Lambda_0$) is such that

$$W_{ij}\Big|_{\text{central point}} = 0, \tag{3.2.10}$$

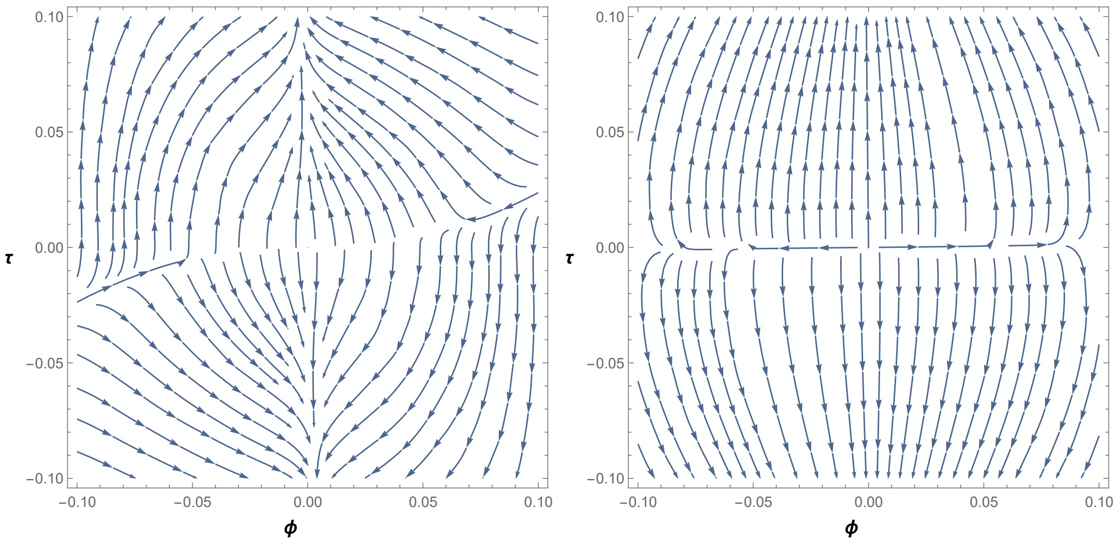


Figure 3.1: The figure shows the stream-plots of the (negative) gradient of $V$ at $t = 0.1$ (on the left hand side) and $t = 1$ (on the right hand side) restricted to the $\phi$ and $\tau$ directions, after consistently setting $\chi$ and $\sigma$ to zero, for $\xi_{\rm UV} = 1$. The origin has positive energy and has one tachyonic direction immediately, while a second tachyonic direction develops as early as $t \gtrsim 0.35$. The behaviour of the masses does not change for larger $t$.

which means that the fermions are bound to remain massless on the central point, located at $T = 0 = X$. If we now assume that, for some reason, a Kähler potential that gives positive masses to the scalars $X$ and $T$ has been generated during the flow, then, because the effective masses typically increase as RG time passes, there would be a scale $\Lambda_1$ below which we can remove the scalars $T$ and $X$ and work with a new set of constrained superfields satisfying

$$\mu \leq \Lambda_1 < \Lambda_0 : \quad X^2 = 0 = XT. \tag{3.2.11}$$

Let us note that, in contrast to the scalar masses, the masses of the fermions are protected by R-symmetry: therefore, they remain zero. Indeed, the R-symmetry assignments are

$$[G_\alpha]_R = -1, \quad [\lambda_\alpha]_R = 3, \tag{3.2.12}$$

and there is no mass combination that can yield a R-invariant. The only way in which a scalar could remain massless is if it was a R-axion, but on the central critical point the R-symmetry is unbroken.

Let us repeat the same procedure by decreasing the energy scale $\mu$ below $\Lambda_1$ and find the new effective theory. Initially, the superpotential takes the form

$$W = fX + \frac{1}{2} Y X^2 + \Phi T X, \tag{3.2.13}$$

and the Kähler potential is

$$\mu = \Lambda_1 : \quad K = |X|^2 + |T|^2, \tag{3.2.14}$$

making explicit the role of $Y$ and $\Phi$ as Lagrange multipliers. This being fixed, we lower the energy below $\Lambda_1$ and characterize the resulting effective theory. We can again observe that the superpotential has the appropriate vertices to generate the kinetic terms for the superfields $Y$ and $\Phi$ and we further notice that, because of the Yukawa couplings, the condition (3.2.10) is still valid on the central critical point, which is now located at $T = 0 = X = Y = \Phi$. As before, on this configuration the fermions are massless by virtue of the R-symmetry and the scalars are bound to become heavy as we go to lower energies. We can consequently integrate out once more *all* the heavy scalars below some energy scale $\Lambda_2$ defining a new effective theory characterized by

$$\mu \leq \Lambda_2 < \Lambda_1 : \quad X^2 = 0 = XT = XY = X\Phi. \tag{3.2.15}$$

The problem is then manifest. Unless there is a dynamical reason to stop this procedure, we could get infinite new states in the deep IR, which lead to a possible series of inconsistencies. Therefore, the flow self-consistently terminates itself by introducing tachyons that, once appearing, can not be decoupled in a consistent way and, as a consequence, this *domino* effect stops.

### 3.2.3 Limitations of the SLPA

The results that we have obtained above have all been derived in our supersymmetric rendition of the local potential approximation (SLPA), which ignores the generation and feedback of higher derivative terms in the exact renormalization group equations. The LPA is a well-motivated and tested approximation and it is a common practice in ERG calculations, while the SLPA is a minimal modification of it, motivated by supersymmetry. However, it is important to discuss its regime of validity and the possible corrections that one could expect to our results because of its use.

Since we start from a UV model that has vanishing non-Kähler interactions, the higher derivative terms have to be generated before feeding back into the flow for the scalar potential and their effect is expected to appear at higher order in the RG time. The SLPA can be regarded as accurately giving the RG flow for a small decrease in the energy scale. A way to see this consists of solving the full exact renormalization group recursively by discretizing $t$ and starting from the UV values of the couplings, that is by making reference to the Kähler potential $K = |X|^2$ and the standard superpotential (3.1.8). In addition, for concreteness, we will consider the flow with the use of the optimized regulator $c(\hat{p}^2) = (1 - \hat{p}^2)\,\Theta\,(1 - \hat{p}^2)$ and we will keep the propagator pieces for both $T$ and $X$. In the first step one would generate the SLPA terms $\int d^4\theta |X|^4$ and $\int d^4\theta |X|^2|T|^2$ and, on top of them, the higher derivative term $\int d^4\theta T\partial^2\overline{T}$, which is quadratic in the auxiliary fields, together with the higher derivative term $\int d^2\theta X^2 \partial^2 T$, which is linear in the auxiliary fields. Such higher derivative terms are ignored in the SLPA. The next recursive step would immediately give the wavefunction renormalization of $X$ and $T$, due to the SLPA effect of $\int d^4\theta |X|^4$ and $\int d^4\theta |X|^2|T|^2$, thus making the composite states manifest. Conversely, the effect of the higher derivative terms would still be inconsequential for the Kähler potential. Indeed, it takes additional steps in this recursive approach until the higher derivative terms start to backreact on the dominant SLPA contributions including the wavefunction renormalization. This is a reflection of the fact that our SLPA approach does not keep track of the anomalous dimension[6], and thus one may not trust the approximation quantitatively for large $t$, where its effect might possibly alter the flow. If one instead considers an infinitesimal $t$, the effective theory of the composite states with a new cut-off, which is infinitesimally near the one corresponding to the start of the flow (where the SLPA dominates),

$$\Lambda_{\text{New}} \lesssim \Lambda_{\text{VA}} \,, \tag{3.2.16}$$

can be derived. This means that, in any case, we get the description of the Volkov–Akulov model in terms of a new EFT defined with a slightly lower cut-off.

Even though we are only slightly moving away from the UV point along the RG flow, there are two features of our SLPA results that we can argue remaining robust even for larger $t$. These are the dynamic nature of the superfield $T$ and its tachyonic behaviour near the central critical point.

The tachyonic nature of the central point has actually two sources. The first one is the off-diagonal $X$-$T$ terms of the scalar mass matrix, due to the superpotential, which always give tachyons and dominate at small $t$. On top of that, the second source of instability lies in the positivity of the couplings $\zeta$ and $\gamma$, as it can be seen from (3.2.7), which dominates at large $t$. For both $\zeta$ and $\gamma$ the flow derived in the framework of the SLPA depends only on themselves and results in a monotonically increasing flow. Thus, the only way to remove the central tachyons would be for the higher order corrections to overpower the SLPA contribution. This would indicate that the theory has reached a point in the RG flow where higher derivative terms are large enough to compete with lower derivative ones. As far as the contribution to the instability of the central critical point due to the superpotential (which does not change during the SLPA flow) is concerned, it has to be overcome by some stabilizing contribution from $\zeta$ and $\gamma$. This will again lead to the aforementioned intricacies.

A similar argument can be made regarding the dynamic nature of $T$. Given that at small

[6] A version of the LPA that incorporates the anomalous dimension has been suggested for example in [211]. It would be interesting to see if a similar modification can be made for the SLPA.

$t$, where the SLPA can be trusted, the $\gamma$ coupling is positive, $T$ clearly acquires a positive kinetic term. Once this happens, it is impossible for the flow to bring this kinetic term back down for any finite energy, because it would require significant effects from higher derivative terms, breaking the EFT description. Moreover, if the corrections managed to pull the kinetic term of $T$ back to zero at any finite energy, at energies below that we would run the risk of obtaining ghosts.

A possibility that is harder to rule out is that additional couplings appear due to the effects of higher derivative terms that manage to stabilize the potential *away* from the central critical point. The SLPA results already indicate the presence of additional (still tachyonic) critical points and additional terms, which could arise from higher derivative contributions, could help in stabilizing them. For instance, a $|T|^4$ term in the Kähler potential can be generated via higher derivative terms with its coupling constant parametrically suppressed relative to $\zeta$ and $\gamma$. We stress that, even in this case, the conclusion that a goldstino condensate forms remains valid and the dynamics of the theory around this *new* vacuum will need to be re-examined.

## 3.3 Coupling to Supergravity

### 3.3.1 Coupling to pure Supergravity

Let us now briefly discuss the supergravity embedding of the model that we have presented so far in the framework of global supersymmetry. As a preliminary important comment, we would like to emphasize that the impact of the quantum effects that are related to the supergravity sector is not taken into account here. In other words, we are simply considering the Kähler potential and the superpotential (with a possible addition of a constant term) of the composite supersymmetric theory, namely (3.1.55) and (3.1.56) (or, more specifically, (3.2.1) and (3.2.2)), coupled to classical Supergravity. If the composite fields take values that are parametrically smaller than the cut-off, then the following analysis can be trusted as a first approximation.

Since we are accessing Supergravity and, therefore, the Planck mass enters as an additional energy scale, we have to deal with it: we will do it in following way. We can set

$$M_P = \Lambda \times P, \tag{3.3.1}$$

where a realistic value for the dimensionless parameter $P$ could be, for instance, $P \simeq 10^4$. After writing all expressions in terms of dimensionless fields, couplings and momenta, this translates to replacing every instance of $M_P$ by $e^t P$, which is the value of the Planck mass in units of $\mu$. The exponential $e^t$ is actually the "classical" flow of $M_P$, simply because it has mass dimension $[M_P] = 1$, as it is the case for any dimensionful coupling that does not flow due to the leading quantum effects that we investigate here.

As mentioned above, we also slightly modify the superpotential by introducing a constant term, which is related to the Lagrangian gravitino mass

$$m_{3/2} = e^{\frac{K}{2M_P^2}} \frac{W}{M_P^2} \tag{3.3.2}$$

for a Kähler potential $K$ and a superpotential $W$.
In accordance with our conventions, we write the superpotential as

$$\begin{aligned} W =& e^{3t} P^3 W_0 + \frac{e^{2t}\xi_{\text{UV}}}{\left[1 - \frac{1}{16\pi^2} + \frac{1}{8\pi^2}(t + \frac{1}{2}e^{-2t})\right]^{1/2}} X + \\ &+ \frac{1}{2} \frac{1}{\left[1 - \frac{1}{16\pi^2} + \frac{1}{8\pi^2}(t + \frac{1}{2}e^{-2t})\right] \left[-\frac{1}{32\pi^2} + \frac{1}{16\pi^2}(t + \frac{1}{2}e^{-2t})\right]^{1/2}} X^2 T. \end{aligned} \tag{3.3.3}$$

Typically, the superpotential constant term is chosen to be independent of $M_P$, so that the gravity decoupling limit $M_P \to +\infty$ is captured by $m_{3/2}$ approaching 0; here, however, we measure it directly in Planck units: therefore, $[W_0] = 0$.

We then make use of (3.2.1) and (3.3.3) to calculate the scalar potential

$$V = e^{e^{-2t}\frac{K}{P^2}} \left( g^{i\bar{j}} D_i W \, \overline{D_{\bar{j}} W} - 3e^{-2t} \frac{W \, \overline{W}}{P^2} \right), \tag{3.3.4}$$

where $D_i W$ is the Kähler covariant derivative of $W$,

$$D_i W = \partial_i W + e^{-2t} \frac{\partial_i K}{P^2} W. \tag{3.3.5}$$

If and only if $W_0 = 0$, (3.3.4) has a de Sitter critical point at

$$\phi\Big|_* = \chi\Big|_* = \tau\Big|_* = \sigma\Big|_* = 0 \text{ with } V\Big|_* = \frac{e^{4t}}{1 - \frac{1}{16\pi^2} + \frac{1}{8\pi^2}(t + \frac{1}{2}e^{-2t})} \ (\text{for } \xi_{\text{UV}} = 1). \tag{3.3.6}$$

Computing the scalar mass matrix at such critical configuration, we find that, as in the rigid case, it is *highly tachyonic* with masses similar to (3.2.7), in accordance with the refined de Sitter Conjecture [100–103]. If $W_0$ is small, the critical point moves away from the origin $(X,T) = (0, 0)$ and it still remains highly unstable. Meanwhile, for large enough $W_0$, the potential remains tachyonic, but also gets pulled down to negative energy.

It is also worth noting that our conclusions agree with other results in the literature, and in particular with [178, 184], where a tachyon shows up in the central critical point. Here, however, we obtain these results in a manifestly supersymmetric setup.

### 3.3.2 Consequences for anti-brane uplifts

In this section we explore some consequences of the new composite state dynamics and their RG flow for string theory constructions involving anti-brane uplifts of AdS vacua to meta-stable de Sitter critical points. The effect of the anti-brane in these constructions is meant to be captured by adding the Volkov–Akulov Kähler potential and superpotential to those of the supergravity model describing the pre-uplift system. In this way, we assume that the Volkov–Akulov system couples to the other ingredients only via supergravity interactions and the scalar potential for the fields $X$ and $T$, as well as their RG flow, should only be affected by Planck suppressed corrections[7]. With these assumptions, we can spell out at least two important consequences for models involving nilpotent chiral multiplets.

First, we should expect the tachyonic behaviour near the origin of the ($X$,$T$) field space to remain (and we will verify this explicitly within the KKLT setup). The end-point of this instability will depend on the specifics of additional non-renormalizable operators in the theory. In principle, one expects such corrections to appear from String Theory as well as from corrections to the local potential approximation. This may stabilize the system, but the final configuration is likely to lie at large values of $X$ and $T$ and its physical interpretation is therefore unclear and deserves further investigation.

Second, regardless of the location of the final configuration, one can ask whether it has any chance of remaining at positive energy values. In models with anti-brane uplifts, we note that the final vacuum energy is typically the result of a competition between two dominant terms in the scalar potential

$$V \sim e^{\frac{K}{M_P^2}} \left(f^2 - M_P^4 V_0\right) + \dots \tag{3.3.7}$$

with other contributions being $M_P$-suppressed. Here, $f$ is the coefficient of the linear term in $X$ in the Volkov–Akulov superpotential, while $V_0$ is the pre-uplift contribution to the energy that is independent of the fields $X$ and $T$, but may depend on the other fields in the model. Incorporating the RG flow and working in terms of dimensionless and canonically normalized fields as in the previous sections, the $f^2$ term will flow as

$$f^2 \sim \frac{e^{4t}\xi_{\mathrm{UV}}^2}{\alpha(t)}, \tag{3.3.8}$$

[7]More generally, it is possible that some of the light degrees of freedom that are present in the EFT have more direct coupling to the Volkov–Akulov sector. This includes scenarios which incorporate the effects of warping on the anti-brane studied in [212] or the presence of the light complex structure modulus studied in [213,214]. In this case, the effect of the light fields on the RG flow would, in principle, have to be taken into account. That said, as discussed in Section 3.1.2, the appearance of the $T$ kinetic term and the tachyonic behaviour that we have described are expected to remain.

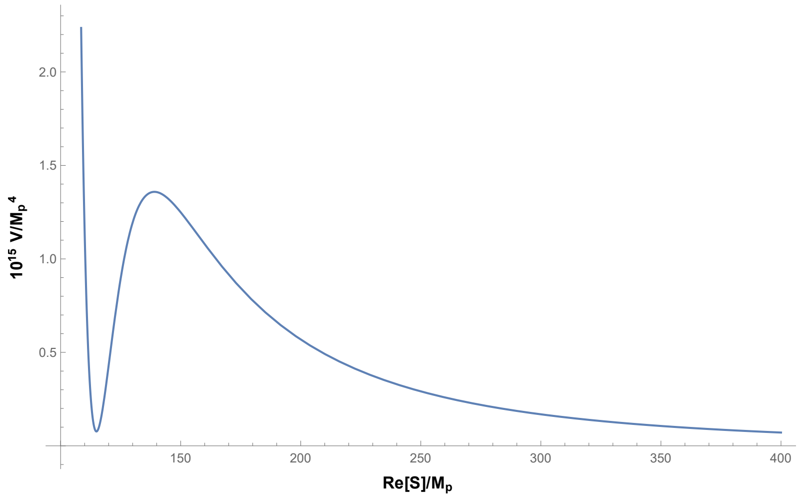


Figure 3.2: The figure shows the KKLT scalar potential obtained from (3.3.13) for $X = T = 0$ and $A = 1$, $a = 0.1$, $W_0 = -10^{-4}$, $P = 80\sqrt{\xi_{\rm UV}}$ at the UV cut-off of the effective theory ($t = 0$).

where the exponential behaviour corresponds to the "classical" RG flow and the $1/\alpha(t)$ factor results from the wavefunction renormalization of the field $X$. The $V_0$ term will, of course, also have the "classical" exponential growth; however, it will not inherit the wavefunction renormalization of the fields $X$ or $T$. At this point we note that $\alpha(t)$ is a monotonically increasing function, and thus the uplift term in the potential becomes suppressed at lower energies. If the superpotential contains a constant term, as it is typical in most models of moduli stabilization, its contribution to the potential will not receive any additional suppression. This means that there is a tendency for $V_0$ to dominate over the uplift term at lower energies, possibly resulting in an AdS vacuum. It therefore appears that, in order for the uplift term to remain "competitive" at lower energies, the superpotential can not contain terms that are independent of any degrees of freedom in the low energy effective theory. This also means that any heavy moduli that are integrated out must not have VEVs contributing to the superpotential.

Note that in our approach $\alpha(t)$ grows linearly at large $t$. This behaviour, however, will be corrected by the anomalous dimension of $X$, which our approach ignores. As a first check, we can naively insert a small anomalous dimension $\delta$ into the flow equation for $\alpha$, giving

$$\dot{\alpha} = -\delta\alpha - 2N(\gamma + \zeta). \tag{3.3.9}$$

For negative anomalous dimension, the growth at large $t$ will be more rapid, exacerbating the problem described above, while for positive anomalous dimension the linear growth is expected to stop and approach a finite value, that is of order $(\gamma_{\rm IR} + \zeta_{\rm IR})/\delta$. In this last case, maintaining positive energy might remain possible, but it requires very small $V_0$.

There is another interesting possible *caveat* to the above argument arising from the fact that in a quasi de Sitter state the Hubble scale provides an IR cut-off, which could potentially halt the RG flow before the negative contributions to the scalar potential overtake the uplift term. Let us imagine a scenario where the scalar potential, evaluated for a particular value of the RG time $t$, has a critical point whose energy is given by an expression of the form (3.3.7), with the uplift term flowing as (3.3.8). The Hubble scale measured in units of the RG scale $\mu$, sourced by this potential, is

$$H(t)^2 M_P^2 \sim e^{\frac{K}{M_P^2}} \left(f(t)^2 - M_P^4 V_0\right) \text{ or } H(t)^2 P^2 \sim \left(\frac{e^{2t}\xi_{\text{UV}}^2}{\alpha(t)} - e^{2t} P^4 V_0\right)\left[1 + \mathcal{O}\left(\frac{e^{-2t}}{P^2}\right)\right] \tag{3.3.10}$$

for large (enough) $t$. Consistency requires that our renormalization scale is above the apparent Hubble scale derived from this potential which can be written as

$$H(t) = e^{t-t_*} \tag{3.3.11}$$

with $t_* > t$. The combination of these expressions gives

$$e^{-2t_*} = \left(\frac{\xi_{\text{UV}}^2}{P^2\alpha(t)} - P^2 V_0\right)\left[1 + \mathcal{O}\left(\frac{e^{-2t}}{P^2}\right)\right]. \tag{3.3.12}$$

The condition that $t_* > t$ can potentially put a stop to the RG flow, provided that the above equation can be satisfied when setting $t = t_*$. For $V_0 = 0$ there is always a solution as long as $\alpha(t)$ does not grow exponentially, since the exponential on the left hand side of (3.3.12) overpowers the sub-exponential growth of $\alpha(t)$ and both sides of the equation asymptote to zero. For small enough $V_0$, a solution continues to exist and pushes $t_*$ higher. In either case, the RG flow will eventually stop at a finite value of $t$, resulting in a de Sitter critical point with a Hubble scale that is exponentially suppressed relative to the UV cut-off.

On the other hand, for sufficiently large $V_0$ the large $t$ solution to (3.3.12) disappears entirely, meaning that the IR cut-off disappears, and the effects of the anomalous dimension of $X$ remain the only potential mechanism of staying at positive energy.

We can investigate these effects in the familiar KKLT setup. We couple our description of the Volkov–Akulov model to an additional chiral multiplet $S$ governed by the pre-uplift KKLT Kähler potential and superpotential. This means that we have $K = -3M_P^2 \log[(S + \bar{S})/M_P] + |X|^2$ and $W = W_0 + Ae^{-aS/M_P} + fX$, where $X^2 = 0$. As mentioned above, we assume that the additional supergravity couplings do not greatly affect the RG flow for the $X$ and $T$ couplings as well as only include the "classical" running for the couplings in the $S$ sector.

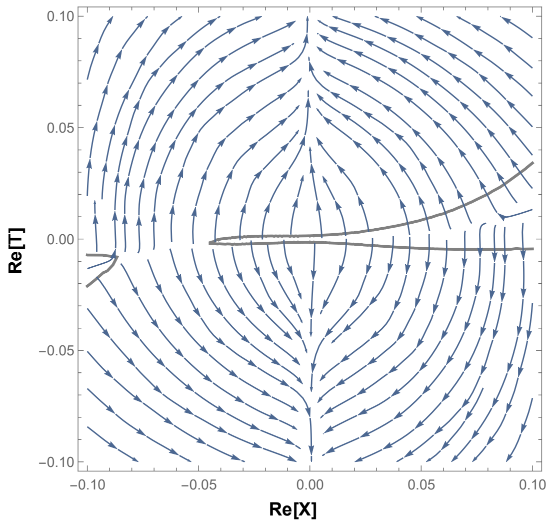


Figure 3.3: This figure presents the stream-plot of the (negative) gradient of the scalar potential with RG time $t = 0.1$ for $S = 114.92$ and $A = 1$, $a = 0.1$, $W_0 = -10^{-4}$, $\xi_{\text{UV}} = 1$ and $P = 80$ near the origin of the $(X,T)$ field space. The black contour shows the location where the Re$S$ component of the gradient vanishes as well. The de Sitter critical point is slightly shifted to $(S = 114.9,\ X = -0.046,\ T = -0.0017)$ and is tachyonic in the $T$ direction.

In our dimensionless conventions the total Kähler potential and superpotential now take the form

$$\begin{aligned} K &= -3P^2 e^{2t} \log\left(\frac{S+\bar{S}}{Pe^t}\right) + K_{\text{norm.}}, \\ W &= P^3 e^{3t}\left(W_0 + Ae^{-\frac{aS}{Pe^t}}\right) + W_{\text{norm.}} \end{aligned} \tag{3.3.13}$$

with $K_{\text{norm.}}$ and $W_{\text{norm.}}$ given in (3.1.55) and (3.1.56), respectively. In this form, the parameters $W_0$, $A$ and $a$ are expressed in Planck units. The strength of the uplift is governed by the ratio of $P/\sqrt{\xi_{\text{UV}}}$. For instance, the original example given in [29], where the uplift term was $D/(\text{Re}S)^3$ with $D = 3 \times 10^{-9}$ corresponds in our conventions to $P = 80.34\sqrt{\xi_{\text{UV}}}$. The value of $\xi_{\text{UV}}$ itself expresses the supersymmetry breaking scale in units of the starting UV cut-off, where we impose that the kinetic term of $T$ vanishes.
At our UV cut-off, where the field $T$ becomes non-dynamical and imposes the nilpotency condition on $X$, we recover the standard KKLT scenario, with the uplift realized via nilpotent superfields. For suitable choices of parameters, one obtains a potential with the familiar de Sitter meta-stable minimum (see Figure 3.2). However, evolving the theory down the RG flow even slightly, the field $T$ becomes dynamical and the nilpotency condition on $X$ is relaxed. The de Sitter critical point moves slightly in the $(X,T)$ plane. More importantly, this critical point is not stable in the $X$ and $T$ directions, but develops a tachyon roughly along the $T$ direction (see Figure 3.3). As in the rigid case, the tachyonic

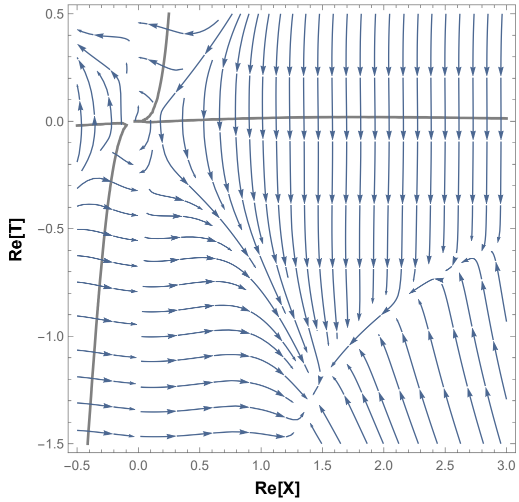


Figure 3.4: The figure shows the stream-plot of the (negative) gradient of the scalar potential with RG time $t = 0.1$ for $S = 114.9$ and $A = 1$, $a = 0.1$, $W_0 = -10^{-4}$, $\xi_{\rm UV} = 1$ and $P = 80$ near the origin of the $(X$,$T)$ field space. The black contour shows the location where the Re$S$ component of the gradient vanishes as well. The "minimum" at $X = 1.42$, $T = -1.27$ no longer has a non-vanishing gradient along the $S$ direction.

behavior appears for all values of $\xi_{\rm UV}$ and only becomes worse as we increase it.

The endpoint of this tachyonic instability is unclear; however, even by considering the potential for fixed $S$, we can see that it rolls down to a configuration with negative energy at field values of $\mathcal{O}(1)$, where possible higher order terms in the Kähler potential become important (see Figure 3.4). The relation of the tachyonic instability that we are finding here to the "goldstino evaporation" setup of [124] or the KPV scenario [215], which end in supersymmetric points, is yet unknown.

In the above analysis we have tuned the parameters so as to produce the de Sitter critical point in the UV, where the field $X$ is nilpotent. Following the RG flow to a lower energy scale, we find precisely the second problem described above. The linear term in $X$ in the superpotential, which is responsible for the uplift, has the form

$$W_{\rm up.} = \frac{e^{2t}\xi_{\rm UV}}{\left[1 - \frac{1}{16\pi^2} + \frac{1}{8\pi^2}(t + \frac{1}{2}e^{-2t})\right]^{1/2}} X \tag{3.3.14}$$

and, together with the $W_0$ terms, it represents the main contribution to the scalar potential:

$$V \sim \frac{1}{|S|^3}\left(\frac{e^{4t}\xi_{\rm UV}^2}{1 - \frac{1}{16\pi^2} + \frac{1}{8\pi^2}(t + \frac{1}{2}e^{-2t})} - 3P^4 e^{4t}|W_0|^2\right) + \dots . \tag{3.3.15}$$

As anticipated, we observe the common $e^{4t}$ factor describing the classical running of the potential, and the monotonically increasing denominator due to the wavefunction renormalization of $X$ for the uplift term. The second term receives no such correction

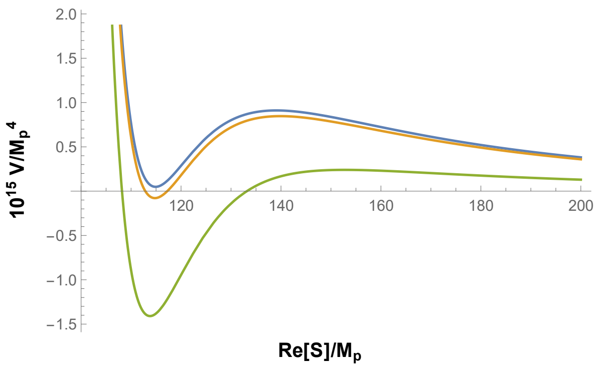


Figure 3.5: The figure shows the KKLT potential for RG time $t = 0.1,\ 0.3,\ 1$, using the parameter values $A = 1$, $a = 0.1$, $W_0 = -10^{-4}$, $P = 80$, $\xi_{\rm UV} = 1$ and with $(X,T)$ restricted to their critical values. The potential at the critical point becomes negative as early as $t \simeq 0.3$.

by virtue of not containing any dynamical fields that have not been integrated out of the effective theory. Solving (3.3.12) using the typical parameter values $A = 1$, $a = 0.1$, $W_0 = -10^{-4}$ and $P = 80\sqrt{\xi_{\rm UV}}$ suggests that a solution might exist for $t \sim 193$, but this is well beyond the small $t$ regime where the SLPA can be trusted. A more careful analysis of the full form of the potential shows that the KKLT critical point reaches negative energies very early in the RG flow, where the SLPA is still valid, as illustrated in Figure 3.5. Choosing higher values of $\xi_{\rm UV}$ only makes the transition to negative energy happen earlier in the flow. It thus appears that at least for the usual values of the parameters, neither the anomalous dimension of $X$ nor the mechanism for stopping the RG flow via the Hubble scale seem to be able to save the original critical point from reaching negative energies.

## 3.4 Comments

In the previous sections we have analyzed the possible composite states that can be generated from the goldstino in the Volkov–Akulov model due to the fermionic self-interaction.

Following an exact renormalization group flow we have studied the low energy effective theory and we have recast it in a form where supersymmetry becomes linearly realized, albeit spontaneously broken. The field space central point that would correspond to the original vacuum turns out to be unstable, and the same happens when we couple the low energy effective theory to 4D N=1 Supergravity. Similarly, when we couple the system to an additional chiral superfield representing the Kähler modulus of KKLT, the instability persists. At this stage it is unclear what the origin of this instability in a full string theory setup is, and whether there are alternative models where a stable de Sitter vacuum can exist. However, our analysis shows that, when an anti-brane is used for uplift purposes, if the effective description is truly four-dimensional such that 4D N=1 non-linear supersymmetry is invoked, stability should not be taken for granted unless the composite goldstino states are firstly analyzed.

## 3.A ERG equations for chiral supermultiplets

We derive here the ERG equations for a supersymmetric model involving chiral multiplets. Note that certain conventions and normalizations differ from those of [190], which we have used in the bulk of the chapter. These will be pointed out when they will occur.

Let us consider the generating functional

$$\mathcal{Z}(\mu) = \int \mathcal{D}\Phi\, e^{L[\Phi;\mu]} = \int \mathcal{D}\Phi\, e^{L[\Phi;\Lambda]} = \mathcal{Z}(\Lambda), \tag{3.A.1}$$

where $\Lambda$ represents the UV cut-off and $\mu$ is an energy scale of interest below $\Lambda$; $\Phi$ is a collective symbol denoting all the fields of our theory, with individual fields labeled as $\Phi_A$, and $L[\Phi;\mu]$ is the (Euclidean) action at a given energy scale $\mu$.
Since n-point correlation functions have to be insensitive to modifications of the energy scale, we require that

$$\dot{\mathcal{Z}}(\mu) \equiv -\mu\partial_\mu \mathcal{Z}(\mu) = \int \mathcal{D}\Phi \dot{L}[\Phi;\mu] e^{L[\Phi;\mu]} = 0, \tag{3.A.2}$$

where $\dot{L}[\Phi;\mu] \equiv -\mu\partial_\mu L[\Phi;\mu]$. In order for (3.A.2) to hold, we require that

$$\int \mathcal{D}\Phi \dot{L}[\Phi;\mu] e^{L[\Phi;\mu]} = \int \mathcal{D}\Phi \frac{\delta}{\delta\Phi_A}\left(\Psi_A[\Phi;\mu] e^{L[\Phi;\mu]}\right), \tag{3.A.3}$$

where our convention for the variational derivatives is defined by, say,

$$\frac{\delta\Phi_A(p)}{\delta\Phi_B(k)} = (2\pi)^4 \delta_A^B \delta^{(4)}(p-k) \tag{3.A.4}$$

(with the normalization of the momentum matching $\delta$-function including a factor of $(2\pi)^4$ relative to the normalization used in the previous sections), and $\{\Psi_A[\Phi;\mu]\}_A$ are some functionals of the fields. Note that different choices for $\{\Psi_A\}_A$ lead to different parametrizations of the RG flow, but do not change the physics. We then require that

$$\dot{L}[\Phi;\mu] = e^{-L[\Phi;\mu]}\frac{\delta}{\delta\Phi_A}\left(\Psi_A[\Phi;\mu] e^{L[\Phi;\mu]}\right), \tag{3.A.5}$$

and try to choose the $\{\Psi_A\}_A$ that result in a convenient expression for the couplings of the various interactions.

### 3.A.1 Bosons

To proceed, we split the action into a propagator and an interaction piece. Namely, we write the action $L[\Phi;\mu]$ as

$$\begin{aligned} L[\Phi;\mu] &= L_{\text{prop.}}[\Phi;\mu] + L_{\text{int.}}[\Phi;\mu] = \\ &= \int \frac{d^4p}{(2\pi)^4}\left[\frac{1}{2}\Phi_A(-p)\left(C^{(\Phi)}_{AB}(p,\mu)\right)^{-1}\Phi_B(p)\right] + L_{\text{int.}}[\Phi;\mu], \end{aligned} \tag{3.A.6}$$

where $C^{(\Phi)}_{AB}(p,\mu)$ defines the propagator including a regulator function. We now restrict our attention to $\{\Phi_A\}_A$ being the bosonic fields $\{\phi^a\}_a$ or $\{F^a\}_a$. Since we are assuming massless propagators, we can diagonalize them through appropriate field redefinitions so that

$$\begin{aligned} C^{(\Phi)}_{AB}\Big|_{\Phi=\phi} &= C^{(\phi)}_{ab} = -\frac{c(p^2/\mu^2)}{p^2}\delta_{ab} \quad \text{for real scalar fields } \phi^a\,; \\ C^{(\Phi)}_{AB}\Big|_{\Phi=F} &= C^{(F)}_{ab} = c(p^2/\mu^2)\delta_{ab} \quad \text{for real auxiliary fields } F^a\,. \end{aligned} \tag{3.A.7}$$

Notice that there is a factor of $\sqrt{2}$-difference in the normalization of the auxiliary fields compared to that used in the bulk of the chapter. For the real scalar and auxiliary fields this diagonalization allows us to omit the Kronecker-$\delta$ and the species index from the $C^{(\Phi)}$'s. The scaling dimensions of these propagators determine the (classical) dimensions of the corresponding fields, which we will denote as $\Delta_\Phi$. In momentum space they are $\Delta_\phi = -3$ and $\Delta_F = -2$. This, in turn, determines the scaling dimensions of the various couplings in $L_{\text{int.}}$. We will denote such couplings as $\{g_\lambda\}_\lambda$ and their scaling dimensions as $\{\Delta_\lambda\}_\lambda$.

At this point, let us observe that only the couplings and the propagator have an internal $\mu$ dependence, while the fields and the momenta are $\mu$-independent. We wish, however, to work with dimensionless fields, couplings and momenta and so we carefully track their $\mu$ dependence. The dimensionless quantities are defined as

$$\hat{p} = p\mu^{-1}, \quad \hat{\Phi}_A(\hat{p}) = \Phi_A(\mu\hat{p})\mu^{-\Delta_A} \quad \text{and} \quad \hat{g}_\lambda(\mu) = g_\lambda(\mu)\mu^{-\Delta_\lambda}. \tag{3.A.8}$$

We also define the dimensionless propagators as

$$\hat{C}^{(\Phi)}(\hat{p}) = \mu^{-4-2\Delta_A}C^{(\Phi)}(p,\mu) \tag{3.A.9}$$

in such a way that the bosonic action propagator term can also be written as

$$L_{\text{prop.}}[\hat{\Phi};\mu] = \int \frac{d^4\hat{p}}{(2\pi)^4}\left[\frac{1}{2}\hat{\Phi}_A(-\hat{p})\left(\hat{C}^{(\Phi)}(\hat{p})\right)^{-1}\hat{\Phi}_A(\hat{p})\right], \tag{3.A.10}$$

where

$$\begin{aligned} \hat{C}^{(\Phi)}\Big|_{\Phi=\phi} &= \hat{C}^{(\phi)} = -\frac{c(\hat{p}^2)}{\hat{p}^2} \quad \text{for real scalar fields } \phi^a\,; \\ \hat{C}^{(\Phi)}\Big|_{\Phi=F} &= \hat{C}^{(F)} = c(\hat{p}^2) \quad \text{for real auxiliary fields } F^a. \end{aligned} \tag{3.A.11}$$

To compute the left hand side of (3.A.5), we will need the $\mu$-derivatives of all the dimensionless quantities that we have introduced. From (3.A.8) we obtain

$$-\mu\partial_\mu \hat{p} = -p\mu\partial_\mu \mu^{-1} = p\mu^{-1} = \hat{p} \tag{3.A.12}$$

and similarly for the momentum space measure, $-\mu\partial_\mu d\hat{p} = d\hat{p}$. We also have

$$\begin{aligned} -\mu\partial_\mu \hat{\Phi}_A(\hat{p}) &= -\mu\partial_\mu\left(\Phi_A(\mu\hat{p})\mu^{-\Delta_A}\right) = \Delta_A\mu^{-\Delta_A}\Phi_A(\mu\hat{p}) - \mu\Phi'_A(\mu\hat{p})\partial_\mu(\mu\hat{p})\mu^{-\Delta_A} \\ &= \Delta_A\mu^{-\Delta_A}\Phi_A(\mu\hat{p}) - \mu\Phi'_A(\mu\hat{p})(\hat{p}+\mu\partial_\mu\hat{p})\mu^{-\Delta_A} \\ &= \Delta_A\mu^{-\Delta_A}\Phi_A(\mu\hat{p}) - \mu\Phi'_A(\mu\hat{p})(\hat{p}-\hat{p})\mu^{-\Delta_A} \\ &= \Delta_A\hat{\Phi}_A(\hat{p}); \\ -\mu\partial_\mu c(\hat{p}) &= (-\mu\partial_\mu\hat{p})\partial_{\hat{p}}c(\hat{p}) = \hat{p}\partial_{\hat{p}}c(\hat{p}), \end{aligned} \tag{3.A.13}$$

where $\Phi'_A$ denotes a regular derivative of $\Phi_A$ with respect to its argument.

Armed with these expressions we can compute

$$\begin{aligned} \dot{\hat{C}}^{(\phi)}(\hat{p}) &= -(\mu\partial_\mu\hat{p})\partial_{\hat{p}}\hat{C}^{(\phi)}(\hat{p}) = \hat{p}\partial_{\hat{p}}\left(-\frac{c(\hat{p})}{\hat{p}^2}\right) = -\frac{\left(\hat{p}\partial_{\hat{p}}c(\hat{p})\right)}{\hat{p}^2} + 2\frac{c(\hat{p})}{\hat{p}^2}; \\ \dot{\hat{C}}^{(F)}(\hat{p}) &= -(\mu\partial_\mu\hat{p})\partial_{\hat{p}}\hat{C}^{(F)}(\hat{p}) = \hat{p}\partial_{\hat{p}}c(\hat{p}). \end{aligned} \tag{3.A.14}$$

Defining, then,

$$\tilde{C}^{(\phi)}(\hat{p}) = -\frac{\left(\hat{p}\partial_{\hat{p}}c(\hat{p})\right)}{\hat{p}^2}, \quad \tilde{C}^{(F)}(\hat{p}) = \hat{p}\partial_{\hat{p}}c(\hat{p}), \tag{3.A.15}$$

we are allowed to write

$$\dot{\hat{C}}^{(\Phi)}(\hat{p}) = \tilde{C}^{(\Phi)}(\hat{p}) + (4+2\Delta_A)\hat{C}^{(\Phi)}(\hat{p}). \tag{3.A.16}$$

From this we can compute

$$\begin{aligned} &\dot{L}^{(\Phi_A)}_{\text{prop.}} = -\mu\partial_\mu\int\frac{d^4\hat{p}}{(2\pi)^4}\left[\frac{1}{2}\hat{\Phi}_A(-\hat{p})\left(\hat{C}^{(\Phi)}(\hat{p})\right)^{-1}\hat{\Phi}_A(\hat{p})\right] = \\ &= \frac{1}{2}\int\frac{d^4\hat{p}}{(2\pi)^4}\left[(4+2\Delta_A)\hat{\Phi}_{A,-\hat{p}}\left(\hat{C}^{(\Phi)}(\hat{p})\right)^{-1}\hat{\Phi}_{A,\hat{p}} - \hat{\Phi}_{A,-\hat{p}}\frac{\dot{\hat{C}}^{(\Phi)}(\hat{p})}{(\hat{C}^{(\Phi)})^2}\hat{\Phi}_{A,\hat{p}}\right] = \\ &= -\frac{1}{2}\int\frac{d^4\hat{p}}{(2\pi)^4}\left[\hat{\Phi}_A(-\hat{p})\frac{\tilde{C}^{(\Phi)}(\hat{p})}{(\hat{C}^{(\Phi)}(\hat{p}))^2}\hat{\Phi}_A(\hat{p})\right], \end{aligned} \tag{3.A.17}$$

so that the left hand side of (3.A.5) is

$$\dot{L} = -\int \frac{d^4\hat{p}}{(2\pi)^4} \left[ \frac{1}{2} \hat{\Phi}_A(-\hat{p}) \frac{\tilde{C}^{(\Phi)}(\hat{p})}{(\hat{C}^{(\Phi)}(\hat{p}))^2} \hat{\Phi}_A(\hat{p}) \right] + \dot{L}^{(\chi)}_{\text{prop.}} + \dot{L}_{\text{int.}}\,, \tag{3.A.18}$$

where in the first term a sum over all scalar and auxiliary fields is understood.

We now show that for a suitable choice of $\{\Psi_A\}_{A=\phi,F}$ this first term also appears on the right hand side of (3.A.5) and do not take part to the final ERG equation. A similar procedure, which we will describe further, allows to get rid of the fermionic propagator terms.

The appropriate choice of $\Psi_A[\hat{\Phi};\mu]$ is as follows:

$$\Psi_A[\hat{\Phi};\mu] = \int \frac{d^4\hat{k}}{(2\pi)^4} \frac{1}{2} \tilde{C}^{(\Phi)}(\hat{k}) \frac{\delta \tilde{L}[\hat{\Phi};\mu]}{\delta \hat{\Phi}_A(\hat{k})} \quad \text{with} \quad \tilde{L} = -L_{\text{prop.}} + L_{\text{int.}}, \tag{3.A.19}$$

and $\tilde{C}^{(\Phi)}$ being defined by (3.A.15). Omitting the functional and function variables of the action, the right-hand side of (3.A.5) consequently becomes

$$\begin{aligned} e^{-L} \frac{\delta}{\delta \hat{\Phi}_A} \left( \Psi_A e^L \right) &= \int \frac{d^4\hat{k}}{(2\pi)^4} \frac{1}{2} \tilde{C}^{(\Phi)}(\hat{k}) \left( \frac{\delta^2 \tilde{L}}{\delta \hat{\Phi}_A(-\hat{k}) \delta \hat{\Phi}_A(\hat{k})} + \frac{\delta \tilde{L}}{\delta \hat{\Phi}_A(\hat{k})} \frac{\delta L}{\delta \hat{\Phi}_A(-\hat{k})} \right) = \\ &= \int \frac{d^4\hat{k}}{(2\pi)^4} \frac{1}{2} \tilde{C}^{(\Phi)}(\hat{k}) \left( \frac{\delta^2 L_{\text{int.}}}{\delta \hat{\Phi}_A(-\hat{k}) \delta \hat{\Phi}_A(\hat{k})} + \frac{\delta L_{\text{int.}}}{\delta \hat{\Phi}_A(\hat{k})} \frac{\delta L_{\text{int.}}}{\delta \hat{\Phi}_A(-\hat{k})} + \right. \\ &\qquad \left. - \frac{\delta^2 L_{\text{prop.}}}{\delta \hat{\Phi}_A(-\hat{k}) \delta \hat{\Phi}_A(\hat{k})} - \frac{\delta L_{\text{prop.}}}{\delta \hat{\Phi}_A(\hat{k})} \frac{\delta L_{\text{prop.}}}{\delta \hat{\Phi}_A(-\hat{k})} \right), \end{aligned} \tag{3.A.20}$$

where no sum over $A$ is implied. The evaluation of the terms in the last line gives

$$\begin{aligned} -\int \frac{d^4\hat{k}}{(2\pi)^4} \frac{1}{2} \tilde{C}^{(\Phi)}(\hat{k}) \frac{\delta^2 L_{\text{prop.}}}{\delta \hat{\Phi}_A(-\hat{k}) \delta \hat{\Phi}_A(\hat{k})} &= -\int \frac{d^4\hat{k}}{(2\pi)^4} \frac{1}{2} \tilde{C}^{(\Phi)}(\hat{k}) \left( \hat{C}^{(\Phi)}(\hat{k}) \right)^{-1} (2\pi)^4 \delta^{(4)}(0) = \\ &= -\frac{1}{2} \int \frac{d^4\hat{k}}{(2\pi)^4} \frac{\hat{k} \partial_{\hat{k}} c(\hat{k})}{c(\hat{k})} (2\pi)^4 \delta^{(4)}(0), \end{aligned} \tag{3.A.21}$$

which can be absorbed into the measure, as well as

$$\begin{aligned} -\int \frac{d^4\hat{k}}{(2\pi)^4} \frac{1}{2} \tilde{C}^{(\Phi)}(\hat{k}) \frac{\delta L_{\text{prop.}}}{\delta \hat{\Phi}_A(\hat{k})} \frac{\delta L_{\text{prop.}}}{\delta \hat{\Phi}_A(-\hat{k})} &= \\ = -\int \frac{d^4\hat{k}}{(2\pi)^4} \left[ \frac{1}{2} \hat{\Phi}_A(-\hat{k}) \frac{\tilde{C}^{(\Phi)}(\hat{k})}{(\hat{C}^{(\Phi)}(\hat{k}))^2} \hat{\Phi}_A(\hat{k}) \right], \end{aligned} \tag{3.A.22}$$

which is precisely the term that we need to cancel its analogous on the other side. Putting things together, we obtain the ERG equation

$$\begin{aligned} \dot{L}_{\text{int.}}[\hat{\Phi};\mu] = \sum_A \int \frac{d^4\hat{k}}{(2\pi)^4} \frac{1}{2} \tilde{C}^{(\Phi)}(\hat{k}) \left( \frac{\delta^2 L_{\text{int.}}[\hat{\Phi};\mu]}{\delta \hat{\Phi}_A(-\hat{k}) \delta \hat{\Phi}_A(\hat{k})} + \frac{\delta L_{\text{int.}}[\hat{\Phi};\mu]}{\delta \hat{\Phi}_A(\hat{k})} \frac{\delta L_{\text{int.}}[\hat{\Phi};\mu]}{\delta \hat{\Phi}_A(-\hat{k})} \right) + \\ + \text{fermionic contributions.} \end{aligned} \tag{3.A.23}$$

We can now expand $\dot{L}_{\text{int.}}$ so that we can isolate the behaviour of individual coupling constants. As already mentioned, we denote these couplings as $\{g_\lambda\}_\lambda$ and their classical scaling dimensions as $\{\Delta_\lambda\}_\lambda$, where $\lambda$ is an index labelling the various interaction terms. A non-derivative interaction term will have the form

$$L_{\text{int.},\lambda} = \int \left(\prod_A \frac{d^4 p_A}{(2\pi)^4}\right) g_\lambda \prod_A \Phi_A(p_A) \times (2\pi)^4 \delta^{(4)}\left(\sum_A p_A\right), \tag{3.A.24}$$

where $\Delta_\lambda - 4 + \sum_A (4 + \Delta_A) = 0$. (Note that the momentum $\delta$-function has dimension $-4$, in order to properly integrate to unit over momentum space). The conversion to dimensionless quantities gives

$$\begin{aligned} L_{\text{int.},\lambda} &= \int \left(\prod_A \frac{\mu^4 d^4 \hat{p}_A}{(2\pi)^4}\right) \mu^{\Delta_\lambda} \hat{g}_\lambda \prod_A \mu^{\Delta_A} \hat{\Phi}_A(\hat{p}_A) \times \frac{(2\pi)^4}{\mu^4} \delta^{(4)}\left(\sum_A \hat{p}_A\right) = \\ &= \int \left(\prod_A \frac{d^4 \hat{p}_A}{(2\pi)^4}\right) \hat{g}_\lambda \prod_A \hat{\Phi}_A(\hat{p}_A) \times (2\pi)^4 \delta^{(4)}\left(\sum_A \hat{p}_A\right), \end{aligned} \tag{3.A.25}$$

which is now manifestly dimensionless and all the quantities have the $\mu$ dependence derived above. Denoting $-\mu\partial_\mu Q$ as $\dot{Q}$ for some quantity $Q$ and using (3.A.12) and (3.A.13) we obtain

$$\begin{aligned} \dot{L}_{\text{int.},\lambda} &= \int \left(\prod_A \frac{d^4 \hat{p}_A}{(2\pi)^4} \hat{\Phi}_A(\hat{p}_A)\right) \left[\left(\sum_A (4 + \Delta_A) - 4\right) \hat{g}_\lambda + \dot{\hat{g}}_\lambda\right] (2\pi)^4 \delta^{(4)}\left(\sum_A \hat{p}_A\right) = \\ &= \int \left(\prod_A \frac{d^4 \hat{p}_A}{(2\pi)^4} \hat{\Phi}_A(\hat{p}_A)\right) \left(-\Delta_\lambda \hat{g}_\lambda + \dot{\hat{g}}_\lambda\right) \times (2\pi)^4 \delta^{(4)}\left(\sum_A \hat{p}_A\right). \end{aligned} \tag{3.A.26}$$

The above expression also holds for couplings involving fermions and it is straightforward to show that a similar final expression also holds for higher derivative interactions, but we do not use these in the present work. Overall we have

$$\dot{L}_{\text{int.}} = \sum_\lambda \dot{L}_{\text{int.},\lambda} \tag{3.A.27}$$

and we can isolate the RG flow of individual couplings by matching terms involving the same field combinations on each side of (3.A.23).

### 3.A.2 Fermions

For a single Weyl fermion we can write the propagator part of the action as

$$L^{(\chi)}_{\text{prop.}} = \int \frac{d^4 \hat{k}}{(2\pi)^4} \chi^\alpha(\hat{k}) \hat{C}^{-1}_{\alpha\dot{\alpha}}(\hat{k}) \overline{\chi}^{\dot{\alpha}}(-\hat{k}), \tag{3.A.28}$$

where the fields are dimensionless (and are related to their dimensionful partners by $\Delta_\chi = -5/2$), but we omit the hats on them to avoid notation clutter, and we also have

$$\hat{C}^{-1}_{\alpha\dot{\alpha}}(\hat{k}) = -ic^{-1}(\hat{k})\sigma^m_{\alpha\dot{\alpha}}\hat{k}_m. \tag{3.A.29}$$

Note that the Euclidean $\sigma$-matrices satisfy $(\sigma^m\overline{\sigma}^n + \sigma^n\overline{\sigma}^m)_\alpha^{\ \beta} = 2\delta^{mn}\delta_\alpha^\beta$. The propagator is then

$$\hat{C}^{\dot{\alpha}\alpha}(\hat{k}) = c(\hat{k})\frac{i\hat{\not{k}}^{\dot{\alpha}\alpha}}{\hat{k}^2} \tag{3.A.30}$$

and, as a consequence,

$$\dot{\hat{C}}^{\dot{\alpha}\alpha} = i\big(\hat{k}\partial_{\hat{k}}c(\hat{k})\big)\frac{\hat{\not{k}}^{\dot{\alpha}\alpha}}{\hat{k}^2} - ic(\hat{k})\frac{\hat{\not{k}}^{\dot{\alpha}\alpha}}{\hat{k}^2}. \tag{3.A.31}$$

Therefore,

$$\begin{aligned}\dot{L}^{(\chi)}_{\text{prop.}} = \int \frac{d^4\hat{k}}{(2\pi)^4}\Bigg[&\left(4 + 2\cdot\left(-\frac{5}{2}\right) + 1\right)\chi^\alpha(\hat{k})\hat{C}^{-1}_{\alpha\dot{\alpha}}(\hat{k})\overline{\chi}^{\dot{\alpha}}(-\hat{k}) + \\ &- \chi^\alpha(\hat{k})\hat{C}^{-1}_{\alpha\dot{\beta}}(\hat{k})\tilde{C}^{\dot{\beta}\beta}(\hat{k})\hat{C}^{-1}_{\beta\dot{\alpha}}(\hat{k})\overline{\chi}^{\dot{\alpha}}(-\hat{k})\Bigg],\end{aligned} \tag{3.A.32}$$

where

$$\tilde{C}^{\dot{\alpha}\alpha} = \left[\big(\hat{k}\partial_{\hat{k}}c(\hat{k})\big)\frac{i\hat{\not{k}}}{\hat{k}^2}\right]^{\dot{\alpha}\alpha}. \tag{3.A.33}$$

As in the bosonic case, this term will need to be cancelled by terms coming from

$$\begin{aligned}&\int \frac{d^4\hat{p}}{(2\pi)^4}\left[\Psi_1^\alpha\frac{\delta L_{\text{prop.}}}{\delta\chi^\alpha(\hat{p})} + \frac{\delta L_{\text{prop.}}}{\delta\overline{\chi}^{\dot{\alpha}}(-\hat{p})}\Psi_2^{\dot{\alpha}}\right] = \\ &= \int \frac{d^4\hat{p}}{(2\pi)^4}\left[\Psi_1^\alpha\left(\hat{C}^{-1}_{\alpha\dot{\alpha}}(\hat{p})\overline{\chi}^{\dot{\alpha}}(-\hat{p})\right) - \left(\chi^\alpha(\hat{p})\hat{C}^{-1}_{\alpha\dot{\alpha}}(\hat{p})\right)\Psi_2^{\dot{\alpha}}\right].\end{aligned} \tag{3.A.34}$$

An appropriate choice of $\Psi_{1,2}$ appears to be

$$\Psi_1^\beta(\hat{k}) = -\tfrac{1}{2}\chi^\alpha(\hat{k})\hat{C}^{-1}_{\alpha\dot{\beta}}(\hat{k})\tilde{C}^{\dot{\beta}\beta} + \Psi_{1,\text{int.}}; \tag{3.A.35}$$

$$\Psi_2^{\dot{\beta}}(\hat{k}) = \tfrac{1}{2}\tilde{C}^{\dot{\beta}\beta}\hat{C}^{-1}_{\beta\dot{\alpha}}(\hat{k})\overline{\chi}^{\dot{\alpha}}(-\hat{k}) + \Psi_{2,\text{int.}} \tag{3.A.36}$$

and by analogy with the bosonic case we consequently arrive at a complete ansatz for $\Psi_{1,2}$:

$$\begin{aligned}\Psi_1^\beta(\hat{k}) &= -\frac{1}{2}\frac{\delta\tilde{L}}{\delta\overline{\chi}^{\dot{\beta}}(-\hat{k})}\tilde{C}^{\dot{\beta}\beta}(\hat{k});\\ \Psi_2^{\dot{\beta}}(\hat{k}) &= -\frac{1}{2}\tilde{C}^{\dot{\beta}\beta}(\hat{k})\frac{\delta\tilde{L}}{\delta\chi^\beta(\hat{k})},\end{aligned} \tag{3.A.37}$$

where as before $\tilde{L} = -L_{\text{prop.}} + L_{\text{int.}}$. With these expressions, we have

$$\begin{aligned}
e^{-L}\frac{\delta}{\delta\chi^\alpha}\left(\Psi_1^\alpha e^L\right) &= -\frac{1}{2}\tilde{C}^{\dot{\alpha}\alpha}\left(\frac{\delta^2\tilde{L}}{\delta\chi^\alpha\delta\overline{\chi}^{\dot{\alpha}}} - \frac{\delta\tilde{L}}{\delta\overline{\chi}^{\dot{\alpha}}}\frac{\delta L}{\delta\chi^\alpha}\right) = \\
&= \frac{1}{2}\tilde{C}^{\dot{\alpha}\alpha}\left(\frac{\delta^2\tilde{L}}{\delta\overline{\chi}^{\dot{\alpha}}\delta\chi^\alpha} + \frac{\delta\tilde{L}}{\delta\overline{\chi}^{\dot{\alpha}}}\frac{\delta L}{\delta\chi^\alpha}\right) = \\
&= \frac{1}{2}\tilde{C}^{\dot{\alpha}\alpha}\left(\frac{\delta^2(-L_{\text{prop.}} + L_{\text{int.}})}{\delta\overline{\chi}^{\dot{\alpha}}\delta\chi^\alpha} + \frac{\delta(-L_{\text{prop.}} + L_{\text{int.}})}{\delta\overline{\chi}^{\dot{\alpha}}}\frac{\delta(L_{\text{prop.}} + L_{\text{int.}})}{\delta\chi^\alpha}\right) \qquad (3.A.38) \\
&= \frac{1}{2}\tilde{C}^{\dot{\alpha}\alpha}\Bigg(-\hat{C}^{-1}_{\alpha\dot{\alpha}}(2\pi)^4\delta^{(4)}(0) + \frac{\delta^2 L_{\text{int.}}}{\delta\overline{\chi}^{\dot{\alpha}}\delta\chi^\alpha} - \frac{\delta L_{\text{prop.}}}{\delta\overline{\chi}^{\dot{\alpha}}}\frac{\delta L_{\text{prop.}}}{\delta\chi^\alpha} + \frac{\delta L_{\text{int.}}}{\delta\overline{\chi}^{\dot{\alpha}}}\frac{\delta L_{\text{int.}}}{\delta\chi^\alpha} + \\
&\qquad - \frac{\delta L_{\text{prop.}}}{\delta\overline{\chi}^{\dot{\alpha}}}\frac{\delta L_{\text{int.}}}{\delta\chi^\alpha} + \frac{\delta L_{\text{int.}}}{\delta\overline{\chi}^{\dot{\alpha}}}\frac{\delta L_{\text{prop.}}}{\delta\chi^\alpha}\Bigg)
\end{aligned}$$

and

$$\begin{aligned}
-e^{-L}\frac{\delta}{\delta\overline{\chi}^{\dot{\alpha}}}\left(\Psi_2^{\dot{\alpha}} e^L\right) &= \frac{1}{2}\tilde{C}^{\dot{\alpha}\alpha}\left(\frac{\delta^2\tilde{L}}{\delta\overline{\chi}^{\dot{\alpha}}\delta\chi^\alpha} - \frac{\delta\tilde{L}}{\delta\chi^\alpha}\frac{\delta L}{\delta\overline{\chi}^{\dot{\alpha}}}\right) = \\
&= \frac{1}{2}\tilde{C}^{\dot{\alpha}\alpha}\left(\frac{\delta^2\tilde{L}}{\delta\overline{\chi}^{\dot{\alpha}}\delta\chi^\alpha} + \frac{\delta L}{\delta\overline{\chi}^{\dot{\alpha}}}\frac{\delta\tilde{L}}{\delta\chi^\alpha}\right) = \\
&= \frac{1}{2}\tilde{C}^{\dot{\alpha}\alpha}\left(\frac{\delta^2(-L_{\text{prop.}} + L_{\text{int.}})}{\delta\overline{\chi}^{\dot{\alpha}}\delta\chi^\alpha} + \frac{\delta(L_{\text{prop.}} + L_{\text{int.}})}{\delta\overline{\chi}^{\dot{\alpha}}}\frac{\delta(-L_{\text{prop.}} + L_{\text{int.}})}{\delta\chi^\alpha}\right) \qquad (3.A.39) \\
&= \frac{1}{2}\tilde{C}^{\dot{\alpha}\alpha}\Bigg(-\hat{C}^{-1}_{\alpha\dot{\alpha}}(2\pi)^4\delta^{(4)}(0) + \frac{\delta^2 L_{\text{int.}}}{\delta\overline{\chi}^{\dot{\alpha}}\delta\chi^\alpha} - \frac{\delta L_{\text{prop.}}}{\delta\overline{\chi}^{\dot{\alpha}}}\frac{\delta L_{\text{prop.}}}{\delta\chi^\alpha} + \frac{\delta L_{\text{int.}}}{\delta\overline{\chi}^{\dot{\alpha}}}\frac{\delta L_{\text{int.}}}{\delta\chi^\alpha} + \\
&\qquad + \frac{\delta L_{\text{prop.}}}{\delta\overline{\chi}^{\dot{\alpha}}}\frac{\delta L_{\text{int.}}}{\delta\chi^\alpha} - \frac{\delta L_{\text{int.}}}{\delta\overline{\chi}^{\dot{\alpha}}}\frac{\delta L_{\text{prop.}}}{\delta\chi^\alpha}\Bigg),
\end{aligned}$$

where we see that the last lines of each of the above equations will cancel each other when added together; the first term in each equation can again be absorbed into the measure and the third terms, when summed up, will, by construction, cancel $\dot{L}^{(\chi)}_{\text{prop.}}$, which we have computed earlier. The invariance of the partition function requires

$$\dot{L} = -e^{-L}\left\{\frac{\delta}{\delta\chi^\alpha}\left(\Psi_1^\alpha e^L\right) - \frac{\delta}{\delta\overline{\chi}^{\dot{\alpha}}}\left(\Psi_2^{\dot{\alpha}} e^L\right)\right\} + \text{ bosonic contributions}, \qquad (3.A.40)$$

and therefore the final contribution to the flow of $L_{\text{int.}}$ is

$$\begin{aligned}
\dot{L}_{\text{int.}} = -\int \frac{d^4\hat{k}}{(2\pi)^4}\tilde{C}^{\dot{\alpha}\alpha}(\hat{k})\left(\frac{\delta^2 L_{\text{int.}}}{\delta\overline{\chi}^{\dot{\alpha}}(-\hat{k})\delta\chi^\alpha(\hat{k})} + \frac{\delta L_{\text{int.}}}{\delta\overline{\chi}^{\dot{\alpha}}(-\hat{k})}\frac{\delta L_{\text{int.}}}{\delta\chi^\alpha(\hat{k})}\right) + \\
+ \text{ bosonic contributions.}
\end{aligned} \qquad (3.A.41)$$

The $(2\pi)^4$ in the denominator in (3.A.41) appears, instead, in the numerator in (3.1.64) because, as we have already explained, in (3.A.41) we have variational derivatives whereas in (3.1.64) we have partial derivatives, and they differ by a factor of $(2\pi)^4$ when it comes to momentum matching.

We make a final observation. For each massless chiral multiplet, which consists of a complex scalar field, a Weyl fermion and a complex auxiliary field, it is straightforward to check that the terms that would otherwise have to be absorbed into the measure in the right hand side of the ERG equation actually sum up to zero, which is a non trivial relation required by supersymmetry.

# Chapter 4

# Goldstino condensation at large N

As the reader knows or may have learnt from the discussion above, fermionic condensation can play an important role in understanding the vacuum structure of a quantum theory. An aspect that makes the basic systems where fermionic condensation has been understood, namely the Nambu–Jona-Lasinio model [174, 175] or the Gross–Neveu model [216][1], stand out is that, due to a large N number of fermion species, the quantum effective potential can be evaluated with arbitrary precision in a 1/N expansion, and the stationary points can be analyzed with confidence.
In Chapter 3 we have brought to the forefront a new type of fermionic condensation, namely the condensation of the N=1 goldstino [61]. This effect has a crucial impact on string flux compactifications and signals an intrinsic instability that may be generically present in anti-brane uplifts to de Sitter vacua [29, 147–151]. This instability also adds a new obstacle to obtaining 4D long-lived de Sitter critical configurations from supersymmetric string theories [10, 67, 153, 158, 218][2], and may further restrict de Sitter solutions in 4D N=1 Supergravity, thus extending some previous results [60, 98, 111, 135, 222]. Furthermore, as we have observed in Section 3.3, within N=1 Supergravity the goldstino condensation effect seems to persist [61] and it should be related to the gravitino condensation, which again shows a tachyonic instability [178, 181–184][3].

Besides the discussion of Chapter 3, where the functional renormalization group [190] was needed because a perturbative loop expansion would not be trustable, in the following pages we would like to provide a demonstration of the goldstino condensation phenomenon by exploiting a well-understood method of QFT: a large N analysis.

[1] For textbook discussions see e.g. [209, 217].

[2] Stringy de Sitter vacua that do not make use of non-linear supersymmetry, e.g. [219–221], could possibly evade the goldstino condensation instability. However, this is not certain since the effect may still take place within theories with linearly realized supersymmetry. Such a question needs to be addressed separately.

[3] The condensation of gravitini has also been studied in 4D N=2 Supergravity as the sole source for a de Sitter uplift [126], but with full stability still being an open question.

Even though it is of course the 4D N=1 goldstino condensation that has the most phenomenological value, this investigation will put the generic existence of the goldstino composite state on firmer theoretical grounds, being supportive from a different perspective[4].

We will work with a single and well-established model that includes only fermions: the Volkov–Akulov (VA) model with N non-linearly realized supersymmetries [172, 204, 224]. Since these supersymmetries are non-linearly realized, their number can be arbitrarily large, and the same holds for the number of the accompanying Goldstone fermions, the goldstini. This means that we are not restricted to the typical $\mathrm{N} \leq 4$ of linearly realized supersymmetry (or $\mathrm{N} \leq 8$ for Supergravity).

The aim of the upcoming chapter is neither to extract nor to study a phenomenological result, but simply to answer one and only one question, whether a system with N goldstini can have a well-controlled stationary point described by goldstino condensation. Our main result will be an affirmative answer to this question, thus giving complementary support to the findings of Chapter 3. We will further suggest that such a configuration corresponds to the restoration of supersymmetry, and we will also give some arguments in favour of its protection from higher-order corrections.

Finally, we will analyze a system that is more relevant for string flux compactifications, where, among N fermions, only one is a 4D N=1 goldstino, with all the others becoming pseudo-goldstini, once they acquire a mass.

## 4.1 N goldstini

### 4.1.1 The effective action and large N

We work with a system that has N non-linearly realized supersymmetries and focus explicitly on the goldstino sector, which is described by the Lagrangian

$$\mathcal{L} = -\frac{\mathrm{N}f^2}{2} \det\left[A_m{}^a\right], \tag{4.1.1}$$

where the goldstino vielbein $A_m{}^a$ is defined as

$$A_m{}^a = \delta_m^a + \frac{i}{\mathrm{N}f^2} \sum_{J=1}^{\mathrm{N}} \partial_m G_J \sigma^a \overline{G}^J - \frac{i}{\mathrm{N}f^2} \sum_{I=1}^{\mathrm{N}} G_I \sigma^a \partial_m \overline{G}^I, \tag{4.1.2}$$

[4] In [223] a *discontinuity* is discussed for the goldstino condensation of [61]. Such discontinuities can indeed appear when condensations take place, but they do not signal an inconsistency *per se*. For example, in the so called $\mathrm{CP}^{N-1}$ model (see e.g. [217]), the original classical critical point is not a critical point of the quantum theory at all, for any finite value of the coupling. Interestingly, the specific bosonic model has a classically spontaneously broken symmetry which is dynamically restored. Then, fields transforming linearly under the restored symmetry are built from the classical Goldstone fields, as it also happens in [61].

and $f$ is the supersymmetry breaking order parameter whose mass dimension is $[f] = 2$. Note that the actual supersymmetry breaking scale is $N^{1/4}\sqrt{f}$.
The system has a global U(N) R-symmetry under which the spinors $G_I$ and $\overline{G}^I$ transform in the fundamental and anti-fundamental representation, respectively. The goldstino vielbein ${A_m}^a$ is, instead, a singlet under such U(N).
This theory is defined with a cut-off $\Lambda$ for which we typically assume

$$\mathrm{N}f^2 > \Lambda^4. \tag{4.1.3}$$

For later convenience, and as it is typical in large N models, we have already extracted the N coefficient in front of the starting Lagrangian (4.1.1): the 't Hooft limit [225] thus corresponds to $\mathrm{N} \to +\infty$ while keeping $f$ fixed. Note that, when N is large, both $\sqrt{f} > \Lambda$ and $\sqrt{f} < \Lambda$ can satisfy (4.1.3).
Let us also observe that the number N of non-linear supersymmetries clearly matches the number of goldstini $I, J = 1, \ldots, \mathrm{N}$, whose transformations are

$$\delta G_{I\,\alpha} = \sqrt{\mathrm{N}} f \epsilon_{I\,\alpha} + \frac{i}{\sqrt{\mathrm{N}} f} \sum_{J=1}^{\mathrm{N}} \left( G_J \sigma^m \,\overline{\epsilon}^J - \epsilon_J \sigma^m \overline{G}^J \right) \partial_m G_{I\,\alpha}. \tag{4.1.4}$$

Making use of the definition (4.1.2), the leading-order terms of the Lagrangian are

$$\mathcal{L} = -\frac{\mathrm{N}f^2}{2} + i \sum_{I=1}^{\mathrm{N}} G_I \sigma^m \partial_m \overline{G}^I + \mathcal{O}\left(\frac{1}{\mathrm{N}f^2}\right). \tag{4.1.5}$$

The constant $-\mathrm{N}f^2/2$ can be always removed so that there is no divergent term in the large N limit.
From now on we will stop inserting the explicit $\sum_{I=1}^{\mathrm{N}}$ and summation over the same $I, J$ indices will be implied unless otherwise specified.

For the sake of completeness, note that we are using the conventions of [209]: therefore, $\eta_{ab} = \mathrm{diag}(+1, -1, -1, -1)$, $\sigma^a = (\mathbb{1}_2, \vec{\sigma})$, and, because it will be useful later on, $\gamma^a = \begin{pmatrix} 0 & \sigma^a \\ \overline{\sigma}^a & 0 \end{pmatrix}$, with $\overline{\sigma}^a = (\mathbb{1}_2, -\vec{\sigma})$.

To pursue our aim, namely to get access to stationary points that possibly describe goldstino condensation, we write the theory (4.1.1) as

$$\mathcal{L} = -\frac{\mathrm{N}f^2}{2} \det[{e_m}^a] + \frac{\mathrm{N}f^2}{2} {C_a}^m ({e_m}^a - {A_m}^a). \tag{4.1.6}$$

Once we integrate out ${C_a}^m$, we get

$$\frac{\delta \mathcal{L}}{\delta {C_a}^m} = 0 \;\longrightarrow\; {e_m}^a = {A_m}^a \tag{4.1.7}$$

and we recover the model with N goldstini (4.1.1). Equivalently, the path integration over ${C_a}^m$ yields a $\delta$-function at each point of spacetime that enforces the constraint.

The advantage of (4.1.6) is that the action becomes Gaussian in the fermions and their path integration can be performed, leaving behind only a bosonic theory. In doing so, we will also be able to explicitly demonstrate the large N behaviour of the model under consideration.

We start with the path integral

$$Z = \frac{1}{N_0} \int D[e_m{}^a] D[C_b{}^n] D[G_I] D[\overline{G}^J] \exp \left[ i \int d^4x \, \mathcal{L} \right], \tag{4.1.8}$$

where we can split the Lagrangian as $\mathcal{L} = \mathcal{L}_B + \mathcal{L}_F$ with

$$\mathcal{L}_B = -\frac{\mathrm{N} f^2}{2} \det[e_m{}^a] + \frac{\mathrm{N} f^2}{2} C_a{}^m (e_m{}^a - \delta_m{}^a) \tag{4.1.9}$$

and

$$\mathcal{L}_F = \frac{i}{2} C_a{}^m \left( G_I \sigma^a \partial_m \overline{G}^I - \partial_m G_J \sigma^a \overline{G}^J \right). \tag{4.1.10}$$

The $N_0$ stands for the overall normalization of the path integral. To evaluate the fermionic contribution to the path integral we pair the N Weyl goldstini into N/2 Dirac spinors as follows

$$\Psi^A = \begin{pmatrix} G^{(A)} \\ \overline{G}^{(A+\mathrm{N}/2)} \end{pmatrix}, \quad A = 1, \dots, \mathrm{N}/2. \tag{4.1.11}$$

This seems as if we were assuming that N is even. However, we would like to stress that this is just a formality that allows us to easily evaluate the fermion path integral: for odd N the end result would be the same. Thus, the fermionic contribution to (4.1.8) formally reduces to

$$\begin{aligned} Z_F &= \int D[\Psi^A] D[\overline{\Psi}^A] \exp \left[ i \int d^4x \, \frac{i}{2} C_a{}^m \sum_{A=1}^{\mathrm{N}/2} \left( \overline{\Psi}^A \gamma^a \partial_m \Psi^A - \partial_m \overline{\Psi}^A \gamma^a \Psi^A \right) \right] \\ &= \left( \det[i C_a{}^m \gamma^a \partial_m] \right)^{\mathrm{N}/2}, \end{aligned} \tag{4.1.12}$$

up to the overall $N_0$ factor that we will shortly discuss and specify.

If we now bring the determinants into the exponential and then into the effective action we get the full bosonic theory

$$S_{\mathrm{N}} = \mathrm{N} \times \left\{ -\frac{f^2}{2} \int d^4x \left[ \det[e_n{}^b] - C_a{}^m (e_m{}^a - \delta_m{}^a) \right] - \frac{i}{2} \operatorname{tr} \log \left[ i C_a{}^m \gamma^a \partial_m \right] \right\}, \tag{4.1.13}$$

where N crucially appears as a global factor, being at the exponent of the determinant in (4.1.12).

The power of the large N construction is now manifest: by taking N parametrically large we can make the classical effects dominant with arbitrary precision over the quantum effects because higher loop contributions are always suppressed by N factors compared to the tree level term. This happens because of the overall N in front of the Lagrangian and means that in the large N limit the quantum effective potential and any stationary points are controlled by the scalar potential of (4.1.13).

Let us notice that the fermion functional determinant contribution is clearly missing a dimensionful normalization inside the logarithm. This is related to the choice of $N_0$ in the path integral normalization, which we should specify. For instance, one can either introduce a scale to match the dimensions or insert the inverse propagator $i\partial\!\!\!/$. Here we will choose $N_0$ to be

$$N_0 = \left(\det\left[i\partial\!\!\!/\right]\right)^{\frac{\mathrm{N}}{2}} . \tag{4.1.14}$$

Other choices of $N_0$ reflect the different ways that one can use to express the determinant in perturbation theory and should ultimately not matter.

### 4.1.2 The effective potential and stationary points

Let us now search for stationary points of the bosonic theory (4.1.13).

From the form of its effective action we directly see that this theory has tensor fields that signal the presence of massive higher-spin excitations[5]. However, since we would like to search for translation-invariant and Lorentz-invariant stationary points, we are interested only in the trace parts of these tensors and, consequently, in the resulting scalar potential. We then describe the VEVs of $C_a{}^m$ and $e_m{}^a$ as follows:

$$C_a{}^m = (1+h)\,\delta_a{}^m , \quad e_m{}^a = (1+\phi)\,\delta_m{}^a . \tag{4.1.15}$$

In order to get the full scalar potential we need to reduce the functional determinant that includes $C_a{}^m$ to a convenient form by treating $C_a{}^m$, and so $h$, as a constant *background* field.

The reader could be concerned that by ignoring the tensor modes' contribution to the scalar potential we may be missing some non-trivial constraint on the trace parts of the tensors. However, this does not happen for the following reason. One can think of splitting both $C_a{}^m$ and $e_m{}^a$ into traceful and traceless parts as $C_a{}^m = (1+h)\,\delta_a{}^m + X_a{}^m$ and $e_m{}^a = (1+\phi)\,\delta_m{}^a + Y_m{}^a$, where $X_a{}^a \equiv 0$ and $Y_a{}^a \equiv 0$. Then, it is easy to see that there can never exist a linear term containing either $X_a{}^m$ or $Y_m{}^a$ in the scalar potential

[5] An analysis of the spectrum shows that around the original VA point the system has massive excitations, including spin-2 fields, along the lines of [226–229]. Understanding the precise spectrum may be interesting *per se*, but we will not go into the specifics of these excitations here. Let us only note that the interpretation of the excitations should be done with care as pointed out e.g. in [229, 230].

simply because there is nothing to contract them with: all terms with $X_a{}^m$ and $Y_m{}^a$ in the potential are directly quadratic in these fields and, therefore, they can be set to vanish consistently when we are searching for a background solution. There can be kinetic mixing of all sorts, of course, but here we are discussing neither the dynamics of the system nor its spectrum.

To proceed, we normalize the fermionic determinant in (4.1.12) with $i\partial\!\!\!/$, and treat the bosons as a background. Focusing directly on the relevant contributions from (4.1.15) we can see that (4.1.12) becomes

$$\begin{aligned} Z_F &= \left(\det\left[\frac{iC_a{}^m\gamma^a\partial_m}{i\partial\!\!\!/}\right]\right)^{\frac{N}{2}} = \left(\det\left[\frac{i(1+h)\gamma^m\partial_m}{i\gamma^n\partial_n}\right]\right)^{\frac{N}{2}} = \\ &= \left(\det\left[\frac{-(1+h)^2\partial^2\mathbb{1}_4}{-\partial^2\mathbb{1}_4}\right]\right)^{\frac{N}{4}} = \left[\left(\det\left[\frac{-(1+h)^2\partial^2}{-\partial^2}\right]\right)^4 \times \det[\mathbb{1}_4]\right]^{\frac{N}{4}} = \\ &= \left(\det\left[\frac{-(1+h)^2\partial^2}{-\partial^2}\right]\right)^{N} = \exp\left[N\,\mathrm{tr}\,\log\left[\frac{-(1+h)^2\partial^2}{-\partial^2}\right]\right]. \end{aligned} \tag{4.1.16}$$

We then evaluate the trace of the operator as the sum of its eigenvalues, that is

$$\mathrm{tr}\,\log\left[\frac{-(1+h)^2\partial^2}{-\partial^2}\right] = \sum_k \log\left[\frac{(1+h)^2k^2}{k^2}\right] = (VT)\int\frac{d^4k}{(2\pi)^4}\log\left[(1+h)^2\right], \tag{4.1.17}$$

where $VT$ is the four-dimensional volume, which we write as $\int d^4x$. The expression for the fermionic path integral can thus be brought to the form

$$Z_F = \exp\left[N\int d^4x\int\frac{d^4k}{(2\pi)^4}\log\left[(1+h)^2\right]\right], \tag{4.1.18}$$

which, once it is evaluated in Euclidean momentum space with a cut-off, gives

$$\int d^4x\, V_F(h,\phi) = i\log Z_F = \int d^4x\left(-\frac{N\Lambda^4}{32\pi^2}\log\left[(1+h)^2\right]\right). \tag{4.1.19}$$

$V_F$ constitutes the full correction to the bosonic effective potential from the fermion Gaussian integral.

For completeness, in the appendix of this chapter we will also discuss such a calculation from the perspective of dimensional regularization.

We conclude that the stationary points of the theory (4.1.1) are described by the critical points of the scalar potential

$$V_{\text{eff.}}(h,\phi) = N\times\left\{\frac{f^2}{2}\left[(1+\phi)^4 - 4(1+h)\phi\right] - \frac{\Lambda^4}{32\pi^2}\log\left[(1+h)^2\right]\right\}. \tag{4.1.20}$$

The scalars $h$ and $\phi$ on which $V_{\text{eff.}}$ depends should *not* be directly thought of as standard scalar fields, because they are physically parts of the tensor fields $C_a{}^m$ and $e_m{}^a$: $h$ is related to the trace of $C_a{}^m$ and $\phi$ to the trace of $e_m{}^a$. In spite of this, the critical values of $h$ and $\phi$ do correspond to stationary points of the system and their potential energies at the stationary points can be legitimately identified as actual energy densities [209,217]. The equations for the critical points are

$$2f^2\phi = -\frac{\Lambda^4}{16\pi^2(1+h)}, \quad (1+h) = (1+\phi)^3, \tag{4.1.21}$$

and they can be combined into a single equation for $\phi$, which in turn directly gives the value of $h$. The resultant equation for $\phi$ is

$$\phi(1+\phi)^3 + \frac{\Lambda^4}{32\pi^2 f^2} = 0. \tag{4.1.22}$$

This equation has at least two real solutions that can be easily found if we solve the system numerically, but can also be determined analytically if we solve the equations perturbatively.

Willing to adopt the latter approach, we assume that

$$\sqrt{f} > \Lambda. \tag{4.1.23}$$

Under such requirement, we find

$$\phi_{\rm VA} \simeq -\frac{\Lambda^4}{32\pi^2 f^2} + \mathcal{O}\left[\left(\frac{\Lambda^4}{32\pi^2 f^2}\right)^2\right] \tag{4.1.24}$$

and

$$\begin{aligned}\phi_{\rm GC} \simeq -1 + \left(\frac{\Lambda^4}{32\pi^2 f^2}\right)^{\frac{1}{3}} + \frac{1}{3}\left(\frac{\Lambda^4}{32\pi^2 f^2}\right)^{\frac{2}{3}} + \frac{1}{3}\left(\frac{\Lambda^4}{32\pi^2 f^2}\right) + \\ + \mathcal{O}\left[\left(\frac{\Lambda^4}{32\pi^2 f^2}\right)^{\frac{4}{3}}\right].\end{aligned} \tag{4.1.25}$$

The first solution corresponds to the original VA point and the second one is related to goldstino condensation (GC). On these critical configurations $h$ takes the values $h = -1 + (1+\phi)^3\big|_{\phi=\phi_{\rm VA},\phi_{\rm GC}}$.
The potential energies corresponding to the stationary points of (4.1.20) that we have just found can then be extracted: they are

$$\mathcal{E}_{\rm VA} = \frac{{\rm N}f^2}{2}\left\{1 + \mathcal{O}\left[\left(\frac{\Lambda^4}{32\pi^2 f^2}\right)^2\right]\right\} \tag{4.1.26}$$

and

$$\mathcal{E}_{\rm GC} = 2{\rm N}f^2\left\{\frac{\Lambda^4}{32\pi^2 f^2} - \frac{\Lambda^4}{32\pi^2 f^2}\log\left[\frac{\Lambda^4}{32\pi^2 f^2}\right] + \mathcal{O}\left[\left(\frac{\Lambda^4}{32\pi^2 f^2}\right)^{\frac{4}{3}}\right]\right\}, \tag{4.1.27}$$

respectively. We see that the configuration $(\phi_{\rm GC}, h_{\rm GC})$, which describes the large condensate, has lower energy than the VA point $(\phi_{\rm VA}, h_{\rm VA})$ (which seems to correspond to a small condensate; we will shortly come back to this point). We would like to highlight that, since N is assumed to be very large, the scalar potential (4.1.20) is arbitrarily close to the full *quantum effective potential*. Therefore, its stationary points $(\phi_{\rm cl}, h_{\rm cl})$ (but only those) correspond to actual quantum states of the theory and the corresponding value of the scalar potential captures the energy density of the state with energy $E = (VT)V_{\rm eff.}(\phi_{\rm cl}, h_{\rm cl})$. As an aside, let us also note that, since the overall value of a potential can always be shifted by a constant, these energy densities have to be considered as *relative* one to the other. We will come to this point in a while.

It is also worth observing that classically, from (4.1.7), we have

$$\phi = \frac{i}{4{\rm N}f^2}\left(\partial_m G_J \sigma^m \overline{G}^J - G_I \sigma^m \partial_m \overline{G}^I\right). \tag{4.1.28}$$

This justifies the interpretation of a non-trivial $\phi$ background value as a signal of goldstino condensation. We also see that, when $\phi$ approaches unit, the VEV of the condensate approaches ${\rm N}f^2$ and possibly jeopardises the control over the possible higher-order terms. Indeed, since we are working with the pure VA term (4.1.1), we are certainly ignoring higher order goldstino self-interactions. We will see shortly that, as long as the Lagrangian is written in terms of ${A_m}^a$ and its derivatives, we maintain full control of the vacua at large N. Therefore, no matter what power of $\phi$ we have in the higher-order terms, the condensate is robust even when $\phi$ goes near unit.

For completeness, let us mention that, contrary to what we have done so far, due to the large number N we could also work under the assumption that $\sqrt{f} < \Lambda$. After further assuming $\sqrt{f} \ll \Lambda$, while still respecting ${\rm N}f^2 > \Lambda^4$, in order to extract an analytic result, we find two almost degenerate solutions: $\phi_\pm \simeq \pm\frac{\Lambda}{2^{5/4}\sqrt{\pi f}}$ and $h_\pm \simeq \phi_\pm^3$. The original VA point is not a part of the stationary configurations any more and only solutions corresponding to large condensation correspond to stationary points. Since the condensate (4.1.28) takes parametrically large values in this case, we will not pursue this limit further here.

### 4.1.3 The stationary points in the deep IR

In this subsection we would like to understand what flowing towards the IR means and how $f$ changes while going to lower energies, assuming that we are already in a low energy regime where the VA model is weakly coupled, that is

$$\sqrt{f} \gg \Lambda. \tag{4.1.29}$$

Since the VA model has a single coupling, $f$, it would be *enough* to evaluate the flow of any specific term or interaction, and the other interactions would change accordingly. This in principle would require to have a regularization scheme that respects non-linear supersymmetry (as, for example, in [61]). Within such a setup one could deduce the flow of $f$ by considering the 4-Fermi derivative interaction. However, such a calculation would be quite involved for various reasons. For instance, it would firstly require the identification of a proper regularization scheme, and then the evaluation of both the fermionic wave-function renormalization and the actual running of the 4-Fermi vertex. Nevertheless, because of (4.1.29), one expects that the loop contributions would be subdominant (i.e. $\mathcal{O}\left(\frac{\Lambda}{\sqrt{f}}\right)$) with respect to the classical running due to the mass dimensionality of $f$. Therefore, without going into the loop calculation we can focus on the vacuum energy, $V_{\text{vac.}} = \frac{\mathrm{N}f^2}{2}$, as a tool to infer the classical flow of $f$ towards the IR. Even though this is a crude analysis, we will not only be able to capture the dominant running, but we will also see that the assumption (4.1.29) is enforced by the flow itself.

Following [209], let us consider the VA model in the form (4.1.1) and the Euclidean path integral

$$Z = \frac{1}{N_0} \int [DGD\overline{G}]_\Lambda \exp\left\{ - \int d^4x \left[ V_{\text{vac.}} + iG_I\sigma^m\partial_m\overline{G}^I + \mathcal{O}\left(\frac{1}{\mathrm{Nf}^2}\right)\right]\right\}, \quad (4.1.30)$$

where

$$[DGD\overline{G}]_\Lambda = \prod_{|k|<\Lambda} dG(k)d\overline{G}(k), \quad (4.1.31)$$

$\Lambda$ representing the momentum cut-off for the quantum field fluctuations (and without specifying the species index, for simplicity). We then distinguish the integration variables into two groups: the "high-momentum" degrees of freedom $(\hat{G}, \hat{\overline{G}})$ that have $b\Lambda \leq |k| < \Lambda$, and the "low-momentum" modes $(\tilde{G}, \tilde{\overline{G}})$ carrying a momentum $|k| < b\Lambda$ (the parameter $b$ being a fraction $b < 1$). We thus have

$$Z = \frac{1}{N_0} \int D\tilde{G}D\tilde{\overline{G}}\, e^{-\int d^4x\left[V_{\text{vac.}} + i\tilde{G}_I\sigma^m\partial_m\tilde{\overline{G}}^I + \dots\right]} \int D\hat{G}D\hat{\overline{G}}\, e^{-\int d^4x\left[i\hat{G}_I\sigma^m\partial_m\hat{\overline{G}}^I + \dots\right]}, \quad (4.1.32)$$

and we further split the path integral normalization factor $N_0$ into the high-momenta and the low-momenta contributions as

$$N_0 = N_0\big|_{\tilde{G}} \times N_0\big|_{\hat{G}} = (\det\left[i\sigma^n\partial_n\right])^{\mathrm{N}}\Big|_{|k|<b\Lambda} \times (\det\left[i\sigma^n\partial_n\right])^{\mathrm{N}}\Big|_{b\Lambda\leq|k|<\Lambda}. \quad (4.1.33)$$

Let us now integrate over the high-momentum modes and focus on the vacuum energy change. The leading contribution to it comes from the Gaussian kinetic term of $\hat{G}$ and $\hat{\overline{G}}$. Once we integrate over such degrees of freedom, we find

$$\Delta V_{\text{vac.}}\Big|_{\text{Gaussian}} = 0. \quad (4.1.34)$$

Higher-order contributions coming, for instance, from quartic terms of $\hat{G}$ or $\hat{\overline{G}}$ are further suppressed by powers of $f$. Then, we rescale distances and momenta according to

$$x = \frac{x'}{b} \quad \text{and} \quad k = bk', \tag{4.1.35}$$

so that the variable $k'$ is still integrated over the range $|k'| < \Lambda$, and the fermionic path integral has once more $[D\tilde{G}D\tilde{\overline{G}}]_\Lambda$ as its measure. Therefore, we have

$$\int [D\tilde{G}D\tilde{\overline{G}}]_\Lambda \, e^{-\int d^4x' b^{-4}(V_{\text{vac.}}+\Delta V_{\text{vac.}}+\dots)} = \int [D\tilde{G}D\tilde{\overline{G}}]_\Lambda \, e^{-\int d^4x' V'_{\text{vac.}}+\dots}, \tag{4.1.36}$$

where, in light of (4.1.34),

$$V'_{\text{vac.}} = V_{\text{vac.}} \times \frac{1}{b^4} + \text{ sub-leading contributions.} \tag{4.1.37}$$

Because of the rescaling (4.1.35), while considering (4.1.37), a decrease of the parameter $b$ represents how much the system flows towards the IR. The expression (4.1.37) clearly shows that the vacuum energy tends to increase as $b$ decreases.
From $V_{\text{vac.}} = \frac{\mathrm{N}f^2}{2}$ we deduce that

$$f = \frac{f_0}{b^2} + \text{ sub-leading contributions}\,, \tag{4.1.38}$$

where $f_0$ is the starting value of $f$, before we integrate out any high-momentum modes (i.e. for $b = 1$), and the sub-leading contributions are of order $\frac{\Lambda}{\sqrt{f_0}} \ll 1$. This in turn implies that the coupling accompanying the higher-order interactions of the VA model becomes more and more irrelevant as one flows to the IR.
We thus conclude that in the deep IR regime $b \to 0$ and gives

$$\frac{\Lambda}{\sqrt{f}} = \frac{\Lambda}{\sqrt{f}_0} \times b \to 0. \tag{4.1.39}$$

In such a limit we can check the relative difference between the energy densities of the two critical points: we obtain

$$\frac{\mathcal{E}_{\text{VA}} - \mathcal{E}_{\text{GC}}}{\mathcal{E}_{\text{VA}}} \to 1 \quad \text{and} \quad \mathcal{E}_{\text{VA}} \to \frac{\mathrm{N}f^2}{2}. \tag{4.1.40}$$

We see that the goldstino condensation point has parametrically lower energy than the VA configuration. Moreover, under the limit (4.1.39) the VA point recovers its classical energy density and all corrections to it vanish.
We see that (4.1.40) is quite suggestive in favour of interpreting the GC point as a supersymmetry restoring field configuration. Such interpretation is further corroborated by the properties of the kinetic terms of the fermions. Indeed, in the deep IR regime (4.1.39), where a scalar VEV is properly defined, we observe that

$$\phi_{\text{VA}} \to 0, \quad h_{\text{VA}} \to 0, \tag{4.1.41}$$

which means that one recovers the classical stationary point for the VA model, and

$$\phi_{\rm GC} \to -1, \quad h_{\rm GC} \to -1. \tag{4.1.42}$$

Since we know from (4.1.10) that the kinetic terms of the fermions on a background defined by the stationary points are

$$\mathcal{L}_{\rm kin.} = \frac{i}{2}(1+h)\left(G_I\sigma^m\partial_m\overline{G}^I - \partial_m G_J\sigma^m\overline{G}^J\right), \tag{4.1.43}$$

we conclude that at the VA point the fermions have canonical kinetic terms, whereas the kinetic terms of the fermions vanish on the GC point in the deep IR. This absence of appropriate Goldstone modes when $h \to -1$ is consistent with the restoration of supersymmetry[6].

As we mentioned earlier, in Appendix 4.A we are going to present the same analysis by using dimensional regularization. We will see that in the deep IR regime that we just studied, defined by the limit $b \to 0$, the results from the two different regularization methods nicely match.

Our findings also connect with the ERG analysis of the previous chapter, for the 4D N=1 system that used superfields [61]. There, the system is driven to an asymptotic supersymmetric point where the derivatives of the superpotential vanish, and so does the vacuum energy. In particular, the asymptotic supersymmetric point of the 4D N=1 system satisfies $G^2\partial^2\overline{G}^2 \sim f^4$, where $G$ is the N=1 goldstino and $\sqrt{f}$ the N=1 supersymmetry breaking scale. We can interestingly observe that (4.1.28) for $\phi \sim -1$ corresponds to a similar limit. For completeness, let us also note that the growth of $f$ in [61] is controlled in the IR by its mass dimension.

## 4.2 Robustness against higher-order terms

In this section we would like to understand how much the stationary points $(\phi_{\rm VA}, h_{\rm VA})$ and $(\phi_{\rm GC}, h_{\rm GC})$ are influenced by the higher-order terms that our starting model is ignoring. Even though there are terms that we can not account for and may change the solutions, especially if they describe R-symmetry breaking, we will provide a simple rule of thumb for the circumstances when higher order interactions could be dangerous. More precisely: (A) When higher-order terms appear only through the goldstino vielbein $A_m{}^a$ and its derivatives, the goldstino condensation is *always* robust for large N; (B) When higher-order terms also include explicit $(A^{-1})_a{}^m\partial_m G^I$ terms, the goldstino condensation

[6] A similar effect takes place in [124], where the goldstino stops propagating on the supersymmetric background. One could expect that on such a background the massive spin-2 excitations organize themselves in a supersymmetric way, e.g. along the lines of [231].

*may* be jeopardized.

We will prove (A) and we will give two different examples of (B).

The reader should keep in mind that, if the goldstino condensation does restore supersymmetry, then higher-order corrections to the treatment above do not threat its existence; only non-perturbative corrections could do that. This is possibly the reason why it is easy to readily control a large class of higher order corrections of the form (A). Despite of the lack of a proof, we will also see that the corrections of the form described in (B) seem to remain innocuous most of the times.

### 4.2.1 Corrections from goldstino vielbeins and matter

Let us start by considering the case in which the higher-order terms are expressed only by the goldstino vielbein $A_m{}^a$ and its derivatives. Schematically, they have the form

$$\frac{1}{M^{R-4}}(\partial_n)^R (A_m{}^a)^T \longrightarrow \frac{1}{M^{R-4}}(\partial_n)^R (e_m{}^a)^T \longrightarrow \frac{1}{M^{R-4}}(\partial_n)^R (1+\phi)^T, \qquad (4.2.1)$$

for some scale $M$ and some powers $R$ and $T$. Since we are focused on the stationary points of a scalar potential, such derivative terms do not change the outcome. More importantly, when the higher-order terms take the form (4.2.1), even if they are not only derivative interactions, they are always parametrically sub-leading in the large N limit simply because they have no N factor in front of them. We can conclude that for large N the goldstino condensation is not spoilt by such higher-order corrections.

Let us now make the discussion a bit more precise by assuming that we have some massive scalars coupled to the system in a way that preserves the existing non-linear supersymmetry. We assume that such scalars are in their VEVs so that we can restrict ourselves to consider the Gaussian piece of their action. These scalar fields could represent some degrees of freedom that have been removed from the spectrum to deduce the low energy goldstino theory. Their impact can serve as a proxy for the higher-order corrections.

We consider $n$ real scalars $b_i$ with

$$\Delta\mathcal{L} = \frac{1}{2}\det[e_l{}^c]\eta^{ab}E_a{}^m(\partial_m b_i)E_b{}^n(\partial_n b_i) - \frac{1}{2}M^2\det[e_m{}^a]b_i^2, \quad \text{for } i = 1,\ldots,n. \quad (4.2.2)$$

We are interested in evaluating the contribution of the functional determinant of the scalars $b_i$ to the effective potential. To this end, we expand $e_m{}^a$ once more as $e_m{}^a = (1+\phi)\delta_m{}^a$, treating $\phi$ as a *background* field. We get

$$\begin{aligned}\int d^4x \Delta V_{\text{eff.}}(\phi) &= -\frac{i\,n}{2}\log\det\left[\det[e_l{}^c]\eta^{ab}E_a{}^m\partial_m E_b{}^n\partial_n + M^2\det[e_m{}^a]\right] = \\ &= -\frac{i\,n}{2}(VT)\int\frac{d^4k}{(2\pi)^4}\log\left[-(1+\phi)^2k^2 + M^2(1+\phi)^4\right].\end{aligned} \qquad (4.2.3)$$

If we now assume that the $N_0$ for each of the scalar fields $b_i$ corresponds to a free massive scalar, after a Wick rotation, we obtain

$$\Delta V_{\text{eff.}}(\phi) = \frac{n}{2} \int \frac{d^4 k_E}{(2\pi)^4} \log \left[ \frac{(1+\phi)^2 k_E^2 + M^2 (1+\phi)^4}{k_E^2 + M^2} \right], \tag{4.2.4}$$

and, explicitly,

$$\begin{aligned} \Delta V_{\text{eff.}}(\phi) = \frac{n}{64\pi^2} \Big\{ & \Lambda^4 \log[(1+\phi)^2] + M^2 \Lambda^2 \left[ (1+\phi)^2 - 1 \right] + \\ & + M^4 \log \left[ \frac{M^2 + \Lambda^2}{M^2} \right] + \Lambda^4 \log \left[ \frac{M^2 (1+\phi)^2 + \Lambda^2}{M^2 + \Lambda^2} \right] + \\ & + M^4 (1+\phi)^4 \log \left[ \frac{M^2 (1+\phi)^2}{M^2 (1+\phi)^2 + \Lambda^2} \right] \Big\}. \end{aligned} \tag{4.2.5}$$

We would like to see how this new contribution to the effective scalar potential changes the stationary points. Even though it is not necessary, we can assume that the scalars $b_i$ are heavy, that is $M^2 > \Lambda^2$. Without actually performing any further calculation, but simply exploiting the large N limit, we observe that the contribution (4.2.5) to the total scalar potential is parametrically subdominant with respect to (4.1.20) as long as

$$\text{N} \gg n. \tag{4.2.6}$$

Indeed, the analysis here falls under the general arguments for the robustness of the stationary points under higher-order deformations of the form (4.2.1). In particular, as far as (4.1.21) is concerned, the left-hand-side equation does not change, whereas the right-hand-side equation becomes

$$(1+h) = (1+\phi)^3 + \frac{1}{2\text{N} f^2} \frac{\partial(\Delta V_{\text{eff.}}(\phi))}{\partial \phi}. \tag{4.2.7}$$

From here it is evident that the deviation of (4.2.7) from (4.1.21) is arbitrarily small at large N.

Let us notice that we can also extend the above conclusion to a more general matter-coupled VA system. Consider a Lagrangian that has a matter part (made by scalars, vectors, spinors) of the form [224, 232]

$$\mathcal{L}_{\text{matter}}(A_m{}^a, b, v_m, \chi_\alpha). \tag{4.2.8}$$

Ultimately, its induced contribution to the quantum effective potential boils down to some $\Delta V_{\text{eff}}(\phi)$ and, as a consequence, the deviation from the original system is controlled by (4.2.7), therefore being arbitrarily small at large N.

We can conclude that goldstino condensation is quite a robust prediction of the large N non-linear supersymmetric theory, assuming that matter is coupled to the starting VA system via (4.2.8) (which inevitably also preserves the R-symmetry).

### 4.2.2 Explicit goldstini under derivatives

We now discuss terms where the goldstini explicitly appear under derivatives,

$$D_a G^I = E_a{}^m \partial_m G^I, \tag{4.2.9}$$

thus breaking the assumption (4.2.1). These terms can possibly jeopardise goldstino condensation even at large N.

For example, one can consider a term like

$$g \det[e_m{}^c]\, D_a G^I \sigma^{ab} D_b G^I + \text{c.c.}, \tag{4.2.10}$$

for some complex dimensionful coupling $g$. Such a term potentially has a non-trivial impact: not only it is not of the form (4.2.8), but it also contributes to the large N functional determinant because it contains the N goldstini. However, since we are interested in scalar backgrounds, (4.2.10) takes the form

$$g\,(1+\phi)^2\, \partial_a G^I \sigma^{ab} \partial_b G^I + \text{c.c.}, \tag{4.2.11}$$

and it is then clear that it never contributes to the quantum effective potential of $\phi$. One can in fact perform an integration by parts, treating $\phi$ as a constant *background* field (because we are interested in the properties of the effective potential), to obtain

$$\int d^4x\, g\,(1+\phi)^2\, \partial_a G^I \sigma^{ab} \partial_b G^I \;\longrightarrow\; -\int d^4x\, g\,(1+\phi)^2\, G^I \sigma^{ab} \partial_a \partial_b G^I = 0. \tag{4.2.12}$$

Let us notice that the same manipulation can be done by going to momentum space and assigning zero momentum to $\phi$, as the standard procedure to evaluate the contributions to the quantum effective potential requires. We conclude that terms like (4.2.10), if present, do not jeopardize the new stationary point associated to goldstino condensation.

As a further example, we can consider the term

$$g'_{IJ} \det[e_m{}^c]\, D_a G^I D^a G^J + \text{c.c.}, \tag{4.2.13}$$

for some complex dimensionful couplings $g'_{IJ}$. This term contributes to the quantum effective potential with a large N coefficient. However, it manifestly changes the number of degrees of freedom because it leads to $G^I \partial^2 G^J$ terms, which induce an additional massive fermion in the spectrum for each goldstino. For a consistent EFT such terms should be, in any case, independently highly suppressed.

Even so, let us analyze the impact of (4.2.13), assuming that one uses it only as an interaction vertex. The easiest way to handle such a term is to package the goldstini once again into Dirac spinors $\Psi^A$ of the form (4.1.11), where $A = 1, \dots, \mathrm{N}/2$. For our analysis we also assume that the only non-zero contributions to (4.2.13) come from

$$g'_{IJ} = g' \times \delta_{A,A+\mathrm{N}/2}, \tag{4.2.14}$$

with $g' \in \mathbb{R}$ now. Then, considering only the background $h$ and $\phi$ contributions from (4.2.13), the Gaussian fermionic sector is

$$\mathcal{L} = i(1+h)\overline{\Psi}^A \gamma^m \partial_m \Psi^A - g'(1+\phi)^2 \overline{\Psi}^A (i\gamma^m\partial_m)^2 \Psi^A, \tag{4.2.15}$$

where the first term originates from (4.1.10). The functional determinant for a single Dirac spinor of (4.2.15) becomes

$$\det\left[i(1+h)\partial\!\!\!/ - g'(1+\phi)^2 (i\partial\!\!\!/)^2\right] \equiv \det\left[i(1+h)\partial\!\!\!/\right] \times \det\left[\mathbb{1}_4 - g'\frac{(1+\phi)^2}{1+h} i\partial\!\!\!/\right]. \tag{4.2.16}$$

We readily see why such a deformation changes the degrees of freedom and introduces new massive fermions. However, as far as our purpose is concerned, we simply need to treat the new contribution to the effective potential from the new massive fermionic functional determinant, having in mind that $\det\left[i(1+h)\partial\!\!\!/\right]$ is already included in the effective potential and corresponds to the original goldstini. One way to do this calculation is to recast the overall functional determinant in the form

$$\begin{aligned} &\det\left[i(1+h)\partial\!\!\!/ - g'(1+\phi)^2(i\partial\!\!\!/)^2\right] \equiv \\ &\equiv \det\left[-g'(1+\phi)^2 i\partial\!\!\!/\right] \times \det\left[i\partial\!\!\!/ - \frac{(1+h)}{g'(1+\phi)^2}\mathbb{1}_4\right]. \end{aligned} \tag{4.2.17}$$

The first term is similar to that which we have already calculated, but with $1+h$ replaced by $-g'(1+\phi)^2$. The second factor corresponds, instead, to the contribution of a massive fermion with canonical kinetic term[7]. We conclude that the potential that we have to extremize is (up to constants)

$$\begin{aligned} V_{\text{eff.}}(h,\phi) =& \frac{\mathrm{N}f^2}{2}\left[(1+\phi)^4 - 4(1+h)\phi\right] - \frac{\mathrm{N}\Lambda^4}{32\pi^2}\log\left[(1+\phi)^4\right] + \\ &- \frac{\mathrm{N}m^4}{32\pi^2}\left(\frac{\Lambda^2}{m^2} + \log\left[\frac{m^2}{m^2+\Lambda^2}\right] - \frac{\Lambda^4}{m^4}\log\left[\frac{\Lambda^2}{m^2+\Lambda^2}\right]\right), \end{aligned} \tag{4.2.18}$$

where

$$m = \frac{(1+h)}{g'(1+\phi)^2}. \tag{4.2.19}$$

[7]To evaluate the contribution of the massive fermions we notice that $\det\left[i\partial\!\!\!/ - m\mathbb{1}_4\right] = \det\left[(i\partial\!\!\!/ - m\mathbb{1}_4)\gamma_5^2\right] = \det\left[(i\partial\!\!\!/ - m\mathbb{1}_4)\gamma_5\right]\det[\gamma_5] = \det\left[\gamma_5(i\partial\!\!\!/ - m\mathbb{1}_4)\gamma_5\right] = \det\left[-i\partial\!\!\!/ - m\mathbb{1}_4\right]$. This allows us to express the functional determinants of massive Dirac fermions in terms of functional determinants of massive scalars, giving the known result: $\det\left[i\partial\!\!\!/ - m\mathbb{1}_4\right] = \left(\det\left[\partial^2 + m^2\right]\right)^2$.

| $f$ | $M$ | $\phi_{\rm GC}$ | $h_{\rm GC}$ | $\mathcal{E}_{\rm GC}/{\rm N}$ | $\phi_{\rm VA}$ | $h_{\rm VA}$ |
|---|---|---|---|---|---|---|
| 591 | $17.1\times10^{3}$ | $-1+10^{-2}$ | $-1+4\times10^{-9}$ | $6\times10^{-2}$ | $-3\times10^{-8}$ | $-6\times10^{-8}$ |
| 24 | 670 | $-1+5\times10^{-2}$ | $-1+3\times10^{-6}$ | $4\times10^{-2}$ | $-6\times10^{-6}$ | $-2\times10^{-5}$ |
| 6 | 162 | $-1+10^{-1}$ | $-1+4\times10^{-5}$ | $3\times10^{-2}$ | $-9\times10^{-5}$ | $-3\times10^{-4}$ |

Table 4.1: Few instances of numerical solutions for stationary points of (4.2.18) with $\Lambda = 1$, but without making any approximation on the effective potential. The numerical solutions approach the analytic ones as we go closer to the parametric limits that allow our approximations. The vacuum energy at the VA point is always in very good agreement with ${\rm N}f^2/2$ and therefore we do not write it explicitly. Note that, because we are interested in the orders of magnitude and in the possible existence of a solution, we have rounded-up the presented numerical results.

Contrary to the previous case, we can not use the large N limit any more to eliminate the new terms. However, the theory still has a valid large N limit and the stationary points of (4.2.18) correspond to stationary points of the full quantum effective potential (in such a limit).

We would like to investigate the existence of a goldstino condensate for the scalar potential (4.2.18). In order to be able to continue analytically we make the assumption that there is a hierarchy between the scales at play, namely

$$M \gg \Lambda, \quad \text{once } M \equiv \frac{1}{g'}, \tag{4.2.20}$$

and we furthermore assume that

$$m = \frac{M(1+h)}{(1+\phi)^2} \gg \Lambda. \tag{4.2.21}$$

We will check that this condition holds on the solutions. We can already see that it is satisfied for the VA point, if the latter persists. Under such assumption, and up to constants, the potential that we need to extremize takes exactly the form (4.1.20) at leading order in the $\Lambda^2/m^2$-expansion. In particular, the corrections are all of the form $\Lambda^4 \times (\Lambda^2/m^2 + \Lambda^4/m^4 + \dots)$. Therefore, both the VA configuration and the GC solution remain intact. Finally, one can check that both solutions satisfy (4.2.21), as long as

$$M^3\Lambda \gg f^2. \tag{4.2.22}$$

This further implies that $M \gg \sqrt{f}$.

Let us observe that we have worked with a parametric separation between the various scales ($M$, $\sqrt{f}$ and $\Lambda$) that enter the problem so that we can easily deduce analytic results. Clearly, the solutions still exist for weaker assumptions but they have to be found numerically: we provide few numerical solutions for more conservative values of the coefficients in Table 4.1. We do not know under which conditions the solutions will seize to exist and if they seize to exist at all. When $M$ becomes smaller than $\sqrt{f}$, the extremization problem can not be approached easily by the adoption of analytical methods, and also the numerical analysis seems to require stronger machines or more refined techniques.

We conclude that higher-order terms with explicit derivatives of the goldstini may seem harmful at first sight, but it is not obvious that they actually have an impact on the system after all. As the reader has appreciated, we have analyzed few such terms and we have seen that the properties of the stationary points do not considerably change. Nonetheless, we do not have a general argument to state that the higher-order corrections lying under the circumstance (B) can not threat the goldstino condensate. As an aside final remark, let us note that other higher-order terms of a similar form can in principle be reduced to the Gaussian terms that we have studied by using Lagrange multipliers.

## 4.3 A single goldstino and N−1 pseudo-goldstini

In this section we wish to take advantage of the large number of fermions in the system to deduce a result for a model that has only N=1 non-linear supersymmetry. To do this, we make all the fermions massive but one, which corresponds to the single goldstino that the theory has in the low energy regime.

We split the goldstini as

$$G^I = (G^0, G^i), \tag{4.3.1}$$

where $G^0$, which we will denote as $G$ from now on, represents the goldstino for the N=1 non-linear supersymmetry, and the $G^i$s are 2n pseudo-goldstini for a reason that will be clarified in a while. For convenience we pair the 2n pseudo-goldstini into n Dirac spinors $\Psi^A$ following (4.1.11). We then explicitly break the extended non-linear supersymmetries down to 1 by introducing a Dirac mass term

$$M \det[A_m{}^a]\overline{\Psi}^A\Psi^A. \tag{4.3.2}$$

Note that, if each Dirac spinor is split into two Majorana fermions, then one gets a Majorana mass term for each one of them. Since we get back the full N goldstini system when these masses vanish, we call the massive fermions $\Psi^A$ pseudo-goldstini.
We have

$$\mathcal{L} = -\frac{\mathrm{N}f^2}{2}\det[e_m{}^a] + \frac{\mathrm{N}f^2}{2}C_a{}^m(e_m{}^a - A_m{}^a) + M\det[e_m{}^a]\overline{\Psi}^A\Psi^A, \tag{4.3.3}$$

with

$$\begin{aligned} A_m{}^a = \;& \delta_m{}^a + \frac{i}{\mathrm{N}f^2}\partial_m G\sigma^a\overline{G} - \frac{i}{\mathrm{N}f^2}G\sigma^a\partial_m\overline{G} + \\ & - \frac{i}{\mathrm{N}f^2}\overline{\Psi}^A\gamma^a\partial_m\Psi^A + \frac{i}{\mathrm{N}f^2}\partial_m\overline{\Psi}^A\gamma^a\Psi^A . \end{aligned} \tag{4.3.4}$$

As elsewhere throughout this chapter, we are interested in stationary points that are translation-invariant and Lorentz-invariant: we consider directly the trace parts of $C_a{}^m$ and $e_m{}^a$ as in (4.1.15).

We want once more to perform the Gaussian integral over the fermions and derive the contribution to the effective potential for $h$ and $\phi$. To deduce the relevant modifications to it we perform two formal steps that allow us to get the result directly from the formulas that we already have at our disposal. First, we redefine all the fermions as follows

$$G^I \longrightarrow \frac{1}{\sqrt{1+h}}G^I , \tag{4.3.5}$$

treating $h$, as always, as a constant (because we are only interested in the effective potential critical points). As a consequence, the Lagrangian takes the form

$$\begin{aligned} \mathcal{L} = & - \frac{\mathrm{N}f^2}{2}(1+\phi)^4 + \frac{\mathrm{N}f^2}{2}4\phi(1+h) + \frac{M(1+\phi)^4}{1+h}\overline{\Psi}^A\Psi^A + \\ & - \frac{i}{2}(\partial_m G\sigma^m\overline{G} - G\sigma^m\partial_m\overline{G} - \overline{\Psi}^A\gamma^m\partial_m\Psi^A + \partial_m\overline{\Psi}^A\gamma^m\Psi^A) + \\ & + \frac{\mathrm{N}\Lambda^4}{32\pi^2}\log\left[(1+h)^2\right] . \end{aligned} \tag{4.3.6}$$

The last term appears from the fermionic measure in the path integral because of the redefinition of the fermions. It has to be so in light of the fact that, if no field redefinition is performed, such a contribution appears from the Gaussian integral over the massless fermions. After (4.3.5) the massless fermion $G$ decouples and it can be eliminated without any effect. This implies that the fermion redefinition has to contribute to the Lagrangian through the path integral measure. Before proceeding, as a reminder for the reader, let us note that $\mathrm{N} = 2\mathrm{n} + 1$.

As we have mentioned just above, the goldstino $G$ can now be integrated over without any effect, except of an overall shift in the vacuum energy, which we ignore. After eliminating $G$ right away, the remaining 2n fermions are combined into n massive Dirac spinors with canonical kinetic terms and mass $M(1+\phi)^4/(1+h)$. Knowing from the previous section how to evaluate the functional integral of these massive fermions, once we integrate out all the fermions by performing the corresponding Gaussian integral, we explicitly find

$$\begin{aligned} V_{\text{eff.}}(h,\phi) = & \frac{\mathrm{N}f^2}{2}\left[(1+\phi)^4 - 4(1+h)\phi\right] - \frac{\mathrm{N}\Lambda^4}{32\pi^2}\log\left[(1+h)^2\right] + \\ & - \frac{(\mathrm{N}-1)m^4}{32\pi^2}\left(\frac{\Lambda^2}{m^2} + \log\left[\frac{m^2}{m^2+\Lambda^2}\right] - \frac{\Lambda^4}{m^4}\log\left[\frac{\Lambda^2}{m^2+\Lambda^2}\right]\right) , \end{aligned} \tag{4.3.7}$$

where

$$m = \frac{M(1+\phi)^4}{1+h}. \tag{4.3.8}$$

We see that the large N limit still gives us a reliable approximation to the full quantum effective potential with arbitrary precision. To derive the stationary points we extremize (4.3.7) with respect to $h$ and $\phi$. To avoid clutter we also assume that in the large N limit we can have $\mathrm{N} \simeq \mathrm{N} - 1$. Then, combining the equations for $h$ and $\phi$ we see that we can readily eliminate $h$ because it is bound to satisfy

$$1 + h = \frac{8\pi^2 f^2 (1+\phi)^4 - \Lambda^4}{8\pi^2 f^2 (1+5\phi)}. \tag{4.3.9}$$

The system of the equations of extremization of (4.3.7) therefore reduces to a single equation for $\phi$. It is possible to search for stationary point solutions without making any assumption on a hierarchy among the various scales at work, but, in this case, one has to proceed numerically. For completeness, we give some numerical results in Table 4.2. However, we can easily proceed analytically by first invoking the typical hierarchy $\Lambda \ll \sqrt{f}$ and consequently observing that, under such requirement, (4.3.9), for the goldstino condensation point, gives

$$1 + h_{\mathrm{GC}} \simeq \frac{\Lambda^4}{32\pi^2 f^2}, \tag{4.3.10}$$

assuming that

$$\phi_{\mathrm{GC}} \simeq -1, \quad f^2 (1+\phi_{\mathrm{GC}})^4 \ll \Lambda^4. \tag{4.3.11}$$

These equations are in complete agreement with (4.1.25) and we will also verify them on the solution. Always within such limits the equation for $\phi_{\mathrm{GC}}$ is

$$\begin{aligned} 2f^2(1+\phi)^3 - \frac{\Lambda^4}{16\pi^2} - \Lambda^4 \frac{512 f^4 M^2 \pi^2 (1+\phi)^7}{\Lambda^{10}} &= \\ = -2(1+\phi)\pi^2\Lambda^4 \left( \frac{512 f^4 M^2 \pi^2 (1+\phi)^7}{\Lambda^{10}} \right)^2 &\times \\ \times \log \left[ 1 + \frac{1}{2(1+\phi)\pi^2} \frac{\Lambda^{10}}{512 f^4 M^2 \pi^2 (1+\phi)^7} \right]&. \end{aligned} \tag{4.3.12}$$

From this relation we can recover the goldstino condensation solution (4.1.25), if we assume that

$$\frac{512 f^4 M^2 \pi^2 (1+\phi)^7}{\Lambda^{10}} \ll 1, \tag{4.3.13}$$

which (for (4.1.25)) reduces to $M^3 \ll \Lambda f$ and therefore also to $M^2 \ll f$. Consistently with these bounds we can still have $M > \Lambda$ or $M < \Lambda$: the original goldstino condensation

| $f$ | $M$ | $\phi_{\rm GC}$ | $h_{\rm GC}$ | $\mathcal{E}_{\rm GC}/{\rm N}$ | $\phi_{\rm VA}$ | $h_{\rm VA}$ |
|---|---|---|---|---|---|---|
| $8.8\times10^6$ | 0.16 | $-1+10^{-4}$ | $-1+2\times10^{-17}$ | 0.25 | $-4\times10^{-17}$ | $-10^{-16}$ |
| 4595 | 4 | $-1+10^{-3}$ | $-1+2\times10^{-10}$ | 0.15 | $-6\times10^{-12}$ | $-6\times10^{-10}$ |
| 1122 | 0.017 | $-1+10^{-2}$ | $-1+2\times10^{-11}$ | 0.15 | $-3\times10^{-9}$ | $-8\times10^{-9}$ |

Table 4.2: Few instances of numerical solutions for stationary points of (4.3.7) with $\Lambda = 1$, but without making any approximation at the level of the effective potential. The numerical solutions approach the analytic solutions as we go closer to the parametric limits that allow our approximations. Since the vacuum energy at the VA point is always in very good agreement with ${\rm N}f^2/2$, we do not write it explicitly. Here again, because we are interested only in the orders of magnitude and in the existence of a solution, we have rounded-up the presented numerical results.

solution is intact for arbitrarily light or for quite heavy pseudo-goldstini. The VA point also remains intact when we have $\sqrt{f} \gg \Lambda \gg M$. It is of course good that the limit $M \to 0$ can be taken smoothly: in this limit the N$-$1 pseudo-goldstini become goldstini and we recover the results for the original N goldstini model of Section 4.1.

This result is relevant for string flux compactifications that include anti-D3/O3 systems as, for example, KKLT does (where supersymmetry is non-linear [165, 167, 169, 233–236]). For the sake of the discussion, let us extrapolate our large N results to the case where N = 4. In [167] the masses of the extra fermions living on the anti-brane world-volume are discussed, in particular for the three massive fermions belonging to the anti-brane, which are pseudo-goldstini. The mass of these fermions is determined by the $n^{(2,1)}$ ISD flux. As we have seen here a small $M$ or, equivalently, a small $n^{(2,1)}$ ISD flux may lead to goldstino condensation and further support the existence of such effects on an anti-brane. Conversely, if $n^{(2,1)}$ is large, then $M$ becomes large as well, and the system has further issues due to large tadpoles [237, 238].

## 4.4 Comments

In this chapter we have investigated the existence of new stationary points in the standard 4D Volkov–Akulov fermionic system in the presence of N non-linear supersymmetries.

An intuitive way to think of such an investigation and of the findings that we have just exposed is the following.
From the standard Lagrangian $\det[A_m{}^a]$ describing the VA system (see (4.1.1)), one can derive the classical equations of motion

$$\det[A^a_m](A^{-1})_b{}^n\sigma^b\partial_n G = 0\,, \tag{4.4.1}$$

and suspect that these equations have two types of vacuum solutions:

$$\langle G_\alpha \rangle = 0 \quad \text{or} \quad \langle \det[A_m{}^a] \rangle = 0\,. \tag{4.4.2}$$

Clearly, the vacuum solution where the goldstino vanishes corresponds to the original VA point that describes supersymmetry breaking: there, in fact, $\langle \det[A_m{}^a] \rangle = 1$. The solution where the goldstino vielbein determinant vanishes, instead, corresponds to a condensation of the goldstini (see (4.1.2) for the form of $A_m{}^a$), and implies that supersymmetry is restored, because the vacuum energy is now vanishing.

The actual computation is more involved than simply solving $\langle \det[A_m{}^a] \rangle = 0$ for the goldstini and proceeds with path integral methods [209, 217], which allow to properly treat fermionic condensates. However, the naive intuitive expectation turns out to be correct and a solution of the form $\langle \det[A_m{}^a] \rangle = 0$ does actually exist, as the path integral method that we have followed verifies.

As we already mentioned, even though we have worked directly in the component form and we have exploited large N methods, our results lend further support to the goldstino condensation analysis of Chapter 3 (or [61]) that was performed with the ERG technique for superfields. These two approaches can be considered complementary and it is gratifying to see that they agree.

It is also important to bring to the reader's attention the fact that bosonic systems with bosonic Goldstone modes can have a similar behaviour where the classically broken symmetry is restored by quantum effects (see [239–242]; or [209, 217], for textbook analysis). The fact that something similar happens for fermionic systems should not come as a big surprise, then. Moreover, this does not mean that supersymmetry can not be broken, but it signals that the breaking of supersymmetry is more intricate than what one naively expects and has to be studied with care.

It is also worth noting that our results give further evidence that the anti-D3-brane/O3-plane system is inherently unstable on a flux-less Minkowski background: such a system corresponds, in fact, to N=4 [233]. It is true that N=4 is not large, but the large N results may still persist. In this respect, evaluating the leading 1/N corrections to the potential will be illuminating.

We have also discussed a system where all but one goldstini get masses. Such a setup corresponds to placing the anti-D3-brane/O3-plane system on a flux background [167]. In this setup we have seen that the goldstino condensation persists. Such a model can also be studied with the use of constrained superfields satisfying $X^2 = 0 = XY^i$ [234], and exploiting the ERG technique to analyze the existence of condensates. The resulting (supersymmetric) backgrounds corresponding to the condensation may be ultimately related to some kind of brane-flux annihilation [215, 243], but we can not know if this

is indeed the case yet. This is one of the important questions that we leave for future studies.
In addition, the impact of including gravitation in our analysis is not necessarily trivial.

Another path that deserves to be investigated is how the goldstino condensation behaves in different dimensions. It is worth performing a similar analysis for example in 2D or in 3D, especially taking into account that spin-2 fields and gauge fields behave differently compared to the 4D case. An analysis of the condensate directly in 10D would also be illuminating, and especially interesting for the BSB models [244]. However, such a study seems more challenging compared to that of the lower dimensional systems.

## 4.A Dimensional regularization and stationary points

The reader may ask what happens if we utilized dimensional regularization when evaluating, for instance, the momentum integral of (4.1.18) in Section 4.1. Here we are then going to work out the pure large N Volkov–Akulov model and the corresponding integrals by means of dimensional regularization (instead of using cut-off regularization as we did in the bulk of the chapter). Once done, we will compare the results.

We directly consider the calculation of the relevant integral for (4.1.18), that is

$$i\mathrm{N}\log[(1+h)^2]\int\frac{d^4k}{(2\pi)^4}. \tag{4.A.1}$$

The cut-off prescription gives $\int\frac{d^4k}{(2\pi)^4}=\frac{i\Lambda^4}{32\pi^2}$, whereas the integral vanishes within dimensional regularization. Let us then evaluate the integral

$$\int\frac{d^4k}{(2\pi)^4}\frac{(k^2)^2}{(k^2-M^2)^2}, \tag{4.A.2}$$

using dimensional regularization and taking, only in the very end, the limit $M\to 0$ to make contact to (4.A.1). We find

$$\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\frac{(k^2)^2}{(k^2-M^2)^2}=\frac{3iM^4}{8\pi^2}\Gamma[-2+\epsilon]\left(\frac{4\pi\mu^2}{M^2}\right)^{\epsilon}, \tag{4.A.3}$$

where $d=4-2\epsilon$. When sending $M\to 0$, we recover the known result that the integral (4.A.1) vanishes within dimensional regularization.

If we went through all the analysis that we did in the bulk of the chapter, we would find that

$$\begin{aligned}\phi_{\mathrm{VA}}&=-\frac{3M^4\Gamma[-2+\epsilon]}{8\pi^2f^2}\left(\frac{4\pi\mu^2}{M^2}\right)^{\epsilon}+\dots\quad\text{and}\\ \phi_{\mathrm{GC}}&=-1+\left[\frac{3M^4\Gamma[-2+\epsilon]}{8\pi^2f^2}\left(\frac{4\pi\mu^2}{M^2}\right)^{\epsilon}\right]^{\frac{1}{3}}+\dots,\end{aligned} \tag{4.A.4}$$

the first solution corresponding to the original VA point and the second one to the goldstino condensation configuration. As far as the energy densities of the stationary points are concerned, we have

$$\mathcal{E}_{\rm VA} = \frac{{\rm N}f^2}{2}\left\{1+6\left[\frac{3M^4\Gamma[-2+\epsilon]}{8\pi^2 f^2}\left(\frac{4\pi\mu^2}{M^2}\right)^{\epsilon}\right]^2+\dots\right\} \tag{4.A.5}$$

and

$$\mathcal{E}_{\rm GC} = \frac{3{\rm N}M^4\Gamma[-2+\epsilon]}{4\pi^2}\left(\frac{4\pi\mu^2}{M^2}\right)^{\epsilon}\left\{1-\log\left[\frac{3M^4\Gamma[-2+\epsilon]}{8\pi^2 f^2}\left(\frac{4\pi\mu^2}{M^2}\right)^{\epsilon}\right]+\dots\right\}. \tag{4.A.6}$$

Let us now send $M\to 0$. We obtain that

$$\text{Original VA point:}\quad \phi_{\rm VA}\to 0,\quad \mathcal{E}_{\rm VA}\to\frac{{\rm N}f^2}{2}, \tag{4.A.7}$$

and

$$\text{Goldstino condensation configuration:}\quad \phi_{\rm GC}\to -1,\quad \mathcal{E}_{\rm GC}\to 0. \tag{4.A.8}$$

A few comments are in order. First of all, any dependence on the regularization scheme has dropped out. In addition, the VA stationary point has the original vacuum energy value and there is no condensate appearing at that point. In the new stationary point the condensate reaches its maximum value which is independent of the regularization, namely $\phi_{\rm GC}\to -1$, while its energy density vanishes. Due to the exact vanishing of the vacuum energy we can deduce, giving further support to what we state in the bulk, that supersymmetry has to be restored at that point. As we already mentioned, let us notice also that the goldstini stop propagating in such a limit at the goldstino condensation point.

Even though, because of the freedom to shift energies, the true value of the energy density in a QFT is a relative matter, we see that the original VA stationary point reaches its original energy density in the limit $M\to 0$. We can therefore define the energy density of the supersymmetry breaking point with respect to that limit as

$$\rho_{\rm VA} = \langle{\rm VA}||Q|^2|{\rm VA}\rangle = \frac{{\rm N}f^2}{2}; \tag{4.A.9}$$

and in the same limit we also find

$$\rho_{\rm GC} = \langle{\rm GC}||Q|^2|{\rm GC}\rangle = 0 \;\longrightarrow\; Q|{\rm GC}\rangle = \mathbf{0}, \tag{4.A.10}$$

thus interpreting the GC point as a supersymmetry restoration point. We see that this analysis agrees exactly with the analysis that we did in Section 4.1 for the deep IR limit. It is gratifying to see that the results do not change depending on the regularization scheme. For this reason we work only with the cut-off regularization prescription in the bulk of the chapter.

# Outlook

As the reader has certainly appreciated, the present thesis work places itself mainly within the Swampland Research Program and, coherently with its spirit, describes the use of Supergravity as a powerful effective tool both to gain insight on the properties that a theory that can be consistently coupled to quantum gravity should have, and more specifically on the interconnected conjectures distilling the Landscape out of the Swampland (such as Weak Gravity Conjecture), and to investigate the intensively discussed de Sitter constructions in a string theory framework.

In Chapter 1 we built evidence that asymptotically flat BPS black holes in four spacetime dimensions should be characterized by a differential constraint on their ADM mass of the form (1.5.1) which, once combined with the use of the WGC, allows to obtain a general constraint on the scalar-dependent masses of the various fields. In particular, for a generic charged black hole in the presence of scalar fields (1.5.2) holds and, by putting it together with our relation, we found that $M^2 + \Sigma^2 - Q^2_\infty = n\, M^2 + \Sigma^2 - D^2 M^2 \geq 0$. This implies that the particle needed to discharge the black hole should satisfy the opposite inequality, which is a rather strong constraint on the possible moduli dependence of the masses of particles in effective theories. We therefore started inspecting the requirements and the limits to interpret such a relation as a novel scalar WGC: studying conjugate BPS configurations in the N=2 STU model, we concluded that the generalization of our relation to theories where the moduli do not come in complex form is not straightforward. In addition, because a sum over all the complex scalars contributing to the BPS configuration is required, we were not able to extract a strong form of the inequality, i.e. valid for each scalar.

After the analysis of the electric WGC and its possible extension to forces mediated by light scalar fields, willing to start attacking the out-standing problem of realizing de Sitter vacua in String Theory, in Chapter 2 we focused on the WGC but in its magnetic formulation and we tried to uncover the difficulties of finding de Sitter critical configurations at the level of four-dimensional Supergravity. We argued that de Sitter critical points in extended (two-derivative) supergravity theories violate the magnetic WGC when they have charged, light gravitini. We gave a general proof of this claim in

N=2 and N=8 gauged Supergravity. We furthermore illustrated this claim with several N=2 models with hypermultiplets, whose scalar potentials admit de Sitter vacua, both stable and unstable.

Despite of the possibility to evade such an argument, as the critical points that have either massive gravitini or no U(1) gauge symmetry at the critical point do, we decided to put forward the idea that stringy de Sitter constructions, as the KKLT model, can be challenged while adopting a low energy four-dimensional perspective.
In Chapter 3 we then focused on the Volkov–Akulov model and, following an exact renormalization group flow, we analyzed the resulting low energy effective theory that we were able to recast in a form where supersymmetry becomes linearly realized, albeit spontaneously broken. We showed that the field space central point that would correspond to the original vacuum turned out to be unstable, and the same happened when we coupled the low energy effective theory to 4D N=1 Supergravity. Interestingly, we gave evidence that the same instability persists when the system is coupled to an additional chiral superfield representing the Kähler modulus of KKLT. Even though we were not able to completely understand and clarify what the origin of this instability in a full string theory setup is, and whether there are alternative models where a stable de Sitter vacuum can exist, we could conclude that, when an anti-brane is used for uplift purposes, if the effective description is truly four-dimensional such that 4D N=1 non-linear supersymmetry is invoked, stability should not be taken for granted unless the composite goldstino states are first analyzed.

In order to put the discovery of the tachyonic instability in Chapter 3 on firmer ground, we studied the component-form 4D Volkov–Akulov model in the presence of N non-linear supersymmetries in Chapter 4. By exploiting large N methods we showed that the effective scalar potential, written in terms of two composite scalar fields, exhibits at least two critical points, one corresponding to the original supersymmetry breaking Volkov–Akulov point and the other one representing goldstino condensation, where supersymmetry is restored in the deep IR. This result is nicely consonant with the conclusions of Chapter 3 and indicates a path to possibly shed light on the reasons and properties (e.g. its endpoint) of the instability that we are putting to the forefront[8].

As we already stressed at the end of each chapter, there are many interesting and worth exploring directions that may help deepening the interconnections among the various Swampland conjectures and targeting their response to well-known physical problems, as the accelerated expansion of our Universe "today". For instance, the study of black holes in extended Supergravity in the presence of a Proca field could suggest interesting

[8]It is matter of current investigation whether such instability is an inherent IR effect or it can be visible in a UV ten-dimensional framework, making it specific, for instance, in a stringy brane construction.

extensions of the Weak Gravity Conjecture to forces mediated by massive vector fields or to composite particles (which may be as well composite vectors) and its possible relations to the "festina lente" proposal, which sets lower bounds to the mass of particles on a de Sitter background. A better understanding of condensation phenomena, as the formation of goldstino condensates, within Supergravity may be a florid ground to explore to challenge de Sitter constructions of String Theory or to build up stringy models consistently accounting for the accelerated expansion that our Universe primordially faced according to the inflationary paradigm and is facing "today" because of dark energy. Both circumstances will improve our understanding of Nature and help targeting research activities more and more accurately towards its description.

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