Asymptotic behavior of the solutions of a transmission problem for the Helmholtz equation. A functional analytic approach

Running title: A singularly perturbed transmission problem

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Abstract: Let Ω^i, Ω^o be bounded open connected subsets of $\mathbb{R}^n$ that contain the origin. Let $\Omega(\epsilon) \equiv \Omega^o \setminus \epsilon\overline{\Omega^i}$ for small $\epsilon > 0$. Then we consider a linear transmission problem for the Helmholtz equation in the pair of domains $\epsilon\Omega^i$ and $\Omega(\epsilon)$ with Neumann boundary conditions on $\partial\Omega^o$. Under appropriate conditions on the wave numbers in $\epsilon\Omega^i$ and $\Omega(\epsilon)$ and on the parameters in-

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volved in the transmission conditions on $\epsilon\partial\Omega^i$, the transmission problem has a unique solution $(u^i(\epsilon,\cdot), u^o(\epsilon,\cdot))$ for small values of $\epsilon > 0$. Here $u^i(\epsilon,\cdot)$ and $u^o(\epsilon,\cdot)$ solve the Helmholtz equation in $\epsilon\Omega^i$ and $\Omega(\epsilon)$, respectively. Then we prove that if $x \in \Omega^o \setminus \{0\}$, then $u^o(\epsilon, x)$ can be expanded into a convergent power expansion of ϵ, $\kappa_n \epsilon \log \epsilon$, $\delta_{2,n} \log^{-1} \epsilon$ for ϵ small enough. Here $\kappa_n = 1$ if n is even and $\kappa_n = 0$ if n is odd and $\delta_{2,2} \equiv 1$ and $\delta_{2,n} \equiv 0$ if $n \geq 3$.

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1 Introduction

In this paper we consider a linear transmission problem for the Helmholtz equation in a domain with a small inclusion. Problems of this type are motivated by the analysis of time-harmonic Maxwell's Equations (see Vogelius and Volkov [32]). For related problems for the Helmholtz equation, we refer to the papers [2] of Ammari, Vogelius and Volkov, [1] of Ammari, Iakovleva and Moskow, [3] of Ammari and Volkov, and [16] of Hansen, Poignard and Vogelius. First we introduce a problem with no hole (and no transmission), and then we consider the case with the hole. We consider $m \in \mathbb{N} \setminus \{0\}$, $n \in \mathbb{N} \setminus \{0, 1\}$, $\alpha \in]0, 1[$ and the following assumption.

Let Ω be a bounded open connected subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Let $\mathbb{R}^n \setminus \overline{\Omega}$ be connected. Let $0 \in \Omega$. (1.1)

Now let Ω^o be as in (1.1). Let

$$k_o \in \mathbb{C} \setminus]-\infty, 0] , \qquad \Im k_o \geq 0 . \tag{1.2}$$

We also assume that k_o^2 is not a Neumann eigenvalue for $-\Delta$ in Ω^o. Then if

$$g^o \in C^{m-1,\alpha}(\partial\Omega^o) , \tag{1.3}$$

and if ν_{Ω^o} is the outward unit normal to $\partial\Omega^o$, the Neumann problem

$$\begin{cases} \Delta u^o + k_o^2 u^o = 0 & \text{in } \Omega^o , \\ \frac{\partial}{\partial \nu_{\Omega^o}} u^o = g^o & \text{on } \partial\Omega^o \end{cases} \tag{1.4}$$

has a unique solution $\tilde{u}^o \in C^{m,\alpha}(\overline{\Omega^o})$ (see for example Colton and Kress [8, Thm. 3.20] and classical Schauder regularity theory). Since the Helmholtz equation in (1.4) is even in k_o, if $k_o \in \mathbb{C} \setminus \{0\}$, possibly replacing k_o by $-k_o$, we can always assume that assumption (1.2) is fulfilled.

We now perturb singularly our problem. To do so, we consider another subset Ω^i of $\mathbb{R}^n$ as in (1.1). Then there exists

$$\epsilon_0 \in]0,1[\text{ such that } \epsilon\overline{\Omega^i} \subseteq \Omega^o \qquad \forall \epsilon \in [-\epsilon_0, \epsilon_0]\,. \tag{1.5}$$

A known topological argument shows that $\Omega(\epsilon) \equiv \Omega^o \setminus \epsilon\overline{\Omega^i}$ is connected, and that $\mathbb{R}^n \setminus \overline{\Omega(\epsilon)}$ has exactly the two connected components $\epsilon\Omega^i$ and $\mathbb{R}^n \setminus \overline{\Omega^o}$, and that

$$\partial\Omega(\epsilon) = (\epsilon\partial\Omega^i) \cup \partial\Omega^o \qquad \forall \epsilon \in]-\epsilon_0, \epsilon_0[\setminus\{0\}\,.$$

Moreover the outward unit normal ν_ϵ to $\partial\Omega(\epsilon)$ satisfies the equality

$$\nu_\epsilon(x) = -\nu_{\Omega^i}(x/\epsilon)\,\mathrm{sgn}(\epsilon) \qquad \forall x \in \epsilon\partial\Omega^i\,, \tag{1.6}$$
$$\nu_\epsilon(x) = \nu_{\Omega^o}(x) \qquad \forall x \in \partial\Omega^o\,, \tag{1.7}$$

for all $\epsilon \in]-\epsilon_0, \epsilon_0[\setminus\{0\}$, where $\mathrm{sgn}(\epsilon) = 1$ if $\epsilon > 0$, $\mathrm{sgn}(\epsilon) = -1$ if $\epsilon < 0$. Then we introduce the constants

$$m^i, m^o \in]0, +\infty[\,, \qquad a \in]0,+\infty[\,,\; b \in \mathbb{R}\,,$$

and

$$k_i \in \mathbb{C}\setminus]-\infty, 0]\,, \qquad \Im k_i \geq 0\,, \tag{1.8}$$

and the datum

$$g^i \in C^{m-1,\alpha}(\partial\Omega^i)\,. \tag{1.9}$$

Then we consider the transmission problem

$$\begin{cases} \Delta u^i + k_i^2 u^i = 0 & \text{in } \epsilon\Omega^i\,, \\ \Delta u^o + k_o^2 u^o = 0 & \text{in } \Omega(\epsilon)\,, \\ u^o(x) - a u^i(x) = b & \forall x \in \epsilon\partial\Omega^i\,, \\ -\frac{1}{m^i}\frac{\partial}{\partial\nu_{\epsilon\Omega^i}} u^i(x) + \frac{1}{m^o}\frac{\partial}{\partial\nu_{\epsilon\Omega^i}} u^o(x) = g^i(x/\epsilon) & \forall x \in \epsilon\partial\Omega^i\,, \\ \frac{\partial}{\partial\nu_{\Omega^o}} u^o = g^o & \text{on } \partial\Omega^o\,, \end{cases} \tag{1.10}$$

in the unknown $(u^i, u^o) \in C^{m,\alpha}(\epsilon\overline{\Omega^i}) \times C^{m,\alpha}(\overline{\Omega(\epsilon)})$ for $\epsilon \in]0, \epsilon_0[$, and we plan to show that for $\epsilon \in]0, \epsilon_0[$ small enough, problem (1.10) has a unique solution $(u^i(\epsilon,\cdot), u^o(\epsilon,\cdot)) \in C^{m,\alpha}(\epsilon\overline{\Omega^i}) \times C^{m,\alpha}(\overline{\Omega(\epsilon)})$ and to understand the behavior of $(u^i(\epsilon,\cdot), u^o(\epsilon,\cdot))$ as ϵ approaches 0. More precisely, we plan to answer the following question.

Let x be fixed in $\overline{\Omega^o} \setminus \{0\}$. What can be said on the map (1.11)

$$\epsilon \mapsto u^o(\epsilon, x) \text{ when } \epsilon > 0 \text{ is close to } 0?$$

In a sense, question (1.11) concerns the 'macroscopic' behavior of $u^o(\epsilon, \cdot)$. We are also interested in the 'macroscopic' behavior of $\{u^i(\epsilon, \cdot)\}_{\epsilon\in]0,\epsilon'[}$ as ϵ is close to 0. Since the only point which belongs to the domain of all the functions $u^i(\epsilon, \cdot)$ as $\epsilon \in]0, \epsilon'[$ is $x = 0$, here we mean that we are interested in the behavior of $\{u^i(\epsilon, 0)\}_{\epsilon\in]0,\epsilon'[}$ as ϵ is close to 0. Such a behavior is a specific case of the 'microscopic' behavior of the family $\{u^i(\epsilon, \cdot)\}_{\epsilon\in]0,\epsilon'[}$, *i.e.*, of the behavior of $\{u^i(\epsilon, \epsilon\xi)\}_{\epsilon\in]0,\epsilon'[}$ in case $\xi = 0$, a case that we plan to analyze in a forthcoming paper.

Questions of this type have long been investigated for linear problems on domains with small holes with the methods of asymptotic analysis, which aim at proving complete asymptotic expansions in terms of the parameter ϵ. It is perhaps difficult to provide a complete list of the contributions. Here we mention the early results in the monographs of Cherepanov [5] and [6] on the formation of cracks and in the books of Nayfeh [29], Van Dyke [31], and Cole [7], which present an extensive review of the expansion methods known at the time. For the rigorous description of the method of matching outer and inner asymptotic expansions we refer to the book of Il'in [18], and for the Compound Expansion Method (also known as Multi-Scale Expansion Method) we mention the two volumes of Mazya, Nazarov and Plamenewskii [27] where, among other results, the authors introduce a systematic approach for analyzing general Douglis and Nirenberg elliptic boundary value problems in domains with perforations and corners.

To analyze the problem and answer the above question, we resort to the Functional Analytic Approach (see [10]). Accordingly, we first convert the transmission problem (1.10) into a system of integral equations by exploiting classical Potential Theory. Then we observe that, by changing the variables appropriately, we can obtain a functional equation that can be analyzed by means of the Implicit Function Theorem around the degenerate case where $\epsilon = 0$. Then we prove that we can represent the unknown densities of the integral equations in terms of real analytic functions of ϵ when $n \geq 3$ is odd, of real analytic functions of ϵ, $\epsilon \log \epsilon$ when $n \geq 3$ is even and of ϵ, $\epsilon \log \epsilon$, $\log^{-1} \epsilon$ for $n = 2$. Next we go back to the integral representation of the solutions of problem (1.10) and we deduce both the existence of $u^i(\epsilon, \cdot)$ and of $u^o(\epsilon, \cdot)$ and the representation formulas that describe their dependence upon ϵ. Thus we prove our main result, *i.e.*, Theorem 5.1 that answers question (1.11) (see also the following comment).

For related problems for the Laplace equation, we refer to [21] and to the paper [28] of Molinarolo. For an existence result in the case of the Laplace equation and of a big inclusion (that is, for $\epsilon > 0$ fixed) we refer to the paper [11] of Dalla Riva and Mishuris and for a local uniqueness result for the solutions of (1.10) themselves, rather than for the family of solutions, we mention the paper [12] of Dalla Riva, Molinarolo and Musolino.

2 Preliminaries and notation

We denote the norm on a normed space $\mathcal{X}$ by $\|\cdot\|_{\mathcal{X}}$. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. We endow the product space $\mathcal{X} \times \mathcal{Y}$ with the norm defined by $\|(x, y)\|_{\mathcal{X}\times\mathcal{Y}} \equiv \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{Y}}$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$, while we use the Euclidean norm for $\mathbb{R}^n$. We denote by I the identity operator. For standard definitions of Calculus in normed spaces, we refer to Cartan [4] and to Prodi and Ambrosetti [30]. The symbol $\mathbb{N}$ denotes the set of natural numbers including 0. Throughout the paper,

$$n \in \mathbb{N} \setminus \{0, 1\} .$$

The inverse function of an invertible function f is denoted $f^{(-1)}$, as opposed to the reciprocal of a complex-valued function g, or the inverse of a matrix A, which are denoted g^{-1} and A^{-1}, respectively. A dot '$\cdot$' denotes the inner product in $\mathbb{R}^n$, or the matrix product between matrices with real entries. Let $\mathbb{D} \subseteq \mathbb{R}^n$. Then $\overline{\mathbb{D}}$ denotes the closure of $\mathbb{D}$ and $\partial\mathbb{D}$ denotes the boundary of $\mathbb{D}$. For all $R > 0$, $x \in \mathbb{R}^n$, x_j denotes the j-th coordinate of x, $|x|$ denotes the Euclidean modulus of x in $\mathbb{R}^n$, and $\mathbb{B}_n(x, R)$ denotes the ball $\{y \in \mathbb{R}^n : |x - y| < R\}$. Let Ω be an open subset of $\mathbb{R}^n$. Then we find convenient to set

$$\Omega^+ \equiv \Omega, \qquad \Omega^- \equiv \mathbb{R}^n \setminus \overline{\Omega} .$$

The space of m times continuously differentiable complex-valued functions on Ω is denoted by $C^m(\Omega, \mathbb{C})$, or more simply by $C^m(\Omega)$. Let $r \in \mathbb{N} \setminus \{0\}$, $f \in (C^m(\Omega))^r$. The s-th component of f is denoted f_s and the Jacobian matrix of f is denoted Df. Let $\eta \equiv (\eta_1, \dots, \eta_n) \in \mathbb{N}^n$, $|\eta| \equiv \eta_1 + \cdots + \eta_n$. Then $D^\eta f$ denotes $\frac{\partial^{|\eta|} f}{\partial x_1^{\eta_1} \dots \partial x_n^{\eta_n}}$. The subspace of $C^m(\Omega)$ of those functions f such that f and its derivatives $D^\eta f$ of order $|\eta| \leq m$ can be extended with continuity to $\overline{\Omega}$ is denoted $C^m(\overline{\Omega})$. The subspace of $C^m(\overline{\Omega})$ whose functions have m-th order derivatives that are Hölder continuous with exponent $\alpha \in]0, 1]$ is denoted $C^{m,\alpha}(\overline{\Omega})$, (cf. *e.g.* [10, §2.11].) Let $\mathbb{D} \subseteq \mathbb{R}^n$. Then $C^{m,\alpha}(\overline{\Omega}, \mathbb{D})$ denotes the set $\left\{f \in \left(C^{m,\alpha}(\overline{\Omega})\right)^n : f(\overline{\Omega}) \subseteq \mathbb{D}\right\}$. We say that a bounded open subset of $\mathbb{R}^n$ is of class C^m or of class $C^{m,\alpha}$, if it is a manifold with boundary imbedded in $\mathbb{R}^n$ of class C^m or $C^{m,\alpha}$, respectively (cf. *e.g.*, [10, §2.13].) For standard properties of the functions of class $C^{m,\alpha}$ both on a domain of $\mathbb{R}^n$ or on a manifold imbedded in $\mathbb{R}^n$ we refer to [10, §2.11, 2.12, 2.14, 2.20] (see also [19, §2, Lem. 3.1, 4.26, Thm. 4.28], [24, §2].) We retain the standard notation of L^p spaces and of corresponding norms. We note that throughout the paper 'analytic' means 'real analytic'.

3 Some basic facts in potential theory.

Next we turn to introduce the fundamental solution of $\Delta + k^2$ when $k \in \mathbb{C}\setminus]-\infty, 0]$. In the sequel, arg and log denote the principal branch of the argument and of the logarithm in $\mathbb{C}\setminus]-\infty, 0]$, respectively. Then we have

$$\arg(z) = \Im \log(z) \in]-\pi, \pi[\qquad \forall z \in \mathbb{C}\setminus]-\infty, 0]\,.$$

Then we set

$$J_\nu^\sharp(z) \equiv \sum_{j=0}^{\infty} \frac{(-1)^j z^j (1/2)^{2j} (1/2)^\nu}{\Gamma(j+1)\Gamma(j+\nu+1)} \qquad \forall z \in \mathbb{C}\,, \tag{3.1}$$

for all $\nu \in \mathbb{C} \setminus \{-j : j \in \mathbb{N} \setminus \{0\}\}$. Here $(1/2)^\nu = e^{\nu \log(1/2)}$. As is well known, if $\nu \in \mathbb{C} \setminus \{-j : j \in \mathbb{N} \setminus \{0\}\}$ then the function $J_\nu^\sharp(\cdot)$ is entire and

$$J_\nu^\sharp(z^2) \equiv e^{-\nu \log z} J_\nu(z) \qquad \forall z \in \mathbb{C}\setminus]-\infty, 0]\,, \tag{3.2}$$

where $J_\nu(\cdot)$ is the Bessel function of the first kind of index ν (cf. *e.g.*, Lebedev [26, Ch. 1, §5.3].) One could also consider case $\nu \in -\mathbb{N}$, but we do not need such a case in this paper. If $\nu \in \mathbb{N}$, we set

$$\begin{aligned} N_\nu^\sharp(z) \equiv & -\frac{2^\nu}{\pi} \sum_{0 \le j \le \nu-1} \frac{(\nu-j-1)!}{j!} z^j (1/2)^{2j} \qquad\qquad (3.3) \\ & -\frac{z^\nu}{\pi} \sum_{j=0}^{\infty} \frac{(-1)^j z^j (1/2)^{2j} (1/2)^\nu}{j!(\nu+j)!} \left(2 \sum_{0<l\le j} \frac{1}{l} + \sum_{j<l\le j+\nu} \frac{1}{l} \right) \quad \forall z \in \mathbb{C}\,. \end{aligned}$$

As one can easily see, the $N_\nu^\sharp(\cdot)$ is an entire holomorphic function of the variable $z \in \mathbb{C}$ and

$$N_\nu(z) = \frac{2}{\pi}(\log(z) - \log 2 + \gamma) J_\nu(z) + z^{-\nu} N_\nu^\sharp(z^2) \qquad \forall z \in \mathbb{C}\setminus]-\infty, 0]\,, \tag{3.4}$$

where γ is the Euler-Mascheroni constant, and where $N_\nu(\cdot)$ is the Neumann function of index ν, also known as Bessel function of second kind and index

ν (cf. *e.g.*, Lebedev [26, Ch. 1, §5.5].) Let $k \in \mathbb{C} \setminus]-\infty, 0]$, $n \in \mathbb{N} \setminus \{0,1\}$, $a_n \in \mathbb{C}$. Then we set

$$b_n \equiv \begin{cases} \pi^{1-(n/2)} 2^{-1-(n/2)} & \text{if } n \text{ is even}\,, \\ (-1)^{\frac{n-1}{2}} \pi^{1-(n/2)} 2^{-1-(n/2)} & \text{if } n \text{ is odd}\,, \end{cases} \tag{3.5}$$

and

$$\tilde{S}_{k,a_n}(x) = \begin{cases} k^{n-2} \Big\{ a_n + \frac{2b_n}{\pi} (\log\, k - \log 2 + \gamma) + \frac{2b_n}{\pi} \log |x| \Big\} \\ \quad \times J^{\sharp}_{\frac{n-2}{2}}(k^2|x|^2) + b_n |x|^{2-n} N^{\sharp}_{\frac{n-2}{2}}(k^2|x|^2) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } n \text{ is even}, \\ a_n k^{n-2} J^{\sharp}_{\frac{n-2}{2}}(k^2|x|^2) + b_n |x|^{2-n} J^{\sharp}_{-\frac{n-2}{2}}(k^2|x|^2) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } n \text{ is odd}, \end{cases} \tag{3.6}$$

for all $x \in \mathbb{R}^n \setminus \{0\}$. As it is known and can be easily verified, the family $\{\tilde{S}_{k,a_n}\}_{a_n \in \mathbb{C}}$ coincides with the family of all radial fundamental solutions of $\Delta + k^2$.

Now we need to consider two specific fundamental solutions. For the first, we need to choose a_n so that the resulting fundamental solution can be extended to an entire holomorphic function of the variable $k \in \mathbb{C}$. For the second, we need to choose a_n so as to obtain the fundamental solution which satisfies the Bohr-Sommerfeld outgoing radiation condition corresponding to k. We start by introducing the holomorphic family, which we denote by $S_{h,n}$. Here the subscript h stands for 'holomorphic'. For a proof we refer to the paper [25, Prop. 3.3] with Rossi.

Theorem 3.7 *Let $n \in \mathbb{N} \setminus \{0,1\}$. Let $S_{h,n}(\cdot,\cdot)$ be the map from $(\mathbb{R}^n \setminus \{0\}) \times \mathbb{C}$ to $\mathbb{C}$ defined by*

$$S_{h,n}(x,k) \equiv \begin{cases} b_n \Big\{ \frac{2}{\pi} k^{n-2} J^{\sharp}_{\frac{n-2}{2}}(k^2|x|^2) \log |x| \\ \qquad\qquad + |x|^{-(n-2)} N^{\sharp}_{\frac{n-2}{2}}(k^2|x|^2) \Big\} & \textit{if } n \textit{ is even}\,, \\ b_n |x|^{-(n-2)} J^{\sharp}_{-\frac{n-2}{2}}(k^2|x|^2) & \textit{if } n \textit{ is odd}\,, \end{cases} \tag{3.8}$$

for all $(x,k) \in (\mathbb{R}^n \setminus \{0\}) \times \mathbb{C}$. Then the following statements hold.

(i) $S_{h,n}(\cdot,k)$ is a fundamental solution of $\Delta + k^2$ for all $k \in \mathbb{C}$ and $S_{h,n}(\cdot,0)$

coincides with the classical fundamental solution S_n of Δ, i.e.,

$$S_{h,n}(x,0) = S_n(x) \equiv \begin{cases} \frac{1}{s_n}\log|x| & \forall x \in \mathbb{R}^n \setminus \{0\}, \quad \text{if } n = 2\,, \\ \frac{1}{(2-n)s_n}|x|^{2-n} & \forall x \in \mathbb{R}^n \setminus \{0\}, \quad \text{if } n > 2\,, \end{cases} \tag{3.9}$$

where s_n denotes the $(n-1)$ dimensional measure of $\partial\mathbb{B}_n(0,1)$.

(ii) $S_{h,n}(\cdot,k)$ is real analytic in $\mathbb{R}^n \setminus \{0\}$. Moreover, if $x \in \mathbb{R}^n \setminus \{0\}$, then the map $S_{h,n}(x,\cdot)$ is holomorphic in $\mathbb{C}$.

Now let $k \in \mathbb{C}\setminus]-\infty,0]$, $\Im k \geq 0$. As well known in scattering theory, a function $u \in C^1(\mathbb{R}^n \setminus \{0\})$ satisfies the outgoing k-radiation condition provided that

$$\lim_{x\to\infty} |x|^{\frac{n-1}{2}}(Du(x)\frac{x}{|x|} - iku(x)) = 0\,. \tag{3.10}$$

Now by writing the fundamental solution of (3.6) in terms of the Hankel functions, and by exploiting the asymptotic behavior at infinity of the Hankel functions, one finds classically that the fundamental solution of (3.6) satisfies the outgoing k-radiation condition if and only if

$$a_n \equiv \left\{ \begin{array}{ll} -ib_n & \text{if } n \text{ is even} \\ -e^{-i\frac{n-2}{2}\pi}b_n & \text{if } n \text{ is odd} \end{array} \right\} = -i\pi^{1-(n/2)}2^{-1-(n/2)} \tag{3.11}$$

Then we introduce the following definition.

Definition 3.12 *Let $n \in \mathbb{N} \setminus \{0,1\}$. Let $k \in \mathbb{C}\setminus]-\infty,0]$. We denote by $S_{r,n}(\cdot,k)$ the function from $\mathbb{R}^n \setminus \{0\}$ to $\mathbb{C}$ defined by*

$$S_{r,n}(x,k) \equiv \tilde{S}_{k,a_n}(x) \qquad \forall x \in \mathbb{R}^n \setminus \{0\}\,,$$

with a_n as in (3.11) (cf. (3.6).)

As we have said above, if $k \in \mathbb{C}\setminus]-\infty,0]$ and $\Im k \geq 0$, then $S_{r,n}(\cdot,k)$ satisfies the outgoing k-radiation condition and $S_{r,n}(\cdot,k)$ is the fundamental solution that is classically used in scattering theory. In particular, for $n = 3$ we have $S_{r,3}(x,k) = -\frac{1}{4\pi|x|}e^{ik|x|}$ for all $x \in \mathbb{R}^n \setminus \{0\}$. The subscript r stands for 'radiation'. Now we introduce the function γ_n from $\mathbb{C}$ to $\mathbb{C}$ defined by setting

$$\gamma_n(z) \equiv \begin{cases} [-i + \frac{2}{\pi}(z - \log 2 + \gamma)]b_n & \text{if } n \text{ is even}\,, \\ -e^{-i\frac{n-2}{2}\pi}b_n & \text{if } n \text{ is odd}\,, \end{cases} \tag{3.13}$$

for all $z \in \mathbb{C}$. Then we have

$$S_{r,n}(x,k) = S_{h,n}(x,k) + \gamma_n(\log k)k^{n-2}J^\sharp_{\frac{n-2}{2}}(k^2|x|^2) \qquad \forall x \in \mathbb{R}^n \setminus \{0\}\,, \tag{3.14}$$

for all $k \in \mathbb{C}\setminus]-\infty,0]$, which is a formula that shows the relation between the fundamental solution $S_{r,n}$ that is normally used in scattering theory and the fundamental solution $S_{h,n}$ that we need here for technical purposes. Next we introduce the layer potentials corresponding to a fundamental solution or to a smooth kernel and the corresponding boundary operators.

Definition 3.15 *Let $n \in \mathbb{N} \setminus \{0,1\}$, $k \in \mathbb{C}$. Let S be either a fundamental solution of $\Delta + k^2$ or a real analytic function from $\mathbb{R}^n$ to $\mathbb{C}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $\mu \in C^0(\partial\Omega)$. Then we introduce the following notation.*

(i) We denote by $v_\Omega[\mu,S]$ the function from $\mathbb{R}^n$ to $\mathbb{C}$ defined by

$$v_\Omega[\mu,S](x) \equiv \int_{\partial\Omega} S(x-y)\mu(y)\,d\sigma_y \qquad \forall x \in \mathbb{R}^n\,. \tag{3.16}$$

Then we denote by $v^+_\Omega[\mu,S]$, by $v^-_\Omega[\mu,S]$ and by $V_\Omega[\mu,S]$, the restriction of $v_\Omega[\mu,S]$ to $\overline{\Omega}$, to $\overline{\Omega^-}$ and to $\partial\Omega$, respectively.

(ii) We denote by $W^t_\Omega[\mu,S]$ the function from $\partial\Omega$ to $\mathbb{C}$ defined by

$$W^t_\Omega[\mu,S](x) \equiv \int_{\partial\Omega} \frac{\partial}{\partial\nu_{\Omega,x}} S(x-y)\mu(y)\,d\sigma_y \qquad \forall x \in \partial\Omega\,, \tag{3.17}$$

where

$$\frac{\partial}{\partial\nu_{\Omega,x}} S(x-y) \equiv DS(x-y)\nu_\Omega(x) \qquad \forall (x,y) \in \partial\Omega\times\partial\Omega, x \neq y\,.$$

If $k \in \mathbb{C}\setminus]-\infty,0]$, we set

$$v_\Omega[\mu,k] \quad \equiv \quad v_\Omega[\mu, S_{r,n}(\cdot,k)]\,, \tag{3.18}$$

and we use corresponding abbreviations for V_Ω, $v^\pm_\Omega$, W^t_Ω. If $k \in \mathbb{C}$, we set

$$v_{\Omega,h}[\mu,k] \quad = \quad v_\Omega[\mu, S_{h,n}(\cdot,k)]\,, \tag{3.19}$$

and we use corresponding abbreviations for $V_{\Omega,h}$, $v^\pm_{\Omega,h}$, $W^t_{\Omega,h}$. If $\lambda \in \mathbb{C}$, we set

$$v_{\Omega,J}[\mu,\lambda] \quad = \quad v_\Omega[\mu, J^\sharp_{\frac{n-2}{2}}(\lambda|\cdot|^2)]\,, \tag{3.20}$$

and we use corresponding abbreviations for $V_{\Omega,J}$, $v^{\pm}_{\Omega,J}$, $W^t_{\Omega,J}$. Next we introduce the following result on acoustic layer potentials, that is a consequence of [9] (which is a generalization of [25]). For a proof we refer to [22, Thm. A.3].

Theorem 3.21 *Let $n \in \mathbb{N}\setminus\{0,1\}$, $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0,1[$. Let Ω be a bounded open connected subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$ such that $\mathbb{R}^n \setminus \overline{\Omega}$ is connected. Let $r \in]0,+\infty[$ be such that $\overline{\Omega} \subseteq \mathbb{B}_n(0,r)$. Then the following statements hold.*

(i) The map $v^+_{\Omega,h}$ from $C^{m-1,\alpha}(\partial\Omega) \times \mathbb{C}$ to $C^{m,\alpha}(\overline{\Omega})$ which takes (μ,k) to the function $v^+_{\Omega,h}[\mu,k]$ is real analytic.

(ii) The map $v^-_{\Omega,h}[\cdot,\cdot]_{|\overline{\mathbb{B}_n(0,r)}\setminus\Omega}$ from $C^{m-1,\alpha}(\partial\Omega) \times \mathbb{C}$ to $C^{m,\alpha}(\overline{\mathbb{B}_n(0,r)} \setminus \Omega)$ which takes (μ,k) to the function $v^-_{\Omega,h}[\mu,k]_{|\overline{\mathbb{B}_n(0,r)}\setminus\Omega}$ is real analytic.

(iii) The map $W^t_{\Omega,h}$ from $C^{m-1,\alpha}(\partial\Omega)\times\mathbb{C}$ to $C^{m-1,\alpha}(\partial\Omega)$ which takes (μ,k) to the function $W^t_{\Omega,h}[\mu,k]$ is real analytic.

Then we have the following (*cf.* [22, Thm. 4]).

Theorem 3.22 *Let $n \in \mathbb{N}\setminus\{0,1\}$, $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0,1[$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Let $r \in]0,+\infty[$ be such that $\overline{\Omega} \subseteq \mathbb{B}_n(0,r)$. Then the following statements hold.*

(i) The map $V_{\Omega,J}[\cdot,\cdot]$ from $C^{m-1,\alpha}(\partial\Omega)\times\mathbb{C}$ to $C^{m,\alpha}(\partial\Omega)$ which takes (μ,λ) to the function $V_{\Omega,J}[\mu,\lambda]$ is real analytic.

(ii) The map $v^+_{\Omega,J}[\cdot,\cdot]$ from $C^{m-1,\alpha}(\partial\Omega) \times \mathbb{C}$ to $C^{m,\alpha}(\overline{\Omega})$ which takes (μ,λ) to the function $v^+_{\Omega,J}[\mu,\lambda]$ is real analytic.

(iii) The map $v^-_{\Omega,J}[\cdot,\cdot]_{|\overline{\mathbb{B}_n(0,r)}\setminus\Omega}$ from $C^{m-1,\alpha}(\partial\Omega) \times \mathbb{C}$ to $C^{m,\alpha}(\overline{\mathbb{B}_n(0,r)} \setminus \Omega)$ which takes (μ,λ) to the function $v^-_{\Omega,J}[\mu,\lambda]_{|\overline{\mathbb{B}_n(0,r)}\setminus\Omega}$ is real analytic.

(iv) The map $\tilde{W}^t_{\Omega,J}[\cdot,\cdot]$ from $C^{m-1,\alpha}(\partial\Omega) \times \mathbb{C}$ to $C^{m-1,\alpha}(\partial\Omega)$ which takes (μ,λ) to the function $\tilde{W}^t_{\Omega,J}[\mu,\lambda]$ from $\partial\Omega$ to $\mathbb{C}$ defined by

$$\tilde{W}^t_{\Omega,J}[\mu,\lambda](x) \equiv 2\int_{\partial\Omega} (J^{\sharp}_{\frac{n-2}{2}})'(\lambda(x-y)(x-y))(x-y)\nu_\Omega(x)\mu(y)\,d\sigma_y \qquad \forall x \in \partial\Omega\,, \tag{3.23}$$

for all $(\mu, \lambda) \in C^{m-1,\alpha}(\partial\Omega) \times \mathbb{C}$ *is real analytic. Moreover,*

$$W^t_{\Omega,J}[\mu, \lambda](x) = \lambda \tilde{W}^t_{\Omega,J}[\mu, \lambda](x) \qquad \forall x \in \partial\Omega\,, \tag{3.24}$$

for all $(\mu, \lambda) \in C^{m-1,\alpha}(\partial\Omega) \times \mathbb{C}$.

Corollary 3.25 *Let* $n \in \mathbb{N} \setminus \{0, 1\}$*,* $m \in \mathbb{N} \setminus \{0\}$*,* $\alpha \in]0, 1[$*,* $k \in \mathbb{C} \setminus] - \infty, 0]$*. Let* Ω *be an open subset of* $\mathbb{R}^n$ *of class* $C^{m,\alpha}$*. Then the following statements hold.*

(i) $v_\Omega[\mu, k] = v_{\Omega,h}[\mu, k] + \gamma_n(\log k)k^{n-2}v_{\Omega,J}[\mu, k^2]$ *on* $\mathbb{R}^n$ *for all* μ *in the space* $C^{m-1,\alpha}(\partial\Omega)$*.*

(ii) $v^\pm_\Omega[\mu, k] = v^\pm_{\Omega,h}[\mu, k] + \gamma_n(\log k)k^{n-2}v^\pm_{\Omega,J}[\mu, k^2]$ *on* $\overline{\Omega^\pm}$ *for all* μ *in* $C^{m-1,\alpha}(\partial\Omega)$*.*

(iii) $W^t_\Omega[\mu, k] = W^t_{\Omega,h}[\mu, k] + \gamma_n(\log k)k^n \tilde{W}^t_{\Omega,J}[\mu, k^2]$ *on* $\partial\Omega$ *for all* μ *in* $C^{m-1,\alpha}(\partial\Omega)$*.*

Proof. Statements (i)–(iii) are immediate consequences of equality (3.14), of the definition of the layer potentials involved, and of Theorem 3.22. □

Next we observe that the fundamental solution $S_{r,n}$ satisfies the following scaling property, which can be verified by exploiting the definition of $S_{r,n}$ and elementary computations.

Lemma 3.26 *Let* $n \in \mathbb{N} \setminus \{0, 1\}$*,* $k \in \mathbb{C} \setminus] - \infty, 0]$*. Then the following equalities hold*

$$\epsilon^{n-2} S_{r,n}(\epsilon x, k) = S_{r,n}(x, \epsilon k)\,, \tag{3.27}$$

$$\epsilon^{n-1} DS_{r,n}(\epsilon x, k) = DS_{r,n}(x, \epsilon k)\,, \tag{3.28}$$

$$\epsilon^n D^2 S_{r,n}(\epsilon x, k) = D^2 S_{r,n}(x, \epsilon k) \tag{3.29}$$

for all $x \in \mathbb{R}^n \setminus \{0\}$*,* $\epsilon \in]0, +\infty[$*.*

Then we note that the following elementary equality holds

$$\gamma_n(\log(\epsilon k)) = \frac{2b_n}{\pi}\kappa_n \log\epsilon + \gamma_n(\log k)\,, \tag{3.30}$$

for all $k \in \mathbb{C} \setminus] - \infty, 0]$ and $\epsilon \in]0, +\infty[$ (cf. (3.13).)

4 Formulation of the transmission problem in terms of integral equations

As a first step, we wish to convert the transmission problem (1.10) into a system of integral equations. By exploiting the representation Theorem A.1 for the solutions of the Helmholtz equation, the solutions u^i, u^o of (1.10) can be written as a single layer acoustic potential. Then by exploiting the classical jump formulas for single layer potentials, we obtain a system of integral equations on the ϵ-dependent domain $\epsilon\partial\Omega^i \cup \partial\Omega^o$. In order to get rid of ϵ in the domain, we rescale our functional variables on $\epsilon\partial\Omega^i$ and finally obtain an ϵ-dependent system of integral equations on the boundaries $\partial\Omega^i$ and $\partial\Omega^o$, which do not depend on ϵ. We present such a change of variable in the following Theorem 4.1.

Theorem 4.1 *Let $m \in \mathbb{N}\setminus\{0\}$, $n \in \mathbb{N}\setminus\{0,1\}$, $\alpha \in]0,1[$. Let Ω^i, Ω^o be as in (1.1). Let m^i, m^o, $a \in]0,+\infty[$, $b \in \mathbb{R}$. Let g^i, g^o be as in (1.3), (1.9). Let k_i, k_o be as in (1.2), (1.8). Assume that k_o^2 is not a Neumann eigenvalue for $-\Delta$ in Ω^o.*

Then there exists $\epsilon_* \in]0,\epsilon_0[$ such that the map from the set of (ψ,θ^i,θ^o) in $C^{m-1,\alpha}(\partial\Omega^i)^2 \times C^{m-1,\alpha}(\partial\Omega^o)$, which solve the following system of integral equations

$$\begin{aligned}
&\epsilon \int_{\partial\Omega^i} S_{h,n}(\xi-\eta,\epsilon k_o)\theta^i(\eta)\,d\sigma_\eta \\
&\quad +\epsilon^{n-1}k_o^{n-2}\left[\frac{2b_n}{\pi}\kappa_n \log\epsilon + \gamma_n(\log k_o)\right] V_{\Omega^i,J}[\theta^i,\epsilon^2 k_o^2](\xi) \\
&\quad +\int_{\partial\Omega^o} S_{r,n}(\epsilon\xi - y, k_o)\theta^o(y)\,d\sigma_y = a\epsilon \int_{\partial\Omega^i} S_{h,n}(\xi-\eta,\epsilon k_i)\psi(\eta)\,d\sigma_\eta \\
&\quad +a\epsilon^{n-1}k_i^{n-2}\left[\frac{2b_n}{\pi}\kappa_n \log\epsilon + \gamma_n(\log k_i)\right] V_{\Omega^i,J}[\psi,\epsilon^2 k_i^2](\xi) + b \quad \forall \xi \in \partial\Omega^i\,,
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
&-\frac{1}{m^i}\Big\{ -\frac{1}{2}\psi(\xi) + \int_{\partial\Omega^i} DS_{h,n}(\xi-\eta,\epsilon k_i)\nu_{\Omega^i}(\xi)\psi(\eta)\,d\sigma_\eta \\
&\quad +\epsilon^n k_i^n \left[\frac{2b_n}{\pi}\kappa_n \log\epsilon + \gamma_n(\log k_i)\right] \tilde{W}^t_{\Omega^i,J}[\psi,\epsilon^2 k_i^2](\xi)\Big\} \\
&-\frac{1}{m^o}\Big\{ -\frac{1}{2}\theta^i(\xi) - \int_{\partial\Omega^i} DS_{h,n}(\xi-\eta,\epsilon k_o)\nu_{\Omega^i}(\xi)\theta^i(\eta)\,d\sigma_\eta \\
&\quad -\epsilon^n k_o^n \left[\frac{2b_n}{\pi}\kappa_n \log\epsilon + \gamma_n(\log k_o)\right] \tilde{W}^t_{\Omega^i,J}[\theta^i,\epsilon^2 k_o^2](\xi)
\end{aligned} \tag{4.3}$$

$$
\begin{aligned}
& \left. - \int_{\partial\Omega^o} DS_{r,n}(\epsilon\xi - y, k_o)\nu_{\Omega^i}(\xi)\theta^o(y)\, d\sigma_y \right\} = g^i(\xi) \qquad \forall \xi \in \partial\Omega^i\,, \\
& -\frac{1}{2}\theta^o(x) + \int_{\partial\Omega^i} DS_{r,n}(x - \epsilon\eta, k_o)\nu_{\Omega^o}(x)\theta^i(\eta)\, d\sigma_\eta \epsilon^{n-1} \qquad (4.4) \\
& \quad + \int_{\partial\Omega^o} DS_{r,n}(x - y, k_o)\nu_{\Omega^o}(x)\theta^o(y)\, d\sigma_y = g^o(x) \qquad \forall x \in \partial\Omega^o\,,
\end{aligned}
$$

to the set of pairs (u^i, u^o) *in* $C^{m,\alpha}(\epsilon\overline{\Omega^i}) \times C^{m,\alpha}(\overline{\Omega(\epsilon)})$*, which satisfy problem (1.10) and that takes* $(\psi, \theta^i, \theta^o)$ *to the pair*

$$
(v_{\epsilon\Omega^i}[\omega, k_i], v_{\Omega(\epsilon)}[\mu, k_o]) \tag{4.5}
$$

where

$$
\begin{aligned}
\omega(x) &\equiv \psi(x/\epsilon) \qquad \forall x \in \epsilon\partial\Omega^i\,, \qquad (4.6) \\
\mu(x) &\equiv \theta^i(x/\epsilon) \qquad \forall x \in \epsilon\partial\Omega^i\,, \qquad \mu(x) \equiv \theta^o(x) \qquad \forall x \in \partial\Omega^o\,,
\end{aligned}
$$

is a bijection for all $\epsilon \in]0, \epsilon_*[$.

Proof. By a known result, we can choose $\epsilon_* \in]0, \epsilon_0[$ so that k_i^2 and k_o^2 are neither Neumann nor Dirichlet eigenvalues for $-\Delta$ in $\epsilon\Omega^i$ for all $\epsilon \in]0, \epsilon_*[$ (cf. *e.g.*, Colton and Kress [8, Lem. 3.26], [22, Prop. 9]).

Next we assume that $(u^i, u^o) \in C^{m,\alpha}(\epsilon\overline{\Omega^i}) \times C^{m,\alpha}(\overline{\Omega(\epsilon)})$ solves problem (1.10) and we show that there exists $(\psi, \theta^i, \theta^o)$ in $C^{m-1,\alpha}(\partial\Omega^i)^2 \times C^{m-1,\alpha}(\partial\Omega^o)$, which solves system (4.2)–(4.4) and such that (u^i, u^o) equals the pair in (4.5).

By the representation Theorem A.1, there exists one and only one (ω, μ) in $C^{m-1,\alpha}(\epsilon\partial\Omega^i) \times C^{m-1,\alpha}(\partial\Omega(\epsilon))$ such that

$$
u^i = v_{\epsilon\Omega^i}[\omega, k_i]\,, \qquad u^o = v_{\Omega(\epsilon)}[\mu, k_o]\,.
$$

Then the jump formulas for the acoustic single layer potential and the boundary conditions of problem (1.10) imply that the pair (ω, μ) satisfies the following system of integral equations

$$
\begin{aligned}
& \int_{\partial\Omega(\epsilon)} S_{r,n}(x - y, k_o)\mu(y)\, d\sigma_y \qquad (4.7) \\
& \qquad = a \int_{\epsilon\partial\Omega^i} S_{r,n}(x - y, k_i)\omega(y)\, d\sigma_y + b \qquad \forall x \in \epsilon\partial\Omega^i\,, \\
& -\frac{1}{m^i}\left\{ -\frac{1}{2}\omega(x) + \int_{\epsilon\partial\Omega^i} \frac{\partial}{\partial\nu_{\epsilon\Omega^i,x}} S_{r,n}(x - y, k_i)\omega(y)\, d\sigma_y \right\} \qquad (4.8)
\end{aligned}
$$

$$
\begin{aligned}
-\frac{1}{m^o}\Big\{-\frac{1}{2}\mu(x)+\int_{\partial\Omega(\epsilon)}\frac{\partial}{\partial\nu_{\Omega(\epsilon),x}}S_{r,n}(x-y,k_o)\,\mu(y)\,d\sigma_y\Big\}& \\
= g^i(x/\epsilon) \qquad \forall x\in\epsilon\partial\Omega^i\,,& \\
-\frac{1}{2}\mu(x)+\int_{\epsilon\partial\Omega^i}\frac{\partial}{\partial\nu_{\Omega^o,x}}S_{r,n}(x-y,k_o)\,\mu(y)\,d\sigma_y& \qquad (4.9)\\
+\int_{\partial\Omega^o}\frac{\partial}{\partial\nu_{\Omega^o,x}}S_{r,n}(x-y,k_o)\,\mu(y)\,d\sigma_y = g^o(x) \qquad \forall x\in\partial\Omega^o\,.&
\end{aligned}
$$

Next we define (ψ,θ^i,θ^o) by means of (4.6). By the definition of norm in Schauder spaces, (ψ,θ^i,θ^o) belongs to $C^{m-1,\alpha}(\partial\Omega^i)^2\times C^{m-1,\alpha}(\partial\Omega^o)$. Then we can rewrite system (4.7)–(4.9) as follows.

$$
\begin{aligned}
&\int_{\partial\Omega^i}S_{r,n}(\epsilon\xi-\epsilon\eta,k_o)\theta^i(\eta)\,d\sigma_y\epsilon^{n-1}+\int_{\partial\Omega^o}S_{r,n}(\epsilon\xi-y,k_o)\theta^o(y)\,d\sigma_y\\
&\qquad = a\int_{\partial\Omega^i}S_{r,n}(\epsilon\xi-\epsilon\eta,k_i)\psi(\eta)\,d\sigma_\eta\epsilon^{n-1}+b \qquad \forall\xi\in\partial\Omega^i\,,\\
&-\frac{1}{m^i}\Big\{-\frac{1}{2}\psi(\xi)+\int_{\partial\Omega^i}DS_{r,n}(\epsilon\xi-\epsilon\eta,k_i)\nu_{\Omega^i}(\xi)\psi(\eta)\,d\sigma_\eta\epsilon^{n-1}\Big\}\\
&\qquad -\frac{1}{m^o}\Big\{-\frac{1}{2}\theta^i(\xi)-\int_{\partial\Omega^i}DS_{r,n}(\epsilon\xi-\epsilon\eta,k_o)\nu_{\Omega^i}(\xi)\,\theta^i(\eta)\,d\sigma_\eta\epsilon^{n-1}\\
&\qquad -\int_{\partial\Omega^o}DS_{r,n}(\epsilon\xi-y,k_o)\nu_{\Omega^i}(\xi)\theta^o(y)\,d\sigma_y\Big\}=g^i(\xi) \qquad \forall\xi\in\partial\Omega^i\,,\\
&-\frac{1}{2}\theta^o(x)+\int_{\partial\Omega^i}DS_{r,n}(x-\epsilon\eta,k_o)\nu_{\Omega^o}(x)\theta^i(\eta)\,d\sigma_\eta\epsilon^{n-1}\\
&\qquad +\int_{\partial\Omega^o}DS_{r,n}(x-y,k_o)\nu_{\Omega^o}(x)\theta^o(y)\,d\sigma_y=g^o(x) \qquad \forall x\in\partial\Omega^o\,.
\end{aligned}
$$

By the homogeneity properties of the fundamental solution $S_{r,n}$ of Lemma 3.26, such a system is equal to the following.

$$
\begin{aligned}
&\int_{\partial\Omega^i}S_{r,n}(\xi-\eta,\epsilon k_o)\theta^i(\eta)\,d\sigma_y\epsilon^{n-1+2-n}+\int_{\partial\Omega^o}S_{r,n}(\epsilon\xi-y,k_o)\theta^o(y)\,d\sigma_y\\
&\qquad = a\int_{\partial\Omega^i}S_{r,n}(\xi-\ \eta,\epsilon k_i)\psi(\eta)\,d\sigma_\eta\epsilon^{n-1+2-n}+b \qquad \forall\xi\in\partial\Omega^i\,,\\
&-\frac{1}{m^i}\Big\{-\frac{1}{2}\psi(\xi)+\int_{\partial\Omega^i}DS_{r,n}(\xi-\eta,\epsilon k_i)\nu_{\Omega^i}(\xi)\psi(\eta)\,d\sigma_\eta\Big\}\\
&\qquad -\frac{1}{m^o}\Big\{-\frac{1}{2}\theta^i(\xi)-\int_{\partial\Omega^i}DS_{r,n}(\xi-\eta,\epsilon k_o)\nu_{\Omega^i}(\xi)\,\theta^i(\eta)\,d\sigma_\eta\\
&\qquad -\int_{\partial\Omega^o}DS_{r,n}(\epsilon\xi-y,k_o)\nu_{\Omega^i}(\xi)\theta^o(y)\,d\sigma_y\Big\}=g^i(\xi) \qquad \forall\xi\in\partial\Omega^i\,,
\end{aligned}
$$

$$-\frac{1}{2}\theta^o(x) + \int_{\partial\Omega^i} DS_{r,n}(x-\epsilon\eta, k_o)\nu_{\Omega^o}(x)\theta^i(\eta)\, d\sigma_\eta \epsilon^{n-1} + \int_{\partial\Omega^o} DS_{r,n}(x-y, k_o)\nu_{\Omega^o}(x)\theta^o(y)\, d\sigma_y = g^o(x) \qquad \forall x \in \partial\Omega^o\,.$$

Next we write the fundamental solution $S_{r,n}$ in terms of $S_{h,n}$ and we resort to Corollary 3.25 and formula (3.30) and obtain the system (4.2)–(4.4).

On the other hand if $(\psi, \theta^i, \theta^o) \in C^{m-1,\alpha}(\partial\Omega^i)^2 \times C^{m-1,\alpha}(\partial\Omega^o)$ solves system (4.2)–(4.4), then by reading backwards the above computations we realize that the pair $(v_{\epsilon\Omega^i}[\omega, k_i], v_{\Omega(\epsilon)}[\mu, k_o])$ solves problem (1.10). Moreover, Theorem A.1 ensures that the map of the statement is injective and the above argument shows that such a map is also surjective. □

By Theorem 4.1 we have transformed problem (1.10) into a problem for integral equations on the boundaries $\partial\Omega^i$ and $\partial\Omega^o$, which do not depend on ϵ. In order to analyze the solutions of system (4.2)–(4.4) as ϵ approaches zero with the methods of the Functional Analytic Approach, we wish to solve system (4.2)–(4.4) with respect to $(\psi, \theta^i, \theta^o)$ in terms of ϵ. To do so one could try by applying the Implicit Function Theorem around a specific point $(0, \psi, \theta^i, \theta^o)$ that solves system (4.2)–(4.4).

However, system (4.2)–(4.4) makes no sense if $\epsilon = 0$. Thus we now look for a suitable change of the functional variable $(\epsilon, \psi, \theta^i, \theta^o)$ so that the problem we obtain makes sense for $\epsilon = 0$ and is not singular in the involved functional variables. In particular, we should be able to take the limit as ϵ tends to 0 in the new problem and obtain a problem that we can solve.

Since we can collect a factor ϵ in all terms of equation (4.2), we may try to set

$$\psi = \frac{\zeta}{\epsilon}\,, \qquad \theta^i = \frac{\varsigma^i}{\epsilon} \tag{4.10}$$

with ζ, $\varsigma^i \in C^{m-1,\alpha}(\partial\Omega^i)$. If we do so, we realize that we obtain a problem that has a limit as ϵ tends to 0 and that the resulting problem does not look degenerate in case $n \geq 3$ (and as we shall see, it is actually well-posed), but not in case $n = 2$, because of the presence of the logarithmic term $\log\epsilon$ that diverges as ϵ tends to zero.

Thus at least in case $n = 2$, we must figure out something different. We first note that the term $\log\epsilon$ appears as a coefficient of $V_{\Omega^i,J}[\theta^i, \epsilon^2 k_o^2]$ and of $V_{\Omega^i,J}[\psi, \epsilon^2 k_i^2]$. Since we can write

$$V_{\Omega^i,J}[\theta^i, \epsilon^2 k_o^2]\log\epsilon = (V_{\Omega^i,J}[\theta^i, \epsilon^2 k_o^2] - V_{\Omega^i,J}[\theta^i, 0])\log\epsilon + V_{\Omega^i,J}[\theta^i, 0]\log\epsilon \qquad \forall\epsilon \in]0, \epsilon_0[\,,$$

and Theorem 3.22 (i) implies that $\epsilon^{-1}(V_{\Omega^i,J}[\theta^i,\epsilon^2 k_o^2] - V_{\Omega^i,J}[\theta^i,0])$ has an analytic extension around $\epsilon = 0$, the term $V_{\Omega^i,J}[\theta^i,\epsilon^2 k_o^2]\log\epsilon$ is no longer divergent as ϵ tends to 0 provided that $V_{\Omega^i,J}[\theta^i,0]$ equals zero.

Now we note that if we choose θ^i with integral equal to zero on $\partial\Omega^i$, then $V_{\Omega^i,J}[\theta^i,0] = 0$ and thus also $V_{\Omega^i,J}[\theta^i,\epsilon^2 k_o^2]\log\epsilon$ is no longer divergent as ϵ tends to 0.

Unfortunately however, not all elements of $C^{m-1,\alpha}(\partial\Omega^i)$ have integral equal to zero. Thus we now try to write the space $C^{m-1,\alpha}(\partial\Omega^i)$ as a direct sum of the subspace of functions that do have integral equal to zero and of a one dimensional subspace of functions that are multiples of a function with integral equal to 1. So we now choose $\theta^\sharp \in C^{m-1,\alpha}(\partial\Omega^i)$ such that

$$\int_{\partial\Omega^i} \theta^\sharp \, d\sigma = 1\,,$$

and we set

$$C^{m-1,\alpha}(\partial\Omega^i)_0 \equiv \left\{\theta \in C^{m-1,\alpha}(\partial\Omega^i) : \int_{\partial\Omega^i} \theta \, d\sigma = 0\right\}. \tag{4.11}$$

Then we have

$$C^{m-1,\alpha}(\partial\Omega^i) = C^{m-1,\alpha}(\partial\Omega^i)_0 \oplus < \theta^\sharp >\,,$$

where the sum is both algebraic and topological. As a consequence, if $\epsilon \in]0,\epsilon_0[$, then the map from

$$C^{m-1,\alpha}(\partial\Omega^i)_0 \times \mathbb{R}$$

to $C^{m-1,\alpha}(\partial\Omega^i)$ that takes (θ, c) to

$$\frac{\theta}{\epsilon} + \frac{c}{\epsilon}\theta^\sharp$$

is an isomorphism. Then we can introduce the new functional variables

$$\psi = \frac{\zeta}{\epsilon} + \frac{c^o}{\epsilon}\theta^\sharp\,, \qquad \theta^i = \frac{\varsigma^i}{\epsilon} + \frac{c^i}{\epsilon}\theta^\sharp\,, \tag{4.12}$$

for (ζ, c^o), $(\varsigma^i, c^i) \in C^{m-1,\alpha}(\partial\Omega^i)_0 \times \mathbb{R}$. Once we do so, we find convenient to write c^o in terms of c^i and of another constant c so that the diverging terms containing $\log\epsilon$ in system (4.2)–(4.4) cancel out and it turns out that a right choice is

$$c^o = a^{-1}(k_o^{n-2}/k_i^{n-2})c^i + \frac{c}{(\log\epsilon)^{\delta_{2,n}}}\,. \tag{4.13}$$

So we now turn to show that by exploiting the above new functional variables, we can desingularize system (4.2)–(4.4) both in case $n \geq 3$ and in case $n = 2$, while in case $n \geq 3$, the change of functional variables of (4.10) would also serve the purpose. In order to simplify our notation and computations, we find convenient to set

$$Y_{m-1,\alpha} \equiv C^{m-1,\alpha}(\partial\Omega^i)_0 \times \mathbb{R} \times C^{m-1,\alpha}(\partial\Omega^i)_0 \times \mathbb{R} \times C^{m-1,\alpha}(\partial\Omega^o) \quad (4.14)$$

and to mention that we can choose $\theta^\sharp \in C^{m-1,\alpha}(\partial\Omega^i)$ such that

$$\int_{\partial\Omega^i} \theta^\sharp \, d\sigma = 1 \,, \qquad -\frac{1}{2}\theta^\sharp + W^t_{\Omega^i,h}[\theta^\sharp, 0] = 0 \text{ on } \partial\Omega^i \,, \quad (4.15)$$

and that accordingly

$$v^\sharp \equiv V_{\Omega^i,h}[\theta^\sharp, 0] \text{ is constant on } \partial\Omega^i \quad (4.16)$$

(cf. *e.g.*, [10, Prop. 6.18, Thms. 6.24, 6.25], [20, Thm. 5.1]). Then we also have

$$V_{\Omega^i,J}[\theta^\sharp, 0] = J^\sharp_{\frac{n-2}{2}}(0) \qquad \text{on } \partial\Omega^i \quad (4.17)$$

and we are now ready to prove the following statement.

Theorem 4.18 *Let $m \in \mathbb{N} \setminus \{0\}$, $n \in \mathbb{N} \setminus \{0,1\}$, $\alpha \in]0,1[$. Let Ω^i, Ω^o be as in (1.1). Let m^i, m^o, $a \in]0,+\infty[$, $b \in \mathbb{R}$. Let g^i, g^o be as in (1.3), (1.9). Let k_i, k_o be as in (1.2), (1.8). Assume that k_o^2 is not a Neumann eigenvalue for $-\Delta$ in Ω^o. Let $\theta^\sharp \in C^{m-1,\alpha}(\partial\Omega^i)$ be as in (4.15). Let $\epsilon_* \in]0,\epsilon_0[$ be as in Theorem 4.1. Let*

$$(u^i[\epsilon,\zeta,c^i,\varsigma^i,c,\theta^o], u^o[\epsilon,\zeta,c^i,\varsigma^i,c,\theta^o]) \quad (4.19)$$

be the pair of functions of $C^{m,\alpha}(\epsilon\overline{\Omega^i}) \times C^{m,\alpha}(\overline{\Omega(\epsilon)})$ defined by

$$\begin{aligned} u^i[\epsilon,\zeta,c^i,\varsigma^i,c,\theta^o](x) &= \frac{1}{\epsilon} v^+_{\epsilon\Omega^i}[\zeta(\cdot/\epsilon), k_i](x) \quad (4.20)\\ &+ \frac{1}{\epsilon}\Big(a^{-1}\big(k_o^{n-2}/k_i^{n-2}\big)c^i + \frac{c}{(\log\epsilon)^{\delta_{2,n}}}\Big) v^+_{\epsilon\Omega^i}[\theta^\sharp(\cdot/\epsilon), k_i](x) \qquad \forall x \in \epsilon\overline{\Omega^i}\,,\\ u^o[\epsilon,\zeta,c^i,\varsigma^i,c,\theta^o](x) &= v^+_{\Omega^o}[\theta^o, k_o](x) + \frac{1}{\epsilon} v^-_{\epsilon\Omega^i}[\varsigma^i(\cdot/\epsilon), k_o](x)\\ &+ \frac{c^i}{\epsilon} v^-_{\epsilon\Omega^i}[\theta^\sharp(\cdot/\epsilon), k_o](x) \qquad \forall x \in \overline{\Omega(\epsilon)}\,, \end{aligned}$$

for all $(\epsilon,\zeta,c^i,\varsigma^i,c,\theta^o) \in]0,\epsilon_*[\times Y_{m-1,\alpha}$.

If $\epsilon \in]0, \epsilon_*[$, then the map $(u^i[\epsilon, \cdot, \cdot, \cdot, \cdot, \cdot], u^o[\epsilon, \cdot, \cdot, \cdot, \cdot, \cdot])$ is a bijection from the subset of $Y_{m-1,\alpha}$ consisting of the 5-tuples $(\zeta, c^i, \varsigma^i, c, \theta^o)$ that solve the following system of integral equations

$$
\begin{aligned}
&\int_{\partial\Omega^i} S_{h,n}(\xi-\eta, \epsilon k_o)\varsigma^i(\eta)\, d\sigma_\eta \qquad (4.21)\\
&\quad + \epsilon^n k_o^n \left[\frac{2b_n}{\pi}\kappa_n \log\epsilon + \gamma_n(\log k_o)\right] \int_0^1 \frac{\partial}{\partial\lambda} V_{\Omega^i,J}[\varsigma^i, t\epsilon^2 k_o^2](\xi)\, dt\\
&+ \int_{\partial\Omega^i} S_{h,n}(\xi-\eta, \epsilon k_o) c^i \theta^\sharp(\eta)\, d\sigma_\eta\\
&\quad + \epsilon^n k_o^n \left[\frac{2b_n}{\pi}\kappa_n \log\epsilon + \gamma_n(\log k_o)\right] c^i \int_0^1 \frac{\partial}{\partial\lambda} V_{\Omega^i,J}[\theta^\sharp, t\epsilon^2 k_o^2](\xi)\, dt\\
&\quad + \epsilon^{n-2} k_o^{n-2} \gamma_n(\log k_o) c^i V_{\Omega^i,J}[\theta^\sharp, 0](\xi) + \int_{\partial\Omega^o} S_{r,n}(\epsilon\xi - y, k_o)\theta^o(y)\, d\sigma_y\\
&= a\int_{\partial\Omega^i} S_{h,n}(\xi-\eta, \epsilon k_i)\zeta(\eta)\, d\sigma_\eta\\
&\quad + a\epsilon^n k_i^n \left[\frac{2b_n}{\pi}\kappa_n \log\epsilon + \gamma_n(\log k_i)\right] \int_0^1 \frac{\partial}{\partial\lambda} V_{\Omega^i,J}[\zeta, t\epsilon^2 k_i^2](\xi)\, dt\\
&+ (k_o^{n-2}/k_i^{n-2}) c^i \int_{\partial\Omega^i} S_{h,n}(\xi-\eta, \epsilon k_i)\theta^\sharp(\eta)\, d\sigma_\eta\\
&\quad + a\int_{\partial\Omega^i} S_{h,n}(\xi-\eta, \epsilon k_i) \frac{c}{(\log\epsilon)^{\delta_{2,n}}}\theta^\sharp(\eta)\, d\sigma_\eta\\
&\quad + \epsilon^n k_o^{n-2} c^i k_i^2 \left[\frac{2b_n}{\pi}\kappa_n \log\epsilon + \gamma_n(\log k_i)\right] \int_0^1 \frac{\partial}{\partial\lambda} V_{\Omega^i,J}[\theta^\sharp, t\epsilon^2 k_i^2](\xi)\, dt\\
&\quad + a\epsilon^n k_i^n \left[\frac{2b_n}{\pi}\kappa_n \log\epsilon + \gamma_n(\log k_i)\right] \frac{c}{(\log\epsilon)^{\delta_{2,n}}} \int_0^1 \frac{\partial}{\partial\lambda} V_{\Omega^i,J}[\theta^\sharp, t\epsilon^2 k_i^2](\xi)\, dt\\
&\quad + \epsilon^{n-2} k_o^{n-2} c^i \gamma_n(\log k_i) V_{\Omega^i,J}[\theta^\sharp, 0](\xi)\\
&\quad + a\epsilon^{n-2} k_i^{n-2} \left[\frac{2b_n}{\pi}\kappa_n \frac{\log\epsilon}{(\log\epsilon)^{\delta_{2,n}}} + \frac{1}{(\log\epsilon)^{\delta_{2,n}}}\gamma_n(\log k_i)\right] c V_{\Omega^i,J}[\theta^\sharp, 0](\xi) + b\\
&\qquad\qquad \forall \xi \in \partial\Omega^i,
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{m^i}\Bigg\{ -\frac{1}{2}\left(\zeta(\xi) + a^{-1}(k_o^{n-2}/k_i^{n-2}) c^i \theta^\sharp(\xi) + \frac{c}{(\log\epsilon)^{\delta_{2,n}}}\theta^\sharp(\xi)\right) \qquad (4.22)\\
&\quad + \int_{\partial\Omega^i} DS_{h,n}(\xi-\eta, \epsilon k_i)\nu_{\Omega^i}(\xi)\\
&\quad \times \left(\zeta(\eta) + a^{-1}(k_o^{n-2}/k_i^{n-2}) c^i \theta^\sharp(\eta) + \frac{c}{(\log\epsilon)^{\delta_{2,n}}}\theta^\sharp(\eta)\right) d\sigma_\eta
\end{aligned}
$$

$$
\begin{aligned}
&+\epsilon^n k_i^n \left[\frac{2b_n}{\pi}\kappa_n \log\epsilon + \gamma_n(\log k_i)\right] \\
&\times \tilde{W}^t_{\Omega^i,J}[\zeta + a^{-1}(k_o^{n-2}/k_i^{n-2})c^i\theta^\sharp + \frac{c}{(\log\epsilon)^{\delta_{2,n}}}\theta^\sharp, \epsilon^2 k_i^2](\xi)\Big\} \\
&-\frac{1}{m^o}\Big\{ -\frac{1}{2}\Big(\varsigma^i(\xi) + c^i\theta^\sharp(\xi)\Big) \\
&\quad -\int_{\partial\Omega^i} DS_{h,n}(\xi-\eta,\epsilon k_o)\nu_{\Omega^i}(\xi)\Big(\varsigma^i(\eta) + c^i\theta^\sharp(\eta)\Big)\, d\sigma_\eta \\
&\quad -\epsilon^n k_o^n \left[\frac{2b_n}{\pi}\kappa_n \log\epsilon + \gamma_n(\log k_o)\right] \tilde{W}^t_{\Omega^i,J}[\varsigma^i + c^i\theta^\sharp, \epsilon^2 k_o^2](\xi) \\
&\quad -\epsilon\int_{\partial\Omega^o} DS_{r,n}(\epsilon\xi - y, k_o)\nu_{\Omega^i}(\xi)\theta^o(y)\, d\sigma_y\Big\} = \epsilon g^i(\xi) \qquad \forall \xi \in \partial\Omega^i\,, \\
&-\frac{1}{2}\theta^o(x) + \int_{\partial\Omega^i} DS_{r,n}(x-\epsilon\eta, k_o)\nu_{\Omega^o}(x)\Big(\varsigma^i(\eta) + c^i\theta^\sharp(\eta)\Big)\, d\sigma_\eta \epsilon^{n-2} \qquad (4.23) \\
&\quad + \int_{\partial\Omega^o} DS_{r,n}(x-y,k_o)\nu_{\Omega^o}(x)\theta^o(y)\, d\sigma_y = g^o(x) \qquad \forall x \in \partial\Omega^o\,.
\end{aligned}
$$

onto the set of solutions (u^i, u^o) *in* $C^{m,\alpha}(\epsilon\overline{\Omega^i}) \times C^{m,\alpha}(\overline{\Omega(\epsilon)})$*, which satisfy problem (1.10). Here* $\frac{\partial}{\partial\lambda}V_{\Omega^i,J}$ *denotes the partial differential of* $V_{\Omega^i,J}$ *with respect to its second argument.*

Proof. The maps from $C^{m-1,\alpha}(\partial\Omega^i)_0 \times \mathbb{R}$ to $C^{m-1,\alpha}(\partial\Omega^i)$ defined by the equalities in (4.12) are isomorphisms. Thus equation (4.2) in the unknown $(\psi, \theta^i, \theta^o)$ is equivalent to the following equation in the unknown $(\zeta, c^i, \varsigma^i, c^o, \theta^o)$

$$
\begin{aligned}
&\int_{\partial\Omega^i} S_{h,n}(\xi-\eta,\epsilon k_o)\varsigma^i(\eta)\, d\sigma_\eta + \epsilon^{n-2}k_o^{n-2}\left[\frac{2b_n}{\pi}\kappa_n\log\epsilon + \gamma_n(\log k_o)\right] \qquad (4.24) \\
&\quad \times \Big(V_{\Omega^i,J}[\varsigma^i, \epsilon^2 k_o^2](\xi) - V_{\Omega^i,J}[\varsigma^i, 0](\xi)\Big) \\
&\quad +\epsilon^{n-2}k_o^{n-2}\left[\frac{2b_n}{\pi}\kappa_n\log\epsilon + \gamma_n(\log k_o)\right] V_{\Omega^i,J}[\varsigma^i, 0](\xi) \\
&+\int_{\partial\Omega^i} S_{h,n}(\xi-\eta,\epsilon k_o)c^i\theta^\sharp(\eta)\, d\sigma_\eta + \epsilon^{n-2}k_o^{n-2}\left[\frac{2b_n}{\pi}\kappa_n\log\epsilon + \gamma_n(\log k_o)\right] \\
&\quad \times \Big(V_{\Omega^i,J}[c^i\theta^\sharp, \epsilon^2 k_o^2](\xi) - V_{\Omega^i,J}[c^i\theta^\sharp, 0](\xi)\Big) \\
&\quad +\epsilon^{n-2}k_o^{n-2}\left[\frac{2b_n}{\pi}\kappa_n\log\epsilon + \gamma_n(\log k_o)\right] V_{\Omega^i,J}[c^i\theta^\sharp, 0](\xi)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \int_{\partial\Omega^o} S_{r,n}(\epsilon\xi - y, k_o)\theta^o(y)\, d\sigma_y \\
&= a \int_{\partial\Omega^i} S_{h,n}(\xi - \eta, \epsilon k_i)\zeta(\eta)\, d\sigma_\eta \\
&\quad + a\epsilon^{n-2} k_i^{n-2} \left[\frac{2b_n}{\pi} \kappa_n \log\epsilon + \gamma_n(\log k_i) \right] \\
&\qquad \times \left(V_{\Omega^i,J}[\zeta, \epsilon^2 k_i^2](\xi) - V_{\Omega^i,J}[\zeta, 0](\xi) \right) \\
&\quad + a\epsilon^{n-2} k_i^{n-2} \left[\frac{2b_n}{\pi} \kappa_n \log\epsilon + \gamma_n(\log k_i) \right] V_{\Omega^i,J}[\zeta, 0](\xi) \\
&+ a \int_{\partial\Omega^i} S_{h,n}(\xi - \eta, \epsilon k_i) c^o \theta^\sharp(\eta)\, d\sigma_\eta \\
&\quad + a\epsilon^{n-2} k_i^{n-2} \left[\frac{2b_n}{\pi} \kappa_n \log\epsilon + \gamma_n(\log k_i) \right] \\
&\qquad\qquad \times \left(V_{\Omega^i,J}[c^o\theta^\sharp, \epsilon^2 k_i^2](\xi) - V_{\Omega^i,J}[c^o\theta^\sharp, 0](\xi) \right) \\
&\quad + a\epsilon^{n-2} k_i^{n-2} \left[\frac{2b_n}{\pi} \kappa_n \log\epsilon + \gamma_n(\log k_i) \right] V_{\Omega^i,J}[c^o\theta^\sharp, 0](\xi) + b \quad \forall \xi \in \partial\Omega^i \,.
\end{aligned}
$$

Equation (4.24) can actually be simplified. Indeed

$$
V_{\Omega^i,J}[\theta, 0] = \int_{\partial\Omega^i} J^\sharp_{\frac{n-2}{2}}(0)\theta\, d\sigma = 0 \qquad \forall \theta \in C^{m-1,\alpha}(\partial\Omega^i)_0 \tag{4.25}
$$

and

$$
\begin{aligned}
&V_{\Omega^i,J}[\theta, k^2](\xi) - V_{\Omega^i,J}[\theta, 0](\xi) \\
&\qquad\qquad = k^2 \int_0^1 \frac{\partial}{\partial\lambda} V_{\Omega^i,J}[\theta, tk^2](\xi)\, dt \qquad \forall \xi \in \partial\Omega^i \,,
\end{aligned} \tag{4.26}
$$

for all $(\theta, k) \in C^{m-1,\alpha}(\partial\Omega^i) \times \mathbb{C}$. Next we want to introduce a further unknown by replacing c^o with the sum

$$
c^o = c_1 + \frac{c}{(\log\epsilon)^{\delta_{2,n}}} \,, \tag{4.27}
$$

where c_1 is a real number such that the following equality is satisfied

$$
\epsilon^{n-2} k_o^{n-2} \frac{2b_n}{\pi} \kappa_n \log\epsilon V_{\Omega^i,J}[c^i\theta^\sharp, 0] = a\epsilon^{n-2} k_i^{n-2} \frac{2b_n}{\pi} \kappa_n \log\epsilon V_{\Omega^i,J}[c_1\theta^\sharp, 0] \tag{4.28}
$$

and where c is a new unknown that replaces c^o by means of the change of variables (4.27). Such a choice is in order to replace the divergent logarithmic terms in the left and right hand sides of (4.24) by a single term in the right hand side containing the factor

$$\epsilon^{n-2}\frac{c}{(\log\epsilon)^{\delta_{2,n}}}\kappa_n \log\epsilon$$

that still contains $\log\epsilon$ precisely in case $n>2$ is even, but with the advantage that $\epsilon^{n-2}\frac{c}{(\log\epsilon)^{\delta_{2,n}}}\kappa_n\log\epsilon$ does not diverge as ϵ tends to zero. Since

$$V_{\Omega^i,J}[\theta^\sharp,0] = J^\sharp_{\frac{n-2}{2}}(0)\int_{\partial\Omega^i}\theta^\sharp\,d\sigma = J^\sharp_{\frac{n-2}{2}}(0)\neq 0\,,$$

equality (4.28) is satisfied provided that we choose

$$c_1 = a^{-1}(k_o^{n-2}/k_i^{n-2})c^i. \tag{4.29}$$

Since the map from $Y_{m-1,\alpha}$ to itself that takes $(\zeta, c^i, \varsigma^i, c, \theta^o)$ to

$$(\zeta, c^i, \varsigma^i, a^{-1}(k_o^{n-2}/k_i^{n-2})c^i + \frac{c}{(\log\epsilon)^{\delta_{2,n}}}, \theta^o) \tag{4.30}$$

is an isomorphism, the equation (4.24) in the unknown $(\zeta, c^i, \varsigma^i, c^o, \theta^o)$ is equivalent to the equation in the unknown $(\zeta, c^i, \varsigma^i, c, \theta^o)$ that we obtain by replacing $(\zeta, c^i, \varsigma^i, c^o, \theta^o)$ by $(\zeta, c^i, \varsigma^i, a^{-1}(k_o^{n-2}/k_i^{n-2})c^i + \frac{c}{(\log\epsilon)^{\delta_{2,n}}}, \theta^o)$, *i.e.*, the following equation (see also (4.25), (4.26))

$$\begin{aligned}
&\int_{\partial\Omega^i} S_{h,n}(\xi-\eta,\epsilon k_o)\varsigma^i(\eta)\,d\sigma_\eta \\
&\qquad +\epsilon^{n-2}k_o^{n-2}\left[\frac{2b_n}{\pi}\kappa_n\log\epsilon+\gamma_n(\log k_o)\right]\epsilon^2k_o^2\int_0^1\frac{\partial}{\partial\lambda}V_{\Omega^i,J}[\varsigma^i,t\epsilon^2k_o^2](\xi)\,dt \\
&+\int_{\partial\Omega^i} S_{h,n}(\xi-\eta,\epsilon k_o)c^i\theta^\sharp(\eta)\,d\sigma_\eta \\
&\qquad +\epsilon^{n-2}k_o^{n-2}\left[\frac{2b_n}{\pi}\kappa_n\log\epsilon+\gamma_n(\log k_o)\right]\epsilon^2k_o^2c^i\int_0^1\frac{\partial}{\partial\lambda}V_{\Omega^i,J}[\theta^\sharp,t\epsilon^2k_o^2](\xi)\,dt \\
&\qquad +\epsilon^{n-2}k_o^{n-2}\left[\frac{2b_n}{\pi}\kappa_n\log\epsilon+\gamma_n(\log k_o)\right]V_{\Omega^i,J}[c^i\theta^\sharp,0](\xi) \\
&\qquad +\int_{\partial\Omega^o} S_{r,n}(\epsilon\xi-y,k_o)\theta^o(y)\,d\sigma_y \\
&= a\int_{\partial\Omega^i} S_{h,n}(\xi-\eta,\epsilon k_i)\zeta(\eta)\,d\sigma_\eta \\
&\qquad +a\epsilon^{n-2}k_i^{n-2}\left[\frac{2b_n}{\pi}\kappa_n\log\epsilon+\gamma_n(\log k_i)\right]\epsilon^2k_i^2\int_0^1\frac{\partial}{\partial\lambda}V_{\Omega^i,J}[\zeta,t\epsilon^2k_i^2](\xi)\,dt
\end{aligned} \tag{4.31}$$

$$
\begin{aligned}
&+a\int_{\partial\Omega^i} S_{h,n}(\xi-\eta,\epsilon k_i)a^{-1}(k_o^{n-2}/k_i^{n-2})c^i\theta^\sharp(\eta)\,d\sigma_\eta\\
&\quad +a\int_{\partial\Omega^i} S_{h,n}(\xi-\eta,\epsilon k_i)\frac{c}{(\log\epsilon)^{\delta_{2,n}}}\theta^\sharp(\eta)\,d\sigma_\eta\\
&\quad +a\epsilon^{n-2}k_i^{n-2}\left[\frac{2b_n}{\pi}\kappa_n\log\epsilon+\gamma_n(\log k_i)\right]\\
&\qquad \times a^{-1}(k_o^{n-2}/k_i^{n-2})c^i\epsilon^2k_i^2\int_0^1\frac{\partial}{\partial\lambda}V_{\Omega^i,J}[\theta^\sharp,t\epsilon^2k_i^2](\xi)\,dt\\
&\quad +a\epsilon^{n-2}k_i^{n-2}\left[\frac{2b_n}{\pi}\kappa_n\log\epsilon+\gamma_n(\log k_i)\right]\\
&\qquad \times\frac{c\epsilon^2k_i^2}{(\log\epsilon)^{\delta_{2,n}}}\int_0^1\frac{\partial}{\partial\lambda}V_{\Omega^i,J}[\theta^\sharp,t\epsilon^2k_i^2](\xi)\,dt\\
&\quad +a\epsilon^{n-2}k_i^{n-2}\left[\frac{2b_n}{\pi}\kappa_n\log\epsilon+\gamma_n(\log k_i)\right]a^{-1}(k_o^{n-2}/k_i^{n-2})c^iV_{\Omega^i,J}[\theta^\sharp,0](\xi)\\
&\quad +a\epsilon^{n-2}k_i^{n-2}\left[\frac{2b_n}{\pi}\kappa_n\log\epsilon+\gamma_n(\log k_i)\right]\frac{c}{(\log\epsilon)^{\delta_{2,n}}}V_{\Omega^i,J}[\theta^\sharp,0](\xi)+b
\end{aligned}
$$

for all $\xi\in\partial\Omega^i$, which is equivalent to equation (4.21) of the statement.

Since the maps from $C^{m-1,\alpha}(\partial\Omega^i)_0\times\mathbb{R}$ to $C^{m-1,\alpha}(\partial\Omega^i)$ defined by the equalities in (4.12) are isomorphisms and the map of (4.30) is an isomorphism in $Y_{m-1,\alpha}$, equations (4.3), (4.4) in the unknown (ψ,θ^i,θ^o) are equivalent to the equations (4.22), (4.23) in the unknown $(\zeta,c^i,\varsigma^i,c,\theta^o)$ of the statement.

Thus we have proved that system (4.21), (4.22), (4.23) in the unknown $(\zeta,c^i,\varsigma^i,c,\theta^o)$ is equivalent to system (4.2), (4.3), (4.4) in the unknown (ψ,θ^i,θ^o). Then Theorem 4.1 implies that the map of the statement that takes $(\zeta,c^i,\varsigma^i,c,\theta^o)$ to the pair of functions in (4.5) with ω, μ as in (4.6), with ψ, θ^i as in (4.12) and c^o as in (4.13), *i.e.* to the pair of functions in (4.20) is a bijection.

On the other hand if $(\zeta,c^i,\varsigma^i,c,\theta^o)\in Y_{m-1,\alpha}$ solves system (4.21)-(4.23), then by reading backwards the above computations we realize that the pair of functions in (4.20) solves problem (1.10). Moreover, Theorem A.1 ensures that the map of the statement is injective and the above argument shows that such a map is also surjective. □

Hence, we are now reduced to analyze system (4.21)–(4.23). Our first step is to note that by letting ϵ tend to zero, we obtain system (4.33) below, which we address to as the 'limiting system', and we now prove the following theorem, which shows the unique solvability of the limiting system and its link with a boundary value problem which we address to as the 'limiting boundary value problem'.

Theorem 4.32 *Let $m \in \mathbb{N} \setminus \{0\}$, $n \in \mathbb{N} \setminus \{0,1\}$, $\alpha \in]0,1[$. Let Ω^i, Ω^o be as in (1.1). Let m^i, m^o, $a \in]0,+\infty[$, $b \in \mathbb{R}$. Let g^i, g^o be as in (1.3), (1.9). Let k_i, k_o be as in (1.2), (1.8). Assume that k_o^2 is not a Neumann eigenvalue for $-\Delta$ in Ω^o. Let $\theta^\sharp \in C^{m-1,\alpha}(\partial\Omega^i)$ be as in (4.15). Then the following statements hold*

(i) The limiting system

$$
\begin{aligned}
& V_{\Omega^i,h}[\varsigma^i,0](\xi) + c^i V_{\Omega^i,h}[\theta^\sharp,0](\xi) \\
& \quad +\delta_{2,n} k_o^{n-2}\gamma_n(\log k_o) c^i V_{\Omega^i,J}[\theta^\sharp,0](\xi) + v^+_{\Omega^o}[\theta^o,k_o](0) = a V_{\Omega^i,h}[\zeta,0](\xi) \\
& \quad + \left(k_o^{n-2}/k_i^{n-2}\right) c^i V_{\Omega^i,h}[\theta^\sharp,0](\xi) + a(1-\delta_{2,n}) c V_{\Omega^i,h}[\theta^\sharp,0](\xi) \\
& \quad +\delta_{2,n} k_o^{n-2}\gamma_n(\log k_i) c^i V_{\Omega^i,J}[\theta^\sharp,0](\xi) \\
& \quad +a\delta_{2,n} k_i^{n-2}\frac{2b_n}{\pi} c V_{\Omega^i,J}[\theta^\sharp,0](\xi) + b \qquad \forall \xi \in \partial\Omega^i\,, \\
& -\frac{1}{m^i}\Big\{ -\frac{1}{2}\Big(\zeta(\xi) + \big(a^{-1}\left(k_o^{n-2}/k_i^{n-2}\right) c^i + c(1-\delta_{2,n})\big)\theta^\sharp(\xi)\Big) \\
& \qquad +W^t_{\Omega^i,h}\Big[\zeta + \big(a^{-1}\left(k_o^{n-2}/k_i^{n-2}\right) c^i + c(1-\delta_{2,n})\big)\theta^\sharp, 0\Big](\xi)\Big\} \\
& \qquad -\frac{1}{m^o}\Big\{ -\frac{1}{2}\Big(\varsigma^i(\xi) + c^i\theta^\sharp(\xi)\Big) \\
& \qquad -W^t_{\Omega^i,h}\Big[\varsigma^i + c^i\theta^\sharp, 0\Big](\xi)\Big\} = 0 \qquad \forall \xi \in \partial\Omega^i\,, \\
& -\frac{1}{2}\theta^o(x) + \int_{\partial\Omega^o} DS_{r,n}(x-y,k_o)\nu_{\Omega^o}(x)\theta^o(y)\,d\sigma_y \\
& \qquad +\delta_{2,n} DS_{r,n}(x,k_o)\nu_{\Omega^o}(x) c^i = g^o(x) \qquad \forall x \in \partial\Omega^o
\end{aligned}
\tag{4.33}
$$

has one and only one solution $(\tilde{\zeta}, \tilde{c}^i, \tilde{\varsigma}^i, \tilde{c}, \tilde{\theta}^o)$ in $Y_{m-1,\alpha}$. Moreover, $\tilde{c}^i = 0$.

(ii) The limiting boundary value problem

$$
\begin{cases}
\Delta u_1^{i,r} = 0 & in\ \Omega^i\,,\\
\Delta u_1^{o,r} = 0 & in\ \Omega^{i-}\,,\\
\Delta u^o + k_o^2 u^o = 0 & in\ \Omega^o\,,\\
u_1^{o,r}(x) + u^o(0) - a u_1^{i,r}(x) = b & \forall x \in \partial\Omega^i\,,\\
-\frac{1}{m^i}\frac{\partial}{\partial\nu_{\Omega^i}} u_1^{i,r}(x) + \frac{1}{m^o}\frac{\partial}{\partial\nu_{\Omega^i}} u_1^{o,r}(x) = 0 & \forall x \in \partial\Omega^i\,,\\
\frac{\partial}{\partial\nu_{\Omega^o}} u^o(x) = g^o & on\ \partial\Omega^o\,,\\
\lim_{\xi\to\infty} u_1^{o,r}(\xi) = 0\,,
\end{cases}
\tag{4.34}
$$

has one and only one solution $(\tilde{u}_1^{i,r}, \tilde{u}_1^{o,r}, \tilde{u}^o)$ *in*

$$C^{m,\alpha}(\overline{\Omega^i}) \times C^{m,\alpha}_{\mathrm{loc}}(\overline{\Omega^{i-}}) \times C^{m,\alpha}(\overline{\Omega^o})\,,$$

which is delivered by the following formulas

$$
\begin{aligned}
&\tilde{u}_1^{i,r} = v^+_{\Omega^i,h}[\tilde{\zeta},0] + \tilde{C} \quad in\ \overline{\Omega^i}\,, \qquad \tilde{u}_1^{o,r} = v^-_{\Omega^i,h}[\tilde{\varsigma}^i,0] \quad in\ \overline{\Omega^{i-}}\,,\\
&\tilde{u}^o = v^+_{\Omega^o}[\tilde{\theta}^o,k_o] \qquad in\ \overline{\Omega^o}
\end{aligned}
\tag{4.35}
$$

where $(\tilde{\zeta},\tilde{c}^i,\tilde{\varsigma}^i,\tilde{c},\tilde{\theta}^o)$ *is the only solution in* $Y_{m-1,\alpha}$ *of the limiting system (4.33) and* $\tilde{C} = \left(\frac{\delta_{2,n}}{2\pi} + (1-\delta_{2,n})\,\upsilon^\sharp\right)\tilde{c}$ *(see (4.16) for the constant* $\upsilon^\sharp \equiv V_{\Omega^i,h}[\theta^\sharp,0]$*).*

Proof. (i) We first assume that the limiting system (4.33) has a solution

$$(\zeta, c^i, \varsigma^i, c, \theta^o)$$

and we show that $c^i = 0$. By integrating the second equality in (4.33), we obtain

$$
\begin{aligned}
-\frac{1}{m_i}\Big\{ \int_{\partial\Omega^i} \left(-\frac{1}{2}I_{\Omega^i} + W^t_{\Omega^i,h}[\cdot,0]\right)[\zeta]d\sigma + \left(a^{-1}\left(k_o^{n-2}/k_i^{n-2}\right)c^i\right.\\
\left.+c\,(1-\delta_{2,n})\right)\int_{\partial\Omega^i}\left(-\frac{1}{2}I_{\Omega^i} + W^t_{\Omega^i,h}[\cdot,0]\right)[\theta^\sharp]d\sigma\Big\}\\
+\frac{1}{m^o}\Big\{\int_{\partial\Omega^i}\left(\frac{1}{2}I_{\Omega^i} + W^t_{\Omega^i,h}[\cdot,0]\right)\left[\varsigma^i + c^i\theta^\sharp\right]d\sigma\Big\} = 0\,.
\end{aligned}
\tag{4.36}
$$

By equality (B.4) of the Appendix, we have

$$\int_{\partial\Omega^i} W^t_{\Omega^i,h}[\zeta,0]d\sigma = \frac{1}{2}\int_{\partial\Omega^i}\zeta d\sigma\,, \qquad \int_{\partial\Omega^i} W^t_{\Omega^i,h}[\theta^\sharp,0]d\sigma = \frac{1}{2}\int_{\partial\Omega^i}\theta^\sharp\, d\sigma \tag{4.37}$$

and

$$\int_{\partial\Omega^i}\left(\frac{1}{2}I_{\Omega^i}+W^t_{\Omega^i,h}[\cdot,0]\right)\left[\varsigma^i+c^i\theta^\sharp\right]d\sigma=\int_{\partial\Omega^i}\varsigma^i+c^i\theta^\sharp d\sigma\,. \tag{4.38}$$

By (4.36), (4.37) and (4.38), we obtain

$$\int_{\partial\Omega^i}\left(\varsigma^i+c^i\theta^\sharp\right)d\sigma=\int_{\partial\Omega^i}\varsigma^i d\sigma+c^i\int_{\partial\Omega^i}\theta^\sharp d\sigma=0\,.$$

Then (4.15) implies that

$$c^i=0\,. \tag{4.39}$$

Since k_o^2 is not a Neumann eigenvalue for $-\Delta$ in Ω^o, the Fredholm Alternative and classical Schauder regularity theory imply that the last equality in (4.33) has a unique solution (cf. Colton and Kress [8, Thm. 3.17], [22, Thm. B.1]). By a simple computation, the coefficient

$$A^\sharp\equiv\delta_{2,n}k_i^{n-2}\frac{2b_n}{\pi}V_{\Omega^i,J}[\theta^\sharp,0]+(1-\delta_{2,n})V_{\Omega^i,h}[\theta^\sharp,0]=\delta_{2,n}\frac{1}{2\pi}+(1-\delta_{2,n})\,\upsilon^\sharp \tag{4.40}$$

of ac in the right hand side of the first equation of (4.33) is not equal to 0. Indeed, $\upsilon^\sharp\neq 0$ if $n\geq 3$ (see [10, Prop. 6.19]). Then by (4.15) and Theorem B.1, the first two equations of (4.33) with c^i as in (4.39) have a unique solution (ζ,ς^i,c) in $C^{m-1,\alpha}(\partial\Omega^i)_0\times C^{m-1,\alpha}(\partial\Omega^i)_0\times\mathbb{R}$.

Hence, we have proved that the limiting system (4.33) has at most one solution. On the other hand, if we set (ζ,ς^i,c) equal to the only solution of the first two equations of (4.33) with c^i is as in (4.39) and if we set θ^o equal to the only solution of the last equation of (4.33) with c^i is as in (4.39), we can verify that $(\zeta,c^i,\varsigma^i,c,\theta^o)$ solves the limiting system (4.33) by reading backward the above argument.

We now consider statement (ii) and we show that the limiting boundary value problem (4.34) has a unique solution. From the third and sixth equation of (4.34), we obtain that u^o satisfies problem (1.4). Since k_o^2 is not a Neumann eigenvalue for $-\Delta$ in Ω^o, we have already pointed out that the Neumann problem (1.4) has the unique solution $\tilde{u}^o\in C^{m,\alpha}(\overline{\Omega^o})$. By system (4.34) with $u^o=\tilde{u}^o$, we have

$$\begin{cases}\Delta u_1^{i,r}=0 & \text{in }\Omega^i\,,\\ \Delta u_1^{o,r}=0 & \text{in }\Omega^{i-}\,,\\ u_1^{o,r}(x)+\tilde{u}^o(0)-au_1^{i,r}(x)=b & \forall x\in\partial\Omega^i\,,\\ -\frac{1}{m^i}\frac{\partial}{\partial\nu_{\Omega^i}}u_1^{i,r}(x)+\frac{1}{m^o}\frac{\partial}{\partial\nu_{\Omega^i}}u_1^{o,r}(x)=0 & \forall x\in\partial\Omega^i\,,\\ \lim_{\xi\to\infty}u_1^{o,r}(\xi)=0\,, & \end{cases}$$

that is a linear transmission problem in unknown $\left(u_1^{i,r}, u_1^{o,r}\right)$, which is known to have at most one solution in $C^{m,\alpha}(\overline{\Omega^i}) \times C^{m,\alpha}_{\mathrm{loc}}(\overline{\Omega^{i-}})$ (cf. *e.g.*, [10, Prop. 6.54, Thm. 6.55 and following comment]). Finally, if $(\tilde{\zeta}, \tilde{c}^i, \tilde{\varsigma}^i, \tilde{c}, \tilde{\theta}^o)$ is the only solution of the limiting system (4.33), then the standard jump properties of the single layer potential imply that $(\tilde{u}_1^{i,r}, \tilde{u}_1^{o,r}, \tilde{u}^o)$ as in (4.35) with $\tilde{C} = A^\sharp \tilde{c}$ solves the first six equations of the limiting boundary value problem (4.34). Also, condition $\int_{\partial\Omega^i} \varsigma^i \, d\sigma = 0$ implies that

$$\lim_{\xi \to \infty} v^-_{\Omega^i,h}[\varsigma^i, 0](\xi) = 0$$

both in dimension $n = 2$ and in dimension $n \geq 3$ (cf. *e.g.*, [10, Thm. 4.23]). □

We now introduce an abstract formulation of system (4.21), (4.22), (4.23). We note that in case n even system (4.21), (4.22) contains logarithmic terms, which are not analytic around $\epsilon = 0$. Namely the terms

$$\begin{array}{lll} \epsilon^n \kappa_n \log \epsilon \,, & \log^{-\delta_{2,n}} \epsilon \,, & \epsilon^{n-2} \log^{-\delta_{2,n}} \epsilon \,, \\ \epsilon^n \log^{-\delta_{2,n}} \epsilon \,, & \epsilon^n \kappa_n \dfrac{\log \epsilon}{\log^{\delta_{2,n}} \epsilon} \,, & \epsilon^{n-2} \kappa_n \dfrac{\log \epsilon}{\log^{\delta_{2,n}} \epsilon} \,. \end{array} \tag{4.41}$$

Next we note that

$$\begin{aligned} \epsilon^n \kappa_n \log \epsilon &= \epsilon^{n-1}(\kappa_n \epsilon \log \epsilon) \,, \qquad \log^{-\delta_{2,n}} \epsilon = (1 - \delta_{2,n}) + \frac{\delta_{2,n}}{\log \epsilon} \,, \\ \epsilon^n \kappa_n \frac{\log \epsilon}{\log^{\delta_{2,n}} \epsilon} &= \epsilon^{n-1}(\kappa_n \epsilon \log \epsilon) \left[(1 - \delta_{2,n}) + \frac{\delta_{2,n}}{\log \epsilon} \right] \,, \\ \epsilon^{n-2} \kappa_n \frac{\log \epsilon}{\log^{\delta_{2,n}} \epsilon} &= [(1 - \delta_{2,n})\epsilon^{n-3} + \delta_{2,n}][(1 - \delta_{2,n})(\kappa_n \epsilon \log \epsilon) + \kappa_n \delta_{2,n}] \end{aligned} \tag{4.42}$$

and that $(1 - \delta_{2,n})\epsilon^{n-3} = 0$ if $n = 2$. Then all terms in (4.41) can be written as sums and products of the following terms

$$1 \,, \qquad \epsilon \,, \qquad \kappa_n \epsilon \log \epsilon \,, \qquad \frac{\delta_{2,n}}{\log \epsilon} \,. \tag{4.43}$$

Here the idea is to write system (4.21), (4.22), (4.23) by replacing the terms that contain $\log \epsilon$, *i.e.*, the terms in (4.41) by the corresponding expressions as sums of products of the terms of (4.43). Then we consider the resulting system and introduce the new independent variables

$$\epsilon_1 = \kappa_n \epsilon \log \epsilon \,, \qquad \epsilon_2 = \frac{\delta_{2,n}}{\log \epsilon}$$

and we obtain a new system that depends on ϵ, ϵ_1, ϵ_2 and that contains no logarithmic terms and that displays no singularities in the variables ϵ, ϵ_1, ϵ_2 around the degenerate points where $\epsilon = \epsilon_1 = \epsilon_2 = 0$ and that is accordingly easier to analyze. We do so by means of the following theorem. In order to shorten our notation, we find convenient to introduce the polynomial function ϱ_n from $\mathbb{R}^2$ to $\mathbb{R}$ defined by

$$\varrho_n(\epsilon, \epsilon_1) \equiv [(1-\delta_{2,n})\epsilon^{n-3} + \delta_{2,n}][(1-\delta_{2,n})\epsilon_1 + \kappa_n \delta_{2,n}] \qquad \forall (\epsilon, \epsilon_1) \in \mathbb{R}^2 \,, \tag{4.44}$$

so that

$$\varrho_n(\epsilon, \kappa_n \epsilon \log \epsilon) = \epsilon^{n-2} \kappa_n \frac{\log \epsilon}{\log^{\delta_{2,n}} \epsilon} \qquad \forall \epsilon \in]0,1[\,. \tag{4.45}$$

Then here and in what follows we find convenient to set

$$Z_{m-1,\alpha} \equiv C^{m,\alpha}(\partial\Omega^i) \times C^{m-1,\alpha}(\partial\Omega^i) \times C^{m-1,\alpha}(\partial\Omega^o)\,. \tag{4.46}$$

We are now ready to introduce the following, that can be easily verified.

Theorem 4.47 *Let $m \in \mathbb{N}\setminus\{0\}$, $n \in \mathbb{N}\setminus\{0,1\}$, $\alpha \in]0,1[$. Let Ω^i, Ω^o be as in (1.1). Let m^i, m^o, $a \in]0,+\infty[$, $b \in \mathbb{R}$. Let g^i, g^o be as in (1.3), (1.9). Let k_i, k_o be as in (1.2), (1.8). Assume that k_o^2 is not a Neumann eigenvalue for $-\Delta$ in Ω^o. Let $\theta^\sharp \in C^{m-1,\alpha}(\partial\Omega^i)$ be as in (4.15). Let $\mathcal{M} \equiv (\mathcal{M}_l)_{l=1,2,3}$ be the map from $]-\epsilon_0,\epsilon_0[\times\mathbb{R}^2\times Y_{m-1,\alpha}$ to $Z_{m-1,\alpha}$ defined by*

$$\begin{aligned}
\mathcal{M}_1[\epsilon,\epsilon_1,\epsilon_2,\zeta,c^i,\varsigma^i,c,\theta^o](\xi) &\equiv \int_{\partial\Omega^i} S_{h,n}(\xi-\eta,\epsilon k_o)\varsigma^i(\eta)\,d\sigma_\eta \\
&\quad + \epsilon^{n-1}k_o^n\left[\frac{2b_n}{\pi}\epsilon_1 + \epsilon\gamma_n(\log k_o)\right]\int_0^1 \frac{\partial}{\partial\lambda}V_{\Omega^i,J}[\varsigma^i, t\epsilon^2 k_o^2](\xi)\,dt \\
&+ \int_{\partial\Omega^i} S_{h,n}(\xi-\eta,\epsilon k_o)c^i\theta^\sharp(\eta)\,d\sigma_\eta \\
&\quad + \epsilon^{n-1}k_o^n\left[\frac{2b_n}{\pi}\epsilon_1 + \epsilon\gamma_n(\log k_o)\right]c^i\int_0^1 \frac{\partial}{\partial\lambda}V_{\Omega^i,J}[\theta^\sharp, t\epsilon^2 k_o^2](\xi)\,dt \\
&\quad + \epsilon^{n-2}k_o^{n-2}\gamma_n(\log k_o)c^iV_{\Omega^i,J}[\theta^\sharp,0](\xi) + \int_{\partial\Omega^o} S_{r,n}(\epsilon\xi-y,k_o)\theta^o(y)\,d\sigma_y \\
&-a\int_{\partial\Omega^i} S_{h,n}(\xi-\eta,\epsilon k_i)\zeta(\eta)\,d\sigma_\eta \\
&\quad - a\epsilon^{n-1}k_i^n\left[\frac{2b_n}{\pi}\epsilon_1 + \epsilon\gamma_n(\log k_i)\right]\int_0^1 \frac{\partial}{\partial\lambda}V_{\Omega^i,J}[\zeta, t\epsilon^2 k_i^2](\xi)\,dt \\
&-(k_o^{n-2}/k_i^{n-2})c^i\int_{\partial\Omega^i} S_{h,n}(\xi-\eta,\epsilon k_i)\theta^\sharp(\eta)\,d\sigma_\eta
\end{aligned} \tag{4.48}$$

$$
\begin{aligned}
&-a\int_{\partial\Omega^i} S_{h,n}(\xi-\eta,\epsilon k_i)c[(1-\delta_{2,n})+\epsilon_2]\theta^\sharp(\eta)\,d\sigma_\eta\\
&-\epsilon^{n-1}k_o^{n-2}c^ik_i^2\left[\frac{2b_n}{\pi}\epsilon_1+\epsilon\gamma_n(\log k_i)\right]\int_0^1\frac{\partial}{\partial\lambda}V_{\Omega^i,J}[\theta^\sharp,t\epsilon^2k_i^2](\xi)\,dt\\
&-a\epsilon^{n-1}k_i^n\left[\frac{2b_n}{\pi}\epsilon_1+\epsilon\gamma_n(\log k_i)\right]c[(1-\delta_{2,n})+\epsilon_2]\\
&\times\int_0^1\frac{\partial}{\partial\lambda}V_{\Omega^i,J}[\theta^\sharp,t\epsilon^2k_i^2](\xi)\,dt-\epsilon^{n-2}k_o^{n-2}c^i\gamma_n(\log k_i)V_{\Omega^i,J}[\theta^\sharp,0](\xi)\\
&-ak_i^{n-2}\left[\frac{2b_n}{\pi}\varrho_n(\epsilon,\epsilon_1)+\epsilon^{n-2}[(1-\delta_{2,n})+\epsilon_2]\gamma_n(\log k_i)\right]cV_{\Omega^i,J}[\theta^\sharp,0](\xi)\\
&-b\qquad\forall\xi\in\partial\Omega^i\,,
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{M}_2[\epsilon,\epsilon_1,\epsilon_2,\zeta,c^i,\varsigma^i,c,\theta^o](\xi) \qquad (4.49)\\
&\equiv-\frac{1}{m^i}\biggl\{-\frac{1}{2}\Bigl(\zeta(\xi)+a^{-1}(k_o^{n-2}/k_i^{n-2})c^i\theta^\sharp(\xi)+c[(1-\delta_{2,n})+\epsilon_2]\theta^\sharp(\xi)\Bigr)\\
&+\int_{\partial\Omega^i}DS_{h,n}(\xi-\eta,\epsilon k_i)\nu_{\Omega^i}(\xi)\\
&\times\Bigl(\zeta(\eta)+a^{-1}(k_o^{n-2}/k_i^{n-2})c^i\theta^\sharp(\eta)+c[(1-\delta_{2,n})+\epsilon_2]\theta^\sharp(\eta)\Bigr)\,d\sigma_\eta\\
&+\epsilon^{n-1}k_i^n\left[\frac{2b_n}{\pi}\epsilon_1+\epsilon\gamma_n(\log k_i)\right]\\
&\times\tilde{W}^t_{\Omega^i,J}[\zeta+a^{-1}(k_o^{n-2}/k_i^{n-2})c^i\theta^\sharp+c[(1-\delta_{2,n})+\epsilon_2]\theta^\sharp,\epsilon^2k_i^2](\xi)\biggr\}\\
&-\frac{1}{m^o}\biggl\{-\frac{1}{2}\Bigl(\varsigma^i(\xi)+c^i\theta^\sharp(\xi)\Bigr)\\
&\qquad-\int_{\partial\Omega^i}DS_{h,n}(\xi-\eta,\epsilon k_o)\nu_{\Omega^i}(\xi)\Bigl(\varsigma^i(\eta)+c^i\theta^\sharp(\eta)\Bigr)\,d\sigma_\eta\\
&-\epsilon^{n-1}k_o^n\left[\frac{2b_n}{\pi}\epsilon_1+\epsilon\gamma_n(\log k_o)\right]\tilde{W}^t_{\Omega^i,J}[\varsigma^i+c^i\theta^\sharp,\epsilon^2k_o^2](\xi)\\
&-\epsilon\int_{\partial\Omega^o}DS_{r,n}(\epsilon\xi-y,k_o)\nu_{\Omega^i}(\xi)\theta^o(y)\,d\sigma_y\biggr\}-\epsilon g^i(\xi)\qquad\forall\xi\in\partial\Omega^i\,,
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{M}_3[\epsilon,\epsilon_1,\epsilon_2,\zeta,c^i,\varsigma^i,c,\theta^o](x) \qquad (4.50)\\
&\equiv-\frac{1}{2}\theta^o(x)+\int_{\partial\Omega^i}DS_{r,n}(x-\epsilon\eta,k_o)\nu_{\Omega^o}(x)\Bigl(\varsigma^i(\eta)+c^i\theta^\sharp(\eta)\Bigr)\,d\sigma_\eta\epsilon^{n-2}\\
&\quad+\int_{\partial\Omega^o}DS_{r,n}(x-y,k_o)\nu_{\Omega^o}(x)\theta^o(y)\,d\sigma_y-g^o(x)\qquad\forall x\in\partial\Omega^o\,,
\end{aligned}
$$

for all $(\epsilon,\epsilon_1,\epsilon_2,\zeta,c^i,\varsigma^i,c,\theta^o)\in]-\epsilon_0,\epsilon_0[\times\mathbb{R}^2\times Y_{m-1,\alpha}$. *Then the following* *statements hold.*

(i) *If* $\epsilon = \epsilon_1 = \epsilon_2 = 0$, *then equation*

$$\mathcal{M}[0,0,0,\zeta,c^i,\varsigma^i,c,\theta^o] = 0$$

is equivalent to the limiting system (4.33) and has one and only one solution $(\tilde{\zeta},\tilde{c}^i,\tilde{\varsigma}^i,\tilde{c},\tilde{\theta}^o)$ *in* $Y_{m-1,\alpha}$. *Moreover,* $\tilde{c}^i = 0$.

(ii) *If* $\epsilon \in]0,\epsilon_0[$, $\epsilon_1 = \kappa_n \epsilon \log \epsilon$, $\epsilon_2 = \frac{\delta_{2,n}}{\log \epsilon}$, *then the equation*

$$\mathcal{M}[\epsilon,\epsilon_1,\epsilon_2,\zeta,c^i,\varsigma^i,c,\theta^o] = 0 \tag{4.51}$$

is equivalent to the system (4.21)–(4.23).

Proof. If $(\epsilon,\epsilon_1,\epsilon_2,\zeta,c^i,\varsigma^i,c,\theta^o)$ belongs to the domain of $\mathcal{M}$, then the classical Schauder regularity properties of the acoustic potentials of Theorems 3.21 and 3.22 ensure that $\mathcal{M}[\epsilon,\epsilon_1,\epsilon_2,\zeta,c^i,\varsigma^i,c,\theta^o]$ belongs to $Z_{m-1,\alpha}$. Statements (i) and (ii) hold true by the definition of $\mathcal{M}$. □

The main advantages of equation (4.51) with respect to system (4.21), (4.22), (4.23), is that equation (4.51) displays no singularity in the variables ϵ, ϵ_1, ϵ_2, and that equation (4.51) makes also sense for $\epsilon \in]-\epsilon_0, 0]$, while system (4.21), (4.22), (4.23) does not. So we now plan to analyze equation (4.51) around the degenerate points where $\epsilon = \epsilon_1 = \epsilon_2 = 0$. To do so, we note that the definition (4.44) of ϱ_n implies that

$$\varrho_n(0,0) = \delta_{2,n}\kappa_n\delta_{2,n} = \delta_{2,n} \tag{4.52}$$

and we prove the following theorem.

Theorem 4.53 *Let* $m \in \mathbb{N}\setminus\{0\}$, $n \in \mathbb{N}\setminus\{0,1\}$, $\alpha \in]0,1[$. *Let* Ω^i, Ω^o *be as in (1.1). Let* m^i, m^o, $a \in]0,+\infty[$, $b \in \mathbb{R}$. *Let* g^i, g^o *be as in (1.3), (1.9). Let* k_i, k_o *be as in (1.2), (1.8). Assume that* k_o^2 *is not a Neumann eigenvalue for* $-\Delta$ *in* Ω^o. *Let* $\epsilon_* \in]0,\epsilon_0[$ *be as in Theorem 4.1. Let* $\mathcal{M} \equiv (\mathcal{M}_l)_{l=1,2,3}$ *be the map from* $]-\epsilon_0,\epsilon_0[\times\mathbb{R}^2\times Y_{m-1,\alpha}$ *to* $Z_{m-1,\alpha}$ *defined by (4.48)–(4.50). Then the following statements hold.*

(i) *The map* $\mathcal{M}$ *is real analytic and the differential*

$$\partial_{(\zeta,c^i,\varsigma^i,c,\theta^o)}\mathcal{M}[0,0,0,\tilde{\zeta},\tilde{c}^i,\tilde{\varsigma}^i,\tilde{c},\tilde{\theta}^o]$$

of $\mathcal{M}$ at $(0,0,0,\tilde{\zeta},\tilde{c}^i,\tilde{\varsigma}^i,\tilde{c},\tilde{\theta}^o)$ with respect to the variable $(\zeta,c^i,\varsigma^i,c,\theta^o)$ is a linear homeomorphism from $Y_{m-1,\alpha}$ onto $Z_{m-1,\alpha}$. Moreover, if $(\bar{f}^i,\bar{g}^i,\bar{f}^o)\in Z_{m-1,\alpha}$ and $(\bar{\zeta},\bar{c}^i,\bar{\varsigma}^i,\bar{c},\bar{\theta}^o)\in Y_{m-1,\alpha}$ satisfy equality

$$\partial_{(\zeta,c^i,\varsigma^i,c,\theta^o)}\mathcal{M}[0,0,0,\tilde{\zeta},\tilde{c}^i,\tilde{\varsigma}^i,\tilde{c},\tilde{\theta}^o](\bar{\zeta},\bar{c}^i,\bar{\varsigma}^i,\bar{c},\bar{\theta}^o)=(\bar{f}^i,\bar{g}^i,\bar{f}^o)\,, \quad (4.54)$$

then

$$\bar{c}^i=m^o\int_{\partial\Omega^i}\bar{g}^i d\sigma\,. \quad (4.55)$$

(ii) There exists $\epsilon'\in]0,\epsilon_*[$, an open neighbourhood $\tilde{U}$ of $(0,0)$ in $\mathbb{R}^2$ and an open neighbourhood $\tilde{V}$ of $(\tilde{\zeta},\tilde{c}^i,\tilde{\varsigma}^i,\tilde{c},\tilde{\theta}^o)$ in $Y_{m-1,\alpha}$ and a real analytic map

$$\big(Z,C^i,S^i,C,\Theta^o\big)$$

from $]-\epsilon',\epsilon'[\times\tilde{U}$ to $\tilde{V}$ such that

$$\left(\kappa_n\epsilon\log\epsilon,\frac{\delta_{2,n}}{\log\epsilon}\right)\in\tilde{U},\quad\forall\epsilon\in]0,\epsilon'[,$$

and such that the set of zeros of $\mathcal{M}$ in $]-\epsilon',\epsilon'[\times\tilde{U}\times\tilde{V}$ coincides with the graph of the map (Z,C^i,S^i,C,Θ^o). In particular,

$$\big(Z[0,0,0],C^i[0,0,0],S^i[0,0,0],C[0,0,0],\Theta^o[0,0,0]\big)=(\tilde{\zeta},\tilde{c}^i,\tilde{\varsigma}^i,\tilde{c},\tilde{\theta}^o).$$

Proof. By Theorem 3.21, $V_{\Omega^i,h}$ and $W^t_{\Omega^i,h}$ are real analytic. By Theorem 3.22, $V_{\Omega^i,J}$ and $\tilde{W}^t_{\Omega^i,J}$ are real analytic. Since $S_{r,n}(\epsilon\xi-y,k_o)$ is real analytic in the variable (ξ,y,ϵ) in an open neighbourhood of $\partial\Omega^i\times\partial\Omega^o\times]-\epsilon_0,\epsilon_0[$, a result of paper [23, Prop. 4.1] with Musolino on the properties of the integral operators with real analytic kernel implies that the function from the set $]-\epsilon_0,\epsilon_0[\times L^1(\partial\Omega^i)$ to $C^{m,\alpha}(\partial\Omega^i)$ which takes (ϵ,f) to the function $\int_{\partial\Omega^o}S_{r,n}(\epsilon\cdot-y,k_o)f(y)d\sigma_y$ is analytic.

Since $J^{\sharp'}_{\frac{n-2}{2}}$ is holomorphic in $\mathbb{C}$, then Proposition 4.1 (ii) of [23] on integral operators with a real analytic kernel implies that the map from $C^{m-1,\alpha}(\partial\Omega^i)\times]-\epsilon_0,\epsilon_0[$ to $C^{m,\alpha}(\partial\Omega^i)$ that takes (ς^i,ϵ) to the function

$$\int_0^1\frac{\partial}{\partial\lambda}V_{\Omega^i,J}[\varsigma^i,t\epsilon^2k_o^2](\xi)\,dt$$
$$=\int_{(\partial\Omega^i)\times]0,1[}J^{\sharp'}_{\frac{n-2}{2}}(t\epsilon^2k_o^2|\xi-\eta|^2)|\xi-\eta|^2\varsigma^i(\eta)\,d\sigma_\eta\otimes dt$$

of the variable $\xi \in \partial\Omega^i$ is real analytic. Then the second term of the first component of $\mathcal{M}$ is analytic. Similarly, one can prove the analyticity of the other terms of the first component of $\mathcal{M}$ that contain either $\frac{\partial}{\partial\lambda}V_{\Omega^i,J}$ or $\frac{\partial}{\partial\lambda}V_{\Omega^o,J}$ and thus the first component of $\mathcal{M}$ is analytic.

Next we consider the second and third components of $\mathcal{M}$. Since $\frac{\partial}{\partial\xi_j}S_{r,n}(\epsilon\xi - y, k_o)$ for $j = 1, ..., n$ is a real analytic function in (ξ, y, ϵ) in an open neighbourhood of $\partial\Omega^i \times \partial\Omega^o \times]-\epsilon_0, \epsilon_0[$, then by a result of [23, Prop. 4.1] on the properties of integral operators with real analytic kernel, the function from $]-\epsilon_0, \epsilon_0[\times L^1(\partial\Omega^o)$ to $C^{m-1,\alpha}(\partial\Omega^i)$ which takes (ϵ, f) to the function

$$\sum_{j=1}^{n}(\nu_{\Omega^i})_j(\xi)\int_{\partial\Omega^o}\frac{\partial}{\partial\xi_j}S_{r,n}(\epsilon\xi - y, k_o)f(y)d\sigma_y \qquad \forall\xi \in \partial\Omega^i$$

is real analytic. Similarly, the map from $]-\epsilon_0, \epsilon_0[\times L^1(\partial\Omega^i)$ to $C^{m-1,\alpha}(\partial\Omega^o)$ which takes (ϵ, f) to the function

$$\sum_{j=1}^{n}(\nu_{\Omega^o})_j(x)\int_{\partial\Omega^i}\frac{\partial}{\partial x_j}S_{r,n}(x - \epsilon\eta, k_o)f(\eta)d\sigma_\eta \qquad \forall x \in \partial\Omega^o$$

is real analytic. Then by the same arguments above one proves that the second and third components of $\mathcal{M}$ are analytic. In particular, $W^t_{\Omega^o}[\cdot, k_o]$ is linear and continuous by Corollary 3.25 (iii). Thus, the operator $\mathcal{M}$ is real analytic. Since the differential of a linear and continuous operator is the operator itself, standard differentiation rules and equality $\delta_{2,n}V_{\Omega^i,J}[\theta^\sharp, 0] = \delta_{2,n}$ imply that the differential of $\mathcal{M}$ at $(0, 0, 0, \tilde{\zeta}, \tilde{c}^i, \tilde{\varsigma}^i, \tilde{c}, \tilde{\theta}^o)$ with respect to the variable $(\zeta, c^i, \varsigma^i, c, \theta^o)$ is delivered by the following formula

$$\begin{aligned}
&\partial_{(\zeta,c^i,\varsigma^i,c,\theta^o)}\mathcal{M}_1[0,0,0,\tilde{\zeta},\tilde{c}^i,\tilde{\varsigma}^i,\tilde{c},\tilde{\theta}^o](\bar{\zeta},\bar{c}^i,\bar{\varsigma}^i,\bar{c},\bar{\theta}^o)(\xi) = V_{\Omega^i,h}[\bar{\varsigma}^i,0](\xi) \qquad (4.56)\\
&\quad +V_{\Omega^i,h}[\theta^\sharp,0](\xi)\bar{c}^i + \delta_{2,n}k_o^{n-2}\gamma_n(\log k_o)\bar{c}^i + v^+_{\Omega^o}[\bar{\theta}^o,k_o](0) - aV_{\Omega^i,h}[\bar{\zeta},0](\xi)\\
&\quad -\left(k_o^{n-2}/k_i^{n-2}\right)V_{\Omega^i,h}[\theta^\sharp,0](\xi)\bar{c}^i - a(1-\delta_{2,n})V_{\Omega^i,h}[\theta^\sharp,0](\xi)\bar{c}\\
&\qquad -\delta_{2,n}k_o^{n-2}\gamma_n(\log k_i)\bar{c}^i - ak_i^{n-2}\frac{2b_n}{\pi}\delta_{2,n}\bar{c} \qquad \forall\xi \in \partial\Omega^i\,,\\
&\partial_{(\zeta,c^i,\varsigma^i,c,\theta^o)}\mathcal{M}_2[0,0,0,\tilde{\zeta},\tilde{c}^i,\tilde{\varsigma}^i,\tilde{c},\tilde{\theta}^o](\bar{\zeta},\bar{c}^i,\bar{\varsigma}^i,\bar{c},\bar{\theta}^o)(\xi)\\
&\quad = -\frac{1}{m^i}\Big\{-\frac{1}{2}\left(\bar{\zeta}(\xi) + a^{-1}\left(k_o^{n-2}/k_i^{n-2}\right)\theta^\sharp(\xi)\bar{c}^i + (1-\delta_{2,n})\theta^\sharp(\xi)\bar{c}\right)\\
&\qquad +W^t_{\Omega^i,h}\left[\bar{\zeta} + a^{-1}\left(k_o^{n-2}/k_i^{n-2}\right)\theta^\sharp\bar{c}^i + (1-\delta_{2,n})\theta^\sharp\bar{c}, 0\right](\xi)\Big\}
\end{aligned}$$

$$-\frac{1}{m^o}\Big\{-\frac{1}{2}\left(\bar{\varsigma}^i(\xi)+\bar{c}^i\theta^\sharp(\xi)\right)-W^t_{\Omega^i,h}\left[\bar{\varsigma}^i+\bar{c}^i\theta^\sharp,0\right](\xi)\Big\} \qquad \forall \xi\in\partial\Omega^i\,,$$

$$\partial_{(\zeta,c^i,\varsigma^i,c,\theta^o)}\mathcal{M}_3[0,0,0,\tilde{\zeta},\tilde{c}^i,\tilde{\varsigma}^i,\tilde{c},\tilde{\theta}^o](\bar{\zeta},\bar{c}^i,\bar{\varsigma}^i,\bar{c},\bar{\theta}^o)(x)$$
$$=-\frac{1}{2}\bar{\theta}^o(x)+\delta_{2,n}DS_{r,n}(x,k_o)\nu_{\Omega^o}(x)\bar{c}^i+W^t_{\Omega^o}[\bar{\theta}^o,k_o](x) \qquad \forall x\in\partial\Omega^o$$

for all $(\bar{\zeta},\bar{c}^i,\bar{\varsigma}^i,\bar{c},\bar{\theta}^o)\in Y_{m-1,\alpha}$. We now prove that the linear and continuous operator

$$\partial_{(\zeta,c^i,\varsigma^i,c,\theta^o)}\mathcal{M}[0,0,0,\tilde{\zeta},\tilde{c}^i,\tilde{\varsigma}^i,\tilde{c},\tilde{\theta}^o]$$

is a homeomorphism. By the Open Mapping Theorem it suffices to show that $\partial_{(\zeta,c^i,\varsigma^i,c,\theta^o)}\mathcal{M}[0,0,0,\tilde{\zeta},\tilde{c}^i,\tilde{\varsigma}^i,\tilde{c},\tilde{\theta}^o]$ is a bijection from $Y_{m-1,\alpha}$ onto $Z_{m-1,\alpha}$. Let $(\bar{f}^i,\bar{g}^i,\bar{f}^o)\in Z_{m-1,\alpha}$. We must show that there exists a unique

$$(\bar{\zeta},\bar{c}^i,\bar{\varsigma}^i,\bar{c},\bar{\theta}^o)\in Y_{m-1,\alpha}$$

such that equation (4.54) is satisfied. In particular, we note that the second equation of (4.54) can be written as follows

$$\frac{1}{m^i}\left\{\frac{1}{2}\bar{\zeta}-W^t_{\Omega^i,h}\left[\bar{\zeta},0\right]\right\}+\frac{1}{m^o}\left\{\frac{1}{2}\bar{\varsigma}^i+W^t_{\Omega^i,h}\left[\bar{\varsigma}^i,0\right]\right\}=\bar{g}^i-\frac{1}{m^o}\bar{c}^i\theta^\sharp \quad \text{on } \partial\Omega^i \tag{4.57}$$

(cf. (4.15)). We first assume that (4.54) has a solution $(\bar{\zeta},\bar{c}^i,\bar{\varsigma}^i,\bar{c},\bar{\theta}^o)$ and we show that $\bar{c}^i$ is uniquely determined. By integrating the second equation of (4.54), *i.e.*, equation (4.57), on $\partial\Omega^i$ and by exploiting equality (4.37), we obtain

$$\frac{1}{m^o}\int_{\partial\Omega^i}\bar{\varsigma}^i\,d\sigma=\int_{\partial\Omega^i}\bar{g}^i-\frac{1}{m^o}\bar{c}^i\theta^\sharp\,d\sigma\,.$$

Since $\int_{\partial\Omega^i}\bar{\varsigma}^i\,d\sigma=0$ and $\int_{\partial\Omega^i}\theta^\sharp\,d\sigma=1$, equality (4.55) holds true. By (4.16) and (4.17), $v^\sharp=V_{\Omega^i,h}[\theta^\sharp,0]$ is constant on $\partial\Omega^i$ and $V_{\Omega^i,J}[\theta^\sharp,0]=J^\sharp_{\frac{n-2}{2}}(0)$ on $\partial\Omega^i$. Since k_o^2 is not a Neumann eigenvalue for $-\Delta$ in Ω^o, the Fredholm Alternative Theorem and classical Schauder regularity theory imply that the last equation of (4.54) with $\bar{c}^i$ as in (4.55) has a unique solution $\bar{\theta}^o$ in $C^{m-1,\alpha}(\partial\Omega^o)$ (cf. Colton and Kress [8, Thm. 3.17], [22, Thm. B.1]). Since the coefficient $A^\sharp$ of $-a\bar{c}$ in the first equation of (4.54) is different from 0 (cf. (4.40) and following comment), Proposition B.1 implies that the first two equations of (4.54) with $\bar{c}^i$ as in (4.55) have a unique solution $(\bar{\zeta},\bar{\varsigma}^i,\bar{c})$ in $C^{m-1,\alpha}(\partial\Omega^i)_0\times C^{m-1,\alpha}(\partial\Omega^i)\times\mathbb{R}$. By equality (4.55) we know that the integral of right hand side of equation (4.57) on $\partial\Omega^i$ is equal to 0 and thus Proposition B.1 ensures that $\int_{\partial\Omega^i}\bar{\varsigma}^i\,d\sigma=0$.

On the other hand, if we take $\bar{c}^i$ as in (4.55) and if we set $(\bar{\zeta}, \bar{\varsigma}^i, \bar{c})$ equal to the only solution in $C^{m-1,\alpha}(\partial\Omega^i)_0 \times C^{m-1,\alpha}(\partial\Omega^i) \times \mathbb{R}$ of the first two equations of (4.54) in which $\bar{c}^i$ is as in (4.55) and $\bar{\theta}^o$ equal to the only solution of the last equation of (4.55) in which $\bar{c}^i$ is as in (4.55), we can verify that $(\bar{\zeta}, \bar{c}^i, \bar{\varsigma}^i, \bar{c}, \bar{\theta}^o)$ solves (4.54) and satisfies equality $\int_{\partial\Omega^i} \bar{\varsigma}^i \, d\sigma = 0$ by reading backwards the above argument.

Hence, we have proved that $\partial_{(\zeta,c^i,\varsigma^i,c,\theta^o)}\mathcal{M}[0,0,0,\tilde{\zeta},\tilde{c}^i,\tilde{\varsigma}^i,\tilde{c},\tilde{\theta}^o]$ is bijection from $Y_{m-1,\alpha}$ to $Z_{m-1,\alpha}$ and statement (i) holds true.

Then the existence of $\epsilon', \tilde{U}, \tilde{V}, (Z, C^i, S^i, C, \Theta^o)$ as in (ii) and the last equality of statement (ii) are a consequence of Theorem 4.47 (i), of statement (i) and of the Implicit Function Theorem in Banach Spaces around the point $(0,0,0,\tilde{\zeta},\tilde{c}^i,\tilde{\varsigma}^i,\tilde{c},\tilde{\theta}^o)$. Possibly shrinking ϵ', we can assume that $\left(\kappa_n \epsilon \log \epsilon, \frac{\delta_{2,n}}{\log \epsilon}\right)$ belongs to $\tilde{U}$ for ϵ in $]0,\epsilon'[$. □

Remark 4.58 Under the assumptions of Theorem 4.53, the known formula for the differential of an implicitly defined function implies that

$$
\begin{aligned}
& d\left(Z, C^i, S^i, C, \Theta^o\right)[0,0,0](\bar{\epsilon}, \bar{\epsilon_1}, \bar{\epsilon_2}) \\
&\qquad = -\left(\partial_{(\zeta,c^i,\varsigma^i,c,\theta^o)}\mathcal{M}[0,0,0,\tilde{\zeta},\tilde{c}^i,\tilde{\varsigma}^i,\tilde{c},\tilde{\theta}^o]\right)^{(-1)} \\
&\qquad\qquad \circ \partial_{(\epsilon,\epsilon_1,\epsilon_2)}\mathcal{M}[0,0,0,\tilde{\zeta},\tilde{c}^i,\tilde{\varsigma}^i,\tilde{c},\tilde{\theta}^o](\bar{\epsilon}, \bar{\epsilon_1}, \bar{\epsilon_2}) \qquad \forall (\bar{\epsilon}, \bar{\epsilon_1}, \bar{\epsilon_2}) \in \mathbb{R}^3 .
\end{aligned}
$$

Then Theorem 4.53 (i) implies that

$$
\begin{aligned}
& dC^i[0,0,0](\bar{\epsilon}, \bar{\epsilon_1}, \bar{\epsilon_2}) \\
&\qquad = -m^o \int_{\partial\Omega^i} \partial_{(\epsilon,\epsilon_1,\epsilon_2)}\mathcal{M}_2[0,0,0,\tilde{\zeta},\tilde{c}^i,\tilde{\varsigma}^i,\tilde{c},\tilde{\theta}^o](\bar{\epsilon}, \bar{\epsilon_1}, \bar{\epsilon_2}) \, d\sigma \\
&\qquad = -m^o \int_{\partial\Omega^i} \partial_{\epsilon}\mathcal{M}_2[0,0,0,\tilde{\zeta},\tilde{c}^i,\tilde{\varsigma}^i,\tilde{c},\tilde{\theta}^o] \, d\sigma \bar{\epsilon} \qquad \forall (\bar{\epsilon}, \bar{\epsilon_1}, \bar{\epsilon_2}) \in \mathbb{R}^3 .
\end{aligned}
$$

Indeed, $\partial_{\epsilon_j}\mathcal{M}_2[0,0,0,\tilde{\zeta},\tilde{c}^i,\tilde{\varsigma}^i,\tilde{c},\tilde{\theta}^o] = 0$ for $j \in \{1,2\}$. As a consequence,

$$
\frac{\partial C^i}{\partial \epsilon_1}[0,0,0] = dC^i[0,0,0](0,1,0) = 0, \;\; \frac{\partial C^i}{\partial \epsilon_2}[0,0,0] = dC^i[0,0,0](0,0,1) = 0. \tag{4.59}
$$

In order to simplify our notation, we set

$$
\Xi_n[\epsilon] \equiv \left(\kappa_n \epsilon \log \epsilon, \frac{\delta_{2,n}}{\log \epsilon}\right), \qquad \forall \epsilon \in]0,1[. \tag{4.60}
$$

Then we can prove the following existence and uniqueness theorem for problem (1.10) for $\epsilon \in]0,\epsilon'[$.

Theorem 4.61 *Let $m \in \mathbb{N} \setminus \{0\}$, $n \in \mathbb{N} \setminus \{0,1\}$, $\alpha \in]0,1[$. Let Ω^i, Ω^o be as in (1.1). Let m^i, m^o, $a \in]0,+\infty[$, $b \in \mathbb{R}$. Let g^i, g^o be as in (1.3), (1.9). Let k_i, k_o be as in (1.2), (1.8). Assume that k_o^2 is not a Neumann eigenvalue for $-\Delta$ in Ω^o. Let $\epsilon' \in]0,\epsilon_0[$ be as in Theorem 4.53 (ii). If $\epsilon \in]0,\epsilon'[$, then the transmission problem (1.10) has one and only one solution $(u^i(\epsilon,\cdot), u^o(\epsilon,\cdot)) \in C^{m,\alpha}(\epsilon\overline{\Omega^i}) \times C^{m,\alpha}(\overline{\Omega(\epsilon)})$ and the following formula holds*

$$
\begin{aligned}
& u^i(\epsilon,\cdot) && (4.62)\\
&\quad = u^i[\epsilon, Z[\epsilon,\Xi_n[\epsilon]], C^i[\epsilon,\Xi_n[\epsilon]], S^i[\epsilon,\Xi_n[\epsilon]], C[\epsilon,\Xi_n[\epsilon]], \Theta^o[\epsilon,\Xi_n[\epsilon]]](\cdot)\\
& u^o(\epsilon,\cdot)\\
&\quad = u^o[\epsilon, Z[\epsilon,\Xi_n[\epsilon]], C^i[\epsilon,\Xi_n[\epsilon]], S^i[\epsilon,\Xi_n[\epsilon]], C[\epsilon,\Xi_n[\epsilon]], \Theta^o[\epsilon,\Xi_n[\epsilon]]](\cdot)
\end{aligned}
$$

for all $\epsilon \in]0,\epsilon'[$ (cf. (4.20)).

Proof. Theorems 4.1, 4.18 and 4.53 ensure that the pair $(u^i(\epsilon,\cdot), u^o(\epsilon,\cdot))$ of (4.62) belongs to $C^{m,\alpha}(\epsilon\overline{\Omega^i}) \times C^{m,\alpha}(\overline{\Omega(\epsilon)})$ and solves the transmission problem (1.10). On the other hand (1.10) is a linear problem and thus if $\epsilon \in]0,\epsilon'[$ and $(u^i_\epsilon, u^o_\epsilon) \in C^{m,\alpha}(\epsilon\overline{\Omega^i}) \times C^{m,\alpha}(\overline{\Omega(\epsilon)})$ is another solution of the transmission problem (1.10), then Theorems 4.1, 4.18 and 4.53 ensure that there exists a unique $(\zeta_\epsilon, c^i_\epsilon, \varsigma^i_\epsilon, c_\epsilon, \theta^o_\epsilon) \in Y_{m-1,\alpha}$ such that

$$
u^i_\epsilon = u^i[\epsilon,\zeta_\epsilon,c^i_\epsilon,\varsigma^i_\epsilon,c_\epsilon,\theta^o_\epsilon]\,, \qquad u^o_\epsilon = u^o[\epsilon,\zeta_\epsilon,c^i_\epsilon,\varsigma^i_\epsilon,c_\epsilon,\theta^o_\epsilon]
$$

and $\mathcal{M}[\epsilon,\zeta_\epsilon,c^i_\epsilon,\varsigma^i_\epsilon,c_\epsilon,\theta^o_\epsilon] = 0$. Since $\mathcal{M}[\epsilon,\cdot,\cdot,\cdot,\cdot,\cdot]$ is affine, we have

$$
\begin{aligned}
\mathcal{M}[\epsilon, & Z[\epsilon,\Xi_n[\epsilon]] + t(\zeta_\epsilon - Z[\epsilon,\Xi_n[\epsilon]]), C^i[\epsilon,\Xi_n[\epsilon]] + t(c^i_\epsilon - C^i[\epsilon,\Xi_n[\epsilon]]),\\
& S^i[\epsilon,\Xi_n[\epsilon]] + t(\varsigma^i_\epsilon - S^i[\epsilon,\Xi_n[\epsilon]]), C[\epsilon,\Xi_n[\epsilon]] + t(c_\epsilon - C[\epsilon,\Xi_n[\epsilon]]),\\
& \Theta^i[\epsilon,\Xi_n[\epsilon]] + t(\theta^i_\epsilon - \Theta^i[\epsilon,\Xi_n[\epsilon]])] = 0
\end{aligned}
$$

for all $t \in \mathbb{R} \setminus \{0\}$. On the other hand, if we choose t sufficiently small, we have

$$
\begin{aligned}
\big(& Z[\epsilon,\Xi_n[\epsilon]] + t(\zeta_\epsilon - Z[\epsilon,\Xi_n[\epsilon]]), C^i[\epsilon,\Xi_n[\epsilon]] + t(c^i_\epsilon - C^i[\epsilon,\Xi_n[\epsilon]]),\\
& S^i[\epsilon,\Xi_n[\epsilon]] + t(\varsigma^i_\epsilon - S^i[\epsilon,\Xi_n[\epsilon]]), C[\epsilon,\Xi_n[\epsilon]] + t(c_\epsilon - C[\epsilon,\Xi_n[\epsilon]]),\\
& \Theta^i[\epsilon,\Xi_n[\epsilon]] + t(\theta^i_\epsilon - \Theta^i[\epsilon,\Xi_n[\epsilon]])\big) \in \tilde{V}
\end{aligned}
$$

Then Theorem 4.53 (ii) implies that

$$
\begin{aligned}
&\zeta_\epsilon - Z[\epsilon,\Xi_n[\epsilon]] = 0\,, \qquad c^i_\epsilon - C^i[\epsilon,\Xi_n[\epsilon]] = 0\,, \qquad \varsigma^i_\epsilon - S^i[\epsilon,\Xi_n[\epsilon]] = 0\,,\\
&c_\epsilon - C[\epsilon,\Xi_n[\epsilon]] = 0\,, \qquad \theta^i_\epsilon - \Theta^i[\epsilon,\Xi_n[\epsilon] = 0\,.
\end{aligned}
$$

Hence,

$$u^i_\epsilon = u^i(\epsilon, \cdot)\,, \qquad u^o_\epsilon = u^o(\epsilon, \cdot)$$

and the proof is complete. □

Remark 4.63 The above Theorem 4.61 provides an existence and uniqueness theorem for problem (1.10) for $\epsilon \in]0, \epsilon_0[$ small enough. By following the arguments of the proof of the existence and uniqueness theorem for the transmission problem of Kress and Roach [17] one could prove existence and uniqueness for problem (1.10) for all $\epsilon \in]0, \epsilon_0[$ under some additional assumptions on m^i, m^o, k_i and k_o.

5 A representation formula for $\{u^o(\epsilon, \cdot)\}_{\epsilon \in]0, \epsilon'[}$

We now turn to prove the following theorem, that clarifies the behavior of the solution $u^o(\epsilon, \cdot)$ as ϵ tends to 0.

Theorem 5.1 *Let the assumptions of Theorem 4.53 hold. Let Ω_M be a bounded open subset of $\Omega^o \setminus \{0\}$ such that $0 \notin \overline{\Omega_M}$. Then there exist $\epsilon_M \in]0, \epsilon'[$ and a real analytic map $\mathcal{U}_M$ from $] - \epsilon_M, \epsilon_M[\times \tilde{U}$ to $C^{m,\alpha}(\overline{\Omega_M})$ such that*

$$\overline{\Omega_M} \subseteq \overline{\Omega(\epsilon)} \qquad \forall \epsilon \in] - \epsilon_M, \epsilon_M[\tag{5.2}$$

$$u^o(\epsilon, \cdot)_{|\overline{\Omega_M}} = \mathcal{U}_M[\epsilon, \Xi_n[\epsilon]] \qquad \forall \epsilon \in]0, \epsilon_M[\tag{5.3}$$

Moreover,

$$\mathcal{U}_M[0, 0, 0](x) = \tilde{u}^o(x) \qquad \forall x \in \overline{\Omega_M}\,, \tag{5.4}$$

where $(\tilde{u}^{i,r}_1, \tilde{u}^{o,r}_1, \tilde{u}^o)$ is the only solution of the limiting boundary value problem (4.34) (see Theorem 4.32 (ii)).

Proof. By the second formulas of (4.20) and (4.62), we have

$$\begin{aligned}
&u^o(\epsilon, x) \\
&\quad = u^o[\epsilon, Z[\epsilon, \Xi_n[\epsilon]], C^i[\epsilon, \Xi_n[\epsilon]], S^i[\epsilon, \Xi_n[\epsilon]], C[\epsilon, \Xi_n[\epsilon]], \Theta^o[\epsilon, \Xi_n[\epsilon]](x) \\
&\quad = v^+_{\Omega^o}[\Theta^o[\epsilon, \Xi_n[\epsilon]], k_o](x) \\
&\quad + \epsilon^{-1} v^-_{\epsilon\Omega^i}\left[S^i[\epsilon, \Xi_n[\epsilon]](\cdot/\epsilon), k_o\right](x) + \epsilon^{-1} v^-_{\epsilon\Omega^i}\left[C^i[\epsilon, \Xi_n[\epsilon]]\theta^\sharp(\cdot/\epsilon), k_o\right](x)
\end{aligned}$$

for all $x \in \overline{\Omega(\epsilon)}$ and $\epsilon \in]0, \epsilon'[$. Then Corollary 3.25 (ii) implies that

$$u^o(\epsilon, x) \tag{5.5}$$

$$= v^+_{\Omega^o,h}[\Theta^o[\epsilon,\Xi_n[\epsilon]],k_o](x) + \gamma_n(\log k_o)k_o^{n-2}v^+_{\Omega^o,J}[\Theta^o[\epsilon,\Xi_n[\epsilon]],k_o^2](x)$$
$$+\epsilon^{n-2}\Big(\int_{\partial\Omega^i} S_{r,n}(x-\epsilon\eta,k_o)S^i[\epsilon,\Xi_n[\epsilon]](\eta)d\sigma_\eta$$
$$+C^i[\epsilon,\Xi_n[\epsilon]]\int_{\partial\Omega^i} S_{r,n}(x-\epsilon\eta,k_o)\theta^\sharp(\eta)d\sigma_\eta\Big) \qquad \forall x\in\overline{\Omega(\epsilon)}$$

for all $\epsilon\in]0,\epsilon'[$. Since $0\notin\overline{\Omega_M}$ and $\overline{\Omega_M}$ is compact, 0 has a positive distance from $\overline{\Omega_M}$ and thus there exists $\epsilon_M\in]0,\epsilon'[$ such that

$$\overline{\Omega_M}\subseteq\overline{\Omega(\epsilon)} \qquad \forall\epsilon\in[-\epsilon_M,\epsilon_M].$$

By equality (5.5), we find natural to define a map $\mathcal{U}_M$ from $]-\epsilon_M,\epsilon_M[\times\tilde{U}$ to $C^{m,\alpha}(\overline{\Omega_M})$ by setting

$$\mathcal{U}_M[\epsilon,\epsilon_1,\epsilon_2](x) \equiv v^+_{\Omega^o,h}\left[\Theta^o[\epsilon,\epsilon_1,\epsilon_2],k_o\right](x) \tag{5.6}$$
$$+\gamma_n(\log k_o)k_o^{n-2}v^+_{\Omega^o,J}\left[\Theta^o[\epsilon,\epsilon_1,\epsilon_2],k_o^2\right](x)$$
$$+\epsilon^{n-2}\left(\int_{\partial\Omega^i} S_{r,n}(x-\epsilon\eta,k_o)S^i[\epsilon,\epsilon_1,\epsilon_2](\eta)d\sigma_\eta\right)$$
$$+\epsilon^{n-2}\left(C^i[\epsilon,\epsilon_1,\epsilon_2]\int_{\partial\Omega^i} S_{r,n}(x-\epsilon\eta,k_o)\theta^\sharp(\eta)d\sigma_\eta\right) \qquad \forall x\in\overline{\Omega_M}$$

for all $(\epsilon,\epsilon_1,\epsilon_2)\in]-\epsilon_M,\epsilon_M[\times\tilde{U}$ (see Theorem 4.53 (ii)). It clearly suffices to show that the right-hand side of (5.6) defines a real analytic map from $]-\epsilon_M,\epsilon_M[\times\tilde{U}$ to $C^{m,\alpha}(\overline{\Omega_M})$. By Theorem 3.21 (i), $v^+_{\Omega^o,h}[\cdot,\cdot]$ defines a real analytic map from $C^{m-1,\alpha}(\partial\Omega^o)\times\mathbb{C}$ to $C^{m,\alpha}(\overline{\Omega^o})$. Then Theorem 4.53 implies that the map from $]-\epsilon_M,\epsilon_M[\times\tilde{U}$ to $C^{m,\alpha}(\overline{\Omega^o})$ which takes $(\epsilon,\epsilon_1,\epsilon_2)$ to $v^+_{\Omega^o,h}\left[\Theta^o[\epsilon,\epsilon_1,\epsilon_2],k_o\right]$ is real analytic. By Theorem 3.22 (ii), $v^+_{\Omega^o,J}[\cdot,\cdot]$ defines a real analytic map from $C^{m-1,\alpha}(\partial\Omega^o)\times\mathbb{C}$ to $C^{m,\alpha}(\overline{\Omega^o})$. Then Theorem 4.53 implies that the map from $]-\epsilon_M,\epsilon_M[\times\tilde{U}$ to $C^{m,\alpha}(\overline{\Omega^o})$ which takes $(\epsilon,\epsilon_1,\epsilon_2)$ to $v^+_{\Omega^o,J}\left[\Theta^o[\epsilon,\epsilon_1,\epsilon_2],k_o\right]$ is real analytic.

Since $\overline{\Omega_M}\subseteq\overline{\Omega^o}$, the first and second summand in the right side of (5.6) define real analytic maps from $]-\epsilon_M,\epsilon_M[\times\tilde{U}$ to $C^{m,\alpha}(\overline{\Omega_M})$.

Since $S_{r,n}(x-\epsilon\eta,k_o)$ is real analytic in the variable (x,η,ϵ) in an open neighbourhood of $\overline{\Omega_M}\times\partial\Omega^i\times]-\epsilon_M,\epsilon_M[$, then by a result of paper [23, Prop. 4.1 (i)] with Musolino on the properties of integral operators with real analytic kernel, the map from $]-\epsilon_M,\epsilon_M[\times L^1(\partial\Omega^i)$ to $C^{m,\alpha}(\overline{\Omega_M})$ which takes (ϵ,f) to the map $\int_{\partial\Omega^i} S_{r,n}(\cdot-\epsilon\eta)f(\eta)d\sigma(\eta)$ is real analytic. Since S^i is real analytic and $C^{m-1,\alpha}(\partial\Omega^i)$ is continuously imbedded into $L^1(\partial\Omega^i)$, we conclude that the map from $]-\epsilon_M,\epsilon_M[\times\tilde{U}$ to $C^{m,\alpha}(\overline{\Omega_M})$ which takes $(\epsilon,\epsilon_1,\epsilon_2)$ to the third summand of the right-hand side of (5.6) is real analytic.

Similarly, the map from $]-\epsilon_M, \epsilon_M[\times\tilde{U}$ to $C^{m,\alpha}(\overline{\Omega_M})$ which takes $(\epsilon, \epsilon_1, \epsilon_2)$ to $\int_{\partial\Omega^i} S_{r,n}(\cdot - \epsilon\eta, k_o)\theta^\sharp(\eta)d\sigma_\eta$ is real analytic. Finally, C^i is real analytic by Theorem 4.53. Hence $\mathcal{U}_M$ is real analytic. Moreover, Theorems 4.32, 4.47, 4.53 imply that

$$\begin{aligned}
\mathcal{U}_M[0,0,0](x) &= v^+_{\Omega^o,h}\left[\Theta^o[0,0,0], k_o\right] \\
&\quad +\gamma_n(\log k_o)k_o^{n-2}v^+_{\Omega^o,J}\left[\Theta^o[0,0,0], k_o^2\right] \\
&\quad +\delta_{2,n}S_{r,n}(x,k_o)\left(\int_{\partial\Omega^i} S^i[0,0,0](\eta)d\sigma_\eta\right) \\
&\quad +\delta_{2,n}S_{r,n}(x,k_o)C^i[0,0,0]\int_{\partial\Omega^i}\theta^\sharp(\eta)d\sigma_\eta = v^+_{\Omega^o}\left[\tilde{\theta}^o, k_o\right] = \tilde{u}^o \quad \forall x\in\overline{\Omega_M}\,.
\end{aligned}$$

□

We note that Theorem 5.1 shows that if $x \in \overline{\Omega^o}\setminus\{0\}$ then for $\epsilon > 0$ small enough we can expand $u^o(\epsilon, x)$ into a convergent power series expansion of powers of ϵ when $n \geq 3$ is odd, of powers of ϵ, $\epsilon\log\epsilon$ when $n \geq 3$ is even and of powers of ϵ, $\epsilon\log\epsilon$, $\log^{-1}\epsilon$ for $n = 2$.

In this paper, we do not provide the algorithms to compute the coefficients of the power series expansion of $u^o(\epsilon, x)$. For this type of computations, we mention the work of Dalla Riva, Musolino and Rogosin [13] for the Laplace operator.

A Appendix A: A representation formula for the solutions of the Helmholtz equation.

Theorem A.1 *Let $n \in \mathbb{N}\setminus\{0,1\}$, $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0,1[$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Let $\{(\Omega^-)_j : j \in \{0,\ldots,\kappa^-\}\}$ denote the (finite) set of connected components of Ω^-. Let $(\Omega^-)_0$ be the only unbounded one. Let $k \in \mathbb{C}\setminus]-\infty,0]$, $\Im k \geq 0$. If for each $j \in \mathbb{N}\setminus\{0\}$ such that $j \leq \kappa^-$, k^2 is not a Dirichlet eigenvalue for $-\Delta$ on $(\Omega^-)_j$, and if k^2 is not a Neumann eigenvalue for $-\Delta$ in Ω, then the map from $C^{m-1,\alpha}(\partial\Omega)$ to the subspace*

$$V^{m,\alpha} \equiv \left\{u \in C^{m,\alpha}(\overline{\Omega}) : \Delta u + k^2 u = 0 \text{ in } \Omega\right\} \tag{A.2}$$

of $C^{m,\alpha}(\overline{\Omega})$, which takes ϕ to $v^+_\Omega[\phi,k]$ is a linear homeomorphism.

Proof. By the regularity properties of the single layer potential of Theorems 3.21 (i), 3.22 (ii), $v^+_\Omega[\cdot,k]$ is linear and continuous. By the Open Mapping Theorem it suffices to show that $v^+_\Omega[\cdot,k]$ is a bijection. If $u \in V^{m,\alpha}$, then the solvability condition for the Neumann problem under our assumptions on k and the Fredholm Alternative Theorem imply the existence of ϕ in $C^{m-1,\alpha}(\partial\Omega)$ such that $-\frac{1}{2}\phi + W^t_\Omega[\phi,k] = \frac{\partial u}{\partial \nu}$ on $\partial\Omega$ (cf. *e.g.*, Colton and Kress [8, Thms. 3.17, 3.20], and the regularity Theorem B.1 (i) in [22]). Then the jump formula implies that $\frac{\partial v^+_\Omega[\phi,k]}{\partial \nu} = \frac{\partial u}{\partial \nu}$. Since both $v^+_\Omega[\phi,k]$ and u solve the same Neumann problem for the Helmholtz equation our assumption on k implies that $u = v^+_\Omega[\phi,k]$ and thus the map of the statement is surjective. To prove the injectivity, we observe that if $\phi \in C^{m-1,\alpha}(\partial\Omega)$ and $v^+_\Omega[\phi,k] = 0$, then our assumptions on k and the uniqueness for the exterior Dirichlet problem with the radiation condition in $(\Omega^-)_0$ imply that $v^-_\Omega[\phi,k] = 0$ in Ω^- and that accordingly $\phi = \frac{\partial v^-_\Omega[\phi,k]}{\partial \nu} - \frac{\partial v^+_\Omega[\phi,k]}{\partial \nu} = 0$. □

In case $k = 0$ (the case of the Laplace equation), the following well known result holds (cf. *e.g.*, [15, Lem. 3.6]).

Theorem A.3 *Let $n \in \mathbb{N} \setminus \{0,1\}$, $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$. Let Ω be a bounded open connected subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Then the map from $C^{m-1,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ to $C^{m,\alpha}(\overline{\Omega})$ which takes (ϕ,ρ) to $v^+_{\Omega,h}[\phi,0] + \rho$ is a linear homeomorphism (cf. (4.11)).*

B Appendix B: existence and uniqueness theorem for a linear system of integral equations

We now prove a technical statement that we need in the paper.

Theorem B.1 *Let $n \in \mathbb{N}\setminus\{0,1\}$, $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0,1[$. Let Ω be a bounded open connected subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$ such that Ω^- is connected. Let $a, A, B \in]0,+\infty[$. Let (f,g) belong to $C^{m,\alpha}(\partial\Omega) \times C^{m-1,\alpha}(\partial\Omega)$. Then the system of integral equations*

$$\begin{cases} V_{\Omega,h}[\theta,0] - a\Big(V_{\Omega,h}[\psi,0] + c_1\Big) = f & \text{on } \partial\Omega \\ A\Big(\frac{1}{2}\theta + W^t_{\Omega,h}[\theta,0]\Big) + B\Big(\frac{1}{2}\psi - W^t_{\Omega,h}[\psi,0]\Big) = g & \text{on } \partial\Omega \end{cases} \tag{B.2}$$

has one and only one solution $(\theta,\psi,c_1) \in C^{m-1,\alpha}(\partial\Omega) \times C^{m-1,\alpha}(\partial\Omega)_0 \times \mathbb{R}$. If $\int_{\partial\Omega} g\, d\sigma = 0$, then we also have $\int_{\partial\Omega} \theta\, d\sigma = 0$.

Proof. We first observe that system (B.2) has a unique solution (θ, ψ, c_1) in $C^{m-1,\alpha}(\partial\Omega) \times C^{m-1,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ if and only if the system

$$\begin{cases} V_{\Omega,h}[\theta, 0] - (aA/B)\Big(V_{\Omega,h}[\psi_2, 0] + c_2\Big) = f \qquad \text{on } \partial\Omega \\ \Big(\frac{1}{2}\theta + W^t_{\Omega,h}[\theta, 0]\Big) + \Big(\frac{1}{2}\psi_2 - W^t_{\Omega,h}[\psi_2, 0]\Big) = g/A \qquad \text{on } \partial\Omega \end{cases} \tag{B.3}$$

has a unique solution $(\theta, \psi_2, c_2) \in C^{m-1,\alpha}(\partial\Omega) \times C^{m-1,\alpha}(\partial\Omega)_0 \times \mathbb{R}$. Indeed, it suffices to set $\psi_2 = B\psi/A$, $c_2 = Bc_1/A$. Now system (B.3) does have a unique solution (θ, ψ_2, c_2) in $C^{0,\alpha}(\partial\Omega) \times C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ by [10, Thm. 11.15]. By exploiting the very same proof of [10, Thm. 11.15] and by replacing the use of Theorem 6.47 of [10] that concerns case $m = 1$ with the above Theorem A.3 that covers case $m \geq 1$, the use of Lemma 11.14 of [10] that concerns case $m = 1$ with (4.15), (4.16) that cover case $m \geq 1$ and Theorem 6.51 of [10] that concerns case $m = 1$ with [10, Thm. 6.51], [20, Thm. 5.1] that cover case $m \geq 1$ we can prove that system (B.3) has a unique solution (θ, ψ_2, c_2) in $C^{m-1,\alpha}(\partial\Omega) \times C^{m-1,\alpha}(\partial\Omega)_0 \times \mathbb{R}$. The last part of the statement is an immediate consequence of the second equation of system (B.2) and of equality

$$\int_{\partial\Omega} \left(\frac{1}{2} I_\Omega + W^t_{\Omega,h}[\cdot, 0]\right)[\phi] d\sigma = \int_{\partial\Omega} \phi \; d\sigma \qquad \forall \phi \in C^{m-1,\alpha}(\partial\Omega) \tag{B.4}$$

(cf. *e.g.*, [10, Lem. 6.11]). □

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