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Semiregularity and connectivity of the non-$\mathfrak{F}$ graph of a finite group

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Given a class $\mathfrak{F}$ of finite groups, we consider the graph $\widetilde{\Gamma}_{\mathfrak{F}}(G)$ whose vertices are the elements of G and where two vertices $g, h \in G$ are adjacent if and only if $\langle g, h\rangle \notin \mathfrak{F}$. Moreover, we denote by $\mathcal{I}_{\mathfrak{F}}(G)$ the set of the isolated vertices of $\widetilde{\Gamma}_{\mathfrak{F}}(G)$ and by $\Gamma_{\mathfrak{F}}(G)$ the graph obtained from $\widetilde{\Gamma}_{\mathfrak{F}}(G)$ by deleting the isolated vertices. We address the following question: to what extent the fact that $\mathcal{I}_{\mathfrak{F}}(H)$ is a subgroup of H for any $H \leq G$, implies that the graph $\Gamma_{\mathfrak{F}}(G)$ is connected?

Keywords: Finite groups; soluble groups; graphs; connectivity.

Mathematics Subject Classification 2020: 20D60, 05E16

1. Introduction

Let $\mathfrak{F}$ be a class of finite groups and G a finite group. Consider the graph $\widetilde{\Gamma}_{\mathfrak{F}}(G)$ whose vertices are the elements of G and where two vertices $g, h \in G$ are adjacent if and only if $\langle g, h\rangle \notin \mathfrak{F}$. Moreover, denote by $\mathcal{I}_{\mathfrak{F}}(G)$ the set of isolated vertices of $\widetilde{\Gamma}_{\mathfrak{F}}(G)$. In [11], we defined the non-$\mathfrak{F}$ graph $\Gamma_{\mathfrak{F}}(G)$ of G as the subgraph of $\widetilde{\Gamma}_{\mathfrak{F}}(G)$ obtained by deleting the isolated vertices. In that paper, we concentrated our attention in the particular case when $\mathfrak{F}$ is a saturated formation. In particular, we addressed the question whether $\mathcal{I}_{\mathfrak{F}}(G)$ is a subgroup of G or not. This is not always the case, however it occurs for several saturated formations and more in general, we called semiregular a class $\mathfrak{F}$ with the property that $\mathcal{I}_{\mathfrak{F}}(G)$ is a subgroup

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of G for every finite group G. In the same paper, we called connected a class $\mathfrak{F}$ with the property that the graph $\Gamma_{\mathfrak{F}}(G)$ is connected for any finite group G. The results obtained in [11] indicate that often a semiregular formation is connected. This occurs for example for the formations of abelian groups, nilpotent groups, soluble groups, supersoluble groups, groups with nilpotent derived subgroup, groups with Fitting length less or equal then t for any $t \in \mathbb{N}$.

This stimulated us to investigate to what extent the semiregularity of $\mathfrak{F}$ implies its connectivity. We addressed the following more general question. Suppose that $\mathfrak{F}$ is a class containing only soluble groups and closed under taking subgroups and that a finite group G has the property that $\mathcal{I}_{\mathfrak{F}}(H)$ is a subgroup of H for any $H \leq G$. Does this implies that $\Gamma_{\mathfrak{F}}(G)$ is connected?

Our main theorem (Theorem 14) says that for a fixed class $\mathfrak{F}$, either the answer is affirmative, or a minimal counterexample G has a very peculiar behavior. Indeed, G is soluble, it cannot be generated with 2 elements and there exists an epimorphism $\pi : G \to V^t \rtimes H$, where H is 2-generated, V is a faithful irreducible H-module, $t = 1 + \dim_{\mathrm{End}_H(V)}(V)$, such that the following properties hold. Denote by $\mathcal{W}$ be the set of the H-submodules of V^t that are H-isomorphic to V^{t-1}, and for any $U \in \mathcal{W}$, let $M_U = \pi^{-1}(UH)$. Then there exists $W \in \mathcal{W}$ such that if $U \in \mathcal{W}\backslash\{W\}$, then there is no edge in $\Gamma_{\mathfrak{F}}(G)$ connecting two elements of M_U and one of the two following situations occurs:

(1) any edge in $\Gamma_{\mathfrak{F}}(G)$ belongs to the subgraph induced by a conjugate of M_W;
(2) H is cyclic of prime order and any egde in $\Gamma_{\mathfrak{F}}(G)$ belongs either to the subgraph induced by a conjugate of M_W or to the subgraph induced by $N = \pi^{-1}(V^t)$.

We construct two examples in which these two situations occur. In the first example, H is a quaternion group, $V \cong C_p \times C_p$, where p is an arbitrary odd prime, and $\Gamma_{\mathfrak{F}}(G)$ has p connected components, corresponding to the non-isolated vertices in the subgraphs of $\Gamma_{\mathfrak{F}}(G)$ induced by the p different conjugates of M_W. In the second example, $H \cong C_3$, $V \cong C_2 \times C_2$ and $\Gamma_{\mathfrak{F}}(G)$ have two connected components: one consisting of the non-isolated vertices of $\Gamma_{\mathfrak{F}}(N)$, the other one consisting of the union of the sets of non-isolated vertices in the four conjugates of M_W.

Despite the presence of these examples, Theorem 14 can be used to prove that the following classes of finite groups are connected: the class of finite groups whose order is divisible by only one prime, the class of finite groups with the property that the derived subgroup is a p-group, for a fixed prime p.

2. Proof of the Main Theorem

We begin this section by recalling some known results and proving some preliminary lemmas needed in the proof.

Proposition 1 ([6]). *Let N be a normal subgroup of a finite group G and suppose that $\langle g_1, \ldots, g_k \rangle N = G$. If $k \geq d(G)$, then there exist $n_1, \ldots, n_k \in N$ so that $\langle g_1 n_1, \ldots, g_k n_k \rangle = G$.*

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Recall that the generating graph of a finite group G is the graph whose vertices are the elements of G and where g_1 and g_2 are adjacent if and only if $\langle g_1, g_2\rangle = G$.

Lemma 2. *Let G be a 2-generated finite soluble group and denote by $\Omega(G)$ the set of non-isolated vertices of the generating graph of G. Assume that N is a minimal normal subgroup of G and that g_1, g_2 are two elements of G with $G = \langle g_1, g_2, N\rangle$. Then there exists $i \in \{1, 2\}$ with the property that $g_iN \subseteq \Omega(G)$.*

Proof. Suppose $g_1n^* \notin \Omega(G)$ for some $n^* \in N$. If $n \in N$, then $\langle g_1n^*, g_2n, N\rangle = G$, while $\langle g_1n^*, g_2n\rangle < G$. By [10, Lemma 1], there exists $n' \in N$ with $\langle g_1n', g_2n\rangle = G$. Hence, $g_2n \in \Omega(G)$. □

Definition 3. Let $\mathfrak{F}$ be a class of finite groups. We define $\mathfrak{F}_2$ as the class of the finite groups G with the property that any 2-generated subgroup of G is in $\mathfrak{F}$.

Assume that $\mathfrak{F}$ is closed under taking subgroups. We say that $\mathfrak{F}$ is 2-recognizable whenever a group G belongs to $\mathfrak{F}$ if all 2-generated subgroups of G belong to $\mathfrak{F}$. If $\mathfrak{F}$ is not 2-recognizable, then $\mathfrak{F}_2 > \mathfrak{F}$. For example, if $\mathfrak{F}$ is the class of the metabelian finite groups, and G is a Sylow 2-subgroup of $\mathrm{Sym}(8)$, then $G \in \mathfrak{F}_2\backslash\mathfrak{F}$. However, the following can be immediately seen.

Lemma 4. *If G is a finite group, then $\Gamma_{\mathfrak{F}}(G) = \Gamma_{\mathfrak{F}_2}(G)$. In particular, $\mathcal{I}_{\mathfrak{F}}(G) = \mathcal{I}_{\mathfrak{F}_2}(G)$.*

Lemma 5. *Let $\mathfrak{F}$ be a class of finite groups. If $\mathcal{I}_{\mathfrak{F}}(G)$ is a subgroup of G and $G = \mathcal{I}_{\mathfrak{F}}(G)\langle g\rangle$ for some $g \in G$, then $G \in \mathfrak{F}_2$.*

Proof. Let x be an arbitrary element of G. We have $x = ig^\alpha$ for some $i \in \mathcal{I}_{\mathfrak{F}}(G)$ and $\alpha \in \mathbb{Z}$. Moreover, $\langle g, ig^\alpha\rangle = \langle g, i\rangle \in \mathfrak{F}$, since $i \in \mathcal{I}_{\mathfrak{F}}(G)$. Hence, $g \in \mathcal{I}_{\mathfrak{F}}(G) = \mathcal{I}_{\mathfrak{F}_2}(G)$, so $G = \mathcal{I}_{\mathfrak{F}_2}(G)$ and therefore, $G \in \mathfrak{F}_2$. □

Let H be a subgroup of G. In the following, we will denote by H_G and H^G, respectively, the normal core and the normal closure of H in G, i.e. $H_G = \cap_{g\in G} H^g$ and $H^G = \langle H^g \mid g \in G\rangle$.

Theorem 6 ([5, Theorem 16.6]). *Let L and M be inconjugate maximal subgroups of a finite soluble group G. If $M_G \not\leq L_G$, then $L \cap M$ is a maximal subgroup of L.*

Definition 7. Let $\mathfrak{F}$ be a class of finite groups. We say that a finite group G is $\mathfrak{F}$-semiregular if $\mathcal{I}_{\mathfrak{F}}(H) \leq H$ for every $H \leq G$.

Lemma 8. *Let G be a finite $\mathfrak{F}$-semiregular soluble group. Suppose that M and L are maximal subgroups of G with $M, L \notin \mathfrak{F}_2$ and that the graphs $\Gamma_{\mathfrak{F}}(L)$ and $\Gamma_{\mathfrak{F}}(M)$ are connected. Denote by Γ_M (respectively, Γ_L) the connected component of $\Gamma_{\mathfrak{F}}(G)$ containing the vertices of $\Gamma_{\mathfrak{F}}(M)$ (respectively, $\Gamma_{\mathfrak{F}}(L)$).*

(a) *If* $L_G \not\leq M_G$, *then either* $\Gamma_L = \Gamma_M$ *or* $L \cap M \subseteq \mathcal{I}_{\mathfrak{F}}(L)$.
(b) *If* $M_G \not\leq L_G$ *and* $L_G \not\leq M_G$, *then* $\Gamma_L = \Gamma_M$.
(c) *If* $M_G < L_G$ *and* $\Gamma_L \neq \Gamma_M$, *then* L_G/M_G *is the unique minimal normal subgroup of* G/M_G.
(d) *If* $M_G < L_G$ *and* $\Gamma_L \neq \Gamma_M$, *then either* L *is normal in* G *or* L/L_G *is cyclic.*

Proof. (a) If $\Gamma_M \neq \Gamma_L$, then $L \cap M \subseteq \mathcal{I}_{\mathfrak{F}}(M) \cup \mathcal{I}_{\mathfrak{F}}(L)$. Thus, either $L \cap M \leq \mathcal{I}_{\mathfrak{F}}(L)$ or $L \cap M \leq \mathcal{I}_{\mathfrak{F}}(M)$. In the second case, by Theorem 6, $L \cap M$ is a a maximal subgroup of M, hence, $M = \langle L \cap M, m \rangle$ for some $m \in M$. But then $M = \mathcal{I}_{\mathfrak{F}}(M)\langle m \rangle$. By Lemma 5, $M \in \mathfrak{F}_2$, against the hypotheses.

(b) If $\Gamma_M \neq \Gamma_L$, then, by (a), $L \cap M \leq \mathcal{I}_{\mathfrak{F}}(L) \cap \mathcal{I}_{\mathfrak{F}}(M)$, but then $L, M \in \mathfrak{F}_2$, against the hypotheses.

(c) Since $L_G \not\leq M_G$, by (a) we have $K := L \cap M \leq \mathcal{I}_{\mathfrak{F}}(L)$. Let $N/M_G := \mathrm{soc}(G/M_G)$. Then $N \leq L_G$, since N/M_G is the unique minimal normal subgroup of G/M_G (see [5, Theorem 15.2]). In particular, $L = L \cap MN = (L \cap M)N = KN$, and therefore, $K^L = K^N$. If $L_G \cap M > M_G$, then $N \leq (L_G \cap M)^N \leq K^N = K^L$, and therefore, $L = KN = K^L \leq \mathcal{I}_{\mathfrak{F}}(L)$ and $L \in \mathfrak{F}_2$, against the assumptions. Hence, $L_G \cap M = M_G$, and consequently $L_G = N$.

(d) Suppose L is not normal in G. Let $U/L_G := \mathrm{soc}(G/L_G)$. As in the proof of (c), we have $L = (L \cap M)N = (L \cap M)L_G$. Since $L \notin \mathfrak{F}_2$ but, by (a) $L \cap M \leq \mathcal{I}_{\mathfrak{F}}(L)$, there exist $l \in L_G \setminus \mathcal{I}_{\mathfrak{F}}(L)$ and $s \in L$ such that $\langle l, s \rangle \notin \mathfrak{F}$. If $U\langle s \rangle < G$, then $U\langle s \rangle \leq S < G$ for a maximal subgroup S with $L_G < U \leq S_G$. Since $\langle l, s \rangle \leq L \cap S$, we have $\Gamma_L = \Gamma_S$ and from (c), $\Gamma_M = \Gamma_S = \Gamma_L$, against the assumptions. Therefore, $U\langle s \rangle = G$ and $G/U \cong L/L_G$ is cyclic. □

The following is immediate.

Lemma 9. *Let* $\mathfrak{F}$ *be a class of groups which is closed under taking subgroups and epimorphic images. Let* $g, h \in G$ *and* $N \trianglelefteq G$.

(a) *If* gN *and* hN *are adjacent vertices of* $\Gamma_{\mathfrak{F}}(G/N)$, *then* g *and* h *are adjacent vertices of* $\Gamma_{\mathfrak{F}}(G)$.
(b) *If* $g \in \mathcal{I}_{\mathfrak{F}}(G)$, *then* $gN \in \mathcal{I}_{\mathfrak{F}}(G/N)$.
(c) $\mathcal{I}_{\mathfrak{F}}(G)^\sigma = \mathcal{I}_{\mathfrak{F}}(G)$ *for every* $\sigma \in \mathrm{Aut}(G)$.

Lemma 10. *Let* $\mathfrak{F}$ *be a class of finite groups with the following properties.*

(1) *All the groups in* $\mathfrak{F}$ *are soluble.*
(2) $\mathfrak{F}$ *is closed under taking subgroups.*

Suppose that G *is* $\mathfrak{F}$*-semiregular and* $\Gamma_{\mathfrak{F}}(H)$ *is connected for any proper subgroup* H *of* G. *If there exists a proper normal subgroup* N *of* G *such that* $G \setminus N$ *is contained in a unique connected component of* $\Gamma_{\mathfrak{F}}(G)$, *then either* $\Gamma_{\mathfrak{F}}(G)$ *is connected or* N *is a maximal subgroup of* G.

Proof. Let Ω be the connected component of $\Gamma_{\mathfrak{F}}(G)$ containing $G \setminus N$.

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If $N \in \mathfrak{F}_2$, then every element of $N \backslash \mathcal{I}_{\mathfrak{F}}(G)$ must be adjacent to an element of $G \backslash N$, so $N \backslash \mathcal{I}_{\mathfrak{F}}(G) \subseteq \Omega$. But this implies $\Omega = G \backslash \mathcal{I}_{\mathfrak{F}}(G)$, and consequently $\Gamma_{\mathfrak{F}}(G)$ is connected.

If $N \notin \mathfrak{F}_2$, then $\mathcal{I}_{\mathfrak{F}}(N) < N$. Let H be a maximal subgroup of G containing N. Since $\Gamma_{\mathfrak{F}}(H)$ is connected, there exists a unique connected component of $\Gamma_{\mathfrak{F}}(G)$, say Δ, containing $H \backslash \mathcal{I}_{\mathfrak{F}}(H)$. Of course $N \backslash \mathcal{I}_{\mathfrak{F}}(N) \subseteq H \backslash \mathcal{I}_{\mathfrak{F}}(H)$, so $N \backslash \mathcal{I}_{\mathfrak{F}}(N) \subseteq \Delta$. Recall that $G \backslash N \subseteq \Omega$. Moreover, if $x \in \mathcal{I}_{\mathfrak{F}}(N) \backslash \mathcal{I}_{\mathfrak{F}}(G)$, then $\langle x, y \rangle \notin \mathfrak{F}$ for some $y \in G \backslash N$, so $\mathcal{I}_{\mathfrak{F}}(N) \backslash \mathcal{I}_{\mathfrak{F}}(G) \subseteq \Omega$. If $\Delta \cap \Omega \neq \emptyset$, then $\Delta = \Omega = G \backslash \mathcal{I}_{\mathfrak{F}}(G)$ and $\Gamma_{\mathfrak{F}}(G)$ is connected. If $\Delta \cap \Omega = \emptyset$, then $(H \backslash \mathcal{I}_{\mathfrak{F}}(H)) \cap (H \backslash N) = \emptyset$, i.e. $H = N \cup \mathcal{I}_{\mathfrak{F}}(H)$. Since $H \notin \mathfrak{F}_2$, $\mathcal{I}_{\mathfrak{F}}(H) \neq H$, and consequently $H = N$. □

Lemma 11. *Let $\mathfrak{F}$ be a class of finite groups with the following properties*:

(1) *All the groups in $\mathfrak{F}$ are soluble.*
(2) *$\mathfrak{F}$ is closed under taking subgroups.*

Suppose that G is $\mathfrak{F}$-semiregular and $\Gamma_{\mathfrak{F}}(H)$ is connected for any proper subgroup H of G. If G is not soluble, then $\Gamma_{\mathfrak{F}}(G)$ is connected.

Proof. Let $R = R(G)$ be the soluble radical of G and let $\mathfrak{S}$ be the class of the finite soluble groups. By our hypothesis, $R < G$. By [7, Theorem 6.4], the only isolated vertex of $\tilde{\Gamma}_{\mathfrak{S}}(G/R)$ is the identity element and the graph $\Gamma_{\mathfrak{S}}(G/R)$ is connected. By Lemma 9, all the elements of $G \backslash R$ belong to the same connected component of $\Gamma_{\mathfrak{S}}(G)$. Since $\mathfrak{F} \subseteq \mathfrak{S}$, there exists a connected component, say Ω, of $\Gamma_{\mathfrak{F}}(G)$ with $G \backslash R \subseteq \Omega$. Since R cannot be a maximal subgroup of G, it follows from Lemma 10 that $\Gamma_{\mathfrak{F}}(G)$ is connected. □

Lemma 12. *Let $\mathfrak{F}$ be a class of finite groups with the following properties*:

(1) *All the groups in $\mathfrak{F}$ are soluble.*
(2) *$\mathfrak{F}$ is closed under taking subgroups.*

Suppose that G is $\mathfrak{F}$-semiregular and $\Gamma_{\mathfrak{F}}(H)$ is connected for any proper subgroup H of G. If G is a 2-generated group, then $\Gamma_{\mathfrak{F}}(G)$ is connected.

Proof. Assume, by contradiction, that the statement is false. By the previous lemma, we may assume that G is soluble. Moreover, we may assume that if $g \in G$, then $\langle g \rangle \in \mathfrak{F}$ (otherwise g would be a universal vertex of $\Gamma_{\mathfrak{F}}(G)$ and consequently $\Gamma_{\mathfrak{F}}(G)$ would be obviously connected). Finally, we may assume $G \notin \mathfrak{F}_2$ (otherwise $\Gamma_{\mathfrak{F}}(G)$ is the empty graph). This implies that the subgraph $\Delta(G)$ of the generating graph of G induced by its non-isolated vertices is a subgraph of $\Gamma_{\mathfrak{F}}(G)$. By [3, Theorem 1], the graph $\Delta(G)$ is connected, so there exists a connected component, say Γ, of $\Gamma_{\mathfrak{F}}(G)$ containing the set $\Omega(G)$ of the vertices of $\Delta(G)$. From now on, for any subgroup H of G, we will denote by $\Omega_{\mathfrak{F}}(H)$ the set $H \backslash \mathcal{I}_{\mathfrak{F}}(H)$ of the vertices of $\Gamma_{\mathfrak{F}}(H)$. Since $\Gamma_{\mathfrak{F}}(G)$ is disconnected, we must have $\Omega_{\mathfrak{F}}(G) \neq \Gamma$.

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Let $g \in \Omega_{\mathfrak{F}}(G)\backslash\Gamma$. Then there exists $\tilde{g} \in G$ such that $\langle g, \tilde{g}\rangle \notin \mathfrak{F}$. Since $g \notin \Omega(G)$, $\langle g, \tilde{g}\rangle \neq G$ and therefore, there exists a maximal subgroup M of G with $g \in \Omega_{\mathfrak{F}}(M)$. We have

$$\Omega_{\mathfrak{F}}(M) \cap \Omega(G) = \emptyset. \tag{2.1}$$

Indeed, assume $m \in \Omega_{\mathfrak{F}}(M) \cap \Omega(G)$. Since $m, g \in \Omega_{\mathfrak{F}}(M)$, then g and m would be contained in the same connected component of $\Gamma_{\mathfrak{F}}(M)$, and consequently in the same connected component of $\Gamma_{\mathfrak{F}}(G)$. However, since $m \in \Omega$, this connected component would coincide with Γ, in contradiction with $g \notin \Gamma$.

We distinguish two cases:

(a) M is a normal subgroup of G. In this case, $\mathcal{I}_{\mathfrak{F}}(M)$ is a characteristic subgroup of M and therefore, it is normal in G. Moreover, $G = \langle y, M\rangle$ for some $y \in G$. Since G is 2-generated, by Proposition 1, there exist $m_1, m_2 \in M$ such that $G = \langle ym_1, m_2\rangle$. Since $\Omega_{\mathfrak{F}}(M) \neq \emptyset$, we must have $M \notin \mathfrak{F}_2$ and therefore, by Lemma 5, $M/\mathcal{I}_{\mathfrak{F}}(M)$ is not cyclic. This implies $m_2 \notin \mathcal{I}_{\mathfrak{F}}(M)$. Hence, $m_2 \in \Omega_{\mathfrak{F}}(M) \cap \Omega(G)$, against (2.1).

(b) M is not a normal subgroup of G. Let M_G be the normal core of M in G. We have $G/M_G \cong A/M_G \rtimes M/M_G$, where A/M_G is a faithful irreducible (M/M_G)-module. By [4, Theorem 7] there exist $k_1, k_2 \in M$ and $a \in A$ such that $G = \langle k_1, k_2^a\rangle M_G$. By Proposition 1, there exist $m_1, m_2 \in M_G$ such that $G = \langle k_1 m_1, k_2^a m_2\rangle$. Hence,

$$G = \langle x, y^a\rangle \quad \text{with } x = k_1 m_1, \quad y = k_2 m_2^{a^{-1}} \in M \quad \text{and} \quad a \in A.$$

Now we claim that M_G is contained in $\mathcal{I}_{\mathfrak{F}}(M)$. Consider a normal series

$$1 = U_0 \leq U_1 \leq \cdots \leq U_t = M_G,$$

where, for $1 \leq i \leq t$, U_i/U_{i-1} is a chief factor of G. We prove by induction on i that $U_i \leq \mathcal{I}_{\mathfrak{F}}(M)$. Since $\mathfrak{F}$ contains all cyclic subgroups of G, $U_0 = 1 \leq \mathcal{I}_{\mathfrak{F}}(M)$. Let $i < t$ and assume $U_i \leq \mathcal{I}_{\mathfrak{F}}(M)$. It follows from Lemma 2 that there are two possibilities: either $xuU_i \in \Omega(G/U_i)$ for any $u \in U_{i+1}$ or $y^a uU_i \in \Omega(G/U_i)$ for any $u \in U_{i+1}$. In the first case, by Proposition 1, for any $u \in U_{i+1}$ there exists $\bar{u} \in U_i$ such that $xu\bar{u} \in \Omega(G)$. Hence, by (2.1), $xu\bar{u} \in \mathcal{I}_{\mathfrak{F}}(M)$. So $\langle xu \mid u \in U_{i+1}\rangle \subseteq \mathcal{I}_{\mathfrak{F}}(M)$ and consequently $U_{i+1} \subseteq \mathcal{I}_{\mathfrak{F}}(M)$. The same argument can be applied in the second case. So we have proved $M_G \leq \mathcal{I}_{\mathfrak{F}}(M)$. Since $x \in \Omega(G)$, it follows from 2.1 that $x \in \mathcal{I}_{\mathfrak{F}}(M)$. But then, since $M_G \leq \mathcal{I}_{\mathfrak{F}}(M)$, $M = \langle x, y, M_G\rangle = \mathcal{I}_{\mathfrak{F}}(M)\langle y\rangle$, hence, $M \in \mathfrak{F}_2$ by Lemma 5, in contradiction with $g \in \Omega_{\mathfrak{F}}(M)$. □

Lemma 13. *Let H be a 2-generated finite soluble group and V a faithful irreducible H-module. Consider the semidirect product $X = V^u \rtimes H$, with $u = \dim_{\mathrm{End}_H(V)}(V)$. Suppose $H = \langle h_1, h_2\rangle$. Then there exists $w \in V^u \rtimes H$ such that $X = \langle h_1, h_2 w\rangle$ if and only if h_1 acts fixed-point-freely on V.*

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Proof. Let $F := \mathrm{End}_H(V)$. We may identify $H = \langle h_1, h_2\rangle$ with a subgroup of the general linear group $\mathrm{GL}(u, F)$. In this identification, for $i = 1, 2$, h_i becomes an $u \times u$ matrix X_i with coefficients in F. Denote by A_i the matrix $I_u - X_i$. Let $w = (v_1, \dots, v_u) \in V^u$. Then every v_j can be viewed as a $1 \times u$ matrix. Denote the $u \times u$ matrix with rows $v_1, \dots, v_u$ by B. By [2, Sec. 4], $\langle h_1, wh_2\rangle = X$ if and only if

$$\det\begin{pmatrix} A_1 & A_2 \\ 0 & B \end{pmatrix} \neq 0. \tag{2.2}$$

A matrix B so that (2.2) is satisfied can be found if and only if A_1 is invertible, i.e. if and only if h_1 acts fixed-point-freely on V. □

Theorem 14. *Let $\mathfrak{F}$ be a class of finite groups with the following properties*:

(1) *All the groups in $\mathfrak{F}$ are soluble.*
(2) *$\mathfrak{F}$ is closed under taking subgroups.*

Suppose that G is a finite group which is minimal with respect to the following properties: *G is $\mathfrak{F}$-semiregular and the graph $\Gamma_{\mathfrak{F}}(G)$ is not connected. Then G is soluble and there exists an epimorphism*

$$\pi : G \to (V_1 \times \cdots \times V_t) \rtimes H,$$

where H is 2-generated and there exists a faithful irreducible H-module V with $V_i \cong_H V$ for $1 \le i \le t$ and $t = 1 + \dim_{\mathrm{End}_H(V)}(V)$.

Moreover, let $\mathcal{W}$ be the set of the H-submodules of $V_1 \times \cdots \times V_t$ that are H-isomorphic to V^{t-1}. There exists one and only one $W \in \mathcal{W}$ with the property that $M = \pi^{-1}(W \rtimes H) \notin \mathfrak{F}_2$. If $g_1, g_2 \in G$ and $\langle g_1, g_2\rangle \notin \mathfrak{F}$, then either $\langle g_1, g_2\rangle \le M^x$, for some $x \in G$, or H is cyclic of prime order and $\langle g_1, g_2\rangle \le \pi^{-1}(V_1 \times \cdots \times V_t)$.

Proof. For a finite group X, let $d(X)$ be the smallest cardinality of a generating set of X. By the previous lemmas, G is soluble and $d(G) \ge 3$. In particular, G contains a normal subgroup N with the property that $d(G/N) = 3$ but $d(G/M) \le 2$ for every normal subgroup M of G properly containing N. By [12] $G/N \cong V^t \rtimes H$, where V is a faithful irreducible H-module and

$$t = 1 + \dim_{\mathrm{End}_H(V)}(V).$$

Consider the epimorphism $\pi : G \to V^t \rtimes H$ and let $\mathcal{M}$ be the set of the maximal subgroups of G containing N and with the property that M^π does not contain V^t. If $M \in \mathcal{M}$, then $M/N \cong V^{t-1} \rtimes H$. Let $F := \mathrm{End}_H(V)$. The multiplicity of V in $FH/J(FH)$ (where FH denote the group algebra and $J(FH)$ its Jacobson radical) is $t - 1$ so it follows from [1, Lemma 1] that the smallest cardinality of a subset of V^t generating V^t as an H-module is $\lceil t/(t-1)\rceil = 2$. In particular, there is no $w \in W$ with the property that $\langle w\rangle_H = V^t$, denoting with $\langle w\rangle_H$ the H-submodule of V^t generated by w. If $g \in G$, then $g^\pi = wh$ for some $w \in V^t$ and $h \in H$, so $g^\pi \le \langle w, h\rangle \le \langle w\rangle_H H$, and therefore, there exists $M \in \mathcal{M}$ with $g \in M$.

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Assume that there exists no $M \in \mathcal{M}$ containing two adjacent vertices of $\Gamma_{\mathfrak{F}}(G)$. Let T be the preimage of H under π. We must have $T > N$ (otherwise if g_1, g_2 are adjacent vertices, then $\langle g_1, g_2\rangle N$ is a proper subgroup of G and it is contained in some $M \in \mathcal{M}$). In particular, $T^G = G$. Fix $l \in T$. For any $g \in G$, we have $g^\pi = wh$, with $w \in V^t$ and $h \in H$. Thus, $\langle l, g\rangle^\pi = \langle l^\pi, wh\rangle \leq \langle w\rangle_H H$. Since, $\langle w\rangle_H \neq V^t$, there exists $M \in \mathcal{M}$ containing $\langle l, g\rangle$. In particular, l and g are not adjacent in $\Gamma_{\mathfrak{F}}(G)$ and therefore, $l \in \mathcal{I}_{\mathfrak{F}}(G)$. Thus, $T \leq \mathcal{I}_{\mathfrak{F}}(G)$. But then $G = T^G = \mathcal{I}_{\mathfrak{F}}(G)$, a contradiction.

So we may assume that there exists $K \in \mathcal{M}$ containing two adjacent vertices g_1, g_2 of $\Gamma_{\mathfrak{F}}(G)$. In particular, $K \notin \mathfrak{F}_2$. Let $\tilde{\mathcal{M}} = \{M \in \mathcal{M} \mid M \notin \mathfrak{F}_2\}$. If $M_1, M_2 \in \tilde{\mathcal{M}}$ and $(M_1)_G \neq (M_2)_G$ then if follows from Lemma 8(b), that $\Gamma_{M_1} = \Gamma_{M_2}$.

Suppose that there exists $K^* \in \tilde{\mathcal{M}}$, with $K^*_G \neq K_G$. By the previous paragraph, $\Gamma_K = \Gamma_{K^*}$. We claim that in this case Γ_K contains all the vertices of $\Gamma_{\mathfrak{F}}(G)$, in contradiction with the assumption that $\Gamma_{\mathfrak{F}}(G)$ is disconnected. Assume that g is a vertex of $\Gamma_{\mathfrak{F}}(G)$. There exists $\tilde{g}$ such that $\tilde{X} = \langle g, \tilde{g}\rangle \notin \mathfrak{F}$. Let L be a maximal subgroup of G containing $\tilde{X}N$. If $L \in \mathcal{M}$, then $g \in \Gamma_L = \Gamma_K = \Gamma_{K^*}$ by Lemma 8(b). If $L \notin \mathcal{M}$, then, up to conjugacy, we may assume $K^\pi = W_1 H$, $(K^*)^\pi = W_2 H$, $L^\pi = V^t X$, with $W_1 \neq W_2$, $W_1 \cong_H W_2 \cong_H V^{t-1}$ and $X \leq H$. By Lemma 8(b), either $\Gamma_L = \Gamma_K = \Gamma_{K^*}$, or $\langle K \cap L, K^* \cap L\rangle \leq \mathcal{I}_{\mathfrak{F}}(L)$. In the latter case, $\langle (K \cap L)^\pi, (K^* \cap L)^\pi\rangle = \langle W_1 X, W_2 X\rangle = V^t X = L^\pi$, but then $\mathcal{I}_{\mathfrak{F}}(L) = L$, a contradiction.

So from now on, we will assume that all the maximal subgroups in $\tilde{\mathcal{M}}$ are conjugate to K. It is not restrictive to assume $K^\pi = (V_1 \times \cdots \times V_{t-1})H$. Suppose now that H is not cyclic of prime order and by contradiction, that there exist $g_1, g_2 \in G$ such that $\langle g_1, g_2\rangle \notin \mathfrak{F}$ and $\langle g_1, g_2\rangle$ is contained in no conjugate of K. There exists a maximal subgroup J of G with $\langle g_1, g_2\rangle \leq J$ and $J^\pi \geq V_1 \times \cdots \times V_t$. We claim that, for any $g \in G$, $\Gamma_J = \Gamma_{K^g}$. Suppose that this is false. By Lemmas 8(c) and 8(d), $J^\pi = V^t C$ where C is a cyclic maximal subgroup of H and $H = CS$ with $S = \operatorname{soc} H$. Let $\mathcal{U}$ be the set of the H-submodules U of $V_1 \times \cdots \times V_t$ with $U \cong_H V$ and $U \not\leq V_1 \times \cdots \times V_{t-1}$. If $\pi^{-1}(U) \leq \mathcal{I}_{\mathfrak{F}}(J)$ for every $U \in \mathcal{U}$, then $R := \langle \pi^{-1}(U) \mid U \in \mathcal{U}\rangle = \pi^{-1}(V_1 \times \cdots \times V_t) \subseteq \mathcal{I}_{\mathfrak{F}}(J)$. Since J/R is cyclic, by Lemma 5, we would have $J \in \mathfrak{F}_2$. So there exist $U \in \mathcal{U}$ and $y_1 \notin \mathcal{I}_{\mathfrak{F}}(J)$ with $y_1^\pi = u \in U$. There exists $y_2 \in J$ with $\langle y_1, y_2\rangle \notin \mathfrak{F}$. Let $y_2^\pi = wc$, with $w \in V^t$ and c in C. If $\langle c\rangle \neq C$, then let L be a maximal subgroup of G containing $\pi^{-1}((V_1 \times \cdots \times V_t)S\langle c\rangle)$. Clearly, $\Gamma_L = \Gamma_J$ since $\{y_1, y_2\} \subseteq \Gamma_L \cap \Gamma_J$. However, by Lemma 8(c), $\Gamma_L = \Gamma_{K^g}$. Hence, $\Gamma_J = \Gamma_{K^g}$, independently of the choice of g. Thus, $\langle c\rangle = C$. There exists a conjugate $\bar{c}$ of c in H such that $H = \langle c, \bar{c}\rangle$. If c acts fixed-point-freely on V, then, up to conjugacy, we may assume $w \in U$, but then $\langle y_1, y_2\rangle \leq \pi^{-1}(UC)$ is contained in a maximal subgroup $M \in \mathcal{M}$, which is not conjugate to K since $U \leq M_G$ but $U \not\leq K_G$. Finally, assume that c does not act fixed-point-freely. We have $w = w_1 + w_2$, with $w_1 \in U$ and $w_2 \in (V_1 \times \cdots \times V_{t-1})$. Since $\bar{c}$, being a conjugate of c, does not act fixed-point-freely of V, by Lemma 13, $\bar{c}$ is an isolated vertex in $(V_1 \times \cdots \times V_{t-1})H$, so $\langle w_2 c, \bar{c}\rangle$ is a proper subgroup of $(V_1 \times \cdots \times V_{t-1})H$ supplementing $V_1 \times \cdots \times V_{t-1}$.

Thus, there exists an H-submodule W of $V_1 \times \cdots \times V_{t-1}$ such that $W \cong V^{t-2}$ and $\langle w_2 c, \bar{c}\rangle \leq WH$. But then $M = \pi^{-1}(UWH) \in \mathcal{M}$, M is not conjugate to K and $\langle y_1, y_2\rangle \leq M$. □

Definition 15. Let $\mathfrak{F}$ be a class of finite groups. We say that a finite group G is strongly $\mathfrak{F}$-semiregular if $\mathcal{I}_{\mathfrak{F}}(X/Y) \leq X/Y$ for every $X \leq G$ and $Y \trianglelefteq X$.

Lemma 16. *Let $\mathfrak{F}$ be a class of finite groups with the following properties*:

(1) *All the groups in $\mathfrak{F}$ are soluble.*
(2) *$\mathfrak{F}$ is closed under taking subgroups.*
(3) *$\mathfrak{F}$ is closed under taking epimorphic images.*

Suppose that G is strongly $\mathfrak{F}$-semiregular and $\Gamma_{\mathfrak{F}}(H)$ is connected for any proper subgroup and any proper quotient H of G. If $\Gamma_{\mathfrak{F}}(G)$ is disconnected, then $G/M \in \mathfrak{F}_2$ for every $1 \neq M \trianglelefteq G$.

Proof. Suppose that there exists $1 \neq M \trianglelefteq G$ such that $G/M \notin \mathfrak{F}_2$. Let $I/M := \mathcal{I}_{\mathfrak{F}}(G/M) \trianglelefteq G/M$. If $g_1M, g_2M \notin I/M$, then by the hypotheses, they belong to the same connected component of the graph $\Gamma_{\mathfrak{F}}(G/M)$. Since, by Lemma 9, a path from g_1M to g_2M in $\Gamma_{\mathfrak{F}}(G/M)$ can be lifted to a path between g_1 and g_2 in $\Gamma_{\mathfrak{F}}(G)$, it follows that $G\backslash I$ is contained in a unique connected component of $\Gamma_{\mathfrak{F}}(G)$. By Lemma 10, either $\Gamma_{\mathfrak{F}}(G)$ is connected or there exists $g \in G$ such that $\langle g\rangle I = G$. However, in the second case, $\langle gM\rangle I/M = \langle gM\rangle \mathcal{I}_{\mathfrak{F}}(G/M) = G/M$ and from Lemma 5 it would follow $G/M \in \mathfrak{F}_2$, against the assumption. □

Lemma 17. *Let $\mathfrak{F}$ be a class of finite groups with the following properties*:

(1) *All the groups in $\mathfrak{F}$ are soluble.*
(2) *$\mathfrak{F}$ is closed under taking subgroups.*
(3) *$\mathcal{I}_{\mathfrak{F}}(G)$ is a subgroup of G for every finite group G.*
(4) *If $G_1 \in \mathfrak{F}_2$ and $|G_2| = |G_1|$, then $G_2 \in \mathfrak{F}_2$.*

Then $\Gamma_{\mathfrak{F}}(G)$ is connected for every finite group G.

Proof. Let G be a counterexample of minimal order. According to the statement of Theorem 14, there exists an epimorphism $\pi : G \to (V_1 \times \cdots \times V_t) \rtimes H$, where H is 2-generated and there exists a faithful irreducible H-module V with $V_i \cong_H V$ for $1 \leq i \leq t$ and $t = 1 + \dim_{\mathrm{End}_H(V)}(V)$. Moreover, there exists $W \in \mathcal{W}$ with $\pi^{-1}(W \rtimes H) \notin \mathfrak{F}_2$, while $\pi^{-1}(W^* \rtimes H) \in \mathfrak{F}_2$ whenever $W \neq W^* \in \mathcal{W}$. Since $|\pi^{-1}(W \rtimes H)| = |\pi^{-1}(W^* \rtimes H)|$, this contradicts (4). □

3. Applications

Proposition 18. *Let $\mathfrak{P}$ be the class of the finite groups whose order is divisible by at most a unique prime. Then for any finite group G the graph $\Gamma_{\mathfrak{P}}(G)$ is connected.*

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Proof. This is an elementary result, that can be easily proved, but it can be also deduced from Lemma 17. We may assume that all the elements of G have prime power order (otherwise $\Gamma_{\mathfrak{F}}(G)$ contains a universal vertex). In this case, let $H \leq G$. We have $\mathcal{I}_{\mathfrak{P}}(H) = H$ if H is a p-group, $\mathcal{I}_{\mathfrak{P}}(H) = 1$ otherwise. In any case, $\mathcal{I}_{\mathfrak{P}}(H)$ is a subgroup of H and therefore, $\Gamma_{\mathfrak{F}}(G)$ is connected by Lemma 17. □

Let $\mathfrak{A}_p$ be the class of the finite groups G with the property that $G/\mathrm{O}_p(G)$ is abelian. It can be easily checked that $\mathfrak{A}_p$ is a subgroup closed formation.

Lemma 19. *Let G be a finite group. Then $\mathrm{O}_p(G) \subseteq \mathcal{I}_{\mathfrak{A}_p}(G)$ and $\mathcal{I}_{\mathfrak{A}_p}(G)/\mathrm{O}_p(G) = Z(G/\mathrm{O}_p(G))$. Hence, $\mathfrak{A}_p$ is semiregular and $(\mathfrak{A}_p)_2 = \mathfrak{A}_p$.*

Proof. Let $Z/\mathrm{O}_p(G) = Z(G/\mathrm{O}_p(G))$. Clearly, $\mathrm{O}_p(G) \leq Z \subseteq \mathcal{I}_{\mathfrak{A}_p}(G)$. Conversely assume $g \in \mathcal{I}_{\mathfrak{A}_p}(G)$. Let $F^*/\mathrm{O}_p(G) = F^*(G/\mathrm{O}_p(G))$ and $F/\mathrm{O}_p(G) = F(G/\mathrm{O}_p(G))$ be, respectively, the generalized Fitting subgroup and the Fitting subgroup of $G/\mathrm{O}_p(G)$. Let $\mathfrak{S}$ be the class of the finite soluble groups and let $y \in F^*$ (see [9, Sec. 9.4] for the definition and the main properties of the generalized Fitting subgroup). Since $\mathcal{I}_{\mathfrak{A}_p}(G) \subseteq \mathcal{I}_{\mathfrak{S}}(G) = R(G)$, we have that $[g, y] \in R(G) \cap F^* = R(F^*) = F$. In particular, $[g, y] \in \mathrm{O}_p(\langle g, y\rangle)$ is a p-element of F. Since $\mathrm{O}_p(G)$ is the Sylow p-subgroup of F, it follows that $[g, y] \in \mathrm{O}_p(G)$. Hence, $g\,\mathrm{O}_p(G) \in C_{G/\mathrm{O}_p(G)}(F^*(G/\mathrm{O}_p(G)))$. By [9, Theorem 9.8],

$$C_{G/\mathrm{O}_p(G)}(F^*(G/\mathrm{O}_p(G))) \leq Z(F^*(G/\mathrm{O}_p(G))) \leq F/\mathrm{O}_p(G),$$

hence, $g\,\mathrm{O}_p(G) \in C_{G/\mathrm{O}_p(G)}(F^*(G/\mathrm{O}_p(G))) \leq F/\mathrm{O}_p(G)$. So, $g \in F$ and therefore, for any $x \in G$, $[g, x]$ is a p-element of F, hence, $[g, x] \in \mathrm{O}_p(G)$ and $g \in Z$. □

Proposition 20. *For any finite group G the graph $\Gamma_{\mathfrak{A}_p}(G)$ is connected.*

Proof. Let G be a counterexample of minimal order. We apply Theorem 14. Let $\pi : G \to (V_1 \times \cdots \times V_t) \rtimes H$ as described in the statement and let $N = \ker \pi$. By Lemma 16, $G/M \in \mathfrak{A}_p$ for every $1 \neq M \trianglelefteq G$. This implies in particular $\mathrm{O}_p(G) = 1$. Let $U \in \mathcal{W}$ with $U \neq W$. Then $Y = \pi^{-1}(U \rtimes H) \in \mathfrak{A}_p$. In particular, $K = \pi^{-1}(U) \in \mathfrak{A}_p$. Since $K \trianglelefteq G$ and $\mathrm{O}_p(G) = 1$, it follows that $\mathrm{O}_p(K) = 1$ and consequently K is abelian and p does not divide $|K|$. In particular, $|V_1|$ is not a p-group. But $V_1 \rtimes H \in \mathfrak{A}_p$ (it is a proper epimorphic image of G), hence, $H = 1$. Thus, p does not divide $|G|$ and $\Gamma_{\mathfrak{A}_p}(G) = \Gamma_{\mathfrak{A}}(G)$ is connected. □

4. Example 1

The statement of Theorem 14 cannot be improved. In this section, we exhibit a class $\mathfrak{F}$ and a finite group G containing a maximal subgroup M with the property that the connected components of $\Gamma_{\mathfrak{F}}(G)$ are precisely the subgraphs induced by the conjugates of M in G.

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Let Q be the quaternion group and $V \cong C_3 \times C_3$ a faithful irreducible Q-module. Consider the semidirect product $X = (V_1 \times V_2 \times V_3) \rtimes Q$, with $V_i \cong_Q V$ for $1 \leq i \leq 3$. Let $T = V_1 \times V_2 \times V_3$ and $\mathcal{U} = \{U_1, \ldots, U_9\}$ be the set of submodules $U \leq_Q T$ with the properties that $T = (V_1 \times V_2) \oplus U$. Fix 9 different primes $p_1, \ldots, p_9$ with the property that 12 divides $p_i - 1$ for $1 \leq i \leq 9$. Then $Y = V^2 \rtimes Q$ has a faithful irreducible action on a vector space of dimension 8 over the field $\mathbb{F}_{p_i}$ with p_i elements. So for $1 \leq i \leq 9$, we may consider an irreducible X-module, say W_i, with $W_i \cong C_{p_i}^8$ and $C_X(W_i) = U_i$. Now consider the semidirect product

$$G = (W_1 \times \cdots \times W_9) \rtimes X.$$

Let $W = W_1 \times \cdots \times W_9$. Let $\mathcal{M}$ be the set of the maximal subgroups M with the property that $W \leq M$ but $T \not\leq M$. The conjugacy classes of maximal subgroups in $\mathcal{M}$ are parametrized by the set $\mathcal{Z}$ of the Q-submodules Z of T with $Z \cong_Q V^2$. Indeed, $\mathcal{M}$ is the disjoint union of the following families:

$$\mathcal{M}_Z = \{WZQ^t \mid t \in T\}, \quad \text{with } Z \in \mathcal{Z}.$$

Let $Z^* = V_1 \times V_2$. Since $Z^* \oplus U_i = T$, it follows that W_i is a faithful irreducible M/W-module for any $M \in \mathcal{M}_{Z^*}$ and $1 \leq i \leq 9$. Suppose that $Z^* \neq Z \in \mathcal{Z}$. There exists $i \in \{1, \ldots, 9\}$ such that $U_i \leq Z$. This implies that if $M \in \mathcal{M}_Z$ then $C_M(W_i) > W$ and consequently the maximal subgroups in $\mathcal{M}_Z$ are not isomorphic to the ones in $\mathcal{M}_{Z^*}$.

Consider now the class $\mathfrak{F}$ of the finite groups with no subgroup isomorphic to $M = WZ^*Q$.

Notice that M is 2-generated, while G is not. In particular, if $g_1, g_2 \in G$, then there is an edge $(g_1, g_2) \in \Gamma_{\mathfrak{F}}(G)$ if and only if $\langle g_1, g_2 \rangle = M^t \in \mathcal{M}_{Z^*}$, for some $t \in V_3$.

For a finite group Y, denote by $\Omega(Y)$ the subset of G consisting of the elements x with the property that $Y = \langle x, y \rangle$ for some $y \in Y$.

Assume $g = wzh \in \Omega(M^t)$ with $w \in W, z \in Z^*$ and $h \in Q^t$. It must be $h \notin F^t$, being $F := \mathrm{Frat}(Q)$. Moreover, it follows from [11, Proposition 2.2] that $wzh \in \Omega(M^t)$ if and only if $zh \in \Omega(Z^*Q^t)$ and since the action of Q^t on V is fixed-point-free, by Lemma 13 the condition $h \notin F^t$ is sufficient to ensure $zh \in \Omega(Z^*Q^t)$. In particular, $\mathcal{I}_{\mathfrak{F}}(M^t) = WZ^*F^t$. If follows that if $H \leq G$, then

$$\mathcal{I}_{\mathfrak{F}}(H) = \begin{cases} WZ^*F^t & \text{if } H = M^t \text{ for some } t \in V_3, \\ H & \text{otherwise.} \end{cases}$$

The set Δ_t of the vertices of $\Gamma_{\mathfrak{F}}(M^t)$ coincides with $WZ^*(Q^t \backslash F^t)$. Moreover, since the action of Q on V is fixed-point-free, if $t_1 \neq t_2$, then $\Delta_{t_1} \cap \Delta_{t_2} = \emptyset$. So the connected components of $\Gamma_{\mathfrak{F}}(G)$ are $WZ^*(Q^t \backslash F^t)$, with $t \in V_3$.

Remark 21. In the previous example, $\Gamma_{\mathfrak{F}}(G)$ has 9 connected components. We may repeat a similar construction starting from a faithful irreducible action of Q on $C_p \times C_p$, where p is an arbitrary odd prime. In this way, we may construct a group G with the property that $\Gamma_{\mathfrak{F}}(G)$ has p^2 connected components.

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5. Example 2

In the previous section, we have constructed an example of a group G satisfying the assumptions of Theorem 14, in which the graph $\Gamma_{\mathfrak{F}}(G)$ is disconnected and contains a maximal subgroup M with the property that each edge in the graph belongs to the subgraph induced by a conjugate of M. However, as it is indicated in the statement of the theorem, there is also the possibility that G contains a proper normal subgroup N in such a way that each edge in the graph belongs either to the subgraph induced by a conjugate of M or to the subgraph induced by N. In this section, we exhibit an example in which this second possibility occurs.

Let $A = \langle a_1 \rangle \times \langle a_2 \rangle$ with $\langle a_1 \rangle \cong \langle a_2 \rangle \cong C_8$ and $Q = \langle b_1, b_2 \rangle \cong Q_8$. Both A and Q admit an automorphism of order 3. So we may consider the semidirect product $X = (A \times Q) \rtimes H$ with $H \cong C_3$. We assume that the map $V_1 = A/\mathrm{Frat}(A) \to V_2 = Q/\mathrm{Frat}(Q)$ sending $a_1 \mathrm{Frat}(A)$ to $b_1 \mathrm{Frat}(Q)$ and $a_2 \mathrm{Frat}(A)$ to $b_2 \mathrm{Frat}(Q)$ is an H-isomorphism. The group X has precisely 6 conjugacy classes of maximal subgroups:

- $M_0 = A \times Q$;
- $M_1 = (A \times \langle c \rangle)H \cong A \rtimes C_6$ with $c = b_1^2 = b_2^2$;
- $M_2 = (A^2 \times Q_8)H$;
- $M_h = \langle a_1^h b_1, a_2^h b_2 \rangle H$, with $h \in H$.

Moreover, QH has an irreducible action on $W \cong C_7 \times C_7$ with the property that Q acts fixed-point-freely on W while $C_W(H) = Z \cong C_7$. This action can be extended to X, with $A \leq C_X(W)$. Let $G = W \rtimes X$. Let

$$B = AH \times C_7, \quad C = W \rtimes \langle a_1 b_1, b_2 \rangle.$$

Moreover, let $\mathfrak{F}$ be the class of finite groups with no subgroup isomorphic to B or C.

Suppose that $g_1 = w_1 x_1, g_2 = w_2 x_2$ have the property that $Y = \langle g_1, g_2 \rangle \notin \mathfrak{F}$. We have two possibilities:

(a) Y has a subgroup isomorphic to B. It must be $\langle x_1, x_2 \rangle \leq M_1^q$ for some $q \in Q$ (since X is not 2-generated and the other maximal subgroups of X with order divisible by 3 do not contain an element of order 8 centralizing a nontrivial element of W). Moreover, $(ZAH)^q$ is the unique subgroup of WM_1^q isomorphic to B.

(b) Y has a subgroup isomorphic to C. Let $J = \langle x_1, x_2 \rangle$ and let M be a maximal subgroup of X containing J. It cannot be that M is a conjugate of M_1, since all the 2-subgroups of M_1 are abelian. It cannot be that M is a conjugate of

M_2, since M_2 has no element of order 8. It cannot be that M is a conjugate of M_h, since all the elements of order 2 in M_h centralize W while b_2^2 does not. So $x_1 = r_1 s_1$, $x_2 = r_2 s_2$ with $r_i \in A, s_i \in Q$. As in the previous case we want to describe when a subgroup $\tilde{C}$ of M is isomorphic to C. This occurs if and only if $\tilde{C} = \langle r_1 s_1, r_2 s_2 \rangle$, with $\langle s_1, s_2 \rangle = Q$, $r_1, r_2 \in A$, $\langle r_1, r_2 \rangle / (\langle r_1, r_2 \rangle \cap A^2) \cong C_2$.

Let $\mathfrak{F}_B$ (respectively, $\mathfrak{F}_C$) the class of finite groups with no subgroup isomorphic to B (respectively, C). Let now K be a proper subgroup of G, and assume $K \notin \mathfrak{F}_2$. Since $M_1^q \cap M_0 = A \times \langle c \rangle$ for any $q \in Q$, K cannot contain both a subgroup isomorphic to B and a subgroup isomorphic to C so we have two possibilities:

(a) $(ZAH)^q \leq K \leq WM_1^q$ for some $q \in Q$. In this case, $\Gamma_{\mathfrak{F}}(K) = \Gamma_{\mathfrak{F}_B}(K)$ and $\mathcal{I}_{\mathfrak{F}}(K) = K \cap (W\,\mathrm{Frat}(A)\langle c \rangle)$.

(b) $K \leq M_0$. In this case, $\Gamma_{\mathfrak{F}}(K) = \Gamma_{\mathfrak{F}_C}(K)$ and $\mathcal{I}_{\mathfrak{F}}(K) = K \cap (WA\langle c \rangle)$.

In particular,

$$\Omega_B = \cup_g WM_1^g \backslash W\,\mathrm{Frat}(A)\langle c \rangle \quad \text{and} \quad \Omega_C = WM_0 \backslash WA\langle c \rangle$$

are, respectively, the set of vertices of $\Gamma_{\mathfrak{F}}(G)$ that are non-isolated in $\Gamma_{\mathfrak{F}}(WM_1^q)$ for some q in Q and the set of vertices of $\Gamma_{\mathfrak{F}}(G)$ that are non-isolated in $\Gamma_{\mathfrak{F}}(WM_0)$. Since $\Omega_B \cap \Omega_C = \emptyset$, the graph $\Gamma_{\mathfrak{F}}(G)$ is disconnected (and indeed Ω_B and Ω_C are the two connected components of the graph). Moreover,

$$\begin{aligned}\mathcal{I}_{\mathfrak{F}}(G) &= (G \backslash \Omega_B) \cap (G \backslash \Omega_C) = W(M_0 \cap (\cap_{q \in Q} M_1^q)) \cup W\,\mathrm{Frat}(A)\langle c \rangle \\ &= W\,\mathrm{Frat}(A)\langle c \rangle.\end{aligned}$$

References

[1] J. Cossey, K. W. Gruenberg and L. G. Kovács, The presentation rank of a direct product of finite groups, *J. Algebra* **28** (1974) 597–603.

[2] E. Crestani and A. Lucchini, d-Wise generation of prosolvable groups, *J. Algebra* **369**(2) (2012) 59–69.

[3] E. Crestani and A. Lucchini, The generating graph of finite soluble groups, *Israel J. Math.* **198**(1) (2013) 63–74.

[4] E. Detomi, A. Lucchini, M. Moscatiello, P. Spiga and G. Traustason, Groups satisfying a strong complement property, *J. Algebra* **535** (2019) 35–52.

[5] K. Doerk and T. Hawkes, *Finite Soluble Groups*, De Gruyter Expositions in Mathematics, Vol. 4 (Walter de Gruyter & Company, Berlin, 1992).

[6] W. Gaschütz, Zu einem von B. H. und H. Neumann gestellten problem, *Math. Nachr.* **14** (1955) 249–252.

[7] R. Guralnick, B. Kunyavskii, E. Plotkin and A. Shalev, Thompson-like characterizations of the solvable radical, *J. Algebra* **300**(1) (2006) 363–375.

[8] H. Heineken, On groups all of whose elements have prime power order, *Math. Proc. R. Ir. Acad.* **106A**(2) (2006) 191–198.

[9] I. Isaacs, *Finite Group Theory*, Graduate Studies in Mathematics, Vol. 92 (American Mathematical Society, Providence, RI, 2008).

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[10] A. Lucchini and F. Menegazzo, Computing a set of generators of minimal cardinality in a solvable group, *J. Symbolic Comput.* **17**(5) (1994) 409–420.

[11] A. Lucchini and D. Nemmi, The non $\mathfrak{F}$-graph of a finite group, *Math. Nachr.* **294**(10) (2021) 1912–1921.

[12] F. D. Volta and A. Lucchini, Finite groups that need more generators than any proper quotient, *J. Aust. Math. Soc. Ser. A* **64**(1) (1998) 82–91.

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