Bright solitons in ultracold atoms

L. Salasnich

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Abstract We review old and recent experimental and theoretical results on bright solitons in Bose-Einstein condensates made of alkali-metal atoms and under external optical confinement. First we deduce the three-dimensional Gross-Pitaevskii equation (3D GPE) from the Dirac-Frenkel action of interacting identical bosons within a time-dependent Hartree approximation. Then we discuss the dimensional reduction of the GPE from 3D to 1D, deriving the 1D GPE and also the 1D nonpolynomial Schrödinger equation (1D NPSE). Finally, we analyze the bright solition solutions of both 1D GPE and 1D NPSE and compare these theoretical predictions with the available experimental data.

Keywords Bright solitons \cdot Ultracold atoms \cdot Gross-Pitaevskii equation \cdot Nonpolynomial Schödinger equation

1 Introduction

In 1995 three experimental groups achieved Bose-Einstein condensation (BEC), i.e. the macroscopic occupation of a single-particle quantum state, cooling very dilute gases of ⁸⁷Rb [1], ⁷Li [2], and ²³Na [3] atoms. For these systems the BEC critical temperature is about $T_c \simeq 100$ nanoKelvin and the gas made of alkalimetal atoms is in a meta-stable state which can survive for minutes. Another ground-breaking result with ultracold atoms was achieved some years later: a stationary optical lattice which traps ultracold atoms was obtained with counter-propagating laser beams inside an optical cavity [4]. The resulting potential confines neutral atoms in the minima of the lattice due to the electric

L. Salasnich

Dipartimento di Fisica e Astronomia "Galileo Galilei" and CNISM, Università di Padova, via Marzolo 8, 35131 Padova, Italy

CNR-INO, via Nello Carrara, 1 - 50019 Sesto Fiorentino, Italy

E-mail: luca.salasnich@unipd.it

dipole of atoms [5]. Nowadays the study of neutral atoms trapped with light is a very hot topic of research because, changing the intensity and shape of the optical lattice, it is possible to confine atoms in very different configurations. One can have many atoms per site but also one atom per site [6].

The main theoretical tool for the study a pure BEC in ultracold and dilute alkali-metal atoms is the Gross-Pitaevskii equation [7], that is a nonlinear Schrödinger equation with cubic nonlinearity. In 1972 Shabat and Zakharov [8] found that the 1D cubic nonlinear Schrödinger equation admits solitonic (i.e. shape invariant) analytical solutions. If the 1D nonlinear strength is repulsive (self-defocusing nonlinearity) one finds the localized dark solitons, while if the nonlinear strength is attractive (self-focusing nonlinearity) one finds bright solitons. Quite remarkably, both dark and bright solitions have been observed experimentally with atomic BECs (see [9] for a comprehensive review). In this paper we concentrate on bright solitons, discussing theoretical and experimental results of this exciting field of research.

2 Gross-Pitaevskii equation

Static and dynamical properties of a pure BEC made of dilute and ultracold atoms are very well described by the Gross-Pitaevskii equation [7]

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{r}) + (N-1)\frac{4\pi\hbar^2 a_s}{m}|\psi(\mathbf{r},t)|^2\right]\psi(\mathbf{r},t) ,\qquad(1)$$

where $U(\mathbf{r})$ is the external trapping potential, m is the mass of each atom, and a_s is the s-wave scattering length of the inter-atomic potential. In this equation $\psi(\mathbf{r}, t)$ is the wavefunction of the BEC normalized to one, i.e.

$$\int |\psi(\mathbf{r},t)|^2 d^3 \mathbf{r} = 1 , \qquad (2)$$

and such that $\rho({\bf r})=N|\psi({\bf r},t)|^2$ is the local number density of the N condensed atoms.

The Gross-Pitaevskii equation (GPE) can be deduced from the many-body quantum Hamiltonian of N identical spinless particles

$$\hat{H} = \sum_{i=1}^{N} \left(-\frac{\hbar^2}{2m} \nabla_i^2 + U(\mathbf{r}) \right) + \frac{1}{2} \sum_{\substack{i,j=1\\i\neq j}}^{N} V(\mathbf{r}_i - \mathbf{r}_j) , \qquad (3)$$

where $V(\mathbf{r} - \mathbf{r'})$ is the inter-atomic potential. The time-dependent Schrödinger equation of this many-body system is given by

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r}_1,...,\mathbf{r}_N,t) = \hat{H}\ \Psi(\mathbf{r}_1,...,\mathbf{r}_N,t),\tag{4}$$

where $\Psi(\mathbf{r}_1, ..., \mathbf{r}_N, t)$ is the time-dependent many-body wavefuction. This timedependent many-body Schrödinger equation is the Euler-Lagrange equation of the following many-body Dirac-Frenkel [10] action functional

$$S = \int dt \ d^3 \mathbf{r}_1 \ \dots \ d^3 \mathbf{r}_N \ \Psi^*(\mathbf{r}_1, \dots, \mathbf{r}_N, t) \left(i\hbar \frac{\partial}{\partial t} - \hat{H}\right) \Psi(\mathbf{r}_1, \dots, \mathbf{r}_N, t) \ . \tag{5}$$

In the case of a pure Bose-Einstein condensate one assumes all bosons in the same time-dependent single-particle orbital (i.e. a time-dependent version of the Hartree approximation [11])

$$\Psi(\mathbf{r}_1,...,\mathbf{r}_N,t) = \prod_{i=1}^N \psi(\mathbf{r}_i,t) .$$
(6)

Inserting this ansatz into the many-body action functional one gets

$$S = N \int dt \ d^{3}\mathbf{r} \ \psi^{*}(\mathbf{r}, t) \Big(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^{2}}{2m} \nabla^{2} - U(\mathbf{r}) \\ - \frac{N-1}{2} \int d^{3}\mathbf{r}' \ |\psi(\mathbf{r}', t)|^{2} V(\mathbf{r} - \mathbf{r}') \Big) \psi(\mathbf{r}, t) \ .$$
(7)

The Euler-Lagrange equation of the previous action functional reads

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{r}) + (N-1)\int d^3\mathbf{r}' \ |\psi(\mathbf{r}',t)|^2 V(\mathbf{r}-\mathbf{r}')\right]\psi(\mathbf{r},t) \ .$$
(8)

This is the time-dependent Hartree equation for N identical bosons in the same single-particle state $\psi(\mathbf{r}, t)$.

In the case of dilute gases one usally assumes (Fermi pseudopotential [12]) that

$$V(\mathbf{r}) \simeq g \ \delta^{(3)}(\mathbf{r}) \tag{9}$$

with $\delta^{(3)}(\mathbf{r})$ the Dirac delta function and, by construction,

$$g = \int V(\mathbf{r}) \ d^3 \mathbf{r} \ . \tag{10}$$

From 3D scattering theory, the s-wave scattering length a_s of the inter-atomic potential can be written (Born approximation [13]) as

$$a_s = \frac{m}{4\pi\hbar^2} \int V(\mathbf{r}) \ d^3\mathbf{r} \ . \tag{11}$$

In this way, from Eq. (8) one obtains the time-dependent 3D GPE, Eq. (1).

3 Dimensional reduction: from 3D to 1D

From the Hartree equation (8) we have obtained the time-dependent 3D GPE. Clearly, this is the Euler-Lagrange equation of the GP action functional

$$S = N \int dt \ d^3 \mathbf{r} \ \psi^*(\mathbf{r}, t) \Big(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 - U(\mathbf{r}) - \frac{N-1}{2} g |\psi(\mathbf{r}, t)|^2 \Big) \psi(\mathbf{r}, t) \ . \tag{12}$$

Let us now consider a very strong harmonic confinement of frequency ω_{\perp} along x and y and a generic confinement $\mathcal{U}(z)$ along z, namely

$$U(\mathbf{r}) = \frac{1}{2}m\omega_{\perp}^{2}(x^{2} + y^{2}) + \mathcal{U}(z) .$$
(13)

On the basis of the chosen external confinement, we adopt the ansatz

$$\psi(\mathbf{r},t) = f(z,t) \frac{1}{\pi^{1/2} a_{\perp}} \exp\left(\frac{x^2 + y^2}{2a_{\perp}^2}\right), \qquad (14)$$

where f(z,t) is the axial wave function and $a_{\perp} = \sqrt{\hbar/(m\omega_{\perp})}$ is the characteristic length of the transverse harmonic confinement. By inserting Eq. (14) into the GP action (12) and integrating along x and y, the resulting effective action functional depends only on the field f(z,t).

One easily finds that the Euler-Lagrange equation of the axial wavefunction $f(\boldsymbol{z},t)$ reads

$$i\hbar\frac{\partial}{\partial t}f(z,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \mathcal{U}(z) + \gamma|f(z,t)|^2\right]f(z,t), \qquad (15)$$

where

$$\gamma = \frac{(N-1)g}{2\pi a_{\perp}^2} \tag{16}$$

is the effective one-dimensional interaction strength and the additive constant $\hbar\omega_{\perp}$ has been omitted because it does not affect the dynamics.

3.1 Bright solitons of 1D GPE

In the absence of axial confinement, i.e. $\mathcal{U}(z) = 0$, the 1D GPE becomes

$$i\hbar\frac{\partial}{\partial t}f(z,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \gamma|f(z,t)|^2\right]f(z,t) .$$
(17)

This is a 1D nonlinear Schrödinger equation with cubic nonlinearity. In 1972 Shabat and Zakharov [8] found that this equation admits solitonic solutions, such that

$$f(z,t) = \phi(z - vt) \ e^{i(mvz - mv^2t/2 - \mu t)/\hbar} , \tag{18}$$



Fig. 1 Probability density $|\phi(\zeta)|^2$ of the bright soliton for three values of the nonlinear strength γ . We set $\hbar = m = 1$.

where v is the arbitrary velocity of propagation of the solution, which has a shape-invariant axial density profile:

$$\rho(z,t) = N|f(z,t)|^2 = N|\phi(z-vt)|^2 .$$
(19)

Setting $\zeta = z - vt$, the 1D stationary GP equation

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{d\zeta^2} + \gamma|\phi(\zeta)|^2\right]\phi(\zeta) = \mu \ \phi(\zeta) \ , \tag{20}$$

with $\gamma < 0$ (self-focusing), admits the bright-soliton solution

$$\phi(\zeta) = \sqrt{\frac{m|\gamma|}{8\hbar^2}} \operatorname{Sech}\left[\frac{m|\gamma|}{4\hbar^2}\zeta\right]$$
(21)

with $Sech[x] = \frac{2}{e^x + e^{-x}}$ and

$$\mu = -\frac{m \gamma^2}{16 \hbar^2} \,. \tag{22}$$

In Fig. 1 we plot the $|\phi(\zeta)|^2$ of the bright soliton for three values of the nonlinear strength γ .

Shabat and Zakharov [8] used the inverse scattering method [14] to find the explicit expression of $\phi(\zeta)$. Here we use a much simpler (but less general) method. Let us assume that $\phi(\zeta)$ is real. Then the 1D stationary Gross-Pitaevskii equation can be rewritten as

$$\phi''(\zeta) = -\frac{\partial W(\phi)}{\partial \phi} , \qquad (23)$$

where

$$W(\phi) = \frac{1}{2} \frac{m|\gamma|}{\hbar^2} \phi^4 + \frac{m\mu}{\hbar^2} \phi^2 .$$
 (24)

Thus, $\phi(\zeta)$ can be seen as the "coordinate" for a fictitious particle at "time" ζ . The constant of motion of the problem reads

$$K = \frac{1}{2}\phi'(\zeta)^2 + W(\phi) , \qquad (25)$$

from which one finds

$$\frac{d\phi}{d\zeta} = \sqrt{2(K - W(\phi))} . \tag{26}$$

Imposing that $\phi(\zeta) \to 0$ as $|\zeta| \to \infty$ one gets K = 0 and consequently

$$\frac{d\phi}{\sqrt{-2W(\phi)}} = d\zeta , \qquad (27)$$

or explicity

$$\frac{d\phi}{\sqrt{-\frac{m|\gamma|}{\hbar^2}\phi^4 + \frac{2m|\mu|}{\hbar^2}\phi^2}} = d\zeta , \qquad (28)$$

with $\mu < 0$. Inserting the integrals one obtains

$$\int_{\phi(0)}^{\phi(\zeta)} \frac{d\phi}{\sqrt{-\frac{m|\gamma|}{\hbar^2}\phi^4 + \frac{2m|\mu|}{\hbar^2}\phi^2}} = \zeta .$$
 (29)

Setting $\phi'(0) = 0$, from the definition of K and using K = 0 one finds $W(\phi(0)) = 0$ and therefore

$$\phi(0) = \sqrt{\frac{2|\mu|}{|\gamma|}} \,. \tag{30}$$

After integration of Eq. (29) one gets

$$\frac{1}{\sqrt{\frac{m|\mu|}{\hbar}}} ArcSech\left[\sqrt{\frac{|\gamma|}{2|\mu|}}\phi(\zeta)\right] = \zeta \tag{31}$$

from which

$$\phi(\zeta) = \sqrt{\frac{2|\mu|}{|\gamma|}} Sech\left[\sqrt{\frac{m|\mu|}{\hbar^2}}\zeta\right] \,. \tag{32}$$

Finally, imposing the normalization condition

$$\int dz \ \phi(\zeta)^2 = 1 , \qquad (33)$$

one obtains

$$\mu = -\frac{m \gamma^2}{16 \hbar^2} \,. \tag{34}$$

4 Improved dimensional reduction: the 1D NPSE

The bright soliton analytical solution has been obtained from the 1D GPE, which is derived from the 3D GPE assuming a transverse Gaussian with a constant transverse width a_{\perp} . A more general assumption [15, 16, 17] is based on a space-time dependent transverse width

$$\psi(\mathbf{r},t) = f(z,t) \frac{1}{\pi^{1/2} a_{\perp} \eta(z,t)} \exp\left(\frac{x^2 + y^2}{2a_{\perp}^2 \eta(z,t)^2}\right),$$
(35)

where f(z,t) is the axial wave function and $\eta(z,t)$ is the adimensional transverse width in units of a_{\perp} . From this ansatz one gets the 1D nonpolynomial Schrödinger equation (1D NPSE)

$$i\hbar\frac{\partial}{\partial t}f = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \mathcal{U}(z) + \frac{\gamma|f|^2}{\eta^2} + \frac{\hbar\omega_{\perp}}{2}\left(\frac{1}{\eta^2} + \eta^2\right)\right]f, \quad (36)$$

$$\eta = \left(1 + \gamma |f|^2\right)^{1/4} \,. \tag{37}$$

In the weak-coupling regime $\gamma |f|^2 \ll 1$ one finds $\eta \simeq 1$ and the 1D NPSE becomes the familiar 1D GPE.

4.1 Bright solitons of 1D NPSE

With $\mathcal{U}(z) = 0$ and assuming $\gamma < 0$ the NPSE admits analytical bright soliton solutions. Setting

$$f(z,t) = \phi(z - vt)e^{i(mv^2/2 - \mu)t/\hbar},$$
(38)

one finds the bright-soliton solution written in implicit form

$$\zeta = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{1-\mu}} \operatorname{arctg} \left[\sqrt{\frac{\sqrt{1-|\gamma|\phi^2 - \mu}}{1-\mu}} \right]$$
(39)

$$-\frac{1}{\sqrt{2}}\frac{1}{\sqrt{1+\mu}}\operatorname{arcth}\left[\sqrt{\frac{\sqrt{1-|\gamma|\phi^2}-\mu}{1+\mu}}\right],\qquad(40)$$

where $\zeta = z - vt$ and $|\gamma| = 2|a_s|(N-1)/a_{\perp}$.

Fig. 2 reports the axial probability density $\rho(\zeta) = |\phi(\zeta)|^2$ of the bright soliton obtained by using the 3D GPE (full line), the 1D NPSE (dotted line), and the 1D GPE (dashed line). In the weak-coupling limit ($\gamma \phi^2 \ll 1$) one finds that the NPSE bright-soliton solution reduces the the 1D GPE one. This is clearly shown in the figure. However, contrary to the 1D GPE bright soliton, the 1D NPSE bright soliton does not exist anymore, collapsing to a Dirac delta function, at

$$\gamma_c = \left(\frac{2a_s(N-1)}{a_\perp}\right)_c = -\frac{4}{3}.$$



Fig. 2 Axial probability density $\rho(\zeta) = |\phi(\zeta)|^2$ of the Bose-condensed bright soliton: 3D GPE (full line), 1D NPSE (dotted line), 1D GPE (dashed line). Length in units $a_{\perp} = (\hbar/m\omega_{\perp})^{1/2}$ and density in units $1/a_{\perp}$. Three values of the interaction strength: a) $\gamma = 0.3$, b) $\gamma = 0.8$, c) $\gamma = 1.3$. Adaped from [16].

This analytical result is in extremely good agreement with the numerical solution of the 3D GPE [16]. Indeed, Fig. 2 shows that up to the collapse the density profile obtained with the NPSE is very close to the 3D GPE one.

5 Bright solitons in experiments with ultracold atoms

In 2002 there were two relevant experiments [19,20] about bright solitons with BECs made of ⁷Li atoms. Both experiments used the technique of Fano-Feshbach resonance [18] to tune the s-wave scattering length a_s of the interatomic potential by means of an external constant magnetic field.

Khaykovich et al. [19] reported the production of bright solitons in an ultracold ⁷Li gas. The interaction was tuned with a Feshbach resonance from repulsive to attractive before release in a one-dimensional optical waveguide, which is attractive in the transverse direction but expulsive in the longitudinal direction. Propagation of the soliton without dispersion over a macroscopic distance of 1.1 millimeter was observed in the case of attractive interaction.



Fig. 3 Root mean square size σ of the longitudinal width of the BEC vs propagation time $t. a_s$ is the s-wave scattering length of the inter-atomic potential, which is experimentally tuned by means of a Fano-Feshbach resonance [18]. Filled circles are experimental data taken from [19]. The dashed line is the ideal gas $(a_s = 0)$ curve. The solid line is obtained solving the time-dependent NPSE. Adapted from [21].

In their experiment, Khaykovich *et al.* [19] measured the root mean square size σ of the longitudinal width versus the propagation time for three values of a_s : $a_s = 0$, $a_s = -0.11$ nm, and $a_s = -0.21$ nm. Fig. 3 shows the experimental data [19] and our numerical results obtained with the time-dependent 1D NPSE [21]. The agreement between experiment and theory is quite good. The numerical results are obtained by using a finite-difference Crank-Nicolson scheme with predictor-corrector [22]. Very recently, various open access and high performance codes have been developed for the solution of the time-dependent GPE [23].

Strecker et al. [20] reported the formation of a train of bright solitons of ⁷Li atoms in a quasi-one-dimensional optical trap by a sudden change in the sign of the scattering length from positive to negative. The solitons were set in motion by offsetting the optical potential, and were observed to propagate in the longitudinal harmonic potential for many oscillatory cycles without spreading.

We successfully simulate the soliton train formation of Ref. [20] by using the time-dependent 3D GPE [24]. In Fig. 4 we plot the probability density in the longitudinal direction $\rho(z)$ of the BEC made of 10⁴ ⁷Li atoms. Initially there is a stable condensate of ⁷Li atoms with a large positive scattering length



Fig. 4 Axial density profile $\rho(z)$ of the BEC made of 10^4 ⁷Li atoms obtained by solving the 3D GPE. For t < 0 the scattering length is $a_s = 100a_B$, while for $t \ge 0$ we set $a_s = -3a_B$ with a_B the Bohr radius. Length z is in units of the characteristic length $a_z = \sqrt{\hbar/(m\omega_z)}$ of the weak axial harmonic confinement of frequency ω_z . Time t in units of $1/\omega_z$. Density ρ in units of $1/a_z$. Adapted from [24].

 a_s but at time t = 0 the scattering length a_s is switched to a negative value. A number N_s of bright solitons is produced and this can be interpreted in terms of the modulational instability of the time-dependent macroscopic wave function of the Bose condensate [25] An estimate of the number N_s of bright solitons which are generated is

$$N_s = \frac{\sqrt{N|a_s|L}}{\pi a_\perp} , \qquad (41)$$

where a_s is the final negative scattering length, N is the total number of atoms, a_{\perp} is the characteristic length of transverse harmonic confinement and L is the initial longitudinal length of the quasi-1D BEC [24]. This formula, based on the analysis of the imaginary Bogoliubov spectrum of elementary excitations (see [24] for details), is in good agreement with both experimental results and numerical simulations.

Very recently, the formation of matter-wave soliton trains by modulational instability was experimentally reexamined by Nguyen *et al.* [26]. They used a nearly nondestructive imaging technique to follow the dynamics of these trains finding that the modulation instability is driven by noise and neighboring solitons interact repulsively during the initial formation of the soliton train. These findings are indeed in full agreement with our theoretical predictions based on the numerical simulation of 3D GPE and 1D NPSE [24].

6 Conclusions

In this paper we have explicitly derived the analytical solution of the 1D bright soliton from the one-dimensional Gross-Pitaevskii equation (1D GPE), which, in turn, is obtained from the 3D GPE assuming a transverse Gaussian with a constant width a_{\perp} . We have then shown that a more general assumption, with a space-time dependent transverse width, gives rise the 1D nonpolynomial Schrödinger equation (1D NPSE). 1D NPSE admits bright solitons which collapse at a critical interaction strength, in good agreement with the findings of full 3D GPE. Both 3D GPE and 1D NPSE are reliable tools to reproduce the available experimental data [19,20,26] of BEC bright solitons made of alkalimetal atoms. The experimental study of bright solitons in ultracold atoms is still a hot topic, as is evident considering the very recent experiments on the train of bright solitons [26] and on attractive two-component bosonic mixtures [27]. Moreover, in the last few years, it is has been suggested that atomic bright solitons can be produced with attractive two-component mixtures [28, 29, with space-dependent scattering lengths [30], and also with artificial spinorbit and Rabi couplings [31,32,33]. These remarkable theoretical predictions need experimental confirmation.

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