Inquisitive Logic

Consequence and inference in the realm of questions

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Contents

Preface

This book is an introduction to the main ideas and results of inquisitive logic. Inquisitive logic is a novel approach to the central notions of logic which makes it possible to extend the traditional boundaries of the discipline. In inquisitive logic, we can formalize not only declarative sentences like ‘Alice passed the test’ and ‘every student passed the test’, but also interrogative sentences like ‘whether Alice passed the test’ and ‘which students passed the test’. The central notions of logic—including entailment, connectives, quantifiers, and proofs—are generalized in such a way that they apply uniformly to both kinds of sentences. In the past few years, research on inquisitive logic has flourished, as witnessed by a wealth of recent publications, stemming from different research communities and countries (see the bibliography in Appendix A). This book offers the first book-length introduction to the subject, and aims to fulfil three different roles.

First, it is intended to serve as a standard reference for the growing research community working on inquisitive logic, and as an accessible entry point for young researchers interested in contributing to this field.

Second, it is designed to serve as a textbook in graduate-level courses. In order to make the book suited to this use, special attention has been paid to motivating the enterprise and the basic design choices in detail, and to illustrating the key notions with concrete examples and figures. Moreover, in each chapter a set of exercises is provided. Some of these exercises are designed to familiarize the students with the notions and the techniques covered in the chapter; others are more research-oriented, challenging students to prove by themselves some interesting results that go beyond the content covered in the chapter.

Third, the book will serve as an overview of the field of inquisitive logic for researchers working in neighboring areas, both inside and outside of logic. Within logic, the audience includes logicians working on modal logic, intuitionistic logic, team-based logics, inferential erotetic logic, dynamic epistemic logic, truth-maker logics and information-based approaches to substructural logics—all areas with significant ties to inquisitive logic. Outside of logic, it will be relevant to current research in natural language semantics (for instance, on the semantics of questions and question-embedding verbs), philosophy of language (aboutness, hyper-intensionality), epistemology (contrastive knowledge), philosophy of mind (question-directed attitudes), and metaphysics (supervenience).

The book is a thoroughly revised, extended, and updated version of the first half of my doctoral dissertation (chapters 1-5 of Ciardelli, 2016a). Parts of the material have also been presented in the form of articles, the most important references being Ciardelli and Roelofsen (2011) and Ciardelli (2016b, 2018b). With respect to the dissertation, Chapter 5 has been extended substantially to reflect some of the recent advances on inquisitive first-order logic, mostly due to Gianluca Grilletti (see Grilletti, 2019, 2020, 2021; Grilletti and Ciardelli, 2021).

In preparing the book, much effort has been put in optimizing the coherence and smoothness of the presentation, especially in the light of two experiences teaching the present material as a graduate-level course at LMU Munich.

Inquisitive logic builds on inquisitive semantics, an approach to the modeling of meaning which allows for a uniform analysis of statements and questions. As such, the present book is related to the textbook *Inquisitive Semantics* (Ciardelli, Groenendijk, and Roelofsen, 2018). However, the two books are concerned with different enterprises and directed at different communities. The Inquisitive Semantics book aims to offer a novel framework for the formal analysis of natural language semantics and pragmatics; the core ideas are motivated by linguistic observations, and the focus is on applications in linguistics (indeed, the book is part of the series *Oxford Surveys in Semantics and Pragmatics*, whose target audience consists mainly of linguists and philosophers of language). By contrast, this is a book in formal logic: our primary aim here is not to investigate the structure of natural languages, but rather to develop formal languages that can be used to unambiguously regiment statements and questions, study their semantic relations, and carry out inferences with them. The importance of taking questions into account in logic is motivated here, not based on the presence of questions in natural language, but based on the fact that extending logic to questions allows us to analyze important logical notions and to carry out interesting kinds of inferences. What we will argue is that questions are extremely interesting from the perspective of formal logic. Relatedly, the target audience for the present book consists primarily of logicians and logic students, rather than linguists and philosophers of language. As a consequence, much attention is paid to matters which are not covered in the Inquisitive Semantics book, such as the mathematical properties of various inquisitive logics, the development of proof systems, and the relations between inquisitive logic and other logical formalisms.

This book would not have been possible without the help of many colleagues and friends. The first acknowledgment goes to Jeroen Groenendijk and Floris Roelofsen, my travel companions in the development of inquisitive semantics. Secondly, as reflected by the bibliography in Appendix A, many people have contributed to the development of inquisitive logic, the subject of this book: for the prominence of their contributions I would like to thank especially Vít Punčochář, Gianluca Grilletti, Thom van Gessel, and Salvador Mascarenhas. Throughout the years in which I worked on the material in this book, I have been fortunate to learn from a number of collaborators, especially Maria Aloni, Martin Otto, Fan Yang, Rosalie Iemhoff, Lucas Champollion, and Nadine Theiler. Thanks to Jouko Väänänen, Hannes Leitgeb, Wesley Holliday, Justin Bledin, Yanjing Wang, Andrzej Wiśniewski and Valentin Goranko for fruitful discussions on inquisitive logic and related topics. I am also very grateful to the MA students of the Munich Center for Mathematical Philosophy who took my course on inquisitive logic: their questions and comments have helped me to optimize the presentation of the material. Finally, many thanks to Adrian Ommundsen and two anonymous reviewers for detailed comments on a first draft of this book.

Chapter 1

Introduction

1.1 Motivation

In logic, we study properties of sentences, such as truth, falsity, necessity, and contingency, relations between sentences, such as entailment and consistency, and practices involving sentences, such as argumentation. However, by *sentences*, we normally only mean sentences of a certain particular kind: declarative sentences—*statements*, and their counterparts in formal languages.

There are principled reasons for this restriction, which is worth examining. If we approach logic from the semantic side, the focus is on *truth*: the meaning of logical operators is captured in terms of how the truth conditions of a compound are derived from the truth conditions of its components, and the central notion of logical entailment is construed in terms of truth preservation: an entailment is valid if the truth of the premises guarantees the truth of the conclusion under every interpretation of the non-logical symbols. If we approach logic from the syntactic side, the focus is on *inference*: on the basic rules that govern inference with certain logical constants, on the ways these rules can be used to build proofs, and on the information encoded by these proofs.

In this book, we are concerned with interrogative sentences—*questions*, and their counterparts in formal languages. Questions play a crucial role in language in many ways (see Ciardelli et al., 2018, for an overview), and accordingly, they are a major topic in linguistics. By contrast, in spite of some amount of work, in logic they have remained somewhat marginal. In view of the central concerns of logic as outlined above, this should not surprise us. Take, for instance, the question what the capital of Spain is. It is intuitively unclear what it would mean to ask whether this question is *true* or *false*. Arguably, questions are not the sort of sentences which are capable of being true or false—in technical jargon, they are not *truth-apt*. And given that the semantic notion of logical entailment is construed in terms of truth, entailment claims are not applicable to questions either. Things do not look more promising if we start from the syntactic side. It is intuitively unclear what it would mean to suppose or to conclude a question—say, what the capital of Spain is—as part of an inference, and what it would mean for an inference involving such moves to count as valid. So, there are serious reasons why questions have played only a marginal role in logic: it would seem that those notions which are the central concern of logic—truth, entailment, valid inference—are simply not applicable to questions.

The aim of this book is to show that in spite of these considerations, the scope of logic can in fact be extended to questions in a way which is conceptually natural, formally well-behaved, and theoretically fruitful. We will see that, if we switch from the standard truth-conditional perspective on semantics to an informational perspective, it is possible to give a unified semantic analysis of statements and questions. Building on this foundation, called *inquisitive semantics*, it is possible to design conservative extensions of classical propositional, predicate, and modal logic that encompass questions alongside statements.

A first benefit of this uniform approach is the following. Logic is supposed to provide an analysis of certain logical items in language, including connectives, quantifiers, and modalities. Some of these items can be combined not only with statements, but also with questions:

(1) a. Alice rented a car <u>and</u> she booked a hotel.
 b. What kind of car did Alice rent, <u>and</u> which hotel did she book?

(2) a. <u>If</u> Alice wins a free trip, she'll go to Athens.
 b. <u>If</u> Alice wins a free trip, where will she go?

(3) a. <u>Every</u> student read a book.
 b. What book did <u>every</u> student read?

(4) a. Bob <u>knows</u> that Alice lives in London.
 b. Bob <u>knows</u> where Alice lives.

The standard analysis of these items in terms of truth conditions only captures their role in the a-sentence of each pair. Our approach will yield a more general analysis which, while coming down to the standard analysis in the case of statements, allows us to understand how these items work uniformly across statements and questions.

A second benefit of the new semantic foundation is that it allows us to extend the central notion of logic, the notion of *entailment*, beyond statements, so that we can study entailment relations involving questions as premises or as conclusions. As it turns out, this in fact results in an exciting generalization of the scope of logic: several interesting logical notions turn out to be instances of logical entailment involving questions as premises and/or as conclusions. Among these, perhaps the most interesting case is given by the relation of *dependency*. This is a ubiquitous and important logical relation, which we are going to discuss in detail in the next chapter. For now, let me give an impressionistic illustration.

Here is an example of a standard entailment, involving statements.

(5) $\left\{ \begin{array}{l} \text{Alice and Bob live in the same city} \\ \text{Alice lives in Munich} \end{array} \right\} \models$ Bob lives in Munich

Here is one way to read this claim: the information that Alice and Bob live in the same city, combined with the information that Alice lives in Munich, yields the information that Bob lives in Munich. Once questions are brought into the picture, we are going to have entailments such as the following.

(6) $\left\{ \begin{array}{l} \text{Alice and Bob live in the same city} \\ \text{Where Alice lives} \end{array} \right\} \models$ Where Bob lives

We can read this as follows: the information that Alice and Bob live in the same city, combined with the information where Alice lives, yields the information where Bob lives.[1] Notice the difference between the two examples: in (5) we are concerned with a relation holding between three specific pieces of information. The situation is different in (6): what this entailment amounts to is that, given the information that Alice and Bob live in the same city, *any* information determining where Alice lives yields *some* corresponding information determining where Bob lives. As we will discuss, we can see this as a relation involving two *types* of information—think of them as type 'Alice's place of residence' and 'Bob's place of residence'. Entailment captures the fact that, given the information that Alice and Bob live in the same city, information of the first type yields information of the second type.

Realizing that dependencies are entailments is not just a neat insight, but has very concrete repercussions: it allows us to apply to dependency the many tools and ideas that logicians have developed for entailment. Here is a prominent example: to track entailment, in logic we develop proof systems. Since dependencies are cases of entailment, it will be possible to formally *prove* dependencies in a proof system equipped with questions. In fact, we will see that such proofs have an interesting kind of constructive content, reminiscent of the

[1]Some might wonder why in these illustrations we use the indirect form of the question, "where Alice lives" instead of the direct form, "where does Alice live?". The two forms arguably share the same semantic content (Belnap, 1983; Groenendijk and Stokhof, 1984; Ciardelli, 2021). However, the direct version of the question is conventionally associated not only with a certain semantic content, but also with a certain force, related to the speech act of *asking*. Just like we distinguish the content of a statement from its assertion, it is crucial to distinguish the content of a question from the act of asking it. In our view, the development of a logic of questions has been hindered by a failure to make this distinction. A question content has many roles to play besides providing a content for asking acts: it plays a role in compositional semantics, in thought, and crucially for us, it stands in logical relations to other contents (for detailed discussion of this point see Ciardelli, 2021). Throughout the book we tend to use the indirect form in our examples to emphasize that we are concerned with the semantic content of questions, and not with the speech act of asking the question.

proofs-as-programs interpretation of intuitionistic logic: a proof of a dependency encodes a method for *computing* the dependency, i.e., for turning answers to the question premises into an answer to the question conclusion.

This will take us to consider in more detail the role of questions in proofs. It will turn out that, if understood in the right way, using questions in a proof is far from meaningless: on the contrary, questions turn out to be powerful tools for inference. A question may be used in a proof as a placeholder for arbitrary information of the corresponding type. For instance, in our proof systems we will be able to assume the question *where Alice lives*; in doing so, we are assuming an arbitrary specification of Alice's place of residence, without assuming anything more specific about what this information is. On that basis, we can then make inferences to what other information we are ensured to have. When we conclude, say, the question *where Bob lives*, that means that, given the assumptions, we are guaranteed to have information that determines Bob's place of residence. Thus, in our proofs, a question essentially serves as a placeholder for arbitrary information of a certain type, in much the same way as an individual constant may be used in first-order proofs as a placeholder for an arbitrary individual satifying a certain formula.

Thus, what we aim to show in this book is that extending logic to questions is not only possible, but also very interesting from a purely logical perspective: once we bring questions into play, we can apply the tools of logic to analyze new and important kinds of logical notions, and to regiment new kinds of inferences.[2]

1.2 Content and structure of the book

In this book, we will substantiate the claims made in the previous section by showing how classical propositional and predicate logic can be extended in a principled way with questions. We will illustrate how the resulting systems can be used to formalize various classes of questions and to investigate logical relations involving them. We will explore in detail the meta-theoretical properties of these systems and their philosophical significance, also discussing alternative design choices at some crucial points. We will present complete proof systems for inquisitive propositional logic and for an important fragment of inquisitive first-order logic, and we will use these proof systems to illustrate how questions allow us to manipulate arbitrary information in inferences. We will also highlight some important open problems in the area, with the aim of stimulating further research on these problems.

[2]A further, independent reason why questions are interesting from the perspective of logic comes from looking at modalities. By having questions and question-embedding modalities, we can extend the scope of modal logic to capture a range of interesting question-directed modal notions. We leave a full presentation of this aspect of inquisitive logic for a future occasion, limiting ourselves to a preview in Chapter 8.

We begin in Chapter 2 by laying out the foundations of inquisitive logic, introducing the key notions of the approach and discussing the significance of entailment in our generalized setting. In Chapter 3 we show how classical propositional logic can be extended with questions and study the meta-theoretic properties of the resulting inquisitive propositional logic. In Chapter 4 we develop a natural deduction system for this logic, prove its completeness, and make some more general points about the role of questions in logical inferences. In Chapter 5 we turn to predicate logic, in which a much broader variety of questions becomes expressible; we define and study inquisitive first-order logic, and present some important open problems. In Chapter 6 we present a natural deduction system for inquisitive first-order logic and show that it is complete for an important fragment of the logic, though possibly incomplete for the logic as a whole. In Chapter 7, we relate inquisitive logic to another recently developed field of logic, namely, *dependence logic* (Väänänen, 2007), which extends classical predicate logic with formulas expressing dependencies between variables.[3] Finally, in Chapter 8 we give a preview of the potential of combining inquisitive logic with modal logic—a much larger topic whose full development must be left for another occasion.

[3]There are also deep and well-understood relations between inquisitive logic, intuitionistic logic, and various intermediate logics, especially the Kreisel-Putnam logic and Medvedev's logic of finite problems. These connections will not be covered in the present book, but they are treated in detail in Ciardelli (2009) and Ciardelli and Roelofsen (2011).

Chapter 2

Foundations of inquisitive logic

In this chapter, we lay the foundations for our enterprise. In particular, we explain how an information-based semantics, *inquisitive semantics*, allows us to interpret statements and questions in a uniform way and to define a general notion of entailment in which questions can occur as premises and conclusions. We explore in detail the significance of this generalized notion of entailment, showing in particular that it captures as a special case an important logical notion that we call *dependency*. We explain how questions can be viewed as denoting *information types* and how inquisitive entailment can thus be seen as generalizing entailment from a notion relating pieces of informations to one relating information types. We also show how a logic based on inquisitive semantics can be equipped in a canonical way with an implication connective that internalizes, in a precise sense, the meta-language entailment relations into the object language. At the end of the section we motivate in more detail some of our setup choices and we relate our approach to previous work on questions in logic.

Throughout this chapter, we deliberately leave some notions underspecified. In particular, we will not specify a formal language or a precise notion of model. This will allow us to focus on the main general ideas underlying the approach and on those aspects of the theory that follow from these ideas. The missing details can then be filled in in different ways, thereby instantiating the general picture to many concrete logical systems. Thus, what we are describing in this chapter can be seen as a general template that underlies the different inquisitive logics to be investigated in the subsequent chapters, as well as many other inquisitive systems that we are not going to cover.

Our presentation of inquisitive semantics in this chapter differs from the one to be found in the more language-oriented expositions—in particular, from the one in the inquisitive semantics textbook (Ciardelli, Groenendijk, and Roelofsen, 2018). The difference concerns how the semantics is motivated as well as how the basic notions are introduced. In terms of motivations, the presentation in Ciardelli et al. (2018) is driven by considerations about natural language semantics and discourse. By contrast, in this chapter—and in the book at large—we will argue for the inquisitive approach purely on the basis of motivations stemming from formal logic. Relatedly, the presentation in Ciardelli et al. (2018) proceeds by introducing inquisitive contents in terms of their discourse effects (namely, providing information and raising issues). As a derivative notion, we get a semantic relation called of *support* between information states and sentences. By contrast, here we will take the support relation as primitive and we will understand this notion in a way which is independent of discourse effects, and arguably more fundamental.[1]

2.1 Dependency

Let us start out with a simple example which will help us illustrate the ideas introduced as we go through this chapter. Consider a regular die with six faces. Let us say that the outcomes 1 and 2 are in the low range, 3 and 4 in the middle range, and 5 and 6 in the high range. Now consider the following questions about the outcome of a die roll.

parity Whether the outcome is even or odd

range What the range of the outcome is (low, middle, or high)

outcome What the outcome is

These three questions are logically related in an interesting way: as soon as the first two questions are settled, the third is bound to be settled as well; that is, as soon as we settle the outcome's parity (even or odd) and its range (low, middle, or high), we thereby settle exactly what the outcome is. We say that in the given situation, the questions parity and range *determine* the question outcome, and we refer to this relation as a *dependency*.[2] Dependency is a notion of great importance. Let us briefly examine why.

Take the setting of experimental science. Consider the range of experiments that we can perform in a certain context. We can think of each experiment as a procedure that yields the answer to a certain question. Let us call a question *experimental* if it can be directly settled by performing an experiment. Then to

[1] We will not make any assumptions about the role that this relation plays in communication. Our perspective is compatible with different views about the semantics/pragmatics interface, including ones that depart from inquisitive semantics "orthodoxy".

[2] In previous literature on questions in logic, the same notion has been considered under the name *containment* (Hamblin, 1958; Belnap and Steel, 1976). We use *dependency*, in part for the sake of consistency with other areas, such as dependence logic and database theory. This notion of dependency is connected to, but not the same as, other important notions of dependency, including causal dependency, explanatory dependency, and ontological dependency (grounding). We will not be concerned with these other notions here, but it is an interesting task for future work to explore whether they can also be viewed as involving questions and how exactly they relate to the notion of dependency which we focus on.

ask whether a question is determined by the set of experimental questions is to ask whether it is possible, in principle, to answer that question by our empirical means. And of course, an analogous issue arises not just with experiments, but whenever there is a distinction between a range of "viable" questions, those that can actually be asked, and the "target" questions that one is interested in.[3] One can also start at the other hand: suppose we are interested in resolving a certain target question. Then it is interesting to ask if a certain set of viable questions logically determines the target question: for that tells us whether the set provides an inquiry strategy which is guaranteed to achieve our goal.

Turning to theoretical science, a crucial aspect of a scientific theory is its *predictive power*. We can characterize this as the power to yield answers to certain questions on the basis of answers to other questions. Thus, the predictive power of a theory lies precisely in the dependencies that hold on the basis of it. For example, classical mechanics can be characterized as predictive of a body's position at a time t given (i) the body's position and velocity at a different time t_0, (ii) the body's mass and (iii) the force field in which the body moves. What this amounts to is that against the background of classical mechanics, any way of settling the questions (i)–(iii) yields a corresponding answer to the question of where the body is located at time t.

In fact, much of the enterprise of natural sciences such as physics and chemistry consists in finding out *what* dependencies hold in nature: on the basis of what factors can we predict the trajectory of a planet, the temperature of a gas, or the speed of a certain chemical reaction? For instance, one of the earliest achievements of modern science was the discovery that, absent air resistance, the time that a body dropped near the Earth surface employs to reach the ground is completely determined by the height from which it is dropped. This is an instance of dependency in our sense: one question, *from what height the body is dropped*, determines another question, *how long it takes to hit the ground*.

A further illustration comes from database theory. A relational database (say, the database of a company) consists of entries (say, one for each employee) where each entry gives a value to a number of attributes (social security number, surname, department, salary, etc). We can think of the attributes as questions, with each entry providing an answer to each question. Certain dependencies are expected to hold between different attributes: for instance, the social security number of an employee should uniquely determine their surname, but not the

[3] A perfect example is provided by games such as *Guess Who* and *Mastermind*, where the goal of a player is to find out the answer to a target question by asking only questions of a certain kind. In *Guess Who*, the target question is "who is the person on the opponent's card?" and the admissible questions are *yes/no* questions. In *Mastermind*, the target question is "what is the secret code?", and the admissible questions have the form "how many pegs in this particular code appear in the same position in the secret code, and how many appear in a different position in the secret code?".

other way around. Keeping track of these dependencies plays a crucial role in strategies designed to efficiently organize the data, which is why dependencies have received much attention in database theory (for a survey, see Fagin and Vardi, 1986).

In this chapter, we will show that the relation of dependency is nothing but a facet of the central logical notion of entailment, once this notion is generalized so that it applies not only to statements, but also to questions. The study of dependency thus pertains to logic in the strictest sense, and many standard notions and techniques of logic can be fruitfully applied to study dependencies in the context of a logic equipped with questions.

We begin in the next section by explaining our strategy for bringing questions within the scope of logic.

2.2 From truth conditions to support conditions

The standard approach to logic is centered around the notion of truth. To give a semantics for a logical language is to give a recursive specification of truth conditions—to lay out, for each sentence of the language, what a state of affairs must be like in order for the sentence to count as true. Formally, semantics thus takes the form of a relation

$$w \models \alpha$$

where α is a sentence and w is a semantic object that models a state of affairs. Let us refer to such an object as a *possible world*.

The exact nature of the objects that play the role of possible worlds in this schema varies. Often, a possible world may be identified simply with a model for the language at stake; for instance, if α is a sentence of predicate logic, then w may be a standard relational structure. However, in this book we will build on *intensional semantics*, an approach which is designed to represent a whole variety of states of affairs in a single model. In this approach, a model M comes with an associated set W_M of possible worlds, primitive entities which stand for different states of affairs.

The central notion of logic, *entailment*, is then understood in terms of necessary preservation of truth: an entailment is valid if the conclusion is true whenever the premises are all true. Focusing of the case of a single premise:

$$\alpha \models_{\mathsf{truth}} \beta \iff \text{for all models } M \text{ and worlds } w \in W_M,\ w \models \alpha \text{ implies } w \models \beta.$$

This perspective naturally leads to the view that the notion of entailment—arguably the central concern of the field of logic—is only meaningful for statements. After all, if entailment is defined as necessary preservation of truth, it is only applicable to sentences which are *truth-apt*, i.e., capable of being true or false. And, arguably, being truth-apt is a property that distinguishes statements from other sentence types, like questions and commands.[4]

However, this truth-based construal of entailment is not the only possibility. An alternative conception arises from a more information-oriented perspective on semantics. Rather than taking semantics to specify in what circumstances a sentence is true, we may take it to specify what information it takes to *settle*, or *establish*, the sentence. On this view, semantics takes the form of a relation

$$s \models \varphi$$

where φ is a sentence and s is a semantic object that models a body of information. We will refer to s as an *information state* and we will read the expression $s \models \varphi$ as "*s supports φ*".[5]

As in the case of possible worlds, different options are available as to the formal modeling of information states that play a role in this semantics. Intuitively, an information state encodes certain information about what things are like, and thereby it determines a distinction between two kinds of states of affairs: those that fit the available information—and which are, thus, *live possibilities* according to the state—and those that do not fit the available information—and which are *ruled out* by the state. Thus, at a minimum, we want an information state s to determine a corresponding set of live possibilities, $\mathsf{live}(s) \subseteq W_M$. For our purposes in this book, this is in fact all we need to know about an information state. Therefore, we may simply identify an information state with a set of possible worlds—the corresponding set of live possibilities. Conversely, given a set s of possible worlds, we can think of it as encoding a body of information: the information that the state of affairs corresponds to one of those in s. Thus, throughout this book, information states are simply modeled as sets of possible worlds.[6]

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[4] As a matter of fact, we will argue below that our semantics does suggest a natural way to extend the notion of truth to questions—something that has been proposed before by Belnap (1966) and Belnap and Steel (1976). This, however, does not undermine our argument in this section: the truth conditions of a question, understood in the technical sense below, heavily underdetermine its semantics, and as such they tell us little about the logical relations between the question and other sentences.

[5] Information-oriented semantics have been considered often in the logic literature, especially as a starting point for various non-classical logics (e.g., Beth, 1956; Kripke, 1965; Routley and Meyer, 1973; Fine, 1974; Veltman, 1981), but sometimes also as alternative foundations for classical logics (e.g., Fine, 1975; Humberstone, 1981; van Benthem, 1986; Holliday, 2014, 2018). As far as the treatment of statements is concerned, our system will be somewhat similar to the ones in the latter tradition, though with one difference, discussed in detail in Section 2.8. To my knowledge, no previous attempt has been made to use such a semantic foundation to extend the scope of logic to questions.

[6] This way of modeling information states goes back at least to Hintikka (1962) and is standardly adopted in logic, formal semantics, and philosophy of language. An alternative approach, instantiated by the references in the previous footnote, treats information states

2.2.1. Definition. [Information states]
An *information state* in a model M is a subset $s \subseteq W_M$.

Notice that no state of affairs fits an inconsistent body of information. Therefore, the set of live possibilities corresponding to inconsistent information is the empty set. Conversely, if a body of information is consistent, there is some state of affairs that fits that information; therefore, the corresponding set of live possibilities is non-empty.

2.2.2. Definition. [Inconsistent state]
The *inconsistent information state* is the empty set of worlds, $\emptyset$. An information state is *consistent* if it is non-empty.

Information states can be ordered naturally according to how much information they contain. If t contains at least as much information as s, then every world which is ruled out by s is also ruled out by t; therefore, the set of live possibilities for t is a subset of the set of live possibilities for s, and so $t \subseteq s$. Conversely, if $t \subseteq s$, then t rules out every world that s rules out and possibly more; given that the only aspect of information that we are taking into account is its potential to circumscribe a set of possibilities, we should count t as being at least as strong as s. Thus, we view t as containing at least as much information as s just in case $t \subseteq s$; we then say that t is an *enhancement* of s, or that t *implies* s.

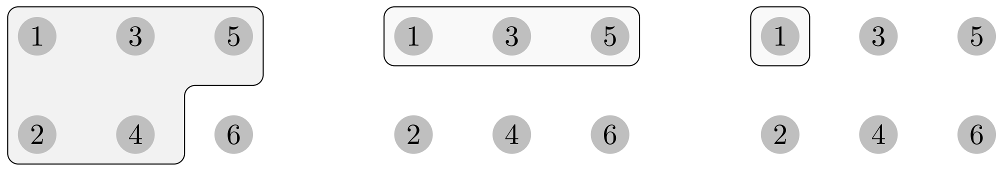
2.2.3. Definition. [Enhancement ordering] Given two information states s and t, we say that t is an *enhancement* of s in case $t \subseteq s$.

Let us illustrate this with an example. We can model our die scenario as involving a logical space of six possible worlds $w_1, \dots, w_6$, corresponding to the six possible outcomes of the die roll. Now here are three things we might know about the outcome of the roll:

(1) a. The outcome is not six.
b. The outcome is odd.
c. The outcome is 1.

If taken as complete descriptions of the available information, these correspond to three information states $s_{\neg\mathsf{six}}, s_{\mathsf{odd}}, s_{\mathsf{one}}$. The corresponding sets of live possibilities are shown in Figure 2.1. Note that these states are ordered from the weakest, $s_{\neg\mathsf{six}}$, to the strongest, s_{one}. The latter is a state of complete information: it determines exactly what the actual state of affairs is, and it is maximally strong among the consistent states.

as primitive entities specified by a model. The different merits of these approaches, and the reasons why we build on the first here, are discussed in detail in Section 2.8.



(a) $s_{\neg\mathsf{six}}$ (b) s_{odd} (c) s_{one}

Figure 2.1: Three information states in the die roll scenario, ordered from the weakest ($s_{\neg\mathsf{six}}$) to the strongest (s_{one}).

Importantly, a connection should obtain between the truth conditions of a statement α and its support conditions: this is because, on the intended understanding of the support relation, to establish that α is just to establish that the world is one where α is true. This means that s should count as supporting α just in case all live possibilities for s are worlds where α is true. To formulate this precisely, it is useful to introduce the following technical notion.

2.2.4. Definition. [Truth set] The truth-set of a statement α in a model M, denoted $|\alpha|_M$, is the set of worlds in M where α is true:

$$|\alpha|_M := \{w \in W_M \mid w \models \alpha\}.$$

Then the intended connection between truth conditions and support conditions can be spelled out as follows.

2.2.5. Constraint. [Truth-Support Bridge] Let α be a statement and M a model. For any information state $s \subseteq W_M$ we should have:

$$s \models \alpha \iff \forall w \in s : w \models \alpha.$$

Or, equivalently:

$$s \models \alpha \iff s \subseteq |\alpha|_M.$$

As an illustration, consider the following statement in the die roll scenario:

(2) The outcome is not two.

This statement should count as supported by the information states s_{one} and s_{odd}, since it is true at every live possibility in these states. But it should not be supported by $s_{\neg\mathsf{six}}$, since it is not true at w_2, which is a live possibility in $s_{\neg\mathsf{six}}$.

The Truth-Support Bridge above implies that the truth conditions of a statement determine its support conditions. Moreover, if we spell out this connection in the special case that s is a singleton state $\{w\}$, we find that the converse is also true, namely, that support conditions determine truth conditions:

$$w \models \alpha \iff \{w\} \subseteq |\alpha|_M \iff \{w\} \models \alpha.$$

Intuitively, this says that α is true at a world w just in case the information that w is the actual world implies that α. Thus, for statements, truth conditions and support conditions are inter-definable.

Let us now turn to entailment. Our informational perspective comes with a natural construal of entailment as preservation of support: an entailment is valid if the conclusion is supported by any information state that supports the premises. Focusing for simplicity on the case of a single premise:

$$\alpha \models_{\mathsf{info}} \beta \iff \text{for all models } M \text{ and info states } s \subseteq W_M,\ s \models \alpha \text{ implies } s \models \beta.$$

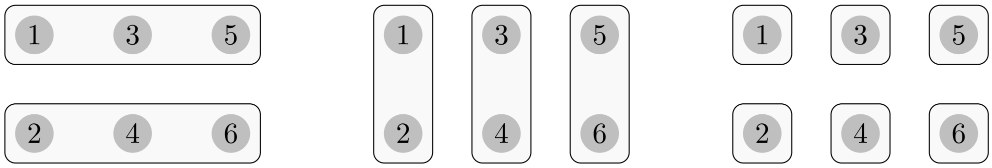
It follows from the Truth-Support Bridge that the two construals of entailment determine the same relation among statements:

$$\alpha \models_{\mathsf{truth}} \beta \iff \alpha \models_{\mathsf{info}} \beta.$$

To see this, suppose $\alpha \models_{\mathsf{truth}} \beta$. Consider an information state s supporting α: by the Truth-Support Bridge, this means that α is true everywhere in s. Since $\alpha \models_{\mathsf{truth}} \beta$, β must be true everywhere in s, too. Thus, using again the Bridge, β must be supported in s. This shows that $\alpha \models_{\mathsf{info}} \beta$. Conversely, suppose $\alpha \models_{\mathsf{info}} \beta$. Consider a world w where α is true. By the Truth-Support Bridge, $\{w\}$ is a state which supports α. Since $\alpha \models_{\mathsf{info}} \beta$, $\{w\}$ must also support β. Thus, using again the Bridge, β is true at w. This shows that $\alpha \models_{\mathsf{truth}} \beta$.

This means that, given our understanding of the support relation, construing entailment as preservation of support does not lead to a non-classical logic; instead, it provides an alternative semantic foundation for classical logic. Given the equivalence between the truth-conditional construal of entailment and the informational one, it is not surprising that the former, which is arguably simpler, has been taken as the standard one. However, the informational approach has a crucial advantage for our purposes: it extends naturally beyond statements to cover also *questions*. Indeed, while it is not clear what it means for a question to be true or false in a given state of affairs, there is a clear sense in which a question can be said to be settled, or not settled, by a given body of information. To illustrate this point, consider again the three questions from our die roll example, repeated below.

parity	Whether the outcome is even or odd
range	What the range of the outcome is (low, middle or high)
outcome	What the outcome is



(a) parity (b) range (c) outcome

Figure 2.2: The maximal supporting states for our three example questions.

Consider the model from Figure 2.1. What information states from this model count as settling each of these questions? The answer is straightforward. To settle the first, we need either enough information to conclude that the outcome is even ($s \subseteq \{w_2, w_4, w_6\}$), or enough information to conclude that the outcome is odd ($s \subseteq \{w_1, w_3, w_5\}$). To settle the second, we need either the information that the outcome is in the low range ($s \subseteq \{w_1, w_2\}$), or that it is in the middle range ($s \subseteq \{w_3, w_4\}$), or that it is in the high range ($s \subseteq \{w_5, w_6\}$). To settle the third we need information that determines exactly which world obtains ($s \subseteq \{w_i\}$ for some i). Thus, the support conditions of these questions in our model are:

(3) a. $s \models \textsf{parity} \iff s \subseteq \{w_1, w_3, w_5\}$ or $s \subseteq \{w_2, w_4, w_6\}$;
b. $s \models \textsf{range} \iff s \subseteq \{w_1, w_2\}$ or $s \subseteq \{w_3, w_4\}$ or $s \subseteq \{w_5, w_6\}$;
c. $s \models \textsf{outcome} \iff s \subseteq \{w_i\}$ for some $i \leq 6$.

These support conditions are visualized in Figure 2.2, which depicts the maximal supporting states for the three questions. In each case, the supporting states are the sets in the picture, as well as their subsets.

This illustrates how support conditions are obviously meaningful for questions. Moreover, there are good reasons to regard support conditions as a natural candidate for the role of semantic contents of questions. Here is one: a key role for questions in communication is that they allow speakers to formulate requests for information. The semantic content of a question should play a crucial role in determining the satisfaction conditions for such a request—i.e., in specifying what information is being requested by asking it. If the content of the question lies in its support conditions, which specify what information must be available for the question to count as settled, then it is clear how this role is played: the request is satisfied just in case a supporting state is established.

2.3 A general notion of entailment

We saw that two different perspectives are possible on the relation of entailment: the standard one based on truth, and an informational one based on support.

We saw that these two perspectives are extensionally equivalent for statements. However we saw that the support relation can be used to interpret not only statements, but also questions. As a consequence, if entailment is characterized in terms of support, then it extends in a natural way to questions. We can thus consider a more general entailment relation, $\Phi \models_{\mathsf{info}} \psi$, holding between a set Φ of sentences, which may include questions as well as statements, and a sentence ψ, which may be either a statement or a question:

$$\Phi \models_{\mathsf{info}} \psi \iff \text{for all models } M \text{ and states } s \subseteq W_M, s \models \Phi \text{ implies } s \models \psi$$

where $s \models \Phi$ abbreviates '$s \models \varphi$ for all $\varphi \in \Phi$'. Since this notion of entailment will be the one we will work with in the rest of the book, we will henceforth drop the subscript info whenever there is no risk of ambiguity. As usual, in terms of entailment we can also define notions of logical equivalence and logical validity:

- φ and ψ are *logically equivalent*, denoted $\varphi \equiv \psi$, if $\varphi \models \psi$ and $\psi \models \varphi$;
- φ is *logically valid*, denoted $\models \varphi$, if φ is entailed by the empty set.

Spelling out the definitions, we find that φ and ψ are logically equivalent if they are supported by the same information states in every model, and that φ is logically valid if it is supported by every state in every model.

2.3.1 Significance of entailments involving questions

What is the significance of this more general entailment relation? Focusing for now on the case of a single premise, we have four possible entailment patterns. Let us examine and illustrate briefly each of them.

- Statement-to-statement. If α and β are statements, then $\alpha \models \beta$ expresses the fact that settling that α implies settling that β. As we have already discussed, given the Truth-Support Bridge, this coincides with the familiar, truth-conditional notion of entailment: $\alpha \models \beta$ holds in case β is true whenever α is.

- Statement-to-question. If α is a statement and μ is a question, then $\alpha \models \mu$ means that settling that α implies settling the question μ. Thus, we may regard $\alpha \models \mu$ as expressing the fact that α *logically resolves* μ.

  Example: the statement 'Galileo discovered Ganymede in 1610' entails the question 'In what year did Galileo discover Ganymede?', as well as the question 'Did Galileo discover anything in 1610?'.

- Question-to-statement. If μ is a question and α is a statement, then $\mu \models \alpha$ means that settling the question μ, no matter how, implies settling that α. We thus regard $\mu \models \alpha$ as expressing the fact that μ *logically presupposes* α.

Example: the question 'In what year did Galileo discover Ganymede?' entails the statement 'Galileo discovered Ganymede'.

- Question-to-question. If μ and ν are both questions, then $\mu \models \nu$ expresses the fact that settling μ implies settling ν. This is just the relation of dependency that we discussed in the previous section, but now in its purely logical version, since all models—not just the intended one—are taken into account. Thus, $\mu \models \nu$ expresses the fact that μ *logically determines* ν.

  Example: the question 'In what year did Galileo discover Ganymede?' entails the question 'In what century did Galileo discover Ganymede?'.

Thus, support semantics gives rise to an interesting general notion of entailment, which covers questions as well as statements and which unifies four interesting logical notions: (i) a statement being a logical consequence of another; (ii) a statement logically resolving a question; (iii) a question logically presupposing a statement; and, finally, (iv) a question logically determining another.

2.3.2 Entailment in context

In ordinary situations, it is rarely the purely logical notion of consequence that we are concerned with. Rather, we typically take many facts about the world for granted and then assess whether *on that basis*, something follows from something else. We say, for instance, that the fact that Galileo discovered some celestial bodies follows from the fact that he discovered some of Jupiter's moons; in doing so, we take for granted the fact that Jupiter's moons are celestial bodies; worlds in which Jupiter's moons are not celestial bodies are simply disregarded.

The same holds for questions: when we are concerned with dependency, it is rarely purely *logical* dependency that is at stake. Rather, we are usually concerned with the relations that one question bears to another, given certain background facts. In our initial example, the background facts include what the possible outcomes of the roll are, what outcomes count as low, middle, and high, etc. It is against this contextual background that the dependency holds.

In order to capture these relations, besides the purely logical notion of entailment that we discussed, we will also introduce notions of entailment relative to a given model M, and relative to a given *context*. We will model a context simply as an information state s. In assessing entailment relative to s, we take the information embodied by s for granted. This means that, to decide whether an entailment holds or not, only worlds in s, and states consisting of such worlds, are taken into account.

2.3.1. Definition. [Contextual entailment]
Let M be a model and let $s \subseteq W_M$ be an information state. We let:

$$\Phi \models_s \psi \iff \text{for all } t \subseteq s, t \models \Phi \text{ implies } t \models \psi.$$

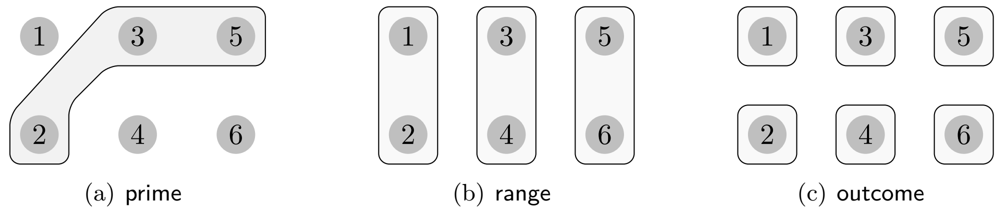


Figure 2.3: Illustration of a conditional dependency: given that the outcome is prime, the range of the outcome determines the outcome.

We write $\varphi \equiv_s \psi$ in case $\varphi \models_s \psi$ and $\psi \models_s \varphi$, i.e., in case φ and ψ are supported by the same states $t \subseteq s$. We write $\models_M$ and $\equiv_M$ instead of $\models_{W_M}$ and $\equiv_{W_M}$ for entailment and equivalence relative to the universe W_M of the model.

Contextual entailment captures relations of consequence, resolution, presupposition, and dependency which hold against the background of a specific context.

For an illustration, consider again our die roll example. Let M be the model formalizing the scenario. We have:

$$\mathsf{parity},\ \mathsf{range} \models_M \mathsf{outcome}.$$

We can see this visually from Figure 2.2: if a state settles parity then it is included in one of the rows of the model; if a state settles range, it is included in one of the columns; thus, if a state settles both parity and range, it must be included in a singleton, which means that it settles outcome.

The fact that this contextual entailment holds amounts precisely to our initial observation that a certain dependency holds in the described context: the outcome's parity and its range jointly determine what the outcome is. Thus, once we extend the notion of entailment to cover questions, dependencies turn out to be entailments—more precisely, question entailments in context.

2.3.3 Conditional dependencies

In our die scenario, the question range does not by itself determine the question outcome. However, *given* the information that the outcome is a prime number, range *does* determine outcome: if the outcome is in the low range it is two, if it is in the middle range it is three, and if it is in the high range it is five. If prime denotes the statement that the outcome is prime, then we can say that in the given context, range determines outcome conditionally on prime (see Fig. 2.3).

Generalizing, we can give the following definition.

2.3.2. Definition. [Conditional dependency]
In a state s, a question μ determines a question ν conditionally on a statement

α if μ determines ν relative to the the α-worlds in s. In symbols:

$$\mu \models_s^\alpha \nu \iff \forall t \subseteq s \cap |\alpha|_M : t \models \mu \text{ implies } t \models \nu.$$

It turns out that conditional dependencies can also be captured as instances of entailment: it suffices to regard the condition α as an additional premise alongside the determining question. That is, we have:

$$\mu \models_s^\alpha \nu \iff \alpha, \mu \models_s \nu.$$

We can show this as follows, using the Truth-Support Bridge, which applies since α is a statement:

$$\begin{aligned}
\mu \models_s^\alpha \nu \quad &\iff \quad \forall t \subseteq s \cap |\alpha|_M : t \models \mu \text{ implies } t \models \nu \\
&\iff \quad \forall t \subseteq s : t \subseteq |\alpha|_M \text{ and } t \models \mu \text{ implies } t \models \nu \\
&\iff \quad \forall t \subseteq s : t \models \alpha \text{ and } t \models \mu \text{ implies } t \models \nu \\
&\iff \quad \alpha, \mu \models_s \nu.
\end{aligned}$$

Thus, for instance, our example of conditional dependency above amounts to the entailment:

$$\mathsf{prime},\ \mathsf{range} \models_M \mathsf{outcome}.$$

The story extends straightforwardly to dependencies involving multiple determining questions and multiple conditions. The upshot is that a contextual entailment

$$\Gamma, \Lambda \models_s \nu$$

where Γ is a set of statements, Λ a set of questions, and ν a question, captures a conditional dependency: relative to s, the questions in Λ jointly determine the question ν given the statements in Γ.

A completely analogous story can be told when we replace contextual entailment by logical entailment: a relation

$$\Gamma, \Lambda \models \nu$$

captures a purely logical conditional dependency: *in any state of any model*, the questions in Λ jointly determine the question ν given the statements in Γ.

2.4 Questions as information types

In this section we show that both statements and questions may be regarded as describing information types: statements describe singleton types, which may be identified with specific pieces of information, while questions describe proper, non-singleton information types. This shows how the support approach may be viewed as generalizing the classical notion of entailment from pieces of information to arbitrary information types.

2.4.1 Inquisitive propositions

In truth-conditional semantics, the content of a sentence α in a model M is encoded by its *truth-set*, that is, by the set of all worlds in M where α is true:

$$|\alpha|_M = \{w \in W_M \mid w \models \alpha\}.$$

Similarly, in support-conditional semantics, the content of a sentence φ in a model M is encoded by its *support-set*, that is, the set of all states in M where φ is supported:

$$[\varphi]_M = \{s \subseteq W_M \mid s \models \varphi\}.$$

The support-set of a formula is a set of information states of a special form. Indeed, suppose an information state s settles a sentence φ: then, any information state t that enhances s will also settle φ. That is, the relation of support is *persistent*.[7]

Persistency: if $t \subseteq s$, $s \models \varphi$ implies $t \models \varphi$.

This implies that the support-set of a sentence $[\varphi]$ is always *downward closed*, that is, if it contains a state s, it also contains all stronger states $t \subseteq s$.

Downward closure: if $t \subseteq s$, $s \in [\varphi]_M$ implies $t \in [\varphi]_M$.

Another way to state this condition uses a downward closure operation $(\cdot)^{\downarrow}$. In words, the downward closure of a set S of info states, denoted $S^{\downarrow}$, is the set of those states which imply some element of S.

2.4.1. Definition. [Downward closure]
If $S \subseteq \wp(W_M)$, the *downward closure* of S is the set:

$$S^{\downarrow} = \{s \subseteq W_M \mid s \subseteq t \text{ for some } t \in S\}.$$

[7] At this point, this may be seen as a stipulation about the support relation, connected to the intended interpretation of this relation. Later on in the book, when we will consider specific formal systems, persistency will be proved as a fact about these systems. Note that in the case of statements, persistency follows from the Truth-Support Bridge.

It is worth pointing out that not all information-based semantics are persistent. E.g., the data semantics of Veltman (1981) contains sentences of the form 'may p' that are accepted by a body of information s, not if some information is available in s, but rather if p is *compatible* with s, i.e., if s fails to establish $\neg p$. Clearly, it is possible for p to be compatible with s but not with a stronger information state t, in which case 'may p' is accepted by s but not by the stronger state t. Many recent semantics for epistemic modals are in the same spirit (see, among others, Yalcin, 2007; Willer, 2013; Hawke and Steinert-Threlkeld, 2020). It is not clear whether the notion of acceptance involved in such accounts should be unified with the notion of support we are considering here. A conservative extension of inquisitive propositional logic with non-persistent operators has been studied by Punčochář (2015), who also provided a complete axiomatization of the resulting logic.

It is easy to see that $S^{\downarrow}$ is always a downward closed set of states, and moreover, it is the *smallest* downward closed set of states which contains S. The fact that the support-set $[\varphi]_M$ of a formula is downward closed may then be expressed succinctly as follows.

Downward closure, restated: $[\varphi]_M = ([\varphi]_M)^{\downarrow}$.

Downward closure is not the only general feature of the support-set of a sentence. Consider the empty information state, $\emptyset$. This represents an inconsistent body of information, which is not compatible with any possible world. It follows from the Truth-Support Bridge that $\emptyset$ supports any statement. This may be seen as a natural semantic version of the *ex-falso quodlibet* principle of classical logic. Similarly, it is natural to assume that $\emptyset$ also trivially supports any question, so that for all φ we have the following.[8]

Semantic ex-falso: $\emptyset \in [\varphi]_M$.

We will refer to a set of states which contains $\emptyset$ and satisfies downward closure as an *inquisitive proposition.* Now, for a downward closed set of states P, we have $\emptyset \in P \iff P \neq \emptyset$. Thus, we can define inquisitive propositions as follows.

2.4.2. Definition. [Inquisitive propositions]
An inquisitive proposition in a model M is a non-empty and downward closed set $P \subseteq \wp(W_M)$ of information states.

We refer to the support-set $[\varphi]_M$ of a sentence φ in a model M as the inquisitive proposition expressed by φ in M.

A special role is played by the maximal elements in an inquisitive proposition P. We will refer to these elements as the *alternatives* in P.

2.4.3. Definition. [Alternatives]
$\textsc{Alt}(P) = \{s \in P \mid \text{there is no } t \supset s \text{ such that } t \in P\}$.

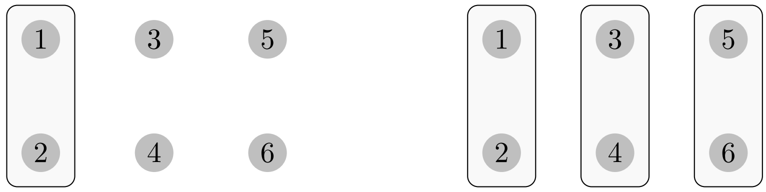
If φ is a sentence, we also refer to the alternatives in $[\varphi]_M$ as the *alternatives for* φ in the model M. Thus, the alternatives for φ are the minimally informed states that support φ, i.e., those that contain *just enough* information to settle φ. We write $\textsc{Alt}_M(\varphi)$ instead of $\textsc{Alt}([\varphi]_M)$.

2.4.2 Pieces and types of information

Consider the proposition expressed by a statement α. The Truth-Support Bridge requires the following connection:

$$s \in [\alpha]_M \iff s \subseteq |\alpha|_M.$$

[8]As for downward closure, at this point this is an assumption about how we want support to behave. Once we turn to concrete formal systems, this assumption will be proved as a fact.



(a) low (b) range

Figure 2.4: Singleton vs. non-singleton information types.

This implies that $[\alpha]_M$ has a unique maximal supporting state—a unique alternative—namely $|\alpha|_M$:

- $\textsc{Alt}_M(\alpha) = \{|\alpha|_M\}$.

This unique alternative for α is naturally regarded as a piece of information: the information that α is true. To settle that α is just to establish this piece of information. By means of downward closure, we can express this as follows:

- $[\alpha]_M = \{|\alpha|_M\}^{\downarrow}$.

Thus, a statement α may be regarded as naming a specific piece of information; the statement is settled in an information state s if this specific piece of information is available in s , i.e., if $s \subseteq |\alpha|_M$.

Things are different for questions. For an illustration, consider the question range of whether the outcome of the die roll is in the low, middle, or high range. This question has three alternatives in the intended model:

- $a_{\mathsf{low}} = \{w_1, w_2\}$;
- $a_{\mathsf{mid}} = \{w_3, w_4\}$;
- $a_{\mathsf{high}} = \{w_5, w_6\}$.

These alternatives correspond to three different pieces of information about the range of the outcome. We can think of them as three pieces of information which instantiate an information type, and we can think of the question range as naming this information type. To settle the question is to establish some information or other of this type. We can express this using the downward closure operation as follows:

- $[\mathsf{range}]_M = \{a_{\mathsf{low}}, a_{\mathsf{mid}}, a_{\mathsf{high}}\}^{\downarrow}$.

The difference between the case of a statement and that of a question is illustrated in Figure 2.4.

Similarly, the question parity—whether the outcome is even or odd—corresponds to an information type comprising the following pieces of information:

- $a_{\mathsf{even}} = \{w_2, w_4, w_6\}$;
- $a_{\mathsf{odd}} = \{w_1, w_3, w_5\}$.

These encode, respectively, the information that the outcome is even, and the information that it is odd.

Finally, the question outcome, what the outcome is, corresponds to a type of information comprising the pieces of information $a_{\mathsf{one}} = \{w_1\}, \ldots, a_{\mathsf{six}} = \{w_6\}$, each providing complete information about what the outcome is.

To make these observations more general, we introduce the following notion.

2.4.4. DEFINITION. [Normality]
We say that an inquisitive proposition P is *normal* in case $P = \textsc{Alt}(P)^{\downarrow}$.
We say that a sentence φ is normal in a model M in case $[\varphi]_M$ is normal.

Note that the inclusion $\textsc{Alt}(P)^{\downarrow} \subseteq P$ holds for any inquisitive proposition P: indeed, if $s \subseteq a$ for some $a \in \textsc{Alt}(P)$, then since $a \in P$ also $s \in P$ by downward closure. Thus, the normality condition amounts to the inclusion $P \subseteq \textsc{Alt}(P)^{\downarrow}$, i.e., to the requirement that any element of P be included in a maximal one.

A statement α is always normal, since as we have seen, the Truth-Support Bridge implies that any state supporting α is included in the truth-set $|\alpha|_M$, which is the unique alternative for α. The questions in our example are also normal, as are many other natural classes of questions. However, there is no reason to suppose that questions will in general be normal. We will encounter some examples of non-normal questions in Chapter 5. If a question μ is normal, then we can naturally think of it as describing, in a model M, a type of information whose instances are the alternatives $a \in \textsc{Alt}_M(\mu)$. To settle the question is to establish the information corresponding to one of these alternatives.

2.4.3 Generators and alternatives

We have seen that, for many examples of sentences φ, we have $[\varphi]_M = \textsc{Alt}_M(\varphi)^{\downarrow}$, and in this case, we can think of φ as describing a type of information $\textsc{Alt}_M(\varphi)$. However, there are many other sets of information states T such that $[\varphi]_M = T^{\downarrow}$. For instance, since $[\varphi]_M$ is downward closed we have $[\varphi]_M = ([\varphi]_M)^{\downarrow}$. So, what is so special about alternatives?

In this section we give an answer to this question having to do with the way in which an inquisitive proposition may be regarded as being *generated* from a given set of information states.

2.4.5. DEFINITION. [Generators for an inquisitive proposition]
A set T of info states is a *generator* for an inquisitive proposition P if $P = T^{\downarrow}$.

If T is a generator for $[\varphi]_M$, this means that a state s supports φ iff s implies some piece of information $a \in T$. Therefore, we can regard φ as standing for the type of information T.

Any inquisitive proposition P admits a trivial generator, namely, P itself. However, the examples discussed in the previous subsection show that many inquisitive propositions admit much smaller generators. For instance, in the case of the statement low that the outcome is low, the proposition $[\mathsf{low}]_M$ admits a singleton generator, namely, $\textsc{Alt}_M(\mathsf{low}) = \{a_{\mathsf{low}}\}$. In the case of the question range, the proposition $[\mathsf{range}]_M$ admits a generator consisting of only three elements, namely, $\textsc{Alt}_M(\mathsf{range}) = \{a_{\mathsf{low}}, a_{\mathsf{mid}}, a_{\mathsf{high}}\}$.

The generators $\textsc{Alt}_M(\mathsf{low})$ and $\textsc{Alt}_M(\mathsf{range})$ are very different from the trivial generators $[\mathsf{low}]_M$ and $[\mathsf{range}]_M$. First, it is easy to check that any element of $\textsc{Alt}_M(\mathsf{low})$ and $\textsc{Alt}_M(\mathsf{range})$ is essential to the representation of the corresponding proposition: if we were to remove it, the resulting set would no longer be a generator for the proposition. We say that these generators are *minimal.* Moreover, the elements of these generators are *pairwise logically independent*, in the sense that one element of the generator never implies another. We will say that these generators are *independent.*

2.4.6. Definition. [Minimal and independent generators]
Let T be a generator for an inquisitive proposition P. We say that T is:

- *minimal* if no proper subset $T' \subset T$ is a generator for P;
- *independent* if there are no $t, t' \in T$ such that $t \subset t'$.

The following proposition says that minimal and independent generators coincide.

2.4.7. Proposition (Minimality and independence are equivalent). *Let T be a generator for a proposition P. T is minimal iff it is independent.*

Proof. Suppose T is independent and consider a proper subset $T' \subset T$. Let $s \in T - T'$. Since $s \in T$ and T is a generator for P, we have $s \in T^{\downarrow} = P$. However, since $T' \subseteq T$, s cannot be a subset of any element of T': otherwise, it would be a subset of some element of T other than itself, contrary to the independence of T. This means that $s \notin T'^{\downarrow}$. Since $s \in P$, this shows that $T'^{\downarrow} \neq P$, i.e., that T' is not a generator for P. Since $T' \subset T$ was arbitrary, this shows that T is minimal.

For the converse, suppose T is not independent. Then, there are two states $s, t \in T$ such that $s \subset t$. Now consider $T' := T - \{s\}$. It is easy to verify that T' is still a generator for P, which shows that T is not minimal. $\square$

We will refer to a minimal generator for a proposition as a *basis*.[9]

2.4.8. Definition. [Basis for an inquisitive proposition]
A *basis* for an inquisitive proposition P is a minimal generator for P.

The next proposition shows that there *is* something special about the alternatives for an inquisitive proposition: whenever a proposition P admits a basis, the unique basis for P is $\textsc{Alt}(P)$.

2.4.9. Proposition.

- *An inquisitive proposition P has a basis iff P is normal;*
- *If P has a basis, then the unique basis for P is* $\textsc{Alt}(P)$.

Proof. First suppose P is normal, i.e., $P = \textsc{Alt}(P)^{\downarrow}$. This means that $\textsc{Alt}(P)$ is a generator for P. Moreover, notice that by definition, $\textsc{Alt}(P)$ is independent. By the previous proposition, it follows that $\textsc{Alt}(P)$ is a basis for P.

Moreover, suppose T is another basis for P. Consider any $s \in \textsc{Alt}(P)$: since $s \in P = T^{\downarrow}$, s must be a subset of some element $t \in T$. But now, since t is in T, t must also be in $T^{\downarrow} = P$. Since $s \in \textsc{Alt}(P)$ is by definition a maximal element in P; so, we must have $s = t$, which implies $s \in T$. Since this holds for any $s \in \textsc{Alt}(P)$, we have $\textsc{Alt}(P) \subseteq T$. By the minimality of T, this implies $T = \textsc{Alt}(P)$. This shows that, if P admits a basis, the unique basis is $\textsc{Alt}(P)$.

Conversely, suppose P is not normal, i.e., $P \neq \textsc{Alt}(P)^{\downarrow}$. Since $\textsc{Alt}(P)^{\downarrow} \subseteq P$ by the downward closure of P, we must have $P \not\subseteq \textsc{Alt}(P)^{\downarrow}$. This means that there must be a state $s \in P$ which is not included in any *maximal* state $t \in P$.

Now consider any generator T for P. Since T is a generator and $s \in P$, we must have $s \subseteq t$ for some $t \in T$. Now, we know that t cannot be a maximal element of P. So, let $u \in P$ be such that $t \subset u$. Since $u \in P$ and T is a generator, u must be a subset of some $t' \in T$. But then we have $t \subset u \subseteq t'$, that is, t is a proper subset of t', and both t and t' are in T. This shows that T is not independent, which by the previous proposition implies that T is not a basis. Since T was arbitrary, this shows that there is no basis for P. □

This result shows that, if P is normal, we can regard it as being generated in a canonical way by the set of its alternatives, while if P is non-normal, there is no minimal choice of a generator for P.[10]

[9] For readers acquainted with linear algebra, there is an analogy here with the notion of a *basis* for a vector space X: this can be characterized as a set T of vectors that (i) generates X, in the sense that $span(T) = X$ and (ii) is minimal, in the sense that no proper subset $T' \subset T$ generates X. Moreover, as in our case, instead of (ii) we can equivalently require that T be *independent* in a suitable sense.

[10] Some previous work in inquisitive semantics (see in particular Section 6.2 of Ciardelli

2.4.4 Entailment as a relation among information types

Consider again our initial example of a dependency. If we think of the questions in the examples as names for information types, we may phrase this relation as follows: information of type parity, combined with information of type range yields information of type outcome.

In this section, we discuss how bringing questions into play may be viewed as taking us from a logic of pieces of information to a logic of information types. To see this, take a specific model M and fix for any sentence φ a generator $T(\varphi)$ of $[\varphi]_M$, so that we can regard φ as describing the type of information $T(\varphi)$. If our sentences are normal, we saw that the canonical and most parsimonoius choice is $T(\varphi) = \textsc{Alt}_M(\varphi)$, but we will not need this assumption.

The following proposition shows that indeed, entailment holds if any piece of information of type φ implies some piece of information of type ψ.

2.4.10. Proposition.
$\varphi \models_M \psi \iff$ *for every* $a \in T(\varphi)$ *there is some* $a' \in T(\psi)$ *such that* $a \subseteq a'$.

Proof. Suppose $\varphi \models_M \psi$. This amounts to the inclusion $[\varphi]_M \subseteq [\psi]_M$. Take an arbitrary $a \in T(\varphi)$. Using the fact that $T(\varphi)$ and $T(\psi)$ are generators for $[\varphi]_M$ and $[\psi]_M$ we have:

$$a \in T(\varphi)^{\downarrow} = [\varphi]_M \subseteq [\psi]_M = T(\psi)^{\downarrow}.$$

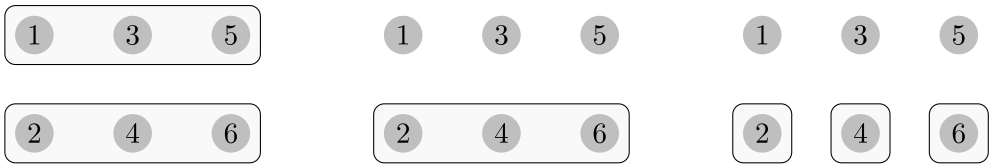
So we have $a \in T(\psi)^{\downarrow}$, which means that $a \subseteq a'$ for some $a' \in T(\psi)$.

Conversely, suppose for all $a \in T(\varphi)$ there is some $a' \in T(\psi)$ such that $a \subseteq a'$. This means that if a state s is a subset of some state in $T(\varphi)$, it is also a subset of some state in $T(\psi)$. So we have $T(\varphi)^{\downarrow} \subseteq T(\psi)^{\downarrow}$. Since $T(\varphi)$ and $T(\psi)$ are generators, it follows that $[\varphi]_M \subseteq [\psi]_M$, which means that $\varphi \models_M \psi$. □

In case at least one of the formulas involved is a statement α, things can be simplified by taking $T(\alpha) = \{|\alpha|_M\}$. First, suppose the formulas α and β at stake are both statements. In this case, the entailment boils down to a relation between pieces of information: the information that α is true implies the information that β is true.

2.4.11. Proposition (Statement-to-statement entailment).
$\alpha \models_M \beta \iff |\alpha|_M \subseteq |\beta|_M$

(2009) and Ciardelli (2010)), may be regarded from the present perspective as being concerned with the task of equipping a formula φ not only with an inquisitive proposition $[\varphi]_M$, but also with a designated generator $T_M(\varphi)$ for this proposition. This, however, will not be needed for our purposes in this book. Here, the focus will be on the semantic relation of support and on the notion of entailment that arises from it.



(a) Even or odd? (b) Outcome is even. (c) Which even number?

Figure 2.5: In this model, the question whether the outcome is even or odd is entailed by the statement that the outcome is even, which in turn is entailed by the question which even number is the outcome.

Next, suppose α is a statement and μ a question. The entailment $\alpha \models \mu$ holds in case the information that α is true yields some piece of information of type μ.

2.4.12. Proposition (Statement-to-question entailment).
$\alpha \models_M \mu \iff |\alpha|_M \subseteq a$ *for some* $a \in T(\mu)$

For an example, consider again the question parity, whether the outcome is even or odd. Then we can take $T(\mathsf{parity}) = \{a_{\mathsf{even}}, a_{\mathsf{odd}}\}$. So a statement α entails parity just in case $|\alpha|_M \subseteq a_{\mathsf{even}}$ or $|\alpha|_M \subseteq a_{\mathsf{odd}}$, i.e., just in case α implies that the outcome is even or it implies that the outcome is odd. This fits the idea that a statement entails a question if it resolves it.

Conversely, if μ is a question and α a statement, the entailment $\mu \models \alpha$ holds in case any information of type μ yields the information that α is true.

2.4.13. Proposition (Question-to-statement entailment).
$\mu \models_M \alpha \iff a \subseteq |\alpha|_M$ *for all* $a \in T(\mu)$

For an example, let μ be the following question:

(4) Which even number is the outcome of the roll?

Let $T(\mu) = \{a_{\mathsf{two}}, a_{\mathsf{four}}, a_{\mathsf{six}}\}$, where $a_{\mathsf{two}} = \{w_2\}$, $a_{\mathsf{four}} = \{w_4\}$, and $a_{\mathsf{six}} = \{w_6\}$. A statement α is entailed by μ just in case we have $a_{\mathsf{two}} \subseteq |\alpha|_M$, $a_{\mathsf{four}} \subseteq |\alpha|_M$, and $a_{\mathsf{six}} \subseteq |\alpha|_M$. This holds just in case $\{w_2, w_4, w_6\} \subseteq |\alpha|_M$. Thus, a statement α is entailed by μ if and only if α is entailed by the statement that the outcome is even. This fits the idea that a question entails a statement if it presupposes it. Figure 2.5 gives a graphical illustration.

Next, let us look at entailments with multiple premises. The following proposition states that an entailment holds if the following is the case: whenever we are given a piece of information for each premise type, the resulting combined body of information implies some piece of information of the conclusion type.

2.4.14. Proposition.

$$\varphi_1, \ldots, \varphi_n \models_M \psi \iff \textit{for every } a_1 \in T(\varphi_1), \ldots, a_n \in T(\varphi_n): \quad a_1 \cap \cdots \cap a_n \subseteq a' \textit{ for some } a' \in T(\psi)$$

We omit the proof, which is a variation of the one given above for the case of a single premise. As an illustration, consider our initial example of a dependency. The questions parity, range, and $\mathsf{outcome}$ are canonically associated with the following information types:

- $T(\mathsf{parity}) = \{a_{\mathsf{even}}, a_{\mathsf{odd}}\}$;
- $T(\mathsf{range}) = \{a_{\mathsf{low}}, a_{\mathsf{mid}}, a_{\mathsf{high}}\}$;
- $T(\mathsf{outcome}) = \{a_{\mathsf{one}}, \ldots, a_{\mathsf{six}}\}$.

The above proposition tells us that the entailment holds if for any way of pairing a piece of information a of type parity with a piece of information a' of type range, the joint information $a \cap a'$ implies some piece of information a'' of type $\mathsf{outcome}$. And this is indeed the case, as the following relations show.

$$a_{\mathsf{even}} \cap a_{\mathsf{low}} \subseteq a_{\mathsf{two}} \qquad a_{\mathsf{odd}} \cap a_{\mathsf{low}} \subseteq a_{\mathsf{one}}$$

$$a_{\mathsf{even}} \cap a_{\mathsf{mid}} \subseteq a_{\mathsf{four}} \qquad a_{\mathsf{odd}} \cap a_{\mathsf{mid}} \subseteq a_{\mathsf{three}}$$

$$a_{\mathsf{even}} \cap a_{\mathsf{high}} \subseteq a_{\mathsf{six}} \qquad a_{\mathsf{odd}} \cap a_{\mathsf{high}} \subseteq a_{\mathsf{five}}$$

This illustrates how dependencies amount to relations between information types: in this case, the entailment $\mathsf{parity}, \mathsf{range} \models_M \mathsf{outcome}$ captures the fact that in the given context, information of type parity and information of type range are guaranteed to jointly yield information of type $\mathsf{outcome}$.

2.5 Inquisitive implication

2.5.1 Internalizing entailment

In a support-based semantics, the contexts to which entailment can be relativized are the same as the objects at which formulas are evaluated, namely, information states. This makes it possible to define an implication connective $\to$ which internalizes the meta-language relation of entailment. The idea is to make $\varphi \to \psi$ supported by an information state s if relative to s, φ entails ψ:

$$s \models \varphi \to \psi \iff \varphi \models_s \psi. \tag{2.1}$$

Simply by making explicit what the condition $\varphi \models_s \psi$ amounts to, we get the support clause governing this operation:

$$s \models \varphi \to \psi \iff \text{for all } t \subseteq s,\ t \models \varphi \text{ implies } t \models \psi.$$

That is, an implication is supported in s in case enhancing s so as to support the antecedent is bound to lead to an information state that supports the consequent. Interestingly, this is, *mutatis mutandis*, precisely the interpretation of implication that we also find in most information-based semantics, such as Beth and Kripke semantics for intuitionistic logic, Veltman's data semantics, and Humberstone's possibility semantics (cf. references in Footnote 5).

Let us first consider the result of applying this implication operation to a pair of statements α and β. Using the Truth-Support Bridge, and denoting by $\supset$ the truth-functional material conditional, we get:

$$\begin{aligned} s \models \alpha \to \beta &\iff \forall t \subseteq s,\ t \models \alpha \text{ implies } t \models \beta \\ &\iff \forall t \subseteq s,\ t \subseteq |\alpha|_M \text{ implies } t \subseteq |\beta|_M \\ &\iff s \cap |\alpha|_M \subseteq |\beta|_M \\ &\iff s \subseteq |\alpha \supset \beta|_M. \end{aligned}$$

Thus, the semantics of $\alpha \to \beta$ that our clause delivers is the same that we would have obtained by lifting the material conditional of classical logic in accordance with the Truth-Support Bridge: $\alpha \to \beta$ is supported at a state if the material conditional is true at each world in the state.

However, our implication connective is applicable not only to statements, but also to questions. For instance, suppose μ and ν are questions. According to the clause above, we have:

$$s \models \mu \to \nu \iff \mu \models_s \nu.$$

The right-hand side amounts to the fact that μ determines ν relative to s. So $\mu \to \nu$ is a formula that expresses within the object language that μ determines ν; it is supported precisely by those information states in which this dependency holds. Similarly, if α is a statement and μ a question, then $\alpha \to \mu$ expresses the fact that α resolves μ, and $\mu \to \alpha$ the fact that μ presupposes α.

Entailment relations involving multiple premises can also be expressed in the object language by means of $\to$, as shown by the following proposition.

2.5.1. Proposition. *For any number n we have:*

$$s \models \varphi_1 \to (\cdots \to (\varphi_n \to \psi)) \iff \varphi_1, \dots, \varphi_n \models_s \psi.$$

Proof. For simplicity, we give the proof for the case of $n = 2$, but the argument generalizes straightforwardly.

$$\begin{aligned} & s \models \varphi_1 \to (\varphi_2 \to \psi) \\ \iff & \forall t \subseteq s : t \models \varphi_1 \text{ implies } \forall t' \subseteq t : (t' \models \varphi_2 \text{ implies } t' \models \psi) \\ \iff & \forall t \subseteq s : t \models \varphi_1 \text{ and } t \models \varphi_2 \text{ implies } t \models \psi \\ \iff & \varphi_1, \varphi_2 \models_s \psi \end{aligned}$$

The crucial step in this derivation is the second biconditional. For the left-to-right direction, notice that we can take $t' = t$. For the converse, note that by the persistency of support, if $t \models \varphi_1$ and $t' \subseteq t$ then also $t' \models \varphi_1$. □

This has interesting repercussions for the expression of conditional dependencies. As we discussed above in Section 2.3.3, the fact that in a state s a question μ determines a question ν conditionally on a statement α is captured by the contextual entailment $\alpha, \mu \models_s \nu$. By what we have just seen, this entailment can be expressed in the object language by the formula

$$\alpha \to (\mu \to \nu).$$

So this formula expresses that the dependency $\mu \to \nu$ holds conditionally on α. This is interesting, as it shows that conditional dependencies can indeed be seen as arising from conditionalizing dependencies in a precise sense. They can be expressed as conditionals having the condition as antecedent, and a dependence formula as consequent.

To summarize: we saw above that the classical meta-language entailment relation can be generalized to questions, allowing us to capture the relations of (conditional) dependency, resolution, and presupposition. Now we have seen that, in a parallel way, the classical implication connective can be generalized to questions, providing us with the linguistic resources to express such relations in the object language.

2.5.2 Generalizing the Ramsey test

Perhaps the most influential insight in theorizing about conditionals is the *Ramsey test* idea, so called after a footnote in Ramsey (1929). According to this idea, interpreting a conditional on the basis of a certain body of information involves supposing the antecedent and then assessing whether (or to what extent) the resulting hypothetical state supports the conclusion.

In this section we are going to see that the semantics of our inquisitive conditional vindicates the Ramsey test idea and generalizes it to questions.

Let us start by asking how we can model the process of supposing a statement α in an information state s.[11] The natural answer in our setting is that to suppose α in s is to suppose that the world is one where α is true, i.e., to enter a hypothetical state which extends s by incorporating the information $|\alpha|_M$. In other words, supposing α takes us from s to the hypothetical state $s \cap |\alpha|_M$.

[11]We are focusing here on the process of supposing α as an epistemic assumption, which is relevant for the interpretation of indicative conditionals such as ‘if Alice left, she went to London’. A different supposition process is involved in interpreting subjunctive conditionals such as ‘if Alice had left, she would have gone to London’.

The following proposition shows that when the antecedent is a statement, our semantics yields a version of the Ramsey test idea: a conditional is supported by an information state just in case the consequent is supported by the hypothetical state obtained by supposing the antecedent.

2.5.2. PROPOSITION (RAMSEY TEST, DETERMINISTIC CASE). *Suppose α is a statement, and ψ a sentence which may be either a statement or a question. Then for any model M and state s:*

$$s \models \alpha \to \psi \iff s \cap |\alpha|_M \models \psi.$$

Proof. We have:

$$\begin{aligned} s \models \alpha \to \psi \quad &\iff \quad \forall t \subseteq s: \ t \models \alpha \text{ implies } t \models \psi \\ &\iff \quad \forall t \subseteq s: \ t \subseteq |\alpha|_M \text{ implies } t \models \psi \\ &\iff \quad \forall t \subseteq s \cap |\alpha|_M : \ t \models \psi \\ &\iff \quad s \cap |\alpha|_M \models \psi \end{aligned}$$

where the second biconditional uses the Truth-Support Bridge for α and the last biconditional uses the persistency of support. □

What about the case in which the antecedent is a question μ? In that case, the antecedent does not identify a single piece of information and, therefore, it does not determine a single supposition. Instead, the antecedent determines an information type $T(\mu)$, which is instantiated by multiple pieces of information $a \in T(\mu)$. Each of these pieces of information can be supposed in s, leading to a corresponding hypothetical state $s \cap a$. Thus, an interrogative antecedent does not determine a single hypothetical state, but multiple hypothetical states. It is then natural to generalize the Ramsey test idea in the following way: interpreting a conditional on the basis of a certain body of information involves supposing information of the antecedent type and then assessing whether each of the resulting hypothetical states supports the conclusion. We can think of an interrogative antecedent as inducing a non-deterministic supposition, and we can think of the conditional as claiming that this supposition is bound to lead to a state that supports the consequent. The following proposition shows that the semantics of our conditional is in line with this generalization of the Ramsey test idea.

2.5.3. PROPOSITION (RAMSEY TEST, GENERAL CASE). *Let φ, ψ be either statements or questions. Let M be a model, $T(\varphi)$ a generator for $[\varphi]_M$, and s an information state in M. Then:*

$$s \models \varphi \to \psi \iff s \cap a \models \psi \textit{ for every } a \in T(\varphi).$$

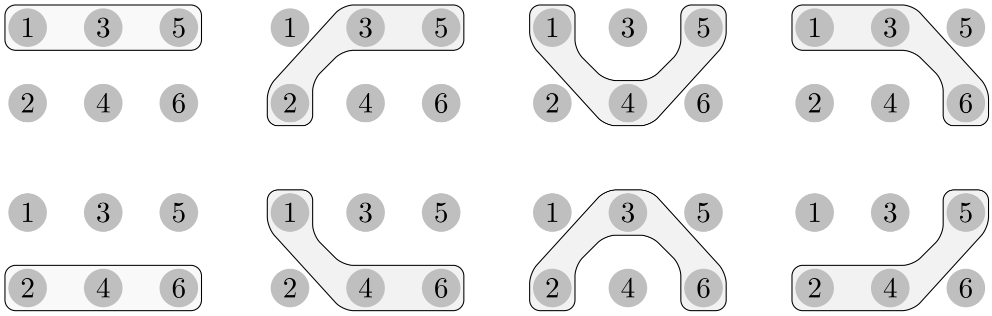


Figure 2.6: The eight alternatives for the implication $\mathsf{range} \to \mathsf{outcome}$. Each alternative represents a maximal state in which range determines $\mathsf{outcome}$, and corresponds to a way for the dependency to obtain.

Proof. Suppose $s \models \varphi \to \psi$. Take any $a \in T(\varphi)$. Since $a \models \varphi$, by persistency $s \cap a$ is a subset of s supporting φ. Since $s \models \varphi \to \psi$, $s \cap a$ must support ψ.

Conversely, suppose $s \cap a \models \varphi$ for every $a \in T(\varphi)$. Take any $t \subseteq s$ which supports φ. Since $T(\varphi)$ is a generator for $[\varphi]_M$, this means that we must have $t \subseteq a$ for some $a \in T(\varphi)$. Therefore, $t \subseteq s \cap a$. By our assumption, $s \cap a \models \psi$, and so by persistency, $t \models \psi$. This shows that $s \models \varphi \to \psi$. $\square$

To illustrate this generalization of the Ramsey test to questions, it is helpful to consider a concrete example.

2.5.4. Example. [Implication among questions] Consider again the questions range and $\mathsf{outcome}$ from our die roll scenario. Recall from Section 2.4.2 that the question range can be associated with the generator

$$T(\mathsf{range}) = \{a_{\mathsf{low}}, a_{\mathsf{mid}}, a_{\mathsf{high}}\}$$

where $a_{\mathsf{low}} = \{w_1, w_2\}$, $a_{\mathsf{mid}} = \{w_3, w_4\}$, and $a_{\mathsf{high}} = \{w_5, w_6\}$. According to the previous proposition, a state s supports the conditional

$$\mathsf{range} \to \mathsf{outcome}$$

just in case the following three conditions hold:

- $s \cap a_{\mathsf{low}} \models \mathsf{outcome}$;
- $s \cap a_{\mathsf{mid}} \models \mathsf{outcome}$;
- $s \cap a_{\mathsf{high}} \models \mathsf{outcome}$.

So, our question antecedent is associated not with a single supposition, but with three suppositions. Graphically, each of these suppositions amounts to restricting the state s to a specific column. In order to support $\mathsf{outcome}$, the resulting hypothetical states are required to contain at most one world. So, the states that support the implication range $\to$ outcome are those that contain at most one world from each column. The maximal such states—the alternatives—are those that select *exactly* one world from each column. Since there are 3 columns and 2 worlds for each, there are $2^3 = 8$ alternatives for the implication range $\to$ outcome. These are visualized in Figure 2.6. Note that each of these alternatives corresponds to one particular way for the question range to determine the question outcome, i.e., to one particular way for the dependency to obtain.

2.6 Truth and question presupposition

In Section 2.2, we saw that for statements, truth conditions and support conditions are inter-definable via the Truth-Support Bridge. In particular, if we are given the support conditions of a statement α, we can recover its truth conditions by means of the following connection: α is true at a world if and only if it is supported at the corresponding singleton state. In symbols:

$$w \models \alpha \iff \{w\} \models \alpha.$$

In a semantics where support is the primitive semantic notion, we can take this relation to provide a definition of truth in terms of support. Since support is defined not only for statements, but also for questions, the resulting notion of truth is then defined for questions as well.

But intuitively, what does it *mean* for a question μ to be true at a world w? The definition says that μ is true at w in case μ is settled by the information state $\{w\}$. Now, $\{w\}$ is a state of complete information: thus, if the question is not settled in this state, this cannot be because not enough information is available: it must be because the question does not admit any truthful resolution at w. We can read $w \models \mu$ as capturing the fact that question μ is *soluble* at w.

This interpretation is backed by the following proposition, which follows immediately from the persistency of support.

2.6.1. Proposition.
For any sentence φ and any world w: $w \models \varphi \iff (w \in s$ for some $s \models \varphi)$.
Equivalently, for any sentence φ and model M we have: $|\varphi|_M = \bigcup[\varphi]_M$.

For a question μ, this proposition states that μ is true at w if there is some body of information s that settles μ ($s \models \mu$) without ruling out w ($w \in s$).

As illustration, consider again the question in (4), repeated below:

(5) Which even number is the outcome of the roll?

(a) $[\text{Which even number?}]_M$ (b) $|\text{Which even number?}|_M$

Figure 2.7: The proposition expressed by a question, and the corresponding truth-set.

This question is supported by the singletons $\{w_2\}$, $\{w_4\}$, $\{w_6\}$, but not by the singletons $\{w_1\}$, $\{w_3\}$, $\{w_5\}$. So, the question is true at w_2, w_4, w_6, but not at w_1, w_3, w_5. In other words, this question is true just in case the outcome is even—it has the same truth conditions as the statement:

(6) The outcome of the roll is even.

Figure 2.7 illustrates the relation between the proposition expressed by (5), $[(5)]_M$, and the set of worlds where (5) is true, $|(5)|_M$.

Interestingly, this way of extending the notion of truth to questions was proposed before by Belnap (1966) (see also Belnap and Steel, 1976), though in the context of a slightly different approach to questions (cf. Section 2.9). Belnap put it as follows:

> I should like in conclusion to propose the following linguistic reform: that we all start calling a question "true" just when some direct answer thereto is true. (Belnap, 1966)

We will also follow (Belnap, 1966) in taking the truth conditions of a question to capture its *presupposition*. Belnap puts forward the following thesis (the exact phrasing is not Belnap's, but it is one he could have subscribed to, given the "linguistic reform" he proposed in the above passage).

Belnap's Thesis
Every question presupposes precisely that its truth conditions obtain.

The qualification *precisely* means that a question presupposes that its truth conditions obtain, and nothing more. Thus, what a question presupposes is captured entirely by its truth conditions.

Like Belnap, we will say that a statement α *expresses the presupposition* of a question μ in case α and μ have the same truth conditions. Thus, for instance, the statement (6) expresses the presupposition of the question (5). In the logical systems that we will develop, it will often be convenient to associate with each question μ in the system a specific statement π_μ which expresses the question's presupposition; for brevity, we will also refer to π_μ as the presupposition of μ.[12]

In the discussion above, we said that, if μ is a question and α is a statement, we can view the entailment $\mu \models \alpha$ as capturing the fact that μ *presupposes* α. The following proposition says that μ presupposes α just in case α follows from the presupposition π_μ of μ. The proof is left as an exercise (Exercise 2.10.5).

2.6.2. PROPOSITION.
Let μ be a question, π_μ a statement that expresses the presupposition of μ, and α an arbitrary statement. Then

$$\mu \models \alpha \iff \pi_\mu \models \alpha.$$

The same holds when logical entailment is replaced by contextual entailment.

To conclude this section, let us address a possible source of confusion. We started by claiming that, in order to bring questions within the scope of logic, we should move away from truth-conditional semantics, since questions cannot be interpreted in terms of truth conditions. But now we are saying that questions have truth conditions after all. Doesn't this, then, undermine our argument?

It does not: although we can define a technical notion of truth that applies to questions, that does not mean that the semantics of questions can be captured in terms of truth conditions. On the contrary, the truth conditions of a question heavily underdetermine its semantics. To see this, consider again the three questions parity, range, and outcome from our die roll example. It is easy to check that these questions have the same truth conditions in our model: they are all true at every world. But obviously they are not equivalent in the model, and the logical relations among them are non-trivial. Thus, we cannot rely on our generalized truth conditions to extend logic to questions in a meaningful way. It is only at the level of support that this extension is possible.

This discussion highlights a fundamental semantic difference between statements and questions. The support conditions of a statement are fully determined by its truth conditions: support at a state just amounts to truth at each world. By contrast, the support conditions of a question are underdetermined by its truth conditions, which only capture the presupposition of the question. We express this by saying that statements, unlike questions, are *truth-conditional*.

2.6.3. DEFINITION. [Truth-conditionality]
We call a sentence φ *truth-conditional* if for all models M and states $s \subseteq W_M$:

$$s \models \varphi \iff w \models \varphi \text{ for all } w \in s.$$

[12]This syntactic notion of presupposition will be in line with the one used in other logical frameworks concerned with questions, such as Hintikka's *interrogative model of inquiry* (Hintikka, 1981, 1999) and Wiśniewski's *inferential erotetic logic* (Wiśniewski, 1994, 1996, 2001).

In the formal systems of the next chapters, we will take truth-conditionality to be the fundamental semantic difference between statements and questions. Our languages will not incorporate a syntactic distinction between statements and questions (though such a distinction is compatible with the project of inquisitive logic; see for instance Groenendijk, 2011; Ciardelli et al., 2015). Rather, we will regard truth-conditional formulas as statements, and non truth-conditional formulas as questions.[13] The following proposition provides an alternative characterization: statements may be seen as describing specific pieces of information, whereas questions need to be regarded as describing proper information types.

2.6.4. PROPOSITION.
φ is truth-conditional $\iff$ $[\varphi]_M$ admits a singleton generator in any model M.

Proof. If φ is truth-conditional, it is easy to see that $[\varphi]_M$ admits the singleton generator $\{|\varphi|_M\}$ in any model. Conversely, suppose $[\varphi]_M$ always admits a singleton generator and consider a model M. Let $\{a_\varphi\}$ be a singleton generator for $[\varphi]_M$, which means that $[\varphi]_M = \{a_\varphi\}^\downarrow$. We have:

$$w \in |\varphi|_M \iff \{w\} \in [\varphi]_M = \{a_\varphi\}^\downarrow \iff \{w\} \subseteq a_\varphi \iff w \in a_\varphi.$$

This shows that $|\varphi|_M = a_\varphi$, that is, the unique element of the generator must be precisely the truth-set of φ. Finally, using this fact we have:

$$s \models \varphi \iff s \in [\varphi]_M = \{a_\varphi\}^\downarrow = \{|\varphi|_M\}^\downarrow \iff s \subseteq |\varphi|_M.$$

That is, φ is supported at a state in M in case it is true everywhere in the state. Since this is true for any M, φ is truth-conditional. □

2.7 Summing up

We have seen that classical logic can be given an alternative, informational semantics in terms of support conditions, which determines when a sentence is settled by a body of information, rather than when it is true at a world. This semantics can be extended to interpret questions in a natural way. Using this semantics, we can generalize the classical notion of entailment to questions. Several interesting logical notions, including the relation of dependency discussed in Section 2.1, turn out to be facets of this general entailment relation.

We have discussed how, based on our semantics, sentences may be regarded as denoting information types: statements denote singleton types, which can

[13] A minor issue is that this criterion does not classify formulas corresponding to tautological questions (e.g., *whether John is John*) as questions. Such formulas are trivially supported by any state, and so, they are trivially truth-conditional. This can be regarded as a consequence of the fact that in intensional semantics, all tautologies are equivalent, and so we cannot expect a semantic difference between tautological questions and tautological statements.

be identified with specific pieces of information; questions denote non-singleton types, which are instantiated by several different pieces of information. An entailment holds if information of the type described by the premises is guaranteed to yield information of the type described by the conclusion.

We saw that a logic formulated within this framework can be equipped with a conditional operator which captures the meta-language entailment relation within the object language—providing us, in particular, with the linguistic resources to express dependencies and conditional dependencies. The semantics of this operator amounts to a generalization of the Ramsey test idea: a conditional is supported by an information state if supposing information of the antecedent type leads to a hypothetical state that supports the consequent.

Finally, we saw that a support-based semantics suggests a natural way to extend the notion of truth to questions and that, under such an extension, the truth conditions of a question may be viewed as capturing its presupposition.

One important part of the conceptual picture is still missing. This has to do with the role of questions in inference. We postpone discussion of this important topic until Chapter 3, since the main points are best illustrated once we have a concrete proof system on the table. However, we may already anticipate the main idea: using questions in proofs allows us to reason with arbitrary information of a given type. For instance, by assuming the question range, one is supposing to have the information whether the outcome is low, middle, or high. One can then reason about what other information one has on that basis, and thereby formally prove that certain dependencies hold.

This completes our presentation of the foundations of inquisitive logic. The rest of this chapter contains two discussion sections: the first (§2.8) concerns some modeling choices we made in this section, the second (§2.9) the relations between the present framework and previous approaches to questions in logic. The contents of these sections are not presupposed in the following chapters, so the reader may choose to skip ahead to the exercises (§2.10) or to the next chapter.

2.8 Setup choices

In this section we discuss in more detail some of the basic setup choices we made in this chapter, which will be reflected in the systems to be developed in the rest of the book. We focus on issues concerning the modeling of information states.

2.8.1 Modeling information states: explicit vs. implicit

We have seen that, by moving from a semantics based on possible worlds to a semantics based on information states, we can interpret both statements and questions in a uniform way, and thereby we obtain an interesting generalization of the classical notion of entailment.

This approach is compatible with different ways to model information states. We have opted here for an *explicit* modeling of information states, which assigns a *content* to information states and then orders information states in terms of their content. In our case, the relevant content of an information state is its power to represent things as being in a certain way—thus circumscribing a set of worlds as live possibilities.[14] This allowed us to identify an information state with the corresponding set of live possibilities and, thus, to model information states in the context of standard possible-world models of the kind commonly used in intensional logic.

Many information-based semantics proceed differently: they take information states and the relation of enhancement between them as primitive objects in the model, possibly making additional assumptions about the resulting ordered set. We may call this an *implicit* modeling of information states, since there is no explicit modeling of the *content* of an information state: in this approach, the model does not explicitly specify what the information available in an information state is. One gets some implicit description of the relevant information *via* the semantics, but even this does not entirely reflect the content of the state, since not all aspects of the available information need to be expressible in the relevant language. For instance, consider Kripke semantics for intuitionistic logic: a model may consist of an infinite chain $s_0, s_1, s_2, \dots,$ of information states, where each state s_{n+1} is stronger than s_n and where each state satisfies the same sentences. The model represents s_1 as being stronger than s_0, but it does not tell what information we have at s_1 but not at s_0.

In principle, the project of inquisitive logic is compatible with both ways of introducing information states into the picture, and in fact, it has been carried out within both kinds of approaches (for studies of inquisitive logics based on the implicit approach, see Punčochář, 2016a,b, 2019, 2020; Holliday, 2020). Both approaches are valuable, for different reasons. For our goals in this book, an explicit modeling of information states has several advantages.

First, it allows us to better motivate and assess the semantics. A crucial feature of the explicit approach is that one can see what the content of an information state is *independently* of the semantics. This allows us to see whether the semantics itself makes reasonable predictions—that is, if it declares a sentence to be supported by those states in which it is intuitively settled in view of the content of the state, which we can grasp independently of the semantics. We made use of this feature above, when we discussed what information states in our die roll scenario should count as supporting our questions. In doing this, we

[14] One could in principle consider other aspects of information as well: one could, for instance, also represent quantitative information about the likelihoods of different states of affairs by equipping an information state with a probability function on the set of live possibilities.

relied on intuitions such as: the question is settled if and only if the information state implies that the state of affairs is such-and-such. An inquisitive semantics can be motivated, and assessed, on the basis of such support intuitions, just like a truth-conditional semantics can be motivated and assessed on the basis of intuitions about truth. By contrast, in the implicit perspective, we have no direct way to motivate and assess the semantics, since we have no independent access to the information that a state is supposed to encode. We can only assess the semantics *indirectly*, via the logic that it yields.

Second, the explicit perspective comes with a natural way to model concrete scenarios of partial information, such as our die example above: just build a model with one world for each way things may be; at least in simple examples, we have a good grasp of what these are. By contrast, the abstract perspective underdetermines which model we are supposed to use to represent a concrete scenario, even one of the simplest kind.

Thirdly, in the implicit approach, the logic one gets depends in part on one's assumptions about the structure of the space of information states. If one already has a logic in mind and simply wants to provide a semantics for it, one just has to find the right set of assumptions. But if one wants to *motivate* a logic on the basis of the semantics, then one has to justify one's choice of a particular set of assumptions; that means not just motivating the assumptions one makes, but also arguing that no further assumptions should be made. This is hard, however, and rarely even attempted. By contrast, in the explicit approach, the structure of the space of information states does not have to be stipulated, but is determined by the contents of the relevant states.

Finally, the explicit approach is more conservative. It is not part of our aim in this book to question the suitability of truth-conditional semantics for statements. At the same time, we argued for a departure from truth-conditional semantics in order to extend logic to questions. Our approach allows us to have our cake and eat it too: we can base our semantics on standard possible-world models for intensional semantics; we interpret sentences in terms of support relative to sets of possible worlds; and then for statements we can retrieve truth conditions relative to worlds in the way described in Section 2.6. One may have independent reason for abandoning truth-conditional semantics for statements; but the task of extending logic to questions does not necessitate this move.

Let me illustrate this point with an example. In Chapter 8 we will discuss modal sentences like $\Box ?p$, which on one interpretation can be read as "the agent knows whether p". Here, the argument of the modality is a question, $?p$, but the entire sentence is a statement. The intended truth conditions of this statement can be specified in the setting of a standard Kripke model for epistemic logic as follows:

$$w \models \Box ?p \iff \forall v, v' \in R[w] : (v \models p \iff v' \models p).$$

In inquisitive modal logic, we can deliver these truth conditions compositionally. The semantics is given in the setting of a standard Kripke model, but now in terms of support conditions. From these we can then derive truth conditions relative to possible worlds. For our statement, these are exactly the ones given above. Interpreting questions need not prevent us from assigning standard truth conditions to statements; on the contrary, it allows us to explain how statements involving question constituents get their truth conditions compositionally.

All this is not to say that the explicit approach is preferable in all respects. The implicit approach has its own benefits. An important one is that it allows us to consider things at a higher level of abstraction, from the perspective of a very general framework that can be further constrained in various ways. In this way, we can see just what assumptions about the space of information states are responsible for certain features of the resulting logic. Moreover, we can study the variety of different logics that can result from different assumptions—as well as what these logics all have in common. This interesting project has been pursued in some detail in a series of papers by Vit Punčochář (2016a; 2016b; 2019; 2020).

2.8.2 "Accessible" information states? Distinguishing semantics and epistemology.

A body of information describes the world as being a certain way, and thus determines a set of worlds $s \subseteq W$. Conversely, given a set of worlds $s \subseteq W$, we can think of it as a body of information: namely, the information that the world is one of those in s.

But—one may object—should this always qualify as an information state? What if it is not actually possible, due to constraints on cognition or on inquiry, to be in a state which determines s as its set of live possibilities?

A first thing to say is that this objection presupposes a different conception of the notion of information state than the one which is relevant for our enterprise. It presupposes that "information state" means something like "possible belief state of the agent" or "possible outcome of inquiry", and therefore must be constrained by considerations about the limits of cognition or inquiry. Our view of information states is more abstract: an information state is just something that partially determines how things are; it makes sense to ask whether such an object settles a question, or a statement, quite regardless of whether it is, or could be, the belief state of an agent in an inquiry scenario.

Moreover, from the point of view of our enterprise, restricting the semantics to information states that are 'accessible' in some epistemic sense would be a bad idea: it would mix semantics and epistemology in an unintended way. Since the suggestion to modify the semantics in this way regularly comes up, it is worth illustrating the consequences of this sort of restriction in some detail.

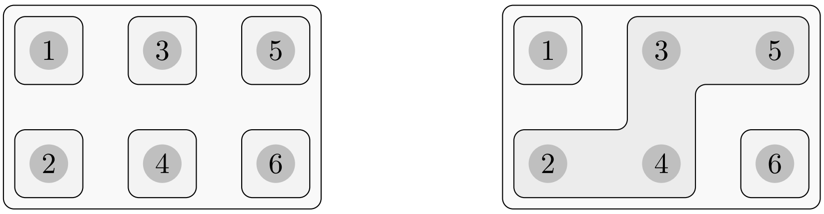


Figure 2.8: The "accessible" information states in the two scenarios discussed in this section.

Suppose that, in the die scenario, things are set up in such a way that the outcome will be revealed to us at once. Given this setup, it is not possible for us to gain partial information about the outcome: the only 'accessible' information states in this scenario are the initial state, where every outcome is possible, and the six complete states in which the outcome is revealed. These are shown in the picture on the left in Figure 2.8.

Let parity denote the question whether the outcome is even or odd, and let range denote the question whether the range is low, middle, or high. If we only accessible information states are taken into account, relative to the model M which represents our state prior to the roll we get the prediction that

$$\mathsf{parity} \models_M \mathsf{range},$$

since any accessible state that supports parity is a complete state, and therefore also supports range. What this relation captures is a pragmatic fact about the inquiry situation: learning the parity of the outcome implies learning the range.

However, by focusing on learning, this approach misses a more basic fact: the parity of the outcome does *not* determine its range: for instance, the outcome being even does not determine whether it is low, middle, or high. This fact has nothing to do with any agent, or with learning. It has to do with what the live possibilities are and with how the two questions are related relative to this set of possibilities. It is a purely *semantic* matter.

On the standard view of logic, it is this sort of basic semantic relation that entailment is supposed to track, and this is so in our view as well.

Now consider a second variation of our die scenario. We insert a coin into a machine that rolls a die, which is hidden from us. Then the following happens:

- if the outcome is 1, we lose and a red light appears;
- if the outcome is 6, we win and a green light appears;
- in all other cases, we get the coin back and a yellow light appears, but the outcome is not revealed.

Thus, the accessible information states are the set $\{w_1, \dots, w_6\}$ (the initial state) and the states $\{w_1\}$ (red light), $\{w_6\}$ (green light), and $\{w_2, w_3, w_4, w_5\}$ (yellow light). These are represented by the picture on the right in Figure 2.8.

Here, restricting the semantics to accessible information states leads to unintended results even for statements. To see this, let even stand for the statement that the outcome is even, and let high be the statement that the outcome is high. If we consider only accessible states we get:

$$\mathsf{even} \models_M \mathsf{high}.$$

However, it is obvious that from the fact that the outcome is even it does *not* follow that the outcome is high, since the outcome may well be two or four. Of course, *learning* that the outcome is even implies *learning* that it is high, but that is a different matter, and not the sort of matter that we normally take logic to be about. This shows that restricting the semantics to accessible information states would take us to a revisionary view of logic, even in the case of statements. Pursuing such a revisionary project is legitimate (in a sense, intuitionistic logic arises precisely from such a project) but our aim here is different: not to revise standard logic, but to generalize it.

Here is another way to put the point: in order to determine if an entailment holds on the basis of an information state, one looks at subsets of the state. In so doing, one is not asking what would happen if an agent were to *learn* something. One is, rather, testing certain features of the available information by exploring what it implies under certain suppositions—looking, for instance, at whether combining the given information with information of one type (say, parity) yields information of another type (range). In sum, whether an entailment holds or not relative to a body of information is an intrinsic property of the information state itself, which turns on what the given information implies when augmented with certain suppositions. So, whether the entailment holds should depend only on the content of the state—the set of live possibilities—and on nothing else.

To distinguish semantic entailments from their pragmatic cousins is especially important since these relations have different logical features. For an example, take again parity, i.e., the question whether the outcome is even or odd. Consider a statement α: in any given context s, the entailment $\alpha \models_s \mathsf{parity}$ captures the fact that relative to the set of possibilities s, the information that α settles the question whether the outcome is even or odd; that happens just in case the information that α implies that the outcome is even, or the information that α implies that the outcome is odd. So, if α is a statement, we must have:

$$\alpha \models_s \mathsf{parity} \iff \alpha \models_s \mathsf{even} \text{ or } \alpha \models_s \mathsf{odd}.$$

As we will see, this is an instance of a general principle, called *Split*, which regulates the interaction of statements with questions. This principle is a tenet of inquisitive logics: it reflects the important idea that statements denote specific pieces of information (as we discussed in Section 2.4). This implies that to check what follows from α in s is to check what follows by strengthening s in a specific way, namely, by supposing that α is true.

Things look different under the pragmatic notion. Consider again the situation in which the only thing one can learn is the exact outcome of the roll. Then, for instance, learning that the outcome is low implies learning whether it is even or odd. However, learning that the outcome is low does not imply learning that it is even (since we may learn that the outcome is 1), and it does not imply learning that it is odd (since we may learn that it is 2). Therefore, if we took entailment to be about *learning*, the split property would fail:

$$\mathsf{low} \models_s \mathsf{parity} \quad \not\Longrightarrow \quad \mathsf{low} \models_s \mathsf{even} \;\text{ or }\; \mathsf{low} \models_s \mathsf{odd}.$$

This is because one may come to learn the statement low in different ways, which lead to different ways of resolving the question parity. This difference has repercussions for the behavior of the implication connective, and thereby leads to a different logic.

Later on, we will see that the fact that implication quantifies over sub-states leads to hard questions—many of which are currently open—about the meta-theoretic properties of inquisitive predicate logic. It is sometimes suggested that a simpler logic may be obtained by revising the semantics so that implication only quantifies over a restricted family of subsets specified by the model. This is analogous to the move that is made by weakening second-order logic by replacing the standard semantics by Henkin semantics. Given the discussion in this section, however, we can see that the simpler logic obtained from this move would not track the intended entailment relation, but rather some kind of pragmatic counterpart of it that has different and weaker logical features.

2.9 Relation with previous work

In this section, we situate the present proposal within the landscape of previous approaches to questions in logic.

2.9.1 Erotetic logic tradition

Although questions have received much less attention than statements in logic, there is nevertheless a rather large literature on them. Most work in this tradition goes under the header *erotetic logic*. Some key references: Prior and Prior (1955); Hamblin (1958); Kubiński (1960, 1980); Harrah (1961, 1963); Åqvist (1965); Belnap and Steel (1976); Tichy (1978); Wiśniewski (1994, 1995, 2013). This is not the place for a detailed survey of this literature (for a valuable survey, see Harrah (2002)). Our aim in this section is merely to explain how our approach relates to previous work in this tradition. A comparison with some more closely related theories is given in the next sections.

We can identify two fundamental differences between inquisitive logic and previous theories in erotetic logic. One difference concerns the way questions are analyzed, the other the aims of the theory.

Different approaches to the interpretation of questions. The most common approach to the interpretation of questions in the erotetic logic tradition is the *answer-set approach*. The idea, which goes back to Hamblin (1958), is that a question is interpreted by specifying what statements count as answers to it. In inquisitive logic, as we saw, questions are instead interpreted by specifying their support conditions. There is an obvious relation between the two approaches: the support conditions of a question capture the conditions in which the question counts as settled—or, we may as well say, *answered*. Clearly, specifying what counts as an answer is tightly related to specifying in what circumstances the question counts as answered (though here one encounters some subtle issues concerning whether information resolving a question is always linguistically expressible, which we do not assume). However, there are also differences. Perhaps most importantly, the set-of-answers approach treats statements and questions differently. Statements are interpreted directly, via standard truth-conditional semantics, while questions are interpreted only derivatively, by interpreting their answers. Our approach, by contrast, treats statements and questions on a par: both are interpreted in terms of the same semantic notion—support relative to an information state. This feature is crucial to get a uniform logic in which statements and questions can participate in the same logical relations and be combined by the same logical operations. Moreover, our approach is similar to the truth-conditional one in that both interpret sentences in terms of a certain satisfaction relation (truth/support) relative to certain evaluation points (worlds/information states), and then define entailment as preservation of this satisfaction relation. As becomes clear by looking at the formal systems, this makes inquisitive logic much more similar to standard logic as compared to logics based on the answer-set approach.[15]

Different aims. Perhaps most importantly, our enterprise here is somewhat different from that of previous theories in erotetic logic. Approaches in the erotetic logic tradition share the idea that dealing with questions in logic means turning attention away from those concerns that take center stage in standard logic, namely, the study of entailment, logical operators (connectives, quantifiers, modalities), and proofs.

[15] For further discussion of the differences between inquisitive semantics and the answer-set approach, see Ciardelli (2017b) and Chapter 6 of Ciardelli et al. (2018).

In the introduction to what is perhaps the most well-developed erotetic logic in the literature, Belnap and Steel (1976) state their aims as follows:

> On the object-language level we want to create a carefully designed apparatus permitting the asking and answering of questions. On the meta-language level we want to elaborate a set of concepts useful for categorizing, evaluating, and relating questions and answers.

This description fits not just Belnap and Steel's own work, but most of the early work in the erotetic logic tradition. Some more recent approaches have focused instead on the role of questions in processes of inquiry, either modeling inquiry itself as a sequence of questioning moves and inference moves, as in the *interrogative model of inquiry* of Hintikka (1999), or characterizing how questions can be arrived at in an inquiry scenario, as in the *inferential erotetic logic* of Wiśniewski (1994, 1995, 1996, 2001, 2013).[16]

Our aim here, by contrast, is to extend logic to questions while staying close to the standard concerns of logic: entailment, logical operators, and proofs.

Start from entailment. The logical relations involving questions that we considered in this section have been considered before in the erotetic logic tradition. The notion of *dependency* that we discussed above has been considered under the name *containment* ever since Hamblin (1958). The two 'mixed' notions—a statement *resolving* a question and a question *presupposing* a statement—are also standard (they are found, e.g., in Belnap and Steel, 1976, where the terminology 'being a complete answer to' is used for the first notion). The novelty of our approach, however, is that it allows us to recognize these notions as being different instances of a single general notion—a notion that can be viewed as a generalization of entailment to questions. This realization allows for a much more thorough deployment of the tools of logic: we already discussed in Section 2.5 how, by internalizing entailment, we get the tools to express the relevant relations in the object language by means of a well-behaved implication connective; moreover, the fact that the relevant relations are entailments means that they can be *formally proved* once we develop a proof system for our logics.

Second, consider logical operators. Standard logic only becomes interesting once we consider complex statements, built up by means of connectives, quantifiers, or modalities. By contrast, in much of the erotetic logic literature, no logical operators can be applied to questions. There are some exceptions: for instance, Belnap and Steel (1976) allow the formation of conjunctive questions and conditional questions. But these are treated on an *ad-hoc* basis: the relevant logical operations are not unified with standard conjunction and implication, and the formal properties of these operations are not investigated at all.

[16] For discussion of the relations between inferential erotetic logic and inquisitive semantics, see Wiśniewski and Leszczyńska-Jasion (2015) and Ciardelli et al. (2015).

Our approach will be very different: we will define our operators so that they can be applied uniformly to statements and questions. Thus, e.g., conditional statements and conditional questions will be treated by using the very same implication connective—the one described above. The study of the properties of these generalized operators is also a central topic in our approach.

Finally, consider proofs. Belnap and Steel are very clear:

> Absolutely the wrong thing is to think [the logic of questions] is a logic in the sense of a deductive system, since one would then be driven to the pointless task of inventing an inferential scheme in which questions, or interrogatives, could serve as premises and conclusions. (Belnap and Steel (1976), page 1)

We disagree: since entailments involving questions are meaningful and interesting, it is natural to ask if they can be established by means of proof systems in which we can manipulate statements and questions. In fact, we will see that questions are very interesting tools for logical inference: they allow us to reason with arbitrary information of a given type. Thus, studying the role of questions in logical proofs turns out to be far from pointless.

2.9.2 The Logic of Interrogation

To my knowledge, the first approach that allows for a generalization of the classical notion of entailment to questions is the Logic of Interrogation (LoI) of Groenendijk (1999), based on the partition theory of questions of Groenendijk and Stokhof (1984). The original presentation of the semantics is a dynamic one, in which the meaning of a sentence is identified with its context-change potential. However, as pointed out by ten Cate and Shan (2007), the dynamic coating is not essential. In essence, the system may be described as follows: both statements and questions are interpreted with respect to pairs $\langle w, w'\rangle$ of possible worlds: a statement is satisfied by such a pair if it is true at both worlds, while a question is satisfied if the complete answer to the question is the same in w and w'. In this approach, the content of a sentence φ is captured by the set of pairs $\langle w, w'\rangle$ satisfying φ; for any φ, this set is an equivalence relation over a subset of W_M, which we will denote as $\sim_\varphi$. Such an equivalence relation may be equivalently regarded as a partition Π_M^φ of a subset of the logical space, where the blocks of the partitions are the equivalence classes $[w]^{\sim_\varphi}$ modulo $\sim_\varphi$ of those worlds in the domain of $\sim_\varphi$. For a statement α, the partition Π_M^α always consists of a unique block, namely, the truth-set $|\alpha|_M$ of the statement. For a question μ, Π_M^μ typically consists of several blocks, which are regarded as the possible complete answers to the question.

Since statements and questions are interpreted by means of a uniform semantics, LoI allows for the definition of a notion of entailment in which both statements and questions can take part:

$$\varphi \models_{\mathsf{LoI}} \psi \iff \text{for all } M \text{ and all } w, w' \in W_M : \langle w, w' \rangle \models \varphi \text{ implies } \langle w, w' \rangle \models \psi.$$

In terms of partitions, this notion of entailment may be cast as follows:

$$\varphi \models_{\mathsf{LoI}} \psi \iff \text{for all } M, \text{ for all } a \in \Pi_M^{\varphi} \text{ there is an } a' \in \Pi_M^{\psi} \text{ such that } a \subseteq a'.$$

This is clearly reminiscent of Proposition 2.4.10: here, too, we can think of a sentence as denoting a (possibly singleton) information type; φ entails ψ if any information of type φ always yields some corresponding information of type ψ.

Groenendijk (1999) applies this approach to a logical language which is an extension of first-order predicate logic with questions. This gives rise to an interesting combined logic of statements and questions, which was investigated and axiomatized by ten Cate and Shan (2007). As we will see in Chapter 4, this can be identified with a fragment of inquisitive predicate logic.

What is the relation between the LoI framework and the present approach? Consider a question μ. Given the LoI perspective, it is natural to assume that μ is settled in an information state s in case s entails some complete answer to μ. This yields the following support conditions:

$$s \models \mu \iff s \subseteq a \text{ for some } a \in \Pi_M^{\mu}.$$

Thus, the set of supporting states is precisely the downward closure of Π_M^{μ}:

$$[\mu]_M = (\Pi_M^{\mu})^{\downarrow}.$$

The same relation holds for a statement α: α is settled in an information state s in case $s \subseteq |\alpha|_M$; given that $\Pi_M^{\alpha} = \{|\alpha|_M\}$, we have that $[\alpha]_M = (\Pi_M^{\alpha})^{\downarrow}$. Thus, for all sentences φ that can be interpreted in LoI, we have:

$$[\varphi]_M = (\Pi_M^{\varphi})^{\downarrow}.$$

That is, the set of blocks of the partition induced by φ is always a *generator* for the inquisitive proposition expressed by φ. This allows us to move from the LoI-representation of a sentence to its support-based representation.

Conversely, the LoI-representation of a sentence φ can be retrieved from its support-based representation by taking the maximal supporting states for φ, i.e., the *alternatives* for the proposition $[\varphi]_M$:

$$\Pi_M^{\varphi} = \textsc{Alt}([\varphi]_M).$$

In sum, for sentences that can be interpreted in LoI, we can go back and forth between the two semantics. Moreover, given that Π_M^{φ} is a generator for $[\varphi]_M$, it follows from Proposition 2.4.10 that the notion of entailment that the two frameworks characterize is the same.

$$\varphi \models \psi \iff \varphi \models_{\mathsf{LoI}} \psi.$$

Thus, the logic of interrogation and our own approach essentially agree with respect to those sentences that can be interpreted in LoI. However, the support approach that we discussed in this section is strictly more general than the LoI approach based on pairs of worlds. In order for a question to be analyzable in LoI, it must be a unique-answer question, in the sense of the following definition.

2.9.1. Definition. [Unique-answer questions]
A question μ is a *unique-answer question* if any $w \in W_M$ is contained in at most one alternative for μ.

The class of unique-answer questions includes many natural kinds of questions, such as the questions parity, range, and outcome from our initial example. But there are also important classes of questions that are *not* unique-answer and that can be analyzed in inquisitive semantics but not in partition semantics.

For an example, imagine a game in which a player has picked a secret two-digits code, where each digit is 1, 2, or 3. So, there are $3 \times 3 = 9$ possible codes (11, 12, 13, etc.), corresponding to 9 possible worlds. Now consider:

(7) What is one digit that occurs in the code?

This question is completely resolved just in case one of the following pieces of information is available:

(8) a. The digit 1 occurs in the code.
b. The digit 2 occurs in the code.
c. The digit 3 occurs in the code.

Thus, we have three alternatives for the question, depicted in Figure 2.9. As the image shows, these alternatives overlap: this corresponds to the fact that the three pieces of information above are not mutually exclusive—on the contrary, they are pair-wise compatible. So, (7) is not a unique-answer question and cannot be analyzed in partition semantics.

Questions like (7), which ask for an instance of a given property, are known as *mention-some* question. Mention-some questions are not exotic—on the contrary, they occur frequently both in ordinary situations and in specialized scientific discourse. The following are examples.

(9) a. Who could serve on this committee?
b. What is a present that Alice would really like?
c. What is a mammalian species that lays eggs?

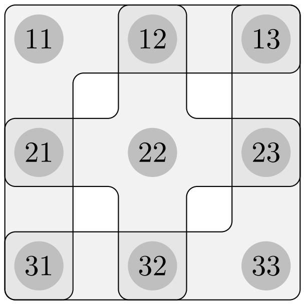


Figure 2.9: Overlapping alternatives for a mention-some question.

d. What is an example of an arithmetic theorem not provable in PA?

Mention-some questions are not the only class of questions that can be analyzed in inquisitive semantics but not in partition semantics. Other examples are *approximate value questions* like (10-a) (cf. Yablo (2014)), which ask for a value with a certain margin of error, and *conditional questions* like (10-b), which ask for the answer to a question under a supposition. We will see in the next chapter how conditional questions can be analyzed naturally as conditionals in inquisitive semantics.

(10) a. How many stars are there, give or take ten?
b. If Axton had an accomplice, who is it?

Summing up, the extra generality of inquisitive semantics allows us to represent, and reason with, a broader class of questions than is treatable in LoI.

However, there is also a second reason why this generality is important. We saw above that inquisitive semantics can be equipped with an implication operator $\to$ that allows us to express the meta-language entailment relation within the object language. This operator plays a key role in inquisitive logics. But, as we will see in the next chapter, the inquisitive proposition expressed by implications is typically one that does not correspond to a partition of the logical space—it involves overlapping alternatives.

In partition semantics, it is provably impossible to define such a connective. For instance, consider the questions parity (whether the outcome is even or odd), and outcome (what the outcome is) in our die roll example. One can prove, for instance, that in the model M of our running example there is no way to assign a partition to an implication ($\mathsf{parity} \to \mathsf{outcome}$) in such a way that for any other unique-answer question λ we have the desired connection:

$$\lambda, \mathsf{parity} \models_M \mathsf{outcome} \iff \lambda \models_M \mathsf{parity} \to \mathsf{outcome}.$$

In other words, due to its semantic assumptions, LoI does not allow us to access some important expressive means that will play an important role in the development of our inquisitive logics.

2.9.3 Inquisitive pair semantics

Starting with the work of Velissaratou (2000) on conditional questions, the pursuit of greater generality led to the development of successors of LoI in which formulas are still evaluated relative to pairs of worlds, but the set of pairs satisfying a given formula is not necessarily an equivalence relation. In this setting, the natural way to read the relation $\langle w, w' \rangle \models \mu$, where μ is a question, is no longer "*the* complete answer to μ is the same in w as in w'", but rather "*some* complete answer to μ is true at both w and w'". This approach, laid out in Groenendijk (2009) and Mascarenhas (2009), was originally dubbed *inquisitive semantics*; it is now referred to as *inquisitive pair semantics* to distinguish it from the present support-based approach.

While Groenendijk (2011) showed that this sort of semantics can indeed deal adequately with conditional questions, Ciardelli (2008, 2009), and later Ciardelli et al. (2015) argued that no pair semantics can provide a satisfactory general framework for question semantics. To get an idea of the problem, consider again our mention-some question (7), and consider the set of possible worlds $s = \{w_{12}, w_{13}, w_{23}\}$ in the model of Figure 2.9 (i.e., the upper-right corner in the figure). In s, our mention-some question is *not* settled: the information available in s implies neither that 1 occurs in the code, nor that 2 occurs, not that 3 occurs. However, this cannot be detected by looking at pairs of worlds, since each pair of worlds in s *does* settle the question. So if our semantics only looks at pairs of worlds, it fails to see that (7) is not settled in s and, thus, it fails to distinguish (7) from a different question which *is* settled at s. Examples of this kind motivated a shift from pairs to sets of worlds as points of evaluation, leading to the modern support-based version of inquisitive semantics.

2.9.4 Nelken and Shan's modal approach

A different uniform approach to statements and questions was proposed by Nelken and Shan (2006). In this approach, questions are translated as modal sentences, and they are interpreted by means of truth conditions: a question is true at a world w in case it is settled by an information state $R[w]$ associated with the world (i.e., the set of successors given by an accessibility relation R). Thus, for instance, Nelken and Shan render the question *whether* p by the modal formula $?p := \Box p \vee \Box \neg p$.

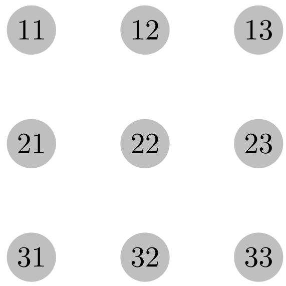
In one respect, this approach is similar to the approach proposed here, since the meaning of a question is essentially taken to be encoded by the conditions under which the question is settled. And indeed, if we consider entailments which involve only questions, the approach of Nelken and Shan would make the same predictions as ours. However, in their approach an important asymmetry between statements and questions is maintained: for questions, what matters is whether they are settled by a relevant information state, while for statements, what matters is whether they are true at the world of evaluation. This asymmetry creates problems the moment we start considering cases of entailment involving both statements and questions. It is easy to see that, if such entailments are to be meaningful at all, entailment cannot just amount to preservation of truth. Nelken and Shan propose to fix this by re-defining entailment as *modal consequence*: $\varphi \models \psi$ if, whenever φ is true at *every* possible world in a model, so is ψ. However, this move has the unintended consequence of changing the consequence relation for modal statements. For instance, if our declarative language indeed contains a Kripke modality, say a knowledge modality K, then if our notion of entailment is redefined as modal consequence, we make undesirable predictions, such as $p \models Kp$. Thus, this approach does not really allow us to extend classical logics with questions in a conservative way.[17]

The asymmetry from which the problem originates can be eliminated by letting statements, too, be interpreted in terms of when they are settled by the state $R[w]$, rather in terms of when they are true at w: that is, we may render a basic statement not as a propositional formula α, but as a modal formula $\Box\alpha$. If we made this move, we would arrive at a framework with a sensible logic, but with some unnecessary complexity: while sentences are interpreted with respect to a world w equipped with an information state $R[w]$, it is only the state $R[w]$ which matters for the interpretation of both statements and questions. We could thus get rid of the worlds altogether and interpret formulas directly relative to states. This would also allow us to work with simpler models and with a simpler syntax, leading to an approach similar to the one taken here.

[17]Besides this, there are other difficulties, too. First, it is hard to make sense of the proposed truth-conditions for questions. For instance, take the question "What is the capital of Spain?". Is this questions actually *true*? According to Nelken and Shan that depends on an accessibility relation, but it is not clear what the relevant relation is. It should not be the epistemic accessibility relation of a particular agent, say Alice, since that would predict that the question means the same as the statement "Alice knows what the capital of Spain is", which is surely wrong—the question is not about Alice at all.

2.10 Exercises

2.10.1. Exercise. [Support conditions]
Imagine a game in which a player has picked a secret two-digits code, where each digit is 1, 2, or 3. So, there are $3 \times 3 = 9$ possible codes, corresponding to 9 possible worlds, as follows:



1. For each of the following statements and questions, determine and draw a picture of the maximal information states in which it is supported.

   (11) a. The code is 12
   b. The first digit is 1.
   c. The code contains a 1.
   d. If the first digit is 1, the second is 2.
   e. Is the code 12?
   f. Is the first digit 1?
   g. Is the first digit 1, or is it 2?
   h. Does the code contain a 1?
   i. What is the code?
   j. What is the first digit?
   k. What is one digit that does not occur in the code?
   l. If the first digit is a 1, what is the second digit?

2. Find questions in English having the following sets as alternatives:

   (12) a. $\{11, 22, 33\}, \{12, 13, 21, 23, 31, 32\}$;
   b. $\{12, 13, 23\}, \{11, 22, 33\}, \{21, 31, 32\}$;
   c. $\{11\}, \{12, 21\}, \{31, 22, 13\}, \{23, 32\}, \{33\}$.

   Note that one trivial option is to formulate the relevant question as: "is the code among those in $\{\dots\}$, or among those in $\{\dots\}$, etc."; try to avoid such trivial solutions and to come up with more natural formulations.

3. How many non-equivalent yes/no questions can we in principle ask about the code in this model? (Include the trivial yes/no question in the count.)

2.10.2. EXERCISE. [Inquisitive entailment in context]
Consider the model and some of the sentences of the previous exercise, labeled by letters as above. We can draw a table showing at row x and column y whether or not sentence x entails sentence y in the model. For instance:

	a	b	c	f	h	i	j
a	✓	✓	✓				
b	×	✓	✓				
c	×	×	✓				
f							
h							
i							
j							

Determine which of the remaining entailments hold and fill the table.

2.10.3. EXERCISE. [Polar questions]
If α is a statement, let $?\alpha$ denote the polar question whether α, which is settled iff the available information determines whether α is true or false:

$$s \models ?\alpha \iff s \subseteq |\alpha|_M \text{ or } s \cap |\alpha|_M = \emptyset.$$

Let M be an arbitrary model. Let us say that a sentence φ is a tautology in M if φ is supported by every information state in M. Prove the following claims:

1. $?\alpha$ is a tautology in $M \iff \alpha$ or $\neg\alpha$ is a tautology in M.
2. $?\alpha \equiv_M ?\beta \iff \alpha \equiv_M \beta$ or $\alpha \equiv_M \neg\beta$.
3. If $?\alpha \models_M ?\beta$ then either $?\alpha \equiv_M ?\beta$ or $?\beta$ is a tautology in M.

Notice in particular that item 3 implies that no proper entailment can hold among non-trivial polar questions.

2.10.4. EXERCISE. [Inquisitive implication]
Consider again the model corresponding to our die roll scenario, and consider the following statements and questions.

even	The outcome is even.
low	The outcome is low.
parity	Is the outcome even, or odd?
range	Is the outcome low, middle, or high?

Determine and draw the alternatives for the following implications.

1. even $\to$ low
2. even $\to$ range
3. even $\to$ parity
4. range $\to$ even
5. parity $\to$ range
6. range $\to$ parity

Hint. Use the Ramsey test clause give by propositions 2.5.2 and 2.5.3. Draw each alternative in a separate picture of the logical space, as in Figure 2.6.

2.10.5. Exercise. [Entailments towards statements]
Let α be a statement and Φ a set of sentences which may be either statements or questions. Using the Truth-Support Bridge, show that we have:

$$\Phi \models \alpha \iff \text{for all models } M \text{ and worlds } w \in W_M : w \models \Phi \text{ implies } w \models \psi,$$

where $w \models \Phi$ means '$w \models \varphi$ for all $\varphi \in \Phi$'. Using this, prove Proposition 2.6.2.

Chapter 3

Questions in propositional logic

In the previous chapter we introduced the fundamental ideas of inquisitive logic. These ideas are general, and can be used to build many specific logical systems in which we can formalize both statements and questions. One way to build such systems is to start with a system of classical logic and extend it to an inquisitive system in two steps. In the first step, we re-implement the classical system by giving it a support semantics. In executing this step, we make sure our semantics satisfies the Truth-Support Bridge: as we discussed in Section 2.2, this guarantees that the original logic of statements is preserved. In the second step, we extend the language with question-forming operators, which can be interpreted naturally in the context of a support semantics. The result is an inquisitive system which is conservative over the original logic. The strategy is illustrated in Figure 3.1.

In this section, we execute this strategy in the simplest setting, that of propositional logic. Thus, this chapter is devoted to the enterprise of enriching classical propositional logic with questions. There are several ways to do that, depending on what operators we take as primitive. We will present a system which is particularly natural, expressively rich, and well-understood. This system is called InqB, where the letter B stands for *basic*; this is to distinguish it from other systems of inquisitive propositional logic with different sets of primitives (see Groenendijk, 2011; Ciardelli et al., 2015, as well as Exercise 3.11.9). Due to its prominent role, InqB is often just referred to as *inquisitive propositional logic*, though strictly speaking it is just *one* inquisitive propositional logic.[1] In this chapter we will introduce this system, illustrate it with examples, and study its formal properties in some detail.

[1] A historical note: the system InqB was first considered by Groenendijk (2008), and shortly after independently by Ciardelli (2008), who argued that this system should replace the previous implementation of inquisitive semantics based on pairs of worlds (cf. Section 2.9.3). The main sources for the results on InqB presented in this section are Ciardelli (2009), Ciardelli and Roelofsen (2011), and Ciardelli (2018b).

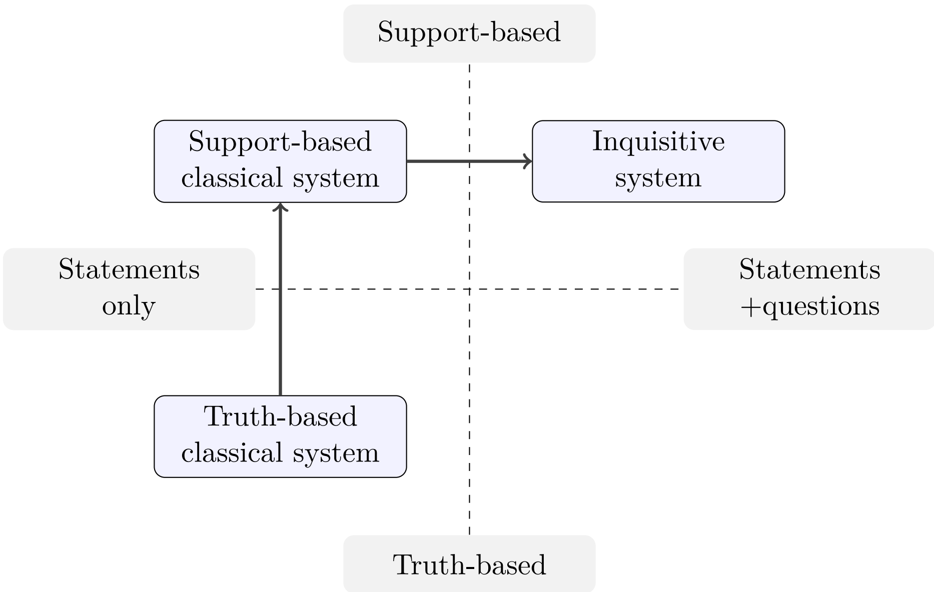


Figure 3.1: From a classical system to an inquisitive system in two steps.

3.1 Support for classical propositional logic

Let $\mathcal{P}$ be a given set of propositional atoms. From the perspective of a propositional language, a state of affairs is characterized completely by a specification of the truth-values of the atoms in $\mathcal{P}$. Thus, a model M for propositional logic will consist simply of a set of possible worlds W and a valuation function V which determines the truth value of each atom at each world.[2]

3.1.1. Definition. [Propositional information models]
A *propositional information model* for $\mathcal{P}$ is a pair $M = \langle W, V \rangle$, where:

- W is a set, whose elements we refer to as *possible worlds*;
- $V : W \times \mathcal{P} \to \{0, 1\}$ is map that assigns to each world w and atom p a truth value $V(w, p)$.

An *information state* in a model $M = \langle W, V \rangle$ is a set $s \subseteq W$ of possible worlds.

[2]Some previous literature on inquisitive propositional logic (for instance, Ciardelli and Roelofsen, 2011) assumes a slightly different setup. It works with a fixed model ω, having the valuation functions $v : \mathcal{P} \to \{0, 1\}$ as possible worlds. Since this model contains a copy of each possible state of affairs, the difference is immaterial to the logic, as the reader is invited to show in Exercise 3.11.7. The present setup makes things slightly more complicated, but it has the advantage of allowing for a smoother transition to predicate logic and modal logic.

We refer to the the empty set of worlds, $\emptyset$, as the *inconstistent state*, and to non-empty sets as *consistent states* (cf. the discussion in Section 2.2).

Let us start by providing a support semantics for classical propositional logic. The set $\mathcal{L}_c^{\mathsf{P}}$ of *classical propositional formulas* is given by the following definition, where $p \in \mathcal{P}$:

$$\alpha \ ::= \ p \mid \bot \mid (\alpha \wedge \alpha) \mid (\alpha \to \alpha).$$

That is, classical propositional formulas are built up from atoms and the falsum constant by means of conjunction and implication. As usual, we omit brackets whenever this causes no confusion. We take negation, disjunction, the bi-conditional, and the constant $\top$ to be defined operators.

3.1.2. Definition. [Defined connectives]

- $\neg\varphi := \varphi \to \bot$
- $\top := \neg\bot$
- $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$
- $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$

Thus, the language $\mathcal{L}_c^{\mathsf{P}}$ is just a standard propositional language with a particular choice of primitives. However, we interpret this language not, as usual, via a recursive definition of truth at a world, but instead via a recursive definition of support at an information state.

3.1.3. Definition. [Support for classical formulas]
Let M be a propositional information model. The relation of support between states s in M and formulas $\varphi \in \mathcal{L}_c^{\mathsf{P}}$ is defined as follows:

- $M, s \models p \iff V(w, p) = 1$ for all $w \in s$;
- $M, s \models \bot \iff s = \emptyset$;
- $M, s \models \varphi \wedge \psi \iff M, s \models \varphi$ and $M, s \models \psi$;
- $M, s \models \varphi \to \psi \iff$ for all $t \subseteq s$, $M, t \models \varphi$ implies $M, t \models \psi$.

We leave M implicit when this is harmless, writing $s \models \varphi$ instead of $M, s \models \varphi$.

Keeping in mind that we read support as capturing the fact that a formula is *settled* in an information state, we can read the clauses as follows. An atom p is settled in s in case the information available in s implies that p is true. The falsum constant $\bot$ is settled only by the state of inconsistent information, $\emptyset$. A conjunction is settled in s in case both conjuncts are settled. Finally, implication internalizes contextual entailment in the way discussed in Section 2.5: $\varphi \to \psi$ is settled in s iff relative to s, φ entails ψ (that is, if any way of strengthening s so as to settle φ results in a state that also settles ψ).

The following basic facts about the semantics can be established simultaneously by a straightforward induction on φ.

3.1.4. Proposition. *For every model M and formula $\varphi \in \mathcal{L}_c^P$ we have:*

- *Persistency: if $s \models \varphi$ and $t \subseteq s$ then $t \models \varphi$;*
- *Empty state property: $\emptyset \models \varphi$.*

Intuitively, persistency says that if a sentence is supported by a state s, then it is also supported by any state that contains at least as much information as s. That is, support is a matter of there being *enough* information in the state. The empty state property says that the inconsistent state trivially supports any formula. As we discussed in Section 2.4, this can be seen as a semantic counterpart of the *ex falso quodlibet* principle.

In order to spell out the clauses for the defined connectives $\neg$ and $\vee$, it will be useful to introduce a derived semantic relation. We will say that *s rules out* φ if s cannot be strengthened consistently to support φ.

3.1.5. Definition. [Ruling out]
A state s rules out a formula φ, denoted $s \perp\!\!\!\perp \varphi$, if there is no consistent $t \subseteq s$ such that $t \models \varphi$.

Using this notion, the derived semantic clauses for the defined operators can be expressed as follows.

3.1.6. Proposition (Support conditions for defined operators).

- $s \models \neg\varphi \iff s \perp\!\!\!\perp \varphi$;
- $s \models \varphi \vee \psi \iff$ *there is no consistent $t \subseteq s$ such that $t \perp\!\!\!\perp \varphi$ and $t \perp\!\!\!\perp \psi$;*
- $s \models \varphi \leftrightarrow \psi \iff \forall t \subseteq s : (t \models \varphi \iff t \models \psi)$;
- $s \models \top$ *always.*

Let us recall some important notions that were defined in the previous chapter. The *support-set* of a formula φ in a model M, $[\varphi]_M$, is the set of states that support φ. The set of *alternatives* for φ, $\textsc{Alt}_M(\varphi)$, is the set of *maximal* states that support φ. Also, recall that, in a support-based semantics, truth at a world is defined as support at the corresponding singleton state:

$$M, w \models \varphi \overset{def}{\iff} M, \{w\} \models \varphi.$$

The truth-set of φ in a model M, $|\varphi|_M$, is the set of worlds in M where φ is true.

The following proposition shows that the notion of truth that we obtain from our support semantics coincides with the notion of truth in classical propositional logic.

3.1.7. PROPOSITION (TRUTH-CONDITIONS FOR CLASSICAL FORMULAS).
For any model $M = \langle W, V \rangle$ and any world $w \in W$ we have:

- $w \models p \iff V(w, p) = 1$;
- $w \not\models \bot$;
- $w \models \varphi \wedge \psi \iff w \models \varphi$ *and* $w \models \psi$;
- $w \models \varphi \to \psi \iff w \not\models \varphi$ *or* $w \models \psi$;
- $w \models \neg\varphi \iff w \not\models \varphi$;
- $w \models \varphi \vee \psi \iff w \models \varphi$ *or* $w \models \psi$.

Proof. It suffices to inspect the support conditions given by Definition 3.1.3 in the case of singleton states. The most interesting case is the one for implication. We have:

$$w \models \varphi \to \psi \iff \{w\} \models \varphi \to \psi \iff \forall t \subseteq \{w\} : t \models \varphi \text{ implies } t \models \psi.$$

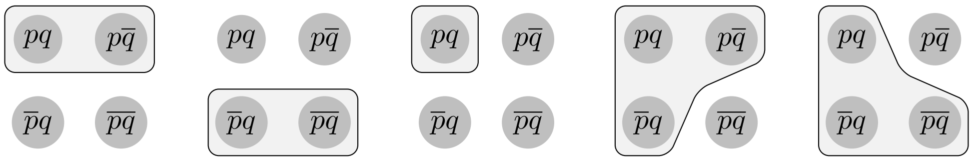
There are only two sets $t \subseteq \{w\}$ to consider: $t = \emptyset$ and $t = \{w\}$. For the case $t = \emptyset$ the condition is trivially satisfied, since $\emptyset$ supports every formula by Proposition 3.1.4. Thus, the above universal requirement boils down to a requirement on the state $\{w\}$, namely, that either $\{w\} \not\models \varphi$ or $\{w\} \models \psi$. By the definition of truth, this amounts to $w \not\models \varphi$ or $w \models \psi$. ☐

Thus, the standard truth conditions for classical formulas can be retrieved from our support semantics. Moreover, the support conditions of a classical formula are related to its truth conditions in accordance with the Truth-Support Bridge. That is, for all classical formulas, to be supported at a state s is just to be true at each world $w \in s$.

3.1.8. PROPOSITION (CLASSICAL FORMULAS ARE TRUTH-CONDITIONAL).
For any model M, any state s, and any formula $\varphi \in \mathcal{L}_c^P$:

$$\begin{aligned} s \models \varphi \quad &\iff \quad w \models \varphi \text{ for all } w \in s \\ &\iff \quad s \subseteq |\varphi|_M. \end{aligned}$$

Proof. By induction on φ. The basic cases for atoms and $\bot$ are immediate, as is the inductive step for conjunction. What remains is the inductive step for



(a) p (b) $\neg p$ (c) $p \wedge q$ (d) $p \vee q$ (e) $p \to q$

Figure 3.2: The alternatives for some classical formulas in a four-world model. Here, pq represents a world where p and q are both true, $p\overline{q}$ a world where p is true and q is false, etc. Proposition 3.1.8 implies that a classical formula φ always has a unique alternative, which coincides with its truth-set $|\varphi|_M$.

implication. So consider a formula $\varphi \to \psi$. We have

$$
\begin{aligned}
s \models \varphi \to \psi \quad &\iff \quad \forall t \subseteq s:\ t \models \varphi \text{ implies } t \models \psi \\
&\iff \quad \forall t \subseteq s:\ t \subseteq |\varphi|_M \text{ implies } t \models \psi \\
&\iff \quad \forall t \subseteq s \cap |\varphi|_M:\ t \models \psi \\
&\iff \quad s \cap |\varphi|_M \models \psi \\
&\iff \quad s \cap |\varphi|_M \subseteq |\psi|_M \\
&\iff \quad s \subseteq (W - |\varphi|_M) \cup |\psi|_M \\
&\iff \quad s \subseteq |\varphi \to \psi|_M
\end{aligned}
$$

where the second and the fifth biconditional use the induction hypotheses, the fourth biconditional uses the persistency of support, and the last biconditional uses the truth conditions for $\varphi \to \psi$ as given by the previous proposition. □

As an immediate corollary of this result, we have that in every model M, a classical formula φ has a unique alternative, which coincides with its truth-set $|\varphi|_M$. Figure 3.2 illustrates this fact by showing the alternatives for some classical formulas in a simple model.

Propositions 3.1.7 and 3.1.8 show that the support-semantics given above and the standard truth-conditional semantics for classical propositional logic are inter-definable. Moreover, we saw in Section 2.2 that once we have the Truth-Support Bridge given by Proposition 3.1.8, we can show that the two semantics yield the same entailment relation. So, what we have given so far is an alternative semantic foundation for classical propositional logic based on support. We have thus executed the first step of the strategy laid out in the introduction to this chapter.

3.2 Adding questions to propositional logic

Having re-implemented classical propositional logic in terms of support, we are now ready for the second step of our strategy: enriching the language of propositional logic with formulas that stand for questions. There are different ways to do this. The way we pursue here is to add to our logical repertoire a new connective, $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$, called *inquisitive disjunction*.[3] Thus, the full language of our system InqB is generated from propositional atoms and $\bot$ by means of the connectives $\wedge, \to$, and $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$.

3.2.1. DEFINITION. [Language $\mathcal{L}^{\mathsf{P}}$]
The language $\mathcal{L}^{\mathsf{P}}$ of propositional inquisitive logic is defined as follows:

$$\varphi \;::=\; p \,|\, \bot \,|\, (\varphi \wedge \varphi) \,|\, (\varphi \to \varphi) \,|\, (\varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \varphi).$$

Intuitively, we regard $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ as a question-forming disjunction. That is, while the classical disjunction $p \vee q$ intuitively stands for the disjunctive statement that p or q is the case (interpreted inclusively), the inquisitive disjunction $p \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} q$ stands for the disjunctive question whether p or q is the case (interpreted inclusively).[4] Similarly, while $p \vee \neg p$ stands for the tautological statement that either p or $\neg p$ is the case, the inquisitive disjunction $p \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \neg p$ stands for the polar question whether p or $\neg p$ is the case, i.e., whether p is true or false. We introduce a defined operator, ?, that allows us to turn a statement into a polar question.

3.2.2. DEFINITION. [Question mark operator] $?\varphi := \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \neg\varphi$

Given the intended interpretation of an inquisitive disjunction $p \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} q$, the support conditions for inquisitive disjunction will be stricter than those of classical disjunction: in order to settle whether p or q is the case, one needs to establish either that p is the case, or that q is the case. This leads naturally to the following support clause.

3.2.3. DEFINITION. [Support for InqB]
The relation of support for InqB is obtained by supplementing Definition 3.1.3 with the following inductive clause:

- $s \models \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi \iff s \models \varphi$ or $s \models \psi$.

[3] A natural alternative is considered in Exercise 3.11.9. As the reader is asked to prove, that alternative leads to a system less expressive than InqB.

[4] It seems plausible that an alternative question like *whether p or q* in English is in fact exclusive, in the sense that it presupposes that exactly one of p and q is true, and it is settled by establishing which one (see, e.g., Biezma and Rawlins, 2012; Aloni et al., 2013). This kind of reading is expressible in our formal language by means of the formula $(p \wedge \neg q) \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} (q \wedge \neg p)$. Nothing in this book hinges on this empirical issue. What matters for our purposes is that both inclusive and exclusive readings can be regimented in our formal language.

(a) $p \mathbin{\vee\!\!\!\vee} q$ (b) $?p$

Figure 3.3: The alternatives for two simple formulas formed by means of $\mathbin{\vee\!\!\!\vee}$.

The derived support clause for the question mark operator is:

- $s \models ?\varphi \iff s \models \varphi$ or $s \perp\!\!\!\perp \varphi$.

That is, s supports $?\varphi$ if it either supports or rules out φ. Let us illustrate the semantics of our new operators by spelling out the support conditions for the formulas $p \mathbin{\vee\!\!\!\vee} q$ and $?p$ that we discussed above.

3.2.4. Example. Consider the formula $p \mathbin{\vee\!\!\!\vee} q$. Since p and q are classical formulas, using Proposition 3.1.8 we have:

$$s \models p \mathbin{\vee\!\!\!\vee} q \iff s \models p \text{ or } s \models q \iff s \subseteq |p|_M \text{ or } s \subseteq |q|_M.$$

That is, as anticipated, a state s settles $p \mathbin{\vee\!\!\!\vee} q$ if it implies that p is true or it implies that q is true. In the toy model we used for our pictures, $p \mathbin{\vee\!\!\!\vee} q$ thus has two alternatives, namely, $|p|_M$ and $|q|_M$, as shown in Figure 3.3(a). From this we can already see a semantic difference between $p \mathbin{\vee\!\!\!\vee} q$ and classical formulas: as we saw above, classical formulas always have a single alternative.

It is instructive to note that the support conditions of $p \mathbin{\vee\!\!\!\vee} q$ differ from those of its classical counterpart $p \vee q$ in the relative scope of universal quantifier and disjunction:

- $s \models p \mathbin{\vee\!\!\!\vee} q \iff (\forall w \in s : w \models p)$ or $(\forall w \in s : w \models q)$;
- $s \models p \vee q \iff \forall w \in s : (w \models p$ or $w \models q)$.

3.2.5. Example. Consider the formula $?p$. Again, since p and $\neg p$ are classical, we have:

$$\begin{aligned} s \models ?p &\iff s \models p \text{ or } s \models \neg p \\ &\iff s \subseteq |p|_M \text{ or } s \subseteq |\neg p|_M \\ &\iff (\forall w \in s : V(w,p) = 1) \text{ or } (\forall w \in s : V(w,p) = 0) \\ &\iff \forall w, w' \in s : V(w,p) = V(w',p). \end{aligned}$$

Thus, a state s settles $?p$ if it implies that p is true, or it implies that p is false. In other words, s settles $?p$ just in case it determines whether p is true or false. These are just the support conditions we expect given our reading of $?p$ as the polar question *whether p*. In the toy model M of our figures, the formula $?p$ has two alternatives, namely, the states $|p|_M$ and $|\neg p|_M$, as shown in Figure 3.3(b).

3.2.6. Example. To conclude this section, let us illustrate how the die roll example from the previous chapter can be formalized in InqB. Imagine that our propositional language has six atomic sentences, standing for the statements "the outcome is n" for $1 \leq n \leq 6$:

$$\mathcal{P} = \{\mathsf{one}, \mathsf{two}, \mathsf{three}, \mathsf{four}, \mathsf{five}, \mathsf{six}\}.$$

We can then define classical formulas formalizing statements such as "the outcome is low" or "the outcome is even", as follows:

- $\mathsf{low} := \mathsf{one} \vee \mathsf{two}$;
- $\mathsf{mid} := \mathsf{three} \vee \mathsf{four}$;
- $\mathsf{high} := \mathsf{five} \vee \mathsf{six}$;
- $\mathsf{even} := \mathsf{two} \vee \mathsf{four} \vee \mathsf{six}$;
- $\mathsf{odd} := \mathsf{one} \vee \mathsf{three} \vee \mathsf{five}$.

So far, everything is familiar from standard propositional logic. But now, using inquisitive disjunction, we can also write formulas formalizing the questions what the outcome is, whether the outcome is even or odd, and whether the outcome is low, middle, or high:[5]

- $\mathsf{outcome} := \mathsf{one} \vee\!\!\!\vee \mathsf{two} \vee\!\!\!\vee \mathsf{three} \vee\!\!\!\vee \mathsf{four} \vee\!\!\!\vee \mathsf{five} \vee\!\!\!\vee \mathsf{six}$;
- $\mathsf{parity} := \mathsf{even} \vee\!\!\!\vee \mathsf{odd}$;
- $\mathsf{range} := \mathsf{low} \vee\!\!\!\vee \mathsf{mid} \vee\!\!\!\vee \mathsf{high}$.

Now consider a model M for our language whose universe contains six worlds, $W = \{w_1, \ldots, w_6\}$ with the obvious valuation (one is true only at w_1, etc.). In this model, the support conditions for the formulas we defined are indeed the ones we gave in the previous chapter, namely:

- $s \models \mathsf{parity} \iff s \subseteq \{w_1, w_3, w_5\}$ or $s \subseteq \{w_2, w_4, w_6\}$;
- $s \models \mathsf{range} \iff s \subseteq \{w_1, w_2\}$ or $s \subseteq \{w_3, w_4\}$ or $s \subseteq \{w_5, w_6\}$;
- $s \models \mathsf{outcome} \iff s \subseteq \{w_i\}$ for some $i \leq 6$.

As a consequence, the alternatives for these formulas are exactly the ones depicted in Figure 2.2 on page 15.

[5] Note that, since $\vee\!\!\!\vee$ is associative, the bracketing of the disjuncts in a disjunction involving more than two disjuncts is irrelevant. This means that we can legitimately write such a disjunction as $\varphi_1 \vee\!\!\!\vee \cdots \vee\!\!\!\vee \varphi_n$.

3.3 Basic properties of InqB

In this section we discuss some basic semantic properties of the system InqB. First of all, it is easy to verify that persistency and the empty state property, given by Proposition 3.1.4 for classical formulas, still hold for our extended language.

3.3.1. Proposition. *For any model M and any formula $\varphi \in \mathcal{L}^{\mathsf{P}}$ we have:*

- *Persistency: if $s \models \varphi$ and $t \subseteq s$, then $t \models \varphi$.*
- *Empty state property: $\emptyset \models \varphi$.*

This ensures that the support-set $[\varphi]_M$ of a formula in a model is always an inquisitive proposition, i.e., a non-empty and downward closed set of information states (cf. Definition 2.4.2).

Another basic feature of the semantics is that it is *local*: support at a state depends exclusively on the worlds in the state, and not on the other worlds present in the model. Let us make this claim precise.

3.3.2. Definition. [Restriction of a model to a state]
The restriction of an information model $M = \langle W, V\rangle$ to a state $s \subseteq W$ is the model $M_{|s} = \langle s, V_{|s}\rangle$, where $V_{|s}$ is the restriction of the map V to the state s.

3.3.3. Proposition (Locality).
For any information model M, any state s in M and any formula $\varphi \in \mathcal{L}$:

$$M, s \models \varphi \iff M_{|s}, s \models \varphi.$$

The proof is straightforward, by induction on φ. Locality is not a property to be taken for granted: it fails, for instance, in inquisitive modal logic (cf. Chapter 8), since the interpretation of a modal formula may well depend on worlds located outside the evaluation state (just like, in standard modal logic, the interpretation of a modal formula depends on worlds different from the evaluation world).

Next, consider the notion of truth for our extended language. All the truth-conditional clauses given in Proposition 3.1.7 still hold for the full language $\mathcal{L}^{\mathsf{P}}$. Moreover, the truth conditions for an inquisitive disjunction coincide with those of the corresponding classical disjunction.

3.3.4. Proposition (Truth conditions for $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$).
$w \models \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi \iff w \models \varphi$ *or* $w \models \psi$.

Finally, a systematic relation holds between the proposition $[\varphi]_M$ expressed by a formula and its truth-set $|\varphi|_M$: the latter always amounts to the union of all the elements in the former. The proof is left as an exercise (see Exercise 3.11.2).

3.3.5. Proposition (Proposition and truth-set).
For any model M and any formula $\varphi \in \mathcal{L}^{\mathsf{P}}$: $|\varphi|_M = \bigcup[\varphi]_M$.

3.4 Truth-conditional formulas

Recall Definition 2.6.3: we call a formula φ *truth-conditional* in case for any model M and state s we have:

$$M, s \models \varphi \iff \forall w \in s : M, w \models \varphi.$$

Proposition 3.1.8 tells us that all classical formulas—i.e., all formulas not containing $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$—are truth-conditional. By contrast, formulas involving $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ are typically not truth-conditional. To see this, note that if φ is truth-conditional then in any model M, φ has a unique alternative, namely, $|\varphi|_M$. This immediately implies that the formulas $p \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} q$ and $?p$ are not truth-conditional, given that in the model of Figure 3.3 they both have two distinct alternatives.

This result is expected, given that we think of these formulas as questions. As we discussed in detail in Section 2.6, we expect statements, but not questions, to be truth-conditional. In fact, we will use truth-conditionality as a criterion to classify formulas of our formal language as being statements or questions.[6]

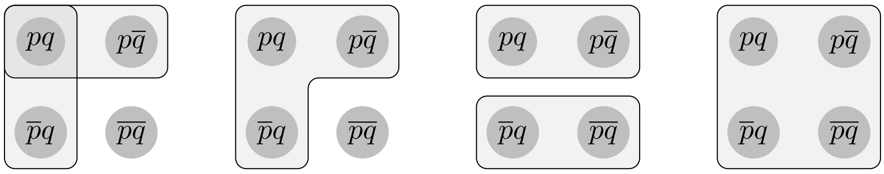
3.4.1. DEFINITION. [Statements and questions]
We call $\varphi \in \mathcal{L}^{\mathsf{P}}$ a *statement* if it is truth-conditional, and a *question* otherwise.

Henceforth, we adopt the following notational convention: we use α, β, γ as meta-variables ranging over statements, λ, μ, ν as meta-variables for questions, and φ, ψ, χ as meta-variables for formulas which may belong to either category.

Let us consider again Proposition 3.1.8, which tells us that every classical formula is truth-conditional. This is as it should be, since we read formulas of classical propositional logic in the usual way, as formalizing statements. We are now going to see that, conversely, any truth-conditional formula in InqB is equivalent to a classical formula. Thus, adding inquisitive disjunction to our language enables our logic to express questions, but not to express new statements.[7] First, we associate to any $\varphi \in \mathcal{L}^{\mathsf{P}}$ a classical formula φ^{cl} that has the same truth conditions as φ.

[6]Note that some work on inquisitive logic (for instance, Groenendijk and Roelofsen, 2009; Ciardelli and Roelofsen, 2011) uses the term *question* in a different way: a sentence φ is called a question if in any model M, $|\varphi|_M = W$. That terminology is based on an interpretation of inquisitive propositions in terms of discourse effects: questions are characterized by the property of being *non-informative*. Here, we do not link support conditions to discourse effects.

[7]This is not always the case: as we will discuss in Chapter 8, in inquisitive modal logic questions may be embedded under specific *inquisitive modalities*, resulting in new truth-conditional formulas expressing, for instance, that an agent *wonders about* a certain question. In such a system, the presence of questions also enables the language to express statements that have no counterpart in the classical fragment of the language. In other words, questions contribute to the range of modal statements that the system can express.



(a) $p \vee\!\!\!\vee q$ (b) $(p \vee\!\!\!\vee q)^{cl}$ (c) $?p$ (d) $(?p)^{cl}$

Figure 3.4: The relation between a formula φ and its classical variant φ^{cl}. The picture for φ^{cl} also illustrates the meaning of the double negation $\neg\neg\varphi$.

3.4.2. Definition. [Classical variant of a formula]
The *classical variant* of a formula $\varphi \in \mathcal{L}^{\mathsf{P}}$, denoted φ^{cl}, is obtained from φ by replacing all occurrences of inquisitive disjunction by classical disjunction.

To give some examples, we have $(p \vee\!\!\!\vee q)^{cl} = p \vee q$ and $(?p)^{cl} = p \vee \neg p$. The following fact follows immediately from Proposition 3.1.7 and Proposition 3.3.4.

3.4.3. Proposition. *For any formula φ and any model M, $|\varphi^{cl}|_M = |\varphi|_M$.*

Thus, φ^{cl} is always a truth-conditional formula sharing the same truth-conditions as φ. The relation between φ and φ^{cl} is illustrated visually in Figure 3.4.

If φ itself is truth-conditional, then φ and φ^{cl} are both truth-conditional formulas with the same truth conditions. Since for truth-conditional formulas truth determines support, φ and φ^{cl} are logically equivalent. Conversely, if φ and φ^{cl} are equivalent, then since φ^{cl} is truth-conditional, so is φ.[8]

3.4.4. Proposition. *For all $\varphi \in \mathcal{L}^{P}$, φ is truth-conditional $\iff \varphi \equiv \varphi^{cl}$.*

This shows in particular that every truth-conditional formula in $\mathcal{L}^{\mathsf{P}}$ is equivalent to a classical formula, which shows that the classical language $\mathcal{L}^{\mathsf{P}}_c$ is, up to equivalence, exactly the truth-conditional fragment of InqB.

3.4.5. Corollary.
For any $\varphi \in \mathcal{L}^{P}$, φ is truth-conditional $\iff \varphi \equiv \alpha$ for some $\alpha \in \mathcal{L}^{P}_c$.

If μ is a question, then μ is not truth-conditional, and therefore it is not equivalent to its classical variant μ^{cl}. In this case, the formula μ^{cl} is a statement that has the same truth-conditions as the question μ: as we discussed in Section 2.6, we can think of μ^{cl} as expressing the presupposition of μ.

[8]Recall from Section 2.3 that φ and ψ are logically equivalent, denoted $\varphi \equiv \psi$, if for every model M and information state s: $M, s \models \varphi \iff M, s \models \psi$.

3.4.6. Definition. [Presupposition of a question]
If μ is a question, its presupposition is the statement $\pi_\mu := \mu^{cl}$.

Thus, e.g., the presupposition of the question $p \mathbin{\vee\!\!\!\vee} q$ is the statement $p \vee q$, while the presupposition of a polar question $?\alpha$ is the tautology $(\alpha \vee \neg\alpha) \equiv \top$.

The classical connectives $\wedge$ and $\to$ preserve truth-conditionality. In fact, in the case of implication, the truth-conditionality of the consequent is sufficient to ensure the truth-conditionality of the implication, regardless of whether the antecedent is truth-conditional.

3.4.7. Proposition ($\wedge$ and $\to$ preserve truth-conditionality).

- *If α and β are truth-conditional, so is $\alpha \wedge \beta$.*
- *If α is truth-conditional, so is $\varphi \to \alpha$ for any φ.*

Proof. We prove the second claim, since the first is straightforward. Suppose α is truth-conditional. We want to show that $\varphi \to \alpha$ is truth-conditional. It suffices to show that whenever $\varphi \to \alpha$ is not supported at a state s, it is false at a world in the state. So, take an arbitrary state s and suppose $s \not\models \varphi \to \alpha$. Then there is some $t \subseteq s$ such that $t \models \varphi$ but $t \not\models \alpha$. Since α is truth-conditional, the fact that $t \not\models \alpha$ implies that there is a world $w \in t$ such that $w \not\models \alpha$. By persistency, since $\{w\} \subseteq t$ and $t \models \varphi$, we have $\{w\} \models \varphi$, that is, $w \models \varphi$. Since truth conditions work in the standard way we have $w \not\models \varphi \to \alpha$. Since $w \in t \subseteq s$, this shows that there is a world in s where $\varphi \to \alpha$ is false. $\square$

The previous proposition also implies that every negation $\neg\varphi$ is truth-conditional, since $\neg\varphi := \varphi \to \bot$ and $\bot$ is truth-conditional.

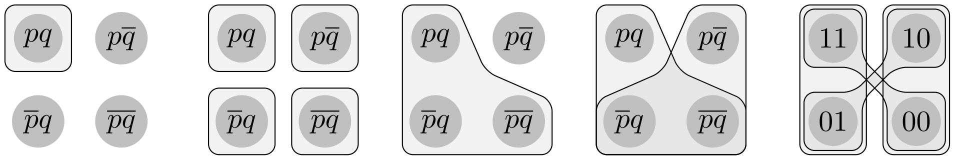
3.4.8. Proposition. *For any $\varphi \in \mathcal{L}^P$, $\neg\varphi$ is truth-conditional.*

Since classical disjunction is defined by letting $\varphi \vee \psi$ abbreviate $\neg(\neg\varphi \wedge \neg\psi)$, this implies in particular that classical disjunctions are always truth-conditional. Moreover, a double negation $\neg\neg\varphi$ is always truth-conditional. Moreover, since truth conditions work in the usual way, $\neg\neg\varphi$ has the same truth conditions as φ. Thus, $\neg\neg\varphi$ is equivalent to φ^{cl}: this is because both formulas are truth-conditional, and their truth conditions coincide with those of φ.

3.4.9. Proposition. *For any $\varphi \in \mathcal{L}^P$, $\neg\neg\varphi \equiv \varphi^{cl}$.*

This means that Figure 3.4(b) also depicts the semantics of $\neg\neg(p \mathbin{\vee\!\!\!\vee} q)$, and Figure 3.4(d) also depicts the semantics of $\neg\neg(?p)$.

An important consequence of these considerations is that a formula is equivalent to its double negation if and only if it is truth-conditional.



(a) $p \wedge q$ (b) $?p \wedge ?q$ (c) $p \to q$ (d) $p \to ?q$ (e) $?p \to ?q$

Figure 3.5: Illustration of the behavior of conjunction and implication in InqB.

3.4.10. Proposition. *For any $\varphi \in \mathcal{L}^P$, $\varphi \equiv \neg\neg\varphi \iff \varphi$ is truth-conditional.*

Proof. If φ is truth-conditional then putting together the previous proposition with Proposition 3.4.4 we have $\varphi \equiv \varphi^{cl} \equiv \neg\neg\varphi$. Conversely, if $\varphi \equiv \neg\neg\varphi$ then since $\neg\neg\varphi$ is truth-conditional, so is φ. □

In other words, the double negation law is the hallmark of truth-conditionality: it holds for statements, but not for questions. Notice also that the entailment $\varphi \models \neg\neg\varphi$ holds for every formula (Exercise 3.11.4), so the above bi-conditional can be written equivalently as: $\neg\neg\varphi \models \varphi \iff \varphi$ is truth-conditional.

3.5 Applying connectives to questions

We saw that, when applied to classical formulas, our connectives $\wedge$ and $\to$ behave just as they do in standard propositional logic. However, in InqB we have placed no syntactic restrictions to the applicability of these connectives. We can thus apply them freely to questions formed by means of $\vee\!\!\!\vee$. This is meaningful, since the semantics of these connectives was given in terms of recursive support clauses. In this section, we will illustrate the results of applying these connectives to questions with some examples.

First, consider conjunction. Proposition 3.4.7 implies that, if α and β are statements, then $\alpha \wedge \beta$ is itself a statement; and Proposition 3.1.7 ensures that this statement has the expected truth conditions: it is true just in case both conjuncts are true. This is illustrated by Figure 3.5(a). However, conjunction can now also be applied to questions. As an example, consider the formula $?p \wedge ?q$: as illustrated by Figure 3.5(b), this is a question that is settled in a state s just when both questions $?p$ and $?q$ are settled, that is, just when the state s determines the truth value of both p and q. More generally, we can use $\wedge$ to form conjunctive questions which are settled just in case both conjuncts are settled. Thus, one and the same conjunction operation can be used to formalize conjunctive statements, such as (1-a), and conjunctive questions, such as (1-b) and (1-c).[9]

(1) a. Sue rented a car and she booked a hotel.
b. Did Sue rent a car, and did she book a hotel?
c. What car did Sue rent, and which hotel did she book?

The situation is similar for implication. Proposition 3.4.7 implies that if α and β are statements then $\alpha \to \beta$ is a statement; and Proposition 3.1.7 implies that this statement has the usual material truth conditions. This is illustrated by Figure 3.5(c). However, as for conjunction, now we can also consider the effect of applying this connective to questions.

Consider first the case of a conditional $\alpha \to \mu$ whose antecedent is a statement and whose consequent is a question. Proposition 2.5.2 tells us that the clause for implication in this case boils down to the Ramsey test clause:

$$s \models \alpha \to \mu \iff s \cap |\alpha|_M \models \mu.$$

That is, $\alpha \to \mu$ is settled if the question μ is settled in restriction to the α-worlds in s. Thus, $\alpha \to \mu$ is a conditional question that asks us to resolve the consequent under the assumption of the antecedent. This seems to be the correct analysis of (at least one class of) conditional questions. Thus, one and the same implication connective can be used to formalize conditional statements like (2-a) and conditional questions like (2-b) and (2-c).[10]

(2) a. If Alice wins a free trip, she'll go to Athens.
b. If Alice wins a free trip, will she go to Athens?
c. If Alice wins a free trip, where will she go?

For a concrete illustration, consider the formula $p \to ?q$. The alternatives for this question in our toy model are shown in Figure 3.5(d): they correspond to the statements $p \to q$ and $p \to \neg q$, and to the two minimal ways to provide information that settles whether q in restriction to the p-worlds.

Next, consider a conditional $\mu \to \alpha$ whose antecedent is a question μ and whose consequent a statement α. It follows from Proposition 3.4.7 that in this

[9]Of course, (1-c) is not expressible in inquisitive propositional logic, since formalizing *wh*-questions requires the resources of predicate logic; but the point we make here is general: given any two questions we can formalize in a system, we can use $\wedge$ to formalize the corresponding conjunctive question.

[10]The point extends to counterfactual conditionals: in the inquisitive setting, it is possible to generalize an account of counterfactuals (for instance the ones of Lewis (1973) and Stalnaker (1968)) in such a way as to get a uniform analysis of statements like "If Alice had won a free trip, she would have gone to Athens" and questions like "If Alice had won a free trip, where would she have gone?". See Ciardelli (2016c) and Chapter 7 of Ciardelli et al. (2018) for detailed discussion.

case the conditional $\mu \to \alpha$ is truth-conditional, and so we have:

$$\begin{aligned} s \models \mu \to \alpha \quad &\iff \quad \forall w \in s : w \models \mu \to \alpha \\ &\iff \quad \forall w \in s : w \not\models \mu \textit{ or } w \models \alpha. \end{aligned}$$

Thus, in this case only the truth conditions of the question μ matter. Since μ and its presupposition π_μ have the same truth-conditions, μ can be replaced by π_μ when the consequent is a statement.

3.5.1. PROPOSITION. *If μ is a question and α a statement, $\mu \to \alpha \equiv \pi_\mu \to \alpha$.*

Thus, for instance, for any statement α we have $(p \vee\!\!\!\vee q) \to \alpha \equiv (p \vee q) \to \alpha$ and $(?p \to \alpha) \equiv (\top \to \alpha) \equiv \alpha$. Notice that as a special case we can take $\alpha = \bot$ and obtain the following corollary.

3.5.2. COROLLARY. *If μ is a question, $\neg\mu \equiv \neg\pi_\mu$.*

That is, applying negation to a question yields the negation of the presupposition of the question. Thus, e.g., $\neg(p \vee\!\!\!\vee q) \equiv \neg(p \vee q)$, while $\neg(?p) \equiv \neg\top \equiv \bot$.

Finally, consider a conditional $\mu \to \nu$ whose antecedent and consequent are both questions. As we saw in Section 2.5, such a conditional is supported in a state s just in case μ determines ν relative to s. As an example, take the formula $?p \to ?q$: this formula is supported in a state s in case within s, the truth value of q is functionally determined by the truth value of p. Indeed, it is easy to see that the support conditions for this formula may be restated as follows:

$$s \models ?p \to ?q \quad \iff \quad \exists f : \{0,1\} \to \{0,1\} \text{ s.t. } \forall w \in s : V(w,q) = f(V(w,p)).$$

The alternatives for this implication are the largest states in which such a dependency obtains. There are four ways in which the truth value of q might be functionally determined by the truth-value of p, given by the four functions $f : \{0,1\} \to \{0,1\}$ which might witness the above existential. In the model of Figure 3.5(e), each of these functions corresponds to one alternative. We will come back to the significance of such conditionals in the next section.[11]

[11]Notice that, having multiple alternatives in at least one model, the formula $?p \to ?q$ is a question in our terminology. We can think of it as a question asking to resolve $?q$ conditionally on an answer to $?p$.

Although the formula $?p \to ?q$ is supported in a state s iff whether p determines whether q relative to s, this formula should *not* be taken to formalize a dependence statement in English like "Whether p determines whether q". Such a statement should instead be formalized as a modal statement $\Box(?p \to ?q)$ which is true or false at a world w depending on whether the dependency holds relative to the set $R[w]$ of successors given by an accessibility relation. See Ciardelli (2018a) for the technical details and the arguments in favor of such an account.

3.6 Resolutions and their applications

3.6.1 Resolutions and normal form

An important feature of the system InqB is that we can compute, recursively on the structure of a formula φ, a set of classical formulas which can be taken to name the different pieces of information of type φ. We refer to these formulas as the *resolutions* of φ.

3.6.1. DEFINITION. [Resolutions]
The set $\mathcal{R}(\varphi)$ of resolutions of a formula $\varphi \in \mathcal{L}^{\mathsf{P}}$ is defined as follows:

- $\mathcal{R}(p) = \{p\}$;
- $\mathcal{R}(\bot) = \{\bot\}$;
- $\mathcal{R}(\varphi \wedge \psi) = \{\alpha \wedge \beta \mid \alpha \in \mathcal{R}(\varphi) \text{ and } \beta \in \mathcal{R}(\psi)\}$;
- $\mathcal{R}(\varphi \vee\!\!\!\vee \psi) = \mathcal{R}(\varphi) \cup \mathcal{R}(\psi)$;
- $\mathcal{R}(\varphi \to \psi) = \{\gamma_f \mid f : \mathcal{R}(\varphi) \to \mathcal{R}(\psi)\}, \quad \text{where } \gamma_f = \bigwedge_{\alpha \in \mathcal{R}(\varphi)}(\alpha \to f(\alpha))$.

The following facts can be immediately verified by induction.

3.6.2. PROPOSITION. *For every φ, $\mathcal{R}(\varphi)$ is a finite set of classical formulas.*

3.6.3. PROPOSITION. *For every classical formula α, $\mathcal{R}(\alpha) = \{\alpha\}$.*

Let us illustrate the recursive clauses by looking at some examples. For graphical convenience, in these examples we write $\overline{p}$ for $\neg p$ and $\overline{q}$ for $\neg q$.

3.6.4. EXAMPLE. [Inquisitive disjunction]
To illustrate the clause for $\vee\!\!\!\vee$, we compute the resolutions of $?p$. By the previous proposition we have $\mathcal{R}(p) = \{p\}$ and $\mathcal{R}(\overline{p}) = \{\overline{p}\}$. Using this we find:

$$\mathcal{R}(?p) = \mathcal{R}(p \vee\!\!\!\vee \overline{p}) = \mathcal{R}(p) \cup \mathcal{R}(\overline{p}) = \{p\} \cup \{\overline{p}\} = \{p, \overline{p}\}.$$

Thus, for $?p$ we have two resolutions, corresponding to the two ways of settling the question.

3.6.5. EXAMPLE. [Conjunction]
To illustrate the clause for $\wedge$, we compute the resolutions of $?p \wedge ?q$. Using the result of the previous example, we have:

$$\begin{aligned} \mathcal{R}(?p \wedge ?q) &= \{\alpha \wedge \beta \mid \alpha \in \{p, \overline{p}\}, \beta \in \{q, \overline{q}\}\} \\ &= \{p \wedge q,\ p \wedge \overline{q},\ \overline{p} \wedge q,\ \overline{p} \wedge \overline{q}\}. \end{aligned}$$

Thus, for $?p \wedge ?q$ we have four resolutions, corresponding to the four ways of settling the conjunctive question.

3.6.6. EXAMPLE. [Implication]
To illustrate the clause for $\to$, we compute the resolutions for $?p \to ?q$. The clause says that we have one resolution γ_f for each function $f : \mathcal{R}(?p) \to \mathcal{R}(?q)$. There are four such functions:

$$f_1 = \begin{cases} p \mapsto q \\ \overline{p} \mapsto q \end{cases} \qquad f_2 = \begin{cases} p \mapsto q \\ \overline{p} \mapsto \overline{q} \end{cases} \qquad f_3 = \begin{cases} p \mapsto \overline{q} \\ \overline{p} \mapsto q \end{cases} \qquad f_4 = \begin{cases} p \mapsto \overline{q} \\ \overline{p} \mapsto \overline{q} \end{cases}$$

For each function f, the corresponding formula γ_f is a conjunction that says that each resolution of the antecedent implies the corresponding resolution of the consequent, as given by the function f. In our case, these formulas are:

$$\begin{aligned} \gamma_{f_1} &= (p \to q) \land (\overline{p} \to q), \qquad & \gamma_{f_2} &= (p \to q) \land (\overline{p} \to \overline{q}), \\ \gamma_{f_3} &= (p \to \overline{q}) \land (\overline{p} \to q), \qquad & \gamma_{f_4} &= (p \to \overline{q}) \land (\overline{p} \to \overline{q}). \end{aligned}$$

These four formulas are the resolutions for our implication: $\mathcal{R}(?p \to ?q) = \{\gamma_{f_1}, \gamma_{f_2}, \gamma_{f_3}, \gamma_{f_4}\}$. Thus, $?p \to ?q$ has four resolutions, corresponding to the four ways for the dependency expressed by $?p \to ?q$ to obtain.

It is interesting to remark that there is a close similarity between the inductive definition of resolutions that we gave, and the inductive definition of *proofs* given in the Brouwer-Heyting-Kolmogorov (BHK) interpretation of intuitionistic logic. In this interpretation, a proof of a conjunction is a pair of two proofs, one for each conjunct; a proof of a disjunction is a proof of either disjunct; and a proof of an implication is a method to turn any proof of the antecedent into a proof of the consequent. Similarly, a resolution of a conjunction is a conjunction of two resolutions, one for each conjunct; a resolution of an inquisitive disjunction is a resolution of either disjunct; and a resolution of an implication corresponds to a function from resolutions of the antecedent to resolutions of the consequent. One difference between the two notions is that, unlike proofs in the BHK interpretation, resolutions are in turn formulas, that is, objects within the same language in which the original formula lives. Another difference is the atomic case: an atom has itself as unique resolution, whereas in the BHK interpretation it may well be associated with multiple proofs.

The crucial property of resolutions is stated by the following Proposition: to support a formula is to support some resolution of it. Thus, the semantics of any formula in InqB can be captured by a corresponding set of classical formulas.

3.6.7. THEOREM. *For any formula $\varphi \in \mathcal{L}^P$, any model M and state s:*

$$s \models \varphi \iff s \models \alpha \text{ for some } \alpha \in \mathcal{R}(\varphi).$$

Proof. By induction on φ. The base cases for atoms and $\bot$ are trivial. The inductive steps for $\land$ and $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ are straightforward, so we only discuss the inductive step for implication: assuming the claim holds for φ and ψ, we show that it holds for $\varphi \to \psi$.

For the left-to-right direction, suppose $s \models \varphi \to \psi$. Consider an arbitrary $\alpha \in \mathcal{R}(\varphi)$. Take the state $s \cap |\alpha|_M$. Since α is a classical formula and thus truth-conditional, we have $s \cap |\alpha|_M \models \alpha$. By the induction hypothesis on φ, this implies that $s \cap |\alpha|_M \models \varphi$. Since $s \cap |\alpha|_M \subseteq s$ and $s \models \varphi \to \psi$, it follows that $s \cap |\alpha|_M \models \psi$. By the induction hypothesis on ψ, we get $s \cap |\alpha|_M \models \beta$ for some resolution $\beta \in \mathcal{R}(\psi)$. By Proposition 2.5.2, this ensures that $s \models \alpha \to \beta$. Since α was an arbitrary resolution of φ, this argument shows that for every $\alpha \in \mathcal{R}(\varphi)$ there is a $\beta \in \mathcal{R}(\psi)$ such that $s \models \alpha \to \beta$. This means that there is a function $f : \mathcal{R}(\varphi) \to \mathcal{R}(\psi)$ such that $s \models \bigwedge_{\alpha \in \mathcal{R}(\varphi)}(\alpha \to f(\alpha))$. By definition, this conjunction is one of the resolutions of $\mathcal{R}(\varphi \to \psi)$.

For the right-to-left direction, suppose that s supports some resolution of $\varphi \to \psi$. This means that there is a function $f : \mathcal{R}(\varphi) \to \mathcal{R}(\psi)$ such that $s \models \gamma_f$. We want to show that $s \models \varphi \to \psi$. So, take any $t \subseteq s$ and suppose $t \models \varphi$. By induction hypothesis, $t \models \alpha$ for some $\alpha \in \mathcal{R}(\varphi)$. By definition, γ_f has a conjunct of the form $\alpha \to f(\alpha)$. Since $s \models \gamma_f$, we have $s \models \alpha \to f(\alpha)$. Since $t \subseteq s$ and $t \models \alpha$, it follows that $t \models f(\alpha)$. Since $f(\alpha) \in \mathcal{R}(\psi)$, by the induction hypothesis on ψ we have $t \models \psi$. This shows that $s \models \varphi \to \psi$. □

The previous theorem has many interesting repercussions. First, it yields an important normal form result for InqB: every formula is equivalent to an inquisitive disjunction of classical formulas.

3.6.8. Proposition (Inquisitive normal form).
For every $\varphi \in \mathcal{L}^P$ we have $\varphi \equiv \alpha_1 \vee\!\!\!\vee \cdots \vee\!\!\!\vee \alpha_n$ where $\mathcal{R}(\varphi) = \{\alpha_1, \ldots, \alpha_n\}$.[12]

Proof. This follows immediately from the previous theorem and the fact that the disjunction $\alpha_1 \vee\!\!\!\vee \cdots \vee\!\!\!\vee \alpha_n$ is supported just in case some α_i is supported. □

This normal form result shows that every formula can be rewritten in such a way that the inquisitive component is made syntactically explicit at the surface layer, and beyond this layer everything works just as in classical propositional logic (since resolutions are classical formulas). It is important to stress that this property is specific to InqB, and not a feature of inquisitive logics in general. For instance, no analogous normal form exists in inquisitive predicate logic.

It is useful to remark explicitly the following corollary, since we will appeal to it later: a formula is entailed by each of its resolutions.

3.6.9. Proposition. *If $\alpha \in \mathcal{R}(\varphi)$, then $\alpha \models \varphi$.*

[12] Note that, since $\vee\!\!\!\vee$ is commutative and associative, the order and bracketing of the disjuncts in the expression $\alpha_1 \vee\!\!\!\vee \cdots \vee\!\!\!\vee \alpha_n$ does not matter.

A further consequence of the above theorem is that for any formula φ, the truth-sets of the resolutions provide a generator for the proposition expressed by φ. This means that the resolutions $\alpha_1, \ldots, \alpha_n$ of φ can be taken to name the different pieces of information of type φ.

3.6.10. Proposition (Resolutions form a generator).
For every formula φ and model M, $[\varphi]_M = \{|\alpha|_M \mid \alpha \in \mathcal{R}(\varphi)\}^{\downarrow}$.

Proof. Spelling out the definitions, what we need to show is that for any state s of any model M we have $s \models \varphi \iff (s \subseteq |\alpha|_M$ for some $\alpha \in \mathcal{R}(\varphi))$. This follows immediately from Theorem 3.6.7 when we consider that a resolution α is a classical formula, and so we have $s \models \alpha \iff s \subseteq |\alpha|_M$. □

This proposition, in turn, allows us to see that in InqB, all formulas are *normal* in the sense of Definition 2.4.4.

3.6.11. Proposition (Normality).
For any formula $\varphi \in \mathcal{L}^P$ and model M: $[\varphi]_M = \textsc{Alt}_M(\varphi)^{\downarrow}$.

Proof. The right-to-left inclusion follows from persistency. For the converse, we need to show that any $s \in [\varphi]_M$ is included in an alternative. Let $\mathcal{R}(\varphi) = \{\alpha_1, \ldots, \alpha_n\}$. Suppose $s \in [\varphi]_M$. By the previous proposition, $s \subseteq |\alpha_{i_1}|_M$ for some $\alpha_{i_1} \in \mathcal{R}(\varphi)$. Now let us ask: is $|\alpha_{i_1}|_M$ an alternative? If so, we are done. If not, then $|\alpha_{i_1}|_M$ is not maximal among the states supporting φ, so $|\alpha_{i_1}|_M \subset t$ for some state $t \in [\varphi]_M$. Using again the previous proposition, we have $t \subseteq |\alpha_{i_2}|_M$ for some $\alpha_{i_2} \in \mathcal{R}(\varphi)$. If $|\alpha_{i_2}|_M$ is an alternative, we are done. Otherwise, iterating the reasoning we find that $|\alpha_{i_2}|_M \subset |\alpha_{i_3}|_M$ for some $\alpha_{i_3} \in \mathcal{R}(\varphi)$. Since the set of resolutions is finite, this process cannot produce an infinite sequence of ever larger sets $|\alpha_{i_1}|_M \subset |\alpha_{i_2}|_M \subset \ldots$. At some point, some element $|\alpha_{i_k}|_M$ must be an alternative. Since $s \subseteq |\alpha_{i_k}|_M$, we have that $s \in \textsc{Alt}_M(\varphi)^{\downarrow}$. □

This proposition implies that the semantics of φ in a model M is fully captured by the set of alternatives $\textsc{Alt}_M(\varphi)$. This makes it possible, e.g., to characterize truth-conditionality in terms of alternatives: a formula φ is truth-conditional if it has a unique alternative in any model, which must then coincide with the truth-set $|\varphi|_M$. The proof is left as an exercise (Exercise 3.11.6).

3.6.12. Proposition.

$$\begin{aligned}
\varphi \text{ is truth-conditional} &\iff \text{for any model } M,\ \textsc{Alt}_M(\varphi) \text{ is a singleton} \\
&\iff \text{for any model } M,\ \textsc{Alt}_M(\varphi) = \{|\varphi|_M\}.
\end{aligned}$$

3.6.2 Local tabularity

Theorem 3.6.7 also provides a way to prove an important fact: InqB is *locally tabular*. This means that, if the set of atoms $\mathcal{P}$ is finite, there are only finitely many non-equivalent formulas. In other words, the range of things that can be expressed in InqB with a finite repertoire of atoms is finite. In this respect, InqB is similar to classical propositional logic, which is also locally tabular, and different from intuitionistic propositional logic, where we can find infinitely many formulas containing only the atom p which are pair-wise non-equivalent (see, e.g., Chagrov and Zakharyaschev, 1997).

3.6.13. Theorem (InqB is locally tabular).
Let $\mathcal{L}^{\mathsf{P}}_{\equiv} := \{[\varphi]_{\equiv} \mid \varphi \in \mathcal{L}^{\mathsf{P}}\}$ be the quotient of our language $\mathcal{L}^{\mathsf{P}}$ under the relation $\equiv$ of logical equivalence. If the set of atoms $\mathcal{P}$ is finite, then $\mathcal{L}^{\mathsf{P}}_{\equiv}$ is finite.

Proof. Suppose $\mathcal{P}$ is finite. In InqB, equivalence among classical formulas is just equivalence in classical propositional logic. Since classical propositional logic is locally tabular, the set $(\mathcal{L}^{\mathsf{P}}_{c})_{\equiv}$ of equivalence classes of classical formulas is finite.

Theorem 3.6.7 tells us that the equivalence class of a formula φ in InqB is fully determined by the set of equivalence classes of its resolutions via the map

$$\{[\alpha_1]_{\equiv}, \dots, [\alpha_n]_{\equiv}\} \quad \mapsto \quad [\alpha_1 \vee\!\!\!\vee \cdots \vee\!\!\!\vee \alpha_n]_{\equiv}$$

which is well-defined since the result does not depend on the choice of representatives. Since there are only finitely many equivalence classes of classical formulas, there are also finitely many sets of such equivalence classes, and therefore only finitely many equivalence classes in $\mathcal{L}^{\mathsf{P}}_{\equiv}$.[13] $\square$

If $\mathcal{P}$ consists of a single atom, then there are only five equivalence classes, as illustrated by Figure 3.6 (see Exercise 3.11.7). Four of these correspond to the statements $\top, \bot, p, \neg p$, and one to the polar question $?p$. However, if $\mathcal{P}$ contains two atoms the number of equivalence classes is already 167; of these, only 16 are equivalence classes of statements. With three atoms the number of equivalence classes is over $5 \cdot 10^{22}$, of which only 256 correspond to statements (see Ciardelli, 2009, Corollary 3.3.5).

[13] The map considered above is not injective: many sets $\Gamma \subseteq (\mathcal{L}^{\mathsf{P}}_{c})_{\equiv}$ induce the same equivalence class. For instance $\{[p]_{\equiv}, [p \wedge q]_{\equiv}\}$ and $\{[p]_{\equiv}\}$ induce the same class, since $p \vee\!\!\!\vee (p \wedge q) \equiv p$. A 1-1 correspondence exists between $\mathcal{L}^{\mathsf{P}}_{\equiv}$ and the set of non-empty *antichains* of the set $(\mathcal{L}^{\mathsf{P}}_{c})_{\equiv}$ ordered by entailment. The number of antichains in a Boolean algebra with m generators is called the Dedekind number $D(m)$ (the sequence of such numbers is Sloane's A000372). The number of InqB-equivalence classes for n atoms is therefore $D(2^n) - 1$. The exact value of this expression is known only for $n \leq 3$. See Corollary 3.3.5 in Ciardelli (2009) for the details.

$$\begin{tikzcd}
& {[\top]_\equiv} & \\
& {[?p]_\equiv} \arrow[u] & \\
{[p]_\equiv} \arrow[ur] & & {[\neg p]_\equiv} \arrow[ul] \\
& {[\bot]_\equiv} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Figure 3.6: The equivalence classes of formulas containing only one atomic sentence p, ordered by entailment.

3.6.3 Implication and dependence functions

We saw above that the truth-sets of the resolutions of φ yield a generator in every model. This allows us to give an alternative characterization of the semantics of implication based on the generalized Ramsey test discussed in Section 2.5.2: a state s supports $\varphi \to \psi$ iff supposing any resolution α of φ yields a hypothetical state $s \cap |\alpha|_M$ that supports ψ.

3.6.14. Proposition (General Ramsey test in InqB).
For every formula φ, model M and state s:

$$s \models \varphi \to \psi \iff s \cap |\alpha|_M \models \psi \text{ for every } \alpha \in \mathcal{R}(\varphi).$$

Proof. Follows immediately from Proposition 2.5.3 and Proposition 3.6.10. □

A further application of resolutions is that they allow us to make the connection between implication and dependency even more explicit. To spell this out, we will introduce the notion of a *dependence function*.

3.6.15. Definition. [Dependence function]
A function $f : \mathcal{R}(\varphi) \to \mathcal{R}(\psi)$ is a dependence function from φ to ψ in a state s of a model M, notation $f : \varphi \rightsquigarrow_s \psi$, in case for any $\alpha \in \mathcal{R}(\varphi)$, $\alpha \models_s f(\alpha)$. We saw that f is a *logical* dependence function from φ to ψ, notation $f : \varphi \rightsquigarrow \psi$, if it is a dependence function in any state of any model.

Thus, f is a dependence function from φ to ψ in s if f can be used, given the information in s, to obtain from any given resolution of φ a corresponding resolution of ψ that follows from it. A *logical* dependence function from φ to ψ is a map that is guaranteed to be a dependence function in any state.

3.6.16. Example. Consider three propositional atoms p, q, r, and a model M having one possible world for each combination of truth values for these atoms. Let us write $p\overline{q}r$ for a world in which p is true, q is false, and r true, and similarly for the other worlds. Now consider the following function $f : \mathcal{R}(?p) \to \mathcal{R}(q \vee r)$ and the following two states s_1 and s_2:

$$f = \left\{ \begin{array}{r} p \mapsto q \\ \neg p \mapsto r \end{array} \right. \qquad s_1 = \left\{ \begin{array}{cc} pqr & \overline{p}qr \\ pq\overline{r} & \overline{p}\overline{q}r \end{array} \right\} \qquad s_2 = \left\{ \begin{array}{cc} p\overline{q}\overline{r} & \overline{p}qr \\ pq\overline{r} & \overline{p}\overline{q}r \end{array} \right\}$$

In s_1, f is a resolution function, since $p \models_{s_1} q$ and $\neg p \models_{s_1} r$. In s_2, f is not a resolution function, since $p \not\models_{s_2} q$. In symbols, we have $f : ?p \rightsquigarrow_{s_1} q \vee r$ and $f : ?p \not\rightsquigarrow_{s_2} q \vee r$.

Recall that the resolutions of an implication $\varphi \to \psi$ are statements of the form

$$\gamma_f = \bigwedge_{\alpha \in \mathcal{R}(\varphi)} (\alpha \to f(\alpha))$$

for a function $f : \mathcal{R}(\varphi) \to \mathcal{R}(\psi)$. The next proposition states that what the statement γ_f expresses is precisely that f is a dependence function.

3.6.17. Proposition.
Let $f : \mathcal{R}(\varphi) \to \mathcal{R}(\psi)$. For any model M and state s, $s \models \gamma_f \iff f : \varphi \rightsquigarrow_s \psi$.

Proof. The claim follows immediately from the definitions and the equivalence $\alpha \models_s f(\alpha) \iff s \models \alpha \to f(\alpha)$. □

Now, Proposition 3.6.8 tells us that $\varphi \to \psi$ is supported in a state s in case some formula $\gamma_f \in \mathcal{R}(\varphi \to \psi)$ is supported. But by the previous proposition, this holds if and only if there exists a dependence function $f : \varphi \rightsquigarrow_s \psi$. We have thus obtained the following result about the support conditions for an implication.

3.6.18. Proposition (Support for implication, restated).
$M, s \models \varphi \to \psi \iff$ there exists a dependence function $f : \varphi \rightsquigarrow_s \psi$.

This shows that a state supports an implication $\varphi \to \psi$ iff it admits a dependence function, i.e., if on the basis of the information in s there is some way of turning any resolution of φ into a resolution of ψ.

3.6.4 Resolutions for sets of formulas

To conclude this section, let us remark that the notion of resolutions, and the results that we have shown about it, can be extended straightforwardly from single formulas to sets of formulas. The idea is that a resolution of a set Φ of formulas is a set Γ of classical formulas obtained by replacing each formula in Φ by a resolution of it.

3.6.19. Definition. [Resolutions for sets]
If Φ is a set of formulas, a *resolution function* for Φ is a map $f : \Phi \to \mathcal{L}_c^{\mathsf{P}}$ such that for each $\varphi \in \Phi$ we have $f(\varphi) \in \mathcal{R}(\varphi)$. We say that a set Γ of formulas is a *resolution* of Φ if it is the image of Φ under some resolution function:

$$\mathcal{R}(\Phi) = \{f[\Phi] \mid f \text{ is a resolution function for } \Phi\}.$$

Thus for instance, the set $\Phi = \{p, ?q, ?r\}$ has the following four resolutions:

- $\{p, q, r\}$,
- $\{p, \neg q, r\}$,
- $\{p, q, \neg r\}$,
- $\{p, \neg q, \neg r\}$.

It is clear from the definition that a resolution of a set of formulas is always a set of classical formulas. Moreover, consider a set Γ of classical formulas: since any $\alpha \in \Gamma$ has itself as unique resolution, there is only one resolution function for Γ, namely, the identity function. As a consequence, Γ is the only resolution of itself.

3.6.20. Proposition. *If* $\Gamma \subseteq \mathcal{L}_c^{P}$, *then* $\mathcal{R}(\Gamma) = \{\Gamma\}$.

Just as a state supports a formula iff it supports some resolution of it, so a state supports a set of formulas iff it supports some resolution of it.[14] Thus, the resolutions of a set capture the ways in which all the formulas in the set may be jointly settled.

3.6.21. Proposition. *For any set of formulas* Φ *and any state* s*:*

$$s \models \Phi \iff s \models \Gamma \textit{ for some } \Gamma \in \mathcal{R}(\Phi).$$

Proof. Suppose $s \models \Phi$. For any $\varphi \in \Phi$, $s \models \varphi$, so by Theorem 3.6.7 we have some resolution $\alpha \in \mathcal{R}(\varphi)$ such that $s \models \alpha$. Now let f be a function which picks for each $\varphi \in \Phi$ a corresponding resolution $f(\varphi) \in \mathcal{R}(\varphi)$ such that $s \models f(\varphi)$: by definition, $f[\Phi] \in \mathcal{R}(\Phi)$ and $s \models f[\Phi]$.[15] The converse direction is immediate, again using Theorem 3.6.7 and the definition of resolutions for sets. □

Similarly, the notion of a *dependence function* can be extended straightforwardly to the case in which we have a set of determining formulas, as follows.

3.6.22. Definition. A function $f : \mathcal{R}(\Phi) \to \mathcal{R}(\psi)$ is a *dependence function* from Φ to ψ in a state s, notation $f : \Phi \rightsquigarrow_s \psi$, in case for all $\Gamma \in \mathcal{R}(\Phi)$ we have $\Gamma \models_s f(\Gamma)$; we say that f is a *logical* dependence function, notation $f : \Phi \rightsquigarrow \psi$, if it is a dependence function in any state of any model.

[14] Recall from the previous chapter that we say that a state supports a set Φ of formulas in case it supports all formulas in the set: $s \models \Phi \iff s \models \varphi$ for all $\varphi \in \Phi$.

[15] Note that the existence of f does not depend on the axiom of choice, since we may fix an enumeration of the formulas in $\mathcal{L}_c^{\mathsf{P}}$ and define $f(\varphi)$ to be the first resolution of φ supported by s in this enumeration.

3.7 Entailment in InqB

Let us now look at the relation of entailment in InqB, which instantiates the general ideas discussed in the previous chapter.

3.7.1. DEFINITION. [Entailment in InqB]
Let $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\mathsf{P}}$. We say that Φ logically entails ψ if in any propositional information model, any state that supports all formulas in Φ also supports ψ:

$$\Phi \models \psi \iff \text{for all models } M \text{ and states } s : M, s \models \Phi \text{ implies } M, s \models \psi.$$

As usual, in this definition $M, s \models \Phi$ is short for '$M, s \models \varphi$ for all $\varphi \in \Phi$'.

We say that Φ contextually entails ψ in an information state s from a model M if the entailment holds in restriction to worlds in s:

$$\Phi \models_s \psi \iff \text{for all states } t \subseteq s : M, t \models \Phi \text{ implies } M, t \models \psi.$$

If $\Phi = \{\varphi_1, \dots, \varphi_n\}$ we write $\varphi_1, \dots, \varphi_n \models \psi$ instead of $\Phi \models \psi$, and if $\Phi = \emptyset$ we simply write $\models \psi$. Similar conventions apply to contextual entailment.

We start by illustrating these relations with some examples, and then turn to their formal properties.

3.7.1 Illustration

In the previous chapter, we discussed in detail how once the relation of entailment is extended to questions, it becomes possible to capture several interesting relations as cases of entailment. We can now make this concrete in the case of propositional logic. To illustrate the point, consider again the propositional formulas and the model M introduced in Example 3.2.6. In our die roll scenario, the information that the outcome is low tells us what the range of the outcome is, but it does not tell us whether the outcome is even or odd. Moreover, in our context the outcome of the roll is jointly determined by its parity and its range, while it is not determined by the parity alone. This is captured formally by the following contextual (non-)entailments:

- $\mathsf{low} \models_M \mathsf{range}$;
- $\mathsf{low} \not\models_M \mathsf{parity}$;
- $\mathsf{parity}, \mathsf{range} \models_M \mathsf{outcome}$;
- $\mathsf{parity} \not\models_M \mathsf{outcome}$.

We can also capture the same facts in terms of *logical*, rather than contextual, entailment, provided we add declarative premises that specify that the outcomes are exhaustive and exclusive. This can be done by the following formulas:

- $\mathsf{exh} \;:=\; (\mathsf{one} \vee \cdots \vee \mathsf{six})$;

- $\mathsf{exc} := \neg(\mathsf{one} \wedge \mathsf{two}) \wedge \neg(\mathsf{one} \wedge \mathsf{three}) \wedge \cdots \wedge \neg(\mathsf{five} \wedge \mathsf{six})$.

Let $\Gamma := \{\mathsf{exh}, \mathsf{exc}\}$. On the basis of these assumptions, the statement low logically resolves the question range, but not the question parity; the questions parity and range logically determine the question outcome, while parity by itself does not determine outcome. This is captured by the following logical facts:

- $\Gamma, \mathsf{low} \models \mathsf{range}$;
- $\Gamma, \mathsf{low} \not\models \mathsf{parity}$;
- $\Gamma, \mathsf{parity}, \mathsf{range} \models \mathsf{outcome}$;
- $\Gamma, \mathsf{parity} \not\models \mathsf{outcome}$.

These examples illustrate how entailment in InqB captures interesting logical facts involving statements and questions in a propositional setting.

Let us now examine the formal properties of this relation, starting with the special cases in which the premises or the conclusion are truth-conditional.

3.7.2 Entailments with truth-conditional conclusions

When the conclusion of an entailment relation is truth-conditional, entailment boils down to preservation of truth.

3.7.2. Proposition (Entailment with a truth-conditional conclusion).
Let $\Phi \cup \{\alpha\} \subseteq \mathcal{L}^P$, where α is truth-conditional. Then:

$$\Phi \models \alpha \iff \textit{for any model } M \textit{ and world } w, w \models \Phi \textit{ implies } w \models \alpha.$$

Proof. The left-to-right direction is immediate, since truth is a special case of support. For the converse, suppose for any model M and any world w in M, $w \models \Phi$ implies $w \models \alpha$. We want to show that $\Phi \models \alpha$. So, consider a model M and an arbitrary state $s \models \Phi$ in M. By persistency, this implies that for all $w \in s$ we have $w \models \Phi$. By our assumption, it follows that for all $w \in s$ we have $w \models \alpha$. Since α is truth-conditional, this implies that $s \models \alpha$. Hence, $\Phi \models \alpha$. □

In particular, since classical formulas are truth-conditional with the standard truth conditions, this implies that entailment restricted to the classical fragment coincides with entailment in classical propositional logic.

3.7.3. Proposition (Conservativity over classical logic).
Let $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}^P_c$. Then $\Gamma \models \alpha \iff \Gamma$ entails α in classical propositional logic.

Thus, InqB is a conservative extension of classical propositional logic.[16] Another immediate consequence of Proposition 3.7.2 is that, when the conclusion is truth-

[16]This will be our perspective on InqB throughout this book. A different take is also possible: if we regard $\vee\!\!\!\vee$ as the "official" disjunction of the system, rather than as a new connective, InqB can also be regarded as a non-standard intermediate logic. This is the perspective taken in some previous literature on inquisitive logic (see, in particular, Ciardelli, 2009; Ciardelli and Roelofsen, 2011).

conditional, any premise may be replaced by its classical variant, which has the same truth-conditions.

3.7.4. Proposition.
Let $\Phi \cup \{\alpha\} \subseteq \mathcal{L}^P$ where α is truth-conditional, and let $\Phi^{cl} = \{\varphi^{cl} \mid \varphi \in \Phi\}$. We have:

$$\Phi \models \alpha \iff \Phi^{cl} \models \alpha.$$

In particular, this tells us that a statement is entailed by a question if and only if it is entailed by the question's presupposition: $\mu \models \alpha \iff \pi_\mu \models \alpha$. Thus, the presupposition of a question can be characterized as being, up to equivalence, the strongest statement which is entailed by the question. Thus, for instance, the question $p \mathbin{\vee\!\!\!\vee} q$ entails its presupposition $p \vee q$, all of its consequences, and no other statements, while the only statements entailed by the polar question $?p$ are tautologies (since the presupposition of $?p$ is the tautology $p \vee \neg p$).

3.7.3 Entailments with truth-conditional premises

Let us now consider the case in which the *premises* of our entailment relation are truth-conditional. In this case, too, the conditions for the entailment to hold can be simplified. To see this, notice first that the property of truth-conditionality is inherited from single formulas to sets of such formulas. Let us write $w \models \Gamma$ as an abbreviation for '$w \models \alpha$ for all $\alpha \in \Gamma$', and let $|\Gamma|_M = \{w \in W \mid w \models \Gamma\}$. Then we have the following proposition (the straightforward proof is omitted).

3.7.5. Proposition. *Let Γ be a set of truth-conditional formulas. Then for any model M and information state s:*

$$\begin{aligned} s \models \Gamma &\iff \forall w \in s : w \models \Gamma \\ &\iff s \subseteq |\Gamma|_M. \end{aligned}$$

When the set of premises of an entailment $\Gamma \models \varphi$ is a set of truth-conditional formulas, to check whether the entailment holds we do not have to check all states in which Γ is supported: it suffices to check whether φ is supported in the state $|\Gamma|_M$, which embodies the information carried by Γ.

3.7.6. Proposition (Entailment with truth-conditional premises).
Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^P$, where all formulas in Γ are truth-conditional. We have:

$$\Gamma \models \varphi \iff \textit{for any model } M,\ |\Gamma|_M \models \varphi$$

.

Proof. Suppose $\Gamma \models \varphi$. Given any model M, by the previous proposition we have $|\Gamma|_M \models \Gamma$, and therefore $|\Gamma|_M \models \varphi$. Conversely, suppose the right-hand side holds. Take any model M and state s with $s \models \Gamma$. By the previous proposition $s \subseteq |\Gamma|_M$; since $|\Gamma|_M \models \varphi$, by persistency also $s \models \varphi$. Hence, $\Gamma \models \varphi$. $\Box$

As a consequence of this proposition, we have the following important property: a set of statements entails an inquisitive disjunction just in case it entails a specific disjunct of it.

3.7.7. Proposition (Logical Split Property for $\vee\!\!\!\vee$).
Let $\Gamma \cup \{\varphi, \psi\} \subseteq \mathcal{L}^P$, where Γ is a set of truth-conditional formulas. We have:

$$\Gamma \models \varphi \vee\!\!\!\vee \psi \iff \Gamma \models \varphi \text{ or } \Gamma \models \psi.$$

Proof. The interesting direction is the left-to-right one (the converse is obvious, since $\varphi \models \varphi \vee\!\!\!\vee \psi$ and $\psi \models \varphi \vee\!\!\!\vee \psi$). We reason by contraposition. Suppose $\Gamma \not\models \varphi$ and $\Gamma \not\models \psi$. By the previous proposition, this means that there are two models $M = \langle W, V\rangle$ and $M' = \langle W', V'\rangle$ such that $|\Gamma|_M \not\models \varphi$ and $|\Gamma|_{M'} \not\models \psi$. We may suppose for simplicity that W and W' are disjoint (otherwise, we can make them disjoint). Now define a new information model $M'' = \langle W'', V''\rangle$, where $W'' = W \cup W'$ and V'' coincides with V on W and with V' on W'. By the locality of the semantics (Proposition 3.3.3), for states $s \subseteq W$ we have $M'', s \models \varphi \iff M, s \models \varphi$, and for states $s' \subseteq W'$ we have $M'', s' \models \varphi \iff M', s' \models \varphi$. Applying this to singleton states, we find that $|\Gamma|_{M''} = |\Gamma|_M \cup |\Gamma|_{M'}$. Since $M, |\Gamma|_M \not\models \varphi$ and $|\Gamma|_M \subseteq W$, by locality we have $M'', |\Gamma|_M \not\models \varphi$, and then by persistency, since $|\Gamma|_M \subseteq |\Gamma|_{M''}$, we obtain $M'', |\Gamma|_{M''} \not\models \varphi$. Reasoning analogously we can conclude $M'', |\Gamma|_{M''} \not\models \psi$. It follows that $|\Gamma|_{M''} \not\models \varphi \vee\!\!\!\vee \psi$. By the previous proposition, this shows that $\Gamma \not\models \varphi \vee\!\!\!\vee \psi$. $\Box$

In particular, if we take $\Gamma = \emptyset$ we obtain the Disjunction Property for $\vee\!\!\!\vee$, analogous to a well-known feature of disjunction in intuitionistic logic.

3.7.8. Corollary (Disjunction Property for $\vee\!\!\!\vee$).
For any $\varphi, \psi \in \mathcal{L}^P$, $\models \varphi \vee\!\!\!\vee \psi \iff\ \models \varphi$ or $\models \psi$.

Notice that an analogous property does not hold for the classical disjunction $\vee$: we have $\models p \vee \neg p$, but $\not\models p$ and $\not\models \neg p$ (recall that on classical formulas, InqB coincides with classical propositional logic).

Putting together Inquisitive Normal Form and the Logical Split Property for $\vee\!\!\!\vee$, we get the following proposition, which ensures that a set of statements entail a formula just in case they entail a particular resolution of it.

3.7.9. Proposition (Logical Resolution Property).
Let Γ be a set of truth-conditional formulas and let φ be an arbitrary formula:

$$\Gamma \models \varphi \iff \Gamma \models \alpha \text{ for some } \alpha \in \mathcal{R}(\varphi).$$

This tells us, for instance, that a statement logically resolves the question $?p$ just in case it logically entails one among p and $\neg p$.

Considering again the special case in which $\Gamma = \emptyset$, we find that a formula is logically valid iff some resolution is logically valid.

3.7.10. Corollary. *For any $\varphi \in \mathcal{L}^{P}$, $\models \varphi \iff \models \alpha$ for some $\alpha \in \mathcal{R}(\varphi)$.*

This immediately implies the following important fact.

3.7.11. Corollary (InqB is decidable). *There is an algorithm to decide whether a given formula $\varphi \in \mathcal{L}^{P}$ is valid in InqB.*

Indeed, here is one way to decide whether $\varphi \in \mathcal{L}^{\mathsf{P}}$ is valid (for a different decision procedure, see Exercise 3.11.7):

1. compute all the resolutions of φ;
2. for each resolution, check whether it is valid: by Proposition 3.7.3, this means checking if α is valid is classical logic, which is a decidable matter (e.g., we can use the standard truth-table method);
3. answer 'yes' if at least one resolution is valid, 'no' otherwise: this is possible since the number of resolutions is always finite.

All the properties that we have seen in this section for logical entailment have counterparts for contextual entailment. We call the contextual versions of these properties *local*, since they hold locally in an information state. First, we show that, when the premises are statements, to check a contextual entailment relative to s is to check support at a specific state: the state $s \cap |\Gamma|_M$ that results from strengthening s with the assumption that all formulas in Γ are true.

3.7.12. Proposition (Specificity).
Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{P}$, where Γ is a set of truth-conditional formulas. For any model M and state s:

$$\Gamma \models_s \varphi \iff s \cap |\Gamma|_M \models \varphi.$$

Proof. We have:

$$\begin{aligned} \Gamma \models_s \varphi \quad &\iff \quad \forall t \subseteq s : t \models \Gamma \text{ implies } t \models \varphi \\ &\iff \quad \forall t \subseteq s : t \subseteq |\Gamma|_M \text{ implies } t \models \varphi \\ &\iff \quad \forall t \subseteq s \cap |\Gamma|_M : t \models \varphi \\ &\iff \quad s \cap |\Gamma|_M \models \varphi, \end{aligned}$$

where the second biconditional uses Proposition 3.7.5 and the last biconditional uses persistency. ☐

From this we immediately get the local version of the Split Property for $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$.

3.7.13. Proposition (Local Split Property for $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$).
Let $\Gamma \cup \{\varphi, \psi\} \subseteq \mathcal{L}^P$, where Γ is a set of truth-conditional formulas. For any model M and state s:

$$\Gamma \models_s \varphi \vee\!\!\!\vee \psi \iff \Gamma \models_s \varphi \text{ or } \Gamma \models_s \psi.$$

Proof. Using the previous proposition, we have:

$$\begin{aligned} \Gamma \models_s \varphi \vee\!\!\!\vee \psi &\iff s \cap |\Gamma|_M \models \varphi \vee\!\!\!\vee \psi \\ &\iff s \cap |\Gamma|_M \models \varphi \text{ or } s \cap |\Gamma| \models \psi \\ &\iff \Gamma \models_s \varphi \text{ or } \Gamma \models_s \psi. \end{aligned}$$

□

Since contextual entailments are internalized by implications, the Local Split Property also amounts to an equivalence in the object language.

3.7.14. Corollary ($\vee\!\!\!\vee$-Split equivalence).
Suppose $\alpha \in \mathcal{L}^P$ is truth-conditional. For any $\varphi, \psi \in \mathcal{L}^P$ we have

$$\alpha \to (\varphi \vee\!\!\!\vee \psi) \equiv (\alpha \to \varphi) \vee\!\!\!\vee (\alpha \to \psi).$$

Proof. Using the previous proposition, we have:

$$\begin{aligned} s \models \alpha \to \varphi \vee\!\!\!\vee \psi &\iff \alpha \models_s \varphi \vee\!\!\!\vee \psi \\ &\iff \alpha \models_s \varphi \text{ or } \alpha \models_s \psi \\ &\iff s \models \alpha \to \varphi \text{ or } s \models \alpha \to \psi \\ &\iff s \models (\alpha \to \varphi) \vee\!\!\!\vee (\alpha \to \psi). \end{aligned}$$

□

Since the $\vee\!\!\!\vee$-Split equivalence plays an important role in inquisitive logic, it is worth pausing for a moment to comment on why it holds. As we saw, this equivalence reflects the Local Split Property, which in turn is a direct consequence of Specificity (Proposition 3.7.12). Specificity says that to check what follows from a statement α in s is to check what follows from strengthening s in a specific way, namely, by supposing that α is true (and similarly for a set of statements). This reflects the fundamental idea that statements, unlike questions, denote specific pieces of information, and that, as a consequence, to suppose a statement α is to make a specific supposition—to extend the available information in a specific way. The split equivalence reflects this all-important idea.

To conclude, notice that, in combination with the normal form result, the previous propositions have the following immediate consequences.

3.7.15. Proposition (Local Resolution Property).
If Γ is a set of truth-conditional formulas, for any $\varphi \in \mathcal{L}^P$ and state s we have:

$$\Gamma \models_s \varphi \iff \Gamma \models_s \alpha \textit{ for some } \alpha \in \mathcal{R}(\varphi).$$

3.7.16. Corollary (Resolution-split equivalence).
Let $\alpha, \varphi \in \mathcal{L}^P$, where α is truth-conditional and $\mathcal{R}(\varphi) = \{\beta_1, \ldots, \beta_m\}$. We have:

$$\alpha \to \varphi \equiv (\alpha \to \beta_1) \vee\!\!\!\vee \cdots \vee\!\!\!\vee (\alpha \to \beta_n).$$

3.7.4 The general case

Let us now consider the general case, where both premises and conclusions are allowed to be questions. In this case, too, an interesting characterization of entailment in terms of resolutions can be given: a set of formulas Φ entails a formula ψ iff every resolution of Φ entails some corresponding resolution of ψ.

3.7.17. Theorem (Resolution Theorem).
For any set Φ of formulas and any formula ψ:

$$\Phi \models \psi \iff \textit{for every } \Gamma \in \mathcal{R}(\Phi) \textit{ there is some } \alpha \in \mathcal{R}(\psi) \textit{ such that } \Gamma \models \alpha.$$

Proof. For the left-to-right direction, suppose $\Phi \models \psi$ and take any $\Gamma \in \mathcal{R}(\Phi)$. By Proposition 3.6.9, Γ entails each formula in Φ, and so $\Gamma \models \psi$. Then, by the Resolution Property (Proposition 3.7.9) we have $\Gamma \models \alpha$ for some $\alpha \in \mathcal{R}(\psi)$.

For the converse, suppose any resolution of Φ entails some resolution of ψ. Consider any model M and state s which supports Φ. By Proposition 3.6.21, $s \models \Phi$ implies $s \models \Gamma$ for some $\Gamma \in \mathcal{R}(\Phi)$. By assumption, for some $\alpha \in \mathcal{R}(\psi)$ we have $\Gamma \models \alpha$, and thus $s \models \alpha$. By Theorem 3.6.7 we can then conclude that $s \models \psi$. This shows that $\Phi \models \psi$. □

Notice that, since resolutions are classical formulas, the entailments occurring on the right-hand side of the biconditional are entailments in classical logic. Thus, the Resolution Theorem can be seen as showing how InqB-entailment is grounded in classical entailment in an interesting way.

Making use of the notion of a dependence function (Definition 3.6.22), the Resolution Theorem can also be stated as follows: an entailment $\Phi \models \psi$ holds iff there exists a logical dependence function from Φ to ψ, i.e., a function which yields for each resolution of Φ a corresponding entailed resolution of ψ.

3.7.18. Corollary.
For any $\Phi \cup \{\psi\} \subseteq \mathcal{L}^P$ we have $\Phi \models \psi \iff \exists f : \Phi \rightsquigarrow \psi$.

In the previous chapter, we saw that an entailment involving questions captures a logical dependency. We may regard a dependence function witnessing this entailment as capturing exactly *how* the dependency is realized. In the next chapter, we are going to see that we can always regard a proof of an entailment $\Phi \models \psi$ in inquisitive logic as encoding a logical dependence function $f : \Phi \rightsquigarrow \psi$.

Both the Resolution Theorem and its formulation in terms of dependence functions have analogues for contextual entailment. The proofs are similar to those of the logical case.

3.7.19. THEOREM (LOCAL RESOLUTION THEOREM).
For any $\Phi \cup \{\psi\} \subseteq \mathcal{L}^P$, for any model M and state s:

$$\Phi \models_s \psi \iff \textit{for every } \Gamma \in \mathcal{R}(\Phi) \textit{ there is some } \alpha \in \mathcal{R}(\psi) \textit{ such that } \Gamma \models_s \alpha.$$

3.7.20. COROLLARY.
For any $\Phi \cup \{\psi\} \subseteq \mathcal{L}^P$ we have $\Phi \models_s \psi \iff \exists f : \Phi \rightsquigarrow_s \psi$.

To illustrate the Resolution Theorem, consider again the example of entailment discussed in Section 3.7.1:

$$\Gamma,\ \mathsf{parity},\ \mathsf{range}\ \models\ \mathsf{outcome},$$

where Γ is a set of statements that lay out the connections between the atomic sentences (see Section 3.7.1) and the questions parity, range, and $\mathsf{outcome}$ are defined as in Example 3.2.6. The resolutions of our questions are:

- $\mathcal{R}(\mathsf{parity}) = \{\mathsf{even}, \mathsf{odd}\}$;
- $\mathcal{R}(\mathsf{range}) = \{\mathsf{low}, \mathsf{mid}, \mathsf{high}\}$;
- $\mathcal{R}(\mathsf{outcome}) = \{\mathsf{one}, \dots, \mathsf{six}\}$.

The Resolution Theorem tells us that our entailment holds just in case every resolution of the premises entails a corresponding resolution of the conclusion. And this is indeed the case, as one can readily verify:

- $\Gamma,\ \mathsf{even},\ \mathsf{low}\ \models\ \mathsf{two}$;
- $\Gamma,\ \mathsf{even},\ \mathsf{mid}\ \models\ \mathsf{four}$;
- $\Gamma,\ \mathsf{even},\ \mathsf{high}\ \models\ \mathsf{six}$;
- $\Gamma,\ \mathsf{odd},\ \mathsf{low}\ \models\ \mathsf{one}$;
- $\Gamma,\ \mathsf{odd},\ \mathsf{mid}\ \models\ \mathsf{three}$;
- $\Gamma,\ \mathsf{odd},\ \mathsf{high}\ \models\ \mathsf{five}$.

3.8 Substitution and the role of atoms

One feature that distinguishes InqB from classical propositional logic and many other well-known logics is that it is not closed under substitution. In this section we discuss this feature and explain why it is conceptually motivated in our setting. This will lead us naturally to a discussion of the treatment of atoms in InqB—an important design choice that we made in setting up the system.

A *substitution function* is a function $(\cdot)^* : \mathcal{P} \to \mathcal{L}^{\mathsf{P}}$ that assigns to each atom a formula. Such a function extends straightforwardly to a map defined on the entire language by letting $\bot^* = \bot$ and $(\varphi \circ \psi)^* = \varphi^* \circ \psi^*$ for $\circ \in \{\wedge, \to, \mathbin{\rangle\!\!\!\vee}\}$. It then further extends to sets of formulas by letting $\Phi^* = \{\varphi^* \mid \varphi \in \Phi\}$.

We can then ask if InqB is closed under uniform substitution, in the sense that for any formulas $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\mathsf{P}}$ and substitution functions $(\cdot)^*$ we have:

$$\Phi \models \psi \quad \Longrightarrow \quad \Phi^* \models \psi^*.$$

The answer is negative.

3.8.1. Proposition. *InqB is not closed under uniform substitution.*

Proof. We just need to give a counterexample. Recall from Proposition 3.4.10 that the entailment $\neg\neg\varphi \models \varphi$ is valid if and only if φ is truth-conditional. In InqB, an atom p is truth-conditional, but the corresponding polar question $?p$ is not. Therefore, we have:

- $\neg\neg p \models p$;
- $\neg\neg ?p \not\models ?p$.

Since the second entailment is obtained from the first via the substitution $p^* = ?p$, this is a counterexample to closure under uniform substitution. □

Closure under uniform substitution is often considered a desideratum for a logic. Why? The reasoning goes like this: propositional atoms are placeholders for arbitrary sentences, whose interpretation is not constrained in any particular way. Thus, whatever holds for atoms should hold for sentences in general.

However, the premise of this reasoning is false for InqB. In InqB, atoms do not stand for arbitrary sentences: they only stand for arbitrary *statements*. Atoms are, by design, not allowed to be questions: instead, questions in InqB have to built up syntactically out of statements by means of inquisitive disjunction. Thus, all we can reasonably expect that whatever validities hold for atoms should hold for arbitrary statements. And this is indeed the case: validity is preserved by substitution functions that map atoms to statements.[17]

[17] It is worth noting that inquisitive logic is not the only example of a logic which is not

3.8.2. DEFINITION. [Legitimate substitutions]
We call $(\cdot)^*$ a *legitimate* substitution if it maps each atom p to a statement p^*.

3.8.3. PROPOSITION (CLOSURE UNDER LEGITIMATE SUBSTITUTIONS).
If $\Phi \models \psi$ and $(\cdot)^*$ is a legitimate substitution, then $\Phi^* \models \psi^*$.

Proof. By contraposition, suppose $\Phi^* \not\models \psi^*$. Then there is a model $M = \langle W, V\rangle$ and a state s such that $M, s \models \Phi^*$ but $M, s \not\models \psi^*$. We can consider a new model $M^* = \langle W, V^*\rangle$ defined by letting $V^*(w, p) = 1 \iff M, w \models p^*$. We claim that for every formula χ:

$$M^*, s \models \chi \iff M, s \models \chi^*.$$

To show this, we proceed by induction on χ. For $\chi = p$ atomic, we use the fact that p^* is truth-conditional:

$$\begin{aligned} M^*, s \models p &\iff \forall w \in s : V^*(w, p) = 1 \\ &\iff \forall w \in s : M, w \models p^* \\ &\iff M, s \models p^*. \end{aligned}$$

The rest of the inductive argument is trivial. Having established the above connection between the two models, we can use it to conclude that $M^*, s \models \Phi$ but $M^*, s \not\models \psi$, which implies that $\Phi \not\models \psi$. □

It is instructive to consider why the argument does not go through if p^* is a question: in that case, the semantics of p^* in a model M can*not* be simulated by the semantics of p in another model M^*, since in our semantics, atoms are truth-conditional in every model.

We discussed how the failure of substitution in InqB reflects a specific design choice: atomic sentences stand for statements, while questions only come in as complex formulas built from statements by means of $\vee\!\!\!\vee$. But we could have easily made a different choice. We could have let atomic sentences denote arbitrary inquisitive propositions. This can be achieved easily: just let a model be a pair $\langle W, V\rangle$ where V assigns to each atom p an inquisitive proposition $V(p)$ over W, that is, a non-empty and downward closed set of states $s \subseteq W$. Then change the

closed under uniform substitution. Other examples are Carnap's modal logic (Carnap, 1946), data logic (Veltman, 1981), public announcement logic (Plaza, 1989), and dependence logic (Väänänen, 2007). In each of these cases, the failure of uniform substitution is linked to the fact that atoms are assumed to stand for a specific kind of sentence: in Carnap's modal logic, atoms stand for sentences which are contingent and mutually independent; in data logic and dynamic epistemic logic, they stand for sentences which are, in a relevant sense, non-epistemic; in dependence logic, they stand for formulas that are flat (roughly, truth-conditional in our sense).

atomic clause to: $M, s \models p \iff s \in V(p)$. It is easy to see that the resulting logic would then be closed under uniform substitution.

So, taking atomic formulas to be statements is a deliberate design choice. This choice has has three advantages for our present purposes.

First, this way of treating atoms allows us to regard our system InqB as a conservative extension of classical propositional logic with a new connective, thus retaining classical logic as a syntactic fragment. This would not be possible if questions were already present at the atomic level, since then even the equivalence $p \equiv \neg\neg p$ would fail (any model in which p is interpreted as a question would be a counterexample).

Second, our setup allows us to associate each question with a recursively defined set of statements—its resolutions—which capture the different ways in which the question may be resolved. The possibility of linking questions and statements in this way plays an important role in some of our most interesting results. In order for this to be possible, it is crucial that every question be syntactically built up from statements.

Finally, it is not hard to see that, if we allowed atoms to be questions, the resulting logic would coincide with the logic of infinite problems of Skvortsov (1979), a variant of Medvedev's logic of finite problems (Medvedev, 1962, 1966). This logic is not finitely axiomatizable (Shehtman and Skvortsov, 1986), and it is a long-standing open problem whether it is decidable. Our logic InqB is more well-behaved: it is decidable (Corollary 3.7.11) and, as we will see in the next chapter, it admits a simple axiomatization, thus providing a natural environment to study inferences involving both statements and questions.

3.9 Expressive completeness

In this section we show that the logical repertoire of InqB is expressively complete in a natural sense: it allows us to express all meanings that we might in principle expect to be able to express in a propositional system of inquisitive logic.

Semantically, a formula φ of InqB defines a property of information states. For instance, $?p$ defines a property that an information state s has in case the truth value of p is constant in s. Let us make this precise.

3.9.1. Definition. [State properties]

- A *state property* is a class $\mathcal{C}$ of pairs $\langle M, s\rangle$ where M is a propositional information model and s is an information state in M.
- The state property defined by $\varphi \in \mathcal{L}^{\mathsf{P}}$ is $[\varphi] = \{\langle M, s\rangle \mid M, s \models \varphi\}$.
- A state property $\mathcal{C}$ is definable in InqB if $\mathcal{C} = [\varphi]$ for some $\varphi \in \mathcal{L}^{\mathsf{P}}$.

Now let us ask: what state properties could we in principle expect to define in a system of propositional inquisitive logic?

First of all, since we expect inquisitive systems to satisfy persistency and the empty state property, we expect to be able to define only state properties that are *inquisitive*, in the sense of the following definition.

3.9.2. DEFINITION. [Inquisitive state properties]
We say that a state property $\mathcal{C}$ is inquisitive if it satisfies:

- Downward closure: $\langle M, s\rangle \in \mathcal{C}$ and $t \subseteq s$ implies $\langle M, t\rangle \in \mathcal{C}$;
- Empty state property: $\langle M, \emptyset\rangle \in \mathcal{C}$ for all models M.

Moreover, since a propositional formula in a finitary language will contain only a finite set $\mathcal{Q}$ of atoms, we expect that the resulting state property will depend only on the information that a state contains about those atoms.

In order to make this idea precise, we assign to each world w an object $[w]_{\mathcal{Q}}$, called the $\mathcal{Q}$-profile of w, which reflects the way things are at w with respect to the atoms in $\mathcal{Q}$. Formally, $[w]_{\mathcal{Q}}$ is just the set of atoms from $\mathcal{Q}$ which are true at w. The information that an information state s contains about $\mathcal{Q}$ is then reflected by the set of $\mathcal{Q}$-profiles which are possible according to the state.

3.9.3. DEFINITION. [$\mathcal{Q}$-profiles]
Let $\mathcal{Q} \subseteq \mathcal{P}$ be a set of atomic sentences, $M = \langle W, V\rangle$ a model.

- If $w \in W$, the $\mathcal{Q}$-profile of w is $[w]^{\mathcal{Q}}_M = \{q \in \mathcal{Q} \mid V(w, q) = 1\}$.
- If $s \subseteq W$, the $\mathcal{Q}$-profile of s is $[s]^{\mathcal{Q}}_M = \{[w]^{\mathcal{Q}}_M \mid w \in s\}$.

Given a state property $\mathcal{C}$, to say that $\mathcal{C}$ depends only on the information about $\mathcal{Q}$ is to say that $\mathcal{C}$ does not distinguish between information states that have the same $\mathcal{Q}$-profile. This is captured by the following definition.

3.9.4. DEFINITION. [Finitely determined properties]
Let $\mathcal{Q} \subseteq \mathcal{P}$ be a set of atoms. A state property $\mathcal{C}$ is $\mathcal{Q}$-*determined* if for any given pairs $\langle M, s\rangle$ and $\langle M', s'\rangle$:

$$[s]^{\mathcal{Q}}_M = [s']^{\mathcal{Q}}_{M'} \text{ implies } (\langle M, s\rangle \in \mathcal{C} \iff \langle M', s'\rangle \in \mathcal{C}).$$

We say that $\mathcal{C}$ is *finitely determined* if it is $\mathcal{Q}$-determined for some finite $\mathcal{Q} \subseteq \mathcal{P}$.

The following theorem says that InqB allows us to define all and only the finitely determined inquisitive state properties.

3.9.5. THEOREM (EXPRESSIVE COMPLETENESS OF INQB).
The following are equivalent for any state property $\mathcal{C}$:

- *$\mathcal{C}$ is a finitely determined inquisitive state property;*
- *$\mathcal{C}$ is definable in InqB.*

Proof. If $\mathcal{C}$ is definable in InqB, then $\mathcal{C} = [\varphi]$ for some formula $\varphi \in \mathcal{L}^{\mathsf{P}}$. Due to persistency and the empty state property (Proposition 3.3.1), $\mathcal{C}$ is an inquisitive property. Moreover, it is easy to check that $\mathcal{C}$ is $\mathcal{Q}$-determined where $\mathcal{Q}$ is the finite set of atoms occurring in φ.

Conversely, suppose that $\mathcal{C}$ is a finitely determined inquisitive state property. Let $\mathcal{Q}$ be a finite set of atoms such that $\mathcal{C}$ is $\mathcal{Q}$-determined. We first define for each world w in a model M a formula χ^w_M which encodes the $\mathcal{Q}$-profile of w:

$$\chi^w_M := \bigwedge\{q \mid q \in [w]^{\mathcal{Q}}_M\} \wedge \bigwedge\{\neg q \mid q \in (\mathcal{Q} - [w]^{\mathcal{Q}}_M)\}.$$

Note that the relevant conjunctions are finite because $\mathcal{Q}$ is finite. It is immediate to check that for any M, w and M', w' we have:

$$M', w' \models \chi^w_M \iff [w']^{\mathcal{Q}}_{M'} = [w]^{\mathcal{Q}}_M. \tag{3.1}$$

Similarly, we associate to each state s in a model M a formula χ^s_M which encodes the $\mathcal{Q}$-profile of s:

$$\chi^s_M := \bigvee\{\chi^w_M \mid w \in s\}.$$

Regardless of whether s is finite, the disjunction is finite since, due to the finiteness of $\mathcal{Q}$, there are only finitely many distinct formulas of the form χ^w_M. In case $s = \emptyset$ we let $\chi^s_M = \bot$. The crucial property of χ^s_M is the following:

$$M', s' \models \chi^s_M \iff [s']^{\mathcal{Q}}_{M'} \subseteq [s]^{\mathcal{Q}}_M. \tag{3.2}$$

Let us show (3.2). If $M', s' \models \chi^s_M$, then for any $w' \in s'$ we have $M', w' \models \chi^s_M$ by persistency. Since truth conditions work as usual, this means that $M', w' \models \chi^w_M$ for some $w \in s$. By (3.1) this means that $[w']^{\mathcal{Q}}_{M'} = [w]^{\mathcal{Q}}_M$ for some $w \in s$, and so $[w']^{\mathcal{Q}}_{M'} \in [s]^{\mathcal{Q}}_M$. Since this is the case for each $w' \in s'$, we have $[s']^{\mathcal{Q}}_{M'} \subseteq [s]^{\mathcal{Q}}_M$.

Conversely, suppose $[s']^{\mathcal{Q}}_{M'} \subseteq [s]^{\mathcal{Q}}_M$. This means that for any $w' \in s'$ there is some $w \in s$ such that $[w']^{\mathcal{Q}}_{M'} = [w]^{\mathcal{Q}}_M$. By (3.1) this means that $M', w' \models \chi^w_M$ and therefore also $M', w' \models \chi^s_M$. So, χ^s_M is true at every world in s'. Since χ^s_M is truth-conditional (as it is a classical formula), it follows that $M', s' \models \chi^s_M$. This completes the proof of (3.2).

Finally, we can write our candidate definition of the class $\mathcal{C}$:

$$\chi_{\mathcal{C}} := \bigvee\!\!\!\bigvee\{\chi^s_M \mid \langle M, s\rangle \in \mathcal{C}\}.$$

Again, this disjunction is finite since there exist only finitely many formulas of the form χ^s_M. Note that the set of disjuncts is non-empty, since $\mathcal{C}$ is an inquisitive property and thus contains $\langle M, \emptyset\rangle$ for each M, which means that $\chi_{\mathcal{C}}$ always contains $\chi_M^{\emptyset} = \bot$ as a disjunct.

It remains to be shown that that $\chi_{\mathcal{C}}$ defines $\mathcal{C}$, i.e., that we have:

$$M, s \models \chi_{\mathcal{C}} \iff \langle M, s\rangle \in \mathcal{C}. \tag{3.3}$$

For the right-to-left direction, suppose $\langle M, s\rangle \in \mathcal{C}$. Then by definition, χ_M^s is a disjunct in $\chi_{\mathcal{C}}$. Since $M, s \models \chi_M^s$ by (3.2), we have $M, s \models \chi_{\mathcal{C}}$.

For the converse, suppose $M, s \models \chi_{\mathcal{C}}$. Then by the clause for $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ we have $M, s \models \chi_{M'}^{s'}$ for some $\langle M', s'\rangle \in \mathcal{C}$. By (3.2), this implies $[s]_M^{\mathcal{Q}} \subseteq [s']_{M'}^{\mathcal{Q}}$. This means that there is $s'' \subseteq s'$ such that $[s]_M^{\mathcal{Q}} = [s'']_{M'}^{\mathcal{Q}}$. Since $\mathcal{C}$ is downward closed, we have $\langle M', s''\rangle \in \mathcal{C}$. Finally, since $\mathcal{C}$ is $\mathcal{Q}$-determined and $[s]_M^{\mathcal{Q}} = [s'']_{M'}^{\mathcal{Q}}$, $\langle M', s''\rangle \in \mathcal{C}$ implies $\langle M, s\rangle \in \mathcal{C}$. $\square$

In fact, we will see in the next section that one can dispense with some of the operators in InqB and obtain a system which is still expressively complete. On the other hand, expressive completeness is by no means a trivial property of an inquisitive logic. To illustrate this point, consider another way we could have added questions to classical propositional logic: instead of introducing questions by means of $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$, we could introduce them by taking the operator ? as a primitive. The resulting system of inquisitive logic is a natural extension of classical propositional logic with questions; but, as the reader is invited to show in Exercise 3.11.9, it is *not* expressively complete in the relevant sense.

3.10 Relations between the connectives

In this section we examine the relation between the connectives of InqB in terms of expressive power and definability. In logics that are closed under uniform substitution, the two issues are connected: a connective is definable if and only if it can be dropped without a loss of expressive power. In InqB, as we will see, the two issues are distinct. We start by introducing the relevant notions.

3.10.1 Preliminaries

Let L be an arbitrary propositional logic with language $\mathcal{L} = \mathcal{L}[C]$ generated by a set C of connectives, giving rise to a relation of logical equivalence $\equiv_L \subseteq \mathcal{L} \times \mathcal{L}$. We assume $\equiv_L$ to be an equivalence relation and a congruence with respect to the connectives: that is, we suppose that for every n-ary connective $\circ \in C$, if $\varphi_i \equiv_L \psi_i$ for $i \leq n$ then $\circ(\varphi_1, \ldots, \varphi_n) \equiv_L \circ(\psi_1, \ldots, \psi_n)$.

3.10.1. Definition. [Context] A *propositional context* $\varphi(p_1, \ldots, p_n)$ consists of a formula $\varphi \in \mathcal{L}$ together with a sequence of designated atomic formulas $p_1, \ldots, p_n$. Note that $\varphi(p_1, \ldots, p_n)$ is allowed to contain other atoms besides $p_1, \ldots, p_n$. If $\varphi(p_1, \ldots, p_n)$ is a context and $\chi_1, \ldots, \chi_n \in \mathcal{L}$, we write $\varphi(\chi_1, \ldots, \chi_n)$ for the result of replacing $p_1, \ldots, p_n$ by $\chi_1, \ldots, \chi_n$ throughout φ.

3.10.2. Definition. [Definability] We say that an n-ary connective $\circ \in C$ is defined by a context $\xi(p_1, \ldots, p_n)$ in case for all $\chi_1, \ldots, \chi_n \in \mathcal{L}[C]$:

$$\circ(\chi_1, \ldots, \chi_n) \equiv_L \xi(\chi_1, \ldots, \chi_n).$$

We say that $\circ$ is definable from a set $C' \subseteq C$ of connectives in case there is a context $\xi(p_1, \ldots, p_n)$, with $\xi \in \mathcal{L}[C']$ which defines $\circ$. If we just say that $\circ$ is definable then we mean that it is definable from $C - \{\circ\}$.

In terms of definability we define the notion of an independent set of connectives.

3.10.3. Definition. [Independence] We say that a set C of connectives is *independent* if no connective $\circ \in C$ is definable.

We also introduce the notion of *eliminability* of a connective, which means that the connective can be omitted without a loss in expressive power.

3.10.4. Definition. [Eliminability] Let $C' \subseteq C$ be a set of connectives. We say that the set of connectives C' is *eliminable* if for each formula $\varphi \in \mathcal{L}[C]$ there is a formula $\varphi^* \in \mathcal{L}[C - C']$ such that $\varphi \equiv_L \varphi^*$. We say that a connective $\circ$ is eliminable if $\{\circ\}$ is eliminable.

Notice that definability implies eliminability.

3.10.5. Proposition. *If a connective $\circ$ is definable, then it is eliminable.*

Proof. Suppose $\circ$ is defined by $\xi(p_1, \ldots, p_n) \in \mathcal{L}[C - \{\circ\}]$. We show by induction that every $\varphi \in \mathcal{L}[C]$ is equivalent to some $\varphi^* \in \mathcal{L}[C - \{\circ\}]$. The only non-trivial case is the one for $\varphi = \circ(\psi_1, \ldots, \psi_n)$. By induction hypothesis there are $\psi_1^*, \ldots, \psi_n^* \in \mathcal{L}[C - \{\circ\}]$ s.t. $\psi_i \equiv_L \psi_i^*$ for $i \leq n$. Then $\varphi \equiv_L \circ(\psi_1^*, \ldots, \psi_n^*)$. Since $\circ$ is defined by ξ we have $\circ(\psi_1^*, \ldots, \psi_n^*) \equiv_L \xi(\psi_1^*, \ldots, \psi_n^*)$. Since $\xi, \psi_1^*, \ldots, \psi_n^* \in \mathcal{L}[C - \{\circ\}]$, we have that $\xi(\psi_1^*, \ldots, \psi_n^*) \in \mathcal{L}[C - \{\circ\}]$. So φ is L-equivalent to some formula in $\mathcal{L}[C - \{\circ\}]$, which completes the inductive step. $\square$

Another important notion in this area is the notion of completeness of a set C' of connectives, which holds when every formula in the language is L-equivalent to one built up using only connectives from C'.

3.10.6. Definition. [Completeness] We say that a set of connectives $C' \subseteq C$ is complete for L if for all $\varphi \in \mathcal{L}[C]$ there exists some $\varphi^* \in \mathcal{L}[C']$ s.t. $\varphi \equiv_L \varphi^*$. We say that a set C' is a *minimal complete* set of connectives for L if C' is complete for L, and no proper subset $C'' \subset C'$ is complete for L.

The notions of definability and eliminability often go hand in hand. This is because the logics L one typically considers are closed under uniform substitution: if an equivalence holds, then it still holds when the atoms are replaced by arbitrary formulas.

3.10.7. PROPOSITION. *If L is closed under uniform substitution and $\circ$ is eliminable, then $\circ$ is definable.*

Proof. Let $p_1, \ldots, p_n$ be n distinct atomic formulas. Suppose L is closed under uniform substitution and $\circ$ is eliminable. Then the formula $\circ(p_1, \ldots, p_n)$ is L-equivalent to some $\xi \in \mathcal{L}[C - \{\circ\}]$. Consider the context $\xi(p_1, \ldots, p_n)$: by closure under uniform substitution, for all $\chi_1, \ldots, \chi_n \in \mathcal{L}$ we have $\circ(\chi_1, \ldots, \chi_n) \equiv_L \xi(\chi_1, \ldots, \chi_n)$. This means that $\circ$ is defined by $\xi(p_1, \ldots, p_n)$. $\square$

As we discussed in Section 3.8, InqB is not closed under uniform substitution. As a consequence, in InqB we find an interesting gap between the eliminability of a connective and its definability. In this section we examine this gap carefully.

3.10.2 Eliminability

In InqB, our repertoire of connectives is $C = \{\bot, \to, \wedge, \vee\!\!\!\vee\}$ (we regard $\bot$ as a 0-ary connective). Let us first consider which connectives are eliminable, i.e., can be dropped without a loss in expressive power. The following three propositions have easy proofs, which are left as an exercise (see Exercise 3.11.11).

3.10.8. PROPOSITION. *$\bot$ is not eliminable in InqB.*

3.10.9. PROPOSITION. *$\vee\!\!\!\vee$ is not eliminable in InqB.*

3.10.10. PROPOSITION. *$\to$ is not eliminable in InqB.*

So, each of $\bot, \to$, and $\vee\!\!\!\vee$ contributes to the expressive power of the language. By contrast, $\wedge$ does not.

3.10.11. PROPOSITION. *$\wedge$ is eliminable in InqB.*

Proof. We need to show that for all $\varphi \in \mathcal{L}^{\mathsf{P}}$ there is an equivalent $\wedge$-free formula φ^*. First take a classical formula $\alpha \in \mathcal{L}^{\mathsf{P}}_c$. We define an equivalent formula α^* which contains only $\bot$ and $\to$ as follows:

- $p^* = p$;
- $\bot^* = \bot$;
- $(\beta \to \gamma)^* = \beta^* \to \gamma^*$;
- $(\beta \wedge \gamma)^* = \neg(\beta^* \to \neg\gamma^*)$.

Since classical formulas obey classical logic, for all classical α we have $\alpha \equiv \alpha^*$. Next, consider an arbitrary $\varphi \in \mathcal{L}^{\mathsf{P}}$. By the inquisitive normal form there are classical formulas $\alpha_1, \dots, \alpha_n$ such that $\varphi \equiv \alpha_1 \vee\!\!\!\vee \cdots \vee\!\!\!\vee \alpha_n$. We can then take $\varphi^* = \alpha_1^* \vee\!\!\!\vee \cdots \vee\!\!\!\vee \alpha_n^*$. Clearly, φ^* is $\wedge$-free and equivalent to φ. □

3.10.12. Corollary. $\{\bot, \to, \vee\!\!\!\vee\}$ *is the only minimal complete set for* InqB.

It is also interesting to consider a slight variant of InqB, denoted $\mathsf{InqB}^{\neg}$, whose set of connectives is $\{\neg, \wedge, \to, \vee\!\!\!\vee\}$. That is, we drop the falsum constant from the language and adopt negation as a primitive, interpreted in accordance with the clause: $s \models \neg\varphi \iff$ there is no consistent $t \subseteq s$ with $t \models \varphi$. The system $\mathsf{InqB}^{\neg}$ is a notational variant of InqB, since negation is definable in InqB by letting $\neg\varphi := \varphi \to \bot$, while $\bot$ is definable in $\mathsf{InqB}^{\neg}$ by letting $\bot := p \wedge \neg p$ for an arbitrary atom p. In particular, InqB and $\mathsf{InqB}^{\neg}$ are equally expressive. Let us now ask about eliminability in $\mathsf{InqB}^{\neg}$. Again, the following two propositions have easy proofs, which are left as an exercise.

3.10.13. Proposition. $\neg$ *is not eliminable in* $\mathsf{InqB}^{\neg}$.

3.10.14. Proposition. $\vee\!\!\!\vee$ *is not eliminable in* $\mathsf{InqB}^{\neg}$.

Thus, $\neg$ and $\vee\!\!\!\vee$ are crucial to the expressive power of $\mathsf{InqB}^{\neg}$. By contrast, $\to$ and $\wedge$ are not: they can be jointly eliminated.

3.10.15. Proposition. *The set* $\{\to, \wedge\}$ *is eliminable in* $\mathsf{InqB}^{\neg}$.

Proof. We need to show that every $\varphi \in \mathcal{L}^{\mathsf{P}}$ is equivalent to some formula φ^* which contains only $\neg$ and $\vee\!\!\!\vee$. First take a classical formula $\alpha \in \mathcal{L}^{\mathsf{P}}_c$. We define an equivalent formula α^* as follows, where p_0 is an arbitrary fixed atom:

- $p^* = p$,
- $\bot^* = \neg(p_0 \vee\!\!\!\vee \neg p_0)$,
- $(\beta \to \gamma)^* = \neg\neg(\neg\beta^* \vee\!\!\!\vee \gamma^*)$,
- $(\beta \wedge \gamma)^* = \neg(\neg\beta^* \vee\!\!\!\vee \neg\gamma^*)$.

We leave it to the reader to check inductively that for all classical α, $\alpha \equiv \alpha^*$. As before, if $\varphi \in \mathcal{L}^{\mathsf{P}}$, we can then take classical formulas $\alpha_1, \dots, \alpha_n$ such that $\varphi \equiv \alpha_1 \vee\!\!\!\vee \cdots \vee\!\!\!\vee \alpha_n$ and let $\varphi^* = \alpha_1^* \vee\!\!\!\vee \cdots \vee\!\!\!\vee \alpha_n^*$. The resulting formula is equivalent to φ and contains only $\neg$ and $\vee\!\!\!\vee$. □

3.10.16. Corollary. $\{\neg, \vee\!\!\!\vee\}$ *is the only minimal complete set for* $\mathsf{InqB}^{\neg}$.

This tells us that one can get a system which is equi-expressive with InqB, and expressively complete in the sense of the previous section, by using only the connectives $\neg$ and $\vee\!\!\!\vee$.

3.10.3 Definability

Let us now we turn to the issue of definability. We start considering the question for InqB. We will prove the following result.

3.10.17. THEOREM. $\{\bot, \wedge, \to, \vee\!\!\!\vee\}$ *is an independent set of connectives.*

That is, none of the connectives is definable from the others. This is obvious for the connectives $\bot, \to$, and $\vee\!\!\!\vee$: we saw in the previous section that these connectives are not eliminable in InqB, which *a fortiori* implies that they are not definable (cf. Proposition 3.10.5). What remains to be proved is that conjunction, while eliminable, is nevertheless not definable from the other connectives.

3.10.18. PROPOSITION. $\wedge$ *is not definable from* $\{\bot, \to, \vee\!\!\!\vee\}$.

Proof. We want to show that $\wedge$ is not defined by any context $\varphi(p_1, p_2)$, where $\varphi \in \mathcal{L}[\bot, \to, \vee\!\!\!\vee]$ (note: φ can contain atoms other than p_1, p_2). Take a candidate context $\varphi(p_1, p_2)$ and fix two atoms p, q that do not occur in φ. We claim that

$$?p \wedge ?q \;\not\equiv\; \varphi(?p, ?q)$$

which implies that $\varphi(p_1, p_2)$ does not in fact define $\wedge$.

To see that $?p \wedge ?q \not\equiv \varphi(?p, ?q)$, we give a model where these formulas have different support conditions. It suffices to take the model M that we used above for our examples (cf., e.g., Fig. 3.5), based on four worlds $\{w_{pq}, w_{p\overline{q}}, w_{\overline{p}q}, w_{\overline{pq}}\}$ corresponding to the four assignments of truth values to the letters p and q. We suppose moreover that every atom $r \neq p, q$ is false at each world in this model.

In this model, $?p \wedge ?q$ is supported only by the singleton states and the empty state (cf. Fig. 3.5(b) on p. 68). By contrast, we will show that $\varphi(?p, ?q)$ is either supported only by the empty state, or it is supported by some state of cardinality 2. In either case, the support conditions of $\varphi(?p, ?q)$ are different from those of $?p \wedge ?q$, which shows that these formulas are not equivalent.

Our claim about the semantics of $\varphi(?p, ?q)$ in this model follows from the following more general claim.

Claim. For every $\wedge$-free context $\xi(p_1, p_2)$ which does not contain the atoms p, q, either of the following holds:[18]

1. $\xi(?p, ?q) \equiv_M \bot$;
2. $\xi(?p, ?q)$ is supported by all singletons in M, and by at least one information state in M of cardinality 2.

[18] Recall that $\varphi \equiv_M \psi$ denotes equivalence relative to the model M, which holds in case φ and ψ are supported by the same information states in M.

We show this by induction on the context ξ. To ease notation, given a context $\xi(p_1, p_2)$, let us write $\xi^\star$ for $\xi(?p, ?q)$.

- $\xi = p_1$. Then $\xi^\star = ?p$ is supported by all singleton states in M, and also by the state $\{w_{pq}, w_{p\overline{q}}\}$, which has cardinality 2. So case 2 holds.
- $\xi = p_2$. Analogous.
- $\xi = \bot$. Then $\xi^\star = \bot$, so case 1 holds.
- $\xi = r$ for $r \neq p_1, p_2, p, q$. Then $\xi^\star = r$. Since r is false in all worlds in s we have $r \equiv_M \bot$, so case 1 holds.
- $\xi = \eta \to \theta$. We distinguish three possibilities:
  - $\eta^\star \equiv_M \bot \equiv_M \theta^\star$. Then $\xi^\star \equiv_M \bot \to \bot \equiv \top$, therefore case 2 holds.
  - $\eta^\star \not\equiv_M \bot \equiv_M \theta^\star$. Then $\xi^\star = \eta^\star \to \theta^\star \equiv_M \eta^\star \to \bot = \neg\eta^\star$. We will show that $\neg\eta^\star \equiv_M \bot$. By the induction hypothesis on η, since $\eta^\star \not\equiv_M \bot$, we have that $\eta^\star$ is supported by every singleton state in M; therefore no singleton state supports $\neg\eta^\star$. By persistency, this implies that $\neg\eta^\star$ is not supported by any non-empty state, i.e., $\neg\eta^\star \equiv_M \bot$. So case 1 holds.
  - $\theta^\star \not\equiv_M \bot$. Then by induction hypothesis $\theta^\star$ is supported by all singletons and by some state of cardinality 2. Since any state that supports $\theta^\star$ also supports $\xi^\star = \eta^\star \to \theta^\star$, case 2 holds.
- $\xi = \eta \mathbin{\vee\!\!\!/} \theta$. If $\eta^\star \equiv_M \theta^\star \equiv_M \bot$ then $\xi^\star \equiv_M \bot$ and case 1 applies. Otherwise, at least one of $\eta^\star$ and $\theta^\star$ is not M-equivalent to $\bot$. Without loss of generality, suppose $\eta^\star \not\equiv_M \bot$. Then by induction hypothesis $\eta^\star$ is supported by all singletons and by some state of cardinality 2. Since any state that supports $\eta^\star$ also supports $\xi^* = \eta^\star \mathbin{\vee\!\!\!/} \theta^\star$, case 2 applies. □

This also completes the proof of Theorem 3.10.17.

Let us now consider the question of definability for $\mathsf{InqB}^\neg$, whose set of connectives is $\{\neg, \wedge, \to, \mathbin{\vee\!\!\!/}\}$. Again, we will show that the connectives are independent.

3.10.19. Theorem. $\{\neg, \wedge, \to, \mathbin{\vee\!\!\!/}\}$ *is an independent set of connectives.*

It is obvious that $\neg$ and $\mathbin{\vee\!\!\!/}$ are not definable in terms of the other connectives, since we saw in the previous section that these connectives are not eliminable. What remains to be shown is that $\wedge$ and $\to$, while eliminable, are nevertheless not definable in terms of the remaining connectives.

3.10.20. Proposition. $\wedge$ *is not definable from* $\{\neg, \to, \vee\!\!\!\vee\}$.

Proof. The proof is a simple adaptation of the one given for Proposition 3.10.18. The details are left to the reader. □

3.10.21. Proposition. $\to$ *is not definable from* $\{\neg, \wedge, \vee\!\!\!\vee\}$.

Proof. The proof is similar to that of Proposition 3.10.18. We want to show that $\to$ is not defined by any context $\varphi(p_1, p_2)$, where $\varphi \in \mathcal{L}[\neg, \wedge, \vee\!\!\!\vee]$. Take a candidate $\varphi(p_1, p_2)$ and fix two atoms p, q that do not occur in φ. We claim that:

$$?p \to ?q \not\equiv \varphi(?p, ?q),$$

which implies that $\varphi(p_1, p_2)$ does not define $\to$.

Again, we show the non-equivalence by showing that the formulas $?p \to ?q$ and $\varphi(?p, ?q)$ come apart in the four-word model M described in the proof of Proposition 3.10.18. The key is to prove the following claim.

Claim. For every $\to$-free context $\xi(p_1, p_2)$ not containing the atoms p, q, the formula $\xi(?p, ?q)$ is equivalent in the model M to a formula in the following set:

$$S \;=\; \{?p,\ ?q,\ \bot,\ \top,\ ?p \wedge ?q,\ ?p \vee\!\!\!\vee ?q\}.$$

In particular, then, $\varphi(?p, ?q)$ is equivalent in M to a formula in S. Since $?p \to ?q$ is not equivalent in M to any formula in S, it follows that $?p \to ?q \not\equiv \varphi(?p, ?q)$.

We show the claim by induction on ξ. As above, to ease notation we write $\xi^\star$ for $\xi(?p, ?q)$.

- $\xi = p_1$. Then $\xi^\star = ?p$ and the claim holds.

- $\xi = p_2$. Then $\xi^\star = ?q$ and the claim holds.

- $\xi = r$ for $r \neq p_1, p_2, p, q$. Then $\xi^\star = r$. Since r is false in all worlds in s we have $r \equiv_M \bot$, so the claim holds.

- $\xi = \neg\eta$. By induction hypothesis, $\eta^\star$ is M-equivalent to some formula in S. If $\eta^\star \equiv_M \bot$ then $\xi^\star = \neg\eta^\star \equiv_M \top$. In all the other cases, $\xi^\star = \neg\eta^\star \equiv_M \bot$.

- $\xi = \eta \wedge \theta$. Then $\xi^\star = \eta^\star \wedge \theta^\star$, and by induction hypothesis, each of $\eta^\star$ and $\theta^\star$ is M-equivalent to some formula in S. So, it suffices to check that the conjunction of two elements of S is equivalent to some element of S. This is straightforward: the 36 cases are summarized by the following table.

$\wedge$	$\top$	$\bot$	$?p$	$?q$	$?p \wedge ?q$	$?p \vee\!\!\vee ?q$
$\top$	$\top$	$\bot$	$?p$	$?q$	$?p \wedge ?q$	$?p \vee\!\!\vee ?q$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
$?p$	$?p$	$\bot$	$?p$	$?p \wedge ?q$	$?p \wedge ?q$	$?p$
$?q$	$?q$	$\bot$	$?p \wedge ?q$	$?q$	$?p \wedge ?q$	$?q$
$?p \wedge ?q$	$?p \wedge ?q$	$\bot$	$?p \wedge ?q$	$?p \wedge ?q$	$?p \wedge ?q$	$?p \wedge ?q$
$?p \vee\!\!\vee ?q$	$?p \vee\!\!\vee ?q$	$\bot$	$?p$	$?q$	$?p \wedge ?q$	$?p \vee\!\!\vee ?q$

- $\xi = \eta \vee\!\!\vee \theta$. This case is analogous to the previous one. We need to check that the inquisitive disjunction of two elements of S is equivalent to some element of S. The following table summarizes the 36 possibilities.

$\vee\!\!\vee$	$\top$	$\bot$	$?p$	$?q$	$?p \wedge ?q$	$?p \vee\!\!\vee ?q$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
$\bot$	$\top$	$\bot$	$?p$	$?q$	$?p \wedge ?q$	$?p \vee\!\!\vee ?q$
$?p$	$\top$	$?p$	$?p$	$?p \vee\!\!\vee ?q$	$?p$	$?p \vee\!\!\vee ?q$
$?q$	$\top$	$?q$	$?p \vee\!\!\vee ?q$	$?q$	$?q$	$?p \vee\!\!\vee ?q$
$?p \wedge ?q$	$\top$	$?p \wedge ?q$	$?p$	$?q$	$?p \wedge ?q$	$?p \vee\!\!\vee ?q$
$?p \vee\!\!\vee ?q$	$\top$	$?p \vee\!\!\vee ?q$	$?p \vee\!\!\vee ?q$	$?p \vee\!\!\vee ?q$	$?p \vee\!\!\vee ?q$	$?p \vee\!\!\vee ?q$

$\square$

3.10.4 Eliminability without definability

Definability and eliminability are about different things. Eliminability turns on the range of meanings that can be expressed by means of a connective: a connective $\circ$ is eliminable if anything that can be expressed by using $\circ$ can also be expressed without using $\circ$. Definability, by contrast, turns on the semantic operation expressed by the connective: $\circ$ is definable if the operation $f_\circ$ associated with $\circ$ can be simulated by a composition of the operations associated with the other connectives. Eliminability is about the outputs that can be produced, while definability is about the operations whereby these outputs are produced.

We saw that in substitution-closed logics, such as classical and intuitionistic logic, the two notions coincide. By contrast, in the setting of a non-substitution logic like InqB, the two notions come apart in an interesting way.

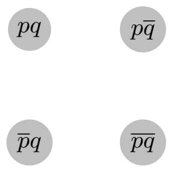
For instance, consider again conjunction. We saw that conjunction is eliminable but not definable in InqB and $\mathsf{InqB}^{\neg}$. This means that, while every formula $\varphi \wedge \psi$ may be rewritten equivalently as a $\wedge-$free formula, there is no schematic rewriting that we can use regardless of what φ and ψ are (as in classical logic, where for all φ, ψ we have $\varphi \wedge \psi \equiv \neg(\neg\varphi \vee \neg\psi)$); instead, the rewriting is necessarily dependent on the specific formulas φ and ψ.

From a semantic perspective, this tells us something about the semantic operation associated with conjunction, which is simply intersection (using the notation introduced in Section 3.9, we have $[\varphi \wedge \psi] = [\varphi] \cap [\psi]$). The fact that conjunction is not definable from the remaining connectives means that there is no way to define the operation of intersection by composing the operations expressed by the connectives, $\neg, \rightarrow$, and $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$. On the other hand, the fact that $\wedge$ is eliminable means that any particular result obtained by means of intersection may also be obtained *without* the use of intersection.

Note that one consequence of the discrepancy between eliminability and definability is that even an expressively complete system such as InqB may be enriched with new connectives expressing operations which are independent of the ones we considered. While adding such connectives does not increase the expressive power of the language, it does lead to a richer logic, which is not a mere notational variant of InqB. Thus, e.g., in Ciardelli (2016b), InqB is extended with a tensor disjunction $\otimes$, imported from dependence logic (Väänänen, 2007) which is shown in Ciardelli and Barbero (2019) not to be definable in InqB. And yet other natural connectives may in principle be added, such as the linear implication $\multimap$ considered by Abramsky and Väänänen (2009). □

3.11 Exercises

3.11.1. Exercise. [Semantics of InqB]
Consider our model with four worlds $w_{pq}, w_{p\overline{q}}, w_{\overline{p}q}, w_{\overline{pq}}$ instantiating the four assignments of truth values to the atoms p and q.



Determine the support conditions of the following InqB formulas in this model and draw their alternatives.

1. $p \wedge ?q$
2. $?p \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} ?q$
3. $?\neg p$
4. $?(p \vee q) \wedge ?(p \wedge q)$
5. $\neg q \rightarrow ?p$
6. $p \vee q \rightarrow ?p \wedge ?q$
7. $?p \rightarrow ?\neg p$
8. $(?p \rightarrow ?q) \rightarrow ?q$

3.11.2. Exercise. [Basic features of InqB]
Prove Proposition 3.3.5. That is, show that for all $\varphi \in \mathcal{L}^{\mathsf{P}}$, $|\varphi|_M = \bigcup[\varphi]_M$.

3.11.3. Exercise. [Truth-conditionality]
Call a formula φ *regular* if for any model M and any family S of states in M:

$$\text{if } M, s \models \varphi \text{ for all } s \in S, \text{ then } M, \bigcup S \models \varphi.$$

Show that for every $\varphi \in \mathcal{L}^{\mathsf{P}}$: φ is regular $\iff$ φ is truth-conditional.

3.11.4. Exercise. [Truth-conditionality]
Show that the entailment $\varphi \models \neg\neg\varphi$ is valid for any formula $\varphi \in \mathcal{L}^{\mathsf{P}}$.

3.11.5. Exercise. [Resolutions]
Compute the sets of resolutions of the following formulas.

1. $p \wedge (q \vee\!\!\!\vee r)$
2. $?(p \vee q) \wedge ?(p \wedge q)$
3. $p \to ?q$
4. $(p \vee\!\!\!\vee q) \to (r \vee\!\!\!\vee s)$

3.11.6. Exercise. [Truth-conditionality and alternatives]
Prove Proposition 3.6.12. That is, show that in InqB, the following are equivalent:

- φ is truth-conditional;
- for every model M, $\textsc{Alt}_M(\varphi) = \{|\varphi|_M\}$;
- for every model M, $\textsc{Alt}_M(\varphi)$ is a singleton.

3.11.7. Exercise. [Standard model and local tabularity]

1. The *standard model* for a set $\mathcal{P}$ of atoms is the model $\omega_{\mathcal{P}} = \langle W_\omega, V_\omega \rangle$, where W_ω is the set of all propositional valuations $w : \mathcal{P} \to \{0, 1\}$, and where $V_\omega(w, p) = w(p)$. Show that for every model M and state s, there is a state $s_\omega \subseteq W_\omega$ such that for every formula $\varphi \in \mathcal{L}^{\mathsf{P}}$: $M, s \models \varphi \iff \omega_{\mathcal{P}}, s_\omega \models \varphi$.
2. Show that for every formula $\varphi \in \mathcal{L}^{\mathsf{P}}$, φ is valid in InqBQ iff $\omega_{\mathcal{P}}, W_\omega \models \varphi$. Use this to conclude that InqBQ is decidable (this gives an alternative proof of Corollary 3.7.11).

3. Show that for every formulas φ, ψ we have: $\varphi \equiv \psi \iff [\varphi]_{\omega_\mathcal{P}} = [\psi]_{\omega_\mathcal{P}}$.

4. Recall that a logic L is called *locally tabular* if, given any finite set $\mathcal{P}$ of atoms, the number of formulas containing these atoms is finite up to logical equivalence. Use the previous item to show that InqB is locally tabular.

5. Use the previous results to show that over a single atom p there are only five formulas up to equivalence. *Hint:* how many worlds does the model $\omega_{\{p\}}$ have? How many distinct support sets are there in such a model?

3.11.8. Exercise. [Coherence]
For $n \geq 1$, we say that a formula is *n-coherent* (cf. Kontinen, 2010) if for every model M and state s we have:

$$s \models \varphi \iff (t \models \varphi \text{ for all } t \subseteq s \text{ with } \#t \leq n)$$

where $\#t$ denotes the cardinality of t. In words, φ is n-coherent if in order to check if φ is supported at a state, we just have to check if it is supported at sub-states of size at most n. Note that the notion of n-coherence is a generalization of truth-conditionality: to be truth-conditional is just to be 1-coherent.

For any formula φ of InqB, the *coherence degree* of φ, denoted $d(\varphi)$, is the least natural number n such that φ is n-coherent, if such a number exists (in fact, we are going to show that for InqB-formulas it always exists).

1. Show that if φ is n-coherent then it is m-coherent for all $m \geq n$. Conclude that, if $d(\varphi)$ is defined, then φ is n-coherent iff $n \geq d(\varphi)$.

2. Show inductively that $d(\varphi)$ is in fact defined for all $\varphi \in \mathcal{L}^\mathsf{P}$ and the following inequalities hold:

   - $d(\varphi) = 1$ if φ is an atom or $\bot$;
   - $d(\varphi \wedge \psi) \leq \max(d(\varphi), d(\psi))$;
   - $d(\varphi \to \psi) \leq d(\psi)$;
   - $d(\varphi \vee\!\!\!\vee \psi) \leq d(\varphi) + d(\psi)$.

3. Assuming $\mathcal{P}$ is infinite, show that for every $n \geq 1$ there is a formula $\varphi \in \mathcal{L}^\mathsf{P}$ such that $d(\varphi) = n$.

3.11.9. Exercise. [The inquisitive propositional logic InqC]
Consider a system InqC where questions are added to classical propositional logic by having ? rather than $\vee\!\!\!\vee$ as a primitive connective, with the semantics:

$$s \models ?\varphi \iff s \models \varphi \text{ or } s \perp\!\!\!\perp \varphi$$

So, the logical repertoire of InqC contains the connectives $\bot, \wedge, \to, ?$.

1. Show that every formula of InqC is equivalent to a formula of InqB.

2. Show that every formula of InqC is 2-coherent. Using this, show that InqC is strictly less expressive than InqB.

For a state property $\mathcal{C}$, say it is n-coherent if for every model M and state s:

$$\langle M, s\rangle \in \mathcal{C} \iff (\langle M, t\rangle \in \mathcal{C} \text{ for all } t \subseteq s \text{ with } 1 \leq \#t \leq n).$$

3. Show that any n-coherent state property is an inquisitive property in the sense of Definition 3.9.2.

4. Show that InqC is expressively complete for finitely determined 2-coherent properties. That is, show that the following are equivalent for any $\mathcal{C}$:

   - $\mathcal{C}$ is finitely determined and 2-coherent;
   - $\mathcal{C} = [\varphi]$ for some $\varphi \in \mathsf{InqC}$.

Note that, from the point of view of InqB, classical logic can be seen as a fragment which, among the finitely determined inquisitive properties, expresses all and only those which are 1-coherent. InqC can be seen as a fragment which expresses all and only the properties which are 2-coherent. This leads naturally to the following question.

Open problem: can we define, for every natural number $n \geq 1$, a syntactic fragment $\mathcal{L}^{\mathsf{P}}_n$ of $\mathcal{L}^{\mathsf{P}}$ (syntactic in the sense of defined by syntactic constrains) which expresses all and only the finitely determined n-coherent properties?

3.11.10. Exercise. [Partitional formulas]
We call a formula φ of InqB *partitional* if in every model M, the elements of $\textsc{Alt}_M(\varphi)$ are a partition of the logical space.

1. Show that φ is partitional iff for every model M there is an equivalence relation $\sim_\varphi$ on W such that for every state $s \subseteq W$:

$$s \models \varphi \iff \forall w, w' \in s : w \sim_\varphi w'.$$

2. Show that for every $\varphi \in \mathcal{L}^{\mathsf{P}}$: φ is partitional $\iff$ $\varphi \equiv {?\alpha_1} \wedge \cdots \wedge {?\alpha_n}$ for some set $\{\alpha_1, \ldots, \alpha_n\}$ of classical formulas.

3.11.11. Exercise. [Eliminability]
Prove the uneliminability claims in Section 3.10.2. That is, prove propositions 3.10.8, 3.10.9, 3.10.10, 3.10.13, and 3.10.14.

3.11.12. Exercise. [Eliminability and undefinability]
Consider the system $\mathsf{InqB}_{\otimes}$ which extends $\mathcal{L}^{\mathsf{P}}$ with the connective $\otimes$, interpreted by the semantic clause:

$$s \models \varphi \otimes \psi \iff \exists s', s'' \text{ such that } s = s' \cup s'', s' \models \varphi, \text{ and } s'' \models \psi.$$

This connective is called tensor disjunction, or split-junction, and it is standard in dependence logic (see Väänänen, 2007; Yang and Väänänen, 2016).

1. Show that persistency and the empty state property still hold for $\mathsf{InqB}_{\otimes}$.
2. Show that, in $\mathsf{InqB}_{\otimes}$, tensor disjunction is eliminable: that is, every formula φ of $\mathsf{InqB}_{\otimes}$ is equivalent to a formula φ^* of InqB.
3. Show that $\otimes$ is not definable from the InqB connectives $\{\bot, \wedge, \to, \vee\!\!\!\vee\}$.

   Hint. Use a strategy similar to the one we used for Proposition 3.10.18. Given a candidate context $\varphi(p_1, p_2)$ and atoms p, q not occurring in φ, show that $(?p \wedge ?q) \otimes (?p \wedge ?q) \not\equiv \varphi(?p \wedge ?q, ?p \wedge ?q)$.

   To prove this, show that in the model we used in the proof of Prop. 3.10.18, $\varphi(?p \wedge ?q, ?p \wedge ?q)$ is guaranteed to be equivalent either to $?p \wedge ?q$, or to $\bot$, or to $\top$, while $(?p \wedge ?q) \otimes (?p \wedge ?q)$ is not equivalent to any of these.

Chapter 4

Inferences with propositional questions

In the previous chapter we have seen how classical propositional logic can be extended with questions, leading to the inquisitive propositional logic InqB. In this section we will describe a natural deduction system for InqB and show this system to be sound and complete. We will also use this system to make some more general points about the role of questions in inference and about the intuitive significance of supposing or concluding a question. Lastly, we will show that proofs in our system have an interesting kind of constructive content: a proof can generally be seen as encoding a method for turning resolutions of the assumptions into a corresponding resolution of the conclusion.

4.1 A natural deduction system for InqB

A natural deduction system for InqB is presented in Figure 4.1. In these rules, the variables φ, ψ, and χ range over all formulas, while α is restricted to classical formulas. We refer to the introduction rule for a connective $\circ$ as ($\circ$i), and to the elimination rule as ($\circ$e). As usual, those rules that discharge assumptions allow us to discharge an arbitrary number of occurrences of the assumption in the relevant sub-proof. We write $P : \Phi \vdash \psi$ to mean that P is a proof whose set of undischarged assumptions is included in Φ and whose conclusion is ψ, and we write $\Phi \vdash \psi$ to mean that a proof $P : \Phi \vdash \psi$ exists. Two formulas φ and ψ are *provably equivalent*, notation $\varphi \dashv\vdash \psi$, in case $\varphi \vdash \psi$ and $\psi \vdash \varphi$. Let us comment briefly on the rules of this system.

Conjunction. Conjunction is handled by the usual introduction and elimination rules: from a conjunction we can infer either conjunct, and from both conjuncts we can infer the conjunction. The soundness of these rules corresponds to the following standard fact.

4.1.1. Proposition. $\Phi \models \varphi \land \psi \iff \Phi \models \varphi$ *and* $\Phi \models \psi$.

Conjunction

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \qquad \frac{\varphi \wedge \psi}{\varphi} \quad \frac{\varphi \wedge \psi}{\psi}$$

Implication

$$\frac{\begin{array}{c}[\varphi] \\ \vdots \\ \psi\end{array}}{\varphi \to \psi} \qquad \frac{\varphi \quad \varphi \to \psi}{\psi}$$

Inquisitive disjunction

$$\frac{\varphi}{\varphi \vee\!\!\!\vee \psi} \quad \frac{\psi}{\varphi \vee\!\!\!\vee \psi} \quad \frac{\varphi \vee\!\!\!\vee \psi \quad \begin{array}{c}[\varphi] \\ \vdots \\ \chi\end{array} \quad \begin{array}{c}[\psi] \\ \vdots \\ \chi\end{array}}{\chi}$$

Falsum

$$\frac{\bot}{\varphi}$$

$\vee\!\!\!\vee$-split

$$\frac{\alpha \to \psi \vee\!\!\!\vee \chi}{(\alpha \to \psi) \vee\!\!\!\vee (\alpha \to \chi)}$$

$\neg\neg$ elimination

$$\frac{\neg\neg\alpha}{\alpha}$$

Figure 4.1: A sound and complete natural-deduction system for InqB. In these rules, the variables φ, ψ, and χ range over arbitrary formulas, while the variable α is restricted to range over classical formulas.

Notice that these rules are not restricted to classical formulas: conjunctive questions such as $?p \wedge ?q$ can be handled in inferences just like standard conjunctions.

Implication. Implication is also handled by the standard inference rules: from an implication together with its antecedent we can infer the consequent; conversely, if from the assumption of φ we can infer ψ, we can discharge the assumption and conclude $\varphi \to \psi$. The soundness of these rules corresponds to the following fact, which captures the tight relation existing between implication and entailment.

4.1.2. PROPOSITION. $\Phi \models \varphi \to \psi \iff \Phi, \varphi \models \psi$.

Again, these rules are not restricted to classical formulas: implications involving questions can also be handled by means of the standard implication rules. This means that, e.g., in order to prove a dependence formula, say $?p \to ?q$, we can proceed by assuming the determinant $?p$ and showing that from it we can

Negation

$$\frac{\begin{matrix}[\varphi]\\ \vdots\\ \bot\end{matrix}}{\neg\varphi} \qquad \frac{\varphi \quad \neg\varphi}{\bot}$$

Classical disjunction

$$\frac{\varphi}{\varphi \vee \psi} \qquad \frac{\psi}{\varphi \vee \psi} \qquad \frac{\varphi \vee \psi \quad \begin{matrix}[\varphi]\\ \vdots\\ \alpha\end{matrix} \quad \begin{matrix}[\psi]\\ \vdots\\ \alpha\end{matrix}}{\alpha}$$

Figure 4.2: Derived rules for $\vee$ and $\neg$, where α is restricted to classical formulas.

derive the determined question, $?q$. Conversely, if we have $?p \to ?q$ as well as the determinant, $?p$, we can on that basis infer the determined question, $?q$.

Falsum. As usual, $\bot$ has no introduction rule, and can be eliminated to infer any formula. This corresponds to the fact that we have $\bot \models \varphi$ for all formulas φ, which in turn is a consequence of the fact that the inconsistent state $\emptyset$ always supports every formula.

Negation. As $\neg\varphi$ is defined as $\varphi \to \bot$, the usual intuitionistic rules for negation, given in Figure 4.2, follow as particular cases of the rules for implication.

Inquisitive disjunction. Inquisitive disjunction is handled by the standard natural deduction rules for disjunction: we can infer a disjunction from either disjunct and, conversely, whatever can be inferred from either disjunct can be inferred from the disjunction. The soundness of these rules corresponds to the following fact.

4.1.3. PROPOSITION. *$\Phi, \varphi \mathbin{\vee\mkern-15mu\vee} \psi \models \chi \iff \Phi, \varphi \models \chi$ and $\Phi, \psi \models \chi$.*

Classical disjunction. Figure 4.2 shows the derived rules for $\vee$. While both rules are standard, the elimination rule is restricted to conclusions that are classical formulas. Without this restriction, the rule would not be sound. E.g., we have $p \models ?p$ and $\neg p \models ?p$, but $p \vee \neg p \not\models ?p$: indeed, the question $?p$ is logically resolved by both statements p and $\neg p$, but not by the disjunction $p \vee \neg p$.

Double negation elimination. We saw in the previous chapter (Proposition 3.4.10) that the double negation law is characteristic of statements. Thus, this rule reflects the fact that classical formulas are statements in InqB.

In fact, to obtain a complete proof system it is not strctly necessary to allow double negation elimination for all classical formulas: it would be sufficient to allow double negation for *atoms*, so as to let the system know that we are taking atomic sentences to be statements (cf. the discussion in Section 3.8). It would then be possible to infer α from $\neg\neg\alpha$ for any classical α on the basis of the rules for $\bot$, $\wedge$, and $\rightarrow$. While this is indeed the choice made in some work on inquisitive logic (e.g., Ciardelli and Roelofsen, 2011), allowing double negation as a primitive rule for all classical formulas is quite natural, given our perspective of viewing InqB as a conservative extension of classical propositional logic; in this way, our system is an extension of a standard natural deduction system for classical logic (as given, e.g., in Gamut, 1991). This ensures that any standard natural deduction proof in classical logic is also a proof in our system.

Split. The $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$-split rule allows us to distribute a classical antecedent over an inquisitive-disjunctive consequent. This rule is backed by the $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$-split equivalence

$$\alpha \rightarrow \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi \;\equiv\; (\alpha \rightarrow \varphi) \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} (\alpha \rightarrow \psi)$$

given by Proposition 3.7.14. As we discussed in detail on page 84, this equivalence is a logical rendering of the fact that statements are *specific*, i.e., they correspond to specific pieces of information (Proposition 3.7.12). As we discussed in Section 2.6, this property can be seen as marking the crucial difference between statements and questions. Thus the $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$-split rule has a clear conceptual significance which may perhaps not be obvious at first.

It is interesting to remark that the double negation rule and the $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$-split rule capture different properties of statements in InqB: the former captures the fact that the logic of statements in InqB is classical, while the latter captures the idea that statements correspond to specific pieces of information. The latter idea seems constitutive of the inquisitive perspective, while the classicality of the underlying logic is not. Indeed, it is possible to build inquisitive logics based on non-classical logics of statements (see especially Punčochář, 2016b, 2019, 2020; Ciardelli et al., 2020). In such logics, the double negation law may fail, but the $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$-split equivalence remains valid.[1,2]

Having discussed the significance of the inference rules, let us now illustrate how these rules can be used to build proofs of inquisitive entailments.

[1]The exception is Holliday (2020), which gives an inquisitive extension of intuitionistic logic which does not validate the $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$-split principle. This violation is unexpected from the standpoint of the conceptual picture developed in Chapter 1. For it means that there is an information state s such that $p \models_s {?q}$, yet $p \not\models_s q$ and $p \not\models_s \neg q$. This means that on the one hand, on the basis of the information in s, p resolves the question $?q$, while on the other hand, on the basis of the same information, p fails to yield either answer to this question (cf. also the discussion on page 42). Even in the context of this approach, however, it is easy to render the split principle valid by imposing extra conditions on the semantics.

[2]Instead of the $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ split rule, some axiomatizations of InqB use the Kreisel-Putnam axiom

4.1.4. Example. As a first example, consider the following InqB-entailment:

$$p \leftrightarrow \neg q \models ?p \to ?q.$$

This is valid: under the assumption that p and q have opposite truth values, it follows that whether q is the case is determined by whether p is the case. Here is a simple proof of this entailment in our natural deduction system (recall that $?p := p \mathbin{\vee\!\!\!\vee} \neg p$ and $?q := q \mathbin{\vee\!\!\!\vee} \neg q$):

$$\dfrac{\dfrac{[?p]_2 \qquad \dfrac{\dfrac{[p]_1 \quad p \leftrightarrow \neg q}{\neg q}}{?q}\,(\mathbin{\vee\!\!\!\vee}\mathsf{i}) \qquad \dfrac{\dfrac{[\neg p]_1 \quad p \leftrightarrow \neg q}{q}}{?q}\,(\mathbin{\vee\!\!\!\vee}\mathsf{i})}{?q}\,(\mathbin{\vee\!\!\!\vee}\mathsf{e}, 1)}{?p \to ?q}\,(\to\mathsf{i}, 2)$$

In this proof, the steps which are not labeled involve only inferences with classical formulas. Since these coincide with inferences in classical propositional logic, we omit the standard details.

4.1.5. Example. As a second example, consider again the conditional dependence discussed in Section 2.3.3: in the die roll scenario, the range of the outcome determines what the outcome is, given that the outcome is prime. This fact can be captured as a logical entailment in InqB. Recall from the previous chapter (cf. Example 3.2.6 on page 63 for the details) that we can formalize the scenario in a propositional language equipped with a set of atoms $\mathcal{P} = \{\mathsf{one}, \dots, \mathsf{six}\}$. We can then define the following statements as classical disjunctions of our atoms:

- $\mathsf{low} := \mathsf{one} \vee \mathsf{two}$;
- $\mathsf{mid} := \mathsf{three} \vee \mathsf{four}$;
- $\mathsf{high} := \mathsf{five} \vee \mathsf{six}$;
- $\mathsf{prime} := \mathsf{two} \vee \mathsf{three} \vee \mathsf{five}$.

(see Ciardelli and Roelofsen, 2011):

$$(\neg\varphi \to \psi \mathbin{\vee\!\!\!\vee} \chi) \to (\neg\varphi \to \psi) \mathbin{\vee\!\!\!\vee} (\neg\varphi \to \chi)$$

In our setting, this plays the same role as $\mathbin{\vee\!\!\!\vee}$-split because, on the basis of the other rules in the system, every classical formula is provably equivalent to a negation, and every negation is equivalent to a classical formula. However, the $\mathbin{\vee\!\!\!\vee}$-split rule has the advantage of generalizing to inquisitive logics based on non-classical logics of statements. If the logic of statement is, say, intuitionistic logic, then double negation will fail even for classical formulas; thus, it will no longer be the case that every statement is equivalent to a negation. In this setting, the Kreisel-Putnam axiom will fail to capture in full generality the assumption that statements are specific; it will capture only a special case of this assumption, for those statements that are equivalent to a negation. Thus, when building inquisitive logics in a non-classical setting, it is crucial to use $\mathbin{\vee\!\!\!\vee}$-split, and not the Kreisel-Putnam axiom (cf. Punčochář, 2016b, 2019, 2020; Ciardelli et al., 2020).

By using inquisitive disjunction we can also define the following questions:

- $\mathsf{range} := \mathsf{low} \mathbin{\vee\!\!\!\vee} \mathsf{mid} \mathbin{\vee\!\!\!\vee} \mathsf{high}$;
- $\mathsf{outcome} := \mathsf{one} \mathbin{\vee\!\!\!\vee} \mathsf{two} \mathbin{\vee\!\!\!\vee} \mathsf{three} \mathbin{\vee\!\!\!\vee} \mathsf{four} \mathbin{\vee\!\!\!\vee} \mathsf{five} \mathbin{\vee\!\!\!\vee} \mathsf{six}$.

The assumptions that the outcomes are jointly exhaustive and mutually exclusive possibilities are captured by the following formulas:

- $\mathsf{exh} \;:=\; (\mathsf{one} \vee \cdots \vee \mathsf{six})$;
- $\mathsf{exc} \;:=\; \neg(\mathsf{one} \wedge \mathsf{two}) \wedge \neg(\mathsf{one} \wedge \mathsf{three}) \wedge \cdots \wedge \neg(\mathsf{five} \wedge \mathsf{six})$.

Let $\Gamma = \{\mathsf{exh}, \mathsf{exc}\}$. Then the following entailment, which amounts to the conditional dependence of $\mathsf{outcome}$ on range given prime, is valid:

$$\Gamma, \mathsf{prime}, \mathsf{range} \models \mathsf{outcome}.$$

Below is a proof of this entailment, where again we omit steps that involve only inferences in classical propositional logic.

$$\dfrac{\mathsf{range} \qquad \dfrac{\dfrac{\Gamma \quad [\mathsf{low}]_1 \quad \mathsf{prime}}{\mathsf{two}}}{\mathsf{outcome}}\,(\mathbin{\vee\!\!\!\vee}\mathsf{i}) \qquad \dfrac{\dfrac{\Gamma \quad [\mathsf{mid}]_1 \quad \mathsf{prime}}{\mathsf{three}}}{\mathsf{outcome}}\,(\mathbin{\vee\!\!\!\vee}\mathsf{i}) \qquad \dfrac{\dfrac{\Gamma \quad [\mathsf{high}]_1 \quad \mathsf{prime}}{\mathsf{five}}}{\mathsf{outcome}}\,(\mathbin{\vee\!\!\!\vee}\mathsf{i})}{\mathsf{outcome}}\,(\mathbin{\vee\!\!\!\vee}\mathsf{e}, 1)$$

4.2 Constructive content of proofs in InqB

Looking again at the two examples of proofs in the previous section, we can see that they are, in a certain sense, constructive. For instance, the second proof we saw does not just witness the fact that, under the given declarative assumptions, information of type range can be used to obtain information of type $\mathsf{outcome}$: it actually describes *how* to use information of type range to obtain information of type $\mathsf{outcome}$. In other words, the proof encodes a logical dependence function f (cf. Definition 3.6.15) that can be used to turn any given information of type range to corresponding information of type $\mathsf{outcome}$. This is not just a feature of this particular proof, but a general fact: any inquisitive proof encodes a logical dependence function. To see how this works, let us write $\overline{\varphi}$ for a sequence $\varphi_1, \ldots, \varphi_n$ of formulas, and $\overline{\alpha} \in \mathcal{R}(\overline{\varphi})$ to mean that $\overline{\alpha}$ is a sequence $\alpha_1, \ldots, \alpha_n$ such that $\alpha_i \in \mathcal{R}(\varphi_i)$. We have the following result (Ciardelli, 2018b).

4.2.1. THEOREM (EXISTENCE OF A RESOLUTION ALGORITHM).
If $P : \overline{\varphi} \vdash \psi$, we can define inductively on P a procedure F_P which maps each $\overline{\alpha} \in \mathcal{R}(\overline{\varphi})$ to a proof $F_P(\overline{\alpha}) : \overline{\alpha} \vdash \beta$ having as conclusion a resolution $\beta \in \mathcal{R}(\psi)$.

Proof. Let us describe how to construct the procedure F_P inductively on P. We distinguish a number of cases depending on the last rule applied in P.

- ψ is an undischarged assumption φ_i. In this case, any resolution $\overline{\alpha} \in \mathcal{R}(\overline{\varphi})$ contains a resolution α_i of φ_i by definition. So, we can just let F_P map $\overline{\alpha}$ to the trivial proof $Q : \overline{\alpha} \vdash \alpha_i$ which consists only of the assumption α_i.

- $\psi = \chi \wedge \xi$ was obtained by ($\wedge$i) from χ and ξ. Then the immediate subproofs of P are a proof $P' : \overline{\varphi} \vdash \chi$ and a proof $P'' : \overline{\varphi} \vdash \xi$, for which the induction hypothesis gives two procedures $F_{P'}, F_{P''}$. Now take any resolution $\overline{\alpha}$ of $\overline{\varphi}$. We have $F_{P'}(\overline{\alpha}) : \overline{\alpha} \vdash \beta$ and $F_{P'}(\overline{\alpha}) : \overline{\alpha} \vdash \gamma$, where $\beta \in \mathcal{R}(\chi)$ and $\gamma \in \mathcal{R}(\xi)$. By extending these proofs with an application of ($\wedge$i), we get a proof $Q : \overline{\alpha} \vdash \beta \wedge \gamma$. Since $(\beta \wedge \gamma) \in \mathcal{R}(\chi \wedge \xi)$, we can let $F_P(\overline{\alpha}) := Q$.

- $\psi = \chi \to \xi$ was obtained by ($\to$i). Then the immediate subproof of P is a proof $P' : \overline{\varphi}, \chi \vdash \xi$, for which the induction hypothesis gives a procedure $F_{P'}$. Now take any resolution $\overline{\alpha}$ of $\overline{\varphi}$. Suppose $\beta_1, \dots, \beta_m$ are the resolutions of χ. For $1 \leq i \leq m$, the sequence $\overline{\alpha}, \beta_i$ is a resolution of $\overline{\varphi}, \chi$, and so we have $F_{P'}(\overline{\alpha}, \beta_i) : \overline{\alpha}, \beta_i \vdash \gamma_i$ for some resolution γ_i of ξ. Extending this proof with an application of ($\to$i), we have a proof $Q_i : \overline{\alpha} \vdash \beta_i \to \gamma_i$. Since this is the case for $1 \leq i \leq n$, by several applications of the rule ($\wedge$i) we obtain a proof $Q : \overline{\alpha} \vdash (\beta_1 \to \gamma_1) \wedge \cdots \wedge (\beta_m \to \gamma_m)$. By construction, $(\beta_1 \to \gamma_1) \wedge \cdots \wedge (\beta_m \to \gamma_m)$ is a resolution of $\chi \to \xi$, and so we can let $F_P(\overline{\alpha}) := Q$.

- $\psi = \chi \vee\!\!\!\vee\, \xi$ was obtained by ($\vee\!\!\!\vee$i) from one of the disjuncts. Without loss of generality, let us assume it is χ. Thus, the immediate subproof of P is a proof $P' : \overline{\varphi} \vdash \chi$, for which the induction hypothesis gives a procedure $F_{P'}$. Now take any resolution $\overline{\alpha}$ of $\overline{\varphi}$. The induction hypothesis gives us a proof $F_{P'}(\overline{\alpha}) : \overline{\alpha} \vdash \beta$ for some $\beta \in \mathcal{R}(\chi)$. Since β is also a resolution of $\chi \vee\!\!\!\vee\, \xi$, we can simply let $F_P(\overline{\alpha}) := F_{P'}(\overline{\alpha})$.

- ψ was obtained by ($\wedge$e) from $\psi \wedge \chi$. Then the immediate subproof of P is a proof $P' : \overline{\varphi} \vdash \psi \wedge \chi$, and the induction hypothesis gives a procedure $F_{P'}$. For any resolution $\overline{\alpha}$ of $\overline{\varphi}$, we have $F_{P'}(\overline{\alpha}) : \overline{\alpha} \vdash \beta$, where $\beta \in \mathcal{R}(\psi \wedge \chi)$. By definition of resolutions for a conjunction, β is of the form $\gamma \wedge \gamma'$ where $\gamma \in \mathcal{R}(\psi)$ and $\gamma' \in \mathcal{R}(\chi)$. Extending $F_{P'}(\overline{\alpha})$ with an application of ($\wedge$e) we have a proof $Q : \overline{\alpha} \vdash \gamma$. Since $\gamma \in \mathcal{R}(\psi)$, we can just let $F_P(\overline{\alpha}) := Q$.

- ψ was obtained by ($\to$e) from χ and $\chi \to \psi$. Then the immediate subproofs of P are a proof $P' : \overline{\varphi} \vdash \chi$, and a proof $P'' : \overline{\varphi} \vdash \chi \to \psi$, for which the induction hypothesis gives procedures $F_{P'}$ and $F_{P''}$. Now consider a resolution $\overline{\alpha}$ of $\overline{\varphi}$. We have $F_{P'}(\overline{\alpha}) : \overline{\alpha} \vdash \beta$ where $\beta \in \mathcal{R}(\chi)$, and a proof

$F_{P''}(\overline{\alpha}) : \overline{\alpha} \vdash \gamma$, where $\gamma \in \mathcal{R}(\chi \to \psi)$. Now, if $\mathcal{R}(\chi) = \{\beta_1, \ldots, \beta_m\}$, then $\beta = \beta_i$ for some i, and by definition of the resolutions of an implication, $\gamma = (\beta_1 \to \gamma_1) \wedge \cdots \wedge (\beta_n \to \gamma_m)$ where $\{\gamma_1, \ldots, \gamma_n\} \subseteq \mathcal{R}(\psi)$. Now, extending $F_{P''}(\overline{\alpha})$ with an application of ($\wedge$e) we obtain a proof $Q'' : \overline{\alpha} \vdash \beta_i \to \gamma_i$. Finally, combining this proof with $F_{P'}(\overline{\alpha})$ and applying ($\to$e), we obtain a proof $Q : \overline{\alpha} \vdash \gamma_i$. Since the conclusion of this proof is a resolution of ψ, we can let $F_P(\overline{\alpha}) := Q$.

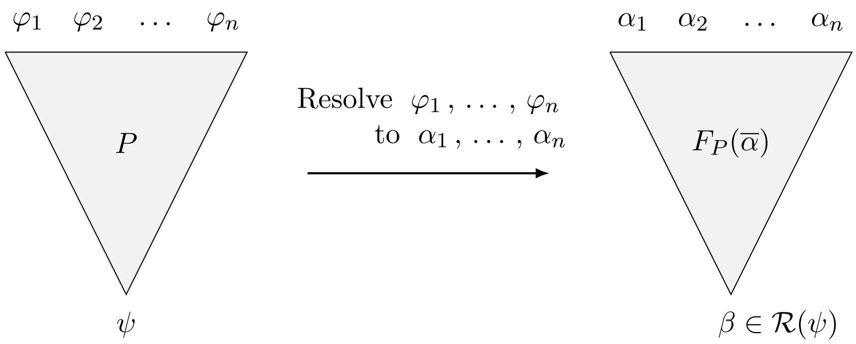
- ψ was obtained by ($\mathbin{\vee\!\!\!\vee}$e) from $\chi \mathbin{\vee\!\!\!\vee} \xi$. Then the immediate subproofs of P are: a proof $P' : \overline{\varphi} \vdash \chi \mathbin{\vee\!\!\!\vee} \xi$; a proof $P'' : \overline{\varphi}, \chi \vdash \psi$; and a proof $P''' : \overline{\varphi}, \xi \vdash \psi$, for which the induction hypothesis gives procedures $F_{P'}$, $F_{P''}$, and $F_{P'''}$. Now take a resolution $\overline{\alpha}$ of $\overline{\varphi}$. We have $F_{P'}(\overline{\alpha}) : \overline{\alpha} \vdash \beta$ for some $\beta \in \mathcal{R}(\chi \mathbin{\vee\!\!\!\vee} \xi) = \mathcal{R}(\chi) \cup \mathcal{R}(\xi)$. Without loss of generality, assume that $\beta \in \mathcal{R}(\chi)$. Then the sequence $\overline{\alpha}, \beta$ is a resolution of $\overline{\varphi}, \chi$. Thus, we have $F_{P''}(\overline{\alpha}, \beta) : \overline{\alpha}, \beta \vdash \gamma$ for some $\gamma \in \mathcal{R}(\psi)$. Now, by substituting any undischarged assumption of β in the proof $F_{P''}(\overline{\alpha}, \beta)$ by an occurrence of the proof $F_{P'}(\overline{\alpha})$, we obtain a proof $Q : \overline{\alpha} \vdash \gamma$ having a resolution of ψ as its conclusion, and we can let $F_P(\overline{\alpha}) := Q$.

- ψ was obtained by ($\bot$e). This means that the immediate subproof of P is a proof $P' : \overline{\varphi} \vdash \bot$, for which the induction hypothesis gives a method $F_{P'}$. Now take any resolution $\overline{\alpha}$ of $\overline{\varphi}$. Since $\mathcal{R}(\bot) = \{\bot\}$, we have $F_{P'}(\overline{\alpha}) : \overline{\alpha} \vdash \bot$. Now take any $\beta \in \mathcal{R}(\psi)$ (notice that, by definition, the set of resolutions of a formula is always non-empty): by extending the proof $F_{P'}(\overline{\alpha})$ with an application of ($\bot$e), we obtain a proof $Q : \overline{\alpha} \vdash \beta$. Since $\beta \in \mathcal{R}(\psi)$, we can let $F_P(\overline{\alpha}) := P$.

- $\psi = (\alpha \to \chi) \mathbin{\vee\!\!\!\vee} (\alpha \to \xi)$ was obtained by an application of the $\mathbin{\vee\!\!\!\vee}$-split rule from $\alpha \to \chi \mathbin{\vee\!\!\!\vee} \xi$, where $\alpha \in \mathcal{L}_c^{\mathsf{P}}$. Then, the immediate subproof of P is a proof $P' : \overline{\varphi} \vdash \alpha \to \chi \mathbin{\vee\!\!\!\vee} \xi$, for which the induction hypothesis gives a method $F_{P'}$. Using the fact that $\mathcal{R}(\alpha) = \{\alpha\}$ (since α is a classical formula) it is easy to verify that $\mathcal{R}(\alpha \to \chi \mathbin{\vee\!\!\!\vee} \xi) = \mathcal{R}((\alpha \to \chi) \mathbin{\vee\!\!\!\vee} (\alpha \to \xi))$. Therefore, we can simply let $F_P := F_{P'}$.

- $\alpha \in \mathcal{L}_c^{\mathsf{P}}$ was obtained by double negation elimination from $\neg\neg\alpha$. In this case, the immediate subproof of P is a proof $P' : \overline{\varphi} \vdash \neg\neg\alpha$, for which the induction hypothesis gives a method $F_{P'}$. Since $\neg\neg\alpha$ is a classical formula, we have $\mathcal{R}(\neg\neg\alpha) = \{\neg\neg\alpha\}$. Thus, for any resolution $\overline{\beta}$ of $\overline{\varphi}$ we have $F_{P'}(\overline{\beta}) : \overline{\beta} \vdash \neg\neg\alpha$. Extending this proof with an application of double negation elimination we obtain a proof $Q : \overline{\beta} \vdash \alpha$. Since α is a classical formula and thus $\mathcal{R}(\alpha) = \{\alpha\}$, we can then let $F_P(\overline{\beta}) := Q$. $\square$

This result shows that a proof P in our system may be seen as a *template* F_P for producing classical proofs, where questions serve as placeholders for generic information of the corresponding type. As soon as the assumptions of the proof are instantiated to particular resolutions, this template can be instantiated to a classical proof which infers some specific resolution of the conclusion. The idea is rendered by the following scheme:



We refer to the proof $F_P(\overline{\alpha})$ as the *resolution* of the proof P on the input $\overline{\alpha}$. Now, let us denote by f_P the function that maps a resolution $\overline{\alpha} \in \mathcal{R}(\overline{\varphi})$ to the conclusion of the proof $F_P(\overline{\alpha})$. By definition, we have $f : \mathcal{R}(\overline{\varphi}) \to \mathcal{R}(\psi)$. Moreover, for any $\overline{\alpha} \in \mathcal{R}(\overline{\varphi})$, we have $F_P(\overline{\alpha}) : \overline{\alpha} \vdash f_P(\overline{\alpha})$, and thus, by the soundness of our proof system, we have $\overline{\alpha} \models f_P(\overline{\alpha})$. This shows that the function f_P determined by the proof P is a logical dependence function from $\overline{\varphi}$ to ψ, in the sense of Definition 3.6.22.

4.2.2. Corollary (Inquisitive proofs encode dependence functions). *If $P : \Phi \vdash \psi$, then inductively on P we can define a logical dependence function $f_P : \Phi \rightsquigarrow \psi$.*

This connection is reminiscent of the proofs-as-programs correspondence known for intuitionistic logic. As discovered by Curry (1934) and Howard (1980), in intuitionistic logic formulas may be regarded as *types* of a certain type theory, extending the simply typed lambda calculus. A proof $P : \varphi \vdash \psi$ in intuitionistic logic may be identified with a term t_P of this type theory which describes a function that maps objects of type φ to objects of type ψ. The situation is similar for InqB, except that now, formulas play double duty. On the one hand, formulas may be still be regarded as types. On the other hand, the elements of a type φ may in turn be identified with certain formulas, namely, the resolutions of φ. As in intuitionistic logic, a proof $P : \varphi \vdash \psi$ determines a function f_P from objects of type φ to objects of type ψ; but since these objects may now be identified with classical formulas, the function f_P is now defined within the language of classical propositional logic, i.e., we have $f_P : \mathcal{L}_c^{\mathsf{P}} \to \mathcal{L}_c^{\mathsf{P}}$.

4.3 Completeness

By showing that each inference rule of our proof system is sound, the discussion in Section 4.1 implies the soundness of our proof system as a whole.

4.3.1. Proposition (Soundness). *If $\Phi \vdash \psi$, then $\Phi \models \psi$.*

In this section, we will be concerned with establishing the converse implication, i.e., with proving the following theorem.

4.3.2. Theorem (Completeness). *If $\Phi \models \psi$, then $\Phi \vdash \psi$.*

There are multiple strategies to obtain this result. One proof (Ciardelli and Roelofsen, 2009; Ciardelli, 2009) follows the format of completeness proofs for intuitionistic and intermediate logics. Another strategy (Ciardelli, 2016b) relies crucially on the normal form result and the fact that our proof system includes a complete system for classical propositional logic. In this section we present yet another proof. The advantages of this proof are that it is completely self-contained and that it can be extended straightforwardly to the setting of inquisitive modal logic (see Ciardelli, 2014, 2018a).

The strategy of the proof can be summarized as follows. First, we will define a canonical model having complete theories of classical formulas as its possible worlds. Second, we will prove an analogue of the truth-lemma, the *support lemma*, which connects support in the canonical model with provability in our system. Finally, we will show that when a formula ψ cannot be derived from a set Φ, we can define a corresponding information state in the canonical model that supports Φ but not ψ.

4.3.1 Preliminary results

Let us start out by establishing a few important facts about our proof system. First, notice that, if we leave out the rules of $\neg\neg$-elimination and $\mathbin{\vee\!\!\!\vee}$-split, what we have is a complete system for intuitionistic propositional logic, with $\mathbin{\vee\!\!\!\vee}$ in the role of intuitionistic disjunction. Thus, we have the following fact.

4.3.3. Lemma (Intuitionistic entailments are provable). *If Φ entails ψ in intuitionistic propositional logic when $\mathbin{\vee\!\!\!\vee}$ is identified with intuitionistic disjunction, then $\Phi \vdash \psi$.*

Second, our proof system allows us to prove the equivalence between a formula and its normal form.

4.3.4. Lemma (Provability of normal form). *For any φ, $\varphi \dashv\vdash \mathbin{\vee\!\!\!\vee}\mathcal{R}(\varphi)$.*

Proof. The proof is by induction on φ. The basic cases for atoms and $\bot$ are trivial, and so is the inductive case for $\veebar$. So, only the inductive cases for conjunction and implication remain to be proved. Consider two formulas φ and ψ, with $\mathcal{R}(\varphi) = \{\alpha_1, \dots, \alpha_n\}$ and $\mathcal{R}(\psi) = \{\beta_1, \dots, \beta_m\}$. Let us make the induction hypothesis that $\varphi \dashv\vdash \alpha_1 \veebar \cdots \veebar \alpha_n$ and $\psi \dashv\vdash \beta_1 \veebar \cdots \veebar \beta_m$, and let us consider the conjunction $\varphi \wedge \psi$ and the implication $\varphi \to \psi$.

- *Conjunction.* From the induction hypothesis and the rules for $\wedge$ we get

  $$\varphi \wedge \psi \dashv\vdash (\alpha_1 \veebar \cdots \veebar \alpha_n) \wedge (\beta_1 \veebar \cdots \veebar \beta_m).$$

  Since the distributivity of conjunction over disjunction is provable in intuitionistic logic, by Lemma 4.3.3 we have

  $$(\alpha_1 \veebar \cdots \veebar \alpha_n) \wedge (\beta_1 \veebar \cdots \veebar \beta_m) \dashv\vdash \bigveebar\{\alpha_i \wedge \beta_j \mid i \leq n, j \leq m\}.$$

  And we are done, since by definition $\mathcal{R}(\varphi \wedge \psi) = \{\alpha_i \wedge \beta_j \mid i \leq n, j \leq m\}$.

- *Implication.* From the induction hypothesis and the rules for $\to$ we get

  $$\varphi \to \psi \dashv\vdash (\alpha_1 \veebar \cdots \veebar \alpha_n) \to (\beta_1 \veebar \cdots \veebar \beta_m).$$

  By intuitionistic reasoning, we obtain the following:

  $$(\alpha_1 \veebar \cdots \veebar \alpha_n) \to (\beta_1 \veebar \cdots \veebar \beta_m) \dashv\vdash \bigwedge_{i \leq n}(\alpha_i \to \ (\beta_1 \veebar \cdots \veebar \beta_m))$$

  Now, since any resolution is a classical formula, it is easy to show using the $\veebar$-split rule that

  $$\alpha_i \to \ (\beta_1 \veebar \cdots \veebar \beta_m) \dashv\vdash (\alpha_i \to \beta_1) \veebar \cdots \veebar (\alpha_i \to \beta_m).$$

  Since this is the case for for $1 \leq i \leq n$, the rules for $\wedge$ yield

  $$\bigwedge_{i \leq n}(\alpha_i \to \ \beta_1 \veebar \cdots \veebar \beta_m) \dashv\vdash \bigwedge_{i \leq n}((\alpha_i \to \beta_1) \veebar \cdots \veebar (\alpha_i \to \beta_m)).$$

  Finally, using again the provable distributivity of $\wedge$ over $\veebar$, we get

  $$\bigwedge_{i \leq n} \bigveebar_{j \leq m} (\alpha_i \to \beta_j) \dashv\vdash \bigveebar_{f : \mathcal{R}(\varphi) \to \mathcal{R}(\psi)} \bigwedge_{i \leq n}(\alpha_i \to f(\alpha_i)).$$

  By definition of resolutions for an implication, the formula on the right is precisely $\bigveebar\mathcal{R}(\varphi \to \psi)$. This completes the inductive proof. $\square$

As a corollary, a formula may always be derived from each of its resolutions.

4.3.5. Corollary. *For every $\varphi \in \mathcal{L}^{\mathsf{P}}$, if $\alpha \in \mathcal{R}(\varphi)$ then $\alpha \vdash \varphi$.*

Proof. Let $\mathcal{R}(\varphi) = \{\alpha_1, \ldots, \alpha_n\}$. By means of the rule ($\mathbin{\rlap{\vee}\,\vee}$i), from α_i we can infer $\alpha_1 \mathbin{\rlap{\vee}\,\vee} \cdots \mathbin{\rlap{\vee}\,\vee} \alpha_n$, and thus, by the previous lemma, we can infer φ. $\square$

Another consequence of Lemma 4.3.4 is that, if ψ in combination with other assumptions fails to yield a conclusion, this failure can always be traced to a specific resolution of ψ.

4.3.6. Lemma. *If $\Phi, \psi \nvdash \chi$, then $\Phi, \alpha \nvdash \chi$ for some $\alpha \in \mathcal{R}(\psi)$.*

Proof. We show the contrapositive: if $\Phi, \alpha \vdash \chi$ for all $\alpha \in \mathcal{R}(\psi)$, then $\Phi, \psi \vdash \chi$. Let $\mathcal{R}(\psi) = \{\alpha_1, \ldots, \alpha_n\}$. The rule ($\mathbin{\rlap{\vee}\,\vee}$e) ensures that if we have $\Phi, \alpha_i \vdash \chi$ for $1 \leq i \leq n$ we also have $\Phi, \alpha_1 \mathbin{\rlap{\vee}\,\vee} \cdots \mathbin{\rlap{\vee}\,\vee} \alpha_n \vdash \chi$. Since the previous lemma gives $\psi \vdash \alpha_1 \mathbin{\rlap{\vee}\,\vee} \cdots \mathbin{\rlap{\vee}\,\vee} \alpha_n$, we also get $\Phi, \psi \vdash \chi$. $\square$

The next lemma extends this result from a single assumption to the whole set.

4.3.7. Lemma (Traceable deduction failure).
If $\Phi \nvdash \psi$, there is some resolution $\Gamma \in \mathcal{R}(\Phi)$ such that $\Gamma \nvdash \psi$.

Proof. Let us fix an enumeration of Φ, say $(\varphi_n)_{n \in \mathbb{N}}$.[3] We are going to define a sequence $(\alpha_n)_{n \in \mathbb{N}}$ of classical formulas in $\mathcal{L}^{\mathsf{P}}$ such that, for all $n \in \mathbb{N}$:

- $\alpha_n \in \mathcal{R}(\varphi_n)$;
- $\{\alpha_i \,|\, i \leq n\} \cup \{\varphi_i \,|\, i > n\} \nvdash \psi$.

Let us apply inductively the previous lemma. Assume we have defined α_i for $i < n$ and let us proceed to define α_n. The induction hypothesis tells us that $\{\alpha_i \,|\, i < n\} \cup \{\varphi_i \,|\, i \geq n\} \nvdash \psi$, that is, $\{\alpha_i \,|\, i < n\} \cup \{\varphi_i \,|\, i > n\}, \varphi_n \nvdash \psi$. Now the previous lemma tells us that we can "specify" the formula φ_n to a resolution, i.e., we can find a formula $\alpha_n \in \mathcal{R}(\varphi_n)$ and $\{\alpha_i \,|\, i < n\} \cup \{\varphi_i \,|\, i > n\}, \alpha_n \nvdash \psi$. This means that $\{\alpha_i \,|\, i \leq n\} \cup \{\varphi_i \,|\, i > n\} \nvdash \psi$, completing the inductive proof.

Now let $\Gamma := \{\alpha_n \,|\, n \in \mathbb{N}\}$. By construction, $\Gamma \in \mathcal{R}(\Phi)$. Moreover, we claim that $\Gamma \nvdash \psi$. To see this, suppose towards a contradiction $\Gamma \vdash \psi$: then for some n it should be the case that $\alpha_1, \ldots, \alpha_n \vdash \psi$; but this is impossible, since by construction we have $\{\alpha_1, \ldots, \alpha_n\} \cup \{\varphi_i \,|\, i > n\} \nvdash \psi$. Thus, $\Gamma \nvdash \psi$. $\square$

Using the existence of the Resolution Algorithm (Theorem 4.2.1) on the one hand, and the Traceable Failure Lemma on the other, we obtain an analogue of the Resolution Theorem (Theorem 3.7.17) for provability: a set of assumptions Φ derives a formula ψ iff any resolution of Φ derives some resolution of ψ.

[3] We assume for simplicity that $\mathcal{P}$, and as a consequence also Φ, is countable, even though this is not strictly needed for the proof, which could be equally run by induction on ordinals.

4.3.8. LEMMA (RESOLUTION LEMMA).
$\Phi \vdash \psi \iff$ *for all* $\Gamma \in \mathcal{R}(\Phi)$ *there is* $\alpha \in \mathcal{R}(\psi)$ *such that* $\Gamma \vdash \alpha$*.*

Proof. The left-to-right direction of the lemma follows immediately from Theorem 4.2.1. Indeed, suppose there is a $P : \Phi \vdash \psi$ and let $\Gamma \in \mathcal{R}(\Phi)$: the theorem describes how to use P and Γ to construct a proof of $\Gamma \vdash \alpha$ for some $\alpha \in \mathcal{R}(\psi)$.

For the converse, suppose $\Phi \nvdash \psi$: the previous lemma tells us that there is a resolution $\Gamma \in \mathcal{R}(\Phi)$ such that $\Gamma \nvdash \psi$. Since from any resolution $\alpha \in \mathcal{R}(\psi)$ we can derive ψ (Corollary 4.3.5), we must also have $\Gamma \nvdash \alpha$ for every $\alpha \in \mathcal{R}(\psi)$. Thus, it is not the case that any resolution of Φ derives some resolution of ψ.□

The next lemma shows that the Split Property is shared by provability, at least for the case in which the assumptions are classical formulas.

4.3.9. LEMMA (PROVABLE SPLIT).
If Γ *is a set of classical formulas and* $\Gamma \vdash \varphi \vee\!\!\!\vee\, \psi$*, then* $\Gamma \vdash \varphi$ *or* $\Gamma \vdash \psi$*.*

Proof. Suppose $\Gamma \vdash \varphi \vee\!\!\!\vee\, \psi$. Since Γ is a set of classical formulas, Proposition 3.6.20 ensures that $\mathcal{R}(\Gamma) = \Gamma$. So, by Lemma 4.3.8 we have $\Gamma \vdash \beta$ for some $\beta \in \mathcal{R}(\varphi \vee\!\!\!\vee\, \psi)$. Since $\mathcal{R}(\varphi \vee\!\!\!\vee\, \psi) = \mathcal{R}(\varphi) \cup \mathcal{R}(\psi)$ we have either $\beta \in \mathcal{R}(\varphi)$ or $\beta \in \mathcal{R}(\psi)$. In the former case, by Corollary 4.3.5 we have $\beta \vdash \varphi$, and thus also $\Gamma \vdash \varphi$. In the latter case, we have $\beta \vdash \psi$ and thus $\Gamma \vdash \psi$. □

4.3.2 Canonical model

Let us now turn to the definition of our canonical model for InqB. As usual in classical and modal logic, we will construct our possible worlds out of complete theories. However, in our setting it is convenient to work with complete theories taken not from the full language, but from its classical fragment, $\mathcal{L}_c^{\mathsf{P}}$.

4.3.10. DEFINITION. [Theories of classical formulas]
A *theory of classical formulas* is a set $\Gamma \subseteq \mathcal{L}_c^{\mathsf{P}}$ which is closed under deduction of classical formulas, that is, if $\alpha \in \mathcal{L}_c^{\mathsf{P}}$ and $\Gamma \vdash \alpha$ then $\alpha \in \Gamma$.

4.3.11. DEFINITION. [Complete theories of classical formulas]
A *complete* theory of classical formulas is a theory of classical formulas Γ s.t.:

- $\bot \notin \Gamma$;
- for any $\alpha \in \mathcal{L}_c^{\mathsf{P}}$, either $\alpha \in \Gamma$ or $\neg\alpha \in \Gamma$.

The following lemma is essentially just Lindenbaum's lemma for classical propositional logic, which can be proved by means of the usual completion procedure.

4.3.12. Lemma. *If $\Gamma \subseteq \mathcal{L}_c^P$ and $\Gamma \not\vdash \bot$ then $\Gamma \subseteq \Delta$ for some complete theory of classical formulas Δ.*

If S is a set of theories of classical formulas, we will denote by $\bigcap S$ the intersection of all the theories $\Gamma \in S$, with the convention that the intersection of the empty set of theories is the set of *all* classical formulas: $\bigcap \emptyset = \mathcal{L}_c^{\mathsf{P}}$. A simple fact that will be useful in our proof is that $\bigcap S$ is itself a theory of classical formulas.

4.3.13. Lemma.
If S is a set of theories of classical formulas, $\bigcap S$ is a theory of classical formulas.

Proof. It is obvious that $\bigcap S$ is a set of classical formulas. Moreover, suppose $\bigcap S \vdash \alpha$ and α is a classical formula. Take any $\Theta \in S$: since $\bigcap S \subseteq \Theta$, we also have $\Theta \vdash \alpha$, and since Θ is a theory of classical formulas, we have $\alpha \in \Theta$. Since this is the case for all $\Theta \in S$, we have $\alpha \in \bigcap S$. □

Our canonical model will have complete theories of classical formulas as worlds, and the canonical valuation will equate truth at a world with membership in it.

4.3.14. Definition. [Canonical model]
The canonical model for InqB is the model $M^c = \langle W^c, V^c \rangle$ defined as follows:

- W^c is the set of complete theories of classical formulas;
- $V^c : W^c \times \mathcal{P} \to \{0, 1\}$ is defined by $V^c(\Delta, p) = 1 \iff p \in \Delta$.

4.3.3 Completeness

Usually, the next step in the completeness proof is to prove the *truth lemma*, a result connecting truth at a possible world in the canonical model with provability from that world. However, in inquisitive semantics the fundamental semantic notion is not truth at a possible world, but support at an information state. Thus, what we need is a *support lemma* that characterizes the notion of support at a state in M^c in terms of provability. What should this characterization be?

We may think of the information available in a state S as being captured by those statements that are true at all the worlds in S. Syntactically, truth at a world will correspond to membership in it. Thus, the information available in a state S is captured syntactically by the theory of classical formulas $\bigcap S$, which consists of those statements that belong to all the worlds in S.

For a formula φ, to be supported at S is to be settled by the information available in S. Syntactically, this would correspond to φ being derivable from $\bigcap S$. Thus, we expect the following connection: $S \models \varphi \iff \bigcap S \vdash \varphi$. The following lemma states that this connection indeed holds.

4.3.15. LEMMA (SUPPORT LEMMA). *For any state $S \subseteq W^c$ and any $\varphi \in \mathcal{L}^{\mathsf{P}}$:*

$$S \models \varphi \iff \bigcap S \vdash \varphi.$$

Proof. The proof is by induction on φ, simultaneously for all $S \subseteq W^c$.

- *Atoms.* By the support clause for atoms, we have $S \models p \iff V^c(\Gamma, p) = 1$ for all $\Gamma \in S$. By definition of the canonical valuation, this is the case if and only if $p \in \Gamma$ for all $\Gamma \in S$, i.e., if and only if $p \in \bigcap S$. Finally, by Lemma 4.3.13 we have $p \in \bigcap S \iff \bigcap S \vdash p$.

- *Falsum.* Suppose $S \models \bot$. This means that $S = \emptyset$. Recalling that we have defined $\bigcap \emptyset$ to be the set $\mathcal{L}^{\mathsf{P}}_c$ of all classical formulas, we have $\bigcap S \vdash \bot$. Conversely, suppose $S \not\models \bot$, that is, $S \neq \emptyset$. Then, take a $\Gamma \in S$: $\bigcap S \subseteq \Gamma$, and since $\Gamma \nvdash \bot$ by definition, also $\bigcap S \nvdash \bot$.

- *Conjunction.* The inference rules for conjunction imply that a conjunction is provable from a set of assumptions iff both of its conjuncts are. Using this fact and the induction hypothesis, we obtain: $S \models \varphi \wedge \psi \iff S \models \varphi$ and $S \models \psi \iff \bigcap S \vdash \varphi$ and $\bigcap S \vdash \psi \iff \bigcap S \vdash \varphi \wedge \psi$.

- *Implication.* Suppose $\bigcap S \vdash (\varphi \to \psi)$. Consider any state $T \subseteq S$ with $T \models \varphi$. By induction hypothesis, this means that $\bigcap T \vdash \varphi$. Since $T \subseteq S$, we have $\bigcap S \subseteq \bigcap T$, and since we are assuming $\bigcap S \vdash (\varphi \to \psi)$, also $\bigcap T \vdash (\varphi \to \psi)$. Now, since from $\bigcap T$ we can derive both φ and $\varphi \to \psi$, by an application of $(\to \mathsf{e})$ we can also derive ψ. Hence, by induction hypothesis we have $T \models \psi$. Since T was an arbitrary substate of S, we have shown that $S \models (\varphi \to \psi)$.

  For the converse, suppose $\bigcap S \nvdash (\varphi \to \psi)$. By the rule $(\to \mathsf{i})$, this implies that $\bigcap S, \varphi \nvdash \psi$. Lemma 4.3.6 then ensures that there is an $\alpha \in \mathcal{R}(\varphi)$ such that $\bigcap S, \alpha \nvdash \psi$.

  Now let $T_\alpha = \{\Gamma \in S \mid \alpha \in \Gamma\}$. First, we have $\alpha \in \bigcap T_\alpha$, whence $\bigcap T_\alpha \vdash \varphi$ by Corollary 4.3.5. By induction hypothesis we then have $T_\alpha \models \varphi$. Now, if we can show that $\bigcap T_\alpha \nvdash \psi$ we are done. For then, the induction hypothesis gives $T_\alpha \not\models \psi$: this would mean that T_α is a substate of S that supports φ but not ψ, showing that $S \not\models (\varphi \to \psi)$.

  So, we are left to show that $\bigcap T_\alpha \nvdash \psi$. Towards a contradiction, suppose $\bigcap T_\alpha \vdash \psi$. Since $\bigcap T_\alpha$ is a set of classical formulas, it is a resolution of itself. Thus, Lemma 4.3.8 tells us that $\bigcap T_\alpha \vdash \beta$ for some resolution $\beta \in \mathcal{R}(\psi)$, which by Lemma 4.3.13 amounts to $\beta \in \bigcap T_\alpha$. So, for any $\Gamma \in T_\alpha$ we have $\beta \in \Gamma$, and thus also $(\alpha \to \beta) \in \Gamma$, since Γ is closed under deduction of classical formulas and $\beta \vdash (\alpha \to \beta)$ by $(\to \mathsf{i})$. Now consider any $\Gamma \in S - T_\alpha$: this means that $\alpha \notin \Gamma$ and so $\neg\alpha \in \Gamma$ since Γ is complete; but then we have $(\alpha \to \beta) \in \Gamma$, because Γ is closed under deduction of classical formulas and $\neg\alpha \vdash (\alpha \to \beta)$ by the rules $(\neg\mathsf{e})$, $(\bot\mathsf{e})$, and $(\to\mathsf{i})$. We have thus shown that $(\alpha \to \beta) \in \Gamma$ for any $\Gamma \in S$, whether $\Gamma \in T_\alpha$ or $\Gamma \in S - T_\alpha$. We can then conclude $(\alpha \to \beta) \in \bigcap S$, whence by $(\to\mathsf{e})$ we have $\bigcap S, \alpha \vdash \beta$. Since $\beta \in \mathcal{R}(\psi)$, by Corollary 4.3.5 we have $\bigcap S, \alpha \vdash \psi$. But this is a contradiction since by assumption α is such that $\bigcap S, \alpha \nvdash \psi$.

- *Inquisitive disjunction.* Suppose $S \models \varphi \mathbin{\vee\!\!\!\vee} \psi$. By the support clause for $\mathbin{\vee\!\!\!\vee}$, this means that either $S \models \varphi$ or $S \models \psi$. The induction hypothesis gives $\bigcap S \vdash \varphi$ in the former case, and $\bigcap S \vdash \psi$ in the latter. In either case, the rule $(\mathbin{\vee\!\!\!\vee}\mathsf{i})$ ensures that $\bigcap S \vdash \varphi \mathbin{\vee\!\!\!\vee} \psi$.

  Conversely, suppose $\bigcap S \vdash \varphi \mathbin{\vee\!\!\!\vee} \psi$. Since $\bigcap S$ is a set of classical formulas, by Lemma 4.3.9 we have either $\bigcap S \vdash \varphi$ or $\bigcap S \vdash \psi$. The induction hypothesis gives $S \models \varphi$ in the former case, and $S \models \psi$ in the latter. In either case, we can conclude $S \models \varphi \mathbin{\vee\!\!\!\vee} \psi$. □

Notice that, if we take our state S to be a singleton $\{\Gamma\}$, then $\bigcap S = \Gamma$, and we obtain the usual Truth Lemma as a special case of the Support Lemma.

4.3.16. Corollary (Truth Lemma). *For any world $\Gamma \in W^c$ and formula φ:*

$$\Gamma \models \varphi \iff \Gamma \vdash \varphi.$$

However, it is really the Support Lemma, and not just the Truth Lemma, that we need in order to establish completeness. This is because many invalid entailments can only be falsified at non-singleton states. For instance, consider the polar question $?p$: although this formula is not logically valid, it is true at any possible world in any model—i.e., supported at any singleton state. Thus, in order to detect the invalidity of $?p$ in the canonical model, we really have to find a non-singleton state S in M^c at which $?p$ is not supported.

In general, given a set of formulas Φ and a formula ψ such that $\Phi \nvdash \psi$, the question is how to produce a state $S \subseteq W^c$ which refutes the entailment $\Phi \models \psi$. In proofs for classical logic, one starts with the observation that if $\Phi \nvdash \psi$, then $\Phi \cup \{\neg\psi\}$ is a consistent set of formulas. But in our logic, this is not true: for instance, the soundness of the logic ensures that $\nvdash ?p$, but it is easy to see that $\neg ?p \vdash \bot$. So, the reasoning at this point needs to be slightly more subtle. The next proof fills in the missing details.

Proof of Theorem 4.3.2. Suppose $\Phi \nvdash \psi$. By the Resolution Lemma, there is a resolution Θ of Φ which does not derive any resolution of ψ. Now let $\mathcal{R}(\psi) = \{\alpha_1, \ldots, \alpha_n\}$ and consider an arbitrary α_i. Since $\Theta \nvdash \alpha_i$, we must have

$\Theta \cup \{\neg\alpha_i\} \nvdash \bot$. For suppose that $\Theta \cup \{\neg\alpha_i\} \vdash \bot$: then by $(\neg\mathsf{i})$ we would also have $\Theta \vdash \neg\neg\alpha_i$ and thus, since α_i is a classical formula, by $\neg\neg$-elimination we would have $\Theta \vdash \alpha_i$, contrary to assumption. Hence, $\Theta \cup \{\neg\alpha_i\}$ is consistent, and thus by Lemma 4.3.12 it can be extended to a complete theory $\Gamma_i \in W^c$.

Now let $S = \{\Gamma_1, \ldots, \Gamma_n\}$: we claim that $S \models \Phi$ but $S \not\models \psi$. To see that $S \models \Phi$, note that by construction we have $\Theta \subseteq \bigcap S$, which implies $S \models \Theta$ by the Support Lemma. But since $\Theta \in \mathcal{R}(\Phi)$, Proposition 3.6.21 implies $S \models \Phi$.

To see that $S \not\models \psi$, suppose towards a contradiction that $S \models \psi$: then by Theorem 3.6.7 we must have $S \models \alpha_i$ for some i. By the Support Lemma, that would mean that $\bigcap S \vdash \alpha_i$. Since $\bigcap S \subseteq \Gamma_i$ and Γ_i is closed under deduction of classical formulas, it follows that $\alpha_i \in \Gamma_i$. But this is impossible, since Γ_i is consistent and contains $\neg\alpha_i$ by construction. Hence, we have $S \models \Phi$ but $S \not\models \psi$, which allows us to conclude $\Phi \not\models \psi$. $\square$

4.4 On the role of questions in proofs

What does it mean, intuitively, to suppose or conclude a question in a proof? To answer, it might be helpful to start from a concrete example. Take again the proof from Example 4.1.5, showing that conditionally on the outcome being prime, the range of the outcome determines what the outcome is.

$$\dfrac{\mathsf{range} \quad \dfrac{\dfrac{\Gamma \quad [\mathsf{low}]_1 \quad \mathsf{prime}}{\mathsf{two}}}{\mathsf{outcome}}(\mathbb{V}\mathsf{i}) \quad \dfrac{\dfrac{\Gamma \quad [\mathsf{mid}]_1 \quad \mathsf{prime}}{\mathsf{three}}}{\mathsf{outcome}}(\mathbb{V}\mathsf{i}) \quad \dfrac{\dfrac{\Gamma \quad [\mathsf{high}]_1 \quad \mathsf{prime}}{\mathsf{five}}}{\mathsf{outcome}}(\mathbb{V}\mathsf{i})}{\mathsf{outcome}}(\mathbb{V}\mathsf{e}, 1)$$

What is the argument encoded by this proof? It may be glossed as follows. Suppose we are given the information about what the range of the outcome is (note: this is where a question is supposed). Then either we have the information that the outcome is low, or we have the information that it is in the middle range, or we have the information that it is high. In the first case, from the assumption that the outcome is prime we can conclude that it is two; therefore, in this case we have the information about what the outcome is (note: this is where a question is concluded). Similarly, in the second case we can conclude that the outcome is three, and in the third case we can conclude that the outcome is five; so in each of these cases we also we have the information about what the outcome is. Thus, in any case, under the given assumptions we are guaranteed to have the information about what the outcome is.

Notice the conceptually natural role that questions play in this argument. When we suppose the question range we are supposing to be given the information whether the outcome is low, middle, or high. We are not, however, supposing anything specific about the range—we are not supposing, say, that the range is low; we are just supposing to have an *arbitrary* specification of the range of the outcome. Similarly, when we conclude the question outcome, what we are concluding is that, under the given assumptions, we are guaranteed to have the information as to what the outcome is—though the specific information we have is bound to depend on the information we are given about the range.

It might be insightful to draw a connection with the arbitrary individual constants used in natural deduction systems for standard first-order logic.[4] For instance, in order to infer ψ from $\exists x\varphi(x)$, one can make a new assumption $\varphi(c)$, where c is fresh in the proof and not occurring in ψ, and then try to derive ψ from this assumption. Here, the idea is that c stands for an *arbitrary* object in the extension of $\varphi(x)$—an arbitrary object "of type φ". If ψ can be inferred from $\varphi(c)$, then it must follow no matter which specific object of type φ the constant c denotes, and thus it must follow from the mere existence of such an object.

Questions allow us to do something similar, except that instead of an arbitrary *individual* of a given type, a question may be viewed as denoting an arbitrary *piece of information* of a given type. For instance, the question range may be viewed as denoting an arbitrary specification of the range of the outcome.

At the outset of their influential book "*The logic of questions*", Belnap and Steel warned their readers:

> Absolutely the wrong thing is to think [the logic of questions] is a logic in the sense of a deductive system, since one would then be driven to the pointless task of inventing an inferential scheme in which questions, or interrogatives, could serve as premises and conclusions. (Belnap and Steel, 1976, page 1)

In this chapter, I hope to have shown that Belnap and Steel were too pessimistic: far from being pointless, questions have a very interesting role to play in logical inference. They can meaningfully serve as premises and conclusions in a proof. In fact, they turn out to be powerful proof-theoretic tools: as we saw, they allow us to reason with arbitrary information of a given type. By making inferences with such arbitrary information, we can provide formal proofs of the validity of certain logical dependencies.

4.5 Exercises

4.5.1. Exercise. Give natural deduction proofs of the following entailments.

1. $?p \wedge ?q \models ?(p \wedge q)$
2. $p \to (?q \to ?r),\ \neg q\ \models\ (p \to r) \vee\!\!\!\vee (p \to \neg r)$

[4]Thanks to Justin Bledin for suggesting this analogy.

4.5.2. Exercise. Miss Marple is investigating a murder. She has concluded that the murderer must be either Alice or Bob. However, Alice has a bullet-proof alibi for the morning, while Bob has one for the rest of the day. So, we may assume the following:

- Either Alice or Bob did it. $a \vee b$ (notice the disjunction is classical)
- If it was Alice, it was not in the morning. $a \to \neg m$
- If it was Bob, it was in the morning. $b \to m$

Given these assumptions, the question who did it is determined by the question whether the murder was committed in the morning. This is captured by the entailment:

$$a \vee b,\ \ a \to \neg m,\ \ b \to m \models\ ?m \to a \mathbin{\vee\!\!\!\vee} b.$$

Give a natural deduction proof of this entailment.

Hint. Recall that $a \vee b$ abbreviates $\neg(\neg a \wedge \neg b)$, so in combination with $\neg a \wedge \neg b$ it can be used to derive $\bot$.

4.5.3. Exercise. Miss Marple is again busy investigating a murder. Her investigation has revealed that during the entire morning, the butler was the only person in the house besides the victim. Therefore, she concluded that:

- If the murder took place in the morning, then if it took place in the house, the culprit is the butler.

Also, it is attested that the butler remained inside the house the whole morning. Therefore, Miss Marple concluded that:

- If the culprit is the butler, then if the murder took place outside the house, it did not take place in the morning.

On the basis of these two conclusions, the first question below determines the second:

- Did the murder take place inside the house?
- If the murder took place in the morning, was it the butler?

Formalize this logical dependency as an entailment in InqB and prove its validity by means of a natural deduction proof.

4.5.4. Exercise. Show that the derived rules for classical disjunction given in Figure 4.2 are admissible on the basis of the primitive rules of our system. That is, show that for any given formulas $\varphi, \psi \in \mathcal{L}^{\mathsf{P}}$, the following facts hold:

1. $\varphi \vdash \varphi \vee \psi$ and $\psi \vdash \varphi \vee \psi$;
2. if Φ is an arbitrary set of formulas and α a classical formula, then if we have $\Phi, \varphi \vdash \alpha$ and $\Phi, \psi \vdash \alpha$, we also have $\Phi, \varphi \vee \psi \vdash \alpha$.

Do not rely on the completeness theorem.

Chapter 5

Questions in predicate logic

In this chapter, we move on from propositional logic to the richer setting of predicate logic. We describe how classical first-order logic can be enriched with questions, leading to a system InqBQ of inquisitive first-order logic (the Q in the acronym stands for *quantification*). With respect to the inquisitive propositional logic of the previous chapters, this extension is interesting not only because first-order logic, as a logic of statements, is a much more expressive system than propositional logic, but also because, through quantification, many important classes of questions can be formalized, in addition to the propositional 'whether…or' questions that we were able to formalize in inquisitive propositional logic. For instance, it will become possible to analyze questions that ask for one or more instances of a property, such as (1-a) and (1-b); questions that ask for the unique individual satisfying of a property, such as (1-c); and questions that ask for the extension of a property, such as (1-d).

(1) a. What is one color that Alice likes?
b. What are two colors that Alice likes?
c. What is the color that Alice likes?
d. What are the colors that Alice likes?

Thus, inquisitive first-order logic provides a rich environment to regiment many classes of questions and study their logic—although there are also some prominent question types, notably *how many* questions like (2), which, while semantically analyzable, are not expressible with the resources of InqBQ (see Grilletti and Ciardelli, 2021).

(2) How many colors does Alice like?

As in the case of propositional logic, we will build our inquisitive system in two steps. In the first step, we will show how classical first-order logic can be given a semantics in terms of support at an information state. In the second step, we will exploit the support semantics to introduce questions into first-order logic, equipping the language with new question-forming operators. In addition to inquisitive disjunction $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$, which will work as in InqB, we will now also have an inquisitive existential quantifier $\exists$, which asks for a witness of a certain property.

With the move to predicate logic, some of the subtleties of intensional semantics also come into play, such as the different ways in which terms may refer to objects (rigidly or variably), the interpretation of identity, and the distinction between the entities to which information is attached and the objects that actually exist in the world. As we will see, the modeling choices one makes about these issues have repercussions for the logic of questions.

While many of the central features of inquisitive propositional logic carry over to the first-order case, there are also some crucial differences. Most importantly, it is no longer the case that a question can be recursively associated with a set of statements that capture the different ways to resolve the question. Mathematically, the full system InqBQ turns out to be a rich and complex system. Indeed, in spite of systematic investigation over the past few years, the main meta-theoretical questions about this logic are currently still open: it is not known whether a complete axiomatization exists, nor whether the logic is entailment-compact (in the sense that whatever follows from a set of premises follows from some finite subset), satisfies analogues of the Löwenheim-Skolem theorems, or has a recursively enumerable set of validities.

At the same time, in recent years there have been important developments in the study of InqBQ, especially due to work by Gianluca Grilletti (see Grilletti, 2019, 2020; Grilletti and Ciardelli, 2021). One exciting recent result that we will cover in detail is the existence of a broad fragment of InqBQ, the *classical antecedent fragment*, which on the one hand contains all the most important classes of questions expressible in InqBQ, and on the other hand turns out to be very well-behaved and to admit an elegant completeness result. This fragment can then be regarded in its own right as a rich logic of questions—much richer than its predecessors, such as the Logic of Interrogation of Groenendijk (1999)—that shares many of the the key features of standard first-order logic. In addition to this, at the end of the chapter we will also survey some other interesting recent results, as well as some open problems.

5.1 Support for classical first-order logic

Let us start out by describing how classical first-order predicate logic may be given a support semantics. For ease of exposition, we focus first on a language without identity, and then turn to the treatment of identity in Section 5.4.

Language. As usual, our language is based on a signature $\mathcal{S}$, consisting of a set $\mathfrak{R}_{\mathcal{S}}$ of relation symbols (also called *predicates*) and a set $\mathcal{F}_{\mathcal{S}}$ of function symbols, where each of these symbols has a certain *arity* $n \geq 0$. Relation symbols of arity 0 are called *propositional atoms*, while function symbols of arity 0 are called *individual constants*. Moreover, we assume that among the function symbols we have a specified set $\mathcal{F}_{\mathcal{S}}^{R}$ of *rigid* function symbols, whose interpretation is required to be fixed across different possible worlds. We will refer to the remaining function symbols, whose interpretation may vary across different possible worlds, as *non-rigid* function symbols. We will use sans serif fonts to mark rigidity: thus, a meta-variable f will range over rigid function symbols, while f will range over all function symbols, rigid or non-rigid.

As usual, we also have a countably infinite stock of first-order variables, $\mathsf{Var} = \{x_0, x_1, x_2, \dots\}$. The set $\mathsf{Ter}(\mathcal{S})$ of *terms* in the language is given as usual by the inductive definition

$$t \ ::= \ x \mid f(t, \dots, t)$$

where $x \in \mathsf{Var}$, $f \in \mathcal{F}_{\mathcal{S}}$, and the number of arguments of f in the inductive clause matches the arity of f. The set of *rigid terms* is defined analogously by

$$\mathsf{t} \ ::= \ x \mid \mathsf{f}(\mathsf{t}, \dots, \mathsf{t})$$

where $\mathsf{f} \in \mathcal{F}_{\mathcal{S}}^{R}$ is a rigid function symbol.

The set of classical first-order formulas in the signature $\mathcal{S}$ is also defined as usual, where we take $\bot, \wedge$, and $\to$ as our primitive propositional connectives, and $\forall$ as our primitive quantifier.

5.1.1. Definition. [Classical formulas]
The set $\mathcal{L}_c^{\mathsf{Q}}(\mathcal{S})$ of classical first-order formulas is defined recursively as follows:

$$\varphi \ ::= \ R(t_1, \dots, t_n) \mid \bot \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \forall x \varphi$$

where R is an n-ary relation symbol in $\mathcal{S}$, $t_1, \dots, t_n \in \mathsf{Ter}(\mathcal{S})$, and $x \in \mathsf{Var}$.

When there is no need to emphasize the signature $\mathcal{S}$, we will drop reference to it and simply refer to the set of classical first-order formulas as $\mathcal{L}_c^{\mathsf{Q}}$. We take the remaining operators of classical first-order logic to be defined as follows:

- $\neg\varphi := \varphi \to \bot$;
- $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$;
- $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$;
- $\exists x \varphi := \neg\forall x \neg\varphi$.

It will be useful to introduce some abbreviations: we will write $\overline{t}$ for a sequence $\langle t_1, \dots, t_n \rangle$ of terms and $\overline{x}$ for a sequence $\langle x_1, \dots, x_n \rangle$ of variables. Moreover, if Q is a quantifier and $\overline{x} = \langle x_1, \dots, x_n \rangle$ a sequence of variables, we will write $Q\overline{x}\varphi$ for $Qx_1 \dots Qx_n\varphi$.

Free and bound occurrences of a variable x in a formula are defined as usual. Given a formula φ, we write $FV(\varphi)$ for the set of variables which are free in φ. Moreover, if $x \in \mathsf{Var}$ and $t \in \mathsf{Ter}(\mathcal{S})$, we write $\varphi[t/x]$ for the formula that results from replacing each free occurrence of x in φ by t. As usual, we say that a term t is *free for* a variable x in a formula φ in case no free occurrence of x in φ lies within the scope of a quantifier which binds a variable y occurring in t.

We allow ourselves to drop parentheses whenever convenient, including in the case of atomic sentences (writing, e.g., Rxy instead of $R(x, y)$). We follow standard conventions about the priority of operators: quantifiers and negation have the highest priority, followed by conjunctions and disjunctions (including inquisitive disjunctions in the full language), while implication has the lowest priority. Thus, e.g., the formula $\varphi \to \neg\forall x\psi \wedge \chi$ should be parsed as $\varphi \to ((\neg\forall x\psi) \wedge \chi)$.

Models. Let us now turn to the structures that are to serve as models for our language. As in the previous chapters, our models will comprise a universe of possible worlds, each representing a certain state of affairs. Moreover, they will comprise a domain D of individuals that our quantifiers range over. These are the individuals that the information represented by the model is about. The state of affairs corresponding to a given world is then characterized by fixing the denotation of the predicate and function symbols.

Note that taking the domain of quantification to be world-independent means that our models will not be able to represent uncertainty about the domain of quantification itself (except insofar as it stems from uncertainty about identities: see Section 5.4). This can be seen as a simplifying assumption that one might want to lift in future work, at the cost of introducing some extra complexity.[1]

5.1.2. DEFINITION. [Relational information models]
A relational information model is a triple $M = \langle W, D, I\rangle$, where:

- W is a set, the elements of which we call *possible worlds*;
- D is a non-empty set, the elements of which we call *individuals*;[2]
- I is a map assigning to each $w \in W$ a function I_w defined on $\mathcal{S}$ such that:
  - $I_w(R) \subseteq D^n$ for an n-ary predicate R; we write R_w for $I_w(R)$.

[1] Without modifying the logic, one could capture uncertainty about the domain of quantification by introducing an existence predicate, whose extension can vary from world to world, and restricting quantifiers explicitly to existing individuals. We will not explore this in detail, but the idea is familiar from modal predicate logic (see, e.g., Fitting and Mendelsohn, 2012).

[2] Later on, when we introduce our treatment of identity, we will want to draw a distinction between the elements of D, which we will call *epistemic* individuals, and the ontic individuals that actually exist in the state of affairs corresponding to a world.

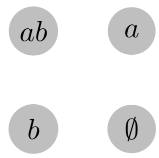


Figure 5.1: A relational information model with two individuals a, b, denoted rigidly by constants a and b, and four possible worlds, corresponding to the four extensions for P. The label ab in the pictures stands for a world w in which $P_w = \{a, b\}$, the label a stands for a world w in which $P_w = \{a\}$, and so on.

- $I_w(f) : D^n \to D$ for an n-ary function symbol f; we write f_w for $I_w(f)$.
  Rigidity constraint: if f is rigid, then for any $w, w' \in W$, $\mathsf{f}_w = \mathsf{f}_{w'}$.

Note that with each world w of such an information model we can associate a standard relational structure of our signature.

5.1.3. Definition. [Relational structure associated with a world]
Let $M = \langle W, D, I \rangle$ be a relational information model and $w \in W$. The relational structure associated with W is $M_w := \langle D, I_w \rangle$.

Thus, a relational information model can be seen alternatively as a collection $\{M_w \mid w \in W\}$ of relational structures sharing the same underlying domain.

For an illustration, consider a signature containing a unary predicate P and two rigid constants, a and b. Consider a simple model M containing two individuals a and b, denoted rigidly by a and b respectively, and four possible worlds, corresponding to the four possible extensions for the predicate P. This model is depicted in Figure 5.1.

Support semantics for classical first-order logic. Let us now see how the standard language of first-order logic can be given a support semantics, which interprets formulas relative to information states $s \subseteq W$ drawn from a relational information model.

As is customary, the semantics is given relative to an assignment function g, which fixes the interpretation of variables. Assignments are defined as usual as functions $g : \mathsf{Var} \to D$. If $d \in D$, we write $g[x \mapsto d]$ for the assignment which maps x to d, and otherwise coincides with g.

We can assign to each term in the language a world-dependent referent in a natural way.

5.1.4. Definition. [Referent of a term]
The referent of a term t in a world w under an assignment g is the individual $[t]^w_g \in D$ defined inductively as follows:

- $[x]^w_g = g(x)$;
- $[f(t_1, \dots, t_n)]^w_g = f_w([t_1]^w_g, \dots, [t_n]^w_g)$.

Note that if t is a closed term (i.e., if t does not contain variables) the referent is independent of g and can be denoted as $[t]^w$, while if $\mathtt{t}$ is rigid, the referent is independent of w and can be denoted as $[\mathtt{t}]_g$. If $\mathtt{t}$ is both closed and rigid we can drop both parameters and write simply $[\mathtt{t}]$.

We are now ready to define the relation of support between states and formulas, which specifies what information it takes to settle a first-order formula.

5.1.5. Definition. [Support for classical first-order formulas]
If M is a relational information model, s an information state in M, and g an assignment, we let:

- $M, s \models_g R(t_1, \dots, t_n) \iff$ for all $w \in s$, $\langle [t_1]^w_g, \dots, [t_n]^w_g \rangle \in R_w$
- $M, s \models_g \bot \iff s = \emptyset$
- $M, s \models_g \varphi \wedge \psi \iff M, s \models_g \varphi$ and $M, s \models_g \psi$
- $M, s \models_g \varphi \to \psi \iff$ for all $t \subseteq s$, $M, t \models_g \varphi$ implies $M, t \models_g \psi$
- $M, s \models_g \forall x \varphi \iff$ for all $d \in D$, $M, s \models_{g[x \mapsto d]} \varphi$.

As usual, atomic formulas $R\bar{t}$ are treated as statements: a state settles that $R\bar{t}$ if the information available in s implies that the tuple of individuals denoted by $\bar{t}$ belongs to the extension of R. The clauses for the propositional connectives are familiar from the previous chapters. A universal $\forall x \varphi(x)$ is settled in s in case $\varphi(x)$ is settled for every value of the variable x.

When no confusion arises, we will drop reference to the model M, and simply write $s \models_g \varphi$. As usual, we refer to the set of information states supporting φ in M (now relative to an assignment g) as the *support-set* of φ, notation $[\varphi]^g_M$. The *alternatives* for φ in M relative to g are the $\subseteq$-maximal elements of $[\varphi]^g_M$, and the set of these alternatives is denoted $\textsc{Alt}^g_M(\varphi)$. It is easy to check that if $FV(\varphi) \subseteq \{x_1, \dots, x_n\}$, only the value of g on $x_1, \dots, x_n$ matters for whether $s \models_g \varphi$; in this case, we may thus write $s \models_{[x_1 \mapsto d_1, \dots, x_n \mapsto d_n]} \varphi$ to mean that $s \models_g \varphi$ where g is an arbitrary assignment mapping x_i to d_i. In particular, if φ is a sentence we may drop reference to the assignment altogether.

Truth at a world w is defined, as usual, as support at the state $\{w\}$:

$$w \models_g \varphi \overset{def}{\iff} \{w\} \models_g \varphi.$$

The truth-set of φ in M relative to g is the set of worlds where φ is true:

$$|\varphi|^g_M := \{w \in W \mid w \models_g \varphi\}.$$

It is straightforward to check that truth at a world w, as given by our semantics, coincides exactly with truth in the structure M_w as given by the standard Tarskian semantics for first-order predicate logic.

5.1.6. Proposition (Truth conditions for classical formulas).
For any information model M, any world w in M, and any assignment g:

- $w \models_g R(t_1, \ldots, t_n) \iff \langle [t_1]^w_g, \ldots, [t_n]^w_g \rangle \in R_w$
- $w \not\models_g \bot$
- $w \models_g \varphi \wedge \psi \iff w \models_g \varphi$ *and* $w \models_g \psi$
- $w \models_g \varphi \to \psi \iff w \not\models_g \varphi$ *or* $w \models_g \psi$
- $w \models_g \forall x\varphi \iff$ *for all* $d \in D$, $w \models_{g[x\mapsto d]} \varphi$.

Thus, our semantics allows us to retrieve standard truth-conditional semantics as a special case. In order to check that our semantics gives a support-based implementation of classical propositional logic, we just have to check that truth and support are related in accordance with the Truth-Support Bridge (Constraint 2.2.5): a formula is supported by a state iff it is true at each world in the state. This is the content of the following proposition.

5.1.7. Proposition (Classical formulas are truth conditional).
For any $\varphi \in \mathcal{L}_c^{\mathbf{Q}}$, state s in an information model M, and assignment g:

$$s \models_g \varphi \iff w \models_g \varphi \text{ for all } w \in s.$$

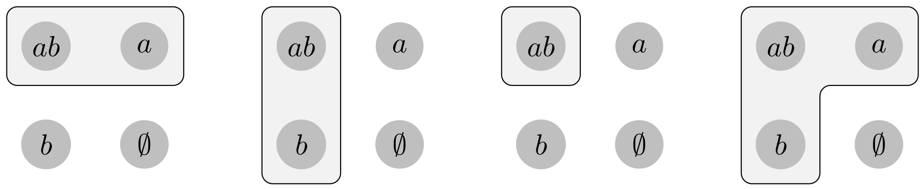
Proof. By induction on φ. The novel case with respect to the proof in the propositional case (Proposition 3.1.8) is the one for $\forall$. We have

$$\begin{aligned} s \models_g \forall x\varphi \quad &\iff \quad \text{for all } d \in D : s \models_g \varphi \\ &\iff \quad \text{for all } d \in D, \text{ for all } w \in s : w \models_{g[x\mapsto d]} \varphi \\ &\iff \quad \text{for all } w \in s, \text{ for all } d \in D : w \models_{g[x\mapsto d]} \varphi \\ &\iff \quad \text{for all } w \in s : w \models_g \forall x\varphi \end{aligned}$$

where the second step uses the induction hypothesis. □

Given that truth and support are related in the appropriate way, it follows from the discussion in Section 2.2 that our support semantics is a semantics for classical first-order logic. This is illustrated by Figure 5.2, which shows the alternatives for some first-order sentences in the model of Figure 5.1. It follows from Proposition 5.1.7 that each statement has a single alternative, consisting of those worlds where it is classically true.

Having thus re-implemented classical predicate logic based on support, we are now ready for the second step of our strategy: bring questions into play by enriching our language with question-forming logical operators.



(a) $P\mathsf{a}$ (b) $P\mathsf{b}$ (c) $\forall x Px$ (d) $\exists x Px$

Figure 5.2: The alternatives for some InqBQ statements in the relational information model of Figure 5.1.

5.2 Adding questions to first-order logic

In the propositional case, questions are introduced into the system by means of a new connective, the inquisitive disjunction $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$. In the first-order case, it is natural to also consider a quantifier counterpart of $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$, denoted $\exists$, which we will call *inquisitive existential quantifier*. The full language of our system is obtained by enriching the language of classical first-order logic with these two operators.

5.2.1. Definition.
The set $\mathcal{L}^{\mathsf{Q}}(\mathcal{S})$ of first-order formulas of InqBQ is defined recursively as follows:

$$\varphi \ ::= \ R(t_1,\dots,t_n) \mid \bot \mid \varphi\wedge\varphi \mid \varphi\to\varphi \mid \forall x\varphi \mid \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \varphi \mid \exists x\varphi$$

where R is an n-ary redicate in $\mathcal{S}$, $t_1,\dots,t_n \in \mathsf{Ter}(\mathcal{S})$, and $x \in \mathsf{Var}$.

As in propositional logic, we also used a derived operator '?', defined by $?\varphi := \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \neg\varphi$. The semantics of $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ is familiar by now: a state s supports an inquisitive disjunction $\varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi$ in case it supports one of the disjuncts. The semantics of $\exists$ is the quantifier analogue of this clause: a state s supports an inquisitive existential $\exists x\varphi(x)$ in case it supports $\varphi(x)$ for some specific value of x.

5.2.2. Definition. [Support for InqBQ]
The relation of support for InqBQ is obtained by augmenting Definition 5.1.5 with the following two clauses:

- $M,s \models_g \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi \iff M,s\models_g \varphi$ or $M,s\models_g \psi$;
- $M,s \models_g \exists x\varphi \iff$ for some $d\in D,\ M,s\models_{g[x\mapsto d]} \varphi$.

By combining the inquisitive operators $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ and $\exists$ with the support-based versions of the classical operators, in InqBQ we can express a wide range of questions. Let us illustrate this with some examples.

First, just like in the propositional setting, we can use the question mark operator ? to turn a statement into the corresponding polar question.

5.2.3. Example. [Polar questions]
Consider the formula $?\forall xPx$, which abbreviates $\forall xPx \mathbin{\vee\!\!\!\vee} \neg\forall xPx$. Using the fact that $\forall xPx$ and $\neg\forall xPx$ are classical formulas, we have:

$$\begin{aligned} s \models_g {?\forall xPx} \quad &\iff \quad s \models_g \forall xPx \text{ or } s \models_g \neg\forall xPx \\ &\iff \quad (\forall w \in s : w \models_g \forall xPx) \text{ or } (\forall w \in s : w \models_g \neg\forall xPx) \\ &\iff \quad (\forall w \in s : P_w = D) \text{ or } (\forall w \in s : P_w \neq D). \end{aligned}$$

In words, $?\forall xPx$ is settled in a state in case the available information determines whether or not all individuals have property P. Thus, $?\forall xPx$ can be seen as a formalization of the polar question *whether everyone has property P*. The two alternatives for this question in our example model of Figure 5.1 are shown in Figure 5.3(a).

In addition to $\mathbin{\vee\!\!\!\vee}$, in InqBQ we can form questions by means of $\exists\!\!\!\exists$. The following example illustrates how this operator allows us to formalize an important class of questions, namely, mention-some questions (cf. the discussion in Section 2.9.2).

5.2.4. Example. [Mention-some questions]
Consider the sentence $\exists\!\!\!\exists xPx$. We have:

$$\begin{aligned} s \models_g \exists\!\!\!\exists xPx \quad &\iff \quad \text{there is a } d \in D \text{ such that } s \models_{g[x\mapsto d]} Px \\ &\iff \quad \text{there is a } d \in D \text{ such that for all } w \in s,\ d \in P_w. \end{aligned}$$

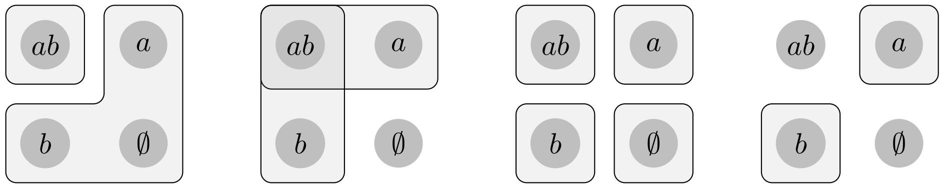
In words, $\exists\!\!\!\exists xPx$ is settled if for some individual d, the available information implies that d has property P. Thus, the formula $\exists\!\!\!\exists xPx$ can be seen as a formalization of the question *what is an instance of P*. In the literature, questions that ask for instances of properties are called *mention-some* questions.

In our model M, which contains just two individuals, the formula $\exists\!\!\!\exists xPx$ has two distinct alternatives, depicted in Figure 5.3(b). These alternatives correspond to the classical formulas $P\mathsf{a}$ and $P\mathsf{b}$, which provide just enough information to establish an instance of property P.

It is interesting to pause to point out that the difference between the semantics of the classical existential $\exists xPx$ and its inquisitive counterpart $\exists\!\!\!\exists xPx$ is one of relative scope of two quantifiers:

$$\begin{aligned} s \models_g \exists xPx \quad &\iff \quad \text{for all } w \in s, \text{ some } d \in D \text{ is such that } d \in I_w(P); \\ s \models_g \exists\!\!\!\exists xPx \quad &\iff \quad \text{some } d \in D \text{ is such that for all } w \in s, d \in I_w(P). \end{aligned}$$

In words, $\exists xPx$ is supported if the available information implies that some individual has property P, while $\exists\!\!\!\exists xPx$ is supported if for some individual, the available information implies that it has property P.



(a) $?\forall x Px$ (b) $\exists x Px$ (c) $\forall x ? Px$ (d) $\exists ! x Px$

Figure 5.3: The alternatives for some InqBQ questions in the relational information model of Figure 5.1.

The idea illustrated by this example generalizes: if R is a binary relation symbol, then the sentence $\exists x \exists y R(x, y)$ is supported in s just in case s establishes of a specific pair $\langle d, d' \rangle$ that it belongs to the extension of R. Thus, $\exists x \exists y R(x, y)$ formalizes a question asking for an instance of a pair which stands in the relation R.

In the previous examples, we considered questions that can be formed *directly* by using the inquisitive operators. But notice that further questions can be expressed by embedding such basic questions under the classical operators $\wedge, \to$, and $\forall$, whose semantics is now generalized in such a way that they can operate on questions as well. In Chapter 2, we have already discussed in detail the effect of embedding questions under conjunction and implication. These operations apply in much the same way to the richer repertoire of questions available in the current setting. By means of conjunction we can, e.g., form conjunctive questions like $\exists x P(x) \wedge \exists x Q(x)$ which asks at once for an instance of property P and an instance of property Q. By means of implication we can form conditional questions like $\exists x P(x) \to \exists x P(x)$, which asks for an instance of property P under the assumption that there is one; we may also form questions such as $\exists x P(x) \to \exists x Q(x)$, which may be seen as asking for a method for turning an instance of property P into an instance of property Q.

The novelty introduced by InqBQ is that it also becomes possible to *universally quantify* over questions. The following example shows that by doing so we can formalize another important class of questions: mention-all questions.

5.2.5. Example. [Mention-all questions]
Consider the sentence $\forall x ? P(x)$: this sentence is supported in a state s in case s settles the polar question $?P(x)$ for all values of x, that is, in case s establishes of every individual $d \in D$ whether or not d has property P. This means that, in order to settle $\forall x ? P(x)$, a state s must settle precisely what the extension of

P is. More formally, using the fact that $P(x)$ is a classical formula we have:

$$\begin{aligned}
s \models_g \forall x?P(x) &\iff \text{for all } d \in D: \; s \models_{g[x\mapsto d]} ?P(x) \\
&\iff \text{for all } d \in D: \; s \models_{g[x\mapsto d]} P(x) \text{ or } s \models_{g[x\mapsto d]} \neg P(x) \\
&\iff \text{for all } d \in D: \; (\text{for all } w \in s : d \in P_w) \text{ or} \\
&\qquad\qquad\qquad\quad (\text{for all } w \in s : d \notin P_w) \\
&\iff \text{for all } d \in D, \text{ for all } w, w' \in s: \; (d \in P_w \iff d \in P_{w'}) \\
&\iff \text{for all } w, w' \in s, \text{ for all } d \in D: \; (d \in P_w \iff d \in P_{w'}) \\
&\iff \text{for all } w, w' \in s : P_w = P_{w'}.
\end{aligned}$$

Thus, $\forall x?P(x)$ can be seen as formalizing the question *which individuals have property P*, or equivalently, *what is the extension of P*. Questions that ask for the extension of a property or relation are known as *mention-all* questions.

In our toy model, this sentence has four alternatives, depicted in figure 5.3(c): each alternative corresponds to one possibility for the extension of P.

The example generalizes: for instance, if R is a binary relation symbol, the sentence $\forall x\forall y?R(x,y)$ is supported by a state s iff s determines the extension of the relation R—i.e., iff the extension is the same at each world in s. Thus, e.g., in a domain consisting only of humans, $\forall x\forall y?R(x,y)$ formalizes the mention-all reading of the question *who is R-related to whom*.

In Section 5.7.3 we will see that the questions that can be obtained by universally quantifying over polar questions are precisely those which ask for the extension of a relation defined by a standard formula of classical first-order logic. As we will see, these coincide exactly with the questions expressible in the *Logic of Interrogation* of Groenendijk (1999).

Summing up, then, in InqBQ we can regiment a range of interesting question types. In particular, we can formalize polar questions like (3-a), mention-some *wh*-questions like (3-b), and mention-all *wh*-questions like (3-c), in addition to conjunctive and conditional questions derived from such questions.

(3)	a.	Did everyone pass the test?	$?\forall xPx$
	b.	Who is an individual who passed the test?	$\exists xPx$
	c.	Which individuals passed the test?	$\forall x?Px$

This discussion does not by any means provide an exhaustive survey of the kinds of questions expressible in InqBQ. On the contrary, many more question types can be expressed,[3] and even more will become expressible once we introduce identity. But hopefully these examples already illustrate the richness of

[3] Just one example: the question $\forall x\exists yRxy$ is settled if we can provide, for each individual d, an instance of an individual d' to which d is R-related. For instance, suppose a teacher wants to give each student, as a present, a keychain in a color the student likes. Then the teacher needs to know for every student x what is a color that x likes. She might express her

InqBQ as a logical framework for regimenting questions, and show how naturally many important question types can be expressed by using only a small set of semantically simple logical operators.[4]

5.3 Basic features of InqBQ

Let us now take a look at some basic features of the system InqBQ which we have defined. We will see that, while many features of the propositional system InqB carry over, there are also some interesting differences.

5.3.1 Support and alternatives

Let us start by examining the features of the support relation. As we expect, support is persistent, and the empty state trivially supports every formula.

5.3.1. PROPOSITION.
For any model M, states s, t, assignment g and formula $\varphi \in \mathcal{L}^{\mathsf{Q}}$, we have:

- *Persistence property: $s \models_g \varphi$ and $t \subseteq s$ implies $t \models_g \varphi$;*
- *Empty state property: $\emptyset \models_g \varphi$.*

As a consequence of persistence, we also get that the truth-set of a formula always coincides with the union of its support set.

5.3.2. PROPOSITION.
For any model M, assignment g, and formula $\varphi \in \mathcal{L}^{\mathsf{Q}}$: $|\varphi|^g_M = \bigcup[\varphi]^g_M$.

Moreover, as in the propositional case, our semantics is *local.* That is, support at a state depends exclusively on the features of the worlds *in* the state.

5.3.3. PROPOSITION (LOCALITY).
Given a relational information model $M = \langle W, D, I\rangle$ and a state s in M, let $M_{|s}$

request for information by means of the question $\forall x(\text{student}(x) \to \exists y(\text{color}(y) \wedge \text{likes}(x, y))$. It is a merit of our formal language that it allows us to express such non-trivial questions in a simple and unambiguous way.

[4] Contrast this with the erotetic logic of Belnap and Steel (1976)—arguably the most ambitious proposal for a general logical language in which to regiment questions. Their language can indeed formalize many question types, but at the cost of introducing a very complex syntactic apparatus of question constructors, one for each question type. This is required because these constructors, unlike the InqBQ operators, are not part of a recursively defined semantics—they only occur as main operators in a sentence. By contrast, in InqBQ we have only two, very basic, question-forming operators, but these operators can be freely embedded within each other and within other logical operators, which leads to a system with considerable expressive power.

be the restriction of M to s, i.e., the model $M_{|s} = \langle s, D, I_{|s}\rangle$, where $I_{|s}$ is the restriction of I to worlds in s. For any assignment g and formula φ we have:

$$M, s \models_g \varphi \iff M_{|s}, s \models_g \varphi.$$

One feature of the propositional system InqB that does *not* carry over to the first-order setting is normality, i.e., the identity $[\varphi]_M = \textsc{Alt}_M(\varphi)^{\downarrow}$.

5.3.4. Proposition (Failure of normality).
There is a model M, a state s, and a sentence $\varphi \in \mathcal{L}^{\mathsf{Q}}$ such that $s \models \varphi$ but s is not included in any alternative for φ in M.

Proof. Consider a signature consisting only of a predicate symbol P, and a model M given as follows:

$$W = \{w_n \,|\, n \in \mathbb{N}\}, \qquad D = \mathbb{N}, \qquad P_{w_n} = \{i \in \mathbb{N} \mid i \geq n\}.$$

For any number $k \in \mathbb{N}$, let us define an information state $s_k = \{w_0, \ldots, w_k\}$. Now consider the formula $\varphi := \exists x Px$. We have:

$$\begin{aligned} s \models \exists x Px \quad &\iff \quad \text{for some } k \in \mathbb{N} : s \models_{x \mapsto k} Px \\ &\iff \quad \text{for some } k \in \mathbb{N}, \text{ for all } w_n \in s : w_n \models_{x \mapsto k} Px \\ &\iff \quad \text{for some } k \in \mathbb{N}, \text{ for all } w_n \in s : k \in P_{w_n} \\ &\iff \quad \text{for some } k \in \mathbb{N}, \text{ for all } w_n \in s : k \geq n \\ &\iff \quad \text{for some } k \in \mathbb{N} : s \subseteq s_k. \end{aligned}$$

Thus, we have a sequence $s_0 \subset s_1 \subset s_2 \subset \ldots$ of information states, each properly included in the next, such that:

- every state s_k in the sequence supports φ;
- every state s supporting φ is included in some element s_k of the sequence.

From this it follows that in our model, there is no maximal supporting state for φ. For suppose s supports φ. Then $s \subseteq s_k$ for some k, and since $s_k \subset s_{k+1}$, s is strictly included in the information state s_{k+1}, which also supports φ. Hence s is not a *maximal* supporting state.

Thus, there are no alternatives for φ in M. Since there *are* supporting states for φ in M (for instance, each state s_k) this is a violation of normality. □

This shows that, unlike in the propositional setting, in the first-order setting the proposition expressed by a sentence in a model is not, in general, fully captured by its set of alternatives.[5] There are, however, significant syntactic fragments of InqBQ for which normality still holds, as we will see in Section 5.7.1

5.3.2 Truth-conditional formulas

Recall that we call a formula φ *truth-conditional* if in any state s of any model, φ is supported at s just in case it is true at every world in s. Also recall that we refer to truth-conditional formulas as *statements*, and to formulas that are not truth-conditional as *questions*.

We saw above that classical formulas $\varphi \in \mathcal{L}_c^{\mathsf{Q}}$ are always truth-conditional. Conversely, as in the propositional case, we can show that any truth-conditional formula in InqBQ is equivalent to a classical formula. In order to show this, we first associate to each formula φ a classical formula φ^{cl} having the same truth conditions as φ.

5.3.5. Definition. [Classical variant of a first-order formula]
If $\varphi \in \mathcal{L}^{\mathsf{Q}}$, the *classical variant* of φ is the formula $\varphi^{cl} \in \mathcal{L}_c^{\mathsf{Q}}$ obtained by replacing each occurrence of $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ by $\vee$, and each occurrence of $\exists\!\!\!\exists$ by $\exists$.

5.3.6. Proposition.
For any $\varphi \in \mathcal{L}^{\mathsf{Q}}$, model M, world w and assignment g: $w \models_g \varphi \iff w \models_g \varphi^{cl}$.

If φ itself is truth-conditional, this implies that φ and φ^{cl} are equivalent. Thus, any truth-conditional formula is equivalent to a classical formula. Conversely, if a formula φ is equivalent to a classical formula, then since classical formulas are truth-conditional, φ must be truth-conditional as well.

5.3.7. Proposition. *The following are equivalent for any $\varphi \in \mathcal{L}^{\mathsf{Q}}$:*

- *φ is truth-conditional;*
- *$\varphi \equiv \varphi^{cl}$;*
- *$\varphi \equiv \alpha$ for some $\alpha \in \mathcal{L}_c^{\mathsf{Q}}$.*

[5] Note that it follows from Proposition 2.4.9 that, in this case, there is no minimal generator for the proposition expressed by φ. However, there is still a natural way to recursively assign to each first-order formula φ a (proper) generator $T_M(\varphi)$ for the proposition $[\varphi]_M$ that it expresses in a model, i.e., a set of states such that $T_M(\varphi)^{\downarrow} = [\varphi]_M$ (cf. Section 6 in Ciardelli, 2009). This line of work is closely related with the definition of exact verification in the intuitionistic truth-maker semantics of Fine (2014).

This shows that, while our question-forming operators $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ and $\exists$ obviously add to the expressive power of the language, enabling us to express questions, they do not allow us to express any new statements.[6]

In Chapter 2, we saw that, like classical formulas, negations are always truth-conditional. Since negation works in exactly the same way in the first-order setting, this is still true in InqBQ.

5.3.8. PROPOSITION. *For any $\varphi \in \mathcal{L}^{\mathsf{Q}}$, $\neg\varphi$ is truth-conditional.*

In particular, for any formula φ, $\neg\neg\varphi$ is truth-conditional. Moreover, since truth conditions work in the standard way, $\neg\neg\varphi$ has the same truth-conditions as φ. Thus, $\neg\neg\varphi$ must be equivalent with the classical variant φ^{cl}.

5.3.9. PROPOSITION. *For any $\varphi \in \mathcal{L}^{\mathsf{Q}}$, $\neg\neg\varphi \equiv \varphi^{cl}$.*

This shows that negations, just like classical formulas, are representative of all truth-conditional formulas in InqBQ.

5.3.10. PROPOSITION. *The following are equivalent for any $\varphi \in \mathcal{L}^{\mathsf{Q}}$:*

- *φ is truth-conditional;*
- *$\varphi \equiv \neg\neg\varphi$;*
- *$\varphi \equiv \neg\psi$ for some $\psi \in \mathcal{L}^{\mathsf{Q}}$.*

Now consider questions, which by definition are not truth-conditional. If μ is a question, the classical variant μ^{cl} is not equivalent to μ: rather, μ^{cl} is a statement which expresses the presupposition of μ (cf. Section 2.6). As in Chapter 3, we will refer to μ^{cl} as *the presupposition* of μ. Let us illustrate this notion by means of two examples.

First, consider the formula $\exists xPx$, which as we saw captures the question *what is an instance of a P*. Its presupposition is the formula $\exists xPx$. This is a statement that captures precisely the conditions under which the question admits a resolution: it is in principle possible to provide an instance of a P if and only if there are objects satisfying P.

Next, consider the formula $\forall x?Px$, which captures the question *what is the extension of P*. Spelling out the question mark operator, the formula is $\forall x(Px \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \neg Px)$, and thus its presupposition is the formula $\forall x(Px \vee \neg Px)$, which is a tautology. This corresponds to the fact that the question $\forall x?Px$ can be settled under any circumstances.

[6] This is not to be taken for granted, as we will see when sketching inquisitive modal logic in Chapter 8. In that setting, by embedding questions under modalities we can express statements which have no counterpart in the classical fragment of the language. In other words, in inquisitive modal logic questions contribute to the expressivity of the language with respect to statements.

5.3.3 Resolutions?

A key property of the propositional logic InqB is that we can associate any formula φ with a set $\mathcal{R}(\varphi)$ of classical formulas such that to settle φ is to establish that α is true for some $\alpha \in \mathcal{R}(\varphi)$. Is something similar possible the first-order case? That is, can we define for each formula $\varphi \in \mathcal{L}^{\mathsf{Q}}$ a set $\mathcal{R}(\varphi)$ of classical formulas with the property that for any M and g, the connection

$$s \models_g \varphi \iff s \models_g \alpha \text{ for some } \alpha \in \mathcal{R}(\varphi)$$

holds? The answer is negative. One reason is that we are not assured a priori to have the means to rigidly designate every individual in the given model. Thus, even if a state s settles of some individual d that it has property P—thus supporting the question $\exists x Px$—we may not be able to trace this to the support of a classical formula, because we may lack a name for the individual d.

We may try to obviate this problem by extending the language with rigid names for all entities in the domain of the given model, and then by giving a set of resolutions $\mathcal{R}_M(\varphi)$ relativized to M, which has the above property for states s in M. But even this does not give us for all formulas φ a set $\mathcal{R}_M(\varphi)$ with the required properties. To see this, consider a mention-all question $\forall x ?Px$, which as we saw asks for the extension of property P. Even if we have names for all individuals in the domain, we do not in general have the syntactic means to describe all possible extensions for P: if the domain D is countably infinite and all possibilities for the extension of P are instantiated in the model, there will be uncountably many such extensions, but only countably many formulas in our language if our initial signature is countable. Now suppose we have no formula in our language stating that the extension of P is a certain set $X \subseteq D$: if s is a non-empty information state containing all and only the worlds in which the extension of P is X, then the question $\forall x ?Px$ will be supported at s, but this will not be traceable to the support of any classical formula in our language.

5.4 Adding identity

Definitions. Let us now see how identity can be introduced into the picture. Syntactically, this is straightforward: given a signature $\mathcal{S}$, we consider languages $\mathcal{L}_c^{\mathsf{Q}=}(\mathcal{S})$ and $\mathcal{L}^{\mathsf{Q}=}(\mathcal{S})$ which are defined just like $\mathcal{L}_c^{\mathsf{Q}}(\mathcal{S})$ and $\mathcal{L}^{\mathsf{Q}}(\mathcal{S})$, but with the addition of atomic formulas of the form $(t = t')$, where $t, t' \in \mathsf{Ter}(\mathcal{S})$.

Semantically, to interpret identity we equip a relational information model with a function $\sim$ which assigns to each world $w \in W$ the extension of the identity relation at w, denoted $\sim_w$. This is required to be an equivalence relation on D and a congruence with respect to the interpretation of function and relation symbols. That is, we require that for each world w in the model:

- for any n-ary function symbol f, if $d_1 \sim_w d'_1, \ldots, d_n \sim_w d'_n$, then

$$f_w(d_1, \ldots, d_n) \sim_w f_w(d'_1, \ldots, d'_n);$$

- for any n-ary relation symbol R, if $d_1 \sim_w d'_1, \ldots, d_n \sim_w d'_n$, then

$$\langle d_1, \ldots, d_n \rangle \in R_w \iff \langle d'_1, \ldots, d'_n \rangle \in R_w.$$

The semantics of identity atoms is then parallel to that of other atomic sentences:

- $M, s \models_g (t = t') \iff$ for all $w \in s$, $[t]^w_g \sim_w [t']^w_g$.

All the facts that we have stated so far about InqBQ carry over immediately to the language extended with identity.

Conceptual motivation. Our formal treatment of identity is built around the idea that there can be uncertainty about the extension identity. At first, this might seem strange. Surely we know a priori what the extension of identity is: every individual is identical to itself, and not identical to any other individual.

Things, however, become more subtle in a setting where the objects to which information is attributed in an information state are not necessarily in a one-to-one correspondence with the objects that actually exist in the world. Let us call the former *epistemic individuals*, and the latter *ontic individuals*.

The importance of distinguishing the two is well illustrated by Frege's puzzle. Consider an ancient astronomer who has just discovered the identity of Hesperus and Phosphorus. Before the discovery, this astronomer had some information about Hesperus (say, that it is visible in the evening), and some information about Phosphorus (say, that it is visible in the morning), yet he did not know whether Hesperus and Phosphorus were in fact two distinct objects, or one and the same object. Thus, our astronomer was in an information state s such that:

$$s \models \text{evening}(\mathsf{h}), \qquad s \models \text{morning}(\mathsf{p}), \qquad s \not\models ?(\mathsf{h} = \mathsf{p}).$$

Note that the astronomer's uncertainty is not about linguistic facts; in fact, we need not even suppose that the astronomer has named the relevant objects. Instead, it is about astronomical facts: he is uncertain about what astronomical objects there actually are—whether there are two of them or just one.

We can conceptualize the astronomer's information state s as involving two epistemic individuals h and p, each of which is known to have certain properties, and as leaving open the issue of whether these individuals are in fact the same. Thus, s contains worlds w such that $h \sim_w p$ (that is, worlds where h and p are in fact the same object) as well as worlds v such that $h \not\sim_v p$ (that is, worlds where h and p are in fact distinct). An example of such an information state is shown in Figure 5.4.

$h \not\sim_w p$	$h \sim_v p$
$\text{evening}_w = \{h\}$	$\text{evening}_v = \{h, p\}$
$\text{morning}_w = \{p\}$	$\text{morning}_v = \{h, p\}$
w	v

Figure 5.4: The information state representing the uncertainty of the ancient astronomer prior to learning whether Hesperus is Phosphorus.

Epistemic vs. ontic individuals. Our perspective in this section involves thinking of the domain D of an information model as a set of *epistemic* individuals: objects to which information is attached and which may be found out to be identical, or to be distinct, as more information about the world is acquired. The actual individuals existing at a possible world w are not the objects in D themselves, but instead the equivalence classes $\{[d]_{\sim_w} \mid d \in D\}$. Thus, even though we think of the domain D as fixed and common to all worlds, it is still possible to model situations in which one is uncertain about how many objects there are, as a result of uncertainty about the identity relation. Notice also that, if two individuals are in fact the same at a world, then they must of course have the same properties at that world, and applying a function to them should give identical results: this is what motivates the requirement that the relation $\sim_w$ be a congruence with respect to predicates and function symbols.

This discussion implies that, in the context of a model $M = \langle W, D, I, \sim\rangle$ for the language including identity, the relational structure associated to a world is not given simply by the structure $M_w = \langle D, I\rangle$, but rather by the quotient of this structure modulo the congruence $\sim_w$. Let us make this precise.

5.4.1. Definition. Let $M = \langle W, D, I, \sim\rangle$ be a model for the language $\mathcal{L}^{\mathsf{Q}=}$. The relational structure associated with a world w is $M_w^{\sim} = \langle D_w^{\sim}, I_w^{\sim}\rangle$, where:

- $D_w^{\sim} = \{[d]_{\sim_w} \mid d \in D\}$ is the set of equivalence classes modulo $\sim_w$;
- $I_w^{\sim}(f)([d_1]_{\sim_w}, \dots, [d_n]_{\sim_w}) = [I_w(f)(d_1, \dots, d_n)]_{\sim_w}$;
- $\langle [d_1]_{\sim_w}, \dots, [d_n]_{\sim_w}\rangle \in I_w^{\sim}(R) \iff \langle d_1, \dots, d_n\rangle \in I_w(R)$.

Since $\sim_w$ is a congruence, this model is well-defined, in the sense that the definitions do not depend on the choice of representatives within an equivalence class. The fact that our semantics generalizes the standard Tarskian semantics can then be stated as follows in the setting of the language $\mathcal{L}^{\mathsf{Q}=}$.

5.4.2. Proposition. *For any world w in a relational information model M, any assignment g into M, and any classical formula $\alpha \in \mathcal{L}_c^{Q=}$:*

$$M, w \models_g \alpha \iff M_w^{\sim} \models_{g_w} \alpha \text{ in standard Tarskian semantics},$$

where g_w is the assignment given by $g_w(x) = [g(x)]_{\sim_w}$.

Proof. Straightforward by induction on φ, using the definition of the structure $M_w^{\sim}$ for the atomic case and Proposition 5.1.6 for complex formulas. □

The following observation about identity will turn out useful in the following: once it is settled that $t = t'$, replacement of t by t' in a formula preserves support.

5.4.3. Proposition (Substitution of known identicals).
Let $\varphi \in \mathcal{L}^{Q=}$ and let t, t' be two terms free for x in φ. For any M, s and g:

$$s \models_g (t = t') \quad \Longrightarrow \quad (\ s \models_g \varphi[t/x] \iff s \models_g \varphi[t'/x]\ ).$$

Proof. Straightforward by induction on φ, using the fact that $\sim_w$ is required to be a congruence. □

id-models. While we allowed for the possibility of uncertainty about the identity relation, it is often the case that one wants to model scenarios where identity is not at stake—situations in which one knows what individuals there are and is merely uncertain about their properties. Such scenarios are captured by models in which the relation $\sim_w$ at each world simply coincides with the identity $id_D = \{\langle d, d\rangle \mid d \in D\}$. If our information model M is of this sort, we will say that M is an id-model.

5.4.4. Definition. [id-models]
An information model $M = \langle W, V, I, \sim\rangle$ is an id-model if for all $w \in W$, $\sim_w = id_D$.

In the setting of id-models, the semantics of identity atoms can be simplified:

- $M, s \models_g t = t' \iff \forall w \in s : [t]_g^w = [t']_g^w$.

As we shall see in Section 5.5.5, restricting to id-models has repercussions on the logic.

Questions involving identity. By means of identity, some interesting classes of questions become expressible, in addition to those discussed in Section 5.2. Let us look at some examples.

5.4.5. EXAMPLE. [Identification questions]
Consider the sentence $\exists x(x = t)$, where t is a term not containing x. We have:

$$\begin{aligned} s \models_g \exists x(x=t) \quad &\iff \quad \text{there is a } d \in D \text{ such that } s \models_{g[x\mapsto d]} (x=t) \\ &\iff \quad \text{there is a } d \in D \text{ such that for all } w \in s,\ [t]_g^w \sim_w d. \end{aligned}$$

Thus, our sentence is supported in a state s in case s establishes of some speicific d that it is identical to the referent of the term t. In an id-model, this clause can be simplified as follows:

$$\begin{aligned} s \models_g \exists x(x=t) \quad &\iff \quad \text{there is a } d \in D \text{ such that for all } w \in s,\ [t]_g^w = d \\ &\iff \quad \text{for all } w, w' \in s,\ [t]_g^w = [t]_g^{w'}. \end{aligned}$$

Thus, $\exists x(x = t)$ is a question which asks to identify the referent of t. For instance, a question such as "who is Bob's sister?" can be rendered formally by the formula $\exists x(x = s(b))$.

It will be convenient to introduce a notation for such identification questions. If t is a term, we let

$$\lambda t := \exists x(x = t)$$

where x is a variable not occurring in t.[7]

Notice that if t is a *rigid* term, then in any model, the referent of t is bound to be the same at every world. So, in this case the support conditions for $\exists x(x = \mathsf{t})$ are always satisfied. Thus, for rigid terms, identification questions are trivial. Note that this does not mean that *identity* questions $?(\mathsf{t} = \mathsf{t}')$ involving such terms are trivial, as illustrated by our discussion of Frege's puzzle above.

5.4.6. EXAMPLE. [Unique-instance questions]
For another interesting example of the sort of questions expressible by means of identity, let us introduce the following abbreviation, analogous to the one commonly used for the classical existential quantifier:

$$\exists! x\varphi(x) \;:=\; \exists x \forall y(\varphi(y) \leftrightarrow y = x).$$

Now consider the formula $\exists! xP(x)$, where P is a predicate symbol. We have:

$$\begin{aligned} s \models_g \exists! xP(x) \quad &\iff \quad \text{there is a } d \in D \text{ such that for all } w \in s: \\ &\qquad\quad d \in P_w \text{ and for all } d' \in P_w, d' \sim_w d \\ &\iff \quad \text{there is a } d \in D \text{ s.t. for all } w \in s:\ I_w^{\sim}(P) = \{[d]_{\sim_w}\}. \end{aligned}$$

[7] Strictly speaking, this underdetermines the formula λt, but this does not matter, since of course formulas which differ only by a renaming of bound variables are equivalent.

In the setting of id-models, this can be simplified as follows:

$$
\begin{aligned}
s \models_g \exists! x P(x) \quad &\iff \quad \text{there is a } d \in D \text{ such that for all } w \in s: \; P_w = \{d\} \\
&\iff \quad \text{the extension of } P \text{ is the same singleton at each } w \in s.
\end{aligned}
$$

Thus, the sentence $\exists! x P(x)$ is supported in a state s in case s establishes of some individual $d \in D$ that d is *the* individual who has property P.[8] Thus, $\exists x! P(x)$ formalizes the question 'who is the P?', which presupposes that exactly one individual has property P and asks for the identity of this individual. Note that, as we expect, the presupposition of this question is $\exists! x Px$ (short for $\exists x \forall y (Px \leftrightarrow x = y)$), the statement that exactly one individual satisfies P.

If we regard the model of Figure 5.3 as an id-model, where the two individuals a and b are distinct at every world, then the formula $\exists! x P(x)$ has two alternatives, as depicted in Figure 5.3(d). These alternatives coincide with the truth-sets of the classical formulas $\forall x(P(x) \leftrightarrow x = \mathsf{a})$ and $\forall x(P(x) \leftrightarrow x = \mathsf{b})$, each of which provides just enough information to establish of some individual that it is the unique P.[9]

5.4.7. Example. [Mention-n questions]
By means of the identity predicate, one can also form questions that ask for more than one instance of objects satisfying a given predicate. For instance, consider the formula

$$\exists x \exists y (Px \land Py \land \neg(x = y)).$$

It is easy to check that this formula is supported at a state s in case there are two individuals d, d' such that s implies that both d and d' have property P ($d, d' \in P_w$ for all $w \in s$) and that d and d' are distinct ($d \not\sim_w d'$ for all $w \in s$). Thus the above sentence is supported by a state s just in case the information in s provides us with at least two instances of property P, and it can be seen as a formalization of the mention-two question 'What are two individuals that

[8] More precisely, in models with variable identity, $\exists! x P(x)$ is supported in s if s establishes of some individual d that it is the only *ontic* individual having property P. That is, s establishes that if any other $d' \in D$ has property P, then d' and d are actually the same individual. This is sensible, as it is possible to have the information that Hesperus is the only planet visible in a certain position in the evening, and thus to have settled the unique-answer question what the relevant planet is ($\exists! x(\text{evening}(x))$), while *not* having settled whether Phosphorus is a planet visible in the evening (as one could be uncertain whether Hesperus is Phosphorus).

[9] With such questions we also touch upon a limitation of the system InqBQ—though not of the inquisitive approach as such: as discussed in detail by Aloni (2001), one and the same *wh*-questions can express different contents depending on the intended method of identification of the relevant individuals. Given two cards lying face down on a table, the question "What is the winning card" means different things depending on whether the intended answer is 'the one on the left/right' or 'the ace of spades/hearts'. Our logic is unequipped to deal with this source of context-dependency, but it can be combined smoothly with Aloni's theory of conceptual covers, as shown in §3 of van Gessel (2020a).

have property P?'. Of course, the idea can be extended straightforwardly to mention-n questions for $n > 2$.

5.4.8. Example. [Cardinality questions]
As a last example of the sort of questions that we can form by using identity, consider the following sentence:

$$?\exists!xPx \;:=\; ?\exists x\forall y(Py \leftrightarrow y = x).$$

This is a polar question that asks whether the statement $\exists!xPx$ is true. The statement is true at a world w just in case there is exactly one (ontic) individual that has property P (i.e., if the actual extension of P at world w, given by the set $P_w/{\sim_w}$, has cardinality 1). Thus, the above question asks whether exactly one object is P. Similarly, for any natural number n it is easy to write polar questions asking whether at most/at least/exactly n objects are P. On the other hand, as we will discuss in Section 5.8, it is *not* possible to write a formula that expresses the related question "How many things are P?/How many P are there?", which is settled in a state s just in case the information in s determines exactly how many individuals have property P.

5.5 Entailment

Let us now turn to the entailment relation in InqBQ, which is defined in the obvious way: Φ logically entails ψ (notation: $\Phi \models \psi$) if for any information model M, information state s and assignment g, if $M, s \models_g \Phi$ then $M, s \models_g \psi$.

Logical equivalence and validity are defined in terms of entailment as usual. Two formulas φ and ψ are *logically equivalent*, denoted $\varphi \equiv \psi$, if they entail each other—which amounts to φ and ψ having the same support conditions. A formula φ is *valid* in InqBQ, denoted $\models \varphi$, if it is entailed by the empty set—which amounts to φ being supported by any information state in any model under any assignment.

Contextual entailment is also defined as usual, modulo a relativization to assignments: Φ entails ψ in the context of an information state s and relative to an assignment g (notation: $\Phi \models_{s,g} \psi$) in case for every information state $t \subseteq s$, if $t \models_g \Phi$ then $t \models_g \psi$. As usual, implication is tightly connected to contextual entailment: $\varphi \to \psi$ is supported in a state s just in case φ entails ψ in the context of s:

$$s \models_g \varphi \to \psi \iff \varphi \models_{s,g} \psi.$$

5.5.1 Illustration

In order to appreciate the sort of logical facts that can be captured as entailments in the logic InqBQ, we will first look at some examples. Then in the next section we will examine more closely the formal properties of the entailment relation.

5.5.1. Example. Consider two unary predicates, P and Q. Given the information that P is the complement of Q, the extension of P determines the extension of Q. This fact is an instance of logical dependency that can be captured as a case of entailment in InqBQ. The assumption that P is the complement of Q can be formalized as usual by the formula $\forall x(Px \leftrightarrow \neg Qx)$. The mention-all questions *what is the extension of* P and *what is the extension of* Q are formalized, as discussed in Section 5.2, by the formulas $\forall x?Px$ and $\forall x?Qx$. Thus, the logical dependency that we observed above amounts to the validity of the following entailment:

$$\forall x(Px \leftrightarrow \neg Qx), \forall x?Px \models \forall x?Qx.$$

We can see that this entailment is valid by reasoning as follows. Take a state s which supports the premises. In order to support the second premise, the extension of P must be the same at every world in s. In order to support the first premise, the extension of Q must be the complement of the extension of P at every world in s. It follows that the extension of Q is the same at every world in s, which means that s supports the conclusion.

5.5.2. Example. Consider a unary predicate P. Given the information which individuals have property P we can in particular determine whether or not all individuals have property P. This is fact is captured by the entailment:

$$\forall x?Px \models ?\forall xPx.$$

We can see that this is valid as follows. Suppose a state s supports $\forall x?Px$. Then the extension of P is the same at every world in s. If this extension of P is the entire domain D at every world, then s supports $\forall xPx$. Otherwise, the extension of P is different from D at every world, and then s supports $\neg\forall xPx$. Either way, s supports $?\forall xPx$.

Notice that the entailment is valid because the domain of quantification is fixed: if uncertainty about the domain of quantification were allowed, then even given a specification of the extension of P, one could still be uncertain about whether or not the relevant extension is the entire domain.

5.5.3. Example. Consider a unary predicate P. Under the assumption that the extension of P is non-empty, from the information about what the extension of P is we can obtain an instance of an object that has property P. Here, the assumption that P is non-empty is captured as usual by $\exists xPx$. The question of what is the extension of P is expressed by $\forall x?Px$, and the question of what is an instance of P is expressed by $\exists xPx$. The above observation then amounts to the entailment:

$$\exists xPx, \forall x?Px \models \exists xPx.$$

We can see that this entailment is valid by reasoning as follows. Take a state s that supports the premises (we may assume $s \neq \emptyset$, since the empty set supports every formula). Then the extension of P must be the same set X of individuals at every world in s (second premise) and this set must be non-empty (first premise). If we then take an object $d \in X$, d is in the extension of P at every world in s, and so the conclusion is supported.

5.5.4. Example. Consider a rigid binary function symbol f and three non-rigid individual constants a, b, c. The rigidity of f means that we are assuming the denotation of f to be known. Then if we are given the information that $a = \mathsf{f}(b, c)$ as well as information identifying b and c, it follows that we can identify a. Recall that we abbreviate the identification question $\exists x(x = a)$ as λa, and similarly for b and c. Then the above fact is captured by the following entailment:

$$a = \mathsf{f}(b, c),\ \lambda b,\ \lambda c \models \lambda a.$$

While the entailment is valid in general, its validity is particularly easy to verify in the setting of an id-model. In this setting, suppose s is a state that supports λb and λc. Then the referent of b and c must be the same individuals d_b and d_c in every world in s. Since f is rigid, it denotes the same function F at all worlds in s. Therefore, the term $\mathsf{f}(b, c)$ must also denote the same individual $d' = F(d_b, d_c)$ at every world w. If s also supports $a = \mathsf{f}(b, c)$, the denotation of a at each world must be d', and thus in particular it must be the same at every world in s. This guarantees that the conclusion λa is supported.

In the case of a model with variable identity, the argument is essentially the same, but the relevant identities have to be computed locally at each world $w \in s$, using the relation $\sim_w$. The details are left to the reader.

In case we are dealing with a non-rigid function symbol f, the above entailment is no longer valid: even if we are given the values of b and c and the information that $a = f(b, c)$, if we do not know what function f denotes we will not in general be able to identify the value of a. However, we can retrieve the entailment if we add the explicit assumption that for every x and y we can identify the value of $f(x, y)$, which is captured by the formula $\forall x \forall x \exists z(z = f(x, y))$. This results in the following valid entailment:

$$\forall x \forall y \exists z(z = f(x, y)),\ a = f(b, c),\ \lambda b,\ \lambda c \models \lambda a.$$

5.5.2 Entailments with truth-conditional conclusions

Many of the features of inquisitive entailment that we discussed in the setting of propositional logic carry over straightforwardly to the first-order case. To start with, entailment towards truth-conditional formulas is truth-conditional.

5.5.5. Proposition (Entailment to a truth-conditional conclusion). *Let $\Phi \cup \{\alpha\} \subseteq \mathcal{L}^{\mathsf{Q}=}$, where α is truth-conditional. We have:*

$$\Phi \models \alpha \iff \textit{for any model } M, \textit{ world } w, \textit{ assignment } g : w \models_g \Phi \textit{ implies } w \models_g \psi.$$

The proof is the same as in the propositional case (cf. Proposition 3.7.2). In particular, since classical formulas are truth-conditional, and since their conditions are the standard ones (Proposition 5.1.6), entailment among classical formulas coincides with entailment in classical first-order logic. Thus, InqBQ is a conservative extension of classical first-order logic.

5.5.6. Proposition (Conservativity over classical first-order logic). *If $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}_c^{\mathsf{Q}=}$, then $\Gamma \models \alpha \iff \Gamma$ entails α in classical first-order logic.*

As in the propositional case, Proposition 5.5.5 implies that, when the conclusion is truth-conditional, any assumption φ may just as well be replaced by its classical variant φ^{cl}.

5.5.7. Proposition. *If α is truth-conditional, for any Φ we have $\Phi \models \alpha \iff \Phi^{cl} \models \alpha$.*

It follows, in particular, that every formula entails its classical variant.

5.5.8. Corollary. *For every $\varphi \in \mathcal{L}^{\mathsf{Q}=}$, $\varphi \models \varphi^{cl}$.*

Moreover, Proposition 5.5.7 implies that a question μ entails all and only the statements that follow from its presupposition. Thus, for instance, the question $\exists xPx$ entails its presupposition $\exists xPx$, any statements that follow from it, and no other statements. For another example, consider the mention-all question $\forall x?Px$: we saw that the presupposition of this question is a tautology; it follows that tautologies are the only statements entailed by $\forall x?Px$.

5.5.3 Entailments with truth-conditional premises

Let us now turn to entailments from truth-conditional premises. The characterization given for propositional logic carries over: a set Γ of truth-conditional formulas entails a formula φ in case, in any model M and with respect to any assignment g, φ is settled in the specific state $|\Gamma|_M^g = \{w \in W_M \mid w \models_g \Gamma\}$ which corresponds to the information that the formulas in Γ are true.

5.5.9. Proposition (Entailment from truth-conditional assumptions). *Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{Q}}$, where all formulas in Γ are truth-conditional. We have:*

$$\Gamma \models \varphi \iff \textit{for all models } M \textit{ and assignments } g, \ |\Gamma|_M^g \models_g \varphi.$$

In particular, as we saw in Chapter 1, to say that a statement α entails a question μ is to say that in any model, the information that α is true suffices to settle the question. For instance, suppose $\mathtt{t}$ is a rigid term: the statement $P\mathtt{t}$ entails the question $\exists x Px$, since in any model, the information that $P\mathtt{t}$ is true suffices to identify an individual which has property P (namely, that individual which is referent of $\mathtt{t}$ in the model), and thus suffices to settle the question $\exists x Px$.

The property that we called *specificity* (cf. Proposition 3.7.12) also carries over: if Γ is a set of truth-conditional formlas, then Γ entails φ in the context s just in case extending s with the information that all formulas in Γ are true leads to a state that supports φ. The proof is the same as in the propositional case.

5.5.10. Proposition (Specificity). *Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{Q=}$, where Γ is a set of truth-conditional formulas. For any model M, state s, and assignment g:*

$$\Gamma \models_{s,g} \varphi \iff s \cap |\Gamma|^g_M \models_g \psi.$$

Using this fact, it is immediate to check that the local split property still holds for $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$, and an analogous property holds for $\exists$ as well.

5.5.11. Proposition (Local split properties).
Let Γ be a set of truth-conditional formulas, and let φ, ψ be arbitrary formulas. Then for any model M, information state s and assignment g we have:

- $\Gamma \models_{s,g} \varphi \vee\!\!\!\vee \psi \iff \alpha \models_{s,g} \varphi$ *or* $\alpha \models_{s,g} \psi$;
- $\Gamma \models_{s,g} \exists x\varphi \iff$ *for some* $d \in D : \Gamma \models_{s,g[x\mapsto d]} \varphi$, *provided* $x \notin FV(\Gamma)$.

Proof. The proof of the first item is identical to the one given in the propositional case. For the second item, we have

$$\begin{aligned}
\Gamma \models_{s,g} \exists x\varphi &\iff s \cap |\Gamma|^g_M \models_g \exists x\varphi \\
&\iff \text{for some } d \in D : s \cap |\Gamma|^g_M \models_{g[x\mapsto d]} \varphi \\
&\iff \text{for some } d \in D : s \cap |\Gamma|^{g[x\mapsto d]}_M \models_{g[x\mapsto d]} \varphi \\
&\iff \text{for some } d \in D : \Gamma \models_{s,g[x\mapsto d]} \varphi
\end{aligned}$$

where the first and last step use Specificity, while the third step uses the fact that $|\Gamma|^g_M = |\Gamma|^{g[x\mapsto d]}_M$ because x does not occur free in Γ. □

As in the propositional case, these properties amount to the validity of certain logical equivalences, which allow us to distribute a truth-conditional antecedent over an inquisitive consequent.

5.5.12. PROPOSITION (SPLIT EQUIVALENCES).
Let α be a truth-conditional formula, and let φ, ψ be arbitrary formulas. We have:

- $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ split: $(\alpha \to \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi) \equiv (\alpha \to \varphi) \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} (\alpha \to \psi)$;
- $\exists$ split: *if* $x \notin FV(\alpha)$, $(\alpha \to \exists x\varphi) \equiv \exists x(\alpha \to \varphi)$.

Proof. Again, we spell out only the case for $\exists$, since the one for $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ is the same as in propositional logic. Take any model M, state s, and assignment g. Using the previous proposition as well as the connection between implication and contextual entailment, we have:

$$\begin{aligned}
s \models \alpha \to \exists x\varphi \quad &\iff \quad \alpha \models_{s,g} \exists x\varphi \\
&\iff \quad \text{for some } d \in D : \alpha \models_{s,g[x\mapsto d]} \varphi \\
&\iff \quad \text{for some } d \in D : s \models_{g[x\mapsto d]} \alpha \to \varphi \\
&\iff \quad s \models_g \exists x(\alpha \to \varphi).
\end{aligned}$$

$\square$

We also have logical counterparts of the split properties. For the case of disjunction, the relevant property is the same as in propositional logic. The proof strategy is also the same: we take two countermodels and combine them into a single one by a disjoint union construction. However, in the current setting the relevant construction must be a bit more subtle, due to the requirement to get a single common domain for the two models being combined.

5.5.13. THEOREM (LOGICAL SPLIT PROPERTY FOR $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ (GRILLETTI, 2019)).
Let $\Gamma \cup \{\varphi, \psi\} \subseteq \mathcal{L}^{\mathsf{Q}=}$, where Γ is a set of truth-conditional formulas. Then:

$$\Gamma \models \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi \iff \Gamma \models \varphi \textit{ or } \Gamma \models \psi.$$

Proof. Given two relational information models, $M^A = \langle W^A, D^A, I^A, \sim^A \rangle$ and $M^B = \langle W^B, D^B, I^B, \sim^B \rangle$, we define a new model $M^B \oplus M^B = \langle W, D, I, \sim \rangle$, where:

- W is the disjoint union of W^A and W^B: $W = W^A \uplus W^B$.
- D is the Cartesian product of D^A and D^B: $D = D^A \times D^B$.
- For every relation symbol R and world w:
  - if $w \in W^A$, $I_w(R)(\langle a_1, b_1 \rangle, \dots, \langle a_n, b_n \rangle) \iff I^A_w(R)(a_1, \dots, a_n)$;
  - if $w \in W^B$, $I_w(R)(\langle a_1, b_1 \rangle, \dots, \langle a_n, b_n \rangle) \iff I^B_w(R)(b_1, \dots, b_n)$.
- Similarly, for any world w:

– if $w \in W^A$, $\langle a_1, b_1\rangle \sim_w \langle a_2, b_2\rangle \iff a_1 \sim_w^A a_2$;
– if $w \in W^B$, $\langle a_1, b_1\rangle \sim_w \langle a_2, b_2\rangle \iff b_1 \sim_w^B b_2$.

- For a non-rigid function symbol f:

$$I_w(f)(\langle a_1, b_1\rangle, \dots, \langle a_n, b_n\rangle) = \begin{cases} \langle I_w^A(f)(a_1, \dots, a_n), b_0\rangle & \text{if } w \in W^A \\ \langle a_0, I_w^B(f)(b_1, \dots, b_n)\rangle & \text{if } w \in W^B \end{cases}$$

  where a_0 is an arbitrary element of D^A and b_0 an arbitrary element of D^B.

- For a rigid function symbol f:

$$I_w(\mathsf{f})(\langle a_1, b_1\rangle, \dots, \langle a_n, b_n\rangle) = \langle F^A(a_1, \dots, a_n), F^B(b_1, \dots, b_n)\rangle$$

  where $F^A = I_{w'}^A(\mathsf{f})$ for an arbitrary world $w' \in W^A$, and $F^B = I_{w''}^B(\mathsf{f})$ for an arbitrary world $w'' \in W^B$ (since f is rigid, the choice of w' and w'' does not matter). Note that the interpretation of f given in this way is indeed rigid, i.e., it yields the same individual for any world w in the model.

It is straightforward to check that $\sim_w$ is indeed a congruence at every world, which guarantees that $M^A \oplus M^B$ is a relational information model.

The crucial feature of the sum model $M^A \oplus M^B$ is that it behaves like the model M^A on states $s \subseteq W^A$, and like the model M^B on states $s \subseteq W^B$. To make this precise, let $\pi_1 : D^A \times D^B \to D^A$ and $\pi_2 : D^A \times D^B \to D^B$ be the natural projection functions. Then for any assignment g into $M^A \oplus M^B$, any information state s in this model, and any formula χ we have:

- if $s \subseteq W^A$ then: $M^A \oplus M^B, s \models_g \chi \iff M^A, s \models_{\pi_1 \circ g} \chi$;
- if $s \subseteq W^B$ then: $M^A \oplus M^B, s \models_g \chi \iff M^B, s \models_{\pi_2 \circ g} \chi$.

The inductive verification of the claim is left as an exercise.

With this model-theoretic construction at hand, we are ready to prove our theorem. By contraposition, suppose $\Gamma \not\models \varphi$ and $\Gamma \not\models \psi$. Then we can find models $M^A = \langle W^A, D^A, I^A, \sim^A\rangle$ and $M^B = \langle W^B, D^B, I^B, \sim^B\rangle$, and corresponding information states s^A, s^B and assignments g^A and g^B such that:

- $M^A, s^A \models_{g^A} \Gamma, \quad M^A, s^A \not\models_{g^A} \varphi$;
- $M^B, s^B \models_{g^B} \Gamma, \quad M^B, s^B \not\models_{g^B} \psi$.

Now consider the model $M^A \oplus M^B$ and the information state $s = s^A \uplus s^B$ obtained as the disjoint union of s^A and s^B. Also, define a valuation function g by setting $g(x) = \langle g^A(x), g^B(x)\rangle$ and notice that $g^A = \pi_1 \circ g$ and $g^B = \pi_2 \circ g$. By the above property of the sum model, we have:

- $M^A \oplus M^B, s^A \models_g \Gamma, \quad M^A \oplus M^B, s^A \not\models_g \varphi$;
- $M^A \oplus M^B, s^B \models_g \Gamma, \quad M^A \oplus M^B, s^B \not\models_g \psi$.

By persistency, since s^A and s^B are substates of $s^A \uplus s^B$ we have $M^A \oplus M^B, s^A \uplus s^B \not\models_g \varphi$ and $M^A \oplus M^B, s^A \uplus s^B \not\models_g \psi$, and therefore also:

$$M^A \oplus M^B, s^A \uplus s^B \not\models_g \varphi \vee\!\!\!\vee \psi.$$

At the same time, consider an arbitrary formula $\alpha \in \Gamma$ and a world $w \in s^A \uplus s^B$. Suppose $w \in s^A$: since $M^A \oplus M^B, s^A \models_g \alpha$, by persistency we have $M^A \oplus M^B, w \models_g \alpha$. The same conclusion can be reached in case $w \in s^B$, using the fact that $M^A \oplus M^B, s^B \models_g \alpha$. Thus, for all $w \in s^A \uplus s^B$ we have $M^A \oplus M^B, w \models_g \alpha$. Since α is truth-conditional, it follows that $M^A \oplus M^B, s^A \uplus s^B \models_g \alpha$. And since this is the case for all $\alpha \in \Gamma$, we have: $M^A \oplus M^B, s^A \uplus s^B \models_g \Gamma$.

Thus, we can conclude that $\Gamma \not\models \varphi \vee\!\!\!\vee \psi$. □

We can also prove a logical split property for the inquisitive existential quantifier: whenever a set Γ of truth-conditional formulas entails an inquisitive existential formula $\exists x\varphi(x)$, this is traceable to the fact that it entails $\varphi(\mathtt{t})$ for some rigid term $\mathtt{t}$. However, the proof of this fact is complex, involving non-trivial model-theoretic constructions. The interested reader is referred to Grilletti (2019) for the details.

5.5.14. Theorem (Logical split property for $\exists$, (Grilletti, 2019)). *Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{Q}=}$, where all formulas in Γ are truth-conditional. Then:*

$$\Gamma \models \exists x\varphi \iff \Gamma \models \varphi[\mathtt{t}/x] \text{ for some rigid term } \mathtt{t} \text{ free for } x \text{ in } \varphi.$$

Note that by taking $\Gamma = \emptyset$ in the above theorems we obtain for the inquisitive operators the disjunction and existence property familiar from intuitionistic logic and arithmetic.

5.5.15. Corollary (Disjunction and existence property). *For all formulas $\varphi, \psi \in \mathcal{L}^{\mathsf{Q}=}$:*

- *if $\varphi \vee\!\!\!\vee \psi$ is valid then φ is valid or ψ is valid;*
- *if $\exists x\varphi$ is valid then $\varphi[\mathtt{t}/x]$ is valid for some rigid term $\mathtt{t}$.*

Note also that if a certain signature Σ does not contain any rigid function symbols, the only rigid terms in the language are variables. Thus, in this case if $\Gamma \models \exists x\varphi$ we can conclude $\Gamma \models \varphi[y/x]$ for some variable y free for x in φ. If additionally Γ is a set of sentences, we have $\Gamma \models \varphi[y/x]$ only in case $\Gamma \models \forall x\varphi$. So, we also have the following corollary.

5.5.16. Corollary. *Let $\Gamma \cup \{\psi\} \subseteq \mathcal{L}^{\mathsf{Q}=}(\Sigma)$, where Σ contains no rigid function symbols and Γ is a set of truth-conditional sentences. Then:*

$$\Gamma \models \exists x\varphi \iff \Gamma \models \forall x\varphi.$$

5.5.4 The role of rigidity

The logical properties of a term depend crucially on whether or not the term is rigid. Rigid terms yield witnesses for the inquisitive existential quantifier, and allow instantiation from the universal quantifier.

5.5.17. Proposition. *For any $\varphi(x) \in \mathcal{L}^{\mathsf{Q}=}$ and any rigid term t which is free for x in φ we have:*

- $\varphi(t) \models \exists x\varphi(x)$;
- $\forall x\varphi(x) \models \varphi(t)$.

Proof. Consider an arbitrary information state s and assignment g. Since t is rigid, there is an object $d_{\mathsf{t}} \in D$ such that in all worlds $w \in s$, $[\mathsf{t}]^g_w = d_{\mathsf{t}}$. This means that in every world $w \in s$, $[\mathsf{t}]^g_w = [x]^{g[x\mapsto d_{\mathsf{t}}]}_w$. Using this fact, it is straightforward to check by induction that for any formula $\varphi(x)$ we have:

$$s \models_g \varphi(\mathsf{t}) \iff s \models_{g[x\mapsto d_{\mathsf{t}}]} \varphi(x).$$

Now suppose $s \models_g \varphi(\mathsf{t})$. Then $s \models_{g[x\mapsto d_{\mathsf{t}}]} \varphi(x)$, and so $s \models_g \exists x\varphi(x)$. This proves the first entailment.

Next, suppose $s \models_g \forall x\varphi(x)$. Then in particular we have $s \models_{g[x\mapsto d_{\mathsf{t}}]} \varphi(x)$, and thus also $s \models_g \varphi(\mathsf{t})$. This shows the second entailment. □

To see that the above proposition does not hold in general if t is non-rigid, consider a unary predicate symbol P and a non-rigid constant c. We have

$$P(c) \not\models \exists xP(x).$$

A counterexample to the entailment is given by the model in Figure 5.5(a) on page 155: the state $\{w, v\}$ in the model supports $P(c)$ but not $\exists xP(x)$. Intuitively, the entailment fails because establishing that c has property P is not sufficient to identify an object with property P, if we do not know what object c refers to. For instance, suppose c stands for 'the thief' and P for the property 'having stolen the jewels'. Having the information that the thief stole the jewels ($P(c)$) does not allow us to settle who stole the jewels ($\exists xP(x)$) unless we know who the thief is (i.e., unless our state supports $\exists x(x = c)$).

Similarly, if c is non-rigid we also have:

$$\forall x?P(x) \not\models ?P(c).$$

A counterexample is given by the model of Figure 5.5(b): the information state $\{w, v\}$ in the model supports $\forall x?P(x)$ but not $?P(c)$. Again, this failure is intuitively motivated: knowing the extension of P is not enough to know whether

(a) $P(c) \not\models \exists x P(x)$ (b) $\forall x ?P(x) \not\models ?P(c)$

Figure 5.5: Two countermodels showing the invalidity of the schemata $\varphi(t) \models \exists x\varphi(x)$ and $\forall x\varphi(x) \models \varphi(t)$ when t is non-rigid. In both models, the domain is $D = \{a, b\}$, c is a non-rigid constant and P a unary predicate.

the object denoted by c is in this extension, unless we know what this object is. For instance, suppose again that c stands for 'the thief' and P for 'being in this room'. We may know that only Alice is in this room, and so know exactly what the extension of P is. However, we may not know whether the thief is in this room, since we may not know whether Alice is the thief.

On the other hand, since we saw that InqBQ is a conservative extension of classical first-order logic, it follows that all terms, rigid or not, bear the standard relation to the classical quantifiers, in the context of classical formulas. Conceptually, this is because the semantics of classical formulas can be assessed in a point-wise way, world by world, and at the level of a single world, there is no difference between rigid and non-rigid terms.

5.5.18. Proposition. *For any classical formula $\alpha(x) \in \mathcal{L}_c^{Q=}$ and any term t which is free for x in φ, whether rigid or not, we have:*

- $\alpha(t) \models \exists x\alpha(x)$; *(notice the classical quantifier!)*
- $\forall x\alpha(x) \models \alpha(t)$.

In fact, for the case of $\exists$ the restriction to classical formulas is inessential, as the following proposition shows.

5.5.19. Proposition. *For any formula $\varphi(x) \in \mathcal{L}^{Q=}$ and any term t free for x in φ we have $\varphi(t) \models \exists x\varphi(x)$.*

Proof. Recall that $\exists x\varphi(x)$ is an abbreviation for $\neg\forall x\neg\varphi(x)$. By Proposition 5.3.8, negations are always truth-conditional, and by Proposition 5.3.7, truth-conditional formulas are logically equivalent to their classical variant. Thus, $\exists x\varphi(x) \equiv (\exists x\varphi(x))^{cl} = \exists x\varphi^{cl}(x)$. By Corollary 5.5.8, $\varphi(t) \models \varphi^{cl}(t)$. Since φ^{cl} is a classical formula, by the previous proposition we have $\varphi^{cl}(t) \models \exists x\varphi(x)$. Putting everything together, we have $\varphi(t) \models \varphi^{cl}(t) \models \exists x\varphi^{cl}(x) \equiv \exists x\varphi(x)$. $\square$

5.5.5 id-entailment

If we are only interested in cases where the extension of identity is not at stake, we will want to work with a stronger notion of entailment, one that only takes id-models into account. We will refer to this stronger notion as id-entailment.

5.5.20. Definition. [id-entailment]
$\Phi \models_{\mathsf{id}} \psi \iff$ for all id-models M, all states s in M, and all assignments g, $M, s \models_g \Phi$ implies $M, s \models_g \psi$.

We refer to the corresponding notions of equivalence and validity as id-equivalence (denoted $\equiv_{\mathsf{id}}$) and id-validity.

In an id-model, matters concerning identities of individuals are assumed to be settled *a priori*. This is captured by the fact that the question $\forall x \forall y ?(x = y)$, which asks to specify the extension of the identity relation, is an id-validity.

5.5.21. Proposition. *$\forall x \forall y ?(x = y)$ is id-valid.*

Proof. Spelling out the support conditions of the question, we have:

$$s \models \forall x \forall y ?(x = y) \iff \text{for all } w, w' \in s : \sim_w = \sim_{w'} .$$

This condition is always satisfied in an id-model, since in such a model $\sim_w$ coincides with the identity relation at every $w \in W$. □

Since universals can always be validly instantiated to a rigid term (Proposition 5.5.17), this also gives the following corollary.

5.5.22. Corollary. *For any rigid terms t, t', $?(\mathsf{t} = \mathsf{t}')$ is id-valid.*

This corollary can be generalized into the following proposition, which says that the truth value of a statement concerning only identities between rigid terms is settled *a priori* in an id-model.

5.5.23. Proposition. *Let α be a classical formula built up from identity atoms of the form $(\mathsf{t} = \mathsf{t}')$ where t, t' are rigid. Then $?\alpha$ is id-valid.*

Proof. Consider any id-model M. If t,t' are rigid then all worlds in M assign the same truth-value to the atom $(\mathsf{t} = \mathsf{t}')$ relative to every assignment. Since truth conditions for classical formulas can be computed recursively in the standard way, all worlds in M assign the same truth value to any classical formula α built up from such atoms. This means that every state s in M supports the polar question $?\alpha$ (recall that $?\alpha$ is supported precisely if all worlds in the state agree on the truth value of α). □

A consequence of this proposition is that, in the setting of an id-model, characterizing a predicate in terms of identities with rigid terms counts as settling its extension.

5.5.24. PROPOSITION.
Let α and β be classical formulas, where α is as in the previous proposition, and let $\overline{x}$ be a sequence of variables. Then:

$$\forall \overline{x}(\alpha \leftrightarrow \beta) \models_{\mathit{id}} \forall \overline{x}?\beta.$$

Proof. It is easy to verify that the following entailment is generally valid, and thus also id-valid:

$$\forall \overline{x}(\alpha \leftrightarrow \beta), \forall \overline{x}?\alpha \models \forall \overline{x}?\beta.$$

By the previous proposition, the formula $?\alpha$ is id-valid, and thus so is its universal closure $\forall \overline{x}?\alpha$. Obviously, adding id-valid premises does not make a difference to the validity of an id-entailment, so our claim follows. □

As notable special cases we have the following, where $\mathsf{t}_1, \dots, \mathsf{t}_n$ are rigid:

- $\forall x(P(x) \ \leftrightarrow \ (x = \mathsf{t}_1 \vee \dots \vee x = \mathsf{t}_n)) \models_{\mathsf{id}} \forall x?P(x)$;
- $\forall x(P(x) \ \leftrightarrow \ (x \neq \mathsf{t}_1 \wedge \dots \wedge x \neq \mathsf{t}_n)) \models_{\mathsf{id}} \forall x?P(x)$.

To get a sense of the differences between general entailment and id-entailment, it is helpful to consider a couple of further examples.

5.5.25. EXAMPLE. Take any natural number $n \geq 1$. Consider the standard first-order sentence that says that there are exactly n individuals in the domain:

$$\chi_n := \exists x_1 \dots \exists x_n (\bigwedge_{i<j} (x_i \neq x_j) \wedge \forall y(y = x_1 \vee \dots \vee y = x_n)).$$

In an id-model, this is either true at every world (if the cardinality of D is n) or false at every world (if the cardinality of D is not n). In the former case, χ_n is supported at any state in the model; in the latter case, $\neg\chi_n$ is supported at any state in the model. Either way, the polar question $?\chi_n$ is supported at any state in the model. This shows that this question is id-valid: $\models_{\mathsf{id}} ?\chi_n$.

By contrast, the polar question $?\chi_n$ is not generally valid in InqBQ. To see this, consider a model M with set of possible worlds $W = \{w_1, w_2, w_3, \dots\}$, domain $D = \mathbb{N}$, and such that $x \sim_{w_n} y$ holds iff x and y are equivalent modulo n. In this model, the sentence χ_n is true at world w_n but false at world w_m for $m \neq n$. Thus, the polar question $?\chi_n$ is not supported by the state W.

5.5.26. EXAMPLE. Let $\mathsf{a}_1, \mathsf{a}_2, \mathsf{b}_1, \mathsf{b}_2$ be rigid constants, and let Γ be the set of the following classical sentences:

- $\forall x(Px \to (x = \mathsf{a}_1 \vee x = \mathsf{a}_2))$,
- $P\mathsf{a}_1 \to Q\mathsf{b}_1$,

- $P\mathsf{a}_2 \to Q\mathsf{b}_2$.

Given Γ, it might at first seem that from a witness for P we can derive a witness for Q, and so, that we should have $\Gamma, \exists xPx \models \exists xQx$. However, this is not the case. For suppose we are given the information that d is a witness of P: we know from Γ that either $d = [\mathsf{a}_1]$, in which case $[\mathsf{b}_1]$ is a witness for Q, or $d = [\mathsf{a}_2]$, in which case $[\mathsf{b}_2]$ is a witness for Q. But if we cannot tell which of these two cases applies, we are in effect unable to produce a witness for Q. And indeed, it is not hard to show that:

$$\Gamma,\ \exists xPx \not\models \exists xQx.$$

On the other hand, in the context of an id-model, the relevant dependency holds, since it is guaranteed that given the information that d has property P, we will be able to tell whether $d = [\mathsf{a}_1]$ or $d = [\mathsf{a}_2]$, and thus to produce a corresponding witness for Q. And indeed, it is easy to check that:

$$\Gamma,\ \exists xPx \models_{\mathsf{id}} \exists xQx.$$

Failure of the logical split properties. It is interesting to note that the disjunction property for $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ and the existence property for $\exists$, given by Corollary 5.5.15 in the context of general entailment, both fail for id-entailment.

For the disjunction property, note that the formula $?(x = y)$ is id-valid and amounts to the disjunction $(x = y) \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \neg(x = y)$, but obviously neither $x = y$ nor $\neg(x = y)$ are id-valid. Alternatively, consider again the cardinality sentence χ_n from Example 5.5.25, which says that there are exactly n distinct individuals: we saw above that $?\chi_n$ is id-valid, but again neither χ_n nor $\neg\chi_n$ is id-valid.

We can also use the same idea to build a counterexample to the existence property for $\exists$. For instance, consider the formula

$$\exists x(((x = y) \land \chi_2) \lor ((x = z) \land \neg\chi_2)).$$

To see that this is valid on id-models, take an arbitrary id-model M, an information state s, and an assignment g. There are two possibilities: either the cardinality of the domain D is 2, or it is different from 2. In the first case, since M is an id-model we have $s \models_g \chi_2$. Then by choosing $d = g(y)$ we have that $s \models_{g[x\mapsto d]} (x = y) \land \chi_2$, from which it follows that $s \models_g \exists x(((x = y) \land \chi_2) \lor ((x = z) \land \neg\chi_2))$. If the cardinality of D is different from 2, then again since M is an id-model we have $s \models_g \neg\chi_2$. Then by choosing $d = g(z)$ we have $s \models_{g[x\mapsto d]} (x = z) \land \neg\chi_2$, and again it follows that $s \models_g \exists x(((x = y) \land \chi_2) \lor ((x = z) \land \neg\chi_2))$. In either case, our formula is supported, which shows that this formula is id-valid.

However, we claim that there is no rigid term t such that the instantiation

$$((\mathsf{t} = y) \land \chi_2) \lor ((\mathsf{t} = z) \land \neg\chi_2)$$

is id-valid. Indeed, let t be any rigid term. If $\mathsf{t} \neq y, z$, it is obvious that the resulting formula is not going to be valid, since then we can easily give an id-model and assignment which satisfy $\neg(\mathsf{t} = y) \wedge \neg(\mathsf{t} = z)$, which ensures that our disjunction is not supported. So, we only have to consider the cases $\mathsf{t} = y$ and $\mathsf{t} = z$. For $\mathsf{t} = y$ we get the formula

$$((y = y) \wedge \chi_2) \vee ((y = z) \wedge \neg\chi_2).$$

This is not a validity: to refute it, take a model M where D contains more than two elements, and an assignment g such that $g(y) \neq g(z)$. Then at any world w in M, χ_2 is false and $(y = z)$ is false, so the above disjunction is false, which implies that this disjunction is not valid.

Similarly, for $\mathsf{t} = z$ we get $((z = y) \wedge \chi_2) \vee ((z = z) \wedge \neg\chi_2)$, which is not a validity either: to refute it, just take a model where D contains exactly two elements and an assignment g such that $g(y) \neq g(z)$.

Thus, the above inquisitive existential is id-valid while no instantiation of it with a rigid term is id-valid—in contrast with the existence property that characterizes our general notion of validity.

General entailment and id-entailment. What is the exact relation between general and id-entailment? To answer this question, let us first introduce the auxiliary notion of a model with decidable identity. This is a model where there is no uncertainty about the extension of the identity relation.

5.5.27. Definition. [Decidable identity]
A model $M = \langle W, D, I, \sim\rangle$ has decidable identity if $\forall w, w' \in W : \sim_w = \sim_{w'}$.

Note that the formula $\forall x \forall y ?(x = y)$ characterizes a model as having decidable identity, in the following sense.

5.5.28. Remark. M has decidable identity $\iff M, W \models \forall x \forall y ?(x = y)$.

Clearly, id-models have decidable identity. On the other hand, a model M with decidable identity is not necessarily an id-model, since $\sim_w$ might be the same relation at every world while being different from the identity relation on D. However, in this case, one can always simplify M turn it into an id-model by taking its quotient modulo $\sim_w$.

5.5.29. Definition. [Turning a model with decidable identity into an id-model]
Let $M = \langle W, D, I, \sim\rangle$ have decidable identity. Let us write $\sim$ for $\sim_w$ where w is an arbitrary world, and let us write $[d]$ for the equivalence class of d modulo $\sim$. The *id-contract* of M is the id-model $M^{\mathsf{id}} = \langle W, D/{\sim}, I^{\sim}, \approx\rangle$, where:

- $D/{\sim} = \{[d] \,|, d \in D\}$;

- $I_w^{\sim}(f)([d_1], \ldots, [d_n]) = [I_w(f)(d_1, \ldots, d_n)]$;
- $\langle [d_1], \ldots, [d_n] \rangle \in I_w^{\sim}(R) \iff \langle d_1, \ldots, d_n \rangle \in I_w(R)$;
- $\approx_w$ is the identity relation on $D/\!\sim$ for any $w \in W$.

The fact that $\sim$ is a congruence at each world guarantees that this is a good definition, i.e., that the definition of $I_w^{\sim}(f)$ and $I_w^{\sim}(R)$ does not depend on the choice of representatives for each equivalence class.

The following proposition ensures that this transformation does not affect the satisfaction of formulas. The straightforward proof is omitted.

5.5.30. PROPOSITION.
Let M have decidable identity and let s be a state in M and g a valuation into M. Let $g^{\mathsf{id}} : \mathsf{Var} \to W/\!\sim$ be the valuation $x \mapsto [g(x)]$. For any formula $\varphi \in \mathcal{L}^{\mathsf{Q}=}$:

$$M, s \models_g \varphi \iff M^{\mathsf{id}}, s \models_{g^{\mathsf{id}}} \varphi.$$

We can now prove that the relation between general entailment and id-entailment is simple: id-entailment can be simulated within general entailment by adding the decidability of identity as an extra premise.

5.5.31. PROPOSITION (SIMULATING ID-ENTAILMENT).
For any $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\mathsf{Q}=}$, $\Phi \models_{\mathsf{id}} \psi \iff \Phi, \forall x \forall y ?(x = y) \models \psi$.

Proof. The right-to-left direction of the theorem follows immediately from the fact that $\forall x \forall y ?(x = y)$ is id-valid. For the converse, we reason by contraposition. Suppose $\Phi, \forall x \forall y ?(x = y) \not\models \psi$. Then, there must be a model $M = \langle W, D, I, \sim \rangle$, a state s, and an assignment g such that $M, s \models_g \Phi$ and $M, s \models_g \forall x \forall y ?(x = y)$, but $M, s \not\models_g \psi$. Now consider the restriction of M to s, $M_{|s}$. By Locality (Proposition 5.3.3) we have $M, s \models_g \chi \iff M_{|s}, s \models_g \chi$ for any formula χ. Since the universe of $M_{|s}$ is s and we have $M_{|s}, s \models_g \forall x \forall y ?(x = y)$, by Remark 5.5.28 the model $M_{|s}$ has decidable identity. Thus, as we just saw, by a quotient construction it can be turned into an id-model $(M_{|s})^{\mathsf{id}}$ based on the same universe that satisfies the same formulas at every state. In particular, we have $(M_{|s})^{\mathsf{id}}, s \models_{g^{\mathsf{id}}} \Phi$ but $(M_{|s})^{\mathsf{id}}, s \not\models_{g^{\mathsf{id}}} \psi$, which shows that $\Phi \not\models_{\mathsf{id}} \psi$. $\square$

Conversely, it is also possible to simulate general entailment within id-entailment. The trick is to treat non-rigid identity as a new predicate, adding axioms ensuring that it is interpreted as a congruence.

Formally, given a signature Σ we can consider a signature $\Sigma^{\asymp}$ which extends Σ with a fresh binary predicate $\asymp$ (we write $t \asymp t'$ instead of $\asymp(t, t')$). Now for any formula $\varphi \in \mathcal{L}^{\mathsf{Q}=}(\Sigma)$, consider the formula $\varphi^{\asymp} \in \mathcal{L}^{\mathsf{Q}}(\Sigma^{\asymp})$ obtained by replacing each identity atom $t = t'$ in φ by a corresponding atom $t \asymp t'$.

Now to an arbitrary model M for the signature Σ we can associate an id-model $M^{\asymp}$ for the extended signature $\Sigma^{\asymp}$ where identity is interpreted rigidly as $\{\langle d, d\rangle \mid d \in D\}$, but where the role of non-rigid identity is taken over by $\asymp$, i.e., for all worlds w we have $I_w(\asymp) = \sim_w$ where $\sim_w$ is the interpretation of identity at w in the model M. It is obvious from the construction that for any information state s in M and any assignment g we have:

$$M, s \models_g \varphi \iff M^{\asymp}, s \models_g \varphi^{\asymp}.$$

It is easy to check that the map $(\cdot)^{\asymp}$ gives a 1-1 correspondence between the class of models for Σ and the class of id-models for $\Sigma^{\asymp}$ where $\asymp$ is interpreted as a congruence at each world.

The fact that $\asymp$ is interpreted as a congruence at each world can be captured by a set of formulas $\text{Cong}^{\asymp}$ in the language $\mathcal{L}^{\mathsf{Q}}(\Sigma^{\asymp})$, containing the axioms of an equivalence relation, namely,

- $\forall x(x \asymp x)$,
- $\forall x \forall y(x \asymp y \ \to\ y \asymp x)$,
- $\forall x \forall y \forall z(x \asymp y \ \wedge\ y \asymp z \ \to\ x \asymp z)$,

in addition to the following formulas for all predicates R and function symbol f in the signature Σ:

- $\forall x_1 \dots x_n \forall y_1 \dots y_n(\bigwedge_{1\leq i\leq n}(x_i \asymp y_i) \to (R(x_1, \dots, x_n) \leftrightarrow R(y_1, \dots, y_n)))$,
- $\forall x_1 \dots x_n \forall y_1 \dots y_n(\bigwedge_{1\leq i\leq n}(x_i \asymp y_i) \to (f(x_1, \dots, x_n) \asymp f(y_1, \dots, y_n)))$.

We can now state exactly how general entailment can be simulated within id-entailment. The proof follows straightforwardly from the preceding discussion.

5.5.32. Proposition (Simulating general entailment).
For any $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\mathsf{Q}=}$, $\Phi \models \psi \iff \Phi^{\asymp} \cup \mathit{Cong}^{\asymp} \models_{\mathsf{id}} \psi^{\asymp}$.

5.5.6 Open problems

Some of the key questions about the meta-theoretical properties of InqBQ are currently open, in spite of much recent progress. In this section we briefly survey some of these questions. As we will see later on, these questions have recently been answered in restriction to several interesting fragments of the logic.

A first major question concerns the compactness of entailment in InqBQ.

5.5.33. Open problem. [Entailment compactness]
Let $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\mathsf{Q}=}$. Is it always the case that if $\Phi \models \psi$ there is a finite subset $\Phi_0 \subseteq \Phi$ such that $\Phi_0 \models \psi$?

It should be noted that compactness is oftentimes formulated not in terms of entailment, but in terms of satisfiability. While the two formulations are equivalent in the context of classical predicate logic, they come apart in the inquisitive setting. In the inquisitive setting, it is natural to call a set Φ of formulas *satisfiable* if there are a model M, a non-empty information state s, and an assignment g such that $M, s \models_g \Phi$. It is then easy to show that InqBQ is compact in the sense of satisfiability, as made precise by the following proposition.

5.5.34. Proposition (Satisfiability compactness).
Let $\Phi \subseteq \mathcal{L}^{\mathsf{Q}=}$. If every finite subset of Φ is satisfiable then Φ is satisfiable.

Proof. First note that by persistency, if $M, s \models_g \Phi$ then for all worlds $w \in s$ we have $M, \{w\} \models_g \Phi$, that is, $M, w \models_g \Phi$. This means that a set Φ is satisfiable just in case it is true at some world relative to some assignment. In particular, for a set Γ of classical formulas, satisfiability is just satisfiability in the sense of classical logic. By Proposition 5.3.6 we know that an arbitrary set of formulas Φ has the same truth conditions as the set Φ^{cl} obtained by replacing each formula $\varphi \in \Phi$ by its classical variant φ^{cl}. So, we have that Φ is satisfiable iff Φ^{cl} is satisfiable in classical first-order logic. Using this fact and the compactness of classical first-order logic, for an arbitrary set $\Phi \subseteq \mathcal{L}^{\mathsf{Q}=}$ we have:

$$\begin{aligned} \Phi \text{ is satisfiable} &\iff \Phi^{cl} \text{ is satisfiable} \\ &\iff \text{for all finite } \Gamma \subseteq \Phi^{cl} : \Gamma \text{ is satisfiable} \\ &\iff \text{for all finite } \Psi \subseteq \Phi : \Psi^{cl} \text{ is satisfiable} \\ &\iff \text{for all finite } \Psi \subseteq \Phi : \Psi \text{ is satisfiable.} \end{aligned}$$

□

A second major open problem concerns whether the set of InqBQ-validities is recursively enumerable, which is a prerequisite for the existence of a complete proof system (under the desideratum that proofs be finite and verifiable).

5.5.35. Open problem. [Recursive enumerability]
Let Σ be a countable signature. Is the set of InqBQ-validities from the language $\mathcal{L}^{\mathsf{Q}=}(\Sigma)$ recursively enumerable?

One way to address the previous question is to ask if there is a translation from InqBQ to classical first-order logic, in the sense made precise below.

5.5.36. Open problem. [Existence of a translation to first-order logic]
Given a signature Σ, is there a map $(\cdot)^* : \mathcal{L}^{\mathsf{Q}=}(\Sigma) \to \mathcal{L}_c^{\mathsf{Q}=}(\Sigma')$ from the language of InqBQ to the language of classical first-order logic based on some signature Σ' such that for all $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\mathsf{Q}=}(\Sigma)$ we have

$$\Phi \models \psi \iff \Phi^* \models \psi^*$$

where $\Phi^* = \{\varphi^* \mid \varphi \in \Phi\}$? (Notice that the entailment on the right-hand side of the biconditional is an entailment in classical first-order logic, since all formulas involved are classical.)

If an entailment-preserving translation to classical first-order logic exists, then since the validities of classical first-order logic are recursively enumerable, so are InqBQ validities.

We then have a number of interesting questions inspired by the Löwenheim-Skolem theorem, about the cardinalities of models and countermodels of a given InqBQ formula. I will just mention a simple example of such questions.

5.5.37. Open problem. [Logic of countable models]
Can any formula φ which is not valid in InqBQ be refuted in a countable model, i.e., in a model $M = \langle W, D, I, \sim\rangle$ with $\#W \leq \aleph_0$ and $\#D \leq \aleph_0$?

The difficulties that one faces in answering these questions have a clear source: they stem from the fact that in inquisitive logic, the implication connective $\to$ introduces a quantification over subsets of the evaluation state. This means that, if we regard an information model as a two-sorted relational structure with domains W and D, the semantic condition on a model defined by a formula φ involving implication is not, at least *prima facie*, a first-order condition. (Whether every φ in fact corresponds to a first-order condition is an open question at this stage.) Due to the presence of implication, the features of InqBQ depend in part of the structure of powerset algebras. Mathematical questions concerning powersets are notoriously difficult—when they admit of definite answers at all. This is well-known from the case of set theory: it is not possible to decide, based on the standard ZFC foundation, what is the cardinality of the powerset of a countably infinite set. But there are also examples of this phenomenon closer to home, in logic. Medvedev's logic (Medvedev, 1962, 1966) arises from a natural formalization of Kolmogorov interpretation of formulas as problems. Model-theoretically, Medvedev's logic can be characterized as the logic of models based on finite powerset structures deprived of the empty set (see Chagrov and Zakharyaschev, 1997). As in inquisitive logic, in Medvedev's logic implication is understood as a quantifier over subsets of the evaluation state. In fact, there are tight connections between Medvedev's logic and inquisitive logic (see Miglioli et al., 1989; Ciardelli, 2009; Ciardelli and Roelofsen, 2011). In spite of a significant amount of research over the last fifty years, which yielded partial results (Maksimova et al., 1979), it is still an open question whether Medvedev's logic admits a recursive axiomatization, and (equivalently) whether its set of validities is recursively enumerable.

5.6 Coherence

In this section, based on Ciardelli and Grilletti (2022), we look at an interesting semantic property that InqBQ-formulas may or may not have: finite coherence. The idea is simple: a formula φ is finitely coherent if there is a natural number n such that, in order to decide whether φ is supported at a state, it is sufficient to look at substates of size at most n. We first define a more general notion of κ-coherence where κ is an arbitrary cardinal.[10]

5.6.1. Definition. [κ-coherence]
For κ a cardinal, we say that a formula $\varphi \in \mathcal{L}^{\mathsf{Q}=}$ is *κ-coherent* if for any model M, state s, and assignment g we have

$$s \models_g \varphi \iff (t \models_g \varphi \text{ for all } t \subseteq s \text{ with } \#t \leq \kappa)$$

where $\#t$ is the cardinality of t. We call φ *coherent* if it is κ-coherent for some cardinal κ, and *finitely coherent* if it is n-coherent for some natural number n.

In the above definition, the left-to-right direction always holds by persistency, so κ-coherence amounts to the requirement that the converse hold as well. Note that truth-conditionality is a special case of n-coherence for $n = 1$. Furthermore, note that if φ is κ-coherent then it is also λ-coherent for all $\lambda \geq \kappa$. This justifies the following definition.

5.6.2. Definition. [Coherence degree]
The coherence degree of a coherent formula φ, denoted d_φ, is the least κ such that φ is κ-coherent (if no such κ exists, φ has no coherence degree).

In many cases, a bound for the coherence degree of a compound can be obtained from the coherence degrees of its components, as the following proposition shows.

5.6.3. Proposition. *Suppose φ is κ-coherent and ψ is κ'-coherent. Then:*

- *$\varphi \wedge \psi$ is λ-coherent for $\lambda = \max(\kappa, \kappa')$ (thus, $d_{\varphi\wedge\psi} \leq \max(d_\varphi, d_\psi)$);*
- *$\varphi \vvee \psi$ is λ-coherent for $\lambda = \kappa + \kappa'$ (thus, $d_{\varphi\vvee\psi} \leq d_\varphi + d_\psi$);*
- *$\chi \to \varphi$ is κ-coherent for any formula χ (thus, $d_{\chi\to\varphi} \leq d_\varphi$);*
- *$\forall x\varphi$ is κ-coherent (thus, $d_{\forall x\chi} \leq d_\chi$).*

[10] The notion of coherence was first considered in the context of dependence logic by Kontinen (2013), who used it to study the complexity of the model checking problem. Exercise 3.11.8 in Chapter 3 asked readers to consider coherence properties in the setting of inquisitive propositional logic InqB.

Proof. We prove the claim for $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ and leave the other cases as exercises to the reader (Exercise 5.9.6). We need to show that $\varphi \vee\!\!\vee \psi$ is λ-coherent for $\lambda = \kappa + \kappa'$. Take an arbitrary model M, information state s, and assignment g. We need to show that if for every subset $t \subseteq s$ of size at most λ we have $t \models_g \varphi \vee\!\!\vee \psi$, then $s \models_g \varphi \vee\!\!\vee \psi$. Contrapositively, suppose $s \not\models_g \varphi \vee\!\!\vee \psi$. Then $s \not\models_g \varphi$ and $s \not\models_g \psi$. Since φ is κ-coherent and ψ is κ'-coherent, there are substates $t, t' \subseteq s$ with $\#t \leq \kappa$, $\#t' \leq \kappa'$ such that $t \not\models_g \varphi$ and $t' \not\models_g \psi$. Now consider the state $t \cup t'$: this is a subset of s of cardinality at most $\kappa + \kappa' = \lambda$, and by persistency we have $t \cup t' \not\models_g \varphi \vee\!\!\vee \psi$.

Note that by definition of coherence degree, φ is d_φ-coherent and ψ is d_ψ-coherent. So by what we have just proved, $\varphi \vee\!\!\vee \psi$ is $d_\varphi + d_\psi$-coherent. Since $d_{\varphi \vee\!\!\vee \psi}$ is defined as the least cardinal for which $\varphi \vee\!\!\vee \psi$ is coherent, $d_{\varphi \vee\!\!\vee \psi} \leq d_\varphi + d_\psi$. □

Using this result, we can show that many formulas of InqBQ are finitely coherent: in particular, all $\exists$-free formulas are (this can be strengthened slightly, as we will see in Section 5.7.1.)

5.6.4. Proposition. *Every $\exists$-free formula of InqBQ is finitely coherent.*

Proof. By induction on the structure of the formula. Atomic formulas and $\bot$ are truth-conditional and so 1-coherent. The previous proposition implies immediately that all operators except for $\exists$ preserve finite coherence. □

Formulas involving $\exists$, on the other hand, are not in general finitely coherent. In fact, even the simplest examples of such formulas are not κ-coherent for any κ, finite or infinite, as the following proposition shows.

5.6.5. Proposition. *The formula $\exists x Px$ is not κ-coherent for any κ. That is, for every cardinal κ, there exists a model M and a state s such that $s \not\models \exists x Px$ and for all $t \subseteq s$ with $\#t \leq \kappa$ we have $t \models \exists x Px$.*

Proof. Consider an arbitrary cardinal κ, and indicate with κ^+ the cardinal successor of κ. Consider the model $M = \langle W, D, I \rangle$ given by:

- $W = \{w_i \mid i < \kappa^+\}$;
- $D = \{d_j \mid j < \kappa^+\}$;
- $d_j \in P_{w_i} \iff i \neq j$.

We have $M, W \not\models \exists x Px$: indeed, for every element $d_j \in D$ we have have $M, W \not\models P(d_j)$, since $d_j \notin P_{w_j}$. However, given any proper subset $t \subset W$ we have $M, t \models \exists x Px$: to see this, let w_j be a world such that $w_j \notin t$; then for any $w_i \in t$ we have $i \neq j$ and so $d_j \in w_i$, which implies $M, t \models P(d_j)$.

Since the cardinality of W is $\kappa^+ > \kappa$, any subset $t \subseteq W$ with $\#t \leq \kappa$ will be a proper subset of W and thus will support $\exists x Px$. Thus, we have found a state where $\exists x Px$ is not supported, while being supported at all subsets of cardinality up to κ. $\square$

The previous result shows that there are formulas that lack a coherence degree. On the other hand, with some work one may produce, for every $n \in \mathbb{N}$, a formula of InqBQ whose coherence degree is exactly n (see Ciardelli and Grilletti, 2022). We conjecture that these are the two only possibilities for formulas in InqBQ.

5.6.6. Conjecture (Dichotomy).
The coherence degree of a formula φ is either finite or undefined.

Finitely coherent formulas have a number of important properties. To start with, they enjoy the following finite universe property.[11]

5.6.7. Proposition. *Let $n \in \mathbb{N}$. If $\Phi \not\models \psi$ and ψ is n-coherent, the entailment can be falsified in a model M based on a finite universe W with $\#W \leq n$.*

Proof. Suppose $\Phi \not\models \psi$. Then there are M, s and g such that $M, s \models_g \Phi$ but $M, s \not\models_g \psi$. If ψ is n-coherent, there exists a state $t \subseteq s$ of size at most n such that $M, t \not\models_g \psi$. By persistency, $M, s \models_g \Phi$ implies $M, t \models_g \Phi$. Then $M_{|t}$, the restriction of M to t, is a model whose universe is t, and thus contains at most n worlds. By locality we have $M_{|t}, t \models_g \Phi$ but $M_{|t}, t \not\models_g \psi$. $\square$

Second, whereas formulas of InqBQ are not in general normal, finitely coherent formulas always are. Thus, it is no coincidence that the same formula $\exists x Px$ that we used in the proof of Proposition 5.3.4 as a counterexample to normality is also a counterexample to finite coherence.

5.6.8. Proposition (Finite coherence implies normality).
If φ is finitely coherent, then it is normal; that is, for every model M, state s and assignment g, if $M, s \models_g \varphi$ then $s \subseteq a$ for some alternative $a \in \textsc{Alt}^g_M(\varphi)$.

Proof. Take an arbitrary model M, information state s, and assignment g such that $M, s \models_g \varphi$. Consider the set S of states containing s and supporting φ,

$$S = \{t \subseteq W \mid s \subseteq t \text{ and } M, t \models_g \varphi\}.$$

We want to show that S contains a maximal element.

[11] Obviously, without further assumptions on the signature we cannot hope for a finite model property with respect to the domain D, since the set of 1-coherent formulas already includes all formulas of classical first-order logic, and we know that some of these formulas can only be falsified over infinite domains D.

For this, we first claim that for every non-empty chain $C \subseteq S$ we have $\bigcup C \in S$. Towards a contradiction, suppose this is not the case. Then we have a non-empty chain $C \subseteq S$ such that $\bigcup C \notin S$. Since $\bigcup C$ does include s, we must have $M, \bigcup C \not\models_g \varphi$. Since φ is finitely coherent, there must be a subset $t \subseteq \bigcup C$ of cardinality at most d_φ such that $M, t \not\models_g \varphi$. Since $t \subseteq \bigcup C$, every $w \in t$ is included in some element of the chain, and since t is finite, there must be an element $s' \in C$ of the chain such that $t \subseteq s'$. By persistency, since $M, t \not\models_g \varphi$ we also have $M, s' \not\models_g \varphi$. But this contradicts the hypothesis that $C \subseteq S$.

We have shown that every non-empty chain from S has an upper bound in S. By Zorn's lemma, S contains a maximal element a. This means that a is a maximal extension of s such that $M, a \models_g \varphi$, i.e., $s \subseteq a$ and $a \in \textsc{Alt}^g_M(\varphi)$. $\square$

Furthermore, InqBQ entailments with finitely coherent conclusions are compact, in the sense specified by the following proposition.

5.6.9. Theorem (Compactness for finitely coherent conclusions). *If $\Phi \models_{\mathsf{InqBQ}} \psi$ and ψ is finitely coherent, there exists a finite subset $\Phi_0 \subseteq \Phi$ such that $\Phi_0 \models_{\mathsf{InqBQ}} \psi$.*

In order to prove this result, we develop a family of maps from the language of InqBQ over the given signature Σ to the language of classical first-order logic over a modified signature Σ^*. These maps allow us to emulate the semantics of InqBQ within standard first-order logic, provided a finite upper bound to the size of information states is fixed in advance. This can then be used in combination with Proposition 5.6.7, which guarantees that given an entailment with a finitely coherent conclusion, such a finite bound on the size of the states can indeed be fixed without affecting the validity of the entailment. The remainder of this section spells out the details of the strategy.

As a first step, we associate to a signature Σ a corresponding signature Σ^* over two sorts, $\mathtt{w}$ for worlds and $\mathtt{e}$ for individuals. Σ^* is given as follows:

- For every n-ary predicate symbol $R \in \Sigma$, Σ^* contains a predicate symbol R^* of arity $n+1$ where the first argument is of sort $\mathtt{w}$ and the remaining arguments of sort $\mathtt{e}$.
- For every non-rigid n-ary function symbol $f \in \Sigma$, Σ^* contains a function symbol f^* of arity $n+1$ where the first argument is of sort $\mathtt{w}$ and the remaining arguments as well as the output are of sort $\mathtt{e}$.
- For every rigid n-ary function symbol $\mathsf{f} \in \Sigma$, Σ^* contains a function symbol f^* of arity n where the arguments and the output are of sort $\mathtt{e}$.

Denote by $\mathcal{L}^{\mathsf{FOL}}_{\mathtt{w},\mathtt{e}}(\Sigma^*)$ the language of two-sorted first-order logic over Σ^*. We use $\mathsf{w}, \mathsf{w}_0, \mathsf{w}_1, \dots$ for variables of type $\mathtt{w}$ in the latter language, and $x, y, \dots$ for variables of type $\mathtt{e}$, which we assume to be the same as the variables of $\mathcal{L}^{\mathsf{Q}=}(\Sigma)$.

Next, we associate to a relational information model $M = \langle W, D, I\rangle$ for the signature Σ a two-sorted relational structure $M^* = \langle W, D, I^*\rangle$ for Σ^*, where:

- For a predicate symbol R: $I^*(R^*)(w, d_1, \dots, d_n) \iff I_w(R)(d_1, \dots, d_n)$;
- For a non-rigid function symbol f: $I^*(f^*)(w, d_1, \dots, d_n) = I_w(f)(d_1, \dots, d_n)$;
- For a rigid function symbol f: $I^*(\mathsf{f}^*)(d_1, \dots, d_n) = I_w(\mathsf{f})(d_1, \dots, d_n)$ for an arbitrary $w \in W$.

It is easy to verify that the map $M \mapsto M^*$ is a bijection between relational information models for Σ and two-sorted relational structures for Σ^*.

The next step is to translate terms. Given a term t of $\mathcal{L}^{\mathsf{Q}=}(\Sigma)$ and a world variable w, we define a corresponding term t_{w} of type e of the language $\mathcal{L}^{\mathsf{Q}=}(\Sigma)$ inductively as follows:

- if t is a variable x then $t_{\mathsf{w}} = x$;
- if $t = f(t^1, \dots, t^n)$ where f is non-rigid then $t_{\mathsf{w}} = f^*(\mathsf{w}, t^1_{\mathsf{w}}, \dots, t^n_{\mathsf{w}})$;
- if $t = \mathsf{f}(t^1, \dots, t^n)$ where f is rigid then $t_{\mathsf{w}} = \mathsf{f}^*(t^1_{\mathsf{w}}, \dots, t^n_{\mathsf{w}})$.

It is straightforward to check that for any relational information model M, assignment g, and term t of $\mathcal{L}^{\mathsf{Q}=}(\Sigma)$ we have

$$[t]^M_{w,g} = [t_{\mathsf{w}}]^{M^*}_{g[\mathsf{w}\mapsto w]}$$

where $g[\mathsf{w} \mapsto w]$ is an arbitrary assignment that coincides with g on the variables of type e and maps the variable w to w.

Finally, let $\mathsf{s} = \{\mathsf{w}_1, \dots, \mathsf{w}_n\}$ be a finite nonempty set of world variables. We define for each formula $\varphi \in \mathcal{L}^{\mathsf{Q}=}(\Sigma)$ a translation $\mathsf{tr}_{\mathsf{s}}(\varphi) \in \mathcal{L}^{\mathsf{FOL}}_{\mathsf{w,e}}(\Sigma)$ as follows:

$$\begin{array}{lcl}
\mathsf{tr}_{\mathsf{s}}(R(t^1, \dots, t^k)) & = & R^*(\mathsf{w}_1, t^1_{\mathsf{w}_1}, \dots, t^k_{\mathsf{w}_1}) \wedge \dots \wedge R^*(\mathsf{w}_n, t^1_{\mathsf{w}_n}, \dots, t^k_{\mathsf{w}_n}) \\
\mathsf{tr}_{\mathsf{s}}(\bot) & = & \bot \\
\mathsf{tr}_{\mathsf{s}}(\varphi \wedge \psi) & = & \mathsf{tr}_{\mathsf{s}}(\varphi) \wedge \mathsf{tr}_{\mathsf{s}}(\psi) \\
\mathsf{tr}_{\mathsf{s}}(\varphi \vee\!\!\!\vee \psi) & = & \mathsf{tr}_{\mathsf{s}}(\varphi) \vee \mathsf{tr}_{\mathsf{s}}(\psi) \\
\mathsf{tr}_{\mathsf{s}}(\varphi \to \psi) & = & \bigwedge\{\mathsf{tr}_{\mathsf{s}'}(\varphi) \to \mathsf{tr}_{\mathsf{s}'}(\psi) \mid \emptyset \neq \mathsf{s}' \subseteq \mathsf{s}\} \\
\mathsf{tr}_{\mathsf{s}}(\forall x\varphi) & = & \forall x\, \mathsf{tr}_{\mathsf{s}}(\varphi) \\
\mathsf{tr}_{\mathsf{s}}(\exists x\varphi) & = & \exists x\, \mathsf{tr}_{\mathsf{s}}(\varphi).
\end{array}$$

We spell out one example by way of illustration. We have

$$\begin{array}{ll}
 & \mathsf{tr}_{\mathsf{s}}(\forall x(Px \vee\!\!\!\vee Qx)) \\
= & \forall x(\mathsf{tr}_{\mathsf{s}}(Px) \vee \mathsf{tr}_{\mathsf{s}}(Qx)) \\
= & \forall x((P^*(\mathsf{w}_1, x) \wedge \dots \wedge P^*(\mathsf{w}_n, x)) \vee (Q^*(\mathsf{w}_1, x) \wedge \dots \wedge Q^*(\mathsf{w}_n, x))).
\end{array}$$

The key property of the map tr_{s} is given by the following proposition. We omit the proof, which is a matter of straightforward case-by-case verification.

5.6.10. PROPOSITION. *Let M be a relational information model, g an assignment, and $s = \{w_1, \ldots, w_n\}$ a finite nonempty state. Let $\mathsf{s} = \{\mathsf{w}_1, \ldots, \mathsf{w}_n\}$ be a set of n world variables and let $g[\mathsf{s} \mapsto s]$ be any two-sorted assignment that coincides with g on variables of type e and which maps the world variable w_i to w_i for $i = 1, \ldots, n$. For any formula $\varphi \in \mathcal{L}^{Q=}(\Sigma)$ we have:*

$$M, s \models_g \varphi \iff M^* \models_{g[\mathsf{s} \mapsto s]} \mathsf{tr}_\mathsf{s}(\varphi).$$

The following proposition shows that, although the maps tr_s are not in general translations from InqBQ to standard first-order logic, they preserve the validity of entailments whose conclusion is n-coherent for $n = \#\mathsf{s}$.

5.6.11. PROPOSITION. *Let $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{Q=}(\Sigma)$ where ψ is n-coherent for $n \in \mathbb{N}$. We have:*

$$\Phi \models_{\mathsf{InqBQ}} \psi \iff \mathsf{tr}_\mathsf{s}(\Phi) \models_{\mathsf{FOL}} \mathsf{tr}_\mathsf{s}(\psi),$$

where $\models_{\mathsf{FOL}}$ denotes entailment in first-order logic, $\mathsf{s} = \{\mathsf{w}_1, \ldots, \mathsf{w}_n\}$ is an arbitrary set of n world variables, and $\mathsf{tr}_\mathsf{s}(\Phi) = \{\mathsf{tr}_\mathsf{s}(\varphi) \mid \varphi \in \Phi\}$.

Proof. Suppose $\Phi \not\models_{\mathsf{InqBQ}} \psi$ and suppose ψ is n-coherent. By Proposition 5.6.7, we can find a model M, an assignment g, and a state s of cardinality at most n such that $M, s \models_g \Phi$ but $M, s \not\models_g \psi$. In fact, we may assume that the cardinality of s is exactly n (if needed, we may always duplicate some worlds in s). By Proposition 5.6.10 we have $M^* \models_{g[\mathsf{s} \mapsto s]} \mathsf{tr}_\mathsf{s}(\Phi)$ but $M^* \not\models_{g[\mathsf{s} \mapsto s]} \mathsf{tr}_\mathsf{s}(\psi)$, which shows that $\mathsf{tr}_\mathsf{s}(\Phi) \not\models_{\mathsf{FOL}} \mathsf{tr}_\mathsf{s}(\psi)$.

For the converse direction, suppose $\mathsf{tr}_\mathsf{s}(\Phi) \not\models_{\mathsf{FOL}} \mathsf{tr}_\mathsf{s}(\psi)$. This means that there is a two-sorted relational structure M' and an assignment g' such that $M' \models_{g'} \mathsf{tr}_\mathsf{s}(\Phi)$ but $M' \not\models_{g'} \mathsf{tr}_\mathsf{s}(\psi)$. Now let M be the relational information model such that $M^* = M'$ (which exists since the map $M \mapsto M^*$ is a bijection between relational information models for Σ and two-sorted structures for Σ^*). Let g be the assignment defined by $g(x) = g'(x)$ for every individual variable x, and let $s = \{w_1, \ldots, w_n\}$ where $w_i = g'(\mathsf{w}_i)$. By the previous proposition, for any $\chi \in \mathcal{L}^{Q=}(\Sigma)$ we have

$$M, s \models_g \chi \iff M^* \models_{g[\mathsf{s} \mapsto s]} \mathsf{tr}_\mathsf{s}(\chi) \iff M' \models_{g'} \mathsf{tr}_\mathsf{s}(\chi),$$

where the last biconditional holds because g' and $g[\mathsf{s} \mapsto s]$ coincide on all variables which occur free in $\mathsf{tr}_\mathsf{s}(\chi)$. This then implies that $M, s \models_g \Phi$ but $M, s \not\models_g \psi$, which shows that $\Phi \not\models_{\mathsf{InqBQ}} \psi$. □

Finally, with this result in place we are able to show the compactness of InqBQ towards finitely coherent conclusions.

Proof of Theorem 5.6.9. Suppose $\Phi \models_{\mathsf{InqBQ}} \psi$ and ψ is finitely coherent. By Proposition 5.6.11, choosing $\mathsf{s} = \{\mathsf{w}_1, \ldots, \mathsf{w}_{d_\psi}\}$ we have $\mathsf{tr}_\mathsf{s}(\Phi) \models_{\mathsf{FOL}} \mathsf{tr}_\mathsf{s}(\psi)$. By

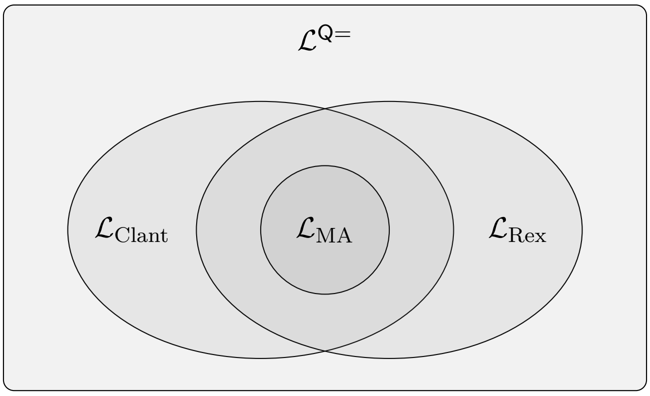


Figure 5.6: Relations between the three fragments discussed in this section: the mention-all fragment $\mathcal{L}_{\text{MA}}$ is included both in the classical antecedent fragment $\mathcal{L}_{\text{Clant}}$ and in the restricted existential fragment $\mathcal{L}_{\text{Rex}}$, while the latter two fragments are incomparable in terms of inclusion.

the compactness of classical first-order logic, there is a finite subset $\Phi_0 \subseteq \Phi$ such that $\mathsf{tr_s}(\Phi_0) \models_{\mathsf{FOL}} \mathsf{tr_s}(\psi)$. Again by Proposition 5.6.11, it follows that $\Phi_0 \models_{\mathsf{InqBQ}} \psi$. $\square$

5.7 Fragments

In this section we discuss three interesting syntactic fragments of InqBQ: the restricted existential fragment $\mathcal{L}_{\text{Rex}}$ (Ciardelli and Grilletti, 2022), the classical antecedent fragment $\mathcal{L}_{\text{Clant}}$ (Grilletti, 2021), and the mention-all fragment $\mathcal{L}_{\text{MA}}$. The first two are interesting as they are the largest syntactic fragments of $\mathcal{L}$ for which the questions posed in Section 5.5.6 have been answered in the positive: entailment in these fragments is compact, validities are recursively enumerable, and every non-entailment can be refuted in a countable model. Moreover, in each case a complete proof system has been established. The mention-all fragment, on the other hand, is interesting in that it has exactly the same expressive power as the logic of interrogation of Groenendijk (1999) (cf. Section 2.9.2)—a predecessor of InqBQ which also extends first-order logic with questions and which was axiomatized by ten Cate and Shan (2007). The inclusions between the three fragments discussed in this section are shown in Figure 5.6.

5.7.1 The restricted existential (rex) fragment

The first fragment of InqBQ that we consider is the restricted existential (in short, *rex*) fragment of InqBQ, obtained by restricting the occurrence of $\exists$ to antecedents of an implication.

5.7.1. Definition. [Rex fragment of InqBQ (Ciardelli and Grilletti, 2022)]
The set $\mathcal{L}_{\text{Rex}}(\Sigma)$ of rex formulas is given by the following syntax:

$$\varphi \ ::= \ p \mid \bot \mid \varphi \wedge \varphi \mid \psi \to \varphi \mid \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \varphi \mid \forall x \varphi$$

where p is an atomic sentence from the signature Σ and ψ an arbitrary sentence from $\mathcal{L}^{\mathsf{Q}=}$, possibly containing occurrences of $\exists$.

The rex fragment is a broad fragment of InqBQ: it contains all classical formulas and it is closed under conjunction, implication, inquisitive disjunction, and the universal quantifier. The key feature of the rex fragment is that every formula is finitely coherent, and an upper bound for its coherence degree is computable from its syntax.

5.7.2. Proposition (Finite coherence property).
For every formula $\varphi \in \mathcal{L}_{Rex}$ there is a natural number n_φ, computable from the syntax of φ, such that φ is n_φ-coherent.

Proof. We define n_φ as follows for $\varphi \in \mathcal{L}_{\text{Rex}}$:

- $n_p = 1$ if p is atomic;
- $n_\bot = 1$;
- $n_{\varphi \wedge \psi} = \max(n_\varphi, n_\psi)$;
- $n_{\varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi} = n_\varphi + n_\psi$;
- $n_{\chi \to \varphi} = n_\varphi$;
- $n_{\forall x \varphi} = n_\varphi$.

To see that φ is n_φ-coherent it suffices to note that atoms and $\bot$ are 1-coherent (i.e., truth-conditional) and to apply inductively Proposition 5.6.3. ☐

Note that the number n_χ is not necessarily equal to the coherence degree d_χ: for instance, if $\chi = (Px \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} Px)$ we have $n_\chi = n_{Px} + n_{Px} = 2$, but since $Px \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} Px \equiv Px$ we have $d_\chi = d_{Px} = 1$. However, since the coherence degree d_χ is defined as the *least* number n for which χ is n-coherent, we have $n_\chi \geq d_\chi$.

An interesting open problem is whether the rex fragment is expressively complete with respect to finitely coherent properties expressible in InqBQ, in the following sense.

5.7.3. Open problem. [Completeness for finitely coherent properties]
Is every finitely coherent $\varphi \in \mathcal{L}^{\mathsf{Q}=}$ logically equivalent to some $\varphi^* \in \mathcal{L}_{\text{Rex}}$?

Since rex formulas are finitely coherent, all the results from the previous section apply to them. In particular, rex formulas are always normal, and entailments with rex conclusions are compact, that is, if $\Phi \models \psi$ and ψ is a rex formula then $\Phi_0 \models \psi$ for some finite $\Phi_0 \subseteq \Phi$. Moreover, using the results in the previous section we can show that the set of rex validities is recursively enumerable.

5.7.4. THEOREM (REX VALIDITIES ARE RECURSIVELY ENUMERABLE).
The set $Val_{\mathsf{rex}} = \{\chi \in \mathcal{L}_{Rex} \mid \chi$ *is valid in* $\mathsf{InqBQ}\}$ *is recursively enumerable.*

Proof. We need to show that there is a method to semi-decide whether a given formula χ belongs to the set $\mathrm{Val}_{\mathsf{rex}}$. This amounts to semi-deciding whether the conjunction ($\chi \in \mathcal{L}_{\mathrm{Rex}}$ and $\models_{\mathsf{InqBQ}} \chi$) holds. For this, we proceed as follows. First, we check whether χ is a rex formula. This is a decidable matter: we just need to check if all occurrences of an inquisitive existential quantifier are within the antecedent of a conditional. If χ is not a rex formula, we do not return any output. Otherwise, we need to semi-decide whether χ is valid in InqBQ. For this, we first compute the number n_χ and then compute the finitary first-order translation $\mathsf{tr}_{\mathsf{s}}(\chi)$ for s a set of n_χ world variables. Since χ is n_χ-coherent, by Proposition 5.6.11 we have $\models_{\mathsf{InqBQ}} \chi \iff \models_{\mathsf{FOL}} \mathsf{tr}_{\mathsf{s}}(\chi)$. Thus, our task reduces to semi-deciding whether $\mathsf{tr}_{\mathsf{s}}(\chi)$ is valid in classical first-order logic. This is possible, since validity in first-order logic is semi-decidable. $\square$

The theorem implies that the set of InqBQ-entailments with finitely many premises and a rex conclusion is also recursively enumerable. This is because we have:

$$\varphi_1, \dots, \varphi_n \models \chi \iff \models \varphi_1 \wedge \dots \wedge \varphi_n \to \chi.$$

Thus, semi-deciding whether $\varphi_1, \dots, \varphi_n \models_{\mathsf{InqBQ}} \chi$ reduces to semi-deciding the validity of the formula $\varphi_1 \wedge \dots \wedge \varphi_n \to \chi$, which is a rex formula if χ is.

The fact that entailments with rex conclusions are compact and recursively enumerable suggests that it may be possible to give a proof system for InqBQ which is complete with respect to such entailments, i.e., such that if $\Phi \models \psi$ and ψ is a rex formula then ψ is provable from Φ in this system. This is indeed the case: one such proof system is obtained by extending the system for InqBQ presented in the next chapter with a *coherence rule* which allows one to freely use certain cardinality formulas in order to infer a finitely coherent formula; the interested reader is referred to Ciardelli and Grilletti (2022) for the details.

5.7.2 The classical antecedent (clant) fragment

The second fragment of InqBQ that we consider is the classical antecedent (in short, *clant*) fragment, obtained by allowing only classical formulas as antecedents of implications.

5.7.5. DEFINITION. [Classical antecedent fragment of InqBQ (Grilletti, 2021)]
The set $\mathcal{L}_{\mathrm{Clant}}(\Sigma)$ of clant formulas in a signature Σ is given by:

$$\varphi \ ::= \ p \mid \bot \mid \varphi \wedge \varphi \mid \alpha \to \varphi \mid \varphi \vee\!\!\!\vee\, \varphi \mid \forall x \varphi \mid \exists x \varphi,$$

where p ranges over atomic formulas in the signature Σ and α over classical formulas.

The clant fragment is a broad fragment of InqBQ: it includes all classical formulas, polar questions of the form $?\alpha$ for α classical, and more generally disjunctive questions of the form $\alpha_1 \vee\!\!\!\vee \cdots \vee\!\!\!\vee \alpha_n$ where the α_i are all classical; mention-some questions of the form $\exists x_1 \ldots \exists x_n \alpha$ with α classical, including as special cases single-instance questions like $\exists! x P(x)$, and identification questions like $\exists x(x = t)$; mention-all questions of the form $\forall x_1 \ldots \forall x_n ?\alpha$ with α classical; and all questions that can be formed by conjoining questions of the above kinds, or conditionalizing such questions to a classical antecedent. In fact, all examples of questions discussed in this chapter, as well as all examples of InqBQ-entailments, involved only clant formulas.

What is *not* included in the fragment are implications of the form $\mu \to \mu'$ where both μ and μ' contain inquisitive operators, for instance the formulas $\forall x ?Px \to \forall x ?Qx$ or $\exists x Px \to \exists x Qx$, as well as compounds including such implications.[12] As we discussed in Section 2.5, implications of this sort capture the fact that in the evaluation state, a certain dependency holds. Thus, what we cannot generally capture in the clant fragment is what follows from premises stating that certain dependencies hold.

The clant fragment neither contains nor is contained in the rex fragment discussed in the previous section. For instance, $\exists x Px$ is a clant formula but not a rex formula, while $\forall x ?Px \to \forall x ?Qx$ is a rex formula but not a clant formula.

Semantically, the key feature of the classical antecedent fragment is that the second-order quantification associated with implication is neutralized. This is because if α is truth-conditional (and thus in particular if α is classical) the clause for implication can be simplified as follows (cf. Proposition 5.5.10 and Proposition 2.5.2):

$$M, s \models_g \alpha \to \varphi \iff M, s \cap |\alpha|_M^g \models_g \varphi.$$

Thus, to check if an implication $\alpha \to \varphi$ holds in a state s it is not necessary to check all subsets of s; it suffices to check one subset of s, namely, $s \cap |\alpha|_M^g$.

One consequence of this fact is that in a clant formula, implication can always be pushed down to the level of classical formulas. Let us make this precise.

5.7.6. Definition. The set of restricted implication (for short, *rimp*) formulas, is the set of formulas where implication occurs only within classical sub-formulas. More formally, the set of rimp formulas is given by the following grammar:

$$\xi \ ::= \ \alpha \mid \xi \wedge \xi \mid \xi \vee\!\!\!\vee \xi \mid \forall x \xi \mid \exists x \xi,$$

where α ranges over classical formulas.

[12] Formulas of the form $\mu \to \alpha$ where μ contains inquisitive operators but α is classical are not included in the fragment either. However, these cases are not especially interesting, since such formulas are equivalent to classical formulas $\mu^{cl} \to \alpha$ (and provably so, in the system introduced in the next chapter).

Note that every rimp formula is a clant formula. The converse is not the case: for instance, $\exists xPx \to \forall x?Px$ is a clant formula but not a rimp formula. However, every clant formula can be turned into an equivalent rimp formula.

5.7.7. PROPOSITION. *Every clant formula is equivalent to a rimp formula.*

Proof. The key to the result lies in the following claim: if ξ is a rimp formula and α is classical, $\alpha \to \xi$ is equivalent to a rimp formula. This is proved by induction on ξ, making crucial use of the split equivalences given by Proposition 5.5.12. We leave this inductive proof as an exercise (Exercise 5.9.7).

Using this claim, we prove our proposition by induction on $\varphi \in \mathcal{L}_{\text{Clant}}$. The interesting case is the one for $\varphi = (\alpha \to \psi)$, where α is a classical formula. By induction hypothesis, $\psi \equiv \psi^*$ for some rimp formula ψ^*, so $\varphi \equiv (\alpha \to \psi^*)$, and by the above claim, the latter formula is equivalent to a rimp formula. $\square$

Using this fact, it is possible to define an entailment-preserving translation $(\cdot)^*$ from the clant fragment of InqBQ to classical first-order logic with two sorts for worlds and individuals.

5.7.8. PROPOSITION. *Let Σ be a signature and let $\mathcal{L}^{\mathsf{FOL}}_{\mathsf{w},\mathsf{e}}(\Sigma^*)$ be the corresponding two-sorted language as defined in Section 5.6. There is a computable translation $\mathsf{tr} : \mathcal{L}_{Clant}(\Sigma) \to \mathcal{L}^{\mathsf{FOL}}_{\mathsf{w},\mathsf{e}}(\Sigma^*)$ such that for all $\Phi \cup \{\psi\} \subseteq \mathcal{L}_{Clant}(\Sigma)$:*

$$\Phi \models \psi \iff \mathsf{tr}(\Phi) \models_{\mathsf{FOL}} \mathsf{tr}(\psi),$$

where $\models_{\mathsf{FOL}}$ is entailment in classical first-order logic.

Proof sketch. We only give the proof strategy, leaving the details as an exercise to the reader (see Exercise 5.9.8). We refer to Section 5.6 for the definition of the two-sorted language $\mathcal{L}^{\mathsf{FOL}}_{\mathtt{w},\mathtt{e}}(\Sigma^*)$ and of the bijection $M \mapsto M^*$ between relational information models for Σ and relational structures for Σ^*.

- *Step 1.* Define for each classical formula $\alpha \in \mathcal{L}^{\mathsf{Q}}_c(\Sigma)$ a two-sorted formula $\alpha^\star(\mathsf{w})$ containing a single free variable w of sort $\mathtt{w}$, in such a way that for every relational information model M, world w, and assignment g we have

  $$M, w \models_g \alpha \iff M^* \models_{\overline{g}[\mathsf{w} \mapsto w]} \alpha^\star,$$

  where $\overline{g}$ is any assignment for the two-sorted language which coincides with g on variables of sort $\mathtt{e}$.

  Example: $(\forall xPx)^\star = \forall x P^*(\mathsf{w}, x)$.

- *Step 2.* Define for each restricted implication formula φ a formula $\mathsf{tr}(\varphi) \in \mathcal{L}^{\mathsf{FOL}}_{\mathtt{w},\mathtt{e}}(\Sigma^*)$, without free variables of sort $\mathtt{w}$, such that

$$M, W \models_g \varphi \iff M^* \models_{\overline{g}} \varphi.$$

Example: $\mathsf{tr}(\exists x Px) = \exists x \forall \mathsf{w} P^*(\mathsf{w}, x)$.

- *Step 3.* Extend tr to all clant formulas using Proposition 5.7.7.
- *Step 4.* Using the fact that $M \mapsto M^*$ is a bijection between relational information models for Σ and two-sorted structures for Σ^*, show that for all $\Phi \cup \{\psi\} \subseteq \mathcal{L}_{\text{Clant}}(\Sigma)$ we have $\Phi \models \psi \iff \mathsf{tr}(\Phi) \models_{\mathsf{FOL}} \mathsf{tr}(\psi)$. □

As consequences of the translation and the properties of classical first-order logic, we obtain the following theorems.

5.7.9. Theorem (Compactness for the clant fragment).
Let $\Phi \cup \{\psi\} \subseteq \mathcal{L}_{Clant}$. If $\Phi \models \psi$ then there is a finite $\Phi_0 \subseteq \Phi$ such that $\Phi_0 \models \psi$.

Proof. Suppose $\Phi \cup \{\psi\} \subseteq \mathcal{L}_{\text{Clant}}$ and $\Phi \models_{\mathsf{InqBQ}} \psi$. By the previous proposition we have $\mathsf{tr}(\Phi) \models_{\mathsf{FOL}} \mathsf{tr}(\psi)$. By the compactness of classical first-order logic, there is a finite subset $\Phi_0 \subseteq \Phi$ such that $\mathsf{tr}(\Phi_0) \models_{\mathsf{FOL}} \mathsf{tr}(\psi)$. Again by the previous proposition, $\Phi_0 \models_{\mathsf{InqBQ}} \psi$. □

5.7.10. Theorem (Recursive enumerability of clant entailments).
The set $\{\langle \varphi_1, \dots, \varphi_n, \psi \rangle \mid n \geq 0, \varphi_1, \dots, \varphi_n, \psi \in \mathcal{L}_{Clant}, \ \varphi_1, \dots, \varphi_n \models \psi\}$ is recursively enumerable.

Proof. We need to show that, given a sequence $\langle \varphi_1, \dots, \varphi_n, \psi \rangle$ of clant formulas, we can semi-decide whether $\varphi_1, \dots, \varphi_n \models \psi$ holds. By Proposition 5.7.8, this boils down to semi-deciding whether $\mathsf{tr}(\varphi_1), \dots, \mathsf{tr}(\varphi_n) \models_{\mathsf{FOL}} \mathsf{tr}(\psi)$ holds, i.e., whether $\mathsf{tr}(\varphi_1) \wedge \dots \wedge \mathsf{tr}(\varphi_n) \to \mathsf{tr}(\psi)$ is a valid formula in first-order logic. Since validity in first-order logic is semi-decidable, the conclusion follows. □

Note that, while this last theorem implies that the set of clant validities is r.e., it is a strictly stronger claim. This is because an entailment $\varphi_1, \dots, \varphi_n \models \psi$ among clant formulas does not always correspond to the validity of a single clant formula: although we have $\varphi_1, \dots, \varphi_n \models \psi$ iff the formula $\varphi_1 \wedge \dots \wedge \varphi_n \to \psi$ is valid, the latter is not in general a clant formula, even when $\varphi_1, \dots, \varphi_n, \psi$ are.

These positive results suggest that it may be possible to find a complete deduction system for the clant fragment of InqBQ. This is indeed the case: following Grilletti (2021), in Section 6.3 we will describe a natural deduction system for the clant fragment and prove its completeness.

5.7.3 The mention-all fragment

The mention-all fragment of InqBQ is given by the following definition.

5.7.11. Definition. [Mention-all fragment]
The set $\mathcal{L}_{\text{MA}}$ of mention-all formulas is defined as the union $\mathcal{L}_c^{\mathsf{Q}=} \cup \mathcal{L}_{\text{MA?}}$, where $\mathcal{L}_c^{\mathsf{Q}=}$ is the set of classical formulas and $\mathcal{L}_{\text{MA?}} = \{\forall x_1 \dots \forall x_n ?\alpha \mid n \geq 0, \alpha \in \mathcal{L}_c^{\mathsf{Q}=}\}$. In words, the formulas in $\mathcal{L}_{\text{MA}}$ are either classical or of the form $\forall \overline{x}?\alpha$, where α is classical and $\overline{x}$ is a possibly empty sequence of variables.

The mention-all fragment is a rather small fragment of InqBQ; it is included both in the rex fragment (since mention-all formulas are $\exists$-free) and in the clant fragment (since implications are only allowed within classical formulas).

One basic feature of the fragment is that every formula in it is 2-coherent.

5.7.12. Proposition. *Every $\varphi \in \mathcal{L}_{MA}$ is 2-coherent.*

Proof. If φ is classical, then it is 1-coherent and thus also 2-coherent. If $\varphi = \forall \overline{x}?\alpha$ where α is classical, by Proposition 5.6.3 and the truth-conditionality of classical formulas we have $d_{\forall \overline{x}?\alpha} \leq d_{?\alpha} \leq d_\alpha + d_{\neg\alpha} = 1 + 1 = 2$. □

The fact that formulas in this fragment are 2-coherent implies that their semantics is completely encoded at the level of states of size at most 2. This opens the way to the possibility of giving an equivalent semantics for the fragment that evaluates formulas with respect to pairs $\langle w, w' \rangle$ of worlds, rather than with respect to states. We will come back to this point when relating the fragment to the Logic of Interrogation. Before turning to that, let us introduce some useful notation.

5.7.13. Definition. Let M be a model and g an assignment. If α is a classical formula and $\overline{x} = \langle x_1, \dots, x_n \rangle$ is a tuple of variables, let $\alpha_g^{\overline{x}}$ be the intensional relation determined by α with respect to $\overline{x}$, i.e., the function which maps any world $w \in W$ to the set of tuples $\overline{d} \in D^n$ that satisfy α in w relative to g.

$$\alpha_g^{\overline{x}}(w) = \{\overline{d} \in D^n \mid w \models_{g[\overline{x} \mapsto \overline{d}]} \alpha\}.$$

We extend this to the case in which the tuple $\overline{x}$ is empty by letting $\alpha_g^{\emptyset}(w)$ be the truth value of α at w relative to g:

$$\alpha_g^{\emptyset}(w) = \begin{cases} 1 & \text{if } w \models_g \alpha \\ 0 & \text{if } w \not\models_g \alpha. \end{cases}$$

Clearly, if α contains no variables besides those in $\overline{x}$ then the assignment g plays no role, so we can drop reference to it.

The following proposition states that, given a classical formula α and a tuple $\overline{x}$ of variables, the question $\forall\overline{x}?\alpha$ asks for the extension of the relation $\alpha_g^{\overline{x}}$. That is, the question is settled in a state s if any two worlds $w, w' \in s$ agree on the extension of $\alpha_g^{\overline{x}}$.

5.7.14. Proposition (Semantics of mention-all questions).
Let $\alpha \in \mathcal{L}_c^{\mathsf{Q}=}$. For any information model M, state s and assignment g:

$$s \models_g \forall\overline{x}?\alpha \iff \textit{for all } w, w' \in s : \alpha_g^{\overline{x}}(w) = \alpha_g^{\overline{x}}(w').$$

Proof. We have the following sequence of equivalences.

$$\begin{aligned}
s \models_g \forall\overline{x}?\alpha \iff & \text{ for all } \overline{d} \in D^n : \ s \models_{g[\overline{x}\mapsto\overline{d}]} \alpha \text{ or } s \models_{g[\overline{x}\mapsto\overline{d}]} \neg\alpha \\
\iff & \text{ for all } \overline{d} \in D^n : \ (\text{for all } w \in s,\, \overline{d} \in \alpha_g^{\overline{x}}(w)) \text{ or} \\
& \qquad\qquad\qquad\ (\text{for all } w \in s,\, \overline{d} \notin \alpha_g^{\overline{x}}(w)) \\
\iff & \text{ for all } \overline{d} \in D^n, \text{ for all } w, w' \in s : \ \overline{d} \in \alpha_g^{\overline{x}}(w) \iff \overline{d} \in \alpha_g^{\overline{x}}(w') \\
\iff & \text{ for all } w, w' \in s, \text{ for all } \overline{d} \in D^n : \ \overline{d} \in \alpha_g^{\overline{x}}(w) \iff \overline{d} \in \alpha_g^{\overline{x}}(w') \\
\iff & \text{ for all } w, w' \in s : \alpha_g^{\overline{x}}(w) = \alpha_g^{\overline{x}}(w').
\end{aligned}$$

It is easy to check that this holds also for the special case in which $\overline{x}$ is empty and the formula under consideration is a polar question $?\alpha$. $\square$

This result allows us to show that questions of the form $\forall\overline{x}?\alpha$ can be seen as inducing partitions of the logical space. Let us make this precise.

5.7.15. Definition. [Partition formulas]
We say that $\varphi \in \mathcal{L}^{\mathsf{Q}=}$ is a *partition formula* if given any information model M and assignment g there is a partition Π^φ of W such that for every state $s \subseteq W$:

$$s \models_g \varphi \iff s \subseteq a \text{ for some } a \in \Pi^\varphi.$$

Equivalently, φ is a partition formula if for any model M and assignment g there is an equivalence relation $\approx^\varphi$ on W such that for every state $s \subseteq W$:

$$s \models_g \varphi \iff \forall w, w' \in s : w \approx^\varphi w'.$$

5.7.16. Proposition. *If α is a classical formula, $\forall\overline{x}?\alpha$ is a partition formula. Indeed, given any model M and assignment g, the set $\textsc{Alt}_M^g(\forall\overline{x}?\alpha)$ forms a partition of the logical space W, and for any information state $s \subseteq W$ we have*

$$s \models_g \forall\overline{x}?\alpha \iff s \subseteq a \textit{ for some } a \in \textsc{Alt}_M^g(\forall\overline{x}?\alpha).$$

The proof of this proposition is left as an exercise to the reader (Exercise 5.9.9).

A natural question is whether $\mathcal{L}_{\mathrm{MA?}}$ is expressively complete with respect to partition formulas. We will leave this as an open problem.

5.7.17. Open problem.
Is every partition formula in InqBQ equivalent to one of the form $\forall\overline{x}?\alpha$ with α classical?

As we discussed in Section 2.9.2, equivalence relations on the logical space—and the partitions they induce—are precisely the objects taken to capture question meanings in the theory of questions of Groenendijk and Stokhof (1984). A logical system based on this theory, called the *Logic of Interrogation* (LoI) has been developed by Groenendijk (1999), and axiomatized by ten Cate and Shan (2007). This system is the most important predecessor of InqBQ. We will now introduce LoI more precisely and show that it is essentially equivalent to the mention-all fragment of InqBQ.

The language $\mathcal{L}_{\mathsf{LoI}}$ of the Logic of Interrogation consists of two kinds of formulas: declaratives, which are simply classical formulas $\alpha \in \mathcal{L}_c^{\mathsf{Q=}}$, and interrogatives, which are of the form $Q\overline{x}\alpha$, where Q is a special question-forming quantifier, $\overline{x}$ is a possibly empty sequence of variables, and α is a classical formula.[13] Intuitively $Q\overline{x}\alpha$ stands for the mention-all question asking which tuples $\overline{x}$ satisfy α. As a special case, when $\overline{x}$ is empty, $Q\alpha$ stands for the polar question asking whether α is true or false.

The semantics of LoI is given relative to models that are essentially the same as our relational information models. The original setup for LoI is slightly less general in two ways: first, only rigid constant symbols are allowed in the language; second, the semantics is restricted to id-models. Since it is straightforward to see how both restrictions can be lifted, I present LoI without these restrictions, so as to bring out the connections to InqBQ in greater generality.

In its original formulation (Groenendijk, 1999), LoI is presented as a dynamic semantics (in the tradition of Groenendijk et al., 1996; Jäger, 1996; Hulstijn, 1997, among others). However, as pointed out by ten Cate and Shan, the dynamic coating is inessential: one can give a simple static semantics for LoI in which formulas are interpreted with respect to ordered pairs of worlds. A classical formula α is satisfied at a pair $\langle w, w'\rangle$ in case w and w' agree that α is true, while a question $Q\overline{x}\alpha$ is satisfied in case the worlds w and w' agree on the extension of the relation defined by α with respect to $\overline{x}$—i.e., agree on the answer to the question $Q\overline{x}\alpha$.

5.7.18. Definition. If M is a relational information model and w, w' are worlds in M (not necessarily distinct) the semantics of LoI is given by:

[13] The original LoI notation for a formula $Q\overline{x}\alpha$ is $?\overline{x}\alpha$. I use the alternative notation $Q\overline{x}\alpha$ to avoid confusion with the way the symbol '?' is used in inquisitive logic.

- $\langle w, w' \rangle \models_g^{\mathsf{LoI}} \alpha \iff w \models_g \alpha$ and $w' \models_g \alpha$;
- $\langle w, w' \rangle \models_g^{\mathsf{LoI}} Q\overline{x}\alpha \iff \alpha_g^{\overline{x}}(w) = \alpha_g^{\overline{x}}(w')$.

Entailment in LoI can be defined in the expected way in terms of this semantics.

5.7.19. Definition. [Entailment in the Logic of Interrogation]
Let $\Phi \cup \{\psi\} \subseteq \mathcal{L}_{\mathsf{LoI}}$. $\Phi \models_{\mathsf{LoI}} \psi$ in case for all models M, all pairs $\langle w, w' \rangle$ of worlds, and all assignments g, if $\langle w, w' \rangle \models_g \varphi$ for all $\varphi \in \Phi$ then $\langle w, w' \rangle \models_g \psi$.

LoI and the mention-all fragment of InqBQ are equivalent in a natural sense. This can be made precise by defining the following translations between $\mathcal{L}_{\mathsf{LoI}}$ and $\mathcal{L}_{\mathrm{MA}}$.

5.7.20. Definition. [Translations]
We define two translations $(\cdot)^\sharp : \mathcal{L}_{\mathrm{MA}} \to \mathcal{L}_{\mathsf{LoI}}$ and $(\cdot)^\flat : \mathcal{L}_{\mathsf{LoI}} \to \mathcal{L}_{\mathrm{MA}}$, as follows:

- $\alpha^\sharp = \alpha$
- $(\forall\overline{x}?\alpha)^\sharp = Q\overline{x}\alpha$
- $\alpha^\flat = \alpha$
- $(Q\overline{x}\alpha)^\flat = \forall\overline{x}?\alpha$

Clearly, the two translations are inverse to each other, i.e., we have $\varphi^{\sharp\flat} = \varphi$ and $\varphi^{\flat\sharp} = \varphi$. The semantic connections between a sentence and its translation are captured by the following proposition. The proof is immediate by inspecting the translations and by Proposition 5.7.14.

5.7.21. Proposition. *Let M be an id-model, g an assignment. Then:*

- *for any $\varphi \in \mathcal{L}_{\mathsf{LoI}}$ and worlds w, w': $\langle w, w' \rangle \models_g^{\mathsf{LoI}} \varphi \iff \{w, w'\} \models_g \varphi^\flat$;*
- *for any $\varphi \in \mathcal{L}_{MA}$ and worlds w, w': $\{w, w'\} \models_g \varphi \iff \langle w, w' \rangle \models_g^{\mathsf{LoI}} \varphi^\sharp$;*
- *for any $\varphi \in \mathcal{L}_{MA}$ and state s: $s \models_g \varphi \iff \forall w, w' \in s : \langle w, w' \rangle \models_g^{\mathsf{LoI}} \varphi^\sharp$.*

This connection ensures that both translations are entailment-preserving.

5.7.22. Proposition (Translations are entailment-preserving).

- *For all $\Phi \cup \{\psi\} \subseteq \mathcal{L}_{MA}$, $\Phi \models \psi \iff \Phi^\sharp \models_{\mathsf{LoI}} \psi^\sharp$.*
- *For all $\Phi \cup \{\psi\} \subseteq \mathcal{L}_{\mathsf{LoI}}$, $\Phi \models_{\mathsf{LoI}} \psi \iff \Phi^\flat \models \psi^\flat$.*

Proof. Consider the first item. If $\Phi \not\models \psi$, then we can find a relational information model M, an assignment g, and an information state s such that $s \models_g \Phi$ and $s \not\models_g \psi$. Since formulas in $\mathcal{L}_{\mathrm{MA}}$ are 2-coherent, we can find worlds $w, w' \in s$ such that such that $\{w, w'\} \not\models_g \psi$. Since $\{w, w'\} \subseteq s$, by persistency we have

$\{w, w'\} \models_g \Phi$. By the previous proposition, in the semantics of LoI we have $\langle w, w' \rangle \models_g^{\mathsf{LoI}} \Phi^\sharp$ and $\langle w, w' \rangle \not\models_g^{\mathsf{LoI}} \psi^\sharp$, which shows that $\Phi^\sharp \not\models_{\mathsf{LoI}} \psi^\sharp$. Reasoning similarly we can show that if $\Phi^\sharp \not\models_{\mathsf{LoI}} \psi^\sharp$ then $\Phi \not\models \psi$.

The second item follows from the first and the fact that the translations are inverse to each other, as we have $\Phi^\flat \models \psi^\flat \iff \Phi^{\flat\sharp} \models_{\mathsf{LoI}} \psi^{\flat\sharp} \iff \Phi \models_{\mathsf{LoI}} \psi$. □

Thus, the Logic of Interrogation can be identified with a fragment of our inquisitive first-order logic InqBQ. One insight that emerges from this connection is that the primitive question quantifier Q of LoI, whose logical features seem quite complex and unfamiliar (cf. the axiomatization in ten Cate and Shan, 2007), can in fact be analyzed in terms of a combination of operators which are logically simple and familiar. Indeed, in InqBQ the formula $Q\overline{x}\alpha$ corresponds to the compound

$$\forall\overline{x}(\alpha \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \neg\alpha)$$

where $\neg$ is standard negation on statements, $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ is a constructive disjunction, and $\forall$ is a generalization to questions of the standard universal quantifier.[14]

Another repercussion of the translation is that it is possible to transform ten Cate and Shan's completeness result for LoI into a completeness result for the mention-all fragment of InqBQ in which proofs use only formulas from the fragment. The interested reader is referred to Section 4.8 of Ciardelli (2016a).[15]

Finally, it is worth noting that InqBQ, and even its well-behaved classical antecedent fragment, is much more expressive than LoI. As we saw, in the clant fragment of InqBQ we can express, among others, mention-some questions such as $\exists x P(x)$, unique-instance questions such as $\exists! x P(x)$, disjunctive questions such as $P(\mathsf{a}) \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} P(\mathsf{b})$, and conditional questions such as $\exists x P(x) \to \forall x ?P(x)$. None of these questions has a counterpart in LoI.

5.8 'How many' questions and generalized quantifiers

An important class of questions that we have not discussed so far is given by 'how many' questions, asking about the number of objects satisfying a certain property. The semantics of such questions can be analyzed in a natural way in our setting. To better see the idea, let us focus on a special case. Suppose P is a unary predicate. Consider first the case of an id-model. In this case, the objects that have property P at a world w are those in the extension P_w. The number

[14] One way to make precise the claim that these operations are "familiar" is to take an algebraic perspective: in the space of inquisitive propositions, which is a Heyting algebra, $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ corresponds to a simple join operation, $\neg$ to the pseudo-complement operation, and $\forall$ to an operation that yields the meet of a family of propositions. See Roelofsen (2013) for the details.

[15] Since ten Cate and Shan (2007) work with id-models, a direct adaptation of their result yields a proof system for id-entailment; however, it seems very likely that the result can be adapted to general entailment as well.

of objects that have property P at w is thus captured by the cardinality $\#P_w$ of this set. An information state s settles how many objects have property P if s implies for some cardinal κ that there are κ objects with property P, i.e., if there is κ such that $\#P_w = \kappa$ for all $w \in s$. This, in turn, happens if and only if the cardinality of the extension P_w is the same in all $w \in s$.

In the context of id-models, we can thus say that a sentence φ_{hm} expresses the question how many objects have property P if it has the following semantics:

$$\begin{aligned} s \models \varphi_{hm} \quad &\iff \quad \text{there is a cardinal } \kappa \text{ such that } \forall w \in s : \#P_w = \kappa \\ &\iff \quad \forall w, w' \in s : \#P_w = \#P_{w'}. \end{aligned}$$

In the more general setting of a model M that has variable identity, things are slightly more subtle. In this case, the set of actual individuals having property P at a world w is not given directly by P_w, but by the quotient modulo the local identity $\sim_w$ (cf. the discussion in Section 5.4). Thus, the number of individuals having property P at w is given by the cardinal:

$$\text{num}_w(P) := \#(P_w/{\sim_w}).$$

In the general setting, we can thus say that a sentence φ_{hm} expresses the question how many objects have property P if it has the following semantics:

$$\begin{aligned} s \models \varphi_{hm} \quad &\iff \quad \text{there is a cardinal } \kappa \text{ such that } \forall w \in s : \text{num}_w(P) = \kappa \\ &\iff \quad \forall w, w' \in s : \text{num}_w(P) = \text{num}_{w'}(P). \end{aligned}$$

It is then natural to ask: is there in fact a sentence of InqBQ with the required semantics? If not, we could make our demands more modest and ask: is there a sentence of InqBQ that has the required semantics at least in restriction to models where D is finite? Or, even more modestly, in restriction to id-models where D is finite?

In order to study questions like this one, concerning the expressive power of InqBQ, Grilletti and Ciardelli (2021) have recently developed an inquisitive counterpart of the classical Ehrenfeucht-Fraïssé game for first-order logic. The game is shown to characterize the expressive power of the logic, in the following sense: if two information states are distinguished by a formula φ of InqBQ, then in the game there is a strategy to bring out the difference between the states within a finite number of moves determined by the number of quantifier nestings and implication nestings in φ; conversely, if there is a strategy to bring out the difference between two information states within a finite number of moves in the game, then there is a formula φ of InqBQ that distinguishes those states.

As in the classical case, the game is a powerful tool to show that certain properties are not expressible in InqBQ. In particular, it can be used to give a negative answer to the question we posed above, even in its less demanding version. We refer to Grilletti and Ciardelli (2021) for the proof.

5.8.1. Theorem (Inexpressibility of 'how many' questions in InqBQ). *There is no sentence $\varphi_{hm} \in \mathcal{L}^{\mathsf{Q}=}$ such that for all models M and states s:*

$$s \models \varphi_{hm} \iff \forall w, w' \in s : num_w(P) = num_{w'}(P).$$

What is more, there is no sentence that satisfies this condition even in restriction to id*-models where D is finite.*

This result shows that InqBQ does not have the resources to express *how many* questions. Given the importance of this class of questions, this result points to an interesting project for future research: add a *how many* inquisitive quantifier to InqBQ and study the resulting logic.

5.8.2. Open problem. [Extending InqBQ with a 'how many' quantifier] Consider a logic InqBQH whose language is like that of InqBQ but with the additional clause that if α is a classical formula and $\overline{x}$ a sequence of variables then $\mathsf{H}\overline{x}\alpha$ is a formula. Intuitively, H is the inquisitive quantifier 'how many', and $\mathsf{H}\overline{x}\alpha$ stands for the question how many values of the sequence $\overline{x}$ satisfy α. The number of such values at a world w is given by:

$$\mathrm{num}^{\overline{x}}_{w,g}(\alpha) \ := \ \#(\alpha^{\overline{x}}_g(w)/\sim_w)$$

where $\alpha^{\overline{x}}_g(w)$ is the extension of α as given by Definition 5.7.13. We can then let the new formulas be interpreted by the following clause:

$$s \models_g \mathsf{H}\overline{x}\alpha \iff \forall w, w' \in s : \mathrm{num}^{\overline{x}}_{w,g}(\alpha) = \mathrm{num}^{\overline{x}}_{w',g}(\alpha).$$

What are the properties of the resulting logic InqBQH?

This open problem is an instance of a more general enterprise that awaits to be pursued: investigate generalized quantifiers in the setting of inquisitive predicate logic. As Grilletti and Ciardelli (2021) discuss, the inquisitive setting gives rise to a new and more general notion of a quantifier, which encompasses not only standard quantifiers like 'some x', 'at least three x' or 'infinitely many x', but also properly inquisitive quantifiers like 'which x', 'whether finitely or infinitely many x', or 'how many x'. Grilletti and Ciardelli (2021) make a first step in the study of such quantifiers, giving a precise characterization of those unary cardinality quantifiers that are expressible in InqBQ. The undefinability of *how many* questions, given by Theorem 5.8.1 above, is a corollary of this characterization. An interesting aspect of the characterization is that, at least with respect to matters of cardinality, InqBQ turns out to pattern with classical first-order logic, and not with second-order logic, in that each formula is only able to draw distinctions up to a fixed finite cardinal n.

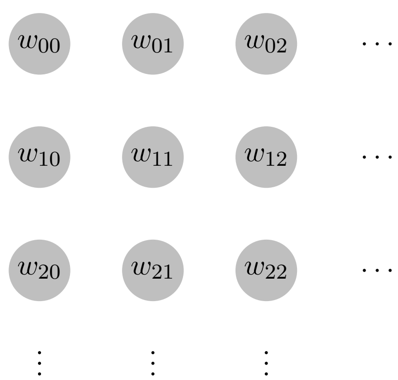
5.9 Exercises

5.9.1. Exercise. [Formalizing questions in InqBQ]
Consider a signature Σ containing a binary predicate D and a rigid constant a. Suppose the domain of quantification consists of guests at a party, $D(x, y)$ stands for "x danced with y" and a denotes Alice. Give a formalization in InqBQ of the following natural language questions:

(4)
- a. Who is someone who danced with Alice?
- b. Which guests danced with Alice?
- c. Was there any guest who did not dance with anybody?
- d. What are two guests who danced with each other?
- e. Which guests danced with everybody?
- f. Who danced with whom?
- g. Which guests danced with at least two people?
- h. Did Alice dance with exactly one person?

5.9.2. Exercise. [Formalizing questions in InqBQ]
Consider a variation of Exercise 2.10.1: somone picked a secret code consisting of two natural numbers. Consider a signature Σ consisting of a rigid constant symbol n for each number $n \in \mathbb{N}$, a rigid binary function symbol $+$, and a binary relation symbol C, where Cxy is read intuitively as "the code is $\langle x, y\rangle$". Suppose we formalize the scenario as a relational id-model M for this signature, where:

- $W = \{w_{ij} \mid i, j \in \mathbb{N}\}$ (so, worlds can be arranged as in the picture below);
- $D = \mathbb{N}$ (the set of natural numbers);
- $I_w(\mathsf{n}) = n$ for all $w \in W$;
- $I_w(+)$ is the sum operation, for all $w \in W$;
- $I_{w_{ij}}(C) = \{\langle i, j\rangle\}$ (that is, w_{ij} is a world where the code is $\langle i, j\rangle$).



Write formulas of $\mathcal{L}^{\mathsf{Q}=}$ that, in the context of this model, express the following statements and questions.

(5)
a. The code is $\langle 1, 2\rangle$.
b. The first number is 1.
c. The code contains the number 1.
d. Is the code $\langle 1, 2\rangle$?
e. Is the first number 1?
f. Does the code contain the number 1 (in either position)?
g. What is the code?
h. What is the first number?
i. What is one number that occurs in the code?
j. If the first number is 1, what is the second number?
k. What is the sum of the two numbers?
l. What is the (absolute) difference of the two numbers?
m. Are the two numbers the same?
n. Is the first number smaller, equal to, or larger than the second?

5.9.3. Exercise. [Formalizing questions in InqBQ]
Given a unary predicate P and a number $n \in \mathbb{N}$, show how to write a formula φ_n that expresses the question whether the number of objects satisfying P is less than n, exactly n, or more than n. Thus, φ_n should be a formula with the following semantics (for the definition of $\mathrm{num}_w(P)$, see page 181):

$$s \models \varphi_n \iff \begin{array}{l} \forall w \in s : \mathrm{num}_w(P) < n, \text{ or} \\ \forall w \in a : \mathrm{num}_w(P) = n, \text{ or} \\ \forall w \in s : \mathrm{num}_w(P) > n. \end{array}$$

5.9.4. Exercise. [Entailment in InqBQ]
Let P, Q be unary predicates. Are the following entailments valid? Give a proof or a countermodel.

1. $\forall x?Px \wedge \forall x?Qx \models \forall x?(Px \wedge Qx)$
2. $\forall x?(Px \wedge Qx) \models \forall x?Px \wedge \forall x?Qx$
3. $\exists x(Px \vee\!\!\!\vee Qx) \models \exists xPx \vee\!\!\!\vee \exists xQx$
4. $\exists xPx \vee\!\!\!\vee \exists xQx \models \exists x(Px \vee\!\!\!\vee Qx)$
5. $\forall x(?Px \to ?Qx) \models \forall x?Px \to \forall x?Qx$
6. $\forall x?Px \to \forall x?Qx \models \forall x(?Px \to ?Qx)$

5.9.5. Exercise. [Entailment in InqBQ]
Show that a necessary condition for an entailment to be valid in InqBQ is that the classical counterpart of the entailment be valid in classical first-order logic. That is, show that for any $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\mathsf{Q}=}$ we have:

$$\Phi \models \psi \implies \Phi^{cl} \models \psi^{cl}$$

where φ^{cl} is the classical variant of a formula, as given by Definition 5.3.5, and if $\Phi^{cl} = \{\varphi^{cl} \mid \varphi \in \Phi\}$. Give an example in which the converse implication fails.

5.9.6. Exercise. [Coherence]
Complete the proof of Proposition 5.6.3 by showing the following claims:

- if φ is κ-coherent and ψ is κ' coherent then $\varphi \wedge \psi$ is $\max(\kappa, \kappa')$-coherent;
- if φ is κ-coherent then so is $\chi \to \varphi$ for any formula χ;
- if φ is κ-coherent then so is $\forall x \varphi$.

5.9.7. Exercise. [Turning clant formulas into rimp formulas]
Complete the proof of Proposition 5.7.7 showing the following claim: if ξ is a rimp formula (cf. Definition 5.7.6) and α is a classical formula, $\alpha \to \xi$ is equivalent to a rimp formula.

Hint: use the split equivalences given by Proposition 5.5.12.

5.9.8. Exercise. [First-order translation of clant formulas]
Execute the strategy outlined in the proof sketch for Theorem 5.7.8 to show that there is a translation from the clant fragment of InqBQ to classical two-sorted first-order logic.

5.9.9. Exercise. [Mention-all questions]
Prove Proposition 5.7.16. Hint: given $\varphi = (\forall \overline{x} ?\alpha)$, take $\approx^\varphi$ to be defined by $w \approx^\varphi w' \iff \alpha_g^{\overline{x}}(w) = \alpha_g^{\overline{x}}(w')$. Clearly, $\approx^\varphi$ is an equivalence relation on the set of worlds. Show that $\textsc{Alt}_M^g(\varphi)$ is precisely the set of equivalence classes of worlds under $\approx^\varphi$ and that a state supports φ just in case it is included in one of these equivalence classes.

Chapter 6

Inferences with first-order questions

In the previous chapter, we saw how classical first-order logic can be enriched with questions, resulting in the system InqBQ of inquisitive first-order logic. We now turn to the task of designing a proof system for InqBQ. As discussed in the previous chapter, it is not currently known whether InqBQ in fact admits a complete proof system, since it is not known whether InqBQ is compact and whether its validities are recursively enumerable. However, in Section 6.1 we will describe a natural deduction system which is sound for InqBQ and which, as demonstrated in Section 6.2, is powerful enough to prove many InqBQ-entailments, including all the examples discussed in the previous chapter. In fact, for all we know this system might be complete, although this seems unlikely.[1] Moreover, we are going to show in Section 6.3 that this system is indeed complete for the classical antecedent fragment of InqBQ, which is in itself a rich extension of first-order logic with questions.

6.1 Natural deduction system for InqBQ

A natural deduction system for InqBQ is described in Figure 6.1. The notational conventions are the usual ones: we write $\Phi \vdash \psi$ to mean that there is a proof in this system whose conclusion is ψ and whose set of undischarged assumptions is included in Φ; we write $\varphi_1, \ldots, \varphi_n \vdash \psi$ instead of $\{\varphi_1, \ldots, \varphi_n\} \vdash \psi$, and write $\varphi \dashv\vdash \psi$ to mean that both $\varphi \vdash \psi$ and $\psi \vdash \varphi$ hold. Before illustrating the system with some examples, let us comment briefly on the various inference rules.

Connectives and quantifiers. As in the propositional system for InqB, each of the connectives $\wedge, \rightarrow, \bot$ and $\vee\!\!\!\vee$ is handled by its standard introduction and elimination rules. The rules for the quantifiers are also standard—with one *caveat*: as we discussed in Section 5.5.4, in general it is only sound to instantiate

[1] See the conclusion section of Ciardelli and Grilletti (2022) for an example of a valid InqBQ-entailment that is conjectured not to be provable in the system.

Conjunction

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \qquad \frac{\varphi \wedge \psi}{\varphi} \quad \frac{\varphi \wedge \psi}{\psi}$$

Implication

$$\frac{\begin{matrix}[\varphi]\\ \vdots \\ \psi\end{matrix}}{\varphi \rightarrow \psi} \qquad \frac{\varphi \quad \varphi \rightarrow \psi}{\psi}$$

Universal quantifier

$$\frac{\varphi[y/x]}{\forall x \varphi} \qquad \frac{\forall x \varphi}{\varphi[\mathsf{t}/x]} \quad \frac{\forall x \alpha}{\alpha[t/x]}$$

Falsum

$$\frac{\bot}{\varphi}$$

Identity

$$\frac{}{t = t} \qquad \frac{\varphi[t/x] \quad t = t'}{\varphi[t'/x]}$$

Classical $\neg\neg$ elimination

$$\frac{\neg\neg\alpha}{\alpha}$$

Inquisitive disjunction

$$\frac{\varphi}{\varphi \mathbin{\vee\!\!\!\vee} \psi} \quad \frac{\psi}{\varphi \mathbin{\vee\!\!\!\vee} \psi} \qquad \frac{\varphi \mathbin{\vee\!\!\!\vee} \psi \quad \begin{matrix}[\varphi]\\ \vdots \\ \chi\end{matrix} \quad \begin{matrix}[\psi]\\ \vdots \\ \chi\end{matrix}}{\chi}$$

Inquisitive existential

$$\frac{\varphi[\mathsf{t}/x]}{\exists\!\!\!\exists x \varphi} \qquad \frac{\exists\!\!\!\exists x \varphi \quad \begin{matrix}[\varphi[y/x]]\\ \vdots \\ \psi\end{matrix}}{\psi}$$

$\mathbin{\vee\!\!\!\vee}$-split

$$\frac{\alpha \rightarrow \varphi \mathbin{\vee\!\!\!\vee} \psi}{(\alpha \rightarrow \varphi) \mathbin{\vee\!\!\!\vee} (\alpha \rightarrow \psi)}$$

$\exists\!\!\!\exists$-split

$$\frac{\alpha \rightarrow \exists\!\!\!\exists x \varphi}{\exists\!\!\!\exists x (\alpha \rightarrow \varphi)} \; x \notin \mathrm{FV}(\alpha)$$

Constant domains (CD)

$$\frac{\forall x (\varphi \mathbin{\vee\!\!\!\vee} \psi)}{(\forall x \varphi) \mathbin{\vee\!\!\!\vee} \psi} \; x \notin \mathrm{FV}(\psi)$$

Classicality of negations (KF)

$$\frac{\forall x \neg\neg\varphi}{\neg\neg\forall x \varphi}$$

Figure 6.1: A sound, but possibly incomplete, natural deduction system for InqBQ. As usual, we denote the introduction and elimination rules for an operator $\circ$ as ($\circ\mathsf{i}$) and ($\circ\mathsf{e}$). In these rules, the variable α ranges over classical formulas, while φ and ψ range over arbitrary formulas of InqBQ; t denotes a rigid term, while t, t' denote arbitrary terms, which may but need not be rigid. In either case, these terms must be free for x in the relevant formula. In the rule ($\forall\mathsf{i}$), y must not occur free in any undischarged assumption. In ($\exists\!\!\!\exists\mathsf{e}$), y must not occur free in ψ or in any undischarged assumption.

Classical disjunction

$$\frac{\varphi}{\varphi \vee \psi} \qquad \frac{\psi}{\varphi \vee \psi} \qquad \frac{\varphi \vee \psi \quad \begin{array}{c}[\varphi]\\ \vdots \\ \alpha\end{array} \quad \begin{array}{c}[\psi]\\ \vdots \\ \alpha\end{array}}{\alpha}$$

Classical existential

$$\frac{\varphi[t/x]}{\exists x\varphi} \qquad \frac{\exists x\varphi \qquad \begin{array}{c}[\varphi[y/x]]\\ \vdots \\ \alpha\end{array}}{\alpha}$$

Figure 6.2: Admissible rules for $\vee$ and $\exists$. Here, α ranges over classical formulas, and φ, ψ over arbitrary formulas; t is an arbitrary term (possibly non-rigid) free for x in φ. The variable y should not occur free in α or any undischarged assumption above the rule

$\forall$ to, and to introduce $\exists$ from, a term which is rigid. As we saw in that section, it is also sound to instantiate a universal to a non-rigid term, but only when the relevant formula is classical. This is given as a second elimination rule for $\forall$.

The rules for $\neg$ are obtained as special cases of the rules for implication. As for classical disjunction and classical existential, the rules shown in Figure 6.2 are implicitly available in the system: whatever can be derived using these rules can in fact be derived from the rules in Figure 6.1. Note that a classical existential can be introduced from an arbitrary term (not only from a rigid term, as in the case of $\exists$); however, as in the case of classical disjunction, it can be eliminated only towards a classical conclusion. Without the restriction to classical conclusions, the elimination rule would not be sound; it would allow us, for instance, to infer the inquisitive existential $\exists xPx$ from the classical existential $\exists xPx$.

Identity predicate. Standard introduction and elimination rules also take care of the identity predicate. The introduction rule is given by the fact that $t = t$ is valid for any t. The elimination rule is backed by Proposition 5.4.3, which ensures the substitutability of terms whose identity is established in the state. Other features of identity, such as symmetry and transitivity, are provable.

Connection with intuitionistic logic. What we have described so far—at least in restriction to formulas containing only rigid terms—is simply a natural deduction system for intuitionistic first-order logic, with $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ and $\exists$ in the role of intuitionistic disjunction and existential quantifier respectively. The soundness of these rules implies that anything that is intuitionistically valid is also valid in InqBQ. This observation is worth stating as a proposition.

6.1.1. Proposition (InqBQ includes intuitionistic logic).
Let $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\mathsf{Q}=}$ consist of formulas containing only rigid terms. If Φ entails ψ in intuitionistic logic when $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ and $\exists$ are identified with intuitionistic disjunction and inquisitive existential respectively, then $\Phi \models \psi$.

On top of this intuitionistic skeleton, InqBQ validates some further principles.

Classical double negation elimination. We saw that the classical fragment of InqBQ coincides with classical first-order logic. To capture this, we endow our system with a rule of double negation elimination restricted to classical formulas. In this way, our proof system includes a complete natural deduction system for classical first-order logic in restriction to classical formulas.

Split rules. In the propositional case, our system contains the $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$-split rule, which allows us to distribute a classical antecedent over an inquisitive-disjunctive consequent. In the first-order case, we have this rule as well as the analogous rule for the inquisitive existential quantifier. The soundness of these rules is given by Proposition 5.5.12. These rules reflect the specificity of statements (Proposition 5.5.10), that is, the crucial fact that statements denote specific pieces of information, and that to suppose a statement is to extend the available information in one particular way.

Constant domains. As is known from work on intuitionistic logic (see Görnemann, 1971; Gabbay, 1981), assuming a constant domain of quantification has repercussions on the logic, rendering valid the schema

$$\forall x(\varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi) \models (\forall x \varphi) \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi$$

where $x \notin \mathrm{FV}(\psi)$. By taking this principle on board as an inference rule, we make our proof system track the assumption that the domain of quantification is fixed and does not vary from world to world.

Classicality of negations. Recall that Proposition 5.3.8 ensures that a negation $\neg\varphi$ is always truth-conditional, and thus equivalent to a classical formula. The KF rule, which allows us to infer $\neg\neg\forall x\varphi$ from $\forall x \neg\neg\varphi$, is precisely what we need in order for our proof system to vindicate this fact. To see this, first recall that Proposition 5.3.9 tells us that the double negation $\neg\neg\varphi$ of a formula is always equivalent with the classical variant, φ^{cl}. Using KF, we can prove this equivalence for every φ.[2]

[2] For those readers who are familiar with intuitionistic logic, the following proposition is related to Glivenko's theorem, which states that $\neg\neg\varphi$ is intuitionistically valid iff φ is classically

6.1.2. PROPOSITION. *For any $\varphi \in \mathcal{L}^{\mathsf{Q}}$, $\neg\neg\varphi \dashv\vdash \varphi^{cl}$.*

Proof. We can show this by induction on the complexity of φ. We focus on the most interesting case, namely, the inductive step for $\forall$. The other cases are left as an exercise (Exercise 6.4.5). So, consider a formula $\forall x\varphi$. Assume the claim holds for all formulas of lower complexity, which include all substitution instances of φ of the form $\varphi[y/x]$. We want to show that $\neg\neg\forall x\varphi \dashv\vdash (\forall x\varphi)^{cl}$.

The two proofs are given below. In these proofs, y is a fresh variable. Observe that by definition of classical variant, $(\forall x\varphi)^{cl}$ coincides with $\forall x\varphi^{cl}$ and $(\varphi[y/x])^{cl}$ coincides with $\varphi^{cl}[y/x]$. The label (IH) marks sub-proofs which exist by the induction hypothesis.

$$
\begin{prooftree}
\AxiomC{$\forall x\varphi^{cl}$}
\RightLabel{$(\forall\mathsf{e})$}
\UnaryInfC{$\varphi^{cl}[y/x]$}
\RightLabel{(IH)}
\UnaryInfC{$\neg\neg\varphi[y/x]$}
\RightLabel{$(\forall\mathsf{i})$}
\UnaryInfC{$\forall x\neg\neg\varphi$}
\RightLabel{(KF)}
\UnaryInfC{$\neg\neg\forall x\varphi$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{$\neg\neg\forall x\varphi$}
\AxiomC{$[\neg\varphi[y/x]]_2$}
\AxiomC{$[\forall x\varphi]_1$}
\RightLabel{$(\forall\mathsf{e})$}
\UnaryInfC{$\varphi[y/x]$}
\RightLabel{$(\to\mathsf{e})$}
\BinaryInfC{$\bot$}
\RightLabel{$(\to\mathsf{i},1)$}
\UnaryInfC{$\neg\forall x\varphi$}
\RightLabel{$(\to\mathsf{e})$}
\BinaryInfC{$\bot$}
\RightLabel{$(\to\mathsf{i},2)$}
\UnaryInfC{$\neg\neg\varphi[y/x]$}
\RightLabel{(IH)}
\UnaryInfC{$\varphi^{cl}[y/x]$}
\RightLabel{$(\forall\mathsf{i})$}
\UnaryInfC{$\forall x\varphi^{cl}$}
\end{prooftree}
$$

Note the crucial role of the KF rule in the first proof. □

Using this fact, we can then prove that all negations in InqBQ are classical.

6.1.3. COROLLARY. *For all $\varphi \in \mathcal{L}^{\mathsf{Q}=}$, $\neg\varphi \dashv\vdash \neg\varphi^{cl}$.*

Proof. We leave it to the reader to check that using the rules for implication we have $\neg\varphi \dashv\vdash \neg\neg\neg\varphi$ for any $\varphi \in \mathcal{L}^{\mathsf{Q}=}$. Putting this together with the previous proposition we have $\neg\varphi \dashv\vdash \neg\neg\neg\varphi \dashv\vdash (\neg\varphi)^{cl} = \neg\varphi^{cl}$. □

We can also show that from a formula $\varphi \in \mathcal{L}^{\mathsf{Q}=}$ we can always infer its classical variant, φ^{cl}.

6.1.4. COROLLARY. *For all $\varphi \in \mathcal{L}^{\mathsf{Q}=}$, $\varphi \vdash \varphi^{cl}$.*

Proof. It is easy to see that for any φ we have $\varphi \vdash \neg\neg\varphi$, and we know that $\neg\neg\varphi \vdash \varphi^{cl}$ by Proposition 6.1.2. □

---

valid. Glivenko's theorem holds in the setting of propositional logic, but not in the setting of predicate logic, essentially due to the fact that in intuitionistic logic, $\forall x\neg\neg\varphi \not\models \neg\neg\forall x\varphi$. The validity of the entailment $\forall x\neg\neg\varphi \models \neg\neg\forall x\varphi$, captured by the KF rule, is exactly what it takes in order for a super-intuitionistic first-order logic to validate Glivenko's theorem (cf. Proposition 2.12.1 in Gabbay et al., 2009).

This result can be used to give an easy proof of the fact that our system is complete for entailments whose conclusion is a classical formula.[3]

6.1.5. THEOREM (COMPLETENESS FOR CLASSICAL CONCLUSIONS).
Let $\Phi \subseteq \mathcal{L}^{Q=}$ and $\alpha \in \mathcal{L}^{Q=}_c$. If $\Phi \models \alpha$, then $\Phi \vdash \alpha$.

Proof. Suppose $\Phi \models \alpha$. As classical formulas are truth-conditional, it follows from Proposition 5.5.7 that $\Phi^{cl} \models \alpha$. Since entailment among classical formulas coincides with entailment in classical first-order logic, and since our proof system includes a complete proof system for classical first-order logic, it follows that $\Phi^{cl} \vdash \alpha$. This means that there must be $\varphi_1, \dots, \varphi_n \in \Phi$ such that $\varphi_1^{cl}, \dots, \varphi_n^{cl} \vdash \alpha$. The previous corollary ensures that $\varphi_i \vdash \varphi_i^{cl}$ for each $i \leq n$. Hence, $\varphi_1, \dots, \varphi_n \vdash \alpha$, which implies $\Phi \vdash \alpha$. □

Before proceeding to show a more interesting completeness result, let us illustrate the deduction system with some examples.

6.2 Illustration

In order to illustrate our proof system, we show how we can use it to prove the examples of logical dependencies that we discussed in Section 5.5.1.

6.2.1. EXAMPLE. Start with the entailment we discussed in Example 5.5.1:

$$\forall x(P(x) \leftrightarrow \neg Q(x)),\ \forall x?P(x)\ \models\ \forall x?Q(x).$$

Here is a proof of this entailment. Recall that $?Px$ and $?Qy$ abbreviate, respectively, the inquisitive disjunctions $Px \vee\!\!\!\vee \neg Px$ and $Qy \vee\!\!\!\vee \neg Qy$.

$$
\begin{prooftree}
\AxiomC{$\forall x?Px$}
\RightLabel{$(\forall\mathsf{e})$}
\UnaryInfC{$?Py$}
\AxiomC{$[Py]_2$}
\AxiomC{$\forall x(Px \leftrightarrow \neg Qx)$}
\RightLabel{$(\forall\mathsf{e})$}
\UnaryInfC{$Py \leftrightarrow \neg Qy$}
\RightLabel{$(\wedge\mathsf{e})$}
\UnaryInfC{$Py \to \neg Qy$}
\RightLabel{$(\to\mathsf{e})$}
\BinaryInfC{$\neg Qy$}
\RightLabel{$(\vee\!\!\!\vee\mathsf{i})$}
\UnaryInfC{$?Qy$}
\AxiomC{$[\neg Py]_2$}
\AxiomC{$[\neg Qy]_1$}
\AxiomC{$\forall x(Px \leftrightarrow \neg Qx)$}
\RightLabel{$(\forall\mathsf{e})$}
\UnaryInfC{$Py \leftrightarrow \neg Qy$}
\RightLabel{$(\wedge\mathsf{e})$}
\UnaryInfC{$\neg Qy \to Py$}
\RightLabel{$(\to\mathsf{e})$}
\BinaryInfC{Py}
\RightLabel{$(\to\mathsf{e})$}
\BinaryInfC{$\bot$}
\RightLabel{$(\to\mathsf{i}, 1)$}
\UnaryInfC{$\neg\neg Qy$}
\RightLabel{$(\neg\neg)$}
\UnaryInfC{Qy}
\RightLabel{$(\vee\!\!\!\vee\mathsf{i})$}
\UnaryInfC{$?Qy$}
\RightLabel{$(\vee\!\!\!\vee\mathsf{e}, 2)$}
\TrinaryInfC{$?Qy$}
\RightLabel{$(\forall\mathsf{i})$}
\UnaryInfC{$\forall x?Qx$}
\end{prooftree}
$$

Note how intuitive and familiar the reasoning is. Even though we manipulate questions, we do so by relatively standard and recognizable logical principles, such as the usual rules for disjunction and the universal quantifier.

[3]This result is an analogue in our setting of a result by Kontinen and Väänänen (2013), who axiomatized the classical consequences of first-order dependence logic.

6.2.2. Example. Next, consider the entailment of Example 5.5.2:

$$\forall x?Px \models ?\forall xPx$$

As we discussed above, this entailment depends crucially on the assumption that the domain of quantification is fixed. It is thus not too surprising that a proof of it makes crucial use of the constant domain rule CD. The last step, marked by (Def), is not a real inference step, but merely a rewriting of the conclusion according to our notational conventions.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x?Px}{?Py}\,(\forall\mathsf{e}) \qquad \dfrac{[Py]_2}{Py \mathbin{\vee\mkern-12mu\vee} \neg\forall xPx}\,(\mathbin{\vee\mkern-12mu\vee}\mathsf{i}) \qquad \dfrac{\dfrac{\dfrac{[\neg Py]_2 \qquad \dfrac{[\forall xPx]_1}{Py}\,(\forall\mathsf{e})}{\bot}\,(\to\mathsf{i})}{\neg\forall xPx}\,(\to\mathsf{i},1)}{Py \mathbin{\vee\mkern-12mu\vee} \neg\forall xPx}\,(\mathbin{\vee\mkern-12mu\vee}\mathsf{i})}{Py \mathbin{\vee\mkern-12mu\vee} \neg\forall xPx}\,(\mathbin{\vee\mkern-12mu\vee}\mathsf{e},2)}{\forall x(Px \mathbin{\vee\mkern-12mu\vee} \neg\forall xPx)}\,(\forall\mathsf{i})}{\forall xPx \mathbin{\vee\mkern-12mu\vee} \neg\forall xPx}\,(\mathrm{CD})}{?\forall xPx}\,(\mathrm{Def})$$

6.2.3. Example. For a more complex example, involving both mention-all and mention-some questions, recall the entailment we discussed in Example 5.5.3:

$$\exists xP(x),\ \forall x?P(x)\ \models\ \exists xP(x).$$

This captures the fact that given the assumption that the extension of P is non-empty, from the information about what the extension of P is we can extract a witness for P. The proof of this fact makes again use of the constant domain rule, CD. Using this rule, we can show that if we are given a specification of the extension of P, from this information we can get either a witness for P, or the information that no object satisfies P. Here is the proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall x?P(x)}{?P(y)}\,(\forall\mathsf{e}) \qquad \dfrac{\dfrac{[P(y)]_1}{\exists xP(x)}\,(\exists\mathsf{i})}{\exists xP(x) \mathbin{\vee\mkern-12mu\vee} \neg P(y)}\,(\mathbin{\vee\mkern-12mu\vee}\mathsf{i}) \qquad \dfrac{[\neg P(y)]_1}{\exists xP(x) \mathbin{\vee\mkern-12mu\vee} \neg P(y)}\,(\mathbin{\vee\mkern-12mu\vee}\mathsf{i})}{\exists xP(x) \mathbin{\vee\mkern-12mu\vee} \neg P(y)}\,(\mathbin{\vee\mkern-12mu\vee}\mathsf{e},1)}{\forall x(\exists xP(x) \mathbin{\vee\mkern-12mu\vee} \neg P(x))}\,(\forall\mathsf{i})}{\exists xP(x) \mathbin{\vee\mkern-12mu\vee} \forall x\neg P(x)}\,(\mathrm{CD})$$

Call this proof P_1. We can then use the conclusion of P_1 in an (inquisitive) disjunctive syllogism: since our assumption $\exists xP(x)$ allows us to rule out $\forall x\neg P(x)$, we can infer the other disjunct, $\exists xP(x)$. Here is the proof in detail (again, the step marked by (Def) is merely a re-writing of the assumption, spelling out the classical existential according to its definition).

$$\dfrac{\dfrac{\forall x?P(x)}{\exists\!\!\!\exists xP(x) \mathbin{\vee\!\!\!\vee} \forall x\neg P(x)}\,(\mathsf{P}_1) \qquad [\exists\!\!\!\exists xP(x)]_1 \qquad \dfrac{\dfrac{[\forall x\neg P(x)]_1 \quad \dfrac{\exists xP(x)}{\neg\forall\neg P(x)}\,(\mathrm{Def})}{\bot}\,(\to\mathsf{e})}{\exists\!\!\!\exists xP(x)}\,(\bot\mathsf{e})}{\exists\!\!\!\exists xP(x)}\,(\mathbin{\vee\!\!\!\vee}\mathsf{e}, 1)$$

6.2.4. Example. Finally, consider the entailment of Example 5.5.4, namely

$$\mathsf{f}(b, c) = a,\ \lambda b,\ \lambda c\ \models\ \lambda a.$$

where f is a rigid function symbol and a, b, c non-rigid constant symbols. Spelling out the definition of the identification questions, this entailment amounts to:

$$\mathsf{f}(b, c) = a,\ \exists\!\!\!\exists x(x = b),\ \exists\!\!\!\exists x(x = c)\ \models\ \exists\!\!\!\exists x(x = a).$$

Here is a proof of this entailment. The premises $\exists\!\!\!\exists x(x = b)$ and $\exists\!\!\!\exists x(x = c)$ allow us to replace the constants b and c in the premise $\mathsf{f}(b, c)$ by variables, obtaining the term $\mathsf{f}(y, z)$; since this term is rigid, the inquisitive existential can be introduced from it.

$$\dfrac{\exists\!\!\!\exists x(x = c) \qquad \dfrac{\exists\!\!\!\exists x(x = b) \qquad \dfrac{\dfrac{[z = c]_2 \quad \dfrac{[y = b]_1 \quad \mathsf{f}(b, c) = a}{\mathsf{f}(y, c) = a}}{\mathsf{f}(y, z) = a}\,(=\mathsf{e})}{\exists\!\!\!\exists x(x = a)}\,(\exists\!\!\!\exists\mathsf{i})}{\exists\!\!\!\exists x(x = a)}\,(\exists\!\!\!\exists\mathsf{e}, 1)}{\exists\!\!\!\exists x(x = a)}\,(\exists\!\!\!\exists\mathsf{e}, 2)$$

Notice once more how intuitive and familiar the proof looks. In this case, reasoning with identification questions involves using familiar rules for identity and the existential quantifier, with the only subtlety of paying attention to rigidity.

6.3 Completeness for the classical antecedent fragment

In this section we show, following Grilletti (2021), that the system described in the previous section is complete for the classical antecedent fragment of InqBQ introduced in Section 5.7.2. In fact, we will prove something slightly stronger: if $\Phi \cup \{\psi\}$ is a set of clant formulas and $\Phi \models \psi$, then there is a proof of this entailment which involves only clant formulas and which moreover does not make use of the KF rule. This completeness result is especially significant as the clant fragment is very rich, including all examples of first-order questions discussed in the previous chapter (though not implications of such questions).

6.3.1 Proof system for the classical antecedent fragment

A sub-system of our natural deduction system for InqBQ is given in Figure 6.3. This system differs from the general system in that it manipulates only clant formulas and in that the KF rule is not present. Throughout this section, we will use the notation $\vdash$ for derivability in this weaker system. A priori, even in restriction to clant formulas, this could be a weaker derivability relation than the one given by the full system in Figure 6.1; however, it will follow from the completeness result that, for clant formulas, the two relations in fact coincide with each other and with the semantic entailment relation.

The remainder of this section is devoted to the proof of the following theorem, due to Grilletti (2021).

6.3.1. Theorem (Completeness for the clant fragment).
Suppose $\Phi \cup \{\psi\} \subseteq \mathcal{L}_{Clant}$. *Then* $\Phi \models \psi \iff \Phi \vdash \psi$.

As usual, the interesting direction is completeness, since soundness follows from the soundness of each inference rule.

6.3.2 Clant-saturated theories

Throughout this section, fix a signature Σ. For simplicity, we suppose Σ to be countable, though this is not strictly required. Let $A = \{\mathsf{a}_0, \mathsf{a}_1, \dots\}$ be a a countably infinite set of rigid constants not included in Σ. Throughout this section we denote by $\mathcal{L}$ and $\mathcal{L}^A$ the set of clant sentences (thus excluding open formulas) in the signatures Σ and $\Sigma \cup \{A\}$. We denote by $\mathcal{L}_c$ and $\mathcal{L}_c^A$ the restrictions of these sets to classical sentences, not containing $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ or $\exists$.

A crucial notion for our completeness proof is the notion of a *clant-saturated theory*. A clant-saturated theory is a set of clant sentences that has the right features to be the set of sentences supported by a non-empty information state, in a model where the constants in A name the individuals in the domain.

6.3.2. Definition. [Clant saturated theories] A set of clant sentences $\Delta \subseteq \mathcal{L}^A$ is called a *clant-saturated theory* over A if it has the following properties:

- Consistency: $\bot \notin \Delta$;
- Deductive closure: for all $\varphi \in \mathcal{L}^A$, if $\Delta \vdash \varphi$ then $\varphi \in \Delta$;
- Inquisitive disjunction property: if $\varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi \in \Delta$ then $\varphi \in \Delta$ or $\psi \in \Delta$;
- Inquisitive existence property: if $\exists x\varphi \in \Delta$ then $\varphi[\mathsf{a}/x] \in \Delta$ for some $\mathsf{a} \in A$;
- Normality: if $\forall x\varphi \notin \Delta$ then $\varphi[\mathsf{a}/x] \notin \Delta$ for some $\mathsf{a} \in A$.

Conjunction

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \qquad \frac{\varphi \wedge \psi}{\varphi} \qquad \frac{\varphi \wedge \psi}{\psi}$$

Implication

$$\frac{\begin{array}{c}[\alpha]\\ \vdots\\ \psi\end{array}}{\alpha \to \psi} \qquad \frac{\alpha \quad \alpha \to \psi}{\psi}$$

Universal quantifier

$$\frac{\varphi[y/x]}{\forall x \varphi} \qquad \frac{\forall x \varphi}{\varphi[\mathsf{t}/x]} \qquad \frac{\forall x \alpha}{\alpha[t/x]}$$

Falsum

$$\frac{\bot}{\varphi}$$

Identity

$$\frac{}{t = t} \qquad \frac{\varphi[t/x] \quad t = t'}{\varphi[t'/x]}$$

Classical $\neg\neg$ elimination

$$\frac{\neg\neg\alpha}{\alpha}$$

Inquisitive disjunction

$$\frac{\varphi}{\varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi} \qquad \frac{\psi}{\varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi} \qquad \frac{\varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi \quad \begin{array}{c}[\varphi]\\ \vdots\\ \chi\end{array} \quad \begin{array}{c}[\psi]\\ \vdots\\ \chi\end{array}}{\chi}$$

Inquisitive existential

$$\frac{\varphi[\mathsf{t}/x]}{\exists x \varphi} \qquad \frac{\exists x \varphi \quad \begin{array}{c}[\varphi[y/x]]\\ \vdots\\ \psi\end{array}}{\psi}$$

$\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$-split

$$\frac{\alpha \to \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi}{(\alpha \to \varphi) \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} (\alpha \to \psi)}$$

$\exists$-split

$$\frac{\alpha \to \exists x \varphi}{\exists x(\alpha \to \varphi)}\ x \notin \mathrm{FV}(\alpha)$$

Constant domains (CD)

$$\frac{\forall x(\varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi)}{(\forall x \varphi) \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi}\ x \notin \mathrm{FV}(\psi)$$

Figure 6.3: A sound and complete natural deduction system for the classical antecedent fragment of InqBQ. In the rules, t denotes a rigid term, while t, t' denote arbitrary terms, which may or may not be rigid. In both cases, the terms must be free for x in the relevant formula. The variable α ranges over classical formulas, while φ and ψ range over clant formulas. The restrictions on variables are the same as in Figure 6.1. The differences with respect to the general proof system of Figure 6.1 are the restriction of all rules to clant formulas and the omission of the KF rule.

An important fact about clant-saturated theories is that extending a clant-saturated theory consistently with a classical sentence leads to a new clant-saturated theory.

6.3.3. Lemma. *Let Δ be a clant-saturated theory over A and α a classical sentence such that $\Delta \nvdash \neg\alpha$. Then the deductive closure*

$$\Delta + \alpha \; := \; \{\varphi \in \mathcal{L}^A \mid \Delta \cup \{\alpha\} \vdash \varphi\}$$

is itself a clant-saturated theory.

Proof. Obviously, $\Delta + \alpha$ is deductively closed. Let us consider the other requirements in turn.

- Consistency. Suppose towards a contradiction that $\bot \in (\Delta + \alpha)$. This means that $\Delta \cup \{\alpha\} \vdash \bot$, and so by the rules for implication $\Delta \vdash \neg\alpha$, contrary to assumption.

- Inquisitive disjunction property. Suppose $\varphi \vee\!\!\!\vee \psi \in (\Delta + \alpha)$. This means that $\Delta \cup \{\alpha\} \vdash \varphi \vee\!\!\!\vee \psi$, so by the rules for implication $\Delta \vdash \alpha \to \varphi \vee\!\!\!\vee \psi$. By the $\vee\!\!\!\vee$-split rule, $\Delta \vdash (\alpha \to \varphi) \vee\!\!\!\vee (\alpha \to \psi)$. By the deductive closure and the disjunction property of Δ, it follows that $(\alpha \to \varphi) \in \Delta$ or $(\alpha \to \psi) \in \Delta$. In the former case we have $\varphi \in (\Delta + \alpha)$, in the latter case $\psi \in (\Delta + \alpha)$.

- Inquisitive existence property. Similar to the previous item, using the $\exists\!\!\!\exists$-split rule and the inquisitive existence property.

- Normality condition. Suppose $\forall x\varphi \notin (\Delta + \alpha)$. This means that $\Delta \cup \{\alpha\} \nvdash \forall x\varphi$, and thus by the rules for implication $\Delta \nvdash \alpha \to \forall x\varphi$. Now, since α is a sentence, $\alpha \to \forall x\varphi$ is provably equivalent in our system to $\forall x(\alpha \to \varphi)$, and so also $\Delta \nvdash \forall x(\alpha \to \varphi)$. By the normality condition on Δ, for some $\mathsf{a} \in A$ we have $\Delta \nvdash (\alpha \to \varphi)[\mathsf{a}/x]$. Since α is a sentence, $(\alpha \to \varphi)[\mathsf{a}/x]$ amounts to $\alpha \to \varphi[\mathsf{a}/x]$. So we have $\Delta \nvdash \alpha \to \varphi[\mathsf{a}/x]$ which by the rules for implication gives $\Delta \cup \{\alpha\} \nvdash \varphi[\mathsf{a}/x]$, that is, $\varphi[\mathsf{a}/x] \notin (\Delta + \alpha)$. □

It is interesting to point out that this is the only place in the completeness proof where the two split rules play a role. We saw that the validity of these rules captures the fact that, semantically, there is always a single minimal way of strengthening a state s to support a classical formula α. Now we saw that these rules enforce the same property on the syntactic side: there is always a single minimal way to extend a clant-saturated theory to one that contains α.

Using the previous lemma, we can establish a very significant fact: a clant-saturated theory is completely determined by its classical part, i.e., by the set of classical formulas it contains.

6.3.4. Definition. Let Δ be a clant-saturated theory. The *classical part* of Δ, Δ_{cl}, is the set of classical formulas in Δ.

6.3.5. Lemma. *If Δ, Δ' are clant-saturated theories over A with $\Delta_{cl} = \Delta'_{cl}$, then $\Delta = \Delta'$.*

Proof. We prove the following claim by induction on φ: for any clant sentence φ, and for any two clant-saturated theories Δ, Δ', if $\Delta_{cl} = \Delta'_{cl}$ then $\varphi \in \Delta \iff \varphi \in \Delta'$.

- Atoms. If φ is an atom or $\bot$ then φ is a classical formula. So if two clant-saturated theories have the same classical fragment, obviously they agree on φ.

- Conjunction. Suppose φ is a conjunction $\psi \wedge \chi$ and take any clant-saturated theories Δ, Δ' with the same classical fragment. By deductive closure of these theories and the induction hypothesis we have:

$$\begin{aligned}(\psi \wedge \chi) \in \Delta &\iff \psi \in \Delta \text{ and } \chi \in \Delta \\ &\iff \psi \in \Delta' \text{ and } \chi \in \Delta' \\ &\iff \psi \wedge \chi \in \Delta'.\end{aligned}$$

- Inquisitive disjunction: this case is similar to the previous one, using the fact that for a clant-saturated theory Δ we have $\psi \vee\!\!\!\vee\, \chi \in \Delta \iff \psi \in \Delta$ or $\chi \in \Delta$, by deductive closure and the inquisitive disjunction property.

- Implication. Suppose φ is an implication $\alpha \to \psi$ (note that since φ is a clant sentence, the antecedent must be classical). Take any clant-saturated theories Δ, Δ' with the same classical fragment. Note that by deductive closure and the rules for implication we have:

$$\begin{aligned}(\alpha \to \psi) \in \Delta &\iff \Delta \vdash \alpha \to \psi \\ &\iff \Delta \cup \{\alpha\} \vdash \psi \\ &\iff \psi \in (\Delta + \alpha).\end{aligned}$$

  and similarly for Δ'. By the previous lemma, $\Delta + \alpha$ and $\Delta' + \alpha$ are clant-saturated theories. Moreover, these theories have the same classical fragment. To see this, take any classical sentence β. Using the rules for implication and the fact that Δ and Δ' have the same classical fragment, we have:

$$\begin{aligned}\beta \in (\Delta + \alpha) &\iff \Delta \vdash \alpha \to \beta \\ &\iff (\alpha \to \beta) \in \Delta \\ &\iff (\alpha \to \beta) \in \Delta' \\ &\iff \Delta' \vdash \alpha \to \beta \\ &\iff \beta \in (\Delta' + \alpha).\end{aligned}$$

Since $\Delta + \alpha$ and $\Delta' + \alpha$ are clant-saturated theories with the same classical fragment, we can then use the induction hypothesis on ψ to conclude that $\psi \in (\Delta + \alpha) \iff \psi \in (\Delta' + \alpha)$, which by what we have seen above implies $(\alpha \to \psi) \in \Delta \iff (\alpha \to \psi) \in \Delta'$.

- Universal quantifier. Suppose φ has the form $\forall x \psi$. Take any clant-saturated theories Δ, Δ' with $\Delta_{cl} = \Delta'_{cl}$. By deductive closure and normality we have $\forall x \psi \in \Delta \iff (\psi[\mathsf{a}/x] \in \Delta$ for all $\mathsf{a} \in A)$, and similarly for Δ'. Then using the inductive hypothesis we have:

$$\begin{aligned} \forall x \psi \in \Delta &\iff \psi[\mathsf{a}/x] \in \Delta \text{ for all } \mathsf{a} \in A \\ &\iff \psi[\mathsf{a}/x] \in \Delta' \text{ for all } \mathsf{a} \in A \\ &\iff \forall x \psi \in \Delta'. \end{aligned}$$

- Inquisitive existential quantifer: this case is similar to the previous one, using deductive closure and the inquisitive existence property. □

It is an interesting open question whether an analogous result holds for the full language of InqBQ.

6.3.6. Open problem. Consider a notion of saturated theory for the full language, defined as in Definition 6.3.2 but without the restriction to clant sentences. Are there two saturated theories Δ, Δ' such that $\Delta_{cl} = \Delta'_{cl}$ but $\Delta \neq \Delta'$?

6.3.3 Canonical model construction

We are now going to show that any clant-saturated theory Δ is the set of clant formulas supported by an information state in some model. To make this claim more precise, let us define the clant-theory of an information state.

6.3.7. Definition. [Clant theory of a state] Let M be a relational information model and s a state in M. The *clant theory* of s is the set of clant sentences supported by s:

$$\mathrm{Th}_{\mathrm{clant}}(M, s) = \{\varphi \in \mathcal{L}^A \mid M, s \models \varphi\}.$$

Our next goal in this section is to show that for any clant-saturated theory Δ there is a model M and a state s such that $\Delta = \mathrm{Th}_{\mathrm{clant}}(M, s)$.

To this end, we will now show how to construct a canonical model based on the clant-saturated theory Δ. As in the propositional case, the worlds in this model will be complete theories of classical formulas. However, in our setting we need complete theories Γ with the extra property that every existential sentence in Γ is witnessed by a constant $\mathsf{a} \in \Gamma$. The idea will be familiar to the reader from the completeness proof for classical first-order logic. We call such theories *classical saturated theories*.

6.3.8. Definition. [Classical saturated theories] A classical saturated theory over A is a set $\Gamma \subseteq \mathcal{L}_c^A$ of classical sentences in the signature $\Sigma \cup \{A\}$ satisfying:

- Consistency: $\bot \notin \Gamma$;
- Deductive closure: for all $\alpha \in \mathcal{L}_c^A$, if $\Gamma \vdash \alpha$ then $\alpha \in \Gamma$;
- Classical disjunction property: if $\alpha \vee \beta \in \Gamma$ then $\alpha \in \Gamma$ or $\beta \in \Gamma$;
- Classical existence property: if $\exists x\alpha \in \Gamma$ then $\alpha[\mathsf{a}/x] \in \Gamma$ for some $\mathsf{a} \in A$.

6.3.9. Lemma (Classical saturated theories are complete).
If Γ is a classical saturated theory, then for any classical sentence $\alpha \in \mathcal{L}_c^A$, exactly one of α and $\neg\alpha$ belongs to Γ.

Proof. Take $\alpha \in \mathcal{L}_c^A$. It is easy to check that the law of excluded middle $\alpha \vee \neg\alpha$ is provable in our system, and so by deductive closure it is in Γ. By the classical disjunction property, one among α and $\neg\alpha$ must be in Γ. Moreover, α and $\neg\alpha$ cannot both be in Γ, since $\alpha, \neg\alpha \vdash \bot$ and then $\bot$ would have to be in Γ as well by deductive closure, violating consistency. □

We now need a standard saturation lemma showing that every consistent set of classical formulas with an extra property can be extended to a classical saturated theory.

6.3.10. Lemma (Classical saturation lemma).
Let $\Theta \subseteq \mathcal{L}_c^A$ be a set of classical sentences satisfying:

- *Consistency: $\Theta \nvdash \bot$;*
- *Normality: for all $\alpha \in \mathcal{L}_c^A$, if $\Theta \nvdash \forall x\alpha$ then $\Theta \nvdash \alpha[\mathsf{a}/x]$ for some $\mathsf{a} \in A$.*

Then Θ can be extended to a classical saturated theory Γ.

Proof. Starting with $\Gamma_0 := \Theta$, we will define a sequence $\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \dots$ of sets, each of which satisfies consistency and normality. To define this sequence, fix an enumeration $\{\alpha_0, \alpha_1, \alpha_2, \dots\}$ of the set $\mathcal{L}_c^A$ of classical sentences. In order to define Γ_{n+1} inductively, we consider sentence α_n and distinguish three cases:

- Case 1: $\Gamma_n \vdash \neg\alpha_n$. In this case $\Gamma_{n+1} := \Gamma_n$. Consistency and normality are obviously preserved.
- Case 2: $\Gamma_n \nvdash \neg\alpha_n$ and α_n is not of the form $\exists x\gamma$. Then we let $\Gamma_{n+1} := \Gamma_n \cup \{\alpha_n\}$.

  Note that Γ_{n+1} is consistent, otherwise by the rules for negation we would have $\Gamma_n \vdash \neg\alpha_n$. Moreover, Γ_{n+1} satisfies normality. For suppose $\Gamma_{n+1} \nvdash$

$\forall x\beta$. By the rules for implication, it follows that $\Gamma_n \nvdash \alpha_n \to \forall x\beta$. Since α_n is a sentence, $\alpha_n \to \forall x\beta$ is provably equivalent to $\forall x(\alpha_n \to \beta)$, so we have $\Gamma_n \nvdash \forall x(\alpha_n \to \beta)$. By the induction hypothesis, $\Gamma_n \nvdash (\alpha_n \to \beta)[\mathsf{a}/x]$ for some $\mathsf{a} \in A$. Since α_n is a sentence, and thus does not contain free occurrences of x, this amounts to $\Gamma_n \nvdash \alpha_n \to \beta[\mathsf{a}/x]$. And again by the rules for implication, it follows that $\Gamma_{n+1} \nvdash \beta[\mathsf{a}/x]$.

- Case 3: $\Gamma_n \nvdash \neg\alpha_n$ and α_n is of the form $\exists x\gamma$. Since in our system $\neg\exists x\gamma$ is inter-derivable with $\forall x\neg\gamma$, it follows that $\Gamma_n \nvdash \forall x\neg\gamma$, and so by normality, $\Gamma_n \nvdash \neg\gamma[\mathsf{a}_i/x]$ for some i. We can then let $\Gamma_{n+1} := \Gamma_n \cup \{\alpha_n, \gamma[\mathsf{a}_i/x]\}$ where i is the least number such that Γ_n.

  Note that Γ_{n+1} is consistent, because if we had $\Gamma_n \cup \{\alpha_n, \gamma[\mathsf{a}_i/x]\} \vdash \bot$, then since $\gamma[\mathsf{a}_i/x] \vdash \alpha_n$ we would have $\Gamma_n \cup \{\gamma[\mathsf{a}_i/x]\} \vdash \bot$, and then by the rules for negation also $\Gamma_n \vdash \neg\gamma[\mathsf{a}_i/x]$, contrary to assumption.

  Moreover, with a reasoning analogous to the one given for Case 2 we can argue that Γ_{n+1} satisfies normality.

We can then let $\Gamma := \bigcup_{n\in\mathbb{N}} \Gamma_n$. We just need to check that Γ is a classical saturated theory.

Clearly, as a limit of a sequence of consistent sets, Γ is consistent. To see that Γ is deductively closed, suppose $\Gamma \vdash \alpha_n$. Since Γ is consistent, this implies $\Gamma \nvdash \neg\alpha_n$. Since $\Gamma_n \subseteq \Gamma$, this implies $\Gamma_n \nvdash \neg\alpha_n$. But then by construction $\alpha_n \in \Gamma_{n+1}$ and thus also $\alpha_n \in \Gamma$.

To see that Γ has the classical disjunction property, we reason by contraposition: suppose that $\alpha_n, \alpha_m \notin \Gamma$. By construction, this is only possible if $\Gamma_n \vdash \neg\alpha_n$ and $\Gamma_m \vdash \neg\alpha_m$, which implies $\Gamma \vdash \neg\alpha_n$ and $\Gamma \vdash \neg\alpha_m$. Since in our proof system we have $\neg\alpha_n, \neg\alpha_m \vdash \neg(\alpha_n \vee \alpha_m)$, by deductive closure we have $\neg(\alpha_n \vee \alpha_m) \in \Gamma$, which implies $\alpha_n \vee \alpha_m \notin \Gamma$.

Finally, to see that Γ has the existence property, suppose $\exists x\gamma \in \Gamma$ and suppose $\exists x\gamma = \alpha_n$. Then by consistency $\Gamma \nvdash \neg\alpha_n$ and thus also $\Gamma_n \nvdash \neg\alpha_n$. Therefore by construction we have $\gamma[\mathsf{a}/x] \in \Gamma_{n+1} \subseteq \Gamma$ for some $\mathsf{a} \in A$. ☐

We are now ready to define a canonical model for the clant-saturated theory Δ.

6.3.11. Definition. [Canonical model] Let Δ be a clant saturated theory. We define the canonical model M^c_Δ for Δ as follows:

- The set of worlds W^c_Δ is the set of classical saturated theories Γ over A such that $\Delta_{cl} \subseteq \Gamma$.
- The domain is A.
- Given a relation symbol R and a world $\Gamma \in W^c_\Delta$, $\langle \mathsf{a}_{i_1}, \dots, \mathsf{a}_{i_n}\rangle \in R_\Gamma \iff R(\mathsf{a}_{i_1}, \dots, \mathsf{a}_{i_n}) \in \Gamma$.

- The identity relation at world Γ is given by $\mathsf{a}_i \sim_\Gamma \mathsf{a}_j \iff (\mathsf{a}_i = \mathsf{a}_j) \in \Gamma$.
- For a non-rigid function symbol f and a world Γ, we let $f_\Gamma(\mathsf{a}_{i_1}, \dots, \mathsf{a}_{i_n}) = \mathsf{a}_j$ where j is the least number such that Γ contains the identity formula $f(\mathsf{a}_{i_1}, \dots, \mathsf{a}_{i_n}) = \mathsf{a}_j$.

  Notice that this j exists: the formula $\exists x(f(\mathsf{a}_{i_1}, \dots, \mathsf{a}_{i_n}) = x)$ is provable in our proof system (Exercise 6.4.2), and since it is a classical formula, by deductive closure it belongs to Γ. Then by the classical existence property Γ must contain $f(\mathsf{a}_{i_1}, \dots, \mathsf{a}_{i_n}) = \mathsf{a}_j$ for some $\mathsf{a}_j \in A$.

  As a special case, for a non-rigid constant symbol c we let $c_\Gamma = \mathsf{a}_j$ where j is the least number such that $(c = \mathsf{a}_j) \in \Gamma$.
- For a rigid function symbol f and a world Γ, we let $\mathsf{f}_\Gamma(\mathsf{a}_{i_1}, \dots, \mathsf{a}_{i_n}) = \mathsf{a}_j$ where j is the least number such that Δ (rather than Γ, as in the previous case) contains the identity formula $\mathsf{f}(\mathsf{a}_{i_1}, \dots, \mathsf{a}_{i_n}) = \mathsf{a}_j$.

  Again, such a j exists: for f rigid, the formula $\exists x(\mathsf{f}(\mathsf{a}_{i_1}, \dots, \mathsf{a}_{i_n}) = x)$ is provable in our proof system (Exercise 6.4.2; note that this time the existential is an inquisitive one), and since it is a clant formula, by deductive closure it belongs to Δ, which is clant saturated. Then by the inquisitive existence property Δ must contain $\mathsf{f}(\mathsf{a}_{i_1}, \dots, \mathsf{a}_{i_n}) = \mathsf{a}_j$ for some $\mathsf{a}_j \in A$.

  Note that the interpretation of f does not depend on the specific world Γ, but only on Δ, and thus the interpretation of f is indeed rigid.

  As a special case, for a rigid constant symbol c we let $\mathsf{c}_\Gamma = \mathsf{a}_j$ where j is the least number such that $(c = \mathsf{a}_j) \in \Delta$.

6.3.12. Lemma. *For any closed term t of the signature $\Sigma \cup \{A\}$ and point $\Gamma \in W^c_\Delta$, if $[t]_\Gamma = \mathsf{a}$ then the formula $(t = \mathsf{a})$ is in Γ.*

Proof. By induction on t.

- If t is atomic, then since t is a closed term it must be a constant c. Then by definition we have $[c]_\Gamma = \mathsf{a}_j$ for some constant a_j such that $(c = \mathsf{a}_j) \in \Gamma$ (this holds also for the case in which c is rigid, since in that case $(c = \mathsf{a}_j) \in \Delta_{cl} \subseteq \Gamma$).
- Suppose t is a complex term $t = f(t_1, \dots, t_n)$ and suppose the claim holds for $t_1, \dots, t_n$. Suppose $[t_i]_\Gamma = \mathsf{a}_{k_i}$ for each i. By induction hypothesis, Γ contains the formula $t_i = \mathsf{a}_{k_i}$ By definition we have $[f(t_1, \dots, t_n)]_\Gamma = f_\Gamma([t_1]_\Gamma, \dots, [t_n]_\Gamma) = f_\Gamma(\mathsf{a}_{k_1}, \dots, \mathsf{a}_{k_1}) = \mathsf{a}_j$ for a constant a_j such that the formula $f(\mathsf{a}_{k_1}, \dots, \mathsf{a}_{k_1}) = \mathsf{a}_j$ is in Γ. By using the rule $(=\mathsf{e})$, i.e., the replacement of identicals, we have:

$$(t_1 = \mathsf{a}_{k_1}), \dots, (t_n = \mathsf{a}_{k_n}), f(\mathsf{a}_{k_1}, \dots, \mathsf{a}_{k_1}) = \mathsf{a}_j \vdash f(t_1, \dots, t_t) = \mathsf{a}_j.$$

Since Γ contains the premises and is closed under deduction of classical formulas, Γ contains the conclusion, which is exactly the formula $t = \mathsf{a}_j$ where $\mathsf{a}_j = [t]_\Gamma$. $\square$

We are now in a position to show the truth lemma: for every classical formula, truth at a world Γ in the canonical model coincides with membership to Γ.

6.3.13. Lemma (Truth lemma). *For any point $\Gamma \in W^c_\Delta$ and any classical sentence $\alpha \in \mathcal{L}^A_c$ we have:*

$$M^c_\Delta, \Gamma \models \alpha \iff \alpha \in \Gamma.$$

Proof. By induction on α.

- α is an atomic formula $R(t_1, \ldots, t_n)$. Suppose $[t_i]_\Gamma = \mathsf{a}_{k_i}$. By the previous lemma, Γ contains the formula $t_i = \mathsf{a}_{k_i}$. We have

$$\begin{aligned} M^c_\Delta, \Gamma \models R(t_1, \ldots, t_n) &\iff \langle \mathsf{a}_{k_1}, \ldots, \mathsf{a}_{k_n} \rangle \in R_\Gamma \\ &\iff R(\mathsf{a}_{k_1}, \ldots, \mathsf{a}_{k_n}) \in \Gamma \\ &\iff R(t_1, \ldots, t_n) \in \Gamma \end{aligned}$$

  where the second biconditional uses the definition of R_Γ, and the last uses the rule (=e) of replacement of identicals and the closure of Γ under classical deduction.

- α is an atomic formula $(t = t')$. Analogous to the previous case.

- $\alpha = \bot$. The claim is obvious, since $\bot$ is not true at Γ, and not contained in Γ by the consistency requirement.

- $\alpha = (\beta \wedge \gamma)$. We have:

$$\begin{aligned} M^c_\Delta, \Gamma \models \beta \wedge \gamma &\iff M^c_\Delta, \Gamma \models \beta \text{ and } M^c_\Delta \models \gamma \\ &\iff \beta \in \Gamma \text{ and } \gamma \in \Gamma \\ &\iff \beta \wedge \gamma \in \Gamma, \end{aligned}$$

  where the second step uses the induction hypothesis and the last step the closure of Γ under classical deduction.

- $\alpha = (\beta \to \gamma)$. We have:

$$\begin{aligned} M^c_\Delta, \Gamma \models \beta \to \gamma &\iff M^c_\Delta, \Gamma \not\models \beta \text{ or } M^c_\Delta \models \gamma \\ &\iff \beta \notin \Gamma \text{ or } \gamma \in \Gamma \\ &\iff \neg\beta \in \Gamma \text{ or } \gamma \in \Gamma \\ &\iff (\neg\beta \vee \gamma) \in \Gamma \\ &\iff (\beta \to \gamma) \in \Gamma. \end{aligned}$$

Here, the second step uses the induction hypothesis. The third step uses Lemma 6.3.9. The fourth step uses the closure of Γ under classical deduction as well as the disjunction property of Γ. And the last step uses again the closure of Γ under classical deduction, since in our system the formulas $\neg\beta \vee \gamma$ and $\beta \to \gamma$ are inter-derivable.

- $\alpha = \forall x\beta$. We have:

$$\begin{aligned} M^c_\Delta, \Gamma \models \forall x\beta \quad &\iff \quad M^c_\Delta, \Gamma \models_{x\mapsto \mathsf{a}_i} \beta \text{ for all } \mathsf{a}_i \in A \\ &\iff \quad M^c_\Delta, \Gamma \models \beta[\mathsf{a}_i/x] \text{ for all } \mathsf{a}_i \in A \\ &\iff \quad \beta[\mathsf{a}_i/x] \in \Gamma \text{ for all } \mathsf{a}_i \in A \\ &\iff \quad \forall x\beta \in \Gamma. \end{aligned}$$

Here, the second step uses the fact that $M^c_\Delta, \Gamma \models_{x\mapsto \mathsf{a}_i} (x = \mathsf{a}_i)$, which justifies the substitution of a_i for x in β and the dropping of referent to the assignment, since the result is a sentence. The third step uses the induction hypothesis. For the left-to-right direction of the last step, suppose towards a contradiction that $\forall x\beta \notin \Gamma$. Then by Lemma 6.3.9, $\neg\forall x\beta \in \Gamma$. Since in our system $\neg\forall x\beta$ is inter-derivable with $\exists x\neg\beta$, by closure under classical deduction we have $\exists x\neg\beta \in \Gamma$. By the classical existence property of Γ, it follows that for some $\mathsf{a}_i \in A$, $\neg\beta[\mathsf{a}_i/x] \in \Gamma$, and therefore by Lemma 6.3.9 $\beta[\mathsf{a}_i/x] \notin \Gamma$. Hence, it is not the case that $\beta[\mathsf{a}_i/x] \in \Gamma$ for all $\mathsf{a}_i \in A$. The converse direction follows simply by closure under classical deduction, since $\forall x\beta \vdash \beta[\mathsf{a}_i/x]$. □

The next step in the proof is to show that the set of classical formulas supported by the entire universe W^c_Δ of the canonical model M^c_Δ is precisely the classical part of Δ.

6.3.14. Lemma. *For any classical sentence $\alpha \in \mathcal{L}^A_c$ we have*

$$M^c_\Delta, W^c_\Delta \models \alpha \iff \alpha \in \Delta_{cl}.$$

Proof. Suppose $\alpha \in \Delta_{cl}$. Then $\alpha \in \Gamma$ for every $\Gamma \in W^c_\Delta$. By the previous lemma, this implies $M^c_\Delta, \Gamma \models \alpha$ for every $\Gamma \in W^c_\Delta$. Since α is classical and thus truth-conditional, this implies $M^c_\Delta, W^c_\Delta \models \alpha$.

For the converse, suppose $\alpha \notin \Delta_{cl}$. We claim that $\Delta_{cl} \cup \{\neg\alpha\} \nvdash \bot$. For suppose towards a contradiction that $\Delta_{cl} \cup \{\neg\alpha\} \vdash \bot$. Then by the rules for implication we would have $\Delta_{cl} \vdash \neg\neg\alpha$, and by classical double negation elimination also $\Delta_{cl} \vdash \alpha$, whence $\alpha \in \Delta_{cl}$ by the deductive closure of Δ, which contradicts our assumption.

Moreover, $\Delta_{cl} \cup \{\neg\alpha\}$ satisfies the normality condition. To see this, suppose $\Delta_{cl} \cup \{\neg\alpha\} \nvdash \forall x\beta$. This means that $\Delta_{cl} \nvdash \neg\alpha \to \forall x\beta$, and since $\neg\alpha$ is a sentence also $\Delta_{cl} \not\vdash \forall x(\neg\alpha \to \beta)$. This implies $\forall x(\neg\alpha \to \beta) \notin \Delta_{cl}$, and thus also $\forall x(\neg\alpha \to \beta) \notin \Delta$ (since $\forall x(\neg\alpha \to \beta)$ is a classical formula). By the normality of Δ, it follows that $\Delta \not\vdash \neg\alpha \to \beta[\mathsf{a}/x]$ for some $\mathsf{a} \in A$, which implies $\Delta \cup \{\neg\alpha\} \not\vdash \beta[\mathsf{a}/x]$. A fortiori also $\Delta_{cl} \cup \{\neg\alpha\} \not\vdash \beta[\mathsf{a}/x]$.

So, we have shown that $\Delta_{cl} \cup \{\neg\alpha\}$ satisfies consistency and normality, By Lemma 6.3.10, this set can be extended to a classical saturated theory Γ. Since $\neg\alpha \in \Gamma$ we have $\alpha \notin \Gamma$, and thus by the previous lemma $M^c_\Delta, \Gamma \not\models \alpha$. Finally, since $\Gamma \in W^c_\Delta$ this implies $M^c_\Delta, W^c_\Delta \not\models \alpha$. □

The last step is to establish that the clant theory of the state W^c_Δ is precisely Δ. For this, we first note that the clant theory of this state is a clant-saturated theory.

6.3.15. LEMMA. *For any non-empty state s, $Th_{clant}(M^c_\Delta, s)$ is a clant-saturated theory.*

Proof. We have consistency since $s \neq \emptyset$. Deductive closure follows since the rules of our proof system are sound and thus anything provable is also a semantic entailment. The remaining properties hold simply by the semantics and the fact that the domain of the model is A. As an illustration, consider normality. If $\forall x\varphi \notin \mathrm{Th}_{\mathrm{clant}}(M^c_\Delta, s)$ this means that $M^c_\Delta, s \not\models \forall x\varphi$. Thus, for some $\mathsf{a} \in A$ we have $M^c_\Delta, s \not\models_{[x\mapsto\mathsf{a}]} \varphi$. By definition of the canonical model we have $M^c_\Delta, s \models_{[x\mapsto\mathsf{a}]} (x = \mathsf{a})$, and therefore by Proposition 5.4.3 $M^c_\Delta, s \not\models \varphi[\mathsf{a}/x]$, where we can drop reference to the assignment because $\varphi[\mathsf{a}/x]$ is a sentence. This means that $\varphi[\mathsf{a}/x] \notin \mathrm{Th}_{\mathrm{clant}}(M^c_\Delta, s)$. □

We can now prove the desired result.

6.3.16. LEMMA. *For any clant-saturated theory Δ, $Th_{clant}(M^c_\Delta, W^c_\Delta) = \Delta$.*

Proof. By the previous lemma, $\mathrm{Th}_{\mathrm{clant}}(M^c_\Delta, W^c_\Delta)$ is a clant saturated theory. By Lemma 6.3.14, $\mathrm{Th}_{\mathrm{clant}}(M^c_\Delta, W^c_\Delta)$ and Δ have the same classical part. Therefore by Lemma 6.3.5, these theories coincide. □

6.3.4 Completeness

In the previous section, we have seen how given a clant-saturated theory Δ we can find an information state that, among the clant sentences, satisfies all and only the sentences in Δ. In order to show completeness, what remains to be shown is the following saturation lemma.

6.3.17. LEMMA. *Let $\Phi \cup \{\psi\} \subseteq \mathcal{L}$ be a set of clant sentences such that $\Phi \not\vdash \psi$. Then there exists clant-saturated theory $\Delta \subseteq \mathcal{L}^A$ over A such that $\Phi \subseteq \Delta$ but $\psi \notin \Delta$.*

In order to show this lemma, it is useful to introduce some new notation. If Δ, Λ are sets of clant formulas with $\Lambda \neq \emptyset$, we write

$$\Delta \vdash \Lambda$$

to mean that there are formulas $\lambda_1, \ldots, \lambda_m \in \Lambda$ such that $\Delta \vdash \lambda_1 \vee\!\!\!\vee \cdots \vee\!\!\!\vee \lambda_n$.[4]

We have the following fact, which amounts to the admissibility of a cut rule for this relation. The proof involves only the rules for inquisitive disjunction, and is left as an exercise (Exercise 6.4.6).

6.3.18. Lemma (Cut admissibility). *If $\Delta \cup \{\chi\} \vdash \Lambda$ and $\Delta \vdash \Lambda \cup \{\chi\}$ then $\Delta \vdash \Lambda$.*

We are now ready to prove our saturation lemma.

Proof of Lemma 6.3.17. The proof adapts a construction by Dov Gabbay (see Gabbay, 1981, Sec. 3.3, Theorem 2). Take $\Phi \cup \{\psi\} \subseteq \mathcal{L}$ with $\Phi \not\vdash \psi$. Fix an enumeration $\varphi_0, \varphi_1, \varphi_2, \ldots$ of all clant sentences in the extended language $\mathcal{L}^A$. We will define inductively two sequences $\Delta_0 \subseteq \Delta_1 \subseteq \ldots$ and $\Lambda_0 \subseteq \Lambda_1 \subseteq \ldots$ making sure that for every n the following conditions are satisfied:

1. $\Delta_n \not\vdash \Lambda_n$;
2. $\Delta_n \cup \Lambda_n$ contains only finitely many constants $\mathsf{a} \in A$.

We start out by setting $\Delta_0 = \Phi$ and $\Lambda_0 = \{\psi\}$. The first condition amounts to $\Phi \not\vdash \psi$, which is true by assumption. The second condition is true since $\Phi \cup \{\psi\}$ is a set in the signature Σ, and so contains no constants from A.

Now inductively, in order to define the $n+1$-th elements of the sequence, we consider the sentence φ_n. Now we distinguish two cases.

- Case 1: $\Delta_n \cup \{\varphi_n\} \not\vdash \Lambda_n$. In this case, we distinguish two sub-cases.
  - Sub-case 1: φ_n is not of the form $\exists x\psi$. In this case we let $\Delta_{n+1} = \Delta_n \cup \{\varphi_n\}$ and $\Lambda_{n+1} = \Lambda_n$. Obviously conditions 1 and 2 are satisfied for the pair $\Delta_{n+1}, \Lambda_{n+1}$.
  - Sub-case 2: φ_n is of the form $\exists x\psi$. In this case, take the first fresh constant $\mathsf{a}_i \in A$ not occurring in $\Delta_n \cup \{\varphi_n\} \cup \Lambda_n$, which exists because by induction hypothesis $\Delta_n \cup \Lambda_n$ contains only finitely many constants from A. We let $\Delta_{n+1} = \Delta_n \cup \{\varphi_n, \psi[\mathsf{a}_i/x]\}$ and $\Lambda_{n+1} = \Lambda_n$.

    Obviously condition 2 is satisfied. To see that condition 1 is satisfied as well, suppose towards a contradiction that $\Delta_{n+1} \vdash \Lambda_{n+1}$, i.e.,

[4] Note that the same relation can also be formulated more symmetrically as follows: $\Delta \vdash \Lambda$ if there are $\delta_1, \ldots, \delta_n \in \Delta$ and $\lambda_1, \ldots, \lambda_m \in \Lambda$ such that $\delta_1 \wedge \cdots \wedge \delta_n \vdash \lambda_1 \vee\!\!\!\vee \cdots \vee\!\!\!\vee \lambda_n$.

$\Delta_n \cup \{\varphi_n, \psi[\mathsf{a}_i/x]\} \vdash \Lambda_n$. This means that there are $\lambda_1, \ldots, \lambda_m \in \Lambda_n$ such that there is a proof $P : \Delta_n \cup \{\varphi_n, \psi[\mathsf{a}_i/x]\} \vdash \lambda_1 \veebar \cdots \veebar \lambda_m$. Now let y be a variable not occurring in the proof P. If we replace every occurrence of a_i in the proof by y, we get a new proof P'. This replacement has no effect on any assumption besides $\psi[\mathsf{a}_i/x]$, nor on the conclusion, since we chose a_i to be such that it does not occur in these formulas. So we have $P' : \Delta_n \cup \{\varphi_n, \psi[y/x]\} \vdash \lambda_1 \veebar \cdots \veebar \lambda_m$. Now since y does not occur free in any assumption besides $\psi[y/x]$ nor in the conclusion, and given that we have the premise $\exists x\psi$ available (which is φ_n), we can use the rule ($\exists$e) to discharge the assumption $\psi[y/x]$ and obtain $P'' : \Delta_n \cup \{\varphi_n\} \vdash \lambda_1 \veebar \cdots \veebar \lambda_m$. But this means that $\Delta_n \cup \{\varphi_n\} \vdash \Lambda_n$, contrary to assumption.

- Case 2: $\Delta_n \cup \{\varphi_n\} \vdash \Lambda_n$. In this case, we must have $\Delta_n \nvdash \Lambda_n \cup \{\varphi_n\}$, otherwise by the previous lemma we would have $\Delta_n \vdash \Lambda_n$, contrary to the induction hypothesis. Once again, we distinguish two sub-cases.
  - Sub-case 1: φ_n is not of the form $\forall x\psi$. In this case we let $\Delta_{n+1} = \Delta_n$ and $\Lambda_{n+1} = \Lambda_n \cup \{\varphi_n\}$. Obviously conditions 1 and 2 are satisfied for the pair $\Delta_{n+1}, \Lambda_{n+1}$.
  - Sub-case 2: φ_n is of the form $\forall x\psi$. In this case, take the first fresh constant $\mathsf{a}_i \in A$ not occurring in $\Delta_n \cup \Lambda_n \cup \{\varphi_n\}$, which exists since by induction hypothesis $\Delta_n \cup \Lambda_n$ contains only finitely many constants from A. We let $\Delta_{n+1} = \Delta_n$ and $\Lambda_{n+1} = \Lambda_n \cup \{\varphi_n, \psi[\mathsf{a}_i/x]\}$.

    Obviously condition 2 is satisfied by the resulting pair. To see that condition 1 is satisfied as well, suppose towards a contradiction that $\Gamma_{n+1} \vdash \Lambda_{n+1}$, that is, $\Gamma_n \vdash \Lambda_n \cup \{\forall x\psi, \psi[\mathsf{a}_i/x]\}$. Then there must be sentences $\lambda_1, \ldots, \lambda_m \in \Lambda_n$ such that there is a proof $P : \Gamma_n \vdash \lambda_1 \veebar \cdots \veebar \lambda_m \veebar \forall x\psi \veebar \psi[\mathsf{a}_i/x]$. Let y be a variable not occurring in this proof. Replacing a_i by y throughout P we get a new proof P'. Note that the substitution does not affect the premises of the proof, nor any of the formulas λ_i or $\forall x\psi$, since a_i was chosen in such a way as not to occur in these sentences. So we have $P' : \Gamma_n \vdash \lambda_1 \veebar \cdots \veebar \lambda_m \veebar \forall x\psi \veebar \psi[y/x]$. Since the variable y does not occur in the assumptions of the proof, we can use the rule ($\forall$i) and get $P'' : \Gamma_n \vdash \forall x(\lambda_1 \veebar \cdots \veebar \lambda_m \veebar \forall x\psi \veebar \psi)$. Since x does not occur free in any disjunct except ψ (as the other disjuncts are sentences), by the (CD) rule we get a proof $P''' : \Gamma_n \vdash \lambda_1 \veebar \cdots \veebar \lambda_m \veebar \forall x\psi \veebar \forall x\psi$. (Note that this is the only point in the completeness proof where the rule (CD) is used.) Recalling that $\forall x\psi$ is simply φ_n, this means that we have $\Gamma_n \vdash \Lambda_n \cup \{\varphi_n\}$, contrary to assumption.

Finally, let $\Delta = \bigcup_{n\in\mathbb{N}} \Delta_n$ and $\Lambda = \bigcup_{n\in\mathbb{N}} \Lambda_n$. Notice that we have $\Delta \nvdash \Lambda$, otherwise we should also have $\Delta_n \vdash \Lambda_n$ for some n, which is not the case by construction. This implies, in particular, that Δ and Λ are disjoint.

Moreover, Δ and Λ partition the set of all clant sentences. For given a clant sentence φ_n, by construction this is going to be included either in Δ_{n+1} (and thus in Δ) or in Λ_{n+1} (and thus in Λ).

We are going to show that Δ is the clant-saturated theory that we need. First, we have $\Phi \subseteq \Delta$ (since $\Phi = \Delta_0$) and $\psi \notin \Delta$ (since ψ is in Λ_0, and so also in Λ, which is disjoint from Δ). It remains to be seen that Δ is clant-saturated.

- Consistency. We have $\bot \notin \Delta$, for if we had $\bot \in \Delta$, then since Λ is non-empty, by the ex-falso rule we would have $\Delta \vdash \Lambda$.
- Deductive closure. If $\Delta \vdash \varphi$ for a clant sentence φ, then φ must be in Δ. Otherwise, φ would have to be in Λ, and then we would have $\Delta \vdash \Lambda$.
- Inquisitive disjunction property. Suppose $\varphi \vee\!\!\!\vee \psi \in \Delta$. Then at least one of φ and ψ must be in Δ. For otherwise, both φ and ψ would be in Λ, and then since $\Delta \vdash \varphi \vee\!\!\!\vee \psi$ with $\varphi, \psi \in \Lambda$ we would have $\Delta \vdash \Lambda$.
- Inquisitive existence property. Suppose $\exists x\psi \in \Delta$. Then $\Delta \cup \{\exists x\psi\} \nvdash \Lambda$. Now suppose $\exists x\psi$ is enumerated as φ_n. A fortiori, since $\Delta_n \subseteq \Delta$ and $\Lambda_n \subseteq \Lambda$, we have $\Delta_n \cup \{\varphi_n\} \nvdash \Lambda_n$. In this case, by construction $\psi[\mathsf{a}/x] \in \Delta_{n+1} \subseteq \Delta$ for some constant $\mathsf{a} \in A$.
- Normality. Suppose $\forall x\psi \notin \Delta$, and suppose $\forall x\psi$ is enumerated as φ_n. Then $\varphi_n \notin \Delta_{n+1}$, and by construction this is only the case if we had $\Delta_n \cup \{\varphi_n\} \vdash \Gamma_n$, i.e., if Case 2 applied to φ_n. In this case, by construction we have $\psi[\mathsf{a}/x] \in \Lambda_{n+1} \subseteq \Lambda$ for some constant $\mathsf{a} \in A$. And since Λ is disjoint from Δ, $\psi[\mathsf{a}/x] \notin \Delta$. □

With this saturation lemma in place, we are finally in a position to prove completeness. We first do so for the case of sentences.

6.3.19. THEOREM. *Suppose $\Phi \cup \{\psi\} \subseteq \mathcal{L}$ be a set of clant sentences. If $\Phi \models \psi$, then $\Phi \vdash \psi$.*

Proof. By contraposition, suppose $\Phi \nvdash \psi$. Then by the previous lemma we can find a clant saturated theory Δ in the extended language $\mathcal{L}^A$ with $\Phi \subseteq \Delta$ and $\psi \notin \Delta$. By Lemma 6.3.16 there is a model M and an information state s in M such that Δ is exactly the set of clant sentences supported at s. In particular, s supports all formulas in Φ but not ψ, which shows that $\Phi \not\models \psi$. □

Finally, we can extend this result easily to open clant formulas.

Proof of Theorem 6.3.1. Suppose $\Phi \cup \{\psi\}$ is a set of clant formulas in a signature Σ such that $\Phi \nvdash \psi$. Let Σ^* be a larger signature obtained by adding a rigid

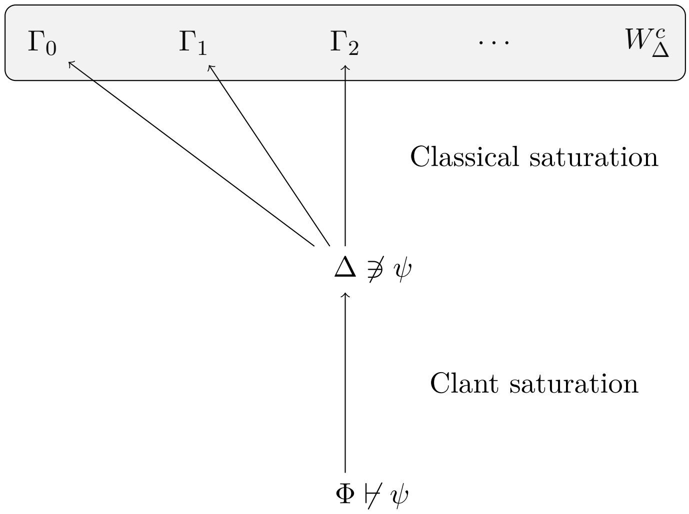


Figure 6.4: Schema showing the proof of completeness for the clant fragment. In the first step, we extend the given set Φ to a clant saturated theory Δ; in the second step, we consider all classical saturations of Δ, and take these as universe for our canonical model M^{c}_{Δ}.

constant c_x for each variable x occurring free in $\Phi \cup \{\psi\}$. Let $\Phi^* \cup \{\psi^*\}$ be the set of sentences obtained from $\Phi \cup \{\psi\}$ by replacing each free occurrence of x by c_x. We have that $\Phi^* \nvdash \psi^*$. For if we had $\Phi^* \vdash \psi^*$, then it would be easy to turn a proof of this into a proof of $\Phi \vdash \psi$. Thus by the previous Theorem, we have a model M and a state s such that $M, s \models \Phi^*$ but $M, s \not\models \psi^*$. Then, defining an assignment g such that $g(x) = [\mathsf{c}_x]$ for all variables x we have $M, s \models_g \Phi$ and $M, s \not\models_g \psi$, which shows that $\Phi \not\models \psi$. □

A graphical illustration of the strategy of our completeness proof is given in Figure 6.4. Note that the completeness theorem gives us an alternative route to two key meta-theoretic proerties of the clant fragment, which we had obtained in Section 5.7.2 via a translation to classical first-order logic. First, entailment among clant formulas is compact (Theorem 5.7.9): if $\Phi \cup \{\psi\} \subseteq \mathcal{L}_{\text{Clant}}$ and $\Phi \models \psi$, the entailment is witnessed by a proof, in which only a finite set Φ_0 of assumptions from Φ can appear; thus, $\Phi_0 \models \psi$ for some finite $\Phi_0 \subseteq \Phi$. Second, since there is a procedure to systematically generate all possibly proofs, it is possible to recursively enumerate the valid finitary entailments among clant formulas (Theorem 5.7.10).

As a further corollary, we also get a completeness theorem for the relation of id-entailment among clant formulas, obtained by restricting to id-models.

Indeed, recall that by Proposition 5.5.31 we have $\Phi \models_{\mathsf{id}} \psi \iff \Phi, \forall x \forall y ?(x = y) \models \psi$. Notice that the formula $\forall x \forall y ?(x = y)$ is a clant formula. Therefore, by our completeness theorem we have $\Phi \models_{\mathsf{id}} \psi$ iff in our proof system for clant, ψ can be derived from Φ with the additional premise $\forall x \forall y ?(x = y)$. Thus, if we extend our proof system by taking $\forall x \forall y ?(x = y)$ as an axiom, we get a system which is sound and complete for id-entailment.

6.3.20. Corollary (Completeness for id-entailment).
Let $\Phi \cup \{\psi\} \subseteq \mathcal{L}_{Clant}$. We have $\Phi \models_{\mathsf{id}} \psi \iff \psi$ is derivable from Φ in the proof system of Figure 6.3 augmented with the axiom $\forall x \forall y ?(x = y)$.

We can also use our results above to show that every non-entailment among clant formulas can be refuted in a countable model, i.e., to establish the following theorem.

6.3.21. Theorem (Existence of countable countermodels).
Suppose $\Phi \cup \{\psi\} \subseteq \mathcal{L}_{Clant}(\Sigma)$ for Σ a countable signature. If $\Phi \not\models \psi$ then there is a model $M = \langle W, D, I, \sim \rangle$ with $\#W \leq \aleph_0$ and $\#D \leq \aleph_0$, and an assignment g, such that $M, W \models_g \Phi$ and $M, W \not\models_g \psi$.

Note that the theorem does not follow directly from our canonical model construction, since the canonical models we constructed above are based on a countable domain D but have an uncountable universe W. To establish our result, we first prove a lemma: in a model with infinite domain D, every formula that can be refuted at all can in fact be refuted on state of size at most $\#D$.

6.3.22. Lemma. *Suppose $M = \langle W, D, I, \sim \rangle$ is a relational information model with D infinite, s a state in M, and g an assignment. For all formulas $\varphi \in \mathcal{L}^{\mathsf{Q}=}$, if $M, s \not\models_g \varphi$, there is a state $t \subseteq s$ with $\#t \leq \#D$ such that $M, t \not\models_g \varphi$.*

Proof. By induction on φ. If φ is an atomic sentence or $\bot$, by truth-conditionality $M, s \not\models_g \varphi$ implies $M, t \not\models_g \varphi$ for some singleton state t, so the claim holds. We only spell out the most interesting case of the inductive step, namely, the one for $\varphi = \exists x \psi$ (for the case of $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ the idea is the same as in the proof of Proposition 5.6.3; for the remaining cases, see Exercise 5.9.6).

So, let $\kappa = \#D$ and suppose $M, s \not\models_g \exists x \psi$. This means that for every $d \in D$ we have $M, s \not\models_{g[x \mapsto d]} \psi$. By the induction hypothesis, we thus have a substate $t_d \subseteq s$ with $\#t_d \leq \#D$ such that $M, t_d \not\models_{g[x \mapsto d]} \psi$. Now let $t = \bigcup_{d \in D} t_d$. Since t is the union of κ sets each of which has cardinality at most κ, we have $\#t \leq \kappa \cdot \kappa$, which is equal to κ since κ is infinite. For any given d, we have $t_d \subseteq t$ and so by persistency $M, t \not\models_{g[x \mapsto d]} \psi$. Therefore, $M, t \not\models_g \exists x \psi$, which completes the inductive step for $\exists$. $\square$

With this lemma at hand, we are now able to prove Theorem 6.3.21.

Proof of Theorem 6.3.21. Suppose $\Phi \cup \{\psi\} \subseteq \mathcal{L}_{\text{Clant}}(\Sigma)$ where Σ is countable, and suppose $\Phi \not\models \psi$. It follows from our canonical model construction above that there is a model $M = \langle W, D, I, \sim \rangle$ with a countable domain D, and an assignment g, such that $M, W \models_g \Phi$ but $M, W \not\models_g \psi$. By Lemma 6.3.22, there is a countable substate $t \subseteq W$ with $M, t \not\models_g \psi$. Moreover, by persistency we have $M, t \models_g \Phi$. Now the restriction of M to t, $M_{|t}$, is a model with countable universe t and countable domain D, and by locality we have $M_{|t}, t \models_g \Phi$ and $M_{|t}, t \not\models_g \psi$. $\square$

6.4 Exercises

6.4.1. Exercise. [Natural deduction for InqBQ]
Consider again the (non)entailments in Exercise 5.9.4. For those entailments that are valid in InqBQ, give natural deduction proofs.

6.4.2. Exercise. [Identity]
Using the rules given in Figure 6.1, prove the following facts:

- Symmetry: if t, t' are any terms, $(t = t') \vdash (t' = t)$.
- Transitivity: if t, t', t'' are any terms, $(t = t'), (t' = t'') \vdash (t = t'')$.
- Existence of referent: if t is a term not containing x, $\vdash \exists x(t = x)$.
- Identifiability of referent: if t is a rigid term not containing x, $\vdash \exists\!\!\!\exists x(t = x)$.

6.4.3. Exercise. [Rigidity]
We saw in Section 5.5.4 that, if t is not rigid, then the entailments $\varphi(t) \models \exists\!\!\!\exists x \varphi(x)$ and $\forall x \varphi(x) \models \varphi(t)$ are not generally valid. However these entailments become generally valid if we add the premise λt stating that we can identify the referent of t (recall that $\lambda t := \exists\!\!\!\exists x(x = t)$ for some x not occurring in t). That is, we have:

- $\varphi(t), \lambda t \models \exists\!\!\!\exists x \varphi(x)$;
- $\forall x \varphi(x), \lambda t \models \varphi(t)$.

Show that these entailments are indeed valid by giving natural deduction proofs.

6.4.4. Exercise. [Classical existential]
Show that the rules for classical existential given in Figure 6.2 are implicitly available in the proof system of Figure 6.1. That is, show that if $\vdash$ denotes derivability in our system of Figure 6.1, we have:

- if $\psi \in \mathcal{L}^{\mathsf{Q}=}$ and t is a term free for x in ψ, $\psi[t/x] \vdash \exists x\psi$;
- if $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\mathsf{Q}=}, \alpha \in \mathcal{L}_c^{\mathsf{Q}=}$, and y is a variable free for x in ψ which does not occur free in $\Phi \cup \{\alpha\}$, then if $\Phi, \psi[y/x] \vdash \alpha$, also $\Phi, \exists x\psi \vdash \alpha$.

6.4.5. Exercise. [Provable classicality of negations]
Complete the inductive proof of Proposition 6.1.2.

6.4.6. Exercise. [Admissibility of cut]
Recall that, given two sets Φ, Ψ of InqBQ formulas, we write $\Phi \vdash \Psi$ if for some $n \geq 0$ there are $\psi_1, \dots, \psi_n \in \Psi$ such that $\Phi \vdash \psi_1 \mathbin{\vee\!\!\!\vee} \cdots \mathbin{\vee\!\!\!\vee} \psi_n$ (such that $\Phi \vdash \bot$, in case $n = 0$). Prove that for any sets Φ, Ψ and any formula χ:

$$\Phi \cup \{\chi\} \vdash \Psi \text{ and } \Phi \vdash \{\chi\} \cup \Psi \text{ implies } \Phi \vdash \Psi.$$

Chapter 7

Relations with Dependence Logic

We saw how by bringing questions into play in logic we can capture dependency relations as cases of (contextual or logical) entailment, and we can analyze such relations using standard tools of logic. The notion of dependency is central to another line of work in logic which has received much attention in recent years, with the rise of *dependence logic* (Väänänen, 2007) and other related logics based on so-called *team semantics* (Hodges, 1997a,b).

In this chapter, we discuss some of the similarities and differences between inquisitive logic and dependence logic, in particular with regard to the treatment of dependencies in these two frameworks. We will focus here on the standard system of dependence logic; for the connections between dependence logic and inquisitive logic in the propositional and modal setting, which are also significant, the reader is referred to Ciardelli (2016b,a) and to Yang and Väänänen (2016).

Dependence logic is similar to inquisitive semantics in many respects. First, like inquisitive logic, it aims to achieve a conservative extension of classical logic with a new kind of formulas; in the case of inquisitive logic, the new formulas are complex formulas expressing questions; in the case of dependence logic, they are atomic formulas expressing dependencies. Moreover, in both cases, the extension is made possible by revising the standard semantics of classical logic, replacing standard points of evaluations by sets of such points: in the case of inquisitive logic, the relevant points are possible worlds, modeling states of affairs; in the case of dependence logic, they are assignment functions fixing the values of variables. Thus, the semantics of dependence logic is given relative to sets of assignments, called *teams*. As in the case of inquisitive logic, this can (although it need not) be seen as a semantics where formulas are evaluated with respect to states of partial information (cf. Galliani, 2012a,b), where this information concerns the values of variables rather than the state of affairs. Moreover, like inquisitive logic, dependence logic (though not its variants, such as independence logic (Grädel and Väänänen, 2013) and inclusion logic (Galliani, 2012a)) satisfies persistency with respect to the information ordering. This means that the logical operators that can be naturally defined in these logics are essentially the same; and indeed, many of the operators we considered in inquisitive logic have also been explored independently in the dependence logic literature (see in particular Abramsky and Väänänen, 2009); at the same time, sometimes the different motivations and history of the two traditions are reflected in different choices of logical repertoire. Nevertheless, the tight formal similarity between the two approaches allows for a fruitful transfer of results and insights between them. In fact, many logical systems can be legitimately regarded either as systems of dependence logic or as systems of inquisitive logic (this applies, e.g., to the system InqBT that we will discuss in Section 7.4); the difference between the two traditions is mostly one of aims and conceptual perspective, which is sometimes, but not always, reflected in different technical setup choices.

One significant difference between dependence logic and inquisitive logic is the conceptualization of the dependency relation. In dependence logic, dependency is viewed as a relation holding between variables, whereas in inquisitive logic, it is viewed as a relation between questions. In this chapter we will explore the connection in detail and we will argue that, while both perspectives are meaningful and natural, the question-based perspective has some important assets to it: it is more general, allowing us to capture a broader spectrum of dependence facts, and it allows us to connect dependency directly to the central notions of logic, including entailment, proofs, and the implication operator.

The chapter is structured as follows. We start in Section 7.1 by introducing the variable-based perspective on dependency, which long predates the rise of dependence logic and has received much attention in database theory. In Section 7.2 we present the standard version of dependence logic, which combines this conception of dependency with the idea of giving a team semantics for predicate logic. In Section 7.3 we show how the key ideas of inquisitive logic apply naturally in the team semantic setting, yielding a question-based perspective on dependency, and we discuss some attractions of this perspective. In Section 7.4 we illustrate this general point by showing that the inquisitive first-order logic of the previous chapter can be naturally adapted to the team semantic setting; we discuss the sort of questions and dependencies that can be captured in this system, and we mention some important open problems about the resulting logic. In Section 7.5 we show how to interpret inquisitive first-order logic in a more general semantic setup, from which both the standard semantics of the previous chapter and the team-based semantics discussed in the present chapter can be obtained as special cases. Finally, in Section 7.6 we conclude with a summary and some further considerations.

7.1 V-dependency in a team

The starting point to understand the analysis of dependence in dependence logic is the notion of a *team*. A team is a set of assignment functions.[1]

7.1.1. DEFINITION. [Teams]
A *team* over a domain D is a set of assignments $g : \mathsf{Var} \to D$.

We can visualize a team as a table, where the columns correspond to the variables, the rows to the assignments in the team, and the cell corresponding to assignment g and variable x contains the value $g(x)$. For instance, Figure 7.1 represents a team of six assignments over the domain of natural numbers; only the values of these assignments on the variables x, y, z are displayed.

T_1	x	y	z
g_1	2	1	1
g_2	2	1	1
g_3	4	2	1
g_4	4	3	2
g_5	6	4	2
g_6	6	4	2

Figure 7.1: An example of a team, where only the values of assignments on the variables x, y, z are displayed.

In the context of such a table, it makes sense to ask whether the value of a variable is or is not determined by the values of other variables. For instance, in the team of Figure 7.1, the value of z is determined by the value of y: if we are told the value of y on a given row in the table, we can infer from it the corresponding value of z. Conversely, the value of y is not determined by the value of z: if we are given the information that the value of z is 1, for instance, we are unable to reconstruct from that the value of y. However, the value of y is jointly determined by the values of x and z: if we are given both the value of x and the value of z on a given row, we can infer the corresponding value of y.

Generalizing, we can view dependency as a relation that may or may not hold between variables in the context of a team. We will refer to this relation here as *v-dependency*, to contrast it with the *q-dependency* relation to be discussed below, which is a relation between questions.

[1] In dependence logic, a team is a set of *partial* assignments, i.e., partial functions from Var to the domain of the given model. While this is convenient in practice, here we will stick with total assignments, simply to avoid having to make stipulations about cases in which the value of a term is undefined. This difference is not essential to the points discussed below.

7.1.2. Definition. [v-dependency]
Let T be a team. A set X of variables determines a variable y in the context of T, denoted $\mathbb{D}_T(X; y)$, if for every $g, g' \in T$, if $g(x) = g'(x)$ for all $x \in X$, then $g(y) = g'(y)$. We write $\mathbb{D}_T(x_1, \ldots, x_n; y)$ as a short-hand for $\mathbb{D}_T(\{x_1, \ldots, x_n\}; y)$. We refer to $\mathbb{D}_T$ as the relation of *v-dependency*.

Focusing for simplicity on the case of a finite set of premises, we can phrase the relation $\mathbb{D}_T(x_1, \ldots, x_n; y)$ equivalently in terms of the existence of a functional dependency f that yields the value of y from the values of $x_1, \ldots, x_n$:

$$\mathbb{D}_T(x_1, \ldots, x_n; y) \iff \exists f : D^n \to D \text{ such that } \forall g \in T : g(y) = f(g(x_1), \ldots, g(x_n)).$$

Thus, for instance, in the team X_1 of Figure 7.1:

- $\mathbb{D}_T(y; z)$ holds;
- $\mathbb{D}_T(z; y)$ does not hold;
- $\mathbb{D}_T(x, z; y)$ holds.

The relation of v-dependency was well-studied long before the rise of dependence logic, especially in the context of database theory (see Fagin and Vardi, 1986, for an overview). The most celebrated result about this relation is that the following three principles, known as Armstrong's axioms, completely characterize the logic of v-dependency in a natural sense (Armstrong, 1974).[2]

1. $\mathbb{D}_T(X; x)$ for any $x \in X$;
2. $\mathbb{D}_T(X; y)$ implies $\mathbb{D}_T(X'; y)$ for all $X' \supseteq X$;
3. $\mathbb{D}_T(Y; z)$ and $\mathbb{D}_T(X; y)$ for all $y \in Y$ implies $\mathbb{D}_T(X; z)$.

These axioms are formally analogous to Tarski's axioms for a consequence relation, with the difference that a consequence relation is defined on formulas rather than variables. We will come back to this point in Section 7.3.4.

[2] In fact, the second property below is implied by the first and the third. Armstrong's axioms are often formulated in terms of a relation $\mathbb{D}_T(X; Y)$ between two sets of variables. However, this relation is distributive in the second component (X determines Y just in case X determines y for each $y \in Y$) and so is reducible to the relation we consider here, which has a single variable in the second component.

7.2 Dependence logic

7.2.1 Historical notes

The line of work leading to dependence logic originates with Henkin's observation that certain patterns of quantification over individuals are not expressible in first-order logic. For instance, it is impossible to write a first-order formula expressing that for every x and x', there exist a y determined only by x and a y' determined only by x', such that a certain formula $\varphi(x, x', y, y')$ holds. To provide the tools to express such patterns, Henkin (1961) introduced so-called *branching quantifiers*, and Hintikka and Sandu (1997) later developed this work in the framework of Independence Friendly (IF) logic, which allows for quantified variables to be explicitly marked as independent of other variables. IF logic was claimed by Hintikka not to allow for a compositional semantics based on a recursive definition of truth/satisfaction. However, Hodges (1997a,b) showed that such a semantics could in fact be given in the framework of *team semantics*, where formulas are evaluated relative to a relational structure and a team—a set of variable assignments (Hodges used the term *trump* instead of *team*, but the latter term has since become standard).

Building on the ideas of team semantics, Väänänen (2007) proposed a new approach to the issue. He noticed that the team semantics context allows us to interpret a new kind of atomic formula $=(x_1, \ldots, x_n; y)$ expressing the fact that the value of y is determined by the values of $x_1, \ldots, x_n$. In this way, dependency and quantification may be disentangled. In the Dependence Logic system that he proposed, the syntax of quantification is standard, and the expression of dependencies between quantified variables is delegated to the new dependence atoms. Thus, e.g., the pattern of quantification mentioned above may be expressed as follows:

$$\forall x \forall x' \exists y \exists y' (\, =(x; y) \,\wedge\, =(x'; y') \,\wedge\, \varphi(x, x', y, y')).$$

Due to the similarity between individual variables in predicate logic and propositional variables in propositional logic, dependence atoms have later been considered also in the setting of propositional and modal logic (Väänänen, 2008; Yang, 2014; Yang and Väänänen, 2016). In this setting, a dependence atom has the form $=(p_1, \ldots, p_n; q)$, and it is interpreted, relative to a set s of possible worlds, as expressing that the truth-value that a world $w \in s$ assigns to q is determined by the truth-values it assigns to $p_1, \ldots, p_n$.

At the same time, it was soon noticed that the basic idea of dependence logic could be used to extend classical predicate logic with other kinds of atoms expressing interesting relations between variables that only become "visible" at the level of teams, such as *independence* (Grädel and Väänänen, 2013) and *inclusion* (Galliani, 2012a). In this way, the study of dependence logic evolved into a more general study of team-based logics which extend predicate logic with formulas expressing global properties of teams.

We cannot do justice here to the large amount of recent literature on these topics; for an overview, a good starting point is the Stanford Encyclopedia entry on Dependence Logic (Galliani, 2021).

7.2.2 The standard system D

In this section, we introduce the standard version of dependence logic, a logical system D introduced by Väänänen (2007) which conservatively extends classical first-order logic with formulas expressing dependencies between variables.

The language $\mathcal{L}^{\mathsf{D}}$ of first-order dependence logic is obtained by introducing, besides the usual atomic formulas of predicate logic, new atomic formulas called *dependence atoms*, having the form $=(x_1, \ldots, x_n, y)$, where $x_1, \ldots, x_n, y \in \mathsf{Var}$. $\mathcal{L}^{\mathsf{D}}$ does not have a primitive negation operator, but instead includes negative versions of the standard atoms of predicate logic, denoted $\neg R(t_1, \ldots, t_n)$ and $t \neq t'$. Complex formulas can be formed by means of conjunction $\wedge$, a "tensor disjunction" $\otimes$, and two quantifiers $\forall^d$ and $\exists^d$.[3] Thus, the language $\mathcal{L}^{\mathsf{D}}$ is given by the following definition, where $\bar{t} = t_1, \ldots, t_n$ is a tuple of terms matching the arity of R, and $\overline{x} = x_1, \ldots, x_n$ is a tuple of variables:

$$\varphi \;:=\; R\bar{t} \mid \neg R\bar{t} \mid t = t' \mid t \neq t' \mid =(\overline{x}; y) \mid \varphi \wedge \varphi \mid \varphi \otimes \varphi \mid \forall^d x\varphi \mid \exists^d x\varphi.$$

Intuitively, formulas without dependence atoms correspond to formulas of classical first-order logic in negation normal form (i.e., where negation only occurs in front of atomic sentences). A dependence atom of the form $=(x_1, \ldots, x_n; y)$ stands for the claim that the values of the variables $x_1, \ldots, x_n$ determine the value of the variable y.

Semantically, the language is interpreted relative to a standard relational structure $M = \langle D, I\rangle$ and a set T of assignments $g : \mathsf{Var} \to D$, i.e., a team over D. In order to state the semantics of D, we first need to introduce some operations on teams.

7.2.1. Definition. [Operations on teams]
Let T be a team over a domain D and let $x \in \mathsf{Var}$, $d \in D$, and $f : T \to \wp^+(D)$, where $\wp^+(D) = (\wp(D) - \{\emptyset\})$. We define:

[3] My notation here diverges from the official one. In the dependence logic literature, tensor disjunction is also called *split-junction* or simply *disjunction*, and often denoted $\vee$. We will use the notation $\otimes$, which, besides avoiding conflict with our defined disjunction, $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$, brings out the fact that, from an algebraic point of view, $\otimes$ is a *quantale multiplication*, as first noted by Abramsky and Väänänen (2009). The two quantifiers are simply denoted $\forall$ and $\exists$ in the dependence logic literature. We add a superscript d to distinguish them from other quantifiers that we consider below.

- $T[x \mapsto d] = \{g[x \mapsto d] \mid g \in T\}$;
- $T[x \mapsto f] = \{g[x \mapsto d] \mid g \in T \text{ and } d \in f(g)\}$;
- $T[x \mapsto D] = \{g[x \mapsto d] \mid g \in T, d \in D\}$.

In words, $T[x \mapsto d]$ is the team that results from setting the value of x to d uniformly throughout the team; $T[x \mapsto f]$ is the team obtained by replacing each $g \in T$ by an x-variant $g[x \mapsto d]$ for each of the values $d \in f(g)$; finally $T[x \mapsto D]$ is the team obtained by taking, for each $g \in T$, all of its x-variants $g[x \to d]$ for $d \in D$.

The semantics of D can then be stated as follows, where the denotation $[t]^M_g$ of a term is defined as usual.

7.2.2. Definition. [Semantics of D]

- $M \models_T R(t_1, \ldots, t_n) \iff$ for all $g \in T$, $\langle [t_1]^M_g, \ldots, [t_n]^M_g \rangle \in I(R)$
- $M \models_T \neg R(t_1, \ldots, t_n) \iff$ for all $g \in T$, $\langle [t_1]^M_g, \ldots, [t_n]^M_g \rangle \notin I(R)$
- $M \models_T (t = t') \iff$ for all $g \in T$, $[t]^M_g = [t']^M_g$
- $M \models_T (t \neq t') \iff$ for all $g \in T$, $[t]^M_g \neq [t']^M_g$
- $M \models_T {=\!\!(}x_1, \ldots, x_n; y) \iff \mathbb{D}_T(x_1, \ldots, x_n; y)$
- $M \models_T \varphi \wedge \psi \iff M \models_T \varphi$ and $M \models_T \psi$
- $M \models_T \varphi \otimes \psi \iff T = T' \cup T''$ for some T', T'' s.t. $M \models_{T'} \varphi$ and $M \models_{T''} \psi$
- $M \models_T \forall^d x \varphi \iff M \models_{T[x \mapsto D]} \varphi$
- $M \models_T \exists^d x \varphi \iff M \models_{T[x \mapsto f]} \varphi$ for some $f : T \to \wp^+(D)$

In words, a literal (positive or negative atom) is satisfied with respect to a team T in case it is true under each assignment $g \in T$. A dependence atom ${=\!\!(}x_1, \ldots, x_n; y)$ is satisfied with respect to T if the variables $x_1, \ldots, x_n$ determine y relative to T, in the sense of Definition 7.1.2. A conjunction is satisfied iff both conjuncts are satisfied. A tensor disjunction is satisfied if the team T can be split into two (not necessarily disjoint) sub-teams with each sub-team supporting one of the disjuncts. The clauses for the quantifiers are perhaps best understood by introducing the notion of x-variant of a team. Intuitively, an x-variant of a team T is a team T' that differs from T only in the column corresponding to x.

7.2.3. DEFINITION. [x-variants]
Two assignments g, g' are x-variants, notation $g \sim_x g'$, if they coincide on every variable except possibly x (i.e., if $g\restriction_{\mathsf{Var}-\{x\}} = g'\restriction_{\mathsf{Var}-\{x\}}$). Given a team T and a set X of variables, the restriction of T to X is obtained by restricting each element of the team:

$$T\restriction_X \;:=\; \{g\restriction_X \mid g \in T\}.$$

We then say that two teams T, T' are x-variants if their restrictions to variables different from x is the same:

$$T \sim_x T' \iff T\restriction_{\mathsf{Var}-\{x\}} = T'\restriction_{\mathsf{Var}-\{x\}}.$$

Equivalently, $T \sim_x T'$ if every $g \in T$ is an x-variant of some $g' \in T'$ and every $g' \in T'$ is an x-variant of some $g \in T$.

The clauses for quantifiers can then be shown to be equivalent (in the context of the present system) to the following ones:

- $M \models_T \forall^d x\varphi \iff$ for every $T' \sim_x T,\ M \models_{T'} \varphi$;
- $M \models_T \exists^d x\varphi \iff$ for some $T' \sim_x T,\ M \models_{T'} \varphi$.

Thus, a team T satisfies $\forall^d x\varphi$ (respectively, $\exists^d x\varphi$) if every (respectively, some) way of re-assigning the interpretation of the variable x leads to a team that satisfies φ.[4]

The satisfaction relation has the same features which are familiar from the support relation, although the relevant information ordering now concerns a team T, rather than a set s of possible worlds: satisfaction is preserved as information grows (persistency property) and in the limit case of inconsistent information, every formula is trivially satisfied (empty team property).

7.2.4. PROPOSITION. *For any relational structure M and formula $\varphi \in \mathcal{L}^{\mathsf{D}}$:*

- Persistency property: $M \models_T \varphi$ *and* $Y \subseteq T$ *implies* $M \models_Y \varphi$.
- Empty team property: $M \models_\emptyset \varphi$.

[4]The equivalence between the two clauses holds in general for the existential quantifier. For the case of the universal quantifier, it holds only in the context of a logic like D which, as we'll see shortly, satisfies a persistency property. In the context of a non-persistent logic, the two clauses give rise to different interpretations of the universal quantifier. While the clause given in Definition 7.2.2 is usually taken as the "official" one in the literature, the clause given in terms of x-variants is arguably more natural from a conceptual point of view.

In analogy to what we did in inquisitive semantics, we can recover a notion of truth relative to single assignment g by defining it in terms of satisfaction by the corresponding singleton team:[5]

$$M \models_g \varphi \overset{def}{\iff} M \models_{\{g\}} \varphi.$$

If we spell out the semantic clauses with respect to singletons, we find the following truth conditions.

7.2.5. Remark. [Truth conditions for D]

- $M \models_g R(t_1, \ldots, t_n) \iff \langle [t_1]_g^M, \ldots, [t_n]_g^M \rangle \in I(R)$
- $M \models_g (t = t') \iff [t]_g^M = [t']_g^M$
- $M \models_g \neg R(t_1, \ldots, t_n) \iff \langle [t_1]_g^M, \ldots, [t_n]_g^M \rangle \notin I(R)$
- $M \models_g (t \neq t') \iff [t]_g^M \neq [t']_g^M$
- $M \models_g {=\!\!(}x_1, \ldots, x_n; y)$ always
- $M \models_g \varphi \wedge \psi \iff M \models_g \varphi$ and $M \models_g \psi$
- $M \models_g \varphi \otimes \psi \iff M \models_g \varphi$ or $M \models_g \psi$
- $M \models_g \forall^d x \varphi \iff M \models_{\{g[x \mapsto d] \,|\, d \in D\}} \varphi$
- $M \models_g \exists^d x \varphi \iff M \models_{g[x \mapsto d]} \varphi$ for some $d \in D$

Here, it is important to notice that the truth conditions for a universal formula $\forall^d \varphi$ relative to g depend on the satisfaction conditions of φ at a non-singleton team obtained by taking all x-variants of g. This means that, unlike in the other systems we encountered so far (but similarly to systems of inquisitive modal logic, see e.g. Ciardelli and Roelofsen (2015)), truth does not admit a direct recursive characterization; rather, computing the truth conditions of some formula φ in general requires computing the satisfaction conditions of some sub-formula ψ with respect to non-singleton teams.

A formula is said to be *flat* if satisfaction at a team T reduces to truth under each assignment $g \in T$. Clearly, the notion of flatness is the counterpart of the notion of *truth-conditionality* that we encountered in inquisitive logic.

7.2.6. Definition. [Flatness]
We call a formula $\varphi \in \mathcal{L}^{\mathsf{D}}$ *flat* if for any model M and team T:

$$M \models_T \varphi \iff M \models_g \varphi \text{ for all } g \in T.$$

[5] In dependence logic there seems to be no special name for this notion; I still find it very useful to have one, and I will use *truth* for the sake of consistency with the previous chapters.

It is easy to check by induction that all formulas of D without dependence atoms are flat. This means that their semantics is fully captured by their truth conditions relative to single assignments.

Moreover, these truth conditions are simply the ones familiar from the Tarskian semantics of first-order logic, when $\otimes$ is identified with disjunction and $\forall^d$ and $\exists^d$ with the quantifiers of first-order logic. This can be proved by induction. The key case is that of a universal formula $\forall^d x\varphi$ (the remaining cases are obvious from Remark 7.2.5). Since we are assuming $\forall^d x\varphi$ does not contain dependence atoms, neither does φ, and so φ is flat. Using Remark 7.2.5 and the flatness of φ we have:

$$\begin{aligned} M \models_g \forall^d x\varphi \quad &\iff \quad M \models_{\{g[x\mapsto d] \mid d\in D\}} \varphi \\ &\iff \quad \forall g' \in \{g[x \mapsto d] \mid d \in D\} : \; M \models_{g'} \varphi \\ &\iff \quad \forall d \in D : \; M \models_{g[x\mapsto d]} \varphi. \end{aligned}$$

This shows that the semantics of formulas not containing dependence atoms is a global counterpart of standard Tarskian semantics (identifying disjunction with $\otimes$ and the first-order quantifiers with $\exists^d$ and $\forall^d$). Moreover, these formulas are representative of all formulas of first-order predicate logic, as every formula φ of first-order logic can be identified with its negation normal form φ^*, obtained recursively by pushing negations in front of atoms. This means, by a reasoning analogous to the one we used for inquisitive semantics (cf. Section 2.2), that in restriction to the "classical fragment" of the language, consisting of formulas without dependence atoms, the above semantics can be seen as a non-standard semantics for classical first-order logic.

Thus, dependence logic and inquisitive first-order logic both extend classical first-order logic by using a similar strategy. We saw that inquisitive first-order logic is a conservative extension of classical first-order logic with questions, obtained by first giving a state-based semantics for classical first-order logic and then exploiting this semantics to interpret new question-forming operators. Similarly, dependence logic can be seen as a conservative extension of classical first-order logic, obtained by first giving a team-based semantics for classical first-order logic and then exploiting this semantics to interpret a new kind of atoms capturing dependencies. Moreover, in both cases the new semantics is obtained in a similar way, by moving from single "points of evaluation" to sets of such points, ensuring that for classical formulas the semantics is distributive, in the sense that satisfaction at a set of points boils down to satisfaction at each element of the set.

One difference is that, whereas in the case of standard inquisitive logic the generalization targets the model of interpretation (thus moving from a single relational structure to a set of possible worlds, each associated with such a structure), in the case of dependence logic it targets the assignment function (thus moving from a single assignment to a set of assignments). This is not an irreconcilable difference, however: as we will see in the next sections, it is possible to implement the key ideas of inquisitive logic in the setting of team semantics, and it is also possible to give a semantic framework that is a common generalization of both information state semantics and team semantics.

Before turning to that, let us look at some of the key features of D. As we expect, the support-conditions for a formula φ relative to a team T are only sensitive to the values that the assignments in T assign to the variables that occur free in φ.

7.2.7. Proposition.
Let $\varphi \in \mathcal{L}^D$. If T and T' are teams such that $T \restriction_{FV(\varphi)} = T' \restriction_{FV(\varphi)}$, then $M \models_T \varphi \iff M \models_{T'} \varphi$.

In particular, consider the case in which φ is a sentence, i.e., $\mathrm{FV}(\varphi) = \emptyset$. Then for every non-empty team T we have $T\restriction_{\mathrm{FV}(\varphi)} = \{\emptyset\}$ (since $\emptyset$ is the only function from the empty set of variables to D). Thus, for any non-empty teams T, T' we have $T\restriction_{\mathrm{FV}(\varphi)} = T'\restriction_{\mathrm{FV}(\varphi)}$, and therefore $M \models_T \varphi \iff M \models_{T'} \varphi$. Thus, if $\varphi \in \mathcal{L}^{\mathsf{D}}$ is a sentence, we can simply write $M \models \varphi$ as a shorthand for $M \models_T \varphi$, where T is an arbitrary non-empty team.

In this way, sentences of dependence logic, just as sentences in classical first-order or second-order logic, define classes of relational structures. This naturally raises the question of how the expressive power of these different systems compares.

This question was answered by Väänänen (2007), who showed that, as far as sentences are concerned, D has the same expressive power as Σ^1_1, the existential fragment of second-order logic, consisting of second-order formulas of the form

$$\exists T_1 \dots \exists T_n \varphi$$

where $T_1, \dots, T_n$ are second-order variables, and φ contains no second-order quantifiers. To state Väänänen's result, let us introduce the following terminology: if φ is a sentence in $\mathcal{L}^{\mathsf{D}}$ and ψ a sentence in Σ^1_1 (over the same signature), we will say that φ and ψ are equivalent, and write $\varphi \equiv \psi$, in case for any model M we have $M \models \varphi \iff M \models \psi$. Then, we have the following theorem.

7.2.8. Theorem (Väänänen, 2007).
There exist computable maps $(\cdot)^{eso} : \mathcal{L}^D \to \Sigma^1_1$ and $(\cdot)^d : \Sigma^1_1 \to \mathcal{L}^D$ such that:

- *for any sentence $\varphi \in \mathcal{L}^D$, $\varphi \equiv \varphi^{eso}$;*
- *for any sentence $\varphi \in \Sigma^1_1$, $\varphi \equiv \varphi^d$.*

This theorem implies that first-order dependence logic is not recursively axiomatizable. If it were, then the set of its valid sentences would be recursively enumerable. But by the previous theorem, this would imply that the set of valid Σ^1_1 sentences is recursively enumerable, which is not the case. In fact, Väänänen (2007) shows that the set of (Gödel numbers of) theorems of D is not only not recursively enumerable, but not even arithmetical, i.e., the property of being a code of a valid D-sentence is not expressible in the language of Peano arithmetic.

For a simple example of a dependence logic sentence that is not equivalent to any sentence in classical first-order logic, consider the following:

$$\exists^d x \forall^d y \exists^d z ({=}(z, y) \wedge z \neq x).$$

Spelling out the semantic clauses, one can verify that this sentence is satisfied in a model $M = \langle D, I \rangle$ iff there exists a function $f : D \to D$ which is injective but not surjective. Such a function exists iff D is infinite. Thus, the above formula is satisfied by exactly those models whose domain is infinite—thus expressing a property which is not expressible in standard first-order logic.

This example also shows that D is not entailment compact: if ξ_n is a first-order formula that says that there are at least n individuals in D, then the above sentence is entailed by the set $\{\xi_n \mid n \in \mathbb{N}\}$ (since if all ξ_n are true, D must be infinite), but not by any finite subset of this set (since the truth of finitely many ξ_n is compatible with the finiteness of D).[6]

7.3 Q-dependency

In the previous sections we discussed the relation of v-dependency in a team and we saw how first-order logic can be extended with formulas that express v-dependencies. We will now see that adopting the ideas of inquisitive logic in the team semantics setting yields another perspective on the notion of dependency. Under this perspective, dependency is viewed, not as a relation between variables, but as a relation between questions (in the way familiar from the previous chapters). We will refer to this notion of dependency as *q-dependency*. In this section, we will consider the relation between the two perspectives on dependency, and we will discuss a number of attractions of the question-based perspective.

[6]However, as shown by Väänänen (2007), D is satisfiability compact: if every finite subset of a set Φ of D formulas is satisfiable, the entire set is satisfiable. The two understandings of compactness are equivalent in classical logic, but they come apart for logics like D and InqBQ, since an entailment claim $\Phi \models \psi$ does not reduce to the claim that a certain set $\Phi \cup \{\psi^\star\}$ is unsatisfiable for some formula $\psi^\star$; this is because, due to persistency, in these systems we cannot find a formula $\psi^\star$ which is supported/satisfied just when ψ is not. See the related discussion of the two notions of compactness in Section 5.5.6.

7.3.1 Inquisitive logic in team semantics

The basic ideas of inquisitive logic, as laid out in Chapter 2, apply straightforwardly to the setting of team semantics. For instance, it is natural to consider a statement α as *supported* in the context of a team T if it is true under all assignments in T. Thus, we expect the following analogue of the Truth-Support Bridge to hold for a team T and a statement α:

$$M \models_T \alpha \iff (M \models_g \alpha \text{ for all } g \in T).$$

For an illustration, consider the team:

T_2	x	y
g_1	2	1
g_2	2	1
g_3	4	2
g_4	4	4
g_5	0	5
g_6	0	5

This team supports the statement α_1 below, but not α_2 or α_3.

(1) α_1: x is even.
α_2: y is even.
α_3: x is larger than y.

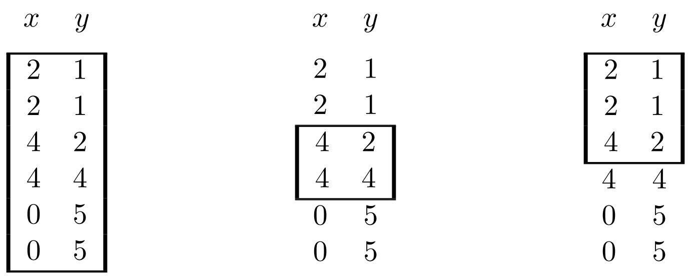
However, α_2 is supported by the sub-team $\{g_3, g_4\}$ and all subsets of this team, while α_3 is supported by $\{g_1, g_2, g_3\}$ and its subsets.

We may refer to the maximal sub-teams of a team T supporting a sentence φ as the *alternatives* for φ in T, denoted $\textsc{Alt}_T(\varphi)$. Then the alternatives for our three statements in our team T_2 are the blocks depicted in Figure 7.2.
It is equally natural to interpret questions involving variables in the context of a team. As an example, let us first consider two particular sorts of questions.

7.3.1. Example. [Identification questions, λx]
With any variable x we can associate an identification question λx, standing for the question what the value of x is. A team T settles λx if it settles what the value of x is, i.e., if every assignment $g \in T$ assigns to x the same value:

$$M \models_T \lambda x \iff \forall g, g' \in T : g(x) = g'(x).$$

7.3.2. Example. [Polar questions, $?\alpha$]
With any statement α we can associate a corresponding polar question $?\alpha$, standing for the question whether α is true or false. A team T settles the



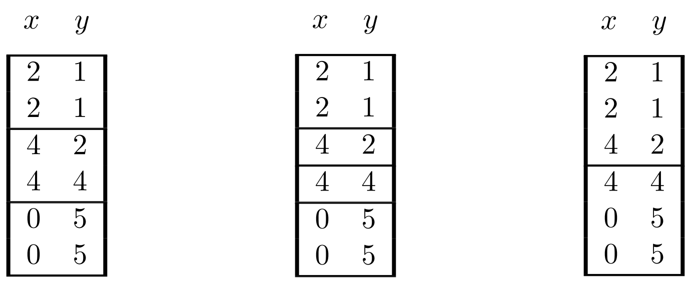
(a) x is even (b) y is even (c) $x > y$

Figure 7.2: The alternatives for three statements within the team T_2.

question $?\alpha$ if it determines what the value of α is, i.e., if every assignment $g \in T$ assigns to α the same truth value:

$$M \models_T ?\alpha \iff \forall g, g' \in T : (M \models_g \alpha \iff M \models_{g'} \alpha).$$

Figure 7.3 illustrates these examples by showing the alternatives for two identification questions and a polar question in the context of the team T_2.



(a) λx (b) λy (c) $?(x > y)$

Figure 7.3: The alternatives for three questions within the team T_2.

All notions, facts, and considerations that we discussed in Chapter 2 carry over straightforwardly to the team semantics setting. We will only restate explicitly those facts and notions that play a special role in our discussion below.

7.3.2 Q-dependency

The inquisitive approach we just described yields a natural analysis of dependency as a relation holding between questions in the context of a team. Within a team T, a question μ is fully determined by a set Λ of questions if μ is settled in any sub-team of T which settles all questions in Λ. This gives us the central notion of *q-dependency*.

7.3.3. Definition. [Q-dependency]
A set Λ of questions determines a question μ in the context of a team T, denoted $\Lambda \models_T \mu$, if for every $T' \subseteq T$, $M \models_{T'} \lambda$ for all $\lambda \in \Lambda$ implies $M \models_{T'} \mu$. We write $\lambda_1, \ldots, \lambda_n \models_T \mu$ as a shorthand for $\{\lambda_1, \ldots, \lambda_n\} \models_T \mu$.

It is easy to see that v-dependency is a special case of q-dependency involving identification questions. This is made precise by the following fact, which the reader is invited to check.

7.3.4. Proposition (v-dependencies are q-dependencies).
Let $X \cup \{y\}$ be a set of variables and T a team. Let λX stand for the set of identification questions $\{\lambda x \mid x \in X\}$. We have:

$$\mathbb{D}_T(X; y) \iff \lambda X \models_T \lambda y.$$

Thus, v-dependencies correspond to a special class of q-dependencies—those which involve only identification questions. However, the notion of q-dependency is much more general: besides v-dependencies, there are many other patterns that can be recognized and captured naturally as cases of q-dependency, as we discuss in the next section.

7.3.3 Generality

Consider the following team, in which the values of x and y always differ by one:

T_3	x	y
g_1	1	0
g_2	1	2
g_3	2	1
g_4	2	3
g_5	3	2
g_6	3	4
g_7	4	3
g_8	4	5

In this team, no non-trivial v-dependencies hold: the value of x neither determines nor is determined by the value of y. Yet, in this table we can still recognize many interesting patterns that one would naturally regard as dependencies. For instance, here are some facts:

- the value of x determines the parity of y (i.e., whether y is even or odd);
- the parity of x determines the parity of y;
- the value of x and whether $x < y$ determines the value of y.

These facts can be captured straightforwardly as q-dependencies involving not only identification questions, but also polar questions, as follows:

- $\lambda x \models_{T_3} ?\text{Even}(y)$;
- $?\text{Even}(x) \models_{T_3} ?\text{Even}(y)$;
- $\lambda x, ?(x < y) \models_{T_3} \lambda y$.

One can also give other examples where the sort of questions involved are of different kinds, for instance mention-some or mention-all *wh*-questions. For instance, consider the following team:

T_4	x	y	z
g_1	1	2	2
g_2	1	4	6
g_3	2	3	3
g_4	2	9	6
g_5	3	6	5
g_6	3	12	10
g_7	3	18	15
g_8	3	24	15

Although the value of x determines neither the value of y nor the value of z, in this table we have the following dependence patterns:

- the value of x determines the set $PF(y)$ of prime factors of y (if $x = 1$ then $PF(y) = \{2\}$; if $x = 2$ then $PF(y) = \{3\}$; if $x = 3$ then $PF(y) = \{2, 3\}$);
- the value of x yields some prime factor of z (if $x = 1$ then $2 \in PF(z)$; if $x = 2$ then $3 \in PF(z)$; if $x = 3$ then $5 \in PF(z)$).

These facts can be captured naturally as cases of q-dependency involving the following *wh*-questions:

- The question $\mu_\forall(y)$:= 'what are the prime factors of y', supported by a team just in case the team determines the exact set of prime factors of y:

$$M \models_T \mu_\forall(y) \iff \forall g, g' \in T : PF(g(y)) = PF(g'(y)).$$

- The question $\mu_\exists(z)$:= 'what is one prime factor of z', supported by a team T just in case T implies of some number n that it is a prime factor of z:
$$M \models_T \mu_\exists(z) \iff \exists n \in \mathbb{N} \forall g \in T : n \in PF(g(z)).$$

The two dependence patterns noticed above amount to the q-dependencies:

- $\lambda x \models_{T_4} \mu_\forall(y)$;
- $\lambda x \models_{T_4} \mu_\exists(z)$.

We will see in the next section how the relevant questions can be expressed in a formal language by adapting the tools of standard inquisitive predicate logic. For now, the important point is that the notion of q-dependency allows us to view v-dependencies as a special case of a much broader spectrum of logical facts that share common features and that are naturally analyzed in a uniform way. This includes all claims of the form 'such-and-such information about x yields such-and-such information about y', where the relevant information need not be the complete information giving the exact value of the variable, but could instead be partial information concerning, e.g., parity, set of prime factors, etc.[7]

In fact, coming back to the idea of information types discussed in detail in Chapter 2, I would like to suggest that we have a dependency whenever information of certain types is guaranteed to yield information of another type. Call these the *input types* and the *output type* of the dependency.

As we discussed in Chapter 2, questions can be seen as names for information types. Thus, e.g., λx stands for information of type 'value of x', while ?Even(x) stands for information of type 'parity of x', and $\mu_\forall(x)$ defined above for information of type 'prime factors of x'. Now to represent an arbitrary dependency as an instance of q-dependency, we just need to find questions $\mu_1, \dots, \mu_n$ that correspond to the input types of the dependency and a question ν that stands for the output type. Then the dependency amounts precisely to the fact that $\mu_1, \dots, \mu_n \models_T \nu$. In this sense, to the extent that we accept the above idea of what a dependency is, the question approach is bound to be a fully general one.[8]

[7]Notice also that the relevant information need not concern different variables: we can also capture dependencies of the form 'such-and-such information about x yields such-and-such other information about x'; nor does each bit of information need to be about a single variable: we can have dependencies of the form 'such-and-such information about the relation between x and y yields such-and-such information about z'.

[8]The claim is not that some given language will give us enough resources to capture all dependency relations. Rather, it is that dependencies can in principle be analyzed as involving questions; which dependencies can be analyzed in some particular formal language then depends on which questions are expressible in that language.

7.3.4 Taking dependencies to the core of logic

Perhaps the greatest merit of the question-based perspective on dependency is that it brings out the deep connections existing between dependency and logical notions like entailment, conjunction, implication, and proof. In this section, we briefly review these manifold connections, with a focus on how they play out in the team semantics setting.

Q-dependency and Tarskian consequence. First, q-dependency in a team is a *Tarskian consequence relation*. This means that it is a relation between formulas which satisfies the following three properties:

- Reflexivity: $\Lambda \models_T \lambda$ for all $\lambda \in \Lambda$;
- Weakening: $\Lambda \models_T \mu$ implies $\Lambda' \models_T \mu$ for $\Lambda' \supseteq \Lambda$;
- Transitivity: $\Lambda \models_T \mu$ and $\Lambda' \models_T \lambda$ for all $\lambda \in \Lambda$ implies $\Lambda' \models_T \mu$.

Thus, q-dependency in a given team is a consequence relation among questions. Since v-dependency can be seen as a special case of q-dependency via the equivalence

$$\mathbb{D}_T(X; y) \iff \lambda X \models_T \lambda y,$$

this means in particular that Armstrong's axioms discussed in Section 7.1 can be seen as a special case of the axioms for consequence.

Q-dependency and logical operators. The fact that the relation of q-dependency connects *formulas*, rather than variables, has important repercussions as well. Formulas, unlike variables, can be combined by means of logical operations, which gives us important tools to manipulate dependency claims. For an illustration, reproducing standard inquisitive semantics in the team setting we can introduce two connectives $\wedge$ and $\rightarrow$ which work as follows:

- $M \models_T \varphi \wedge \psi \iff M \models_T \varphi$ and $M \models_T \psi$;
- $M \models_T \varphi \rightarrow \psi \iff \forall T' \subseteq T : M \models_{T'} \varphi$ implies $M \models_{T'} \psi$.

These connectives interact with the q-dependency relation in the way conjunction and implication standardly interact with a consequence relation. We have:

- $\Lambda, \mu_1, \mu_2 \models_T \nu \iff \Lambda, \mu_1 \wedge \mu_2 \models_T \nu$;
- $\Lambda, \mu \models_T \nu \iff \Lambda \models_T \mu \rightarrow \nu$.

Thus, we can always trade multiple determining questions for a single conjunctive one, and we can always drop one of the determining questions and correspondingly weaken our conclusion to a conditional question. This is just an illustration of the fact that q-dependency is a consequence relation that is naturally related to a set of well-behaved logical operations on questions.

Interestingly, moreover, the relevant operations are generalizations to questions of the standard operations of classical logic: when applied to statements α and β, the connectives $\wedge$ and $\to$ we just defined yield formulas $\alpha \wedge \beta$ and $\alpha \to \beta$ which behave as conjunction and material conditional in classical logic. Thus, by working with questions we can handle the premises and the conclusion of a dependence relation by means of logical operators that obey familiar properties and which, moreover, generalize the familiar operators of classical logic. This is a significant finding.

Notice also that, as we discussed in Section 2.5, implication yields a fully general way to express q-dependencies in the object language. Indeed, one can check that we have:

$$\begin{aligned} \lambda_1, \dots, \lambda_n \models_T \mu \quad &\iff \quad M \models_T \lambda_1 \wedge \dots \wedge \lambda_n \to \mu \\ &\iff \quad M \models_T \lambda_1 \to (\dots \to (\lambda_n \to \mu)). \end{aligned}$$

So, the fact that $\lambda_1, \dots, \lambda_n$ determine μ is expressed in the object language by the formula $\lambda_1 \wedge \dots \wedge \lambda_n \to \mu$, or equivalently by $\lambda_1 \to (\dots \to (\lambda_n \to \mu))$. This brings out a deep connection existing between q-dependency and the implication connective of inquisitive logic.

Conditional q-dependencies for free. Consider the following team:

T_5	x	y
g_1	1	0
g_2	1	2
g_3	2	3
g_4	2	3
g_5	3	2
g_6	3	4
g_7	4	5
g_8	4	5

In this team, the value of x does not generally determine the value of y, but it does so in restriction to those assignments in which the value of x is even. This is an example of a *conditional* dependency.

In the question-based perspective on dependency, conditional dependencies are captured straightforwardly by allowing statements, in addition to questions, as premises of a q-dependence relation. Since statements and questions can both be interpreted in terms of the same notion of support, the definition of the relation $\models_T$ does not need to be generalized to accommodate statements, but can be applied directly. Following a reasoning analogous to the one in Section 2.3.3, we can then verify that the following holds.

7.3.5. Proposition. *Let* Γ *be a set of statements and* $\Lambda \cup \{\mu\}$ *a set of questions. Let* T *be a team and let* $|\Gamma|_M = \{g \mid M \models_g \gamma$ *for all* $\gamma \in \Gamma\}$. *We have:*

$$\Gamma, \Lambda \models_T \mu \iff \Lambda \models_{T \cap |\Gamma|_M} \mu.$$

This means that the relation $\Gamma, \Lambda \models_T \mu$ captures a conditional q-dependency: the questions in Λ determine the question μ, not (necessarily) relative to the entire team T, but relative to those assignments in T that satisfy Γ. We can read the relation $\Gamma, \Lambda \models_T \mu$ as 'Λ determines μ *given* Γ'.

Notice also that the approach vindicates the connection between conditional dependencies and conditionals. Let us focus for simplicity on the case in which a question μ determines a question ν given a statement α, i.e., the case in which $\alpha, \mu \models_T \nu$. This can be expressed in the object language by the formula:

$$\alpha \to (\mu \to \nu).$$

Recall that $\mu \to \nu$ expresses the fact that μ determines ν in the evaluation state. Thus, a conditional dependency is expressed by a conditional having the condition α as antecedent and the dependence formula $\mu \to \nu$ as consequent.

Summing up, there is no need to further generalize the q-dependency relation $\models_T$ in order to capture conditional dependencies: it suffices to allow statements to be plugged in as determinants alongside questions. In addition, conditional dependencies can be expressed smoothly in the language as conditionals having the relevant conditions as antecedents.

Q-dependency and logical entailment. As usual, the inquisitive approach comes with a general notion of entailment, defined in terms of preservation of support, where the premises and the conclusion can be statements or questions. In the team semantics setting, this is given by the following definition.

7.3.6. Definition. [Entailment]
$\Phi \models \psi \iff$ for every model M and team $T : M \models_T \Phi$ implies $M \models_T \psi$.

This general notion of entailment is a conservative extension of the standard entailment relation for statements. That is, if $\Gamma \cup \{\alpha\}$ is a set of statements, for which support amounts to global truth, we have:

$$\Gamma \models \alpha \iff \text{for every model } M \text{ and assignment } g : M \models_g \Gamma \text{ implies } M \models_g \alpha.$$

At the same time, in the case in which we have question assumptions and a question conclusion, this general notion of entailment captures *logical q-dependencies*, i.e., q-dependencies which hold in virtue of the logical form of the sentences involved, regardless of the interpretation of non-logical symbols. That is, suppose Γ is a set of statements and $\Lambda \cup \{\mu\}$ a set of questions. We have:

$$\Gamma, \Lambda \models \mu \iff \Gamma, \Lambda \models_T \mu \text{ for any team } T \text{ in any model } M.$$

Thus, $\Gamma, \Lambda \models \mu$ captures the fact that Λ *logically* determines μ given Γ.

For a simple example, in the team version of inquisitive first-order logic given in the next section, the following are simple examples of logical q-dependencies (where P, Q are unary predicates and f a unary function symbol):

$$?Px,\ Px \leftrightarrow \neg Qy \ \models ?Qy,$$

$$\lambda x,\ y = f(x) \ \models\ \lambda y.$$

Summing up, we saw that q-dependency comes in two versions: a contextual version, relativized to a team, and a logical version, obtained by quantifying over all teams. Logical q-dependencies are those q-dependencies that hold purely on the basis of the logical form of the sentences involved. The main insight of the question-based perspective is that logical q-dependency is nothing but a facet of the central notion of logic, the notion of *entailment*, once this notion is generalized to apply not just to statements, but also to questions.

Q-dependency and logical proofs. One important repercussion of the fact that logical q-dependencies are logical entailments is that such dependencies can be formally *proved* if we have a proof system for (a fragment of) our logic. This brings out the connections between dependency and another central concern of logic, namely, proofs.

For an example, consider the logical q-dependency $?Px,\ Px \leftrightarrow \neg Qy \ \models ?Qy$ discussed above. We can prove the validity of this dependency in exactly the same way as we can prove the entailment $?p, p \leftrightarrow \neg q \models ?q$ in InqB, using standard inference rules for disjunction and implication (recall that in inquisitive logic, polar questions $?\alpha$ are realized as inquisitive disjunctions $\alpha \vee\!\!\!\vee \neg\alpha$). Omitting proof steps which involve only statements, we have the following proof.

$$\dfrac{?Px \qquad \dfrac{\dfrac{[Px]_1 \quad Px \leftrightarrow \neg Qy}{\neg Qy}}{?Qy}\,(\vee\!\!\!\vee\mathsf{i}) \qquad \dfrac{\dfrac{[\neg Px]_1 \quad Px \leftrightarrow \neg Qy}{Qy}}{?Qy}\,(\vee\!\!\!\vee\mathsf{i})}{?Qy}\,(\vee\!\!\!\vee\mathsf{e}, 1)$$

The fact we saw above, that q-dependencies are expressed by implications in the object language, also has repercussions in proofs, since it means that we can make inferences with dependence formulas just as we normally do with implications. In order to show that a dependency $\mu \to \nu$ holds under certain assumptions, we can simply suppose the question μ and try on that basis to derive the question ν. This corresponds to the standard implication introduction rule.[9] Moreover, if we have a dependency $\mu \to \nu$ and we also have the determining question μ, we can conclude the determined question ν. This is just the standard implication elimination rule.[10]

Summing up, then, by manipulating questions in inferences we can prove that certain dependencies are logically valid—i.e., hold merely on the basis of the logical form of the sentences involved. Note that in order for this to be possible, it matters that questions—unlike variables—have syntactic structure to them. Since questions are built up by means of certain logical operators, we can make inferences with them by using the inference rules for these operators, as illustrated by the above example of a proof.

7.3.5 Wrapping up

We saw that the core ideas of inquisitive semantics, as developed in Chapter 2, apply straightforwardly in the setting of team semantics. This allows us to interpret statements and questions involving variables uniformly in the context of a team. This perspective yields a natural notion of dependency as a relation between questions, which encompasses v-dependency as a special case, but which is much more general, capturing not only dependencies of the form 'the value of $x_1, \dots, x_n$ yields the value of y', but also, for instance, all dependencies of the form 'such-and-such information about $x_1, \dots, x_n$ yields such-and-such information about y'. Moreover, since on this view dependency is a relation between questions, and since questions are sentences, we can uncover a number of significant connections between dependency and generalized versions of the classical logical operators, consequence relation, and proof system.

The general view discussed in this section can be implemented in many particular formal systems, differing from each other in their set of primitive logical operators. In the next section, we will make the discussion more concrete by considering one particular implementation of these ideas, which stems from interpreting the language of inquisitive first-order logic in the setting of team semantics.

[9] Recalling from Section 4.4 that questions in proofs can be seen as placeholders for arbitrary information of the corresponding type, this is intuitive: in order to prove that a dependency holds, we suppose to be given information of the input type, and try to infer on that basis that we have information of the output type.

[10] Again, given our conception on the role of questions in proofs, this is intuitive: if we have a dependency as well as information of the input type, we also have information of the output type.

7.4 The system InqBT

In this section, we will see how the language $\mathcal{L}^{\mathsf{Q}=}$ of inquisitive first-order logic can be given a natural interpretation in the setting of team semantics, where formulas are interpreted relative to a single model and a set of assignments. We will refer to this system as InqBT, where the letter T marks the fact that formulas are interpreted with respect to teams. In the dependence logic literature, this system has been considered by Yang (2014) under the name of WID (for *weak intuitionistic dependence logic*), and most of the results mentioned here can already be found in her work. However, we will interpret the system in the light of the conceptual picture of inquisitive semantics, as laid out in Chapter 2 and in the previous section.

7.4.1 Syntax and semantics

The language of InqBT is just the language $\mathcal{L}^{\mathsf{Q}=}$ studied in the previous chapter, given by the following syntax:

$$\varphi \ ::= \ p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \varphi \mathbin{\vee\!\!\!\vee} \varphi \mid \forall x\varphi \mid \exists x\varphi,$$

where p is an atom in the given signature (of the form $R(t_1, \dots, t_n)$ or $(t = t')$). We regard the inquisitive connectives $\mathbin{\vee\!\!\!\vee}$ and $\exists$ as question-forming operators in the way familiar from the previous chapter. Formulas without these operators are called *classical* and identified with formulas of classical first-order logic. The set of classical formulas is denoted $\mathcal{L}_c^{\mathsf{Q}=}$. The operators $\neg, \vee, \exists$, and ? are defined as in the previous chapter.

The semantics of InqBT is given in the setting of teams semantics: formulas are evaluated with respect to a relational structure $M = \langle D, I\rangle$ and a team T. The clauses are identical to those for InqBQ, except that now it is the team that plays the role of the information state.

7.4.1. Definition. [Semantics of InqBT]

- $M \models_T R(t_1, \dots, t_n) \iff$ for all $g \in T$, $\langle [t_1]_g^M, \dots, [t_n]_g^M\rangle \in I(R)$
- $M \models_T (t = t') \iff$ for all $g \in T$, $[t]_g^M = [t']_g^M$
- $M \models_T \bot \iff T = \emptyset$
- $M \models_T \varphi \wedge \psi \iff M \models_T \varphi$ and $M \models_T \psi$
- $M \models_T \varphi \to \psi \iff$ for all $T' \subseteq T : M \models_{T'} \varphi$ implies $M \models_{T'} \psi$
- $M \models_T \varphi \mathbin{\vee\!\!\!\vee} \psi \iff M \models_T \varphi$ or $M \models_T \psi$
- $M \models_T \forall x\varphi \iff M \models_{T[x\mapsto d]} \varphi$ for every $d \in D$

- $M \models_T \exists x\varphi \iff M \models_{T[x\mapsto d]} \varphi$ for some $d \in D$

The clauses for atoms are the same as in D: an atomic sentence is settled with respect to a team T if it is true under any assignment $g \in T$. Notice that unlike in D, we do not need negative atoms in the language, since the same result can be produced compositionally by negating atoms by means of the negation operator, defined as $\neg\varphi := \varphi \to \bot$. The clauses for the connectives are the familiar inquisitive clauses, except that now, the relevant information ordering concerns the team T. The clauses for the quantifiers are also very similar to those we used in InqBQ, except that instead of setting the value of x to d in just one assignment, we have to do this for all assignments $g \in T$. Setting the value of x to d throughout T amounts to stipulating that x denotes d; this allows us to look at what is settled in T about the object d, rather than about the variable x.

7.4.2 Basic properties

Support in InqBT has the usual features.

7.4.2. Proposition. *For any relational structure M and $\varphi \in \mathcal{L}^{\mathsf{Q}=}$ we have:*

- Persistence property: $M \models_T \varphi$ *and* $T' \subseteq T$ *implies* $M \models_{T'} \varphi$.
- Empty team property: $M \models_\emptyset \varphi$.

We define the notion of truth by setting $M \models_g \varphi \iff M \models_{\{g\}} \varphi$. We can then check that all the standard operators have the familiar truth-conditions, while the inquisitive operators $\vee\!\!\!\vee$ and $\exists$ have the same truth-conditions as the corresponding classical operators $\vee$ and $\exists$.

7.4.3. Proposition (Truth-conditions for InqBT).

- $M \models_g R(t_1, \ldots, t_n) \iff \langle [t_1]^M_g, \ldots, [t_n]^M_g \rangle \in I(R)$
- $M \models_g (t = t') \iff [t]^M_g = [t']^M_g$
- $M \not\models_g \bot$
- $M \models_g \varphi \wedge \psi \iff M \models_g \varphi$ *and* $M \models_g \psi$
- $M \models_g \varphi \to \psi \iff M \not\models_g \varphi$ *or* $M \models_g \psi$
- $M \models_g \varphi \vee\!\!\!\vee \psi \iff M \models_g \varphi$ *or* $M \models_g \psi$
- $M \models_g \forall x\varphi \iff M \models_{g[x\mapsto d]} \varphi$ *for all* $d \in D$
- $M \models_g \exists x\varphi \iff M \models_{g[x\mapsto d]} \varphi$ *for some* $d \in D$

Notice that now, unlike in D, the truth-conditions for a formula $\varphi \in \mathcal{L}^{\mathsf{Q}}$ depend only on the truth-conditions for the sub-formulas of φ: computing the truth-conditions for φ never requires moving to non-singleton teams. Also, recall from the previous chapter that φ^{cl} is the classical formula obtained from φ by replacing each occurrence of $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ and $\exists$ with $\vee$ and $\exists$, respectively. Then, it follows from the previous proposition that φ and φ^{cl} always have the same truth-conditions.

7.4.4. Corollary.
For any model M, assignment g, and formula $\varphi \in \mathcal{L}^{\mathsf{Q}}$, $M \models_g \varphi \iff M \models_g \varphi^{cl}$.

Thus, any formula in InqBT has the same truth conditions as some classical formula. This is strikingly different from the situation in D, where some formulas have truth conditions which are not shared by any standard first-order formula; an example is the sentence $\exists^d x \forall^d y \exists^d z(=(z, y) \wedge z \neq x)$ which is true only if the domain of the model is infinite.

We say that φ is *truth-conditional* (or *flat*, in the dependence logic lingo) if φ is supported by a team T whenever it is true relative to each $g \in T$. As in InqBQ, classical formulas are always truth-conditional. This fits the idea that we regard such formulas as statements and that for statements, support is connected to truth via the Truth-Support Bridge.

7.4.5. Proposition. *Every $\alpha \in \mathcal{L}_c^{\mathsf{Q}=}$ is truth-conditional in InqBT.*

Since classical formulas are truth-conditional and their truth conditions are the standard ones, with respect to such formulas what we have given is simply a team semantics for classical first-order logic.

As in the system D, the support conditions for a formula φ depend only on the values that the assignments in the team T give for those variables which actually occur free in φ.

7.4.6. Proposition.
Let $\varphi \in \mathcal{L}^{\mathsf{Q}=}$. If $T\upharpoonright_{FV(\varphi)} = T'\upharpoonright_{FV(\varphi)}$, then $M \models_T \varphi \iff M \models_{T'} \varphi$.

In particular, a sentence, which has no free variables, is not sensitive to the team of evaluation at all, as long as this team is non-empty. If φ is a sentence, we can thus write $M \models \varphi$ as a shorthand for $M \models_T \varphi$, where T is any non-empty team. As a corollary of this fact, we get that all sentences are trivially truth-conditional. Thus, in InqBT, unike in InqBQ, no sentence is a question, even if it contains occurrences of $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ and $\exists$. Notice that, since a sentence φ and its classical variant φ^{cl} have the same truth-conditions (Corollary 7.4.4), and since both are truth-conditional, we always have $\varphi \equiv \varphi^{cl}$.

7.4.7. PROPOSITION. *If φ is a sentence, $\varphi \equiv \varphi^{cl}$ in InqBT.*

Thus, any sentence of InqBT is equivalent to a classical first-order sentence. Again, this is very different from what we find in D, where some sentences are equivalent to properly second-order sentences of standard predicate logic.

On the other hand, in InqBT things become interesting as soon as we consider formulas with free variables. As we show in the next section, by using the inquisitive operators $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ and $\exists$ we can capture many classes of questions concerning the values of variables.

7.4.3 Questions in InqBT

In this section we show how the different sorts of questions concerning variables that came up in our discussion in Section 7.3.3 can be captured by formulas in InqBT. For our illustration we will make use of Figure 7.4, which depicts four teams over a domain of natural numbers. In our examples, the questions we will consider have x as their only free variable. Proposition 7.4.6 ensures that the value of the assignments in T on the variable x is all that matters to decide on the support of these questions, which is why our tables in the figure consist of only one column, the one corresponding to the variable x.

7.4.8. EXAMPLE. [Polar questions]
Given a classical formula α, consider the formula $?\alpha := \alpha \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \neg\alpha$. Using Proposition 7.4.5, we have:

$$\begin{aligned} M \models_T ?\alpha &\iff M \models_T \alpha \text{ or } M \models_T \neg\alpha \\ &\iff [\, M \models_g \alpha \text{ for all } g \in T\,] \text{ or } [\, M \models_g \neg\alpha \text{ for all } g \in T\,] \\ &\iff \forall g, g' \in T : [\, M \models_g \alpha \iff M \models_{g'} \alpha\,]. \end{aligned}$$

Thus, $?\alpha$ captures the polar question *whether* α, which is settled relative to a team T if all the assignments in the team agree on whether α is true or false.

For instance, suppose our domain is the set $\mathbb{N}$ of natural numbers, and Even is a predicate symbol interpreted as the set of even numbers. Then, $?\mathrm{Even}(x)$ expresses the question whether the value of x is even or odd, which is supported by a team just in case the parity of x is constant in the team.

Thus, for instance, the question $?\mathrm{Even}(x)$ is settled in the teams T_a and T_c of Figure 7.4, where it is settled that x is even, and also in T_b, where it is settled that x is odd, but not in T_d where it is unsettled whether x is even or odd.

7.4.9. EXAMPLE. [Identification questions]
Let t be a term, and let y be any variable which does not occur in t. Consider

x
16
16
16
16

(a)

x
7
13
17
25

(b)

x
12
18
24
36

(c)

x
5
10
15
35

(d)

Figure 7.4: Four teams T_a, T_b, T_c, T_d, each consisting of four assignments into the domain $\mathbb{N}$ of natural numbers. For simplicity, we only display the value that each assignment in the team gives for the variable x.

Question	T_a	T_b	T_c	T_d
$?\mathrm{Even}(x)$	✓	✓	✓	
λx	✓			
$\lambda\mathrm{mod}_3(x)$	✓		✓	
$\exists y \mathrm{Pf}(y, x)$	✓		✓	✓
$\forall y ?\mathrm{Pf}(y, x)$	✓		✓	

Figure 7.5: The support conditions for the questions in our examples with respect to the four assignments of Figure 7.4.

the formula $\exists y(t = y)$. We have:

$$\begin{aligned}
M \models_T \exists y(t = y) \quad &\iff \quad \text{there is a } d \in D \text{ s.t. } M \models_{T[y\mapsto d]} (t = y) \\
&\iff \quad \text{there is a } d \in D \text{ s.t. for all } g \in T,\ [t]^M_{g[y\mapsto d]} = [y]^M_{g[y\mapsto d]} \\
&\iff \quad \text{there is a } d \in D \text{ s.t. for all } g \in T,\ [t]^M_g = d \\
&\iff \quad \text{for all } g, g' \in T :\ [t]^M_g = [t]^M_{g'}.
\end{aligned}$$

Thus, $\exists y(t = y)$ is settled in T in case all $g \in T$ assign the same value to t. If, as in the previous chapter, we define the abbreviation

$$\lambda t := \exists y(t = y) \qquad [y \notin FV(t)]$$

we thus have that λt captures precisely the identification question "what is the value of t".

This covers, in particular, identification questions of the form λx about the value of variables, which we considered in Section 7.3.3, which are settled in a team T if all the assignments $g \in T$ agree on the value of x. Among the teams depicted in Figure 7.4, λx is settled only in T_a.

Identification questions about complex terms are also interesting. For instance, suppose our domain is $\mathbb{N}$, and suppose mod_k is a unary function symbol such that $\mathrm{mod}_k(x)$ denotes the remainder of the division of x by k. Then the question

$$\lambda \mathrm{mod}_k(x)$$

is settled in a team T in case all the assignments $g \in T$ agree on the value of $\mathrm{mod}_k(x)$, that is, in case the equivalence class of x modulo k is settled in T. Thus, $\lambda \mathrm{mod}_k(x)$ captures the question "what is the value of x, modulo k?". Note that in the particular case of $k = 2$ this is equivalent to the polar question ?Even(x) discussed in the previous example.

Among the teams in Figure 7.4, the question $\lambda \mathrm{mod}_3(x)$ is settled in teams T_a, where it is settled that $\mathrm{mod}_3(x) = 1$, and also in team T_c, where it is settled that $\mathrm{mod}_3(x) = 0$. The question is not settled in team T_b and T_d, since the value of x modulo 3 is not constant in these teams.

Finally, it is worth pointing out that identification questions can be expressed in a different way as well. Consider the formula $\forall y?(t = y)$, where y does not occur in t. We have:

$$\begin{aligned}
M \models_T \forall y?(t = y) \quad &\iff \quad \text{for all } d \in D : M \models_{T[y\mapsto d]} ?(t = y) \\
&\iff \quad \text{for all } d \in D, \text{ for all } g, g' \in T : \\
&\qquad\quad M \models_{g[y\mapsto d]} (t = y) \iff M \models_{g'[y\mapsto d]} (t = y) \\
&\iff \quad \text{for all } g, g' \in T, \text{ for all } d \in D : \\
&\qquad\quad ([t]^M_g = d) \iff ([t]^M_{g'} = d) \\
&\iff \quad \text{for all } g, g' \in T : [t]^M_g = [t]^M_{g'}.
\end{aligned}$$

This is precisely the semantics of the identification question λt defined above. Thus, the identification question about a term t can be expressed equivalently in InqBT as $\exists y(t = y)$ or as $\forall x?(t = x)$. This double route to identification questions is useful, since the two ways to express such questions use a different set of operators, which gives us two alternative perspectives on such questions.

7.4.10. Example. [Mention-some questions]
Consider the formula $\exists y R(\overline{x}, y)$, where R is a relation symbol. We have:

$$\begin{aligned}
M \models_T \exists y R(\overline{x}, y) \quad &\iff \quad \text{there is a } d \in D \text{ s.t. } M \models_{T[y\mapsto d]} R(\overline{x}, y) \\
&\iff \quad \text{there is a } d \in D \text{ s.t. for all } g \in T : M \models_{g[y\mapsto d]} R(\overline{x}, y) \\
&\iff \quad \text{there is a } d \in D \text{ s.t. for all } g \in T : \langle g(\overline{x}), d\rangle \in I(R).
\end{aligned}$$

Thus, $\exists y R(\overline{x}, y)$ is settled in the team T if there is an object d such that all assignments in T agree that the value of $\overline{x}$ is R-related to d.

To make this more concrete, suppose again that our domain is the set $\mathbb{N}$ of natural numbers, and suppose our language contains a binary relation symbol Pf such that Pf(y, x) holds iff y is a prime factor of x. Then the question

$\exists y \mathrm{Pf}(y,x)$ is settled relative to a team T in case there is a number $n \in \mathbb{N}$ which is a prime factor of x throughout the team. So, this formula captures precisely the mention-some question "what is some prime factor of x?" discussed in Section 7.3.3.

Among the teams of Figure 7.4, the question $\exists y \mathrm{Pf}(y,x)$ is supported in T_a, where it is settled that 2 is a prime factor of x, as well as in T_d, where it is settled that 5 is a prime factor, and in T_c, where it is settled of both 2 and 3 that they are prime factors of x. The question is not supported in T_b, since there is no number which is settled in T_b to be a prime factor of x.

7.4.11. Example. [Mention-all questions]
Consider the formula $\forall y ?R(\overline{x}, y)$, where R is a relation symbol. Using the support conditions for polar questions that we have seen above, we have:

$$\begin{array}{rcl} M \models_T \forall y ?R(\overline{x},y) & \iff & \text{for all } d \in D : M \models_{T[y\mapsto d]} ?R(\overline{x},y) \\ & \iff & \text{for all } d \in D, \text{ for all } g, g' \in T : \\ & & M \models_{g[y\mapsto d]} R(\overline{x},y) \iff M \models_{g'[y\mapsto d]} R(\overline{x},y) \\ & \iff & \text{for all } g, g' \in T, \text{ for all } d \in D : \\ & & \langle g(\overline{x}), d\rangle \in I(R) \iff \langle g'(\overline{x}), d\rangle \in I(R) \\ & \iff & \text{for all } g, g' \in T : \\ & & \{d \mid \langle g(\overline{x}), d\rangle \in R\} = \{d \mid \langle g'(\overline{x}), d\rangle \in R\}. \end{array}$$

Thus, $\exists y R(\overline{x}, y)$ is settled in the team T in case all assignments in T agree on the set of objects d such that $\overline{x}$ is R-related to d.

To make this more concrete, consider the formula $\forall y ?\mathrm{Pf}(y,x)$. This is settled relative to a team T in case all assignments in T agree on the set of prime factors of x. Thus, this formula captures the question "what are the prime factors of x?" which we discussed in Section 7.3.3.

Among the teams of Figure 7.4, the formula $\forall y ?\mathrm{Pf}(y,x)$ is supported in T_a, where it is settled that the only prime factor of x is 2, and in T_c, where it is settled that the prime factors of x are 2 and 3. It is not settled in the remaining teams T_b and T_d, since the assignments in these teams to not agree with each other on the set of prime factors of x.

The examples we saw are just a small sample of the class of questions expressible in InqBT. Yet, these examples hopefully suffice to illustrate that the sort of picture discussed abstractly in the previous section can be made concrete in the setting of a simple formal language in which questions about the values of variables can be formalized.

7.4.4 Dependencies in InqBT

Let us now illustrate how q-dependencies can be analyzed and expressed naturally in InqBT, in accordance with the general idea described in Section 7.3.2. Recall that, if T is a team based on the model M, we obtain a notion of entailment relative to T by letting:

$$\Phi \models_T \psi \iff \forall T' \subseteq T : (M \models_{T'} \Phi \text{ implies } M \models_{T'} \psi).$$

As we saw above, if $\Lambda \cup \{\mu\}$ is a set of questions, then $\Lambda \models_T \mu$ captures a q-dependency relation: in the context of T, the questions in Λ determine the question μ. If moreover Γ is a set of statements, then $\Gamma, \Lambda \models \mu$ captures a conditional q-dependency: Λ determines μ relative to those assignments that make Γ true.

Recall moreover that, as usual in inquisitive logic, contextual entailments are expressed in the object language by implications:

$$M \models_T \varphi \to \psi \iff \varphi \models_T \psi.$$

We can now see that the q-dependencies discussed in Section 7.3.3 can indeed all be captured as relations between questions expressible in the system InqBT, and can be expressed in the object language by corresponding implications. By way of illustration, here are some examples, where we use the abbreviations introduced in the previous section for questions in InqBT.

- The value of $x_1, \ldots, x_n$ determines the value of y.
  - Meta-language: $\lambda x_1, \ldots, \lambda x_n \models_T \lambda y$.
    (Note: this corresponds to the v-dependency $\mathbb{D}_T(x_1, \ldots, x_n; y)$.)
  - Object language: $\lambda x_1 \wedge \cdots \wedge \lambda x_n \to \lambda y$.
    (Note: this is equivalent to the dependence atom $=\!(x_1, \ldots, x_n; y)$.)
- The value of x determines the parity of y.
  - Meta-language: $\lambda x \models_T ?\mathrm{Even}(y)$.
  - Object language: $\lambda x \to ?\mathrm{Even}(y)$.
- The parity of x determines the parity of y.
  - Meta-language: $?\mathrm{Even}(x) \models_T ?\mathrm{Even}(y)$.
  - Object language: $?\mathrm{Even}(x) \to ?\mathrm{Even}(y)$.
- The value of x and whether $x < y$ determines the value of y.
  - Meta-language: $\lambda x,\ ?(x < y) \models_T \lambda y$.

x	y
0	2
0	4
1	3
1	5
2	5
2	7

(a)

x	y
0	2
1	3
2	2
3	3
4	2
5	3

(b)

x	y
3	2
3	4
4	3
4	5
5	4
5	6

(c)

x	y
1	2
1	4
2	6
2	12
3	3
3	9

(d)

x	y
1	6
1	12
1	14
2	6
2	12
2	15

(e)

Figure 7.6: Five teams, each consisting of six assignments into the domain $\mathbb{N}$. Only the value of the assignments on the variables x and y is displayed.

Dependency	T_a	T_b	T_c	T_d	T_e
$\lambda x \to \lambda y$		✓			
$\lambda x \to ?Ey$	✓	✓	✓	✓	
$?Ex \to \lambda y$		✓			
$?Ex \to ?Ey$		✓	✓		
$\lambda x \wedge ?(x<y) \to \lambda y$		✓	✓		
$\lambda x \to \forall z?\mathrm{Pf}(z,y)$		✓		✓	
$\lambda x \to \exists z\mathrm{Pf}(z,y)$		✓		✓	✓

Figure 7.7: A table that shows in which of the teams of Figure 7.6 each of the dependencies we consider holds.

  - Object language: $\lambda x \wedge ?(x < y) \to \lambda y$.

- The value of x determines the prime factors of y.
  - Meta-language: $\lambda x \models_T \forall z?\mathrm{Pf}(z,y)$.
  - Object language: $\lambda x \to \forall z?\mathrm{Pf}(z,y)$.
- The value of x determines some prime factor of y.
  - Meta-language: $\lambda x \models_T \exists z\mathrm{Pf}(z,y)$.
  - Object language: $\lambda x \to \exists z\mathrm{Pf}(z,y)$.

Figure 7.6 depicts several different teams, and Figure 7.7 illustrates the above dependencies by showing in which of these teams each dependency holds.

As these examples illustrate, in InqBT we can capture and express a broad range of dependence facts, of which standard functional dependencies are just a particular case. Notice that among the relations that can be captured as q-dependencies, some are weaker than standard functional dependencies; this holds, for instance, when complete information about x yields only some partial information about y (say, the parity of y, its equivalence class modulo k, its set of prime factors, a single prime factor, etc.) Some other dependencies are stronger than the standard ones. This is the case, e.g., when partial information about x suffices to get complete information about y (when, e.g., we can determine the value of y just based on the parity of x). Many other dependencies are neither weaker nor stronger than the standard ones, but simply incomparable. This holds, e.g., when some kind of partial information about x determines some other kind of partial information about y (say, the parity of x determines the set of prime factors of y).

7.4.5 Higher-order dependencies

Another interesting class of dependencies that can be captured naturally from the inquisitive perspective are what we might call *higher-order dependencies*. To illustrate the idea, consider four variables x, y, z, t and a team T over the set $\mathbb{R}$ of real numbers which contains all assignments of the following form, for $a, b \in \mathbb{R}$:

x	y	z	t
a	b	$2a$	$-b$

In the context of this team, giving a functional dependency f of y on x implies giving a functional dependency h_f of t on z. For suppose we are given the information that y is functionally determined from x via f, i.e., the information that $g(y) = f(g(x))$. Then it follows that $g(t) = -g(y) = -f(g(x)) = -f(g(z)/2)$, and so $g(t) = h_f(g(z))$ for the function $h_f(r) = -f(r/2)$.

This means that in any sub-team $T' \subseteq T$ in which there is a functional dependency of y on x, there is also a functional dependency of t on z. Now, given what we have seen above, in T' there is a functional dependency of y on x just in case $M \models_{T'} \lambda x \to \lambda y$, and there is a functional dependency of t on z just in case $M \models_{T'} \lambda z \to \lambda t$. So, the observation amounts to the fact that:

$$\forall T' \subseteq T : M \models_{T'} \lambda x \to \lambda y \text{ implies } M \models_{T'} \lambda z \to \lambda t.$$

This is nothing but the definition of the q-dependency relation:

$$\lambda x \to \lambda y \;\models_T\; \lambda z \to \lambda t.$$

Thus, the higher-order dependency that we pointed out in the context of the above team can be analyzed straightforwardly in InqBT as a case of standard q-dependency where the premise and the conclusion are both dependence formulas. (Notice that a dependence formula like $\lambda x \to \lambda y$ can be seen as a question asking for a functional dependence of y on x; the question is supported in a team just in case such a functional dependence is established.)

As usual, our higher-order dependence can then be expressed by an implication of the relevant formulas:

$$(\lambda x \to \lambda y) \to (\lambda z \to \lambda t).$$

This illustrates a further advantage of question-based approach over the variable-based one: the former, unlike the latter, can be naturally iterated. We first consider dependencies between certain questions, which amount to contextual entailments. These can be expressed in the object language by corresponding implications. These implications can themselves be seen as questions, which can in turn bear q-dependency relations to each other relative to a team. And these higher-order dependencies can be captured simply by adding another implication among the relevant formulas. And of course, this can be iterated further.

7.4.6 Properties of InqBT and relations to other systems.

As the previous sections illustrate, InqBT is a natural choice for a system that captures a broad range of dependencies in the team semantics setting. As we saw, this system can be viewed naturally as an inquisitive logic, which extends classical first-order logic with formulas expressing questions about the values of variables. The basic operators of the system are essentially the same as those of the inquisitive first-order logic InqBQ, which, as we saw in the previous chapter, have familiar logical properties. In spite of these attractions, InqBT has received relatively little attention in the literature, and its properties are not well-understood. In this section, we mention some facts and some open problems about this logic and its connections to the systems InqBQ and D.

Fundamental open questions. The basic meta-theoretical questions which are open for InqBQ are also open for InqBT. In particular, it is not known whether InqBT is compact in the sense of entailment, i.e., if for every valid entailment $\Phi \models \psi$ there is a finite set $\Phi_0 \subseteq \Phi$ such that $\Phi_0 \models \psi$.[11] Neither is it known whether the set of InqBT-validities is recursively enumerable, or whether the logic admits a sound and complete axiomatization. Finally, it is not known if there exists an entailment-preserving translation from InqBT to classical first-order logic over a suitable signature, nor whether any invalid entailment can be refuted relative to a countable structure and a countable

[11] It is easy to see, with a reasoning analogous to the one given for Proposition 5.5.34, that InqBT *is* compact in the sense of satisfiability: if every finite subset of a set Φ of formulas is satisfiable, then Φ is satisfiable (where Φ is satisfiable in InqBT if there is a relational structure M and a non-empty team T such that $M \models_T \Phi$.)

team. More investigation is needed to settle these important (and interrelated) questions.

On the other hand, most of the positive results that we saw in the previous chapter about fragments of InqBQ, if not all, have counterparts for InqBT. In the case of the classical antecedent fragment (cf. Section 5.7.2), it can be expected that an entailment-preserving translation to classical first-order logic can be given by a strategy analogous to the one discussed in Section 5.7.2; the existence of such a translation then implies that in restriction to the fragment, entailment is compact, and the set of validities (as well as the set of valid entailments with finitely many premises) is recursively enumerable. It seems also likely that the completeness proof for the fragment given in Section 5.7.2 can be adapted to InqBT, given a suitably adapted version of the proof system. In the case of the finitely coherent fragment (cf. Section 5.7.1), it is easy to show by induction that the counterpart of Proposition 5.7.2 also holds in InqBT: every formula in the fragment is n-coherent for some finite n (where coherence is now understood in terms of the team, as in Kontinen (2013)). This fact allows us to use a strategy analogous to the one described in Meißner and Otto (2021) to define an entailment-preserving translation from the finitely coherent fragment to classical first-order logic. Again, the existence of such a translation implies that, in restriction to the fragment, entailment is compact and the set of InqBT-validities is recursively enumerable. Notice that the finitely coherent fragment includes formulas corresponding to the dependence atoms of D: let us abbreviate by κt the formula $\forall x?(t = x)$, where x is a variable not occurring in t; we have seen in example 7.4.9 that κt is equivalent to λt, and expresses the identification question about t; then the formula $\kappa x_1 \wedge \cdots \wedge \kappa x_n \to \kappa y$ has the same semantics as the dependence atom $=(x_1, \ldots, x_n; y)$, and it belongs to the finitely coherent fragment since it does not contain the operator $\exists$.

Relations to the inquisitive first-order logic InqBQ. The system InqBT can be seen as a counterpart of InqBQ in a setting in which the relevant information state is given by a set of assignments instead of a set of possible worlds. Since the semantics is structurally the same, most of the facts about InqBQ which we established in the previous chapter carry over straightforwardly to InqBT. We will not restate the relevant facts here. Instead, we will point out some respects in which the two logics differ as a result of their different setups.

First, in InqBQ a significant role was played by rigid terms, whose interpretation is fixed across different possible worlds in a state. The counterpart of rigid terms in InqBT is given by *closed terms*—terms not involving any variables—whose interpretation is fixed across different assignments in a team. These will be the terms to which a universal can be validly instantiated, and from which an inquisitive existential can be introduced.

7.4.12. Proposition. *If* t *is a closed term then for any formula* $\varphi \in \mathcal{L}^{\mathsf{Q}=}$*, the entailments* $\varphi[\mathsf{t}/x] \models \exists x\varphi$ *and* $\forall x\varphi \models \varphi[\mathsf{t}/x]$ *are valid in InqBT.*

Proof. We show only the first entailment, since the proof of the second is similar. Suppose $M \models_T \varphi[\mathsf{t}/x]$ for some relational structure M and team T. Now let $d \in D$ be the object such that $d = [\mathsf{t}]_M$: crucially, this object is assignment-independent, since t is closed. It is straightforward to show by induction that for every formula ψ we have $M \models_T \psi[\mathsf{t}/x] \iff M \models_{T[x\mapsto d]} \psi$. Since $M \models_T \varphi[\mathsf{t}/x]$, it follows that $M \models_{T[x\mapsto d]} \varphi$, which by the semantics implies $M \models_T \exists x\varphi$. $\square$

Note that if t is not closed, the relevant entailments are not in general valid. For instance, let φ be the formula $(x = y)$. Then $\varphi[y/x]$ is the formula $(y = y)$, which is a validity. However, $\exists x\varphi$ is the formula $\exists x(x = y)$, which is not a validity, but the identification question that we denoted by λy, which is supported by a team if all assignments agree on the value of y. Thus in this case we have $\varphi[y/x] \not\models \exists x\varphi$.

Similarly, if we take φ to be the formula $?P(x)$, then the formula $\forall x\varphi$ is a validity by Proposition 7.4.7, but $\varphi[y/x]$ is the formula $?P(y)$, which is not a validity (cf. Example 7.4.8). This shows that $\forall x\varphi \not\models \varphi[y/x]$.

For analogous reasons, the role of free variables as placeholders for arbitrary individuals is taken over in InqBT by fresh constant symbols.

7.4.13. Proposition. *If* c *is a constant not occurring in the set* $\Phi \cup \{\varphi, \psi\}$*:*

- $\Phi \models \forall x\varphi \iff \Phi \models \varphi[\mathsf{c}/x]$*;*
- $\Phi, \exists x\varphi \models \psi \iff \Phi, \varphi[\mathsf{c}/x] \models \psi$*.*

This proposition also hold in InqBQ, provided the constant c is rigid. However, in InqBQ analogous facts hold if instead of a fresh constant c we use a fresh variable y. This is not the case in InqBT: for instance, as we already mentioned, in InqBT $\forall x?P(x)$ is valid even though $?P(y)$ is not. What underlies this mismatch between free and bound variables is that, in InqBT, bound variables are always interpreted rigidly in the team by the semantic clause for the quantifiers, while free variables may receive different values at different assignments in the team.

This discussion suggests the following strategy to make inferences with quantifiers in InqBT: we first extend the relevant signature with a countably infinite stock of constant symbols (to make sure that we can never run out of fresh variables in a proof) and then we adopt the inference rules for quantifiers given in Figure 7.8.

Another significant difference between InqBT and InqBQ stems from Proposition 7.4.7: in InqBT, every sentence φ is equivalent to its classical variant φ^{cl}. Thus, at the level of sentences inquisitive operators collapse onto the corresponding classical operators. This reflects the fact that in InqBT, we do not

Universal quantifier			Inquisitive existential	
$\dfrac{\varphi[\mathsf{c}/x]}{\forall x\varphi}$	$\dfrac{\forall x\varphi}{\varphi[\mathsf{t}/x]}$	$\dfrac{\forall x\alpha}{\alpha[t/x]}$	$\dfrac{\varphi[\mathsf{t}/x]}{\exists x\varphi}$	$\dfrac{\exists x\varphi \qquad \begin{matrix}[\varphi[\mathsf{c}/x]]\\ \vdots \\ \psi\end{matrix}}{\psi}$

Figure 7.8: Inference rules for quantifiers in InqBT. In these rules, α ranges over classical formulas, while φ, ψ range over arbitrary formulas; t is a closed term, while t is an arbitrary term; in the introduction rule for the universal quantifier, c is a constant symbol which does not occur in any undischarged assumption; in the elimination rule for the inquisitive existential quantifier, c is a constant symbol which does not occur free in ψ or in any undischarged assumption.

model uncertainty about the state of affairs (since the relational structure is fixed), but only uncertainty about the values of variables, which is only relevant for the interpretation of open sentences.

As a consequence of this collapse of inquisitive operators in the case of sentences, InqBT does not share some of the meta-theoretic properties of InqBQ. In particular, we do not have the disjunction property for $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$. To see this, let c be a constant symbol: the formula $?P(c)$, which abbreviates $P(c) \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \neg P(c)$, is equivalent to $P(c) \vee \neg P(c)$ in InqBT by Proposition 7.4.7, and thus logically valid; but obviously, neither $P(c)$ nor $\neg P(c)$ is logically valid. Similarly, we do not have the existence property for $\exists$. To see this, consider the formula

$$\exists x((P(c) \wedge x = c) \vee (\neg P(c) \wedge x = c'))$$

where c, c' are two distinct constant symbols. With a reasoning analogous to the one we gave on page 158 we can show that this formula is logically valid in InqBT, but none of the formulas

$$(P(c) \wedge t = c) \vee (\neg P(c) \wedge t = c')$$

obtained by instantiating the existential to a term t is logically valid.

In spite of these difference existing between InqBQ and InqBT, it seems natural to conjecture that one can give entailment-preserving translations between the two systems. If so, many of the open questions about the properties of InqBT reduce to the corresponding questions about InqBQ, and vice versa. Thus, research on these two systems is tightly connected. We leave it as an open problem to establish (or disprove) this conjecture.

7.4.14. Open problem. [Existence of a translation of InqBT into InqBQ] Given a signature Σ, is there a signature Σ', a decidable set $\Theta \subseteq \mathcal{L}^{\mathsf{Q}=}(\Sigma')$ and a computable map $(\cdot)^* : \mathcal{L}^{\mathsf{Q}=}(\Sigma) \to \mathcal{L}^{\mathsf{Q}=}(\Sigma')$ s.t. for all sets $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\mathsf{Q}=}(\Sigma)$ we have $\Phi \models_{\mathsf{InqBT}} \psi \iff \Phi^*, \Theta \models_{\mathsf{InqBQ}} \psi^*$?

7.4.15. Open problem. [Existence of a translation of InqBQ into InqBT] Given a signature Σ, is there a signature Σ', a decidable set $\Theta \subseteq \mathcal{L}^{\mathsf{Q}=}(\Sigma')$ and a computable map $(\cdot)^* : \mathcal{L}^{\mathsf{Q}=}(\Sigma) \to \mathcal{L}^{\mathsf{Q}=}(\Sigma')$ s.t. for all sets $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\mathsf{Q}=}(\Sigma)$ we have $\Phi \models_{\mathsf{InqBQ}} \psi \iff \Phi^*, \Theta \models_{\mathsf{InqBT}} \psi^*$?

Relations to standard dependence logic D. The systems InqBT and D are defined in the same semantic setting: in both cases, formulas are interpreted relative to a relational structure M and a team T. We can thus ask straightforwardly how the expressive power of these two systems relates. One thing that we can immediately say is the following.

7.4.16. Proposition. *There are formulas in D which are not equivalent to any formula in InqBT.*

Proof. Take the dependence logic sentence $\exists^d x \forall^d y \exists^d z (=(z, y) \wedge z \neq x)$ discussed in Section 7.2.2, which is satisfied relative to a structure M and a non-empty team just in case the domain of M is infinite. Let us denote this sentence by φ_{inf}. We claim that this formula is not equivalent to any formula in InqBT. Towards a contradiction, suppose ψ is an InqBT-formula equivalent to φ_{inf}. Then in particular, ψ and φ_{inf} must be equivalent in terms of truth conditions (i.e., when interpreted relative to single assignments). But we know from Corollary 7.4.4 that ψ has the same truth conditions as its classical variant ψ^{cl}, which are just the truth conditions of ψ^{cl} in classical first-order logic. It follows that φ_{inf} has the same truth conditions as a classical first-order formula, which is not the case. $\square$

What about the converse? That is, in terms of expressive power, can InqBT be seen as a fragment of D, or are the two systems simply incomparable? This is, to the best of my knowledge, an interesting open question.

7.4.17. Open problem. Is every formula in InqBT equivalent to some formula in D?

In particular, one could ask whether the examples of InqBT-dependence formulas discussed in Section 7.4.4 and Section 7.4.5 are expressible in D.

7.5 A general framework for first-order questions and dependencies

As we saw, there is a discrepancy between the semantic framework of standard inquisitive first-order logic, based on sets of possible worlds and a single assignment, and the framework used in work on dependence logic, based on a single relational structure and a set of assignments. We saw that the language of inquisitive first-order logic can be interpreted in both settings, leading to different systems InqBQ and InqBT. In these systems we can capture different classes of questions and dependencies. For instance, in InqBQ we can express the question "what is the extension of P?", and capture the fact that the extension of P determines the extension of Q. In InqBT, by contrast, we can express the question "what is the value of x?" and capture the fact that the value of x determines the value of y. In this section, we show that it is possible to introduce a more general semantic framework, which allows us to capture questions about the state of affairs as well as questions about the values of variables, and questions involving both of these things. As we will see, the systems InqBQ and InqBT can be seen as special cases of a system InqBQ^+ formulated in this general setting, each obtained by restricting to a certain kind of evaluation points.

In order to obtain our general framework, we evaluate formulas with respect to objects that capture partial information about both the state of affairs and the value of variables, as well as their correlations. This can be achieved by taking our points of evaluation to be sets of world-assignment pairs. We will refer to such objects as *information states with referents*, abbreviated as *r-states*.[12,13]

7.5.1. Definition. [Indices and r-states]
Let $M = \langle W, D, I\rangle$ be a first-order information id-model.[14]

[12]A similar semantic setup has been proposed by Väänänen (2014) with a rather different motivation in mind. Väänanen's goal is to develop a logic capable of expressing interesting properties of a set-theoretic *multiverse*, i.e., a structure containing a multitude of distinct models of set theory. In his system, formulas are evaluated with respect to a multiset of first-order models and to a function mapping each of these models to an assignment into the corresponding domain. While seemingly more complex, this setup is essentially equivalent to our setup based on sets of model-assignment pairs, provided that we allow different worlds to have different domains. For simplicity, in this section we stick to the case of a constant domain.

[13]Information states with referents are a fundamental notion in *dynamic semantics* (see, e.g., Heim, 1982; Dekker, 1993; Groenendijk et al., 1996; Aloni, 2000), where they are simply called *information states*. In this line of work, the standard way to think about such an object s is as follows: s encodes not only information about features of the world, but also about the possible values of certain *discourse referents*, which stand for individuals that the discourse is about, but whose identity is not necessarily known. For instance, if we hear that "a girl was running", and if we use variable x to store the new discourse referent that this sentence introduces, the resulting state s will only contain pairs $\langle w, g\rangle$ such that the individual $g(x)$ is a girl who was running in world w.

[14]For simplicity, we spell out the proposal for the case of id-models. It is straightforward

- An *index* is a pair $i = \langle w_i, g_i \rangle$, where $w_i \in W$ and $g_i : \mathsf{Var} \to D$.
- An information state with referents, or *r-state* for short, is a set s of indices.

In the system InqBQ^+, sentences are interpreted relative to r-states. Notice that an r-state determines both an ordinary information state and a team.

7.5.2. Definition. Let s be an information state with referents. Then:

- the state associated with s is $\pi_1[s] = \{w \mid \langle w, g \rangle \in s \text{ for some } g\}$;
- the team associated with s is $\pi_2[s] = \{g \mid \langle w, g \rangle \in s \text{ for some } w\}$.

However, an r-state is not uniquely determined by the state $\pi_1[s]$ and the team $\pi_2[s]$. This is because, in general, s also encodes information about the correlation between the state of affairs and the value of variables, information that is not reflected by the projections $\pi_1[s]$ and $\pi_2[s]$. For instance, the r-states $s_1 = \{\langle w, g \rangle, \langle w', g' \rangle, \langle w, g' \rangle, \langle w', g \rangle\}$ and $s_2 = \{\langle w, g \rangle, \langle w', g' \rangle\}$ have the same projections, but the latter also encodes a certain correlation between the state of affairs and the assignment function, which the former does not encode.

The language of our system InqBQ^+ is the same first-order language $\mathcal{L}^{\mathsf{Q}=}$ that we have for the systems InqBQ and InqBT. The value of a term t at an index i is simply $[t]^i := [t]^{w_i}_{g_i}$. Moreover, given an r-state s and an individual $d \in D$, we write $s[x \mapsto d]$ for the r-state obtained by modifying the valuation at each index in s from g to $g[x \mapsto d]$:

$$s[x \mapsto d] = \{\langle w, g[x \mapsto d] \rangle \mid \langle w, g \rangle \in s\}.$$

The relation of support between r-states s in a model M and formulas $\varphi \in \mathcal{L}^{\mathsf{Q}}$ is defined as follows.

7.5.3. Definition. [Support in InqBQ^+]

- $M, s \models R(t_1, \ldots, t_n) \iff \langle [t_1]^i, \ldots, [t_n]^i \rangle \in I_{w_i}(R)$ for all $i \in s$
- $M, s \models (t = t') \iff [t]^i = [t']^i$ for all $i \in s$
- $M, s \models \bot \iff s = \emptyset$
- $M, s \models \varphi \wedge \psi \iff M, s \models \varphi$ and $M, s \models \psi$
- $M, s \models \varphi \to \psi \iff$ for all $t \subseteq s, M, t \models \varphi$ implies $M, t \models \psi$
- $M, s \models \varphi \vee\!\!\!\vee \psi \iff M, s \models \varphi$ or $M, s \models \psi$

to generalize this system to allow for the more flexible treatment of identity described in the previous chapter.

- $M, s \models \forall x\varphi \iff M, s[x \mapsto d] \models \varphi$ for every $d \in D$
- $M, s \models \exists x\varphi \iff M, s[x \mapsto d] \models \varphi$ for some $d \in D$

This system is similar to InqBQ in many respects. First of all, the semantics is persistent, and every formula is supported by the empty r-state. We can define truth with respect to an index i as support with respect to $\{i\}$, and we can call a formula truth-conditional if support for it always amounts to truth at each world. Then, we still have that all classical formulas are truth-conditional and that, moreover, the truth-conditions for a classical formula at an index $i = \langle w, g\rangle$ are the ones given by classical first-order logic with respect to the assignment g and the relational structure $M_w = \langle D, I_w\rangle$ associated with w.

7.5.4. PROPOSITION. *For every $\alpha \in \mathcal{L}_c^Q$ and every model M:*

- *Truth-conditionality: for every r-state s in M,*

$$M, s \models \alpha \iff M, i \models \alpha \textit{ for all } i \in s.$$

- *Standard truth conditions: for every index $i = \langle w, g\rangle$ in M,*

$$M, i \models \alpha \iff M_w \models_g \alpha \textit{ in standard Tarskian semantics.}$$

This shows that, as far as classical formulas are concerned, InqBQ^+ is yet another informational semantics for classical first-order logic: in restriction to classical formulas, the entailment relation that arises from InqBQ^+ is just the one of classical first-order logic.

Moreover, it is worth remarking that the systems InqBQ and InqBT can both be seen as special cases of InqBQ^+, obtained by restricting the semantics of InqBQ^+ to particular kinds of r-states. The system InqBQ is obtained by restricting the semantics to r-states in which all indices i have the same assignment component g_i.

7.5.5. PROPOSITION.
Suppose $\pi_2[s] = \{g\}$. Then $M, s \models \varphi \iff M, \pi_1[s] \models_g \varphi$ in InqBQ.

Similarly, the system InqBT is obtained by restricting the semantics to r-states in which all indices i have the same world component w_i.

7.5.6. PROPOSITION.
Suppose $\pi_1[s] = \{w\}$. Then $M, s \models \varphi \iff M_w \models_{\pi_2[s]} \varphi$ in InqBT.

As we expect, the interpretation of sentences is insensitive to the assignment component of an r-state, which means that, as far as sentences are concerned, InqBQ^+ coincides with InqBQ, in the following sense.

7.5.7. Proposition.
If φ is a sentence and s is an r-state, $M, s \models \varphi \iff M, \pi_1[s] \models \varphi$ in InqBQ.

Notice all the questions that we discussed in the previous chapter were sentences: we were only interested in variables insofar as these would ultimately get bound. The previous proposition implies that all those questions receive exactly the same interpretation in InqBQ^+ as they did in InqBQ. This includes, e.g., polar questions of the form $?\exists x P(x)$, mention-some questions of the form $\exists x P(x)$, and mention-all questions of the form $\forall x ?P(x)$.

In addition to these questions, however, InqBQ^+ can also interpret questions which concern the value of free variables. This includes, in particular, identification questions of the form λx for a variable x (recall that we defined λx to be $\exists y(y = x)$ for some variable y distinct from x), which receive the same interpretation as in InqBT:

$$M, s \models \lambda x \iff \text{for all } g, g' \in \pi_2[s] : \ g(x) = g'(x).$$

Thus, in InqBQ^+ we can capture questions that concern only the state of affairs, such as $\forall x ?P(x)$, and questions that concern only the value of free variables, such as λx. In addition, we can also capture questions that concern both aspects at once. For an example, consider a polar question $?P(x)$. We have:

$$M, s \models ?P(x) \iff [\, g_i(x) \in P_{w_i} \text{ for all } i \in s] \text{ or } [\, g_i(x) \notin P_{w_i} \text{ for all } i \in s].$$

Thus, whether $?Px$ is supported depends on what s settles about the value of x and about the extension of P. An r-state s might determine exactly the value of x, but fail to support $?Px$ because it does not determine whether the extension of P includes the relevant object; conversely, s might determine exactly the extention of P, yet fail to support $?Px$ because it does not determine the value of x. On the other hand, an r-state s may support $?Px$ without determining of any particular object whether it has property P.

Within the system InqBQ^+, we obtain a uniform analysis of the different sorts of dependencies that we encountered in the previous chapter and in the present one. As in the previous chapter, we can capture, e.g., the fact that the extensions of $P_1, \dots, P_n$ determine the extension of Q. This is expressed by the formula:

$$\forall x ?P_1(x) \wedge \cdots \wedge \forall x ?P_n(x) \ \to \ \forall x ?Q(x).$$

As in the previous section, we can capture the fact that the value of $x_1, \dots, x_n$ determines the value of y. This is expressed by the formula:

$$\lambda x_1 \wedge \cdots \wedge \lambda x_n \ \to \ \lambda y.$$

Moreover, we can express mixed dependencies. For instance, the following formula, which is logically valid in InqBQ^+, expresses the fact that the value of x and the extension of P jointly determine whether x has property P:

$$\forall y?P(y) \wedge \lambda x \;\to\; ?P(x).$$

We will not delve further here into the study of the system InqBQ^+. Our aim in this section was merely to illustrate how one can give a semantic framework which simultaneously represents partial information about the state of affairs, the values of variables, and their correlations, and to show how within such a framework one can define a generalized version of inquisitive first-order logic that encompasses, as special cases, both the standard inquisitive first-order logic InqBQ that we studied in detail in the previous chapter, and its team semantics counterpart InqBT that we discussed in the previous section.

7.6 Summary and final considerations

In this chapter, we looked at some of the tight relations existing between inquisitive logic and dependence logic, with a special focus on the analysis of the notion of dependency. We started out by introducing the standard notion of functional dependency in a team, understood as a relation between variables, and we saw how this relation plays a role in the semantics of the standard system of dependence logic. We then discussed how the basic ideas and notions of inquisitive logic, as laid out in Chapter 2, can be applied naturally in the context of team semantics. Doing so yields a new perspective on the notion of dependency, which can be viewed as a relation between questions, rather than as a relation between variables. We emphasized several virtues of this perspective: it is much more general than the one based on variables, since even in a context where very few variables are at stake, there is a broad spectrum of questions about these variables that can be considered, and thus a broad spectrum of dependencies that can be analyzed in terms of such questions. Standard functional dependency is thus found to be a special case of a broad class of relations which share a common logical core and can be handled by the same logical tools. Moreover, the question-based perspective reveals that dependency can be seen as a facet of the central logical notion of entailment, once this is extended to questions. Dependencies are thus directly connected to the central concerns of logic.

We illustrated these general points by means of a specific logical system, InqBT, which is an adaptation to the team semantics setting of the first-order inquisitive logic of the previous chapter. In exploring this system, our main aim was to illustrate the inquisitive approach in the team semantics setting, of which the system InqBT is only an instance. When deciding on a logic to analyze and reason about dependencies, the particular choice of logical repertoire will depend on one's ultimate purposes. If one's purpose is to capture dependencies between bound variables, of the kind that play a role in Henkin quantifiers, then InqBT is not sufficient, since this system is equivalent to standard first-order logic with respect to sentences. For this purpose, one may want, e.g., to enrich InqBT with the quantifiers of dependence logic. On the other hand, InqBT is a very natural logic to capture dependencies involving free variables, and thus to reason about features of a given team. As we saw, more research is needed to understand the exact meta-theoretic properties of InqBT—to determine, e.g., whether this system is recursively axiomatizable. If this system turns out to be relatively complex, for some purposes one may well want to restrict to a well-behaved fragment, such as the finitely coherent fragment (cf. Section 5.7.1).[15] However, all these different choices concerning the logical repertoire of the system are compatible with the general conception and system architecture that stems from the key ideas of inquisitive logic discussed in Chapter 2.

7.7 Exercises

7.7.1. Exercise. [Formalizing dependencies.]
Consider a team over the domain $\mathbb{R}$ of real numbers which contains all assignments g such that $g(y)$ equals the square of $g(x)$. The following table depicts some rows of this team.

x	y
$\vdots$	$\vdots$
-2	4
-1	1
$-\frac{1}{2}$	$\frac{1}{4}$
0	0
$\frac{1}{2}$	$\frac{1}{4}$
1	1
2	4
$\vdots$	$\vdots$

Consider a language equipped with a relation symbol $<$, two constant sym-

[15]Note that, as we discussed on page 246, the finitely coherent fragment includes in particular formulas corresponding to dependence atoms, and is closed under implication, which means that all dependencies involving questions in the fragment can be expressed within the fragment.

bols $1, 0$ interpreted in the natural way, and with a function symbol $|\cdot|$ interpreted as mapping a number a to its absolute value.

The following (conditional) dependencies hold in the context of this team.

(2) a. The value of x determines the value of y.
b. Given that x is positive, the value of y determines the value of x.
c. The value of y together with the information whether x is positive determines the value of x.
d. The value of y determines the absolute value of x.
e. Whether y is greater than 1 is determined by whether the absolute value of x is greater than 1.

Write down formulas of InqBT that express these dependencies.

7.7.2. Exercise. [Inquisitive logic in team semantics]
Consider a team over the domain $\mathbb{N}$ of natural numbers which contains all assignments g such that $g(z) = g(x)g(y)$. The following table gives the idea.

x	y	z
a	b	ab

Consider a language equipped with predicate symbols Even, $<$, and Pf, interpreted respectively as "is even", "is less than", "is a prime factor of", as well as constant symbols $0, 1$ and a binary function symbol $+$ interpreted in the natural way. Determine whether the following implications are supported in the team.

1. $\lambda x \wedge \lambda y \rightarrow \lambda z$
2. $\lambda x \wedge \lambda z \rightarrow \lambda y$
3. $(x > 0) \wedge \lambda x \wedge \lambda z \rightarrow \lambda y$
4. $?\mathrm{Even}(x) \wedge ?\mathrm{Even}(y) \rightarrow ?\mathrm{Even}(z)$
5. $(x > 0) \wedge ?\mathrm{Even}(x) \wedge ?\mathrm{Even}(z) \rightarrow ?\mathrm{Even}(y)$
6. $\lambda x \wedge \lambda(y + z) \rightarrow \lambda z$
7. $\exists t \mathrm{Pf}(t, x) \rightarrow \exists t \mathrm{Pf}(t, z)$
8. $\exists t \mathrm{Pf}(t, z) \rightarrow \exists t(\mathrm{Pf}(t, x) \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \mathrm{Pf}(t, y))$
9. $\exists t \mathrm{Pf}(t, z) \rightarrow \exists t(\mathrm{Pf}(t, x) \vee \mathrm{Pf}(t, y))$
10. $(x > 1) \wedge \forall t ?\mathrm{Pf}(t, x) \rightarrow \forall t ?\mathrm{Pf}(t, z)$
11. $(x > 1) \wedge (y > 1) \wedge \forall t ?\mathrm{Pf}(t, x) \wedge \forall t ?\mathrm{Pf}(t, y) \rightarrow \forall t ?\mathrm{Pf}(t, z)$

12. $(x > 1) \wedge (z > 1) \wedge \forall t ?\mathrm{Pf}(t, x) \wedge \forall t ?\mathrm{Pf}(t, z) \rightarrow \forall t ?\mathrm{Pf}(t, y)$
13. $(x > 0) \wedge \forall t ?\mathrm{Pf}(t, x) \rightarrow ?\mathrm{Even}(x)$

Chapter 8

Inquisitive modal logic: a preview

Throughout this book, we have emphasized different ways in which questions are relevant in logic. We saw that questions can be seen as names for types of information, and that by generalizing logic to questions we can capture logical relations holding between information types. We also saw that in the inquisitive setting, a more general account of certain logical operators emerges, which boils down to the classical one in the special case of statements, but which also covers the role of these operators in questions. Furthermore, we saw that questions may be used in inferences as placeholders for arbitrary information of a certain type, and that by reasoning with questions we can build formal proofs of the existence of certain logical dependencies (and of certain logical relations of answerhood and presupposition). However, there is another important role for questions in logic that we have not yet touched upon. This role becomes apparent when we equip inquisitive logic with modal operators that capture question-directed modal notions. This takes us into the realm of *inquisitive modal logic*.

Since inquisitive modal logic is the topic of a freshly started research project, significant developments in this area can be expected in the next few years. For this reason, we leave a comprehensive exposition of the topic for a future occasion. In this final chapter, we will however give a preview of this subfield of inquisitive logic. First, in Section 8.1 we will explain why the prospect of adding modalities to inquisitive logic is especially promising. In Sections 8.2 and 8.3 we will sketch how different kinds of modalities have so far been added to inquisitive logic, illustrating the significance of these modalities in the setting of one particular interpretation, and discussing one important aspect that distinguishes inquisitive modal logic from the inquisitive propositional and predicate logics discussed in this book. Finally, in Section 8.4 we mention some directions for future work.

8.1 Issue-directed modal notions

Modal logic is an incredibly versatile sub-field of logic, which is used to formally analyze a range of notions of great interdisciplinary importance. In its various interpretations, it is used to capture notions such as knowledge and belief, permission and obligation, different varieties of necessity and possibility, provability in a theory, truth in the past or in the future, strategic ability, and much more. What do these notions have in common? They can all be analyzed as properties of propositions: it is propositions that can be necessary or possible, known or believed, true in the past or in the future, etcetera. Not surprisingly, in the sort of English sentences which are formalized in standard modal logic, the argument of a modal operator is typically given by a 'that' clause, whose content is a (standard) proposition. This is illustrated by the following examples:

(1) a. It is possible that Smith will win the election.
b. Brown thinks that Smith will win the election.
c. Brown has a strategy to ensure that Smith wins the election.

Now consider the following sentences:

(2) a. Who wins the election is determined by how many votes each candidate gets.
b. Brown wonders who will win the election.
c. Brown doesn't care who wins the election.
d. Brown controls who wins the election.

These sentences express instances of important modal notions. Sentence (2-a) is an example of a *supervenience* claim: given the electoral system, there can be no difference in winner without an underlying difference in the number of votes. Supervenience is a modal notion that plays a key role in all areas of analytic philosophy, at least as important as that of possibility and necessity. Sentences (2-b) and (2-c) ascribe a certain 'inquisitive attitude' to Brown: they characterize her as being in a state that bears a certain relation to a question content, just like the belief ascription in (1-b) characterizes her as standing in a certain relation to a proposition. Inquisitive attitudes have recently come under attention in philosophy of mind; various authors (Friedman, 2013; Carruthers, 2018) have emphasized that such attitudes are just as important to the analysis of inquiry and agency as the much more widely studied propositional attitudes. Finally, sentence (2-c) is a control claim, stating that a certain aspect of the world—in this case the outcome of the election—is under Brown's control. To be able to express and reason about what aspects of a situation each agent controls is important for the analysis of action and strategic reasoning in a multi-agent setting.

What these three examples have in common is that they ascribe a modal property, not to a proposition, but to a question, namely, the embedded interrogative 'who wins the election'. Or rather, more precisely, they ascribe a modal property to the content expressed of this question. Let us refer to the content expressed by a question as an *issue*. We can then say that the notions illustrated by the examples in (2) are *issue-directed*: it is issues that are the relata of a supervenience claim, the objects of wondering and caring, and the sort of things that an agent may or may not have control over.

In standard modal logic, the language does not contain formulas that stand for questions. This is no accident, since standard modal logic builds on truth-conditional semantics, which as we discussed is not suitable to interpret questions. As a consequence, issue-directed modal notions have so far remained outside of the scope of modal logic. This is a significant limitation: we saw three examples of important issue-directed modal notions, and they are not isolated cases: on the contrary, once we look for them, interesting issue-directed notions can be identified in all areas of application of modal logic.

This limitation can be overcome by building a new framework for modal logic based on inquisitive logic. As we saw, in inquisitive logic we have not only formulas that stand for statements, but also formulas that stand for questions. The semantics allows us to model the content of a question as a set of information states—those in which the question is supported. In this setting, we can naturally equip our language with modal operators O that can apply to a question μ to yield a statement $O\mu$, the truth conditions of which are defined in terms of the content of μ. In this way, a broad range of new, interesting modal notions can be formally analyzed and brought within the purview of modal logic. As a result, the domain of application of modal logic can be extended substantially.

In the next two sections, we will make the idea more concrete by looking at two particular kinds of modal operators that can be added to an inquisitive logic and which have received attention in the inquisitive logic literature (see Ciardelli and Roelofsen, 2015; Ciardelli, 2014, 2017a; van Gessel, 2020b, 2021; Ciardelli and Otto, 2020; Meißner and Otto, 2021; Punčochář and Sedlár, 2021a,b).

8.2 Generalizing Kripke modalities to questions

Modalities in Kripke semantics. In standard Kripke semantics, modalities are analyzed as quantifiers over a set of accessible worlds. Formally, this works by extending the language with a new unary operator $\Box$ (a dual is defined by letting $\Diamond\varphi$ abbreviate $\neg\Box\neg\varphi$). Models for the language are obtained by equipping a set W of possible worlds with an accessibility relation $R : W \times W$. The semantics, which is given in terms of truth conditions relative to a world, interprets $\Box$ by means of the following clause, where $R[w] = \{v \in W \mid wRv\}$:

$$M, w \models \Box\varphi \iff \forall v \in R[w] : M, v \models \varphi$$

Conceptually, the relation R can be given many different interpretations. For instance, $R[w]$ could be viewed as the set of worlds that are possible according to what a certain agent knows or believes at w; it could be the set of worlds that conform to a certain normative code, or the set of worlds which are compatible with certain background facts about the world (e.g., the laws of physics), or the set of worlds which are possible future instants relative to w, etcetera. Each of these interpretations gives rise to a corresponding reading for modal formulas. For instance, $\Box p$ could be read as "the agent knows/believes p", "it is obligatory that p", "necessarily p", "it will always be the case that p", and so on.

Kripke modalities in inquisitive logic. Let us see how this treatment of modalities can be extended to the inquisitive setting. Syntactically, we may just extend the language of inquisitive (propositional or predicate) logic with a new unary operator $\Box$. Models are defined simply by extending an information model M with an accessibility relation R on the set of possible worlds. Note that the resulting models coincide with standard Kripke models (in the predicate logic case, Kripke models with a constant domain).

The semantics is obtained by extending the definition of support with the following clause:[1]

- $M, s \models \Box\varphi \iff \forall w \in s : M, R[w] \models \varphi$.

By specializing this clause to singleton states, we get the following truth conditions for $\Box\varphi$:

- $M, w \models \Box\varphi \iff M, R[w] \models \varphi$.

It is clear from these clauses that for any φ—regardless of whether φ is a statement or a question—$\Box\varphi$ is truth-conditional: it supported at a state s iff it is true at each world $w \in s$. As a consequence , the semantics of $\Box\varphi$ is fully determined at the level of truth conditions. In words, these truth conditions say that $\Box\varphi$ is true at a world w iff φ is supported by the set $R[w]$ of successors of w.

In case the argument of our modality is a statement α, our semantics boils down to Kripke semantics. Indeed, if α is truth-conditional, we have:

$$\begin{aligned} M, w \models \Box\alpha &\iff M, R[w] \models \alpha \\ &\iff \forall v \in R[w] : M, v \models \alpha. \end{aligned}$$

[1] In the case of predicate logic, the clause needs to be relativized to an assignment g.

If we restrict the language to classical formulas not containing inquisitive operators ($\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ or, in the predicate logic case, $\exists$), we thus obtain a language that can be fully identified with the language of standard modal logic. All formulas in this fragment are truth-conditional, and the truth conditions for them are the same as in standard modal logic. This means that our inquisitive modal logics (in the plural, since different logics arise from different classes of frames, as usual) are conservative extensions of the corresponding classical systems.

Illustration in the epistemic setting. To appreciate how inquisitive semantics allows us to extend the operator $\Box$ to questions, it is helpful to have in mind the epistemic interpretation of $\Box$ as formalizing the verb 'know'. In this interpretation, the information state $R[w]$ models the epistemic state of the agent at world w—the set of worlds compatible with what the agent knows at w.

Consider the knowledge ascriptions in (3). In inquisitive modal logic, they can be formalized straightforwardly by applying the knowledge modality to the translations of the complements 'that Alice passed the test' (Pa), 'whether Alice passed the test' ($?Pa$) and 'who passed the test' ($\forall x?Px$).

(3) a. The agent knows that Alice passed the test. $\Box Pa$
b. The agent knows whether Alice passed the test. $\Box ?Pa$
c. The agent knows who passed the test. $\Box\forall x?Px$

Let us see what our semantics predicts for these formulas. First, we saw that all of them are truth-conditional, which is in line with the fact that the sentences in (3) are statements. We also saw that for the 'standard' knowledge ascription $\Box Pa$, the truth conditions are the same as given by standard epistemic logic:

$$M, w \models \Box Pa \iff \forall v \in R[w] : M, v \models Pa.$$

In words, $\Box Pa$ is true in case Pa is true in all worlds compatible with the agent's knowledge.

Now let us consider the formula $\Box ?Pa$, where the argument is the polar question $?Pa$. Using the support conditions for polar questions, which are familiar by now, we have:

$$\begin{aligned} M, w \models \Box ?Pa \quad &\iff \quad M, R[w] \models ?Pa \\ &\iff \quad \forall v, v' \in R[w] : (M, v \models Pa \iff M, v' \models Pa). \end{aligned}$$

Thus, $\Box ?Pa$ is true just in case the truth value of Pa is settled in the agent's epistemic state—i.e., if the agent has no uncertainty concerning this truth value. This is intuitively the correct prediction for a knowing-whether ascription such as (3-b). We also have:

$$\begin{aligned} M, w \models \Box ?Pa &\iff M, R[w] \models ?Pa \\ &\iff M, R[w] \models Pa \text{ or } M, R[w] \models \neg Pa \\ &\iff M, w \models \Box Pa \text{ or } M, w \models \Box \neg Pa \\ &\iff M, w \models \Box Pa \vee \Box \neg Pa. \end{aligned}$$

Thus, $\Box ?Pa$ has the same truth conditions as $\Box Pa \vee \Box \neg Pa$, and since the two formulas are both truth-conditional, they are equivalent: $\Box ?Pa \equiv \Box Pa \vee \Box \neg Pa$. This is an intuitive result: knowing whether Alice passed the test amounts to knowing either that Alice passed the test, or that she did not pass the test.

Finally, consider the formula $\Box \forall x ?Px$, where the argument is the mention-all question $\forall x ?Px$ asking for the extension of P. Using the support conditions for mention-all questions (cf. Example 5.2.5 on page 134), we have:

$$\begin{aligned} M, w \models \Box \forall x ?Px &\iff M, R[w] \models \forall x ?Px \\ &\iff \forall v, v' \in R[w] : P_v = P_{v'}. \end{aligned}$$

Thus, $\Box \forall x ?Px$ is true at a world w in case the extension of P is settled in the agent's epistemic state—i.e., if the agent has no uncertainty about this extension. This is the intuitively correct prediction for (3-c): to know who passed the test is to know the extension of the predicate 'having passed the test'.

As these examples illustrate, our generalized semantics for $\Box$ allows us to give a neat uniform account of knowledge ascriptions involving statements ('knowing that') and questions ('knowing whether, who, what, ...'). Note that in this case, the point is not that knowledge ascriptions such as (3-b) and (3-c) cannot be captured in standard modal logic: they can. We already saw that $\Box ?Pa$ is equivalent to the standard modal formula $\Box Pa \vee \Box \neg Pa$, and for (3-b), we have:

$$\Box \forall x ?Px \;\equiv\; \forall x(\Box Px \vee \Box \neg Px).$$

The point is, rather, that in inquisitive modal logic the semantics of knowledge attributions can be derived compositionally in a principled way from a general semantics for 'know' and the semantics of the embedded complements. The fact that, say, knowing whether p amounts to knowing p or knowing $\neg p$ does not have to be stipulated, but is *derived* in the logic, which is a welcome result. Moreover, such an account sheds light on how verbs like 'know' (but also 'remember', 'tell', and many others) work in natural language (for the linguistic relevance of such an account, see Ciardelli and Roelofsen, 2018; Theiler et al., 2018, 2019).

Non-epistemic interpretations. We illustrated the semantics for $\Box$ in the epistemic setting, but there are many other natural interpretations. Just to give a hint, let us consider the pair of formulas $\Box p$ and $\Box ?p$. In a legal setting, if $\Box p$ says that the law mandates that p, then $\Box ?p$ says that the law mandates whether p. In the setting of provability logic, if $\Box p$ says that the theory proves p, then $\Box ?p$ says that the theory decides p. In the setting of temporal logic, if $\Box p$ says that p will henceforth always be the case, then $\Box ?p$ says that p is henceforth immutable. And yet other natural interpretations can easily be given.

Notes on the logic. The logic of the generalized modality $\Box$ turns out to be very simple. The distributivity axiom of standard modal logic is generally valid, including when φ and ψ are questions:

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi).$$

The necessitation principle also holds: if φ is a validity, then so is $\Box\varphi$. From this, additional facts follow, such as the monotonicity of $\Box$ and the commutation of $\Box$ with $\wedge$. In the predicate logic case, the constant domain setup ensures that the Barcan and coverse Barcan formulas also hold, i.e., $\Box$ commutes with the universal quantifier:[2]

$$\forall x \Box\varphi \equiv \Box\forall x\varphi.$$

A distinctive feature of $\Box$ in the inquisitive setting is the validity of the following pseuso-commutation principles, which say that an inquisitive operator under $\Box$ is equivalent to the corresponding classical operator above $\Box$.

$$\Box(\varphi \vee\!\!\!\vee \psi) \equiv \Box\varphi \vee \Box\psi, \qquad \Box\exists x\varphi \equiv \exists x\Box\varphi.$$

These principles allow us to push a $\Box$ modality through an inquisitive operator. In the propositional case, this can used to show that any formula of the form $\Box\varphi$ is equivalent to a formula of standard modal logic (see §6 of Ciardelli, 2016a). This is not the case in the first-order setting, however: there are modal formulas of the form $\Box\varphi$, where φ is a question, which, while being statements, are not equivalent to any formula of standard first-order modal logic.[3] Thus, by extending $\Box$ to questions we obtain a logic which is more expressive than standard modal logic, even in restriction to statements.

In the propositional setting, Ciardelli (2016a) (§6) has described a strategy for turning complete axiomatizations for standard (normal) modal logics into complete axiomatizations of the corresponding inquisitive modal logics. This strategy applies to all the most familiar modal logics, yielding completeness results for their inquisitive extensions.

[2]Of course, this is not a necessary component of inquisitive modal logic as such, but a consequence of our choice to work with models based on a fixed domain.

[3]While this result is not mentioned explicitly, it follows easily from recent results contained in Hamed (2021).

8.3 Properly inquisitive modalities

Basic setup. In the literature on inquisitive modal logic, a different modal operator has also been studied. This modality is standardly denoted $\boxplus$, and as in the case of $\Box$, it can be added to the language of inquisitive propositional or predicate logic. As we will see, this is an example of a modality that allows us to capture issue-directed modal notions that cannot be captured in standard modal logic.

In order to interpret this modality, we equip an information model with a relation $\mathcal{R} : W \times \wp(W)$ between worlds and information states. Such a relation allows us to associate to each world w a set of information states $\mathcal{R}[w] = \{s \subseteq W \mid w\mathcal{R}s\}$.[4,5] Let us call the resulting model an *inquisitive modal model*.

The semantics is obtained by extending the definition of support with the following clause for $\boxplus$:

- $M, s \models \boxplus\varphi \iff \forall w \in s, \forall t \in \mathcal{R}[w] : M, t \models \varphi.$

This clause makes the formula $\boxplus\varphi$ truth-conditional. We can thus study its semantics by looking at its truth conditions, which are as follows:

- $M, w \models \boxplus\varphi \iff \forall t \in \mathcal{R}[w] : M, t \models \varphi.$

Thus, $\boxplus\varphi$ is a statement which is true at a world w just in case φ is supported at all $\mathcal{R}$-successors of w. Note that the relation $\mathcal{R}$ also induces a Kripke-style accessibility relation $R \subseteq W \times W$, given by:

$$R[w] := \bigcup \mathcal{R}[w].$$

This means that in the context of an inquisitive modal model we can also interpret a modality $\Box$, which uses the induced relation R in the way discussed in the previous section.

When applied to a truth-conditional formula, the two modalities $\boxplus$ and $\Box$ coincide with each other and with the universal modality of standard modal logic. That is, when α is truth-conditional we have:

$$\begin{aligned} M, w \models \boxplus\alpha &\iff M, w \models \Box\alpha \\ &\iff \forall v \in R[w] : M, v \models \alpha. \end{aligned}$$

[4]Equivalently, in the literature a map $\Sigma : W \to \wp\wp(W)$ is used. Formally, this is the sort of map used to interpret modalities in neighborhood semantics for standard modal logic. So, a model for our logic is essentially a standard neighborhood model for modal logic. As we will see, however, in the inquisitive setting this map is used to give a Kripke-style semantics for the modality $\boxplus$, and not a neighborhood semantics.

[5]Most previous work also assumes that $\mathcal{R}$ is downward closed, in the sense that if $w\mathcal{R}s$ and $t \subseteq s$ then $w\mathcal{R}t$. However, as pointed out by Meißner and Otto (2021), this assumption is not strictly needed, in that we obtain the same logic without the downward closure requirement.

Thus, the classical fragment of our modal logic is just standard modal logic, with $\boxplus$ collapsing onto $\Box$. However, things become interesting as soon as we consider formulas obtained by applying $\boxplus$ to a question. To appreciate the results, it is helpful to have a concrete interpretation of inquisitive modal models in mind.

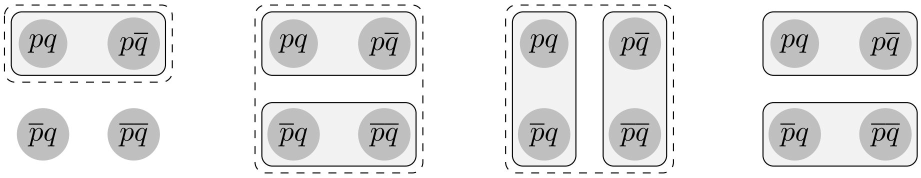
Illustration in the inquisitive-epistemic setting. In the inquisitive epistemic logic proposed by Ciardelli and Roelofsen (2015), an inquisitive modal model is given the following intuitive interpretation: given a world w, we have $w\mathcal{R}s$ just in case all the issues the agent is interested in are settled in the state s—that is, if s is an information state where the agent's curiosity is satisfied. The set of states $\mathcal{R}[w]$ thus captures the *inquisitive state* of the agent at world w, encoding the issues that the agent is interested in. The information state $R[w] = \bigcup \mathcal{R}[w]$ is viewed as reflecting the agent's knowledge at w.

To make this more concrete, consider the three situations depicted in Figure 8.1, where as usual, w_{pq} stands for a world where p and q are both true, $w_{p\overline{q}}$ for a world where p is true and q false, etcetera. In Figures 8.1(a)-8.1(c), the maximal elements of the agent's inquisitive state $\mathcal{R}[w]$ are drawn in solid lines; the corresponding information state $R[w] = \bigcup \mathcal{R}[w]$ is drawn by the dashed line. Here, w stands for an arbitrary world in the model (note that the different sub-figures depict different models, since the accessibility relation $\mathcal{R}$ is different).

- In Figure 8.1(a), the epistemic state of the agent is $R_1[w] = \{w_{pq}, w_{p\overline{q}}\}$. This means that the agent knows that p and does not know whether q. The agent's inquisitive state is $\mathcal{R}_1[w] = \{\{w_{pq}, w_{p\overline{q}}\}\}^{\downarrow}$. This means that the issues of the agent are already settled by the agent's knowledge state (and, therefore, also by any stronger body of information). Thus, Figure 8.1(a) represents a situation where the agent knows that p and has no further open issues.

- In Figure 8.1(b), the epistemic state of the agent is the entire set of worlds $R_2[w] = \{w_{pq}, w_{p\overline{q}}, w_{\overline{p}q}, w_{\overline{pq}}\}$. This means that the agent has no information. The agent's inquisitive state is $\mathcal{R}_2[w] = \{\{w_{pq}, w_{p\overline{q}}\}, \{w_{\overline{p}q}, w_{\overline{pq}}\}\}^{\downarrow}$. This means that the issues of the agent can be settled either by reaching a state at least as strong as $\{w_{pq}, w_{p\overline{q}}\}$—i.e., by establishing that p—or by reaching a state at least as strong as $\{w_{\overline{p}q}, w_{\overline{pq}}\}$—i.e., by establishing that $\neg p$. In other words, the agent's issues are settled just in case the question $?p$ is resolved. Thus, Figure 8.1(b) represents a situation where the agent has no knowledge and is interested (only) in the issue of whether p.

- In Figure 8.1(c), the situation is parallel to the one in the previous case, but with the roles of p and q swapped: the agent has no knowledge and is interested (only) in the issue of whether q.



(a) $\mathcal{R}_1[w]$ (b) $\mathcal{R}_2[w]$ (c) $\mathcal{R}_3[w]$ (d) $[?p]_M$

Figure 8.1: Three inquisitive states for an agent. The maximal elements of the inquisitive state are depicted in solid lines, the corresponding epistemic states in dashed lines. The last figure shows the alternatives for the question $?p$.

With this particular intuitive interpretation of inquisitive modal models in mind, let us now examine the different significance of the modal claims $\Box\mu$ and $\boxplus\mu$ when μ is a question. The truth conditions for $\Box\mu$ are:

- $M, w \models \Box\mu \iff M, R[w] \models \mu$

Thus, $\Box\mu$ is true if what the agent knows suffices to resolve the question μ. As we saw in the previous section, this corresponds to the intuitive truth conditions of the statement 'the agent knows μ'. Now let us consider $\boxplus\mu$. We have:

- $M, w \models \boxplus\mu \iff \forall t \in \mathcal{R}[w] : M, t \models \mu$

Thus, $\boxplus\mu$ is true just in case any information that settles the agent's issues also settles μ—in other words, if settling μ is necessary in order to satisfy the agent's curiosity. This could be the case in a trivial way if the agent's *current* information settles μ, i.e., if $\Box\mu$ is true. But the truth-conditions of $\boxplus\mu$ are more lenient: it could be the case that μ is not settled by the agent's current information, but it is settled by all "target" information states where the agent's issues are settled. This situation is described by the formula $\neg\Box\mu \land \boxplus\mu$, which Ciardelli and Roelofsen (2015) propose to adopt as a formalization of the statement 'the agent wonders about μ'. Let us illustrate this with the pictures in Figure 8.1.

- In Figure 8.1(a), the agent's epistemic state (dashed) supports the question $?p$. Thus, we have $w \models \Box?p$ (and then, by persistency, also $w \models \boxplus?p$). Thus, this is classified as a situation where the agent knows whether p.

- In Figure 8.1(b), the agent's state epistemic state (dashed) does not support $?p$. However, each element of the agent's inquisitive state (the two solid blocks and their subsets) supports $?p$. Thus, in this state we have $w \models \neg\Box?p \land \boxplus?p$. So, this is classified as a situation where the agent wonders whether p.

- In Figure 8.1(c), the agent's epistemic state again does not support $?p$. In this case, however, some elements of the agent's inquisitive state, for instance $\{w_{pq}, w_{\overline{p}q}\}$, do not support $?p$ either. In this situation, we have $w \models \neg\Box?p \wedge \neg\boxplus?p$. So this is classified as a situation where the agent neither knows whether p, nor wonders about it.

This illustrates how in inquisitive modal logic we can express not only facts about the knowledge agents have, but also facts about the issues they entertain. As we are now going to see, in order to express these facts, questions are crucial.

Modal statements about questions. Throughout the book, when setting up a system of inquisitive logic, we started out with a classical logic of statements and we added questions to it. While the addition of questions resulted in a more expressive language, this gain in expressive power did not concern statements: as witnessed by Corollary 3.4.5 and Proposition 5.3.7, any truth-conditional formula in inquisitive propositional or predicate logic is equivalent to a classical formula that does not contain any question operator—and thus equivalent to a formula of classical propositional or predicate logic.

In inquisitive modal logic, the situation is different. Consider again the formula $\boxplus?p$. Like any modal formula, this is a statement, i.e., truth-conditional. However, one can prove (cf. Prop. 7.1.18 in Ciardelli, 2016a) that $\boxplus?p$ is not equivalent to any $\vee\!\!\!\vee$-free formula. Thus, in inquisitive modal logic, the presence of questions has repercussions also on the range of *statements* that the language can express: by embedding questions under modal operators, we can express modal statements that are not expressible without referring to questions.

This is a significant difference: it means that in inquisitive modal logic—unlike in the systems considered in the previous chapters—questions are not merely added on top of a pre-existing logic of statements. Rather, statements and questions are crucially intertwined in the way illustrated in Figure 8.2: questions are built up from statements by means of $\vee\!\!\!\vee$ and $\exists$; at the same time, by embedding questions under $\Box$ and $\boxplus$ we can form new statements.

This brings out a further role for questions in logic, in addition to the ones discussed in detail in this book: questions give us names for issues; by defining modal operators that can apply to questions we can then capture modal facts *about* issues—the kind of issue-directed modal notions mentioned in Section 8.1. We have illustrated this potential in this section by describing the proposed analysis of *wondering* in inquisitive modal logic, but similar ideas can in principle be deployed for many other issue-directed notions. This territory, however, is almost entirely uncharted, and remains to be explored in future research.

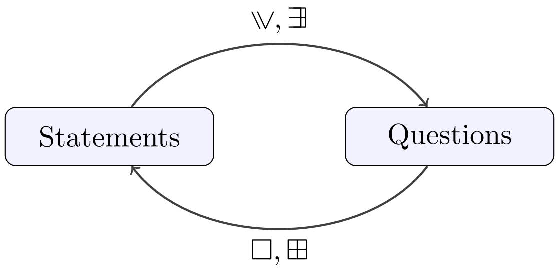


Figure 8.2: In inquisitive modal logic, statements and questions are intertwined in an essential way.

Notes on the logic. Like the modality $\Box$, the inquisitive modality $\boxplus$ also has very familiar features. In this case as well, the distributivity axiom

$$\boxplus(\varphi \to \psi) \to (\boxplus\varphi \to \boxplus\psi)$$

is valid for any formulas φ and ψ, and so is the necessitation principle: if φ is valid then so is $\boxplus\varphi$. As usual, this implies that $\boxplus$ is monotonic and commutes with $\wedge$. In the predicate logic case, $\boxplus$ also commutes with $\forall$ due to the constant domain setup of the semantics. The main difference with $\Box$ is that $\boxplus$ does not validate the pseudo-distributivity over inquisitive operators:

$$\boxplus(p \mathbin{\vee\!\!\!\vee} q) \not\equiv \boxplus p \vee \boxplus q, \qquad\qquad \boxplus\boxplus xPx \not\equiv \exists x \boxplus Px.$$

As shown by Ciardelli (§7 of 2016a), distributivity and necessitation are in fact all we need to completely axiomatize the logic of $\boxplus$ in the propositional case, which shows that $\boxplus$ is an extremely natural generalization of the standard universal modality. This completeness result is also extended to a range of modal logics obtained by imposing some salient conditions on the relation $\mathcal{R}$.

8.4 Looking ahead

The foregoing discussion has, hopefully, given the reader an idea of why combining questions with modalities is especially interesting, and of how extending modal logic into the inquisitive territory has the potential to bring new interesting modal notions within the purview of logical analysis. Although inquisitive modal logic has received some attention in recent years (see the list of references in Appendix A), much more work is needed to bring out this potential.

First, the role of questions in modal logic has to be demonstrated by showing that a range of interesting modal notions can be analyzed in this setting. In addition to the analysis of issue-directed attitudes such as *wondering* illustrated above, the range of natural applications include the analysis of supervenience and strategic control. In the case of supervenience, the idea is, at a first pass, that property P supervenes on property Q in case, in the relevant domain of possibilities, the extension of Q is determined by the extension of P, which can be formalized in inquisitive modal logic by $\Box(\forall x?Px \to \forall x?Qx)$ (Ciardelli, 2018a). Things are somewhat more complicated, however: for reasons that we cannot discuss in detail here, a proper analysis of different kinds of supervenience in fact requires the tools of the logic InqBQ^+ developed in Section 7.5, where information states are modeled as sets of world-assignment pairs. In the case of strategic control, the idea is that, within an appropriate interpretation of inquisitive modal models, a formula such as $\neg\Box?p \wedge \boxplus?p$ says that the truth value of p is not settled before the agent's action, but becomes settled as soon as the agent has acted; this captures the fact that whether p comes about is determined by the agent's choice at a certain point in time. The possibility of this interpretation of inquisitive modalities is mentioned by Ciardelli (§7.5 of 2016a), but its integration in the context of logics of actions like stit logic (Belnap et al., 2001), coalition logic (Pauly, 2001) and alternating-time temporal logic (Alur et al., 2002) remains to be developed and investigated.

Moreover, modal logic has an extremely rich mathematical theory. Once we extend modal logic to questions, such a theory needs to be reconstructed in the generalized setting. Ciardelli and Otto (2020) and Meißner and Otto (2021) have recently made a first step towards a model theory of inquisitive modal logic, characterizing the expressive power of the logic in terms of a suitable notion of bisimulation, defining translations to first-order predicate logic, and proving analogues of the classical van Benthem theorem, which characterizes modal logic as the bisimulation invariant fragment of first-order predicate logic. Much more remains to be done, however. One topic that remains entirely to be studied in the inquisitive setting is *frame definability*, where it is natural to look for analogues of Sahlqvist theory and the Goldblatt-Thomason theorem. Other areas to be explored are the proof theory of inquisitive modal logic (where, for instance, tableaux systems might be fruitfully developed), the range of modal operators definable in the inquisitive setting, and the properties of first-order inquisitive modal logic, which has so far not been systematically investigated.

Appendix A

Overview of the literature on inquisitive logic

This book is intended only as an introduction to inquisitive logic. There are many ideas, results, and applications of inquisitive logic that we were not able to cover in the book. This appendix gives an overview of the publications on inquisitive logic, categorized by topic. It is intended to help the reader orient in the literature on the subject, which has flourished over the past few years.

Work on inquisitive propositional logic.

- Early work on a predecessor of InqB:
  Groenendijk (2009); Mascarenhas (2009)
- Sources for the system InqB:
  Groenendijk (2008); Ciardelli (2009); Ciardelli and Roelofsen (2011)
- Axiomatization and other basic results on InqB:
  Ciardelli (2009); Ciardelli and Roelofsen (2011)
- Definability and eliminability of connectives in InqB:
  Ciardelli (2009); Ciardelli and Barbero (2019)
- A modal companion to InqB:
  Punčochář (2012)
- Extension of InqB with non-persistent operators:
  Punčochář (2015)
- Extension with Stalnaker-Lewis-style conditional operators, application to the logic of unconditionals: Ciardelli (2016c); Bledin (2020)
- Natural deduction system for InqB and constructive content of inquisitive proofs: Ciardelli (2018b)

- A display calculus for InqB:
  Frittella, Greco, Palmigiano, and Yang (2016)

- A labelled sequent calculus for InqB:
  Chen and Ma (2017)

- Structural completeness of InqB:
  Iemhoff and Yang (2016)

- Complexity of model checking for InqB:
  Grilletti and Zeuner (2022)

- Systems of propositional inquisitive logic different from InqB:
  Groenendijk (2011); Ciardelli, Groenendijk, and Roelofsen (2015); Groenendijk and Roelofsen (2015)

- An alternative conceptual interpretation of the system InqB:
  Wiśniewski (2014)

- Inquisitive logics based on a (super)intuitionistic logic of statements:
  Punčochář (2016a,b); Ciardelli, Iemhoff, and Yang (2020); Holliday (2020); Sano (2020)

- An inquisitive logic based on a version of relevance logic:
  Punčochář (2020)

- A general approach to building inquisitive versions of substructural logics:
  Punčochář (2019)

- Studies on inquisitive logic from an algebraic perspective:
  Bezhanishvili, Grilletti, and Holliday (2019); Quadrellaro (2019); Punčochář (2021); Grilletti and Quadrellaro (2021); Bezhanishvili, Grilletti, and Quadrellaro (2021)

Work on inquisitive predicate logic.

- Original sources for InqBQ:
  Ciardelli (2009); Roelofsen (2013); Ciardelli, Groenendijk, and Roelofsen (2018)

- A recent dissertation containing many advances on InqBQ:
  Grilletti (2020)

- Model-theoretic investigations:
  Grilletti (2019); Grilletti and Ciardelli (2021); Ciardelli and Grilletti (2022)

- Axiomatizations of fragments/variants:
  Sano (2011), Ciardelli (§4 of 2016a), Grilletti (2021), Ciardelli and Grilletti (2022)

Work on inquisitive modal logic.

- General framework:
  Ciardelli (§6-7 of 2016a)
- Model-theoretic study and connections to classical first-order logic:
  Ciardelli and Otto (2017, 2020); Meißner and Otto (2021)
- Inquisitive epistemic logic:
  Ciardelli and Roelofsen (2015); Ciardelli (2014); Punčochář and Sedlár (2021a)
- Inquisitive dynamic epistemic logic:
  Ciardelli and Roelofsen (2015); Ciardelli (2017a); van Gessel (2020b); Mellema (2019)
- Inquisitive strict conditionals and the analysis of modal determinacy and supervenience: Ciardelli (2018a); Humberstone (2019)
- Inquisitive modal logics of programs (inquisitive versions of PDL):
  Mellema (2019); Punčochář and Sedlár (2021b)
- Inquisitive modal logics with applications to free choice effects in natural language: Aher (2012); Aloni and Ciardelli (2013); Aher and Groenendijk (2015); Willer (2018, 2019); Nygren (2021); Booth (2021)

Two-dimensional extension of inquisitive logic.

- A two-dimensional framework to capture the role of indexicals in questions and generalize the logic of apriority and necessity: van Gessel (2020a, 2021)

Connections with other logical frameworks.

- Connections to intermediate logics:
  Ciardelli (2009); Ciardelli and Roelofsen (2011)[1]
- Connections to dependence logic:
  Ciardelli (2016b); Yang and Väänänen (2016)

[1]Note: some of these results are anticipated in Miglioli, Moscato, Ornaghi, Quazza, and Usberti (1989), where the logic InqB was already considered from a mathematical standpoint, though not in connection with the ideas of inquisitive semantics.

- Connections to inferential erotetic logic:
  Wiśniewski and Leszczyńska-Jasion (2015)

- Connections to proof-theoretic semantics:
  Stafford (2021)

- Connections to non-standard epistemic logics:
  Ciardelli (§6 of 2016a)

- Connections with possibility semantics for modal logic:
  Ciardelli (§6 of 2016a)

Applications of inquisitive logic.

- Applications in linguistics:
  this part of the literature is too vast to be listed here; the reader is referred to Appendix B in Ciardelli, Groenendijk, and Roelofsen (2018) and to the list of references maintained on the inquisitive semantics webpage (https://projects.illc.uva.nl/inquisitivesemantics).

- Applications in formal epistemology:
  Uegaki (2012); Hamami (2014); Cohen (2019)

- Applications in psychology:
  Koralus and Mascarenhas (2013); Mascarenhas (2014); Mascarenhas and Koralus (2015); Mascarenhas and Picat (2019)

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