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A NEWTON-KANTOROVICH METHOD FOR A FUNCTIONAL EQUATION RELATIVE TO THE CONFORMAL REPRESENTATION

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ABSTRACT: In this paper we consider the problem of computing the Riemann Map $g_{\zeta,w}$ of the plane unit disk $\mathbb{D}$ onto the Jordan domain bounded by the simple closed curve ζ containing the point w in the interior, and normalized by conditions $g_{\zeta,w}(0) = w$, $g'_{\zeta,w}(0) > 0$. To solve such problem, we consider a suitable functional equation involving ζ, w, $g^{(-1)}_{\zeta,w} \circ \zeta$, and we show that one can obtain $g^{(-1)}_{\zeta,w} \circ \zeta$ by applying a version of the Newton-Kantorovich method.

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1. Introduction

In this paper we consider the Riemann map $g_{\zeta,w}$ of the unit disk $\mathbb{D}$ of the complex plane onto the plane domain $\mathbb{I}[\zeta]$ enclosed by the simple Jordan curve ζ, and normalized by conditions

$$g_{\zeta,w}(0) = w, \quad g'_{\zeta,w}(0) > 0,$$

where w is a point of $\mathbb{I}[\zeta]$, and we show that $g_{\zeta,w}$ can be computed by applying the Newton-Kantorovich method in a suitable function space.

By [15], $g^{(-1)}_{\zeta,w} \circ \zeta$ depends real analytically on the pair (ζ, w) in a Schauder space setting (see also [10]), while the dependence of $g_{\zeta,w}$ upon (ζ, w) has a lower degree of regularity (cf. [10], Lanza and Preciso [13, §3], Lanza and Preciso [14]). Thus, instead of trying to obtain $g_{\zeta,w}$ as a limit of Newton iterates, we turn our attention to $g^{(-1)}_{\zeta,w} \circ \zeta$, and we consider a system of functional equations involving $\zeta, w, g^{(-1)}_{\zeta,w} \circ \zeta$ introduced and analyzed in [15] (see also [9]).

As it is well known, the conformal representation $g_{\zeta,w}$ relative to ζ, w can be determined explicitly for certain pairs (ζ_0, w_0), as for example when ζ_0 is a circle, and w_0 is its center. Then for a pair $(\zeta, w) = (\zeta_1, w_1)$ we consider a homotopy $(\zeta_\lambda, w_\lambda)_{\lambda\in[0,1]}$ of (ζ_0, w_0) to (ζ_1, w_1), and we show that there exists a finite set of numbers $\{0 = \lambda_0 < \lambda_1 < \ldots < \lambda_q = 1\}$ of $[0,1]$, such that $g^{(-1)}_{\zeta_{\lambda_k}, w_{\lambda_k}} \circ \zeta_{\lambda_k}$ can be computed by applying the Newton-Kantorovich method to the functional equation mentioned above, provided that $g^{(-1)}_{\zeta_{\lambda_{k-1}}, w_{\lambda_{k-1}}} \circ \zeta_{\lambda_{k-1}}$ is known.

The importance and the influence of the Riemann map in the applications is well known, and for a historical account, we refer to Gray [4] and Ullrich [18]. Several papers have been devoted to the boundary behaviour of the Riemann map, and more generally of univalent functions. For a modern presentation, we refer to Pommerenke [16] and to Wen [19].

The problem of the Riemann map has also been studied extensively by means of integral equations (see Gaier [2]). Probably, the most known among these equations is that of Theodorsen, which applies to star like domains. The Theodorsen equation has been successfully employed to compute numerically the Riemann map (cf. *e.g.*, Gutknecht [5], Hübner [6] and references therein). In particular, in Hübner [6] the Newton method has been applied to the study of a nonlinear system which appears following a discretization of the Theodorsen integral equation.

Among the contributions to the study of the Theodorsen equation we also mention that of von Wolfersdorf [20], who has reduced the Theodorsen equation to a nonlinear Riemann-Hilbert problem in order to prove uniqueness. The system of integral equations which we examine is more complicated than the Theodorsen equation, but has the advantage of requiring no restriction on the geometry of the domain, although it requires some regularity of the boundary of the domain.

2. Technical Preliminaries and Notation

The inverse function of a function f is denoted $f^{(-1)}$ as opposed to the reciprocal of a complex-valued function g, which is denoted g^{-1}. Throughout the paper, we make no formal distinction between complex numbers and pairs of real numbers, so for example if $f = (f_1, f_2)$ is a map of $\mathbb{R}^2$ to $\mathbb{R}^2$, and if $f_1 + if_2$ is holomorphic in the complex variable $x_1 + ix_2$, then f' denotes the complex derivative of $f_1 + if_2$. We denote by $\mathbb{D}$ the open unit disk in $\mathbb{C}$ (or in $\mathbb{R}^2$), by $\mathbb{T}$ the boundary of $\mathbb{D}$, and by $\mathrm{cl}\,\mathbb{D}$ the closure of $\mathbb{D}$. We denote by $\Re z$ and by $\Im z$ the real and the imaginary part of a complex number z. By $\int_{\mathbb{T}} f(s)\,ds$ we understand the line integral of the function f of $\mathbb{T}$ to $\mathbb{C}$ computed with respect to the parametrization $\theta \mapsto e^{i\theta}$, $\theta \in [0, 2\pi]$, of $\mathbb{T}$. By $\int_{\mathbb{T}} f(s)\,|ds|$ we understand the integral of f with respect to the ordinary measure $|ds|$ on $\mathbb{T}$. Let $\mathbb{N}$ be the set of nonnegative integers including 0. Let $m \in \mathbb{N}$. $C^m_*(\mathbb{T}, \mathbb{C})$ denotes the space of m times continuously differentiable functions from $\mathbb{T}$ to $\mathbb{C}$, and $C^{m,\alpha}_*(\mathbb{T}, \mathbb{C})$ denotes the subspace of $C^m_*(\mathbb{T}, \mathbb{C})$ of those functions which have m-th order derivatives that are Hölder continuous with exponent $\alpha \in]0,1]$ (the subscript '$*$' means that the derivatives are being taken with respect to the variable in $\mathbb{T}$).

Let $B \subseteq \mathbb{C}$. We set $C_*^{m,\alpha}(\mathbb{T}, B) \equiv \{f \in C_*^{m,\alpha}(\mathbb{T}, \mathbb{C}) : f(\mathbb{T}) \subseteq B\}$. If $f \in C_*^{0,\alpha}(\mathbb{T}, \mathbb{C})$, we set $|f|_\alpha \equiv \sup\left\{\frac{|f(s)-f(t)|}{|s-t|^\alpha} : s,t \in \mathbb{T}, s \neq t\right\}$. We endow $C_*^{m,\alpha}(\mathbb{T}, \mathbb{C})$ with its usual norm $\|f\|_{m,\alpha} \equiv \sum_{j=0}^{m} \sup_{t\in\mathbf{T}} |D^j f(t)| + |D^m f|_\alpha$. It is well-known that $(C_*^{m,\alpha}(\mathbb{T}, \mathbb{C}), \|f\|_{m,\alpha})$ is a Banach space. Similarly, we define $C^{m,\alpha}(\mathrm{cl}\,\mathbb{D}, \mathbb{R})$ to be the space of m-times continuously differentiable real-valued functions in $\mathbb{D}$ such that all the partial derivatives up to order m admit a continuous extension to $\mathrm{cl}\,\mathbb{D}$, and such that the partial derivatives of order m are α-Hölder continuous. By $C^{m,\alpha}(\mathrm{cl}\,\mathbb{D}, \mathbb{R}^2)$ we understand $(C^{m,\alpha}(\mathrm{cl}\,\mathbb{D}, \mathbb{R}))^2$, and we take as norm of a pair of functions the sum of the norms of the components. It can be readily verified that the trace operator is linear and continuous from $C^{m,\alpha}(\mathrm{cl}\,\mathbb{D}, \mathbb{R}^2)$ to $C_*^{m,\alpha}(\mathbb{T}, \mathbb{C})$ (cf. *e.g.*, Lanza and Preciso [12, Lem. 2.8].) We endow a Cartesian product of normed spaces with the norm given by the sum of the norms of the components. Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ be normed spaces. We denote by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ the space of linear continuous operators of $\mathcal{X}$ to $\mathcal{Y}$ endowed with its usual norm of the uniform convergence on the unit sphere of $\mathcal{X}$. We denote by $\mathcal{B}(\mathcal{X} \times \mathcal{Y}, \mathcal{Z})$ the space of bilinear and continuous maps of $\mathcal{X} \times \mathcal{Y}$ to $\mathcal{Z}$ endowed with its usual norm of the uniform convergence on the product of the unit spheres of $\mathcal{X}$ and $\mathcal{Y}$. For standard definitions of Calculus in normed spaces, we refer for example to Berger [1] and to Prodi and Ambrosetti [17].

We collect in the following Lemma a few facts we need on the space $C_*^{m,\alpha}(\mathbb{T}, \mathbb{C})$. A proof can be effected by elementary computations which we omit.

LEMMA 2.1. *Let $m \in \mathbb{N}, \alpha \in]0,1]$. Then the following statements hold.*

(i) *If $f \in C_*^1(\mathbb{T}, \mathbb{C})$, then*

$$|f(t_1) - f(t_2)| \leq \frac{\pi}{2}\|f'\|_0 |t_1 - t_2| \qquad \forall t_1, t_2 \in \mathbb{T}. \tag{2.2}$$

(ii)

$$\|u\|_{m-1,\alpha} \leq 2^{-\alpha}\pi\|u\|_m \qquad \forall u \in C_*^m(\mathbb{T}, \mathbb{C}). \tag{2.3}$$

(iii) *There exists $c_{1,m,\alpha} > 0$ such that*

$$\|u \cdot v\|_{m,\alpha} \leq c_{1,m,\alpha}\|u\|_{m,\alpha}\|v\|_{m,\alpha} \qquad \forall u, v \in C_*^{m,\alpha}(\mathbb{T}, \mathbb{C}). \tag{2.4}$$

Furthermore, we can take $c_{1,0,\alpha} = 1$, $c_{1,1,\alpha} = 2^{1-\alpha}\pi$.

(iv) *There exists $c_{2,m,\alpha} > 0$ such that*

$$\|\overline{u}\|_{m,\alpha} \leq c_{2,m,\alpha}\|u\|_{m,\alpha}, \quad \|\Re u\|_{m,\alpha} \leq \frac{1 + c_{2,m,\alpha}}{2}\|u\|_{m,\alpha}, \quad \|\Im u\|_{m,\alpha} \leq \frac{1 + c_{2,m,\alpha}}{2}\|u\|_{m,\alpha}, \tag{2.5}$$

for all $u \in C_*^{m,\alpha}(\mathbb{T}, \mathbb{C})$. If $m = 1$, then we can take $c_{2,1,\alpha} = 1 + 2^{2-\alpha}$.

(v) *If $f \in C_*^{1,\alpha}(\mathbb{T}, \mathbb{C})$, then $|f(t_1) - f(t_2) - f'(t_2)(t_1 - t_2)| \leq \frac{\pi}{2}|f'|_\alpha |t_1 - t_2|^{1+\alpha}$ for all t_1, $t_2 \in \mathbb{T}$.*

(vi) *There exists a function* $\varphi_{1,m,\alpha}$ *of* $[0,+\infty[\times]0,+\infty[$ *to* $]0,+\infty[$ *such that*

$$\|1/k\|_{m,\alpha} \leq \varphi_{1,m,\alpha}\left[\|k\|_{m,\alpha}, \min|k|\right],$$

for all $k \in C_*^{m,\alpha}(\mathbb{T},\mathbb{C})$ *which satisfy* $|k| > 0$ *on* $\mathbb{T}$, *and such that* $\varphi_{1,m,\alpha}[x_1,y_1] \leq \varphi_{1,m,\alpha}[x_2,y_2]$, *when* $0 \leq x_1 \leq x_2$, $0 < y_2 \leq y_1$. *In particular, if* $m = 1$ *we can take* $\varphi_{1,1,\alpha}[x,y] = y^{-1}(1+2(x/y)+2(x/y)^3)$.

We note that the value of the constants of the previous lemma is not optimal. We now give a formal definition of a curve. A regular curve is often defined as an equivalence class of regular parametrizations. However, for our purposes, it is necessary to distinguish among the different parametrizations. Thus we define a curve of class C_*^1 to be a map ζ of class C_*^1 from the boundary $\mathbb{T}$ of the unit disk $\mathbb{D}$ to $\mathbb{C}$. By a simple curve of class C_*^1, we understand an injective map of class C_*^1 from $\mathbb{T}$ to $\mathbb{C}$. Also, a curve ζ should not be confused with $\zeta(\mathbb{T})$.

By a simple contradiction argument, it can be readily verified that the following holds (cf. [8, p. 124], Lanza and Antman [11, p. 1201].)

LEMMA 2.6. *The set*

$$\mathcal{Z} \equiv \left\{\zeta \in C_*^1(\mathbb{T},\mathbb{C}) : l[\zeta] \equiv \inf\left\{\frac{|\zeta(s)-\zeta(t)|}{|s-t|} : s,t \in \mathbb{T},\ s \neq t\right\} > 0\right\}$$

coincides with the set of simple curves of class C_*^1 *with everywhere nonvanishing tangent vector. The nonlinear map* $l[\cdot]$ *of* $C_*^1(\mathbb{T},\mathbb{C})$ *to* $]0,+\infty[$ *is continuous, and the set* $\mathcal{Z}$ *is open in* $C_*^1(\mathbb{T},\mathbb{C})$.

If ζ is a simple closed curve of class C_*^1, we denote by $\mathbb{I}[\zeta]$ the bounded open connected component of $\mathbb{C}\setminus\zeta(\mathbb{T})$. We now introduce the following notation. We denote by $\mathrm{ind}[\zeta]$ the Cauchy index (winding number) of the map $\theta \mapsto \zeta(e^{i\theta})$, $\theta \in [0,2\pi[$, with respect to any of the points z of $\mathbb{I}[\zeta]$. Thus

$$\mathrm{ind}[\zeta] \equiv \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\zeta'(s)}{\zeta(s)-z}\,ds \qquad \forall z \in \mathbb{I}[\zeta].$$

The map $\mathrm{ind}[\cdot]$ is well-known to be constant on the open connected components of $\mathcal{Z}$ in $C_*^1(\mathbb{T},\mathbb{C})$. Clearly, $\mathrm{ind}[\zeta] \in \{-1,1\}$ for all $\zeta \in \mathcal{Z}$.

Since we need to estimate the distance of an element of $\mathcal{Z}$ from the boundary of $\mathcal{Z}$, we introduce the following Lemma, whose validity can be readily verified by exploiting the definition of $l[\cdot]$, Lemma 2.1, (i), and the triangle inequality.

LEMMA 2.7. *The following statements hold.*

(i) *If* $h \in \mathcal{Z}$, *then* $l[h] \leq \min_{\mathbb{T}}|h'|$.

(ii) *If* $h \in \mathcal{Z}$ *and* $h(\mathbb{T}) = \mathbb{T}$, *then* $l[h] \geq \frac{2}{\pi}\min_{\mathbb{T}}|h'|$.

(iii) *If* $\tilde{h} \in \mathcal{Z}$, $h \in C_*^1(\mathbb{T},\mathbb{C})$, $\|h'-\tilde{h}'\|_0 < \frac{2}{\pi}l[\tilde{h}]$, *then* $l[h] \geq l[\tilde{h}] - \frac{\pi}{2}\|h'-\tilde{h}'\|_0 > 0$, *and* $\mathrm{ind}[h] = \mathrm{ind}[\tilde{h}]$.

(iv) *If* $\tilde{\zeta} \in \mathcal{Z}$, $\tilde{w} \in \mathbb{I}[\tilde{\zeta}]$, $\zeta \in C^1_*(\mathbb{T},\mathbb{C})$, $w \in \mathbb{C}$, $\|\zeta' - \tilde{\zeta}'\|_0 < \frac{2}{\pi}l[\tilde{\zeta}]$, $\|\zeta - \tilde{\zeta}\|_0 + |w - \tilde{w}| < \min_{\mathbb{T}} |\tilde{\zeta}(\cdot) - \tilde{w}|$, *then* $\zeta \in \mathcal{Z}$, *and* $w \in \mathbb{I}[\zeta]$.

Then we have the following uniform estimate for the Cauchy integral, which we prove by a standard argument developed in Gakhov [3, p. 38]. We first introduce some notation. We set

$$\mathbb{L}(t_1, t_2) \equiv \{s \in \mathbb{T} : |s - t_1| < 2|t_1 - t_2|\}, \tag{2.8}$$

for all t_1, $t_2 \in \mathbb{T}$, and we understand that a line integral on $\mathbb{L}(t_1,t_2)$ or on $\mathbb{T} \setminus \mathbb{L}(t_1,t_2)$ is computed with respect to the parametrization $\theta \mapsto e^{i\theta}$ defined on a suitable interval of the real line of length less or equal to 2π.

<u>PROPOSITION 2.9</u>. *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$. *Then there exists a function* $\varphi_{2,m,\alpha}[\cdot,\cdot]$ *of* $[0,+\infty[\times]0,+\infty[$ *to* $]0,+\infty[$ *such that*

$$\varphi_{2,m,\alpha}[x_1, y_1] \leq \varphi_{2,m,\alpha}[x_2, y_2]$$

if $0 \leq x_1 \leq x_2$, $0 < y_2 \leq y_1$, *and which satisfies the following inequality*

$$\|\mathbf{C}[h,u]\|_{m,\alpha} \leq \varphi_{2,m,\alpha}[\|h\|_{m,\alpha}, l[h]]\, \|u\|_{m,\alpha}, \tag{2.10}$$

for all $u \in C^{m,\alpha}_*(\mathbb{T},\mathbb{C})$, $h \in C^{m,\alpha}_*(\mathbb{T},\mathbb{C}) \bigcap \mathcal{Z}$, *where*

$$\mathbf{C}[h,u](t) \equiv \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{u(s)h'(s)}{h(s) - h(t)}\, ds \qquad \forall t \in \mathbb{T}. \tag{2.11}$$

If $m = 1$, *then we can take*

$$\varphi_{2,1,\alpha}[x,y] = 2^{-1} + x^4 y^{-4}(20p_\alpha + 4), \tag{2.12}$$

where the real number p_α *defined by equality*

$$p_\alpha \equiv \frac{1}{2\pi} \sup_{t_1,t_2 \in \mathbb{T}} \left\{ \int_{\mathbb{T}} \frac{|ds|}{|s - t_1|^{1-\alpha}}, \int_{\mathbb{L}(t_1,t_2)} \frac{|t_1 - t_2|^{-\alpha}}{|s - t_j|^{1-\alpha}} |ds|, \right.$$
$$\left. \int_{\mathbb{T} \setminus \mathbb{L}(t_1,t_2)} \frac{|t_1 - t_2|^{1-\alpha}}{|s - t_2|^{2-\alpha}} |ds|, \left| \int_{\mathbb{T} \setminus \mathbb{L}(t_1,t_2)} \frac{ds}{s - t_2} \right| : j \in \{1,2\} \right\}$$

is finite.

<u>PROOF</u>. The finiteness of p_α can be easily verified. We now estimate $\|\mathbf{C}[h,u]\|_{0,\alpha}$, for $u \in C^{0,\alpha}_*(\mathbb{T},\mathbb{C})$, $h \in C^{1,\alpha}_*(\mathbb{T},\mathbb{C}) \cap \mathcal{Z}$. Since

$$\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{h'(s)}{h(s) - h(t)}\, ds = \frac{1}{2}\mathrm{ind}[h], \tag{2.13}$$

for each fixed $t \in \mathbb{T}$, then inequality (2.5) and the definition of $l[h]$ imply that

$$\|\mathbf{C}[h,u]\|_0 \leq |u|_\alpha \|h'\|_0 (l[h])^{-1} p_\alpha + 2^{-1}\|u\|_0. \tag{2.14}$$

We now estimate the Hölder quotient $|\mathbf{C}[h,u]|_\alpha$. For all t_1, $t_2 \in \mathbb{T}$ with $t_1 \neq t_2$, we have

$$\begin{aligned} (2.15)\quad & \mathbf{C}[h,u](t_1) - \mathbf{C}[h,u](t_2) = \\ &= \frac{1}{2\pi i}\int_{\mathbb{L}(t_1,t_2)} \frac{u(s)-u(t_1)}{h(s)-h(t_1)} h'(s)\,ds - \frac{1}{2\pi i}\int_{\mathbb{L}(t_1,t_2)} \frac{u(s)-u(t_2)}{h(s)-h(t_2)} h'(s)\,ds + \\ &\quad + \frac{1}{2\pi i}\int_{\mathbb{T}\setminus\mathbb{L}(t_1,t_2)} \frac{(u(s)-u(t_1))(h(t_1)-h(t_2))}{(h(s)-h(t_1))(h(s)-h(t_2))} h'(s)\,ds + \\ &\quad + \frac{1}{2\pi i}\int_{\mathbb{T}\setminus\mathbb{L}(t_1,t_2)} \frac{u(t_2)-u(t_1)}{h(s)-h(t_2)} h'(s)\,ds + 2^{-1}u(t_1)\mathrm{ind}[h] - 2^{-1}u(t_2)\mathrm{ind}[h]. \end{aligned}$$

Furthermore

$$\begin{aligned} (2.16)\quad & \frac{1}{2\pi i}\int_{\mathbb{T}\setminus\mathbb{L}(t_1,t_2)} \frac{h'(s)\,ds}{h(s)-h(t_2)} = \\ &= \frac{1}{2\pi i}\int_{\mathbb{T}\setminus\mathbb{L}(t_1,t_2)} \frac{(h'(s)-h'(t_2))(s-t_2) - (h(s)-h(t_2)-h'(t_2)(s-t_2))}{(h(s)-h(t_2))(s-t_2)}\,ds + \\ &\quad + \frac{1}{2\pi i}\int_{\mathbb{T}\setminus\mathbb{L}(t_1,t_2)} \frac{ds}{s-t_2}. \end{aligned}$$

Since $l[h] \leq \|h'\|_0$, and $\frac{2}{3}|s-t_2| \leq |s-t_1| \leq 2|s-t_2|$, for all $t_1, t_2 \in \mathbb{T}$ and for all $s \in \mathbb{T}\setminus\mathbb{L}(t_1,t_2)$, then Lemma 2.1 (i), (v), inequality (2.14), and equalities (2.15), (2.16), imply that

$$\|\mathbf{C}[h,u]\|_{0,\alpha} \leq \|u\|_{0,\alpha}\left(9p_\alpha (l[h])^{-2}\|h'\|^2_{0,\alpha} + p_\alpha + 1\right), \tag{2.17}$$

for all $u \in C^{0,\alpha}_*(\mathbb{T},\mathbb{C})$, $h \in C^{1,\alpha}_*(\mathbb{T},\mathbb{C}) \cap \mathcal{Z}$. We now assume that $u \in C^{m,\alpha}_*(\mathbb{T},\mathbb{C})$ and that $h \in C^{m,\alpha}_*(\mathbb{T},\mathbb{C}) \cap \mathcal{Z}$. By standard computations, we have

$$\frac{d}{dt}\left(\mathbf{C}[h,u](t)\right) = \mathbf{C}\left[h, u'/h'\right](t)h'(t). \tag{2.18}$$

We now observe that by Lemma 2.1 (iii) and by (2.18) the following holds

$$\|\mathbf{C}[h,u]\|_{m,\alpha} \leq \|\mathbf{C}[h,u]\|_0 + c_{1,m-1,\alpha}\left\|\mathbf{C}\left[h,u'/h'\right]\right\|_{m-1,\alpha}\|h'\|_{m-1,\alpha}. \tag{2.19}$$

Then by equality (2.13) and by the definition of $l[h]$, we have

$$\|\mathbf{C}[h,u]\|_0 \leq \|u\|_{m,\alpha}\left(\frac{\pi}{2}(l[h])^{-1}\|h\|_{m,\alpha} + \frac{1}{2}\right). \tag{2.20}$$

If $m = 1$, then Lemma 2.7 (i) and inequality $|1/h'|_\alpha \leq |h'|_\alpha (\min|h'|)^{-2}$, and inequalities (2.17), (2.19), (2.20) imply the validity of inequality (2.10) with $\varphi_{2,1,\alpha}[\cdot,\cdot]$ as in (2.12). To prove the statement for $m > 1$, we proceed by induction on m. Since $\min_{\mathbb{T}} |h'| \geq l[h]$, Lemma 2.1 (ii), (iii), (vi), inequalities (2.19) and (2.20), and the inductive assumption of existence of $\varphi_{2,m-1,\alpha}[\cdot,\cdot]$ imply the existence of $\varphi_{2,m,\alpha}[\cdot,\cdot]$ as in the statement. □

Then we have the following.

COROLLARY 2.21. *Let $m, n \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0,1[$. Let $\tilde{h} \in C^{m,\alpha}_*(\mathbb{T},\mathbb{C}) \cap \mathcal{Z}$. Let $r > 0$ be such that $\left\{h \in C^{m,\alpha}_*(\mathbb{T},\mathbb{C}) : \|h-\tilde{h}\|_{m,\alpha} \leq r\right\} \subseteq \mathcal{Z}$, then $\inf_{\|h-\tilde{h}\|_{m,\alpha}\leq r} l[h] > 0$, and the following two inequalities hold.*

$$\text{(2.22)} \qquad \sup_{\left\{h\in C^{m,\alpha}_*(\mathbb{T},\mathbb{C}):\, \|h-\tilde{h}\|_{m,\alpha}\leq r/2\right\}} \left\| \frac{1}{2\pi i}\int_{\mathbb{T}} u'(s)\prod_{j=1}^{n}\left(\frac{v_j(t)-v_j(s)}{h(t)-h(s)}\right) ds \right\|_{m,\alpha}$$
$$\leq \frac{2^n n^n}{(n-1)!} r^{-n}\varphi_{2,m,\alpha}\left[\|\tilde{h}\|_{m,\alpha}+r, \inf_{\|h-\tilde{h}\|_{m,\alpha}\leq r} l[h]\right]\cdot\|u\|_{m,\alpha}\cdot\prod_{j=1}^{n}\|v_j\|_{m,\alpha},$$

for all $u, v_1, \dots, v_n \in C^{m,\alpha}_*(\mathbb{T},\mathbb{C})$, and

$$\text{(2.23)} \qquad \sup_{\left\{h\in C^{m,\alpha}_*(\mathbb{T},\mathbb{C}):\, \|h-\tilde{h}\|_{m,\alpha}\leq r/2\right\}} \left\| \frac{1}{2\pi i}\int_{\mathbb{T}} w(s)\prod_{j=1}^{n}\left(\frac{v_j(t)-v_j(s)}{h(t)-h(s)}\right) ds \right\|_{m,\alpha} \leq$$
$$\leq (1+\pi)\frac{2^n n^n}{(n-1)!} r^{-n}\varphi_{2,m,\alpha}\left[\|\tilde{h}\|_{m,\alpha}+r, \inf_{\|h-\tilde{h}\|_{m,\alpha}\leq r} l[h]\right]\cdot\|w\|_{m-1,\alpha}\cdot\prod_{j=1}^{n}\|v_j\|_{m,\alpha},$$

for all $w \in C^{m-1,\alpha}_*(\mathbb{T},\mathbb{C})$ such that $\int_{\mathbb{T}} w(s)\,ds = 0$, and for all $v_1, \dots, v_n \in C^{m,\alpha}_*(\mathbb{T},\mathbb{C})$.

PROOF. Since the imbedding of the space $C^{m,\alpha}_*(\mathbb{T},\mathbb{C})$ into the space $C^m_*(\mathbb{T},\mathbb{C})$ is compact, then the set $\left\{h \in C^{m,\alpha}_*(\mathbb{T},\mathbb{C}) : \|h-\tilde{h}\|_{m,\alpha} \leq r\right\}$ is compact in $C^1_*(\mathbb{T},\mathbb{C})$. Then we point out as in Lanza and Preciso [12, p. 389], that the continuous function $l[\cdot]$ of $C^1_*(\mathbb{T},\mathbb{C})$ to $]0,+\infty[$ is strictly positive on $\{h \in C^{m,\alpha}_*(\mathbb{T},\mathbb{C}) : \|h-\tilde{h}\|_{m,\alpha} \leq r\}$, and that accordingly $\inf_{\|h-\tilde{h}\|_{m,\alpha}\leq r} l[h] > 0$.

We now turn to the proof of inequalities (2.22), (2.23). By Lanza and Preciso [12, Prop. 4.1], the nonlinear operator $\mathbf{C}[\cdot,\cdot]$ of $(C^{m,\alpha}_*(\mathbb{T},\mathbb{C}) \cap \mathcal{Z}) \times C^{m,\alpha}_*(\mathbb{T},\mathbb{C})$ defined by (2.11) is complex-analytic. Thus by fixing u, and by exploiting the Cauchy inequalities for holomorphic operators (cf. *e.g.*, Berger [1, p. 88] and Prodi and Ambrosetti [17, p. 85]), one can obtain that

$$(2.24)\qquad \sup_{\|h-\tilde{h}\|_{m,\alpha}\leq r/2}\left\{\sup_{\substack{v_1,\dots,v_n\in C^{m,\alpha}_*(\mathbb{T},\mathbb{C})\\ v_1\neq 0,\dots,v_n\neq 0}}\frac{\|\partial^n_h \mathbf{C}[h,u](v_1,\dots,v_n)\|_{m,\alpha}}{\|v_1\|_{m,\alpha}\cdots\|v_n\|_{m,\alpha}}\right\}\leq$$

$$\leq 2^n n^n r^{-n}\cdot \sup_{\|h-\tilde{h}\|_{m,\alpha}\leq r}\|\mathbf{C}[h,u]\|_{m,\alpha}.$$

Then by Proposition 2.9 and by the formula for $\partial^n_h \mathbf{C}[h,u]$ (cf. Lanza and Preciso [12, Prop. 4.1]), one obtains (2.22). To obtain (2.23) by (2.22) it sufficies to note that if $w\in C^{m-1,\alpha}_*(\mathbb{T},\mathbb{C})$ and $\int_{\mathbb{T}} w(s)\,ds = 0$, then the equality $U(e^{i\theta}) = \int_0^\theta w(e^{i\xi})ie^{i\xi}d\xi$ defines an element $U\in C^{m,\alpha}_*(\mathbb{T},\mathbb{C})$, such that $U' = w$, and for which $\|U\|_{m,\alpha}\leq \pi\|w\|_0 + \|w\|_{m-1,\alpha}\leq (1+\pi)\cdot\|w\|_{m-1,\alpha}$. □

3. The Newton-Kantorovich approximation scheme

As we have anticipated in the Introduction, we plan to obtain $g^{(-1)}_{(\zeta_1,w_1)}\circ\zeta_1$ by a given (ζ_1,w_1) in the set

$$\mathcal{E}_{m,\alpha}\equiv\{(\zeta,w)\in(C^{m,\alpha}_*(\mathbb{T},\mathbb{C})\cap\mathcal{Z})\times\mathbb{C} : w\in\mathbb{I}[\zeta]\}$$

by assuming that $g^{(-1)}_{(\zeta_0,w_0)}\circ\zeta_0$ is known for some $(\zeta_0,w_0)\in\mathcal{E}_{m,\alpha}$, and by assuming that there is a homotopy $(\zeta_\lambda,w_\lambda)_{\lambda\in[0,1]}$, of (ζ_0,w_0) and (ζ_1,w_1), with $\lambda\mapsto(\zeta_\lambda,w_\lambda)$ continuous from $[0,1]$ to $\mathcal{E}_{m,\alpha}$.

To do so, we first introduce a functional equation of the form

$$\mathbf{Q}[\zeta,w,h]=0,$$

and such that the set of solutions of such equation coincides with the set of triples (ζ,w,h), where (ζ,w) is an arbitrary element of $\mathcal{E}_{m,\alpha}$, and where $h\equiv h[\zeta,w]\equiv g^{(-1)}_{\zeta,w}\circ\zeta$. We do so in Theorem 3.1, by taking $\mathbf{Q}$ equal to the operator $\Pi_{\tilde{h}}\circ\mathbf{P}$ introduced in Theorem 3.1 (see (3.2).) Then we show that for each $(\zeta_\lambda,w_\lambda)$ of the homotopy there exist $\sigma_{(\zeta_\lambda,w_\lambda,h_\lambda)}>0$ and $r_{\zeta_\lambda,h_\lambda}>0$, where $h_\lambda\equiv h[\zeta_\lambda,w_\lambda]$, such that if $(\zeta,w)\in\mathcal{E}_{m,\alpha}$ satisfies inequality $\|\zeta-\zeta_\lambda\|_{m,\alpha}+|w-w_\lambda|\leq\sigma_{(\zeta_\lambda,w_\lambda,h_\lambda)}$, then equation $\mathbf{Q}[\zeta,w,h]=0$ has exactly one solution h such that $\|h-h_\lambda\|_{m,\alpha}\leq r_{\zeta_\lambda,h_\lambda}$, which can be expressed as the limit of appropriate Newton iterates (see Theorem 3.20.) To prove the convergence of the Newton iterates, we apply a result of Kantorovich and Akilov [7], which requires estimates of first and second order partial differentials of the operator $\mathbf{Q}$. We obtain such estimates in Theorem 3.4, and in Proposition 3.16. Finally, we show that there exists $0<\sigma\leq\inf_{\lambda\in[0,1]}\sigma_{(\zeta_\lambda,w_\lambda,h_\lambda)}$, and that accordingly there exist finitely many numbers $0=\lambda_0<\lambda_1<\ldots<\lambda_q=1$ such that $\sup_{k=1,\dots,q}\|\zeta_{\lambda_k}-\zeta_{\lambda_{k-1}}\|_{m,\alpha}+|w_{\lambda_k}-w_{\lambda_{k-1}}|<\sigma$, and such that $h[\zeta_{\lambda_k},w_{\lambda_k}]$ can be computed by knowing $(\zeta_{\lambda_{k-1}},w_{\lambda_{k-1}},h[\zeta_{\lambda_{k-1}},w_{\lambda_{k-1}}])$, as limit of Newton iterates. Thus we can obtain $h[\zeta_{\lambda_q},w_{\lambda_q}]$ in up to q steps.

We now introduce the functional equation for $(\zeta, w, h[\zeta, w])$, by means of the following Theorem, which summarizes a part of Propositions 3.5, 3.7, 3.12, 5.3 of [15].

THEOREM 3.1. *Let $\alpha \in]0,1[$, $m \in \mathbb{N} \setminus \{0\}$. Let $\mathcal{A}$ be the set defined by*

$$\mathcal{A} \equiv \left\{ (\zeta, w, h) \in C_*^{m,\alpha}(\mathbb{T}, \mathbb{C}) \times \mathbb{C} \times C_*^{m,\alpha}(\mathbb{T}, \mathbb{C}) : \zeta \in \mathcal{Z}, h \in \mathcal{Z}, \right.$$
$$\left. w \in \mathbb{I}[\zeta], 0 \in \mathbb{I}[h],\ \Re\left\{ \frac{\text{ind}[h]}{2\pi i} \int_{\mathbb{T}} \frac{\zeta(s)h'(s)}{h^2(s)}\, ds \right\} > 0 \right\}.$$

Let $\mathbf{P}$ be the nonlinear operator of $\mathcal{A}$ to $C_*^{m,\alpha}(\mathbb{T}, \mathbb{R}) \times \mathbb{C} \times \mathbb{R} \times C_*^{m,\alpha}(\mathbb{T}, \mathbb{R})$ defined by

$$\mathbf{P}[\zeta, w, h](t) \equiv (\mathbf{P}_l[\zeta, w, h](t))_{l=1,2,3,4} \equiv$$
$$\left(\Re\left\{ \zeta(t) - \frac{\text{ind}[h]}{\pi i} \int_{\mathbb{T}} \frac{\zeta(s)h'(s)}{h(s) - h(t)}\, ds \right\}, \frac{\text{ind}[h]}{2\pi i} \int_{\mathbb{T}} \frac{\zeta(s)h'(s)}{h(s)}\, ds - w, \right.$$
$$\left. \Im\left\{ \frac{\text{ind}[h]}{2\pi i} \int_{\mathbb{T}} \frac{\zeta(s)h'(s)}{h^2(s)}\, ds \right\}, h(t)\overline{h(t)} - 1 \right).$$

If $(\zeta, w, h) \in \mathcal{A}$, and if $\mathbf{P}[\zeta, w, h] = 0$, then $h = g_{\zeta,w}^{(-1)} \circ \zeta \equiv h[\zeta, w]$, and in particular, h is a bijection of $\mathbb{T}$ onto $\mathbb{T}$. Conversely, if $(\zeta, w) \in \mathcal{E}_{m,\alpha}$, then $(\zeta, w, h[\zeta, w]) \in \mathcal{A}$ and $\mathbf{P}[\zeta, w, h[\zeta, w]] = 0$. The domain $\mathcal{A}$ of $\mathbf{P}$ is open in the (real or complex) Banach space $C_*^{m,\alpha}(\mathbb{T}, \mathbb{C}) \times \mathbb{C} \times C_*^{m,\alpha}(\mathbb{T}, \mathbb{C})$.

The nonlinear operator $h[\cdot, \cdot]$ of $\mathcal{E}_{m,\alpha}$ to $C_*^{m,\alpha}(\mathbb{T}, \mathbb{T}) \cap \mathcal{Z}$, which takes (ζ, w) to $h[\zeta, w]$, is real analytic.

Let $(\tilde{\zeta}, \tilde{w}, \tilde{h}) \in \mathcal{A}$, $\tilde{h}(\mathbb{T}) = \mathbb{T}$. Let $\Pi_{\tilde{h}}$ be the map of $C_*^{m,\alpha}(\mathbb{T}, \mathbb{R}) \times \mathbb{C} \times \mathbb{R} \times C_*^{m,\alpha}(\mathbb{T}, \mathbb{R})$ to itself defined by

$$\Pi_{\tilde{h}}[f, \omega, \beta, b] \equiv \left(f - \frac{(\text{ind}[\tilde{h}])^{-1}}{2\pi i} \int_{\mathbb{T}} f(s) \frac{\tilde{h}'(s)}{\tilde{h}(s)}\, ds, \omega, \beta, b \right),$$

for all $(f, \omega, \beta, b) \in C_*^{m,\alpha}(\mathbb{T}, \mathbb{R}) \times \mathbb{C} \times \mathbb{R} \times C_*^{m,\alpha}(\mathbb{T}, \mathbb{R})$. If $(\zeta, w, h) \in \mathcal{A}$, then equation

$$\Pi_{\tilde{h}} \circ \mathbf{P}[\zeta, w, h] = 0, \tag{3.2}$$

is satisfied if and only if $\mathbf{P}[\zeta, w, h] = 0$. The differential $\partial_h \{\Pi_{\tilde{h}} \circ \mathbf{P}\}[\tilde{\zeta}, \tilde{w}, \tilde{h}]$ is a linear homeomorphism of $C_*^{m,\alpha}(\mathbb{T}, \mathbb{C})$ onto the image

$$\mathcal{V}_{\tilde{h}}^{m,\alpha} \equiv \left\{ (f, \omega, \beta, b) \in C_*^{m,\alpha}(\mathbb{T}, \mathbb{R}) \times \mathbb{C} \times \mathbb{R} \times C_*^{m,\alpha}(\mathbb{T}, \mathbb{R}) : \frac{1}{2\pi i} \int_{\mathbb{T}} f(s) \frac{\tilde{h}'(s)}{\tilde{h}(s)}\, ds = 0 \right\}$$

of the operator $\Pi_{\tilde{h}}$.

In view of the previous Theorem, for a given (ζ,w), the corresponding $h[\zeta,w]$ is the only zero of the map $h\mapsto \Pi_{\tilde h}\circ \mathbf{P}[\zeta,w,h]$ in the set of h's such that (ζ,w,h) is still in the domain $\mathcal{A}$ of $\Pi_{\tilde h}\circ\mathbf{P}$. Again, by Theorem 3.1, if (ζ,w) is close enough to $(\tilde\zeta,\tilde w)$, then the solution $h[\zeta,w]$ is close to $h[\tilde\zeta,\tilde w]$. We now show that there exists $r_{\tilde\zeta,\tilde h}>0$ depending only on $(\tilde\zeta,\tilde h)$, and $\sigma_{(\tilde\zeta,\tilde w,\tilde h)}>0$ depending only on $(\tilde\zeta,\tilde w,\tilde h)$, such that if the distance of $(\tilde\zeta,\tilde w)$ and (ζ,w) is less than $\sigma_{(\tilde\zeta,\tilde w,\tilde h)}$, then $h[\zeta,w]$ lies in a ball centered at $\tilde h$ and with radius $r_{\tilde\zeta,\tilde h}$, and $h[\zeta,w]$ is a limit of the Newton iterates relative to the operator $h\mapsto\Pi_{\tilde h}\circ\mathbf{P}[\zeta,w,h]$, and with initial point $\tilde h$. To do so, we will exploit the Newton-Kantorovich Theorem in the form of Kantorovich and Akilov [7, Thm. 6, p. 708], which requires an estimate of the norm of the inverse of the linearized operator $\partial_h\{\Pi_{\tilde h}\circ\mathbf{P}\}[\zeta,w,\tilde h]$, and of the norm of first and second order partial differentials of $\Pi_{\tilde h}\circ\mathbf{P}$ in a neighborhood of $(\tilde\zeta,\tilde w,\tilde h)$. Thus as a first step, we provide such estimates, and we start with that concerning the linearized $\partial_h\{\Pi_{\tilde h}\circ\mathbf{P}\}[\tilde\zeta,\tilde w,\tilde h]$. We observe that if $h\in\mathcal{Z}$, $h(\mathbb{T})=\mathbb{T}$, then the following elementary inequality holds

$$\left|\frac{1}{2\pi i}\int_{\mathbb{T}} f(s)\frac{h'(s)}{h(s)}\,ds\right|\le\sup_{\mathbb{T}}|f|, \tag{3.3}$$

for all $f\in C^{0,\alpha}_*(\mathbb{T},\mathbb{C})$. Then we have the following.

<u>THEOREM 3.4</u>. *Let $\alpha\in]0,1[$, $m\in\mathbb{N}\setminus\{0\}$. Let $(\tilde\zeta,\tilde w,\tilde h)\in\mathcal{A}$ be a zero of $\mathbf{P}$. Then the following inequality holds.*

$$\left\|\left(\partial_h\{\Pi_{\tilde h}\circ\mathbf{P}\}[\tilde\zeta,\tilde w,\tilde h]\right)^{(-1)}\right\|_{\mathcal{L}(\mathcal{V}^{m,\alpha}_{\tilde h},C^{m,\alpha}_*(\mathbb{T},\mathbb{C}))}\le a_1[\tilde\zeta,\tilde w,\tilde h], \tag{3.5}$$

where

$$\begin{aligned}(3.6)\quad a_1[\tilde\zeta,\tilde w,\tilde h]\equiv{}& c_{1,m,\alpha}\{4\pi(l[\tilde h])^{-1}(1+\pi)(c_{2,m,\alpha}+1)\varphi_{2,m,\alpha}\left[\|\tilde h\|_{m,\alpha}+\pi^{-1}l[\tilde h],2^{-1}l[\tilde h]\right]\times\\ &\times c^2_{1,m-1,\alpha}c_{1,m,\alpha}\varphi_{1,m,\alpha}\left[\|\tilde h\|_{m,\alpha},1\right]\|\tilde h\|^2_{m,\alpha}\varphi_{1,m-1,\alpha}\left[\|\tilde\zeta\|_{m,\alpha},l[\tilde\zeta]\right]+3(l[\tilde\zeta])^{-1}\|\tilde h\|_{m,\alpha}+\\ &+\frac{3}{2}+2\pi(l[\tilde h])^{-1}\varphi_{2,m,\alpha}\left[\|\tilde h\|_{m,\alpha}+\pi^{-1}l[\tilde h],2^{-1}l[\tilde h]\right]\|\tilde h\|_{m,\alpha}\}\|\tilde h\|_{m,\alpha}.\end{aligned}$$

<u>PROOF</u>. If $(f,\omega,\beta,b)\in\mathcal{V}^{m,\alpha}_{\tilde h}$, then by [15, Thm. 2.1, Proofs of Prop. 4.3 and 5.3] we have that $\mu\equiv\left(\partial_h\{\Pi_{\tilde h}\circ\mathbf{P}\}[\tilde\zeta,\tilde w,\tilde h](f,\omega,\beta,b)\right)^{(-1)}$ is delivered by the formula

$$\begin{aligned}(3.7)\quad \frac{\mu(t)}{\tilde h(t)}={}&2i\Im\left\{\frac{\operatorname{ind}\left[\tilde h\right]}{2\pi i}\int_{\mathbb{T}}\frac{\tilde h'(s)}{g'_{\tilde\zeta,\tilde w}(\tilde h(s))}\frac{\gamma(t)-\gamma(s)}{(\tilde h(s)-\tilde h(t))}\,ds\right\}+\\ &+\frac{\operatorname{ind}\left[\tilde h\right]}{2\pi}\int_{\mathbb{T}}\frac{\Im\{\gamma(s)/g'_{\tilde\zeta,\tilde w}(\tilde h(s))\}\tilde h'(s)}{\tilde h(s)}\,ds+b(t)-\frac{\operatorname{ind}\left[\tilde h\right]}{4\pi i}\int_{\mathbb{T}}\frac{\tilde h'(s)b(s)}{\tilde h(s)}\,ds+\end{aligned}$$

$$+\frac{\mathrm{ind}\left[\tilde{h}\right]}{2\pi i}\int_{\mathbf{T}}\frac{\tilde{h}'(s)(b(s)-b(t))}{\tilde{h}(s)-\tilde{h}(t)}\,ds-\frac{i}{g'_{\tilde{\zeta},\tilde{\omega}}(0)}\left\{\beta-\Im\left\{\frac{\mathrm{ind}\left[\tilde{h}\right]}{2\pi i}\int_{\mathbf{T}}\frac{\tilde{h}'(s)f(s)}{\tilde{h}^2(s)}\,ds\right\}\right\},$$

where

$$\gamma(t)\equiv\frac{-\omega+\frac{\mathrm{ind}[\tilde{h}]}{2\pi i}\int_{\mathbf{T}}\frac{\tilde{h}'(s)f(s)}{\tilde{h}(s)}\,ds-f(t)}{\tilde{h}(t)}.$$

By Lemma 2.7, by Corollary 2.21 with $r=\pi^{-1}l[\tilde{h}]$, by observing that $\int_{\mathbf{T}}\frac{\tilde{h}'(s)}{g'_{\tilde{\zeta},\tilde{\omega}}(\tilde{h}(s))}\,ds=0$, and by equality $g_{\tilde{\zeta},\tilde{\omega}}\circ\tilde{h}=\tilde{\zeta}$, we have

$$\left\|2i\Im\left\{\frac{\mathrm{ind}\left[\tilde{h}\right]}{2\pi i}\int_{\mathbf{T}}\frac{\tilde{h}'(s)}{g'_{\tilde{\zeta},\tilde{\omega}}(\tilde{h}(s))}\frac{\gamma(t)-\gamma(s)}{\tilde{h}(s)-\tilde{h}(t)}\,ds\right\}\right\|_{m,\alpha}\leq 2\pi(l[\tilde{h}])^{-1}(1+\pi)\times$$
$$\times(c_{2,m,\alpha}+1)\varphi_{2,m,\alpha}\left[\|\tilde{h}\|_{m,\alpha}+\pi^{-1}l[\tilde{h}],2^{-1}l[\tilde{h}]\right]\left\|(\tilde{h}')^2(\tilde{\zeta}')^{-1}\right\|_{m-1,\alpha}\|\gamma\|_{m,\alpha},\tag{3.8}$$

and

$$\left\|\frac{\mathrm{ind}\left[\tilde{h}\right]}{2\pi i}\int_{\mathbf{T}}\frac{\tilde{h}'(s)(b(s)-b(t))}{\tilde{h}(s)-\tilde{h}(t)}\,ds\right\|_{m,\alpha}\leq$$
$$2\pi(l[\tilde{h}])^{-1}\varphi_{2,m,\alpha}\left[\|\tilde{h}\|_{m,\alpha}+\pi^{-1}l[\tilde{h}],2^{-1}l[\tilde{h}]\right]\|\tilde{h}\|_{m,\alpha}\|b\|_{m,\alpha}.\tag{3.9}$$

By applying the Maximum Principle to the function $\left|\frac{1}{g'_{\tilde{\zeta},\tilde{\omega}}(z)}\right|$, we obtain

$$\left|\frac{1}{g'_{\tilde{\zeta},\tilde{\omega}}(0)}\right|\leq\sup_{s\in\mathbf{T}}\left|\frac{1}{g'_{\tilde{\zeta},\tilde{\omega}}(s)}\right|=\sup_{s\in\mathbf{T}}\left|\frac{\tilde{h}'(s)}{\tilde{\zeta}'(s)}\right|\leq(l[\tilde{\zeta}])^{-1}\|\tilde{h}\|_{m,\alpha}.\tag{3.10}$$

By inequality (3.3), and by equality $\tilde{h}(\mathbf{T})=\mathbf{T}$, and by the obvious equality $\|c\|_{m,\alpha}=|c|$, for all $c\in\mathbb{C}$, we have

$$\left\|\frac{\mathrm{ind}\left[\tilde{h}\right]}{2\pi}\int_{\mathbf{T}}\frac{\Im\{\gamma(s)/g'_{\tilde{\zeta},\tilde{\omega}}(\tilde{h}(s))\}\tilde{h}'(s)}{\tilde{h}(s)}\,ds\right\|_{m,\alpha}\leq$$
$$\leq\sup_{s\in\mathbf{T}}\left|\frac{\gamma(s)\tilde{h}'(s)}{g'_{\tilde{\zeta},\tilde{\omega}}(\tilde{h}(s))\tilde{h}'(s)}\right|=\sup_{s\in\mathbf{T}}\left|\frac{\gamma(s)\tilde{h}'(s)}{\tilde{\zeta}'(s)}\right|\leq(l[\tilde{\zeta}])^{-1}\|\tilde{h}\|_{m,\alpha}(|\omega|+2\|f\|_{m,\alpha}),\tag{3.11}$$

and

$$\left\|\beta-\Im\left\{\frac{\mathrm{ind}\left[\tilde{h}\right]}{2\pi i}\int_{\mathbf{T}}\frac{\tilde{h}'(s)f(s)}{\tilde{h}^2(s)}\,ds\right\}\right\|_{m,\alpha}\leq|\beta|+\|f\|_{m,\alpha},\tag{3.12}$$

and

$$\|\gamma\|_{m,\alpha} \le c_{1,m,\alpha} \left\| (\tilde{h})^{-1} \right\|_{m,\alpha} (|\omega| + 2\|f\|_{m,\alpha}) \tag{3.13}$$

Then by Lemma 2.1, and by equality (3.7) and inequalities (3.8)–(3.13), we deduce the validity of inequality (3.5). □

We now turn to estimate the partial differentials of $\Pi_{\tilde{h}} \circ \mathbf{P}$ on a closed ball centered at $(\tilde{\zeta}, \tilde{w}, \tilde{h}) \in \mathcal{A}$, and contained in $\mathcal{A}$. The following elementary Lemma provides an explicit upper bound for a possible radius of such ball.

<u>LEMMA 3.14</u>. *Let* $\alpha \in]0,1[$, $m \in \mathbb{N} \setminus \{0\}$. *Let* $(\tilde{\zeta}, \tilde{w}, \tilde{h}) \in \mathcal{A}$ *be a zero of* $\mathbf{P}$. *Let* $r_{\tilde{\zeta},\tilde{h}} \equiv 2^{-1} \inf \left\{1, \pi^{-1} l[\tilde{h}], 2\nu\right\}$, $\delta_{(\tilde{\zeta},\tilde{w},\tilde{h})} \equiv 2^{-1} \inf \left\{2, 2\pi^{-1} l[\tilde{\zeta}], \min_{t\in\mathbb{T}} |\tilde{\zeta}(t) - \tilde{w}|, \nu\right\}$, *with*

$$\nu \equiv 30^{-1}(\|\tilde{\zeta}\|_0 + 1)^{-1}(\|\tilde{h}'\|_0 + 1)^{-1} \Re \left\{ \frac{\mathrm{ind}[\tilde{h}]}{2\pi i} \int_{\mathbb{T}} \frac{\tilde{\zeta}(s)\tilde{h}'(s)}{\tilde{h}^2(s)}\, ds \right\}.$$

Let

$$\begin{aligned} B_{(\tilde{\zeta},\tilde{w},\tilde{h})} \equiv \ & \Big\{ (\zeta, w, h) \in C^{m,\alpha}_*(\mathbb{T}, \mathbb{C}) \times \mathbb{C} \times C^{m,\alpha}_*(\mathbb{T}, \mathbb{C}) : \\ & \left\| \zeta - \tilde{\zeta} \right\|_{m,\alpha} + |w - \tilde{w}| \le \delta_{(\tilde{\zeta},\tilde{w},\tilde{h})}, \left\| h - \tilde{h} \right\|_{m,\alpha} \le r_{\tilde{\zeta},\tilde{h}} \Big\}. \end{aligned}$$

Then $B_{(\tilde{\zeta},\tilde{w},\tilde{h})} \subseteq \mathcal{A}$, *and*

$$\begin{aligned} & \inf_{\|h-\tilde{h}\|_{m,\alpha} \le 2r_{\tilde{\zeta},\tilde{h}}} l[h] \ge 2^{-1} l[\tilde{h}], \\ & \inf_{\|h-\tilde{h}\|_{m,\alpha} \le r_{\tilde{\zeta},\tilde{h}}} \min_{\mathbb{T}} |h| \ge 2^{-1}, \\ & \inf_{\|\zeta-\tilde{\zeta}\|_{m,\alpha} + |w-\tilde{w}| \le \delta_{(\tilde{\zeta},\tilde{w},\tilde{h})}} \min_{t\in\mathbb{T}} |\zeta(t) - w| \ge 2^{-1} \min_{t\in\mathbb{T}} \left|\tilde{\zeta}(t) - \tilde{w}\right|, \\ & \inf_{\|\zeta-\tilde{\zeta}\|_{m,\alpha} + |w-\tilde{w}| \le \delta_{(\tilde{\zeta},\tilde{w},\tilde{h})}} l[\zeta] \ge 2^{-1} l[\tilde{\zeta}]. \end{aligned}$$

<u>PROOF</u>. Clearly, $\tilde{h}(\mathbb{T}) = \mathbb{T}$. If $\min_{\mathbb{T}} |h| \ge 2^{-1}$, $\|h - \tilde{h}\|_1 \le 2^{-1}$, $\|\zeta - \tilde{\zeta}\|_0 \le 1$, then we have

$$\begin{aligned} & \left\| \frac{\tilde{\zeta}\tilde{h}'}{\tilde{h}^2} - \frac{\zeta h'}{h^2} \right\|_0 \le \\ & \le 4 \left\{ \|\tilde{\zeta} - \zeta\|_0 \|\tilde{h}'\|_0 \|h^2\|_0 + \|\zeta\|_0 \|\tilde{h}' - h'\|_0 \|h^2\|_0 + \|\zeta\|_0 \|h'\|_0 \|h^2 - \tilde{h}^2\|_0 \right\} \\ & \le 10(\|\tilde{\zeta}\|_0 + 1)(\|\tilde{h}'\|_0 + 1)(\|\tilde{\zeta} - \zeta\|_0 + \|\tilde{h} - h\|_1). \end{aligned} \tag{3.15}$$

Thus the statement follows by Lemma 2.7, by inequality (3.15), and by the triangular inequality. □

Then we have the following.

PROPOSITION 3.16. *Let $\alpha \in]0,1[$, $m \in \mathbb{N} \setminus \{0\}$. Let $(\tilde{\zeta}, \tilde{w}, \tilde{h}) \in \mathcal{A}$ be a zero of* $\mathbf{P}$. *Let the constants $r_{\tilde{\zeta},\tilde{h}}$, $\delta_{(\tilde{\zeta},\tilde{w},\tilde{h})}$, and the ball $B_{(\tilde{\zeta},\tilde{w},\tilde{h})}$ be defined as in Lemma 3.14. Then*

$$(3.17)\qquad \sup_{(\zeta,w,h)\in B_{(\tilde{\zeta},\tilde{w},\tilde{h})}} \left\| \partial_{(\zeta,w)}\partial_h \{\Pi_{\tilde{h}} \circ \mathbf{P}\}[\zeta, w, h] \right\|_{\mathcal{B}_1} \leq a_2\left[\tilde{\zeta}, \tilde{w}, \tilde{h}\right],$$

where

$$\mathcal{B}_1 \equiv \mathcal{B}\left((C_*^{m,\alpha}(\mathbb{T},\mathbb{C}) \times \mathbb{C}) \times C_*^{m,\alpha}(\mathbb{T},\mathbb{C}), C_*^{m,\alpha}(\mathbb{T},\mathbb{R}) \times \mathbb{C} \times \mathbb{R} \times C_*^{m,\alpha}(\mathbb{T},\mathbb{R})\right),$$

$$a_2\left[\tilde{\zeta}, \tilde{w}, \tilde{h}\right] \equiv 2r^{-1}_{\tilde{\zeta},\tilde{h}}(c_{2,m,\alpha}+1)\varphi_{2,m,\alpha}\left[\|\tilde{h}\|_{m,\alpha} + 2r_{\tilde{\zeta},\tilde{h}}, 2^{-1}l[\tilde{h}]\right] + 6.$$

Furthermore,

$$(3.18)\qquad \sup_{(\zeta,w,h)\in B_{(\tilde{\zeta},\tilde{w},\tilde{h})}} \left\| \partial_h^2 \{\Pi_{\tilde{h}} \circ \mathbf{P}\}[\zeta, w, h] \right\|_{\mathcal{B}_2} \leq a_3\left[\tilde{\zeta}, \tilde{w}, \tilde{h}\right],$$

where

$$\mathcal{B}_2 \equiv \mathcal{B}\left((C_*^{m,\alpha}(\mathbb{T},\mathbb{C}))^2, C_*^{m,\alpha}(\mathbb{T},\mathbb{R}) \times \mathbb{C} \times \mathbb{R} \times C_*^{m,\alpha}(\mathbb{T},\mathbb{R})\right),$$

$$a_3\left[\tilde{\zeta}, \tilde{w}, \tilde{h}\right] \equiv \left\{8r^{-2}_{\tilde{\zeta},\tilde{h}}(c_{2,m,\alpha}+1)\varphi_{2,m,\alpha}\left[\|\tilde{h}\|_{m,\alpha} + 2r_{\tilde{\zeta},\tilde{h}}, 2^{-1}l[\tilde{h}]\right] + \right.$$
$$\left. +20\right\}(\|\tilde{\zeta}\|_{m,\alpha} + \delta_{(\tilde{\zeta},\tilde{w},\tilde{h})}) + 2c_{1,m,\alpha}c_{2,m,\alpha},$$

and

$$(3.19)\qquad \sup_{\|\zeta-\tilde{\zeta}\|+|w-\tilde{w}|\leq\delta_{(\tilde{\zeta},\tilde{w},\tilde{h})}} \left\| \partial_{(\zeta,w)} \{\Pi_{\tilde{h}} \circ \mathbf{P}\}[\zeta, w, \tilde{h}] \right\|_{\mathcal{L}} \leq a_4\left[\tilde{\zeta}, \tilde{w}, \tilde{h}\right],$$

where

$$\mathcal{L} \equiv \mathcal{L}\left(C_*^{m,\alpha}(\mathbb{T},\mathbb{C}) \times \mathbb{C}, C_*^{m,\alpha}(\mathbb{T},\mathbb{R}) \times \mathbb{C} \times \mathbb{R} \times C_*^{m,\alpha}(\mathbb{T},\mathbb{R})\right),$$

$$a_4\left[\tilde{\zeta}, \tilde{w}, \tilde{h}\right] \equiv 2r^{-1}_{\tilde{\zeta},\tilde{h}}(c_{2,m,\alpha}+1)\varphi_{2,m,\alpha}\left[\|\tilde{h}\|_{m,\alpha} + 2r_{\tilde{\zeta},\tilde{h}}, 2^{-1}l[\tilde{h}]\right]\|\tilde{h}\|_{m,\alpha} + 2.$$

PROOF. Since $B_{(\tilde{\zeta},\tilde{w},\tilde{h})}$ is connected, ind$[h]$ is constant for $(\zeta, w, h) \in B_{(\tilde{\zeta},\tilde{w},\tilde{h})}$, and thus we can assume that it is identically equal to one. Then by Lanza and Preciso [12, Prop. 4.1], by

[15, Lemma 4.1], and by elementary Calculus (see also [15, Prop. 3.7], where $\partial_h \mathbf{P}$ has been computed), we have

$$\partial_{(\zeta,w)}\partial_h\{\Pi_{\tilde{h}} \circ \mathbf{P}\}[\zeta, w, h](\xi, \eta, \mu) =$$

$$\Big(-\Re\Big\{\tfrac{1}{\pi i}\int_{\mathbf{T}} \xi'(s)\tfrac{\mu(\cdot)-\mu(s)}{h(s)-h(\cdot)}\,ds\Big\} + \tfrac{1}{2\pi i}\int_{\mathbf{T}} \Re\Big\{\tfrac{1}{\pi i}\int_{\mathbf{T}} \xi'(s)\tfrac{\mu(t)-\mu(s)}{h(s)-h(t)}\,ds\Big\}\tfrac{\tilde{h}'(t)}{\tilde{h}(t)}dt,$$

$$-\tfrac{1}{2\pi i}\int_{\mathbf{T}} \tfrac{\xi'(s)\mu(s)}{h(s)}\,ds, -\Im\Big\{\tfrac{1}{2\pi i}\int_{\mathbf{T}} \tfrac{\xi'(s)\mu(s)}{h^2(s)}\,ds\Big\}, 0\Big),$$

for all $(\xi, \eta, \mu) \in C_*^{m,\alpha}(\mathbb{T}, \mathbb{C}) \times \mathbb{C} \times C_*^{m,\alpha}(\mathbb{T}, \mathbb{C})$, and for all $(\zeta, w, h) \in B_{(\tilde{\zeta},\tilde{w},\tilde{h})}$, and thus by Lemma 2.1, by Corollary 2.21 with $r = 2r_{\tilde{\zeta},\tilde{h}}$, by inequality (3.3), and by Lemma 3.14, we have

$$\left\|\partial_{(\zeta,w)}\partial_h\{\Pi_{\tilde{h}} \circ \mathbf{P}\}[\zeta, w, h]\right\|_{\mathcal{B}_1} \leq 2r_{\tilde{\zeta},\tilde{h}}^{-1}(c_{2,m,\alpha} + 1)\varphi_{2,m,\alpha}\left[\|\tilde{h}\|_{m,\alpha} + 2r_{\tilde{\zeta},\tilde{h}}, 2^{-1}l[\tilde{h}]\right] + 6.$$

Thus, we can deduce the validity of (3.17). Again by Lanza and Preciso [12, Prop. 4.1], and by elementary Calculus, we have

$$\partial_h^2\{\Pi_{\tilde{h}} \circ \mathbf{P}\}[\zeta, w, h](\mu_1, \mu_2) = \Big(-\Re\Big\{\tfrac{1}{\pi i}\int_{\mathbf{T}} \zeta'(s)\tfrac{(\mu_1(\cdot)-\mu_1(s))(\mu_2(\cdot)-\mu_2(s))}{(h(s)-h(\cdot))^2}\,ds\Big\} +$$

$$+\tfrac{1}{2\pi i}\int_{\mathbf{T}} \Re\Big\{\tfrac{1}{\pi i}\int_{\mathbf{T}} \zeta'(s)\tfrac{(\mu_1(t)-\mu_1(s))(\mu_2(t)-\mu_2(s))}{(h(s)-h(t))^2}\,ds\Big\}\tfrac{\tilde{h}'(t)}{\tilde{h}(t)}dt,$$

$$+\tfrac{1}{2\pi i}\int_{\mathbf{T}} \zeta'(s)\tfrac{\mu_1(s)\mu_2(s)}{h^2(s)}\,ds, 2\Im\Big\{\tfrac{1}{2\pi i}\int_{\mathbf{T}} \zeta'(s)\tfrac{\mu_1(s)\mu_2(s)}{h^3(s)}\,ds\Big\}, \mu_1(\cdot)\overline{\mu_2(\cdot)} + \mu_2(\cdot)\overline{\mu_1(\cdot)}\Big),$$

for all $(\mu_1, \mu_2) \in (C_*^{m,\alpha}(\mathbb{T}, \mathbb{C}))^2$, and for all $(\zeta, w, h) \in B_{(\tilde{\zeta},\tilde{w},\tilde{h})}$. Then by Lemma 2.1, by Corollary 2.21 with $r = 2r_{\tilde{\zeta},\tilde{h}}$, by inequality (3.3), and by Lemma 3.14, we have

$$\left\|\partial_h^2\{\Pi_{\tilde{h}} \circ \mathbf{P}\}[\zeta, w, h]\right\|_{\mathcal{B}_2} \leq 8r_{\tilde{\zeta},\tilde{h}}^{-2}(c_{2,m,\alpha} + 1)\varphi_{2,m,\alpha}\left[\|\tilde{h}\|_{m,\alpha} + 2r_{\tilde{\zeta},\tilde{h}}, 2^{-1}l[\tilde{h}]\right]\|\zeta\|_{m,\alpha} +$$
$$+20\|\zeta\|_{m,\alpha} + 2c_{1,m,\alpha}c_{2,m,\alpha},$$

and thus we have (3.18). Finally, we consider (3.19). By elementary Calculus, we have

$$\partial_{(\zeta,w)}\{\Pi_{\tilde{h}} \circ \mathbf{P}\}[\zeta, w, \tilde{h}](\xi, \eta) = \left(\Re\left\{\xi(\cdot) - \frac{1}{\pi i}\int_{\mathbf{T}} \frac{\xi(s)\tilde{h}'(s)\,ds}{\tilde{h}(s) - \tilde{h}(\cdot)}\right\}\right.$$

$$-\frac{1}{2\pi i}\int_{\mathbf{T}} \Re\left\{\xi(t) - \frac{1}{\pi i}\int_{\mathbf{T}} \frac{\xi(s)\tilde{h}'(s)\,ds}{\tilde{h}(s) - \tilde{h}(t)}\right\}\frac{\tilde{h}'(t)}{\tilde{h}(t)}dt,$$

$$\left.\frac{1}{2\pi i}\int_{\mathbf{T}} \frac{\xi(s)\tilde{h}'(s)\,ds}{\tilde{h}(s)} - \eta, \Im\left\{\frac{1}{2\pi i}\int_{\mathbf{T}} \frac{\xi(s)\tilde{h}'(s)\,ds}{\tilde{h}^2(s)}\right\}, 0\right),$$

for all $(\xi,\eta)\in C_*^{m,\alpha}(\mathbb{T},\mathbb{C})\times\mathbb{C}$, and for all $\|\zeta-\tilde{\zeta}\|+|w-\tilde{w}|\leq\delta_{(\tilde{\zeta},\tilde{w},\tilde{h})}$. Then by equality (2.13), by Lemma 2.1, by Corollary 2.21 with $r=2r_{\tilde{\zeta},\tilde{h}}$, by inequality (3.3), and by Lemma 3.14, we have

$$\left\|\partial_{(\zeta,w)}\{\Pi_{\tilde{h}}\circ\mathbf{P}\}[\zeta,w,\tilde{h}]\right\|_{\mathcal{L}}\leq 2r_{\tilde{\zeta},\tilde{h}}^{-1}(c_{2,m,\alpha}+1)\varphi_{2,m,\alpha}\left[\|\tilde{h}\|_{m,\alpha}+2r_{\tilde{\zeta},\tilde{h}},2^{-1}l[\tilde{h}]\right]\|\tilde{h}\|_{m,\alpha}+2.$$

Then (3.19) follows. □

We are now ready to prove the following.

THEOREM 3.20. *Let $\alpha\in]0,1[$, $m\in\mathbb{N}\setminus\{0\}$. Let $(\tilde{\zeta},\tilde{w},\tilde{h})\in\mathcal{A}$ be a zero of $\mathbf{P}$. Let $\sigma_{(\tilde{\zeta},\tilde{w},\tilde{h})}$ be equal to the minimum of the following four numbers:*

$$\begin{aligned}&\delta_{(\tilde{\zeta},\tilde{w},\tilde{h})}, && \left(2a_1[\tilde{\zeta},\tilde{w},\tilde{h}]a_2[\tilde{\zeta},\tilde{w},\tilde{h}]\right)^{-1},\\ &r_{\tilde{\zeta},\tilde{h}}\left(4a_1[\tilde{\zeta},\tilde{w},\tilde{h}]a_4[\tilde{\zeta},\tilde{w},\tilde{h}]\right)^{-1}, && \left(16a_1[\tilde{\zeta},\tilde{w},\tilde{h}]^2a_3[\tilde{\zeta},\tilde{w},\tilde{h}]a_4[\tilde{\zeta},\tilde{w},\tilde{h}]\right)^{-1}.\end{aligned}\tag{3.21}$$

If $(\zeta,w)\in C_*^{m,\alpha}(\mathbb{T},\mathbb{C})\times\mathbb{C}$, and if $\|\zeta-\tilde{\zeta}\|_{m,\alpha}+|w-\tilde{w}|\leq\sigma_{(\tilde{\zeta},\tilde{w},\tilde{h})}$, then equation

$$\Pi_{\tilde{h}}\circ\mathbf{P}[\zeta,w,h]=0\tag{3.22}$$

in the unknown h has one and only one solution $h[\zeta,w]\in C_*^{m,\alpha}(\mathbb{T},\mathbb{C})$, such that $\|h-\tilde{h}\|_{m,\alpha}\leq r_{\tilde{\zeta},\tilde{h}}$. Furthermore, the Newton iterates

$$\begin{aligned}&H_0\equiv\tilde{h},\\ &H_{j+1}\equiv H_j-\left(\partial_h\{\Pi_{\tilde{h}}\circ\mathbf{P}\}[\zeta,w,\tilde{h}]\right)^{(-1)}\circ\Pi_{\tilde{h}}\circ\mathbf{P}[\zeta,w,H_j],\ \text{for } j\geq 0,\end{aligned}\tag{3.23}$$

converge to $h[\zeta,w]$ in $C_*^{m,\alpha}(\mathbb{T},\mathbb{C})$ and satisfy the inequality

$$\|H_j-h[\zeta,w]\|_{m,\alpha}\leq\left(2a_1[\tilde{\zeta},\tilde{w},\tilde{h}]a_3[\tilde{\zeta},\tilde{w},\tilde{h}]\right)^{-1}\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)^{j+1}\qquad\forall j\in\mathbb{N}.\tag{3.24}$$

PROOF. As we have announced, our result will follow by applying the Newton-Kantorovich 'modified' method in the formulation of Kantorovich and Akilov [7, Thm. 6, p. 708]. We first note that if $\|\zeta-\tilde{\zeta}\|_{m,\alpha}+|w-\tilde{w}|\leq\sigma_{(\tilde{\zeta},\tilde{w},\tilde{h})}$, then by the Mean Value Inequality, and by Theorem 3.4, and by inequality (3.17), and by definition of $\sigma_{(\tilde{\zeta},\tilde{w},\tilde{h})}$, we have

$$\left\|\left(\partial_h\{\Pi_{\tilde{h}}\circ\mathbf{P}\}[\zeta,w,\tilde{h}]\right)^{(-1)}\right\|_{\mathcal{L}(\mathcal{V}_{\tilde{h}}^{m,\alpha},C_*^{m,\alpha}(\mathbb{T},\mathbb{C}))}\leq$$

$$\leq\left\{a_1^{-1}[\tilde{\zeta},\tilde{w},\tilde{h}]-a_2[\tilde{\zeta},\tilde{w},\tilde{h}](\|\zeta-\tilde{\zeta}\|_{m,\alpha}+|w-\tilde{w}|)\right\}^{-1}\leq 2a_1[\tilde{\zeta},\tilde{w},\tilde{h}].$$

Similarly,

$$\left\|\Pi_{\tilde{h}} \circ \mathbf{P}[\zeta, w, \tilde{h}]\right\|_{m,\alpha} \leq a_4[\tilde{\zeta}, \tilde{w}, \tilde{h}]\sigma_{(\tilde{\zeta},\tilde{w},\tilde{h})}. \tag{3.25}$$

By Proposition 3.16, we have

$$\left\|\partial_h^2\{\Pi_{\tilde{h}} \circ \mathbf{P}\}[\zeta, w, h]\right\|_{\mathcal{B}_2} \leq a_3[\tilde{\zeta}, \tilde{w}, \tilde{h}] \qquad \forall (\zeta, w, h) \in B_{(\tilde{\zeta},\tilde{w},\tilde{h})}. \tag{3.26}$$

Thus we have

$$\left\|\left(\partial_h\{\Pi_{\tilde{h}} \circ \mathbf{P}\}[\zeta, w, \tilde{h}]\right)^{(-1)} \circ \Pi_{\tilde{h}} \circ \mathbf{P}[\zeta, w, \tilde{h}]\right\|_{m,\alpha} \times$$
$$\times \left\|\left(\partial_h\{\Pi_{\tilde{h}} \circ \mathbf{P}\}[\zeta, w, \tilde{h}]\right)^{(-1)} \circ \partial_h^2\{\Pi_{\tilde{h}} \circ \mathbf{P}\}[\zeta, w, h]\right\|_{\mathcal{B}\left(\left(C_*^{m,\alpha}(\mathbb{T},\mathbb{C})\right)^2, C_*^{m,\alpha}(\mathbb{T},\mathbb{C})\right)} \leq \beta \leq \tfrac{1}{4},$$

for all ζ, w, h such that $\|\zeta - \tilde{\zeta}\|_{m,\alpha} + |w - \tilde{w}| \leq \sigma_{(\tilde{\zeta},\tilde{w},\tilde{h})}$, $\|h - \tilde{h}\|_{m,\alpha} \leq r_{\tilde{\zeta},\tilde{h}}$, and where the constant β is defined by $\beta \equiv \left(2a_1[\tilde{\zeta}, \tilde{w}, \tilde{h}]a_3[\tilde{\zeta}, \tilde{w}, \tilde{h}]\right)\left(2a_1[\tilde{\zeta}, \tilde{w}, \tilde{h}]a_4[\tilde{\zeta}, \tilde{w}, \tilde{h}]\sigma_{(\tilde{\zeta},\tilde{w},\tilde{h})}\right)$. Furthermore, we have $1 - \sqrt{1 - 2\beta} \leq 2\beta$, and thus

$$\frac{1 - \sqrt{1 - 2\beta}}{\beta} 2a_1[\tilde{\zeta}, \tilde{w}, \tilde{h}]a_4[\tilde{\zeta}, \tilde{w}, \tilde{h}]\sigma_{(\tilde{\zeta},\tilde{w},\tilde{h})} \leq r_{\tilde{\zeta},\tilde{h}}\,. \tag{3.27}$$

Then we can invoke Kantorovich and Akilov [7, Thm. 6, p. 708] to deduce that the Newton iterates of (3.23) converge to a solution of equation $\Pi_{\tilde{h}} \circ \mathbf{P}[\zeta, w, h] = 0$ in the ball $\left\{h \in C_*^{m,\alpha}(\mathbb{T}, \mathbb{C}) : \|h - \tilde{h}\| < r_{\tilde{\zeta},\tilde{h}}\right\}$ and that (3.24) holds. By Lemma 3.14, the triple (ζ, w, h) belongs to $\mathcal{A}$. Thus, by Therorem 3.1 such solution coincides with $h[\zeta, w]$, and the proof is complete. □

Then we have the following.

THEOREM 3.28. *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0, 1[$. Let Λ be a continuous map of $[0,1]$ to $\mathcal{E}_{m,\alpha}$. Let $\Lambda(\lambda) \equiv (\zeta_\lambda, w_\lambda)$, for all $\lambda \in [0,1]$. Then there exist $q \in \mathbb{N}$ and $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_q = 1$, with $\lambda_k - \lambda_{k-1} = q^{-1}$, $k = 1, \cdots, q$, such that $h\left[\zeta_{\lambda_k}, w_{\lambda_k}\right]$ can be obtained by $\left(\zeta_{\lambda_{k-1}}, w_{\lambda_{k-1}}, h\left[\zeta_{\lambda_{k-1}}, w_{\lambda_{k-1}}\right]\right)$, as a limit of the Newton iterates of (3.23), with $\left(\tilde{\zeta}, \tilde{w}, h[\tilde{\zeta}, \tilde{w}]\right) \equiv \left(\zeta_{\lambda_{k-1}}, w_{\lambda_{k-1}}, h\left[\zeta_{\lambda_{k-1}}, w_{\lambda_{k-1}}\right]\right)$.*

PROOF. It suffices to note that the continuity of $h[\cdot,\cdot]$ on $\mathcal{E}_{m,\alpha}$ inferred by Theorem 3.1, the continuity of $\|\cdot\|_{m,\alpha}$ on $C_*^{m,\alpha}(\mathbb{T}, \mathbb{C})$, the continuity and positivity of $l[\cdot]$ on $\mathcal{Z}$, of $\Re\left\{\frac{\mathrm{ind}[h]}{2\pi i}\int_{\mathbb{T}} \frac{\zeta(s)h'(s)}{h^2(s)}\,ds\right\}$ on $\mathcal{A}$, and of $\min_{t\in\mathbb{T}}|\zeta(t) - w|$ on $\mathcal{E}_{m,\alpha}$, imply that $\min_{\lambda\in[0,1]} l[\zeta_\lambda] > 0$, $\min_{\lambda\in[0,1]} l\left[h\left[\zeta_\lambda, w_\lambda\right]\right] > 0$, $\max_{\lambda\in[0,1]}\|\zeta_\lambda\|_{m,\alpha} < +\infty$, $\max_{\lambda\in[0,1]}\left\|h\left[\zeta_\lambda, w_\lambda\right]\right\|_{m,\alpha} < +\infty$, $\min_{\lambda\in[0,1]}\min_{t\in\mathbb{T}}|\zeta_\lambda(t) - w_\lambda| > 0$,

$$\min_{\lambda\in[0,1]} \Re\left\{\frac{\mathrm{ind}\left[h[\zeta_\lambda, w_\lambda]\right]}{2\pi i}\int_{\mathbb{T}} \frac{\zeta_\lambda(s)h[\zeta_\lambda, w_\lambda]'(s)}{h[\zeta_\lambda, w_\lambda]^2(s)}\,ds\right\} > 0.$$

Then by the definition of $a_j[\zeta_\lambda, w_\lambda, h[\zeta_\lambda, w_\lambda]]$, with $j \in \{1, \dots, 4\}$, (see Theorem 3.4 and Proposition 3.16), and by the definition of $\sigma_{(\zeta_\lambda, w_\lambda, h_\lambda)}$ (see Theorem 3.20), and by Lemma 2.7 (i), we deduce that $\inf_{\lambda \in [0,1]} \sigma_{(\zeta_\lambda, w_\lambda, h_\lambda)} > 0$, and the proof is complete. □

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