3D BEC Bright Solitons under Transverse Confinement

—— *Analytical Results with the Nonpolynomial Schrödinger Equation* ——

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The Bose-Einstein condensate (BEC) of a dilute gas of bosons is well described by the three-dimensional Gross-Pitaevskii equation (3D GPE), that is a nonlinear Schrödinger equation. By imposing a transverse confinement the BEC can travel only in the cylindrical axial direction. We show that in this case the BEC with attractive interaction admits a 3D bright soliton solution which generalizes the text-book one, that is valid in the one-dimensional limit (1D GPE). Contrary to the 1D case, the 3D bright soliton exists only below a critical number of Bosons that depends on the extent of confinement. Finally, we find that the 3D bright soliton collapses if its density excedes a critical value. Our results are obtained by using a nonpolynomial Schrödinger equation (NPSE), an effective one-dimensional equation derived from the 3D GPE.

§1. Introduction

In a dilute gas of N Bosons at zero temperature practically all particles are in the same single-particle state of the density matrix $\rho(\boldsymbol{r},\boldsymbol{r}';t)$. [1),2)] This macroscopically occupied single-particle state is called Bose-Einstein condensate (BEC) and it is well described by a complex classical field $\psi(\boldsymbol{r},t)$ (order parameter or macroscopic wavefunction) whose Lagrangian density is given by

$$\mathcal{L} = \psi^* \left[i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\nabla^2 - U \right] \psi - \frac{1}{2} g|\psi|^4 \, , \tag{1·1}$$

where $U(\boldsymbol{r})$ is the external potential, $g = 4\pi\hbar^2 a_s/m$ is the interatomic strength with a_s the s-wave scattering length, and the complex field is normalized to N. By imposing the least action principle one obtains the following Euler-Lagrange equation

$$i\hbar\frac{\partial}{\partial t}\psi = \left[-\frac{\hbar^2}{2m}\nabla^2 + U + g|\psi|^2 \right] \psi \, , \tag{1·2}$$

which is a nonlinear Schrödinger equation (NLSE) and it is called three-dimensional Gross-Pitaevskii equation (3D GPE). [1),2)]

In recent experiments [3),4)] Bose-Einstein condensates have been trapped in quasi-1D cylindrical optical traps to study bright soliton solutions, [5)] which are shape-invariant matter waves with attractive interatomic interaction ($a_s < 0$). In this paper we derive analytical results for the BEC bright solitons by using a one-dimensional nonpolynomial Schrödinger equation (NPSE) we have recently obtained from the 3D GPE. [6),7)]

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§2. From 3D GPE to NPSE

The cylindrical optical confinement of Refs. 3) and 4) can be described by a harmonic potential

$$U(\boldsymbol{r}) = \frac{1}{2} m\omega_\perp^2 (x^2+y^2) \; , \tag{2·1}$$

in the transverse direction of the BEC. This external potential suggests to map the 3D GPE into an effective 1D equation, which simplifies greatly the solution of the 3D GPE. Our approach is to choose for the complex field the following variational ansatz

$$\psi(\boldsymbol{r},t) = N^{1/2}\phi(x,y,t;\sigma(z,t)) \, f(z,t) \; , \tag{2·2}$$

where both ϕ and f are normalized to one and ϕ is represented by a Gaussian:

$$\phi(x,y,t;\sigma(z,t)) = \frac{e^{\frac{-(x^2+y^2)}{2\sigma(z,t)^2}}}{\pi^{1/2}\sigma(z,t)} \; . \tag{2·3}$$

Moreover we assume that the transverse wavefunction ϕ is slowly varying along the axial direction with respect to the transverse direction, i.e. $\nabla^2\phi \simeq \nabla_\perp^2\phi$ where $\nabla_\perp^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. By using the trial wave-function in the Lagrangian density and after spatial integration along x and y variables the Lagrangian density becomes a function of σ, f and f^*. The Euler-Lagrange equation with respect σ is given by

$$\sigma^2 = a_\perp^2 \sqrt{1+2a_s N|f|^2} \; , \tag{2·4}$$

where $a_\perp = \sqrt{\hbar/(m\omega_\perp)}$ is the oscillator length in the transverse direction. Inserting this result in the Euler-Lagrange equation with respect to f^* one finally obtains

$$\begin{aligned} i\hbar\frac{\partial}{\partial t} f = \Big[& -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \frac{gN}{2\pi a_\perp^2}\frac{|f|^2}{\sqrt{1+2a_sN|f|^2}} \\ & + \frac{\hbar\omega_\perp}{2}\left(\frac{1}{\sqrt{1+2a_sN|f|^2}} + \sqrt{1+2a_sN|f|^2} \right) \Big] f \; . \end{aligned} \tag{2·5}$$

This equation is a time-dependent non-polynomial Schrodinger equation (NPSE). Note that under the condition $a_sN|f|^2 \ll 1$ one has $\sigma^2 = a_\perp^2$ and NPSE reduces to

$$i\hbar\frac{\partial}{\partial t} f = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \frac{gN}{2\pi a_\perp^2}|f|^2 \right] f \; , \tag{2·6}$$

where the additive constant $\hbar\omega_\perp$ has been omitted because it does not affect the dynamics. This equation, called one-dimensional Gross-Pitaevskii equation (1D GPE), is the familiar nonlinear cubic Schrödinger equation. The nonlinear coefficient g' of this 1D GPE can be thus obtained from the nonlinear coefficient g of the 3D GPE by setting $g' = g/(2\pi a_\perp^2)$. Note that the limit $a_sN|f|^2 \ll 1$ is precisely the condition for the one-dimensional regime where the healing length $\xi = (8\pi a_s\rho)^{-1/2}$ of the BEC is larger than $a_\perp$. In Refs. 6) and 7) we have verified that NPSE is very accurate in the description of cigar-shaped Bose condensates both in the 3D regime and in the 1D regime. NPSE has been recently applied in the study of an atom laser.[8)]

§3. 3D BEC bright solitons

Under transverse confinement and negative scattering length ($a_s < 0$) a bright soliton [5] sets up when the negative inter-atomic energy of the BEC compensates the positive kinetic energy such that the BEC is self-trapped in the axial direction. The shape of this 3D condensate bright soliton can be deduced from NPSE. Scaling z in units of $a_\perp$ and t in units of $\omega_\perp^{-1}$, with the standard position

$$f(z,t) = \Phi(z-vt)e^{iv(z-vt)}e^{i(v^2/2-\mu)t} \; , \tag{3·1}$$

from NPSE one obtains the Newtonian second-order differential equation

$$\left[\frac{d^2}{d\zeta^2} - 2\gamma\frac{\Phi^2}{\sqrt{1-2\gamma\Phi^2}} + \frac{1}{2}\left(\frac{1}{\sqrt{1-2\gamma\Phi^2}} + \sqrt{1-2\gamma\Phi^2}\right)\right]\Phi = \mu\,\Phi \; , \tag{3·2}$$

where $\zeta = z - vt$ and $\gamma = |a_s|N/a_\perp$. A simple constant of motion of this equation is given by

$$E = \frac{1}{2}\left(\frac{d\Phi}{d\zeta}\right)^2 + \mu\,\Phi^2 - \Phi^2\sqrt{1-2\gamma\Phi^2} \; , \tag{3·3}$$

from which one can write

$$\int \frac{d\Phi}{\sqrt{2(E-\mu\,\Phi^2+\Phi^2\sqrt{1-2\gamma\Phi^2})}} = \int d\zeta \; . \tag{3·4}$$

By imposing the boundary condition $\Phi \to 0$ for $\zeta \to \infty$, which implies that $E = 0$, by quadratures one obtains the solitary bright-soliton solution written in implicit form

$$\zeta = \frac{2^{-1/2}}{\sqrt{1-\mu}}\,\mathrm{arctg}\left[\sqrt{\frac{\sqrt{1-2\gamma\Phi^2}-\mu}{1-\mu}}\right] - \frac{2^{-1/2}}{\sqrt{1+\mu}}\,\mathrm{arcth}\left[\sqrt{\frac{\sqrt{1-2\gamma\Phi^2}-\mu}{1+\mu}}\right] \; . \tag{3·5}$$

Moreover, by imposing the normalization condition to the function Φ, one has

$$(1-\mu)^{3/2} - \frac{3}{2}(1-\mu)^{1/2} + \frac{3}{2\sqrt{2}}\gamma = 0 \; . \tag{3·6}$$

The normalization relates the chemical potential μ to the coupling constant γ, while the velocity v of the bright soliton remains arbitrary. Note that in the 1D limit ($\gamma\Phi^2 \ll 1$), the normalization condition gives $\mu = 1 - \gamma^2/2$ and the bright-soliton solution reads

$$\Phi(\zeta) = \sqrt{\frac{\gamma}{2}}\,\mathrm{sech}\,[\gamma\zeta] \; . \tag{3·7}$$

The above solution is the text-book 1D bright soliton of the 1D nonlinear (cubic) Schrödinger equation (1D GPE).

From Eq. (3·6) it easy to show that for $\gamma > 2/3$ there are no solitary-wave solutions. This is a remarkable result because, contrary to the 3D bright soliton the widely studied 1D bright soliton exists (and it is stable) at any γ. Thus BEC bright solitons exist only below a critical number

$$N_c = \frac{2}{3}\frac{a_\perp}{|a_s|} \tag{3·8}$$

of Bosons. Finally, we observe that Eq. (3·5) is well defined only for $\gamma\Phi^2 < 1/2$; at $\gamma\Phi^2 = 1/2$ the transverse size σ of the BEC soliton is zero. Because Φ^2 is measured in units of $1/a_\perp$, it follows that it exists a critical axial density

$$\rho_c = \frac{1}{2|a_s|} \tag{3·9}$$

above which there is the collapse of the BEC bright soliton. This result is confirmed by the numerical integration of 3D GPE.

§4. Conclusions

It is well known that in one dimension soliton states exist for Bose condensed atoms with negative scattering length a_s described by the one-dimensional Gross-Pitaevskii equation, that is the one-dimensional nonlinear cubic Schrödinger equation. [5] Such states, called bright solitons, are characterized by a time invariant shape. We have studied the existence of similar states in higher dimensions. In two and three dimensions in a flat potential stable bright solitons do not exist. [5] However if an external harmonic potential constrains the motion in two dimensions we have found that such 3D soliton states exist but only below a critical number N_c of bosons. The value of N_c depends on the transverse confinement, more precisely it is proportional to the ratio of the transverse harmonic length to the scattering length. Moreover, we have found that 3D bright solitons collapse if their axial density is larger than $1/(2a_s)$: for this value the transverse width of the bright soliton shrinks to zero. Finally, it is important to stress that our solitary-wave solutions can be called solitonic solutions because they have particle-like properties, e.g. collisional stability, as recently verified studying scattering processes. [9]

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