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# On the stability of asymptotic properties of perturbed $C_0$-semigroups

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**Abstract.** We give conditions on a strongly continuous semigroup $\mathcal{T}$ and an unbounded perturbation $B$ in the class of Miyadera-Voigt such that the perturbed semigroup $\mathcal{S}$ inherits asymptotic properties of $\mathcal{T}$ as boundedness, asymptotic almost periodicity, uniform ergodicity and total uniform ergodicity. A systematic application of the abstract result to partial differential equations with delay is made.

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## 1 Introduction

According to a classical result proved by R. S. Phillips [15] in 1953, if $A$ generates a strongly continuous semigroup $\mathcal{T}$ of bounded linear operators on a Banach space $X$, then so is $A + B$, when $B$ is any bounded linear operator on $X$. This result has been generalized into various directions. On one hand, the hypotheses have been weakened, admitting $B$ to be an unbounded operator. On the other hand, the question whether some properties of $\mathcal{T}$ persist under such perturbations has been studied by several authors.

R. S. Phillips made the start by cataloguing stable und unstable properties under bounded perturbations. He proved, in particular, that immediate norm continuity and immediate compactness of $\mathcal{T}$ are inherited by the perturbed semigroup $\mathcal{S}$, generated by $A + B$. This is not the case for eventual norm continuity, eventual compactness and, as shown only in 1995 by M. Renardy [16], for immediate differentiability. Nevertheless, M. G. Crandall and A. Pazy [6, 14] exhibited growth conditions on the norm of $AT(t)$ as $t \to 0^+$, under which the perturbed semigroup is immediately differentiable. More recently, R. Nagel and the second author of this paper found in [12] other conditions guaranteeing the permanence of these regularity properties under bounded perturbations. Their method, based on the representation

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of $\mathcal{S}$ through the Dyson-Phillips series, can be also applied to study the asymptotic behaviour of $\mathcal{S}$. This is the main purpose of this paper, where we consider, however, the more general case of unbounded perturbations. In particular, the operator $B$, by which we perturb the infinitesimal generator $A$, belongs to a class, which was introduced for the first time by Miyadera and Voigt (see, e.g., [9] and [18]).

We consider a semigroup $\mathcal{T}$ such that every orbit $t \mapsto T(t)x$, $x \in X$, belongs to a closed, translation invariant subspace $\mathcal{E}$ of the space $\mathcal{C}_{ub}(\mathbb{R}_+, X)$ of all uniformly continuous and bounded functions from $\mathbb{R}_+$ to $X$ (see Section 3 for definitions). For example, $\mathcal{T}$ could be strongly asymptotically almost periodic (even in the sense of Eberlein) or uniformly ergodic or such that $t \mapsto T(t)x$ vanishes at infinity for all $x \in X$. In all these cases, the semigroup $\mathcal{T}$ is uniformly bounded, as a consequence of the uniform boundedness principle. Thus, if $\mathcal{S}$ inherits the same property, $\mathcal{S}$ is uniformly bounded as well. This means that, first of all, one has to find conditions under which a bounded perturbation preserves the uniform boundedness of $\mathcal{T}$. We obtain such a result in Corollary 2.2 (see also the results of J. Voigt in [18]). In the case of a positive (or positive dominated) $C_0$-group on a Banach lattice and a positive (or positive dominated) perturbation, this was done by M. Mokhtar Kharroubi in [10] and [11].

The second step consists in proving the permanence of certain asymptotic properties under a perturbation $B$. We use that $\mathcal{S}$ can be obtained as

$$\mathcal{S} = \sum_{n=0}^{\infty} (V^n \mathcal{T}),$$

where $V$ is an abstract "Volterra-type" operator and the series converges in a sufficiently strong topology. Then the permanence problem can be rewritten as follows: if $\mathcal{T}$ is such that $t \mapsto T(t)x$ belongs to the space $\mathcal{E}$ for all $x \in X$, under which conditions does the map $t \mapsto (V^n \mathcal{T})(t)x$ belong to $\mathcal{E}$?

Some recent results of C. J. K. Batty and R. Chill [5] on the convolution product between a bounded strongly continuous map from $\mathbb{R}_+$ to $\mathcal{L}(X)$ and a function belonging to $L^1(\mathbb{R}, X)$ yield an answer to this question. Bounded convolutions were also considered by B. Basit [3]. In particular, he found conditions on the Laplace transform $\hat{f}$ of a measurable function $f$ on $\mathbb{R}_+$, such that $\mathcal{T} * f$ is bounded, when $\mathcal{T}$ is a uniformly bounded, holomorphic semigroup. However, using the results of [5], we do not need any spectral assumption on $\hat{f}$ or $\sigma(A)$. We investigate the particular case of bounded perturbations as well, and then discuss some examples using operator matrices.

Finally, in Section 4 we present an application of our results to partial differential equations with delay. We consider a delay equation of the form

$$\text{(DE)} \quad \begin{cases} u'(t) = Au(t) + \Phi u_t, & t \geq 0, \\ u(0) = x, \\ u_0 = f, \end{cases}$$

where $(A, D(A))$ is the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on $X$, $x \in X$, $f \in L^p([-1, 0], X)$, $1 \leq p < \infty$ and $\Phi$ is a bounded linear operator from $W^{1,p}([-1, 0], X)$ to $X$. We prove, in particular, that the knowledge of the asymptotic behaviour of $(T(t))_{t\geq 0}$ yields full information about the asymptotic behaviour of the solution of (DE).

## 2 The convergence of the Dyson-Phillips series

Let $X$ be a complex Banach space. $\mathscr{L}(X)$ denotes the Banach algebra of all bounded linear operators on $X$.

Let $\mathscr{T} = (T(t))_{t\geq 0}$ be a strongly continuous semigroup of bounded linear operators on $X$ with generator $(A, D(A))$. The symbol $X_1$ will denote the Sobolev space of order one associated to $\mathscr{T}$, that is the Banach space $(D(A), \|\cdot\|_1)$ where $\|x\|_1 := \|x\| + \|Ax\|$ for every $x \in D(A)$.

Let $\mathscr{L}_s(X)$ denote the space $\mathscr{L}(X)$ endowed with the strong operator topology and $\mathscr{X}$ the operator-valued function space $\mathscr{C}_{ub}(\mathbb{R}_+, \mathscr{L}_s(X))$ of all uniformly continuous, bounded functions from $\mathbb{R}_+$ to $\mathscr{L}_s(X)$. Then $F \in \mathscr{X}$ if and only if $F(t) \in \mathscr{L}(X)$ for $t \geq 0$, $t \mapsto F(t)x$ is uniformly continuous for every $x \in X$ and $\sup_{t\geq 0} \|F(t)\| < +\infty$.

The space $\mathscr{X}$ is a Banach space for the norm

$$\|F\|_\infty := \sup_{t\geq 0} \|F(t)\|, \quad F \in \mathscr{X} \text{ (see [7, Proposition A.7])}.$$

Take now a perturbing operator $B \in \mathscr{L}(X_1, X)$. In analogy with [7, Chapter III], we define the abstract Volterra operator $V_B$ by

$$(V_B F)(t)x := \int_0^t F(s)BT(t-s)x\, ds \quad \text{for every } F \in \mathscr{X}, \quad t \geq 0, \ x \in D(A).$$

Observe that, for every $t \geq 0$, $(V_B F)(t) \in \mathscr{L}(X_1, X)$. Always in analogy to [7, Chapter III], we now assume that for every $F \in \mathscr{X}$ the following conditions are satisfied

(1) for all $t \in [0, +\infty)$ the map $(V_B F)(t) : D(A) \subset X \to X$ can be extended to a bounded operator $\overline{(V_B F)(t)} : X \to X$;

(2) the map $t \mapsto \overline{(V_B F)(t)}x$ is uniformly continuous for every $x \in X$;

(3) $\overline{V_B}$ defines a bounded operator on $\mathscr{X}$ satisfying $\|\overline{V_B}\| < 1$.

The class of operators satisfying conditions (1), (2), (3) will be denoted by $\mathscr{S}_\infty$, that is

$$\mathscr{S}_\infty := \{B \in \mathscr{L}(X_1, X) : \overline{V_B} \in \mathscr{L}(\mathscr{X}) \text{ and } \|\overline{V_B}\| < 1\}.$$

In the following we will denote $\overline{V_B}$ by $V$. It is well known ([7, Theorem III.3.14]) that, if $B$ belongs to $\mathscr{S}_\infty$, then $(A + B, D(A))$ generates a strongly continuous semigroup $\mathscr{S}$

on $X$. Moreover, the perturbed semigroup $\mathscr{S}$ is given by the abstract Dyson-Phillips series

$$(2.1)\qquad S(t) = \sum_{n=0}^{\infty} (V^n \mathscr{T})(t), \quad t \geq 0,$$

where $V$ is defined as above.

It is well known that this series converges uniformly on compact intervals of $\mathbb{R}_+$ (see, for example, [7, Corollary III.3.15]). However, in order to deduce asymptotic properties of $\mathscr{S}$ from those of $V^n \mathscr{T}$, the uniform convergence on $\mathbb{R}_+$ of the Dyson-Phillips series is required. Such a convergence holds only if suitable hypotheses are made. For example, in [10], [11] this has been proved in the framework of Banach lattices and for a (not necessarily bounded) one-dimensional perturbation $B$. We present the following result.

**Theorem 2.1.** *Let $\mathscr{T} = (T(t))_{t \geq 0}$ be a strongly continuous semigroup of bounded linear operators on $X$ generated by $A : D(A) \subseteq X \to X$ and satisfying*

$$\|T(t)\| \leq M \quad \textit{for all } t \geq 0 \quad \textit{and some } M \geq 1$$

*and let $B \in \mathscr{L}(X_1, X)$. Assume that there exists a constant $0 < q < 1$ such that*

$$(2.2)\qquad \int_0^t \|BT(s)x\| \, ds \leq q\|x\|$$

*for all $t \geq 0$ and $x \in D(A)$.*

*Then $B$ belongs to $\mathscr{S}_\infty$, and the series* (2.1) *converges uniformly on* $\mathbb{R}_+$.

*Proof.* First of all, we shall prove that $B$ belongs to the perturbation class $\mathscr{S}_\infty$, by showing (1), (2) and (3).

Let $F \in \mathscr{X}$ and $t \geq 0$. It follows from (2.2) that

$$\begin{aligned} \|VF(t)x\| &\leq \|F\|_\infty \int_0^t \|BT(t-s)x\| \, ds \\ &= \|F\|_\infty \int_0^t \|BT(s)x\| \, ds \leq \|F\|_\infty \cdot q \cdot \|x\| \end{aligned}$$

for every $x \in D(A)$. Since $D(A)$ is dense in $X$, $(V_B F)(t)$ can be extended to a bounded operator $\overline{(V_B F)(t)}$ on $X$, such that

$$(2.3)\qquad \|\overline{(V_B F)(t)}\| \leq q \cdot \|F\|_\infty \quad \text{for every } t \geq 0,$$

so that (1) holds.

In order to prove (2), we show, first of all, that the map $t \mapsto (V_B F)(t)x$ is uniformly continuous on $\mathbb{R}_+$ for every $x \in D(A)$. For $t \in \mathbb{R}_+$, $x \in D(A)$ and $h > 0$, then

$$\begin{aligned}
&\|VF(t+h)x - VF(t)x\| \\
&= \left\| \int_0^{t+h} F(s)BT(t+h-s)x\,ds - \int_0^t F(s)BT(t-s)x\,ds \right\| \\
&\leq \left\| \int_0^{t+h} F(s)BT(t+h-s)x\,ds - \int_0^t F(s)BT(t+h-s)x\,ds \right\| \\
&\quad + \left\| \int_0^t F(s)BT(t+h-s)x\,ds - \int_0^t F(s)BT(t-s)x\,ds \right\| \\
&= \left\| \int_t^{t+h} F(s)BT(t+h-s)x\,ds \right\| + \left\| \int_0^t F(s)B(T(t+h-s) - T(t-s))x\,ds \right\| \\
&\leq \|F\|_\infty \int_0^h \|BT(s)x\|\,ds + \|F\|_\infty \cdot \int_0^t \|BT(t-s)(T(h)x - x)\|\,ds \\
&\leq \|F\|_\infty \|B\|_{\mathscr{L}(X_1,X)} \int_0^h (\|T(s)x\| + \|AT(s)x\|)\,ds + \|F\|_\infty \cdot q \cdot \|T(h)x - x\|.
\end{aligned}$$

This last expression converges to zero as $h \to 0^+$, uniformly with respect to $t$.

Take now $x \in X$. Then there exists a sequence $\{x_j\} \subset D(A)$ such that $x_j \to x$, and we have

$$\sup_{t\geq 0} \|\overline{(V_B F)(t)}x - (V_B F)(t)x_j\| = \sup_{t\geq 0} \|\overline{(V_B F)(t)}(x - x_j)\| \leq q \cdot \|F\|_\infty \|x - x_j\|$$

and hence the sequence $(V_B)F(t)x_j)_j$ converges uniformly on $\mathbb{R}_+$ to $(V_B F)(t)x$. This shows that the function $t \mapsto \overline{(V_B F)(t)}x$ is uniformly continuous on $\mathbb{R}_+$. Finally, from (2.3) it follows that

$$\|V\| \leq q < 1$$

and hence the spectral radius $r(V)$ is strictly smaller than 1 and $1 \in \rho(V)$.

Since $\mathscr{T}$ belongs to $\mathscr{X}$ and $I - V$ is bijective on $\mathscr{X}$, there exists a unique $\mathscr{S} \in \mathscr{X}$ such that

$$(I - V)\mathscr{S} = \mathscr{T}.$$

Moreover, the resolvent $R(1, V)$ exists and is given by the Neumann series

$$R(1, V) = \sum_{n=0}^{+\infty} V^n.$$

Thus $\mathscr{S} = R(1, V)\mathscr{T} = \sum_{n=0}^{\infty} V^n \mathscr{T}$, the series being convergent in the norm of $\mathscr{X}$, and

$$\sup_{t\geq 0} \|S(t)\| \leq \sum_{n=0}^{+\infty} \sup_{t\geq 0} \|(V^n \mathscr{T})(t)\| \leq \sum_{n=0}^{+\infty} q^n \cdot M = \frac{M}{1-q}. \qquad \square$$

Since one knows that $\mathscr{S}$ is the semigroup generated by $A + B$, we obtain the following result (see also [18, Theorem 1]).

**Corollary 2.2.** *Under the assumptions of Theorem 2.1, the strongly continuous semigroup $\mathscr{S} = (S(t))_{t\geq 0}$ generated by $(A + B, D(A))$ is uniformly bounded.*

**Remark 2.3**

**1.** Condition (2.2) was introduced for the first time in the Perturbation Theorem of Miyadera and Voigt (see [9], [18] and [7, Chapter III.3.c]).

**2.** Suppose that $(A, D(A))$ is the infinitesimal generator of a strongly continuous, uniformly bounded group $(T(t))_{t\in\mathbb{R}}$ on $X$ and that $B$ belongs to $\mathscr{L}(X_1, X)$. If there exists a constant $0 < q < 1$ such that

$$\int_{-t}^{t} \|BT(s)x\|\, ds \leq q \cdot \|x\|$$

for all $t \in \mathbb{R}$ and $x \in D(A)$, then it is easy to verify that $(A + B, D(A))$ is the infinitesimal generator of a strongly continuous, uniformly bounded group $(S(t))_{t\in\mathbb{R}}$. An analogous remark was made by J. Voigt in [18].

**3.** If the operator $B \in \mathscr{L}(X)$ satisfies condition (2.2) and commutes with every $T(t), t \geq 0$, then

$$\int_0^t \|BT(s)x\|\, ds = \int_0^t \|T(s)Bx\|\, ds$$

for all $x \in X$ and $t > 0$. In particular, if $B$ is surjective, then

$$\int_0^{\infty} \|T(s)x\|\, ds < \infty$$

for all $x \in X$ and therefore, in view of the Theorem of Datko and Pazy (see e.g. [7, Theorem V.1.8] or [13, Theorem 3.1.8]), $\mathscr{T}$ is uniformly exponentially stable.

**Examples 2.4**
Two examples will now be presented in the case in which $B$ is bounded. The first investigates when uniformly exponentially stable semigroups fulfill condition (2.2); the second one shows that, when the semigroup $\mathcal{T}$ is expressed by a matrix, it may happen that the behaviour and the growth of only one entry is essential in order to verify condition (2.2).

**1.** Let $\mathcal{T}$ be a uniformly exponentially stable semigroup, i.e. $\|T(t)\| \le Me^{-\omega t}$ for all $t \ge 0$, some $M \ge 1$ and $\omega > 0$. Let $(A, D(A))$ be the infinitesimal generator of $\mathcal{T}$.

If $B \in \mathcal{L}(X)$, condition (2.2) is satisfied if $B$ satisfies the estimate $\|B\|M/\omega < 1$.

**2.** Let $X_1$ and $X_2$ be complex Banach spaces, $\mathcal{T}_1 = (T_1(t))_{t\ge 0}$ and $\mathcal{T}_2 = (T_2(t))_{t\ge 0}$ be strongly continuous semigroups on $X_1$ and $X_2$, respectively, with generator $(A_1, D(A_1))$ and $(A_2, D(A_2))$. Suppose that $\|T_1(t)\| \le M_1$ and $\|T_2(t)\| \le M_2 e^{-\omega t}$ for every $t \ge 0$ and some $M_1, M_2 \ge 1$ and $\omega > 0$.

Let $(U(t))_{t\ge 0}$ be the strongly continuous semigroup on $X_1 \times X_2$ given by

$$U(t) := \begin{pmatrix} T_1(t) & 0 \\ 0 & T_2(t) \end{pmatrix} \quad \text{with generator } \mathcal{A} := \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}.$$

Take a linear operator $B : X_2 \to X_1$ and the operator matrix

$$\mathcal{B} := \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(X_1 \times X_2).$$

Then

$$\mathcal{A} + \mathcal{B} = \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix} \quad \text{with domain } D(A_1) \times D(A_2)$$

generates a strongly continuous semigroup $\mathcal{S} = (S(t))_{t\ge 0}$ on $X_1 \times X_2$. Since

$$\mathcal{B}U(t) = \begin{pmatrix} 0 & BT_2(t) \\ 0 & 0 \end{pmatrix}$$

and $\mathcal{T}_2$ is uniformly exponentially stable, condition (2.2) is satisfied, independently on $(T_1(t))_{t\ge 0}$, provided that $M_2\|B\|/\omega < 1$. By choosing $B$ with sufficiently small norm, the perturbed semigroup $(S(t))_{t\ge 0}$ is then bounded.

## 3 Permanence of asymptotic properties

Suppose now that certain asymptotic properties of the strongly continuous semigroup $\mathcal{T}$ are known. Is it possible to deduce informations about the asymptotic behaviour of the perturbed semigroup $\mathcal{S}$?

We first consider the stability of asymptotic almost periodicity under bounded perturbations. We need some definitions. If $f : \mathbb{R}_+ \to X$, $H(f)$ is the set of all translates $\{f(\cdot + \omega) : \omega \in \mathbb{R}_+\}$. Let $\mathscr{C}_b(\mathbb{R}_+, X)$ be the Banach space of all bounded continuous functions from $\mathbb{R}_+$ to $X$, endowed with the uniform norm.

A function $f \in \mathscr{C}_b(\mathbb{R}_+, X)$ is called *asymptotically almost periodic* (abbreviated as a.a.p.) if $H(f)$ is relatively compact in $\mathscr{C}_b(\mathbb{R}_+, X)$.

We need the following decomposition theorem, due, in the general case, to W. M. Ruess and to W. H. Summers.

**Theorem 3.1 [17].** *A function $f \in \mathscr{C}_b(\mathbb{R}_+, X)$ is asymptotically almost periodic if and only if one of the following two equivalent conditions is satisfied*:

(1) *there exist a unique almost periodic function $g \in \mathscr{C}_b(\mathbb{R}, X)$ and a unique $h \in \mathscr{C}_b(\mathbb{R}_+, X)$, vanishing at infinity, such that $f = h + g_{|\mathbb{R}_+}$*;

(2) *for every $\varepsilon > 0$ there exist $\Lambda > 0$ and $K \geq 0$ such that every interval of lenght $\Lambda$ contains some $\tau$ for which the inequality*

$$\|f(t+\tau) - f(t)\| \leq \varepsilon$$

*holds whenever $t, t+\tau \geq K$.*

The functions $g$ and $h$ are called, respectively, the *principal term* and the *correction term* of $f$. We recall that $\mathscr{C}_b(\mathbb{R}, X)$ is the space of all bounded continuous functions from $\mathbb{R}$ to $X$ and that a function $g \in \mathscr{C}_b(\mathbb{R}, X)$ is said to be *almost periodic* if the set $\{g(\cdot + \omega) : \omega \in \mathbb{R}\}$ is relatively compact in $\mathscr{C}_b(\mathbb{R}, X)$.

A strongly continuous semigroup is called *strongly asymptotically almost periodic* if the function $t \mapsto T(t)x$, from $\mathbb{R}_+$ to $X$, is a.a.p. for every $x \in X$.

A first simple result about asymptotic almost periodicity follows by the following, well-known proposition (see, for example, [1]).

**Proposition 3.2.** *Let $(A, D(A))$ be the infinitesimal generator of a strongly continuous, uniformly bounded semigroup $\mathscr{T}$. If $A$ has compact resolvent, then $\mathscr{T}$ is strongly asymptotically almost periodic.*

Since $R(\lambda, A+B)$ remains compact if $R(\lambda, A)$ is compact and $B$ is bounded, then the following result can be stated.

**Proposition 3.3.** *Let $(A, D(A))$ be the infinitesimal generator of a strongly continuous, uniformly bounded semigroup $\mathscr{T}$ and let $B \in \mathscr{L}(X)$. Suppose that $R(\lambda, A)$ is compact for some $\lambda \in \rho(A)$.*

*If condition (2.2) of Theorem 2.1 holds, then the semigroup $(S(t))_{t \geq 0}$, generated by $(A+B, D(A))$, is strongly asymptotically almost periodic.*

Indeed, recent results of C. J. K. Batty and R. Chill [5] yield much more information on the asymptotic behaviour of the perturbed semigroup. In particular, this method yields information on the asymptotic behaviour of $(S(t))_{t\geq 0}$ independently on the special class of functions considered.

We recall that a closed subspace $\mathscr{E}$ of $C_{ub}(\mathbb{R}_+, X)$ is said to be *translation invariant* if

$$\mathscr{E} = \{f \in C_{ub}(\mathbb{R}_+, X) : f(\cdot + t) \in \mathscr{E}\} \quad \text{for all } t \geq 0.$$

A closed subspace $\mathscr{E}$ of $C_{ub}(\mathbb{R}_+, X)$ is said to be *operator invariant* if $M \circ f \in \mathscr{E}$ for every $f \in \mathscr{E}$ and $M \in \mathscr{L}(X)$, where $M \circ f$ is defined by $(M \circ f)(t) = M(f(t))$, $t \geq 0$.

As remarked by Batty and Chill [5], the following classes of $X$-valued functions are closed, translation invariant and operator invariant subspaces of $C_{ub}(\mathbb{R}_+, X)$:

- the space $C_0(\mathbb{R}_+, X)$ of all continuous functions vanishing at infinity;
- the class of all asymptotically almost periodic functions from $\mathbb{R}_+$ to $X$;
- the class of all weakly asymptotically almost periodic functions in the sense of Eberlein;
- the class of uniformly ergodic functions from $\mathbb{R}_+$ to $X$;
- the class of totally (uniformly) ergodic functions from $\mathbb{R}_+$ to $X$.

For the sake of completeness we recall the definitions.

A continuous bounded function $f : \mathbb{R}_+ \to X$ is called *weakly asymptotically almost periodic in the sense of Eberlein* if $H(f)$ is weakly relatively compact in $\mathscr{C}_b(\mathbb{R}_+, X)$.

A function $f \in \mathscr{C}_{ub}(\mathbb{R}_+, X)$ is said to be *uniformly ergodic* if the limit

$$\lim_{\alpha\to 0^+} \alpha \int_0^\infty e^{-\alpha s} f(\cdot + s)\, ds \tag{3.1}$$

exists and defines an element of $\mathscr{C}_{ub}(\mathbb{R}_+, X)$.

A function $f \in \mathscr{C}_{ub}(\mathbb{R}_+, X)$ is said to be *totally* (*uniformly*) *ergodic* if the function $e^{i\theta\cdot} f(\cdot)$ is uniformly ergodic for all $\theta \in \mathbb{R}$. Since $f$ is uniformly bounded, this is also equivalent to the existence of the Cesàro limit

$$\lim_{t\to+\infty} \frac{1}{t} \int_0^t e^{i\theta s} f(\cdot + s)\, ds$$

in $\mathscr{C}_{ub}(\mathbb{R}_+, X)$, as remarked in [2].

The following lemma will be used.

**Lemma 3.4 [5].** *Let* $L : \mathbb{R}_+ \to \mathscr{L}(X)$ *be a bounded strongly continuous function. Let* $\mathscr{E}$ *be a closed, translation invariant, operator invariant subspace of* $C_{ub}(\mathbb{R}_+, X)$, *such that the functions* $\mathbb{R}_+ \ni t \mapsto L(t)x \in X$ *belong to* $\mathscr{E}$ *for every* $x \in X$.

*If* $g \in L^1(\mathbb{R}, X)$, *then* $(L * g)_{|\mathbb{R}_+} \in \mathscr{E}$.

Here, the convolution between $L$ and $g$ is defined as

$$(L * g)(t) = \int_0^{+\infty} L(s)g(t-s)\,ds = \int_{-\infty}^{t} L(t-s)g(s)\,ds, \quad t \geq 0.$$

We can now prove the main result of this paper.

**Theorem 3.5.** *Let* $\mathscr{T} = (T(t))_{t\geq 0}$ *be a strongly continuous semigroup on X, generated by* $(A, D(A))$ *and let* $B \in \mathscr{L}(X_1, X)$ *satisfy condition* (2.2) *in Theorem 2.1. Let* $(S(t))_{t\geq 0}$ *be the strongly continuous semigroup generated by* $(A + B, D(A))$.

*If* $\mathscr{E}$ *is a translation invariant and operator invariant closed subspace of* $C_{ub}(\mathbb{R}_+, X)$ *and if* $t \mapsto T(t)x$ *belongs to* $\mathscr{E}$ *for every* $x \in X$, *then* $t \mapsto S(t)x$ *belongs to* $\mathscr{E}$ *for every* $x \in X$.

*Proof.* Let $V$ be the extension of the "Volterra-type" operator introduced in Section 2. We first observe that, as a consequence of the uniform boundedness principle, the semigroup $\mathscr{T}$ is uniformly bounded. Therefore, we can apply Theorem 2.1 and obtain that the semigroup $\mathscr{S} = (S(t))_{t\geq 0}$ is uniformly bounded and is given by

$$\mathscr{S} = \sum_{n=0}^{+\infty} V^n \mathscr{T},$$

the convergence being in the norm of $\mathscr{X} = \mathscr{C}_{ub}(\mathbb{R}_+, \mathscr{L}_s(X))$ (observe that the operator $V^n\mathscr{T}$ belongs to $\mathscr{X}$ for every $n \in \mathbb{N}$, since we proved in Theorem 2.1 that $V$ maps $\mathscr{X}$ into $\mathscr{X}$).

We now show that the functions

$$t \mapsto (V^n\mathscr{T})(t)x$$

belong to $\mathscr{E}$ for every $n \in \mathbb{N}$ and $x \in X$.

We consider first the case $n = 1$. Take $x \in D(A)$ and define

$$g(t) := \begin{cases} BT(t)x & \text{if } t \geq 0 \\ 0 & \text{if } t < 0. \end{cases}$$

Since

$$\int_{-\infty}^{+\infty} \|g(t)\| \, dt = \int_{0}^{+\infty} \|BT(t)x\| \, dt \le q \cdot \|x\| < \|x\|,$$

$q$ belongs to $L^1(\mathbb{R}, X)$. Since $\mathscr{T}$ is bounded, strongly continuous and, by hypothesis, $t \mapsto T(t)x$ belongs to $\mathscr{E}$ for every $x \in X$, Lemma 3.4 entails that $(\mathscr{T} * g)_{|\mathbb{R}_+} \in \mathscr{E}$.

Observe now that

$$(\mathscr{T} * g)_{|\mathbb{R}_+}(t) = \int_{-\infty}^{t} T(t-s)g(s) \, ds = \int_{0}^{t} T(t-s)BT(s)x \, ds.$$

Since $x \in D(A)$

$$(V\mathscr{T})(t)x = (V_B\mathscr{T})(t)x = \int_{0}^{t} T(t-s)BT(s)x \, ds,$$

so that $t \mapsto (V\mathscr{T})(t)x$ belong to $\mathscr{E}$ for every $x \in D(A)$.

Take now $x \in X$. Thus there exists a sequence $\{x_j\} \subset D(A)$, such that $x_j \mapsto x$.

Since $V\mathscr{T}$ belongs to $\mathscr{X}$, $\{(V\mathscr{T})(t) : t \ge 0\}$ is uniformly bounded on $\mathbb{R}_+$, and therefore the sequence $\{(V\mathscr{T})(t)x_j : j \in \mathbb{N}\}$ converges uniformly on $\mathbb{R}_+$ to $(V\mathscr{T})(t)x$. Since $\mathscr{E}$ is a closed subspace of $\mathscr{C}_{ub}(\mathbb{R}_+, X)$, we conclude that the limit function belongs to $\mathscr{E}$ as well, proving the assertion.

Suppose now that the maps $t \mapsto (V^{n-1}\mathscr{T})(t)x$ belong to $\mathscr{E}$ for all $x \in X$.

Then, if $x \in D(A)$ and $t \in \mathbb{R}_+$,

$$\begin{aligned}(V^n\mathscr{T})(t)x &= V(V^{n-1}\mathscr{T})(t)x = V_B(V^{n-1}\mathscr{T})(t)x \\ &= \int_{0}^{t} (V^{n-1}\mathscr{T})(t-s)BT(s)x \, ds.\end{aligned}$$

Since $V^{n-1}\mathscr{T}$ belongs to $\mathscr{X}$, the operator-valued function $V^{n-1}\mathscr{T} : \mathbb{R}_+ \to \mathscr{L}(X)$ is strongly continuous. Moreover, by hypothesis, $(V^{n-1}\mathscr{T})(\cdot)x$ belongs to $\mathscr{E}$ for all $x \in X$.

Thus, we can apply Lemma 3.4, taking as $L$ the operator $V^{n-1}\mathscr{T}$ and choosing $g$ as above. Then

$$t \mapsto (L * g)_{|\mathbb{R}_+}(t) = \int_{0}^{t} (V^{n-1}\mathscr{T})(t-s)BT(s)x \, ds$$

belongs to $\mathscr{E}$ for every $x \in D(A)$.

Therefore, every function $t \mapsto (V^n\mathscr{T})(t)x$ belongs to $\mathscr{E}$ for all $x \in D(A)$.

As in the case $n = 1$ we conclude that $(V^n \mathscr{T})(\cdot)x$ belongs to $\mathscr{E}$ for all $x \in X$.

We can now conclude the proof. Every term of the series

$$S(t)x = \sum_{n=0}^{+\infty} (V^n \mathscr{T})(t)x, \quad x \in X, \ t \geq 0,$$

belongs to $\mathscr{E}$. Since $\mathscr{E}$ is closed in $\mathscr{C}_{ub}(\mathbb{R}_+, X)$ and the above series converges in the norm of $\mathscr{X}$, every map $t \mapsto S(t)x$ belongs to $\mathscr{E}$. □

**Example 3.6.** Define, as in Example 2.4.2, the semigroup

$$U(t) := \begin{pmatrix} T_1(t) & 0 \\ 0 & T_2(t) \end{pmatrix},$$

where $(T_1(t))_{t \geq 0}$ is such that for every $x \in X_1$ the map $t \mapsto T_1(t)x$ is the restriction of an almost periodic function and $(T_2(t))_{t \geq 0}$ is such that $\|T_2(t)\| \leq Me^{-\omega t}$ for every $t \geq 0$, for some $M \geq 1$ and $\omega > 0$.

Then $(U(t))_{t \geq 0}$ is strongly asymptotically almost periodic.

Take a linear operator $B : X_2 \to X_1$ with $\frac{M\|B\|}{\omega} < 1$, and define the operator matrix $\mathscr{B}$ as in Example 2.4.2. Then $\mathscr{A} + \mathscr{B}$ is again the infinitesimal generator of a strongly asymptotically almost periodic semigroup which is given by

$$\begin{pmatrix} T_1(t) & \int_0^t T_1(t-s) B T_2(s)\, ds \\ 0 & T_2(t) \end{pmatrix}_{t \geq 0}.$$

## 4 Application to partial differential equations with delay

Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ on a Banach space $X$. On $E := X \times L^p([-1, 0], X)$, $1 \leq p < \infty$, we consider the operator

$$\mathscr{A} := \begin{pmatrix} A & 0 \\ 0 & \dfrac{d}{d\sigma} \end{pmatrix}$$

with domain

$$D(\mathscr{A}) := \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(A) \times W^{1,p}([-1, 0], X) : f(0) = x \right\}.$$

This operator generates the strongly continuous semigroup $(\mathscr{T}(t))_{t \geq 0}$ given by

$$\mathscr{T}(t) := \begin{pmatrix} T(t) & 0 \\ T_t & T_0(t) \end{pmatrix},$$

where $(T_0(t))_{t\geq 0}$ is the nilpotent left shift semigroup on $L^p([-1,0],X)$ and $T_t : X \to L^p([-1,0],X)$ is defined as

$$(T_t x)(\sigma) := \begin{cases} T(t+\sigma)x, & \sigma \geq -t, \\ 0, & \text{otherwise.} \end{cases}$$

Since the semigroup $(\mathscr{T}(t))_{t\geq 0}$ is essentially given by $(T(t))_{t\geq 0}$, it is reasonable to hope that $(\mathscr{T}(t))_{t\geq 0}$ has the same asymptotic behaviour as $(T(t))_{t\geq 0}$. Indeed this depends on the type of asymptotic behaviour.

**Lemma 4.1.** *Assume that for every $x \in X$ the map $\mathbb{R}_+ \ni t \mapsto T(t)x \in X$ is*

(1) *continuous and vanishing at infinity*, *or*

(2) *asymptotically almost periodic*, *or*

(3) *uniformly ergodic*, *or*

(4) *totally uniformly ergodic.*

*Then the map $\mathbb{R}_+ \ni t \mapsto \mathscr{T}(t)\begin{pmatrix} x \\ f \end{pmatrix} \in E$ has the same property for all $\begin{pmatrix} x \\ f \end{pmatrix} \in E$.*

*Proof.* Since all these classes are translation invariant, it suffices to show the assertions for the map

$$\mathbb{R}_+ \ni t \mapsto \mathscr{T}(t+1)\begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} T(t+1) & 0 \\ T_{t+1} & 0 \end{pmatrix}\begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} T(t+1)x \\ T_{t+1}x \end{pmatrix} \in E.$$

Let $x \in X$.

(1) If $(t \mapsto T(t)x) \in \mathscr{C}_0(\mathbb{R}_+, X)$, then for every $\varepsilon > 0$ there exists a $T > 0$ such that $\|T(t)x\| < \varepsilon$ for all $t > T$. Then, for $t > T$, we have

$$\|T_{t+1}x\|_p < \varepsilon,$$

hence

$$\left\| \mathscr{T}(t+1)\begin{pmatrix} x \\ f \end{pmatrix} \right\| < 2\varepsilon$$

for all $t > T$.

(2) Since $t \mapsto T(t)x$ is asymptotically almost periodic, it follows from Theorem 3.1 that for every $\varepsilon > 0$ there exist $\Lambda > 0$ and $K \geq 0$ such that every interval of lenght $\Lambda$ contains some $\tau$ for which

$$\|T(t+\tau)x - T(t)x\| \leq \varepsilon$$

holds whenever $t, t+\tau \geq K$. Hence

$$\int_{-1}^{0} \|(T_{t+1+\tau}x)(\sigma) - (T_{t+1}x)(\sigma)\|^p \, d\sigma$$

$$= \int_{-1}^{0} \|T(t+1+\tau+\sigma)x - T(t+1+\sigma)x\|^p \, d\sigma$$

$$= \int_{0}^{1} \|T(t+\tau+\sigma)x - T(t+\sigma)x\|^p \, d\sigma \leq \varepsilon^p$$

whenever $t, t+\tau \geq K$, showing that the map $t \mapsto T_{t+1}x$ from $\mathbb{R}_+$ to $L^p([-1,0], X)$ is asymptotically almost periodic for every $x \in X$.

(3) The limit

$$F(\cdot) := \lim_{\alpha \searrow 0} \alpha \int_0^\infty e^{-\alpha t} T(\cdot + t)x \, dt$$

exists in $\mathscr{C}_{ub}(\mathbb{R}_+, X)$. For $s \geq 0$ we define $F_{s+1} : [-1,0] \to X$ by $F_{s+1}(\sigma) := F(s+1+\sigma)$. It is easy to see that the map $\mathbb{R}_+ \ni s \mapsto F_{s+1}$ belongs to $\mathscr{C}_{ub}(\mathbb{R}_+, L^p([-1,0], X))$. Then

$$\lim_{\alpha \searrow 0} \sup_{s \in \mathbb{R}_+} \left\| \left( \alpha \int_0^\infty e^{-\alpha t} T_{\cdot+1+t}x \, dt \right)(s) - F_{s+1} \right\|_p^p$$

$$= \lim_{\alpha \searrow 0} \sup_{s \in \mathbb{R}_+} \int_{-1}^{0} \left\| \alpha \int_0^\infty e^{-\alpha t} T(s+1+t+\sigma)x \, dt - F(s+1+\sigma) \right\|^p d\sigma$$

$$= 0,$$

since $\|\alpha \int_0^\infty e^{-\alpha t} T(s+1+t+\sigma)x \, dt - F(s+1+\sigma)\| \to 0$ as $\alpha \searrow 0$, uniformly for $s \in \mathbb{R}_+$.

(4) The proof is analogous to that of (3). □

We now define an operator $\mathscr{B}$ on $E$ by

$$\mathscr{B} := \begin{pmatrix} 0 & \Phi \\ 0 & 0 \end{pmatrix}, \quad D(\mathscr{B}) := D(\mathscr{A}),$$

where $\Phi$ is a bounded linear operator from $W^{1,p}([-1,0], X)$ to $X$. We then have

$$\int_0^t \left\| \mathscr{B}\mathscr{T}(s) \begin{pmatrix} x \\ f \end{pmatrix} \right\| ds = \int_0^t \|\Phi(T_s x + T_0(s) f)\| \, ds$$

for all $t \geq 0$ and can state the following theorem.

**Theorem 4.2.** *Assume that for every* $x \in X$ *the map* $\mathbb{R}_+ \ni t \mapsto T(t)x$ *belongs to one of the classes* (1)–(4) *of Lemma* 4.1. *Moreover assume that there exists a constant* $0 < q < 1$ *such that*

$$\int_0^t \|\Phi(T_s x + T_0(s)f)\|\, ds \le q \left\| \begin{pmatrix} x \\ f \end{pmatrix} \right\|$$

*for all* $t \ge 0$ *and all* $\begin{pmatrix} x \\ f \end{pmatrix} \in D(\mathscr{A})$.

*Then* $(\mathscr{A} + \mathscr{B}, D(\mathscr{A}))$ *generates a strongly continuous semigroup* $(\mathscr{S}(t))_{t \ge 0}$ *and for all* $\begin{pmatrix} x \\ f \end{pmatrix} \in E$ *the map* $t \mapsto \mathscr{S}(t)\begin{pmatrix} x \\ f \end{pmatrix}$ *belongs to the same class of the map* $t \mapsto T(t)x$.

*Proof.* From Lemma 4.1 we know that $t \mapsto \mathscr{T}(t)\begin{pmatrix} x \\ f \end{pmatrix}$ is in the same class as $t \mapsto T(t)x$ for all $\begin{pmatrix} x \\ f \end{pmatrix} \in E$. Moreover $(\mathscr{T}(t))_{t \ge 0}$ and $\mathscr{B}$ satisfy assumption (2.2) of Theorem 2.1. So by Theorem 3.5 we have that $(\mathscr{A} + \mathscr{B}, D(\mathscr{A}))$ generates a strongly continuous semigroup $(\mathscr{S}(t))_{t \ge 0}$ on $E$ and $t \mapsto \mathscr{S}(t)\begin{pmatrix} x \\ f \end{pmatrix}$ is in the same class as $t \mapsto \mathscr{T}(t)\begin{pmatrix} x \\ f \end{pmatrix}$ for all $\begin{pmatrix} x \\ f \end{pmatrix} \in E$. □

The above operators model abstract linear partial differential equations with delay, i.e., equations of the form

$$\text{(DE)} \qquad \begin{cases} u'(t) = Au(t) + \Phi u_t, & t \ge 0, \\ u(0) = x, \\ u_0 = f, \end{cases}$$

where $x \in X$ and $f \in L^p([-1,0], X)$.

In fact, equation (DE) is "well-posed" if and only if the operator $(\mathscr{A} + \mathscr{B}, D(\mathscr{A}))$ generates a strongly continuous semigroup $(\mathscr{S}(t))_{t \ge 0}$ on $E$. Moreover, if this is the case, the solutions of equation (DE) are given by

$$\begin{pmatrix} u(t) \\ u_t \end{pmatrix} = \mathscr{S}(t)\begin{pmatrix} x \\ f \end{pmatrix}, \quad t \ge 0.$$

Therefore, the asymptotic behaviour of the semigroup $(\mathscr{S}(t))_{t \ge 0}$ yields the asymptotic behaviour of the solutions of equation (DE). For more details about this semigroup approach to partial differential delay equation we refer to [4] and [8].

**Example 4.3.** Let $p > 1$ and $\Phi f := Cf(-1)$, where $C \in \mathscr{L}(X)$ with $\|C\| < 1$. Moreover, assume that there exists $0 < c < 1$ such that

$$\int_0^t \|CT(s)x\|\, ds \leq c\|x\|$$

for all $t \geq 0$ and $x \in D(A)$ (this is true, for example, if $(T(t))_{t\geq 0}$ is as in Example 2.4.2).

Then

$$\int_0^t \|\Phi(T_s x + T_0(s)f)\|\, ds$$

$$= \begin{cases} \int_0^t \|Cf(s-1)\|\, ds, & 0 \leq t \leq 1, \\ \int_0^1 \|Cf(s-1)\|\, ds + \int_1^t \|CT(s-1)x\|\, ds, & t > 1. \end{cases}$$

For $t \in [0,1]$ we have

$$\int_0^t \|\Phi(T_s x + T_0(s)f)\|\, ds \leq \|C\| t^{1/p'} \|f\|_p$$

for $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

If for every $x \in X$ the map $t \mapsto T(t)x$ is in one of the classes (1)–(4) in Lemma 4.1, then the assumptions of Theorem 4.2 are satisfied, and the solutions of the delay equation

$$\begin{cases} u'(t) = Au(t) + Cu(t-1), & t \geq 0, \\ u(0) = x, \\ u_0 = f, \end{cases}$$

are in the same class as $t \mapsto T(t)x$ for every $\begin{pmatrix} x \\ f \end{pmatrix} \in E$.

## References


1. Arendt, W. and Batty, C. J. K.: Almost periodic solutions of first and second order Cauchy problems. J. Diff. Equations **137** (1997), 363–383
2. Arendt, W. and Batty, C. J. K.: Asymptotically almost periodic solutions of inhomogeneous Cauchy problems on the half-line. Bull. London Math. Soc. **31** (1999), 291–304

3. Basit, B.: Harmonic analysis and asymptotic behaviour of solutions of the abstract Cauchy problem. Semigroup Forum **54** (1997), 58–74
4. Bátkai, A., Piazzera, S.: Partial differential equations with delay. Preprint
5. Batty, C. J. K. and Chill, R.: Bounded convolutions and solutions of inhomogeneous Cauchy problems. Forum Math. **11** (1999), 253–277
6. Crandall, M. G. and Pazy, A.: On the differentiability of weak solutions of a differential equation in Banach space. J. Math. Mech. **18** (1969), 1007–1016
7. Engel, K. J. and Nagel, R.: One-Parameter Semigroups for Linear Evolution Equations. To appear in Graduate Texts in Mathematics. Springer-Verlag, 1999
8. Kunisch, K. and Schappacher, W.: Necessary conditions for partial differential equations with delay to generate $\mathscr{C}_0$-semigroups. J. Diff. Equations **50** (1983), 49–79
9. Miyadera, I.: On perturbation theory for semi-groups of operators. Tôhoku Math. J. **18** (1966), 299–310
10. Mokhtar Kharroubi, M.: Limiting absorption principles and wave operators on $L^1(\mu)$ spaces: Applications to transport theory. J. Funct. Anal. **115** (1993), 119–145
11. Mokhtar Kharroubi, M.: On perturbations of positive $C_0$-(semi)groups on Banach lattices and applications. J. Math. Anal. Appl. **202** (1996), 843–861
12. Nagel, R. and Piazzera, S.: On the regularity properties of perturbed semigroups. Serie II, Rend. Circ. Mat. Palermo **56** (1998), 99–110
13. van Neerven, J.: The Asymptotic Behaviour of Semigroups of Linear Operators. Birkhäuser, Basel 1996
14. Pazy, A.: On the differentiability and compactness of semigroups and linear operators. J. Math. Mech. **17** (1968), 1131–1141
15. Phillips, R. S.: Perturbation theory for semigroups of linear operators. Trans. Amer. Math. Soc. **74** (1953), 199–221
16. Renardy, M.: On the stability of differentiability of semigroups. Semigroup Forum **51** (1995), 343–346
17. Ruess, W. M. and Summers, W. H.: Compactness in spaces of vector-valued continuous functions and asymptotic almost periodicity. Math. Nachr. **135** (1988), 7–33
18. Voigt, J.: On the perturbation theory for strongly continuous semigroups. Math. Ann. **229** (1977), 163–171

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