

Multi-center non-BPS black holes — the solution

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Multi-center non-BPS black holes — the solution

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ABSTRACT: We construct multi-center, non-supersymmetric four-dimensional solutions describing a rotating $\overline{\text{D6}}$ -D2 black hole and an arbitrary number of D4-D2-D0 black holes in a line. These solutions correspond to an arbitrary number of extremal non-BPS black rings in a Taub-NUT space with a rotating three-charge black hole in the middle. The positions of the centers are determined by solving a set of “bubble” or “integrability” equations that contain cubic polynomials of the inter-center distance, and that allow scaling solutions even when the total four-dimensional angular momentum of the scaling centers is non-zero.

KEYWORDS: Black Holes in String Theory, D-branes

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1 Introduction

Multi-center BPS black-hole solutions in four dimensions, and their five-dimensional counterparts, [1–6] have played a crucial role in several areas of research aimed at understanding the quantum structure of black holes in string theory. These areas include the relation between five-dimensional black rings and four-dimensional black holes [7], the “proof” and “disproof” [8] of the OSV conjecture [9], the construction of smooth horizonless solutions that describe black-hole microstates in the same regime of parameters where the classical black hole exist [10, 11], the construction of entropy enigmas [8], the calculation of index-jumps when crossing walls of marginal stability [12], and the realization that quantum effects can wipe away a macroscopic region of a smooth horizonless low-curvature solution [11, 13].

Given the large amount of knowledge about BPS black holes that has been obtained by studying multi-center solutions, it is natural to ask whether these solutions can be generalized to non-BPS black holes. This problem appears, *a priori*, rather hopeless, given that one is looking for four- or five-dimensional non-supersymmetric solutions of Einstein’s equations that depend on at least two variables, and that these equations generically do not “factorize” into first-order equation (as they do for BPS systems [14, 15]). Indeed, most of the known solutions have been constructed in an “artisanal” fashion, and are either essentially two-centered [16] or have no $E \times B$ interactions between the centers [17].¹

¹Hence, these configurations are more similar in spirit to Majumdar-Papapetrou multi-center solutions than to the ones of [1]

The best target for a systematic construction of multi-center, non-BPS black holes are extremal solutions. Indeed, for single-center configurations the equations underlying these solutions have been shown to factorize [18, 19]. Furthermore, Goldstein and Katmadas have observed [20] that one can construct a specific class of “almost-BPS” solutions by solving the same linear system of equations as for BPS solutions but on a four-dimensional base space of reverse orientation. This observation has led to the explicit construction, in [21], of the seed solution for the most general rotating extremal black hole in $\mathcal{N} = 8$ supergravity in four dimension, and of a solution describing a non-BPS black ring in Taub-NUT. This latter solution descends into four dimensions as a two-centered solution in which one of the centers is a rotating $\overline{\text{D6}}$ -D2 black hole, and the other center is a D4-D2-D0 black hole.

Our purpose in this paper is to extend this construction and build non-BPS solutions that contain a black hole and an arbitrary number of concentric black rings in Taub-NUT. As in [21], these solutions have a non-trivial four-dimensional angular momentum that comes both from the rotation of the black hole and from the $E \times B$ interactions between the black-hole and the black-ring centers, and between black-ring and black-ring centers. Hence, for generic charges our solution can be described in terms of a quiver that has arrows running between every pair of points.

Just as for BPS multi-center solutions, the locations of the centers are not arbitrary: the absence of closed time-like curves and of Dirac strings imposes certain “bubble” or “integrability” equations that these locations must satisfy. However, unlike the BPS bubble equations, that are linear in the inverse of the inter-center distances, the non-BPS bubble equations have denominators that are cubic polynomials in the inter-center distances. Moreover, both the two-centered and the multi-centered solutions have walls of marginal stability in the moduli space, across which the solutions can disappear.

Another important aspect of the non-BPS bubble equations is that they admit scaling solutions. Furthermore, when one of the scaling centers is the rotating black hole at the center of Taub-NUT, the total four-dimensional angular momentum of the scaling centers can remain large throughout the scaling! The throat of the non-BPS scaling solutions then asymptotes to the (intrinsically non-BPS) throat of a rotating four-dimensional black hole. This makes our scaling solutions more general than the BPS ones (whose four-dimensional angular momentum always goes to zero in the scaling limit).

Interestingly enough, in the scaling regime, the non-BPS bubble equations become identical to the BPS bubble equations. For scaling solutions with vanishing four-dimensional angular momentum, this is to be expected: Indeed, as observed in [20, 21], when the Taub-NUT base space degenerates to \mathbb{R}^4 or $\mathbb{R}^3 \times S^1$, the almost-BPS solutions become identical to the BPS ones.² When the centers are very close to each other, the harmonic function that determines the Taub-NUT base can be approximated either by $1/r$ or by a constant (depending on whether the Taub-NUT center is included in the scaling solution or not). Hence, as the centers scale they see a base space that resembles with increasing accuracy the base of a BPS scaling solution, and the non-BPS bubble

²This fact was used extensively in [21] and will also be used here to obtain solutions to the almost-BPS equations by recycling pieces of the BPS solution.

equations asymptote to the BPS bubble equations. Putting it another way, the throat of a non-BPS, non-rotating scaling solution in which the Taub-NUT center participates increasingly resembles the throat a D2-D2-D2- $\overline{D6}$ extremal non-BPS single-center black hole, which is the same as the throat of its D2-D2-D2-D6 BPS cousin [22]. However, it is rather mysterious why, in the presence of four-dimensional angular momentum, the scaling limit of the non-BPS bubble equations is still the same as that of the BPS ones, or, equivalently, why the addition of four-dimensional angular momentum to the black hole center does not affect the non-BPS bubble equations.

In section 2 we find the metric warp factors, the electric and magnetic field strengths, as well as the angular momentum vector of our multi-center solutions. In section 3 we study the regularity conditions imposed by the absence of closed time-like curves (CTC's), and find the “bubble” or “integrability” equations that the positions of the centers must satisfy. We also study regularity at the black-hole and black-ring horizons, and relate the charges that appear in the supergravity solution to quantized charges. We conclude this section by investigating scaling solutions. We present conclusions and potential future directions of research in section 4.

2 Multi-center non-BPS solutions in Taub-NUT

2.1 The Ansatz and the almost-BPS equations

As observed in [20, 21], both BPS and almost BPS solutions of eleven-dimensional supergravity carrying M2 and M5 charges are of the form:

$$ds^2 = -(Z_1 Z_2 Z_3)^{-2/3} (dt + k)^2 + (Z_1 Z_2 Z_3)^{1/3} ds_4^2 + \left(\frac{Z_2 Z_3}{Z_1^2}\right)^{1/3} (dx_1^2 + dx_2^2) + \left(\frac{Z_1 Z_3}{Z_2^2}\right)^{1/3} (dx_3^2 + dx_4^2) + \left(\frac{Z_1 Z_2}{Z_3^2}\right)^{1/3} (dx_5^2 + dx_6^2) \quad (2.1)$$

$$C^{(3)} = \left(a^1 - \frac{dt+k}{Z_1}\right) \wedge dx_1 \wedge dx_2 + \left(a^2 - \frac{dt+k}{Z_2}\right) \wedge dx_3 \wedge dx_4 + \left(a^3 - \frac{dt+k}{Z_3}\right) \wedge dx_5 \wedge dx_6, \quad (2.2)$$

where ds_4^2 is a hyper-Kähler, four-dimensional metric whose curvature we take to be self-dual.

The almost-BPS solutions are given by:

$$\Theta^{(I)} = - *_4 \Theta^{(I)} \quad (2.3)$$

$$d *_4 dZ_I = \frac{C_{IJK}}{2} \Theta^{(I)} \wedge \Theta^{(J)} \quad (2.4)$$

$$dk - *_4 dk = Z_I \Theta^{(I)}, \quad (2.5)$$

where $*_4$ is the Hodge duality operator for the metric ds_4^2 , and the anti-self-dual dipole field strengths are defined as $\Theta^{(I)} \equiv da^I$. Note that if one considers these equations on a hyper-Kähler base with an anti-self-dual curvature, they describe BPS solutions. Supersymmetry is broken only because the curvature of the base and the two-form dipole field strengths have opposite orientations.

2.2 Solutions with a Taub-NUT base

Our purpose is to construct multi-center solutions with a Taub-NUT base:

$$ds_4^2 = V^{-1}(d\psi + A) + V ds_3^2 \tag{2.6}$$

with

$$V = h + \frac{q}{r}, \quad A = q \cos \theta d\phi, \quad ds_3^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \tag{2.7}$$

Let $a_i, i = 1, \dots, N$ denote a succession of points along the z axis in \mathbb{R}^3 , distinct from the Taub-NUT origin ($a_i \neq 0$). In the \mathbb{R}^3 base of the Taub-NUT space, the distance from a given point (r, θ, ϕ) to any one of these points is

$$\Sigma_i = \sqrt{r^2 + a_i^2 - 2ra_i \cos \theta}, \tag{2.8}$$

and the polar angle of that point with respect to the point i is

$$\cos \theta_i = \frac{r \cos \theta - a_i}{\Sigma_i}. \tag{2.9}$$

As shown in [20, 21], the M5 (magnetic) charges are determined by harmonic functions $K^{(I)}$, and we assume that they have generic poles at the points a_i ³

$$K^{(I)} = \sum_{i=1}^N \frac{d_i^{(I)}}{\Sigma_i}. \tag{2.10}$$

The harmonic functions L_I associated with the M2 (electric) charges can have poles both at the points, a_i , and at the Taub-NUT center:

$$L_I = \ell_I + \frac{Q_0^{(I)}}{r} + \sum_i \frac{Q_i^{(I)}}{\Sigma_i} = \ell_I + \sum_{i=0}^N \frac{Q_i^{(I)}}{\Sigma_i}, \tag{2.11}$$

where $\Sigma_0 \equiv r$. A solution of the almost-BPS equations (2.3), (2.4) and (2.5) can now be constructed from these harmonic functions.

2.3 Dipole field strengths

The two-form field strengths, $\Theta^{(I)}$, are closed and anti-self dual in the Taub-NUT space and have the form:

$$\Theta^{(I)} = d[K^{(I)}(d\psi + A) + b^{(I)}], \tag{2.12}$$

where $K^{(I)}$ is given in (2.10) and $b^{(I)}$ is given by

$$*_3 db^{(I)} = V dK^{(I)} - K^{(I)} dV \Rightarrow b^{(I)} = \sum_i \frac{d_i^{(I)}}{\Sigma_i} \left(h(r \cos \theta - a_i) + q \frac{r - a_i \cos \theta}{a_i} \right) d\phi. \tag{2.13}$$

³Allowing $K^{(I)}$ to have poles at $r = 0$ appears to lead to singular solutions.

2.4 Warp factors

The warp factors, Z_I , which encode the M2 charges, are determined by (2.4), and for the dipole field strengths in (2.12) this equation becomes:

$$\square_3 Z_I = V \frac{|\epsilon_{IJK}|}{2} \square_3 (K^{(J)} K^{(K)}) = \left(h + \frac{q}{r} \right) \sum_{j,k} \frac{|\epsilon_{IJK}|}{2} \square_3 \left(\frac{d_j^{(J)} d_k^{(K)}}{\Sigma_j \Sigma_k} \right), \quad (2.14)$$

where sums over repeated J, K indices are implicit (as they will be throughout this paper). It is completely trivial to solve this equation for the terms proportional to h and for the term proportional to q we use the identity:

$$\square_3 \left[\frac{r}{a_i a_j} \frac{1}{\Sigma_i \Sigma_j} \right] = \frac{1}{r} \square_3 \left[\frac{1}{\Sigma_i \Sigma_j} \right]. \quad (2.15)$$

If one also includes the freedom to add to Z_I a generic harmonic function, L_I , given in (2.11), the complete solution for Z_I is

$$Z_I = L_I + \frac{|\epsilon_{IJK}|}{2} \sum_{j,k} \left(h + \frac{qr}{a_j a_k} \right) \frac{d_j^{(J)} d_k^{(K)}}{\Sigma_j \Sigma_k}. \quad (2.16)$$

2.5 The angular momentum one-form

The angular momentum one-form, k , can be decomposed as

$$k = \mu(d\psi + A) + \omega, \quad (2.17)$$

where ω is a one-form on \mathbb{R}^3 . Equation (2.5) then becomes:⁴

$$\begin{aligned} d(V\mu) + *_3 d\omega &= V Z_I dK^{(I)} \\ &= V \sum_i \ell_I d_i^{(I)} d \frac{1}{\Sigma_i} + \left(h + \frac{q}{r} \right) \sum_{i,i'} Q_i^{(I)} d_{i'}^{(I)} \frac{1}{\Sigma_i} d \frac{1}{\Sigma_{i'}} \\ &\quad + \frac{|\epsilon_{IJK}|}{2} \sum_{i,j,k} d_i^{(I)} d_j^{(J)} d_k^{(K)} \left[h^2 + \frac{q^2}{a_j a_k} + hq \left(\frac{1}{r} + \frac{r}{a_j a_k} \right) \right] \frac{1}{\Sigma_j \Sigma_k} d \frac{1}{\Sigma_i}. \end{aligned} \quad (2.18)$$

It is convenient to rewrite the term cubic in $d_i^{(I)}$ as

$$\begin{aligned} &\frac{|\epsilon_{IJK}|}{2} \sum_{i,j,k} d_i^{(I)} d_j^{(J)} d_k^{(K)} \left[h^2 + \frac{q^2}{a_j a_k} + hq \left(\frac{1}{r} + \frac{r}{a_j a_k} \right) \right] \frac{1}{\Sigma_j \Sigma_k} d \frac{1}{\Sigma_i} \\ &= \sum_{i,j,k} d_i^{(1)} d_j^{(2)} d_k^{(3)} (h^2 T_{ijk}^{(1)} + q^2 T_{ijk}^{(2)} + hq T_{ijk}^{(3)}), \end{aligned} \quad (2.19)$$

where

$$\begin{aligned} T_{ijk}^{(1)} &\equiv \frac{1}{\Sigma_j \Sigma_k} d \frac{1}{\Sigma_i} + \frac{1}{\Sigma_i \Sigma_k} d \frac{1}{\Sigma_j} + \frac{1}{\Sigma_i \Sigma_j} d \frac{1}{\Sigma_k} \\ T_{ijk}^{(2)} &\equiv \frac{1}{a_j a_k} \frac{1}{\Sigma_j \Sigma_k} d \frac{1}{\Sigma_i} + \frac{1}{a_i a_k} \frac{1}{\Sigma_i \Sigma_k} d \frac{1}{\Sigma_j} + \frac{1}{a_i a_j} \frac{1}{\Sigma_i \Sigma_j} d \frac{1}{\Sigma_k} \\ T_{ijk}^{(3)} &\equiv \left(\frac{1}{r} + \frac{r}{a_j a_k} \right) \frac{1}{\Sigma_j \Sigma_k} d \frac{1}{\Sigma_i} + \left(\frac{1}{r} + \frac{r}{a_i a_k} \right) \frac{1}{\Sigma_i \Sigma_k} d \frac{1}{\Sigma_j} + \left(\frac{1}{r} + \frac{r}{a_i a_j} \right) \frac{1}{\Sigma_i \Sigma_j} d \frac{1}{\Sigma_k}, \end{aligned} \quad (2.20)$$

⁴All sums over i, i', j, k are from 0 to N , with the convention that $d_0^{(I)} = 0$.

with a_i, a_j, a_k any three, possibly coincident, non-vanishing points. Note that in (2.20) we have explicitly symmetrized over the three source points and so there is an associated factor of $1/3$ but this is canceled in (2.19) by the explicit replacement of $\frac{1}{2}|\epsilon_{IJK}|$.

One can thus reduce the complete solution for μ and ω to the solution of the following equations:

$$\begin{aligned}
 d(V\mu_i^{(1)}) + *_3d\omega_i^{(1)} &= Vd\frac{1}{\Sigma_i} \\
 d(V\mu_i^{(2)}) + *_3d\omega_i^{(2)} &= \frac{1}{\Sigma_i}d\frac{1}{\Sigma_i} & (i \neq 0) \\
 d(V\mu_{ij}^{(3)}) + *_3d\omega_{ij}^{(3)} &= \frac{1}{\Sigma_i}d\frac{1}{\Sigma_j} & (i \neq j) \\
 d(V\mu_i^{(4)}) + *_3d\omega_i^{(4)} &= \frac{1}{r\Sigma_i}d\frac{1}{\Sigma_i} & (i \neq 0) \\
 d(V\mu_{ij}^{(5)}) + *_3d\omega_{ij}^{(5)} &= \frac{1}{r\Sigma_i}d\frac{1}{\Sigma_j} & (i \neq j, j \neq 0) \\
 d(V\mu_{ijk}^{(6)}) + *_3d\omega_{ijk}^{(6)} &= T_{ijk}^{(1)} & (i, j, k \neq 0) \\
 d(V\mu_{ijk}^{(7)}) + *_3d\omega_{ijk}^{(7)} &= T_{ijk}^{(2)} & (i, j, k \neq 0) \\
 d(V\mu_{ijk}^{(8)}) + *_3d\omega_{ijk}^{(8)} &= T_{ijk}^{(3)} & (i, j, k \neq 0).
 \end{aligned} \tag{2.21}$$

A solution to this is:

$$\begin{aligned}
 V\mu_i^{(1)} &= \frac{V}{2\Sigma_i}, & \omega_i^{(1)} &= \frac{hr \cos \theta - a_i}{2\Sigma_i}d\phi + \frac{qr - a_i \cos \theta}{2a_i\Sigma_i}d\phi \\
 V\mu_i^{(2)} &= \frac{1}{2\Sigma_i^2}, & \omega_i^{(2)} &= 0 \\
 V\mu_{ij}^{(3)} &= \frac{1}{2}\frac{1}{\Sigma_i\Sigma_j}, & \omega_{ij}^{(3)} &= \frac{r^2 + a_i a_j - (a_i + a_j)r \cos \theta}{2(a_j - a_i)\Sigma_i\Sigma_j}d\phi \\
 V\mu_i^{(4)} &= \frac{\cos \theta}{2a_i\Sigma_i^2}, & \omega_i^{(4)} &= \frac{r \sin^2 \theta}{2a_i\Sigma_i^2}d\phi \\
 V\mu_{ij}^{(5)} &= \frac{r^2 + a_i a_j - 2a_j r \cos \theta}{2a_j(a_i - a_j)r\Sigma_i\Sigma_j}, & \omega_{ij}^{(5)} &= \frac{r(a_i + a_j \cos 2\theta) - (r^2 + a_i a_j) \cos \theta}{2a_j(a_i - a_j)\Sigma_i\Sigma_j}d\phi \\
 V\mu_{ijk}^{(6)} &= \frac{1}{\Sigma_i\Sigma_j\Sigma_k}, & \omega_{ijk}^{(6)} &= 0 \\
 V\mu_{ijk}^{(7)} &= \frac{r \cos \theta}{a_i a_j a_k \Sigma_i \Sigma_j \Sigma_k}, & \omega_{ijk}^{(7)} &= \frac{r^2 \sin^2 \theta}{a_i a_j a_k \Sigma_i \Sigma_j \Sigma_k}d\phi \\
 V\mu_{ijk}^{(8)} &= \frac{r^2(a_i + a_j + a_k) + a_i a_j a_k}{2a_i a_j a_k r \Sigma_i \Sigma_j \Sigma_k} \\
 \omega_{ijk}^{(8)} &= \frac{r^3 + r(a_i a_j + a_i a_k + a_j a_k) - (r^2(a_i + a_j + a_k) + a_i a_j a_k) \cos \theta}{2a_i a_j a_k \Sigma_i \Sigma_j \Sigma_k}d\phi.
 \end{aligned} \tag{2.22}$$

One can also add to k a solution of the homogeneous equation $dk - *_4dk = 0$, and we consider a such solution with components:

$$V\mu^{(9)} = M, \quad *_3d\omega^{(9)} = -dM, \tag{2.23}$$

where M is a harmonic function that generically can be of the form:

$$M = m + \sum_{i=0}^N \frac{m_i}{\Sigma_i} + \sum_{i=0}^N \alpha_i \frac{\cos \theta_i}{\Sigma_i^2}. \quad (2.24)$$

Note that we have allowed for the possibility of dipole harmonic functions in M because we know, from the two-center solution [21], that these are necessary to obtain a rotating black hole at the Taub-NUT center. The corresponding $\omega^{(9)}$ is:

$$\omega^{(9)} = \kappa d\phi - \sum_{i=0}^N m_i \cos \theta_i d\phi + \sum_{i=0}^N \alpha_i \frac{r^2 \sin^2 \theta}{\Sigma_i^3} d\phi. \quad (2.25)$$

The complete expression for μ and ω is then

$$\begin{aligned} \mu = & \sum_i \ell_I d_i^{(I)} \mu_i^{(1)} + \sum_i Q_i^{(I)} d_i^{(I)} (h\mu_i^{(2)} + q\mu_i^{(4)}) + \sum_{i \neq i'} Q_i^{(I)} d_{i'}^{(I)} (h\mu_{ii'}^{(3)} + q\mu_{ii'}^{(5)}) \\ & + \sum_{i,j,k} d_i^{(1)} d_j^{(2)} d_k^{(3)} (h^2 \mu_{ijk}^{(6)} + q^2 \mu_{ijk}^{(7)} + hq \mu_{ijk}^{(8)}) + \mu^{(9)} \end{aligned} \quad (2.26)$$

$$\begin{aligned} \omega = & \sum_i \ell_I d_i^{(I)} \omega_i^{(1)} + \sum_i Q_i^{(I)} d_i^{(I)} (h\omega_i^{(2)} + q\omega_i^{(4)}) + \sum_{i \neq i'} Q_i^{(I)} d_{i'}^{(I)} (h\omega_{ii'}^{(3)} + q\omega_{ii'}^{(5)}) \\ & + \sum_{i,j,k} d_i^{(1)} d_j^{(2)} d_k^{(3)} (h^2 \omega_{ijk}^{(6)} + q^2 \omega_{ijk}^{(7)} + hq \omega_{ijk}^{(8)}) + \omega^{(9)}, \end{aligned} \quad (2.27)$$

or, more explicitly,

$$\begin{aligned} \mu = & \sum_i \frac{\ell_I d_i^{(I)}}{2\Sigma_i} + \sum_i \frac{Q_i^{(I)} d_i^{(I)}}{2V\Sigma_i^2} \left(h + \frac{q \cos \theta}{a_i} \right) + \sum_{i \neq i'} \frac{Q_i^{(I)} d_{i'}^{(I)}}{2V\Sigma_i \Sigma_{i'}} \left(h + q \frac{r^2 + a_i a_{i'} - 2a_{i'} r \cos \theta}{a_{i'}(a_i - a_{i'})} \right) \\ & + \sum_{i,j,k} \frac{d_i^{(1)} d_j^{(2)} d_k^{(3)}}{V\Sigma_i \Sigma_j \Sigma_k} \left(h^2 + q^2 \frac{r \cos \theta}{a_i a_j a_k} + hq \frac{r^2(a_i + a_j + a_k) + a_i a_j a_k}{2a_i a_j a_k r} \right) + \frac{M}{V}, \end{aligned} \quad (2.28)$$

$$\begin{aligned} \omega = & \sum_i \frac{\ell_I d_i^{(I)}}{2\Sigma_i} \left(h(r \cos \theta - a_i) + q \frac{r - a_i \cos \theta}{a_i} \right) d\phi + \sum_i Q_i^{(I)} d_i^{(I)} \frac{qr \sin^2 \theta}{2a_i \Sigma_i^2} d\phi \\ & + \sum_{i \neq i'} \frac{Q_i^{(I)} d_{i'}^{(I)}}{2(a_{i'} - a_i) \Sigma_i \Sigma_{i'}} \left(h(r^2 + a_i a_{i'} - (a_i + a_{i'}) r \cos \theta) \right. \\ & \quad \left. - q \frac{r(a_i + a_{i'} \cos 2\theta) - (r^2 + a_i a_{i'}) \cos \theta}{a_{i'}} \right) d\phi \\ & + \sum_{i,j,k} \frac{d_i^{(1)} d_j^{(2)} d_k^{(3)}}{a_i a_j a_k \Sigma_i \Sigma_j \Sigma_k} \left(q^2 r^2 \sin^2 \theta \right. \\ & \quad \left. + hq \frac{r^3 + r(a_i a_j + a_i a_k + a_j a_k) - (r^2(a_i + a_j + a_k) + a_i a_j a_k) \cos \theta}{2} \right) d\phi \\ & + \kappa d\phi - \sum_{i=0}^N m_i \cos \theta_i d\phi + \sum_{i=0}^N \alpha_i \frac{r^2 \sin^2 \theta}{\Sigma_i^3} d\phi. \end{aligned} \quad (2.29)$$

3 Regularity

The solutions constructed above satisfy the equations of motion, but are not necessarily regular. Indeed, the angular momentum one-form ω is proportional to $d\phi$, and can have Dirac-Misner string singularities, and these would lead to closed time-like curves (CTC's). One must therefore require ω to vanish on the z -axis (where the ϕ coordinate degenerates). Furthermore, near the poles of the harmonic functions the warp factor and rotation one-form blow up, and this can also lead to CTC's. We now find the conditions that guarantee the absence of CTC's in these two obvious places.

The conditions we will obtain are necessary but not sufficient; to be absolutely sure of regularity and absence of CTC's one must usually check each solution globally and in practice this is usually done individually and numerically. Nevertheless, in our experience (and that of others [23]), when the charges and dipole charges of the rings have the same signs, and there are no Dirac-Misner strings or CTC's at the horizons, the multi-center black ring solution is regular.

3.1 Removing closed time-like curves

We require ω_ϕ to vanish for $\theta = 0$ or π . Looking at the various terms contributing to ω we see that only $\omega^{(1)}$, $\omega^{(3)}$, $\omega^{(5)}$, $\omega^{(8)}$ and $\omega^{(9)}$ are non-vanishing on the z -axis. Their values are:

$$\begin{aligned} \omega_i^{(1)} &= \frac{s_i^{(-)}}{2} \left(h + \frac{q}{a_i} \right) d\phi, & \omega_{ij}^{(3)} &= \frac{s_i^{(-)} s_j^{(-)}}{2(a_j - a_i)} d\phi, & \omega_{ij}^{(5)} &= \frac{s_i^{(-)} s_j^{(-)}}{2a_j(a_j - a_i)} d\phi, \\ \omega_{ijk}^{(8)} &= \frac{s_i^{(-)} s_j^{(-)} s_k^{(-)}}{2a_i a_j a_k} d\phi, & \omega^{(9)} &= \left(\kappa - m_0 - \sum_{i \neq 0} s_i^{(-)} \right) d\phi, \end{aligned} \quad (3.1)$$

at $\theta = 0$, while for $\theta = \pi$ one has:

$$\begin{aligned} \omega_i^{(1)} &= \frac{s_i^{(+)}}{2} \left(-h + \frac{q}{a_i} \right) d\phi, & \omega_{ij}^{(3)} &= \frac{s_i^{(+)} s_j^{(+)}}{2(a_j - a_i)} d\phi, & \omega_{ij}^{(5)} &= -\frac{s_i^{(+)} s_j^{(+)}}{2a_j(a_j - a_i)} d\phi, \\ \omega_{ijk}^{(8)} &= \frac{s_i^{(+)} s_j^{(+)} s_k^{(+)}}{2a_i a_j a_k} d\phi, & \omega^{(9)} &= \left(\kappa + m_0 + \sum_{i \neq 0} s_i^{(+)} \right) d\phi, \end{aligned} \quad (3.2)$$

where we have defined

$$s_i^\pm \equiv \text{sign}(r \pm a_i). \quad (3.3)$$

Hence the absence of Dirac-Misner strings imposes the constraints

$$\begin{aligned} \sum_i \ell_I d_i^{(I)} \frac{s_i^{(-)}}{2} \left(h + \frac{q}{a_i} \right) + \sum_{i \neq i'} Q_i^{(I)} d_{i'}^{(I)} \frac{s_i^{(-)} s_{i'}^{(-)}}{2(a_{i'} - a_i)} \left(h + \frac{q}{a_{i'}} \right) + \\ + hq \sum_{ijk} d_i^{(1)} d_j^{(2)} d_k^{(3)} \frac{s_i^{(-)} s_j^{(-)} s_k^{(-)}}{2a_i a_j a_k} + \kappa - m_0 - \sum_{i \neq 0} s_i^{(-)} m_i = 0, \end{aligned} \quad (3.4)$$

$$\begin{aligned}
 & - \sum_i \ell_I d_i^{(I)} \frac{s_i^{(+)}}{2} \left(h - \frac{q}{a_i} \right) + \sum_{i \neq i'} Q_i^{(I)} d_{i'}^{(I)} \frac{s_i^{(+)} s_{i'}^{(+)}}{2(a_{i'} - a_i)} \left(h - \frac{q}{a_{i'}} \right) + \\
 & + hq \sum_{ijk} d_i^{(1)} d_j^{(2)} d_k^{(3)} \frac{s_i^{(+)} s_j^{(+)} s_k^{(+)}}{2a_i a_j a_k} + \kappa + m_0 + \sum_{i \neq 0} s_i^{(+)} m_i = 0. \quad (3.5)
 \end{aligned}$$

Note that, taking into account the possible values of the signs $s_i^{(\pm)}$, the conditions above imply $N + 2$ independent constraints. One can make these constraints explicit, for example, by solving them with respect to the $N + 2$ variables κ , m_0 and m_i for $i = 1, \dots, N$. If one considers, for definiteness, a configuration in which all the poles a_i lie to the right of the Taub-NUT center ($0 < a_1 < \dots < a_N$), then the regularity constraints are:

$$\kappa = -q \sum_i \frac{\ell_I d_i^{(I)}}{2a_i} - h \sum_{i \neq i'} \frac{Q_i^{(I)} d_{i'}^{(I)}}{2(a_{i'} - a_i)} - hq \sum_{i,j,k} \frac{d_i^{(1)} d_j^{(2)} d_k^{(3)}}{2a_i a_j a_k}, \quad (3.6)$$

$$m_0 = -q \sum_i \frac{\ell_I d_i^{(I)}}{2a_i} - h \sum_i \frac{Q_0^{(I)} d_i^{(I)}}{2a_i} + q \sum_{i \neq i', i \neq 0} \frac{Q_i^{(I)} d_{i'}^{(I)}}{2a_{i'}(a_{i'} - a_i)} - hq \sum_{i,j,k} \frac{d_i^{(1)} d_j^{(2)} d_k^{(3)}}{2a_i a_j a_k}, \quad (3.7)$$

$$\begin{aligned}
 m_i = & \frac{\ell_I d_i^{(I)}}{2} \left(h + \frac{q}{a_i} \right) + \sum_j \frac{1}{2|a_i - a_j|} \left[Q_j^{(I)} d_i^{(I)} \left(h + \frac{q}{a_i} \right) - Q_i^{(I)} d_j^{(I)} \left(h + \frac{q}{a_j} \right) \right] \\
 & + \frac{hq}{2} \left[\frac{d_i^{(1)} d_j^{(2)} d_k^{(3)}}{a_i^3} + \frac{|\epsilon_{IJK}| d_i^{(I)}}{2 a_i} \sum_{j,k} \text{sign}(a_j - a_i) \text{sign}(a_k - a_i) \frac{d_j^{(J)} d_k^{(K)}}{a_j a_k} \right] \quad (i \geq 1) \quad (3.8)
 \end{aligned}$$

where we have used the convention $\text{sign}(0) = 0$.

When there is no black hole and no rotation at the center of Taub-NUT ($Q_0^{(I)} = 0$ and $\alpha_0 = 0$), the metric around $r = 0$ is expected to describe empty space, and hence be completely regular. As both coordinates ψ and ϕ degenerate at $r = 0$, regularity requires that μ and ω vanish. From (2.26) and (2.27) and the regularity relations (3.6), (3.7) and (3.8), one indeed finds that μ and ω must satisfy:

$$\begin{aligned}
 \mu|_{r=0} = & \sum_i \frac{\ell_I d_i^{(I)}}{2a_i} - \sum_{i \neq i', i \neq 0} \frac{Q_i^{(I)} d_{i'}^{(I)}}{2a_{i'}(a_{i'} - a_i)} + h \sum_{i,j,k} \frac{d_i^{(1)} d_j^{(2)} d_k^{(3)}}{2a_i a_j a_k} + \frac{m_0}{q} = 0, \\
 \omega|_{r=0} = & \left[- \sum_i \frac{\ell_I d_i^{(I)}}{2} \left(h + \frac{q \cos \theta}{a_i} \right) + \sum_{i \neq i', i \neq 0} \frac{Q_i^{(I)} d_{i'}^{(I)}}{2(a_{i'} - a_i)} \left(h + \frac{q \cos \theta}{a_{i'}} \right) \right. \\
 & \left. - hq \sum_{i,j,k} \frac{d_i^{(1)} d_j^{(2)} d_k^{(3)} \cos \theta}{2a_i a_j a_k} + \kappa - m_0 \cos \theta + \sum_{i \neq 0} m_i \right] d\phi = 0, \quad (3.9)
 \end{aligned}$$

which are automatically implied by (3.6), (3.7) and (3.8). Hence, these relations are enough to guarantee the regularity of the solution at the center of Taub-NUT space.

3.2 Regularity at the horizons

It is also important to study the geometry in the vicinity of the poles $\Sigma_i = 0$, where, for generic charges and not-too-large angular momenta, we expect to find regular horizons.

For this purpose it is convenient to define

$$I_4 = Z_1 Z_2 Z_3 V - \mu^2 V^2. \quad (3.10)$$

The volume element of the horizon around $\Sigma_i = 0$ is

$$\sqrt{g_{H,i}} = \Sigma_i (I_4 \Sigma_i^2 \sin^2 \theta_i - \omega_\phi^2)^{1/2}. \quad (3.11)$$

Consider first the black hole horizon at $\Sigma_0 \equiv r = 0$. The near-horizon expansion gives

$$I_4 \approx \frac{Q_0^{(1)} Q_0^{(2)} Q_0^{(3)} q - \alpha_0^2 \cos^2 \theta}{r^4}, \quad \omega_\phi \approx \alpha_0 \frac{\sin^2 \theta}{r}, \quad (3.12)$$

and hence

$$\sqrt{g_{H,0}} \approx (Q_0^{(1)} Q_0^{(2)} Q_0^{(3)} q - \alpha_0^2)^{1/2} \sin \theta. \quad (3.13)$$

Thus we find a horizon of finite area⁵ given by:

$$A_{H,0} = (4\pi q)(4\pi)(Q_0^{(1)} Q_0^{(2)} Q_0^{(3)} q - \alpha_0^2)^{1/2}. \quad (3.14)$$

As expected, the black hole at the center is the four-charge rotating black hole constructed in [21], and the parameter α_0 encodes its four-dimensional angular momentum.

Consider now the limiting form of the metric near the i^{th} point (around $\Sigma_i = 0$). After several highly non-trivial cancelations one obtains:

$$I_4 = -2\alpha_i d_i^{(1)} d_i^{(2)} d_i^{(3)} \left(h + \frac{q}{a_i} \right)^2 \frac{\cos \theta_i}{\Sigma_i^5} + O(\Sigma_i^{-4}) \quad (3.15)$$

and

$$\omega_\phi \sim \Sigma_i^{-1}. \quad (3.16)$$

This would lead to closed timelike curves outside the horizon unless the term of order Σ_i^{-5} in I_4 vanishes, which requires:⁶

$$\alpha_i = 0 \quad (i \geq 1). \quad (3.17)$$

When this condition is imposed, each point $\Sigma_i = 0$ is a black ring horizon of area

$$A_H = 16\pi^2 q J_4^{1/2}, \quad (3.18)$$

where J_4 is the usual $E_{7(7)}$ quartic invariant that appears in the black ring horizon area [25]:

$$J_4 = \frac{1}{2} \sum_{I < J} \hat{d}_i^{(I)} \hat{d}_i^{(J)} Q_i^{(I)} Q_i^{(J)} - \frac{1}{4} \sum_I (\hat{d}_i^{(I)})^2 (Q_i^{(I)})^2 - 2\hat{d}_i^{(1)} \hat{d}_i^{(2)} \hat{d}_i^{(3)} \hat{m}_i. \quad (3.19)$$

⁵As usual, area means the spatial measure of the three-dimensional horizon of the five-dimensional black hole.

⁶In the two-center solution of [21] a non-zero value for α_i was required for regularity at the black ring horizon. However, the parameter α_i in [21] differs from the one used here by a constant coming from the gauge choice for $\mu^{(6)}$, and the two results are consistent.

In order to bring J_4 to its canonical form, we have defined the “effective” dipole and angular momentum parameters;⁷

$$\hat{d}_i^{(I)} = \left(h + \frac{q}{a_i} \right) d_i^{(I)}, \quad \hat{m}_i = \left(h + \frac{q}{a_i} \right)^{-1} m_i. \quad (3.21)$$

Note that the result (3.18) and (3.19) coincides with the one for an isolated BPS black ring carrying charges $\hat{d}_i^{(I)}$, Q_i^I and \hat{m}_i : the area of the i^{th} horizon is not affected by the presence of the other horizons nor by the switch of orientation of the base space that is characteristic of our non-BPS solutions.

If one chooses units such that the five-dimensional Newton’s constant is $G_5 = \frac{\pi}{4}$ and the three tori have equal sizes, the integer M2, M5 and KK momentum charges carried by the i^{th} center are:

$$n_i^{(I)} = -\frac{\hat{d}_i^{(I)}}{2}, \quad N_i^{(I)} = \frac{Q_i^{(I)}}{4}, \quad J_i^{(KK)} = -\frac{\hat{m}_i}{8}. \quad (3.22)$$

One can also construct solutions in which some of the centers do not have three M2 charges and three M5 charges, but only two M2 charges and one M5 charge. These solutions describe now two-charge round supertubes [26], and the geometry near an individual supertube is expected to be smooth in the duality frame in which the dipole charge of the tube corresponds to KK-monopoles, and the electric charges correspond to D1 and D5 brane [27–29]).

For the supertube with dipole charge corresponding to, say, K^3 , this regularity condition is [29]:

$$\lim_{\Sigma_i \rightarrow 0} \Sigma_i^2 (Z_3 V (K^{(3)})^2 - 2\mu V K^{(3)} + Z_1 Z_2) = 0. \quad (3.23)$$

Just as for black rings, this requires that the “dipole” harmonic term in M vanish (otherwise $\mu V K_i^3 \sim \Sigma_i^{-3}$):

$$\alpha_i = 0. \quad (3.24)$$

Furthermore, equation (3.23) implies the usual supertube regularity condition:

$$2\hat{d}_i^3 m_i = Q_i^2 Q_i^1. \quad (3.25)$$

3.3 Scaling solutions

Consider the limit in which the positions of the centers are scaled to zero ($a_i \ll \frac{q}{h}$). In this limit the regularity conditions (3.7) and (3.8), when written in terms of the quantized

⁷The “effective angular momentum” that appears in the J_4 parameter of the non-BPS black ring in Taub-NUT constructed in [21] is not \hat{m}_i but

$$\hat{m}_i^{\text{BDGRW}} \equiv \hat{m}_i - \frac{q}{2a_i^2} \left(h + \frac{q}{a_i} \right)^{-2} Q_0^{(I)} \hat{d}_i^{(I)} \quad \Leftrightarrow \quad m_i^{\text{here}} = m_i^{\text{BDGRW}} + \frac{q}{2a_i^2} Q_0^{(I)} d_i^{(I)}. \quad (3.20)$$

We find here, instead, that J_4 simply depends on \hat{m}_i . The two results are consistent because here we are using a different (and more natural) gauge choice originating from a different definition of $\mu_{0i}^{(5)}$ and reflected in the equation for m_i^{here} .

charge parameters $\hat{d}_i^{(I)}$, Q_i^I and \hat{m}_i , reduce to:

$$m_0 = -\sum_i \frac{\ell_I \hat{d}_i^{(I)}}{2} - \frac{h}{q} \sum_i \frac{Q_0^{(I)} \hat{d}_i^{(I)}}{2} + \sum_{i \neq i', i \neq 0} \frac{Q_i^{(I)} \hat{d}_{i'}^{(I)}}{2(a_{i'} - a_i)} - \frac{h}{q^2} \sum_{i,j,k} \frac{\hat{d}_i^{(1)} \hat{d}_j^{(2)} \hat{d}_k^{(3)}}{2}, \quad (3.26)$$

$$q \frac{\hat{m}_i}{a_i} = \frac{\ell_I \hat{d}_i^{(I)}}{2} + \sum_j \frac{1}{2|a_i - a_j|} \left[Q_j^{(I)} \hat{d}_i^{(I)} - Q_i^{(I)} \hat{d}_j^{(I)} \right] + \frac{h}{2q^2} \left[\hat{d}_i^{(1)} \hat{d}_i^{(2)} \hat{d}_i^{(3)} + \frac{|\epsilon_{IJK}|}{2} \hat{d}_i^{(I)} \sum_{j,k} \text{sign}(a_j - a_i) \text{sign}(a_k - a_i) \hat{d}_j^{(J)} \hat{d}_k^{(K)} \right] \quad (i \geq 1). \quad (3.27)$$

These equations are now linear in the inverse of the inter-center distance, much as they are for BPS solutions.

As the parameters $\hat{d}_i^{(I)}$, Q_i^I and \hat{m}_i with $i > 0$ are associated to quantized charges, their value is to be kept finite while the a_i 's are scaled to zero. Note however that m_0 does not correspond to any quantized charge, but is a parameter needed for regularity, as indicated by (3.6). Hence, one should think about equation (3.26) (or (3.7) in the full solution) as determining the value of a parameter of the solution as a function of the charges and the positions of the centers, and about equations (3.27) (or (3.27) in the full solution) as the ‘‘bubble equations’’ of the system, that determine the inter-center distances as a function of the charges and the moduli.

In the small a_i limit, the non-BPS bubble equations become:

$$\sum_j \frac{1}{2|a_i - a_j|} \left[Q_j^{(I)} \hat{d}_i^{(I)} - Q_i^{(I)} \hat{d}_j^{(I)} \right] = q \frac{\hat{m}_i}{a_i}, \quad (3.28)$$

which coincides with the scaling limit of the BPS bubble equations.

4 Conclusions and future directions

We have constructed almost-BPS multi-center solutions that describe a black hole and an arbitrary number of black rings in Taub-NUT. This solution descends to four dimensions to a multi-center configurations containing one rotating $\overline{\text{D6}}\text{-(D2)}^3$ and an arbitrary number of collinear $(\text{D4})^3\text{-(D2)}^3\text{-D0}$ black holes. These solutions admit scaling regimes where some, or all, of the centers get very close to each other (in \mathbb{R}^3 coordinate distance), and the throats of the black holes that are scaling join into a bigger throat. Furthermore, since the bubble equations are insensitive to the four-dimensional rotation of the black hole, we can obtain scaling solutions that have a non-zero four-dimensional angular momentum.

There are several obvious directions for future research. On a technical level, it should be possible to generalize our results to ‘‘tilted’’ black rings where the centers are not collinear in the \mathbb{R}^3 base. Preliminary calculations suggest that while this should in principle be possible, it is technically quite complicated.

On a more fundamental and physical level it would be interesting to determine whether the non-BPS bubble equations can be derived from a microscopic quiver perspective in the

way the BPS ones were derived in [1]. Given the complicated structure of the bubble equations, and the fact that they do not depend on the four-dimensional angular momentum, this would be quite spectacular. It would also hint at the existence of non-renormalization theorems that apply both to BPS and to non-BPS multi-center solutions, and may allow a moduli-space quantization of these solutions similar to that of [13].

It is also interesting to explore the lines of marginal stability in the moduli space of the almost-BPS solutions. Note that, unlike their BPS counterparts, these solutions are completely independent of some of the moduli: for example, Wilson lines along the Taub-NUT fiber at infinity correspond to adding constants to the $K^{(I)}$ harmonic functions, which does not affect at all the metric or the Maxwell fields.

It is equally important to try to use our multi-center almost-BPS solutions to construct smooth horizonless black hole microstate geometries corresponding to microstates of rotating non-BPS four-dimensional black holes. This is however not as straightforward as for BPS solutions. Indeed, if one considers an almost-BPS solution with a multi-center Gibbons-Hawking or Taub-NUT base, the flux on a two-cycle running between two centers is anti-self-dual, and hence non-normalizable. Such solutions thus tend to be unphysical and so such fluxes should be set to zero. If one then builds solutions with multiple D6 centers but without fluxes, these centers are always mutually local (there is no arrow between them in the quiver description), and the solution one builds is uninteresting.

One way to proceed is to relax at first the requirement of smoothness, and to focus rather on “primitive” centers (that correspond to fluxed D4, fluxed D2, or D0 branes, and that preserve locally 16 supercharges). Then our solutions contain, for example, a four-point quiver that has one D6 and three mutually-nonlocal fluxed D4 centers (which can also be thought of as supertubes that have three different kinds of dipole charges). This quiver has arrows running between all centers, admits scaling solutions, and can have overall charges corresponding to a rotating black hole of macroscopic horizon area. Furthermore, one can argue that upon a chain of dualities⁸ this solution can be brought to a duality frame in which it is completely smooth, much like a fluxed D4 (also known as a supertube) sources a smooth supergravity solution in the duality frame where the supertube has KKM dipole charge [27–29]. We believe this route will yield rather generic smooth microstates of non-BPS extremal black holes, and will help extend the fuzzball proposal [30] beyond supersymmetric settings.

Last, but not least it is important to understand the other circumstances in which Einstein’s equations can factorize and one has the hope of constructing solutions systematically. As we will see in a forthcoming publication [31], the almost-BPS solutions are not the most generic non-BPS solutions that factorize, and there may exist routes to classify and find all the extremal multi-center non-BPS solutions one can build in four or five-dimensional supergravity.

⁸For example three spectral flow transformations of the type explored in [24]

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