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Algebraic entropy for valuation domains

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Abstract: Let R be a non-discrete Archimedean valuation domain, G an R-module, $\phi \in \mathrm{End}_R(G)$. We compute the algebraic entropy $\mathrm{ent}_v(\phi)$, when ϕ is restricted to a cyclic trajectory in G. We derive a special case of the Addition Theorem for ent_v, that is proved directly, without using the deep results and the difficult techniques of the paper by Salce and Virili [8].

Keywords: Algebraic entropy; valuation domains

1 Introduction

A sketchy definition of the concept of algebraic entropy for endomorphisms of Abelian groups was given in a 1965 article by Adler-Konheim-McAndrew [1], dedicated to topological entropy. In 1975 this concept was resumed and developed by Weiss [14], in a paper that related algebraic and topological entropies using Pontryagin duality. In the algebraic context, as well as in other areas of mathematics and physics, entropy is viewed as a measure of the "average disorder" created by a transformation when we repeatedly apply it. Following this philosophy, in [4] Dikranjan, Goldsmith, Salce and Zanardo thoroughly investigated the algebraic entropy of [1] and [14]. In [4] the notion was mainly used to get a better understanding of endomorphism rings of Abelian p-groups. Thereafter, Salce and Zanardo [10] defined the algebraic entropy for R a commutative ring, and used the rank as an invariant to deal with the case of torsion-free Abelian groups. Many more papers devoted to algebraic entropy in its different aspects have appeared since; for an ample list of references, see, for instance, [3].

A central result for an algebraic entropy (however defined) is the Addition Theorem. It states that, given an endomorphism ϕ of a left R-module M (R not necessarily commutative), and a ϕ-invariant submodule N of M, the following formula holds

$$\mathrm{ent}(\phi) = \mathrm{ent}(\phi|_N) + \mathrm{ent}(\bar{\phi})$$

where $\bar{\phi}$ is the endomorphism of M/N induced by ϕ. Starting with results in [4] and [10], Salce, Vámos and Virili [7] proved that every algebraic entropy induced by a discrete length function satisfies the Addition Theorem (see the preliminaries for the unexplained notions).

The present note is mainly motivated by a recent paper of Salce and Virili [8], where the authors proved in full generality the Addition Theorem for the algebraic entropy induced by any length function, either non-discrete or discrete. So, this important new result extends and completes the above recalled one of [7], that was proved using methods applicable only to discrete length functions.

Northcott and Reufel [6] defined a length function for modules over a valuation domain R. In the case where R is Archimedean non-discrete, i.e., its value group is a dense ordered subgroup of $\mathbb{R}$, we get the basic example of a non-discrete length function. In the present paper, we in fact consider a valuation domain R of this type, the length function L_v determined by the valuation v of R, and the algebraic entropy ent_v induced by L_v.

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Let G be an R-module, $\phi \in \mathrm{End}_R(G)$. Our main purpose is to compute $\mathrm{ent}_v(\phi)$, when ϕ is restricted to a cyclic trajectory in G (see the definitions in the first section). Of course, this computation, made in Theorem 4.5, may be helpful to determine $\mathrm{ent}_v(\phi)$ on all of G. It is worth noting that a thorough knowledge of finitely generated modules over valuation domains is needed to get this result. We remark that Proposition 4.4 was used in [8] to prove the Uniqueness Theorem 7.3.

From Theorem 4.5 we also derive a special case of the Addition Theorem for ent_v, namely Proposition 4.7, that is proved directly, without using the deep results and the difficult techniques of [8].

2 Preliminaries and definitions.

For the notions and basic results on the theory of valuation domains and their modules, in particular the finitely generated ones, we refer to the book by Fuchs and Salce [5].

A general notion of length function was introduced by Northcott and Reufel [6]. This concept was investigated by Vámos [11], [12], by the author [15], and by Virili [13], in the context of Grothendieck categories. We recall below the definition.

Let R be any ring. A length function L on $\mathrm{Mod}\,R$ is a map $L : \mathrm{Mod}\,R \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that $L(0) = 0$, $L(M) = L(M')$ whenever $M \cong M'$, and

(1) L is additive, that is, for every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathrm{Mod}\,R$ we have $L(B) = L(A) + L(C)$;

(2) L is upper continuous, that is, for any $M \in \mathrm{Mod}\,R$, $L(M)$ is the sup of the lengths $L(N)$, where N ranges among the finitely generated submodules of M.

L is said to be discrete if its image (without ∞) form an ordered semigroup of $\mathbb{R}_{\geq 0}$ isomorphic to $\mathbb{N}$; otherwise, L is called non-discrete.

In what follows, R is an Archimedean valuation domain, i.e., the value group Γ_v of R is an ordered subgroup of $\mathbb{R}$. We denote by Q the field of fractions of R, by v the valuation on Q determined by R, and by P the maximal ideal of R, i.e., $P = \{x \in Q : v(x) > 0\}$. Of course, under the present circumstances, P is the unique nonzero prime ideal of R. We confine ourselves to non-discrete valuations, i.e., Γ_v is a dense subgroup of $\mathbb{R}$. If I is an ideal of the valuation domain R, we set $v(I) = \inf\{v(r) : r \in I\}$.

A submodule N of the R-module M is said to be *pure* if $N \cap rM = rN$ for every $r \in R$ (see [5], Ch.I). For M an R-module, we denote by $\mathrm{Ann}(M)$ its annihilator. We define $\mathrm{gen}\,M$ to be the minimal cardinality of a generating system of M, when M is finitely generated; otherwise, we set $\mathrm{gen}\,M = \infty$.

The next results on finitely generated modules over a valuation domain R were firstly proved in [9]; a neat discussion on this subject may be found in [5], Ch.V.5. Let X be a finitely generated R-module; then $\mathrm{gen}\,X = \dim_{R/P}(X/PX)$. Say $\mathrm{gen}\,X = n$, and let $\{x_1, \ldots, x_n\}$ be a generating set for X. Then there exists a reordering $z_i = x_{\tau(i)}$ of the generators x_i (τ a suitable permutation of $\{1, \ldots, n\}$) such that, setting $Z_0 = 0$ and $Z_i = \langle z_1, \ldots z_i \rangle$, for $1 \leq i \leq n$, the following properties are satisfied

(a) each Z_i is pure in X;

(b) for $1 \leq i \leq n$, Z_i/Z_{i-1} is isomorphic to R/A_i, where

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n.$$

The above sequence of ideals is determined by X, and is called the *annihilator sequence* of X. Note that $A_i = \mathrm{Ann}(z_i + Z_{i-1})$. We say that

$$0 = Z_0 < Z_1 < \cdots < Z_{n-1} < Z_n = X$$

is a *pure-composition series* of X.

Now we recall some basic facts proved in [6] (see also [15]).

(1) There exists a non-trivial length function L_v on Mod R such that $L_v(R/I) = v(I)$, for every ideal I of R. In particular, $L_v(R) = \infty$.

(2) If X is finitely generated with annihilator sequence $A_1, \dots, A_n$, then $L_v(X) = \sum_{i=1}^{n} v(A_i)$.

Since we assume that Γ_v is a dense subgroup of $\mathbb{R}$, it is clear that L_v is not discrete.

Let us observe that $L_v(M) = \infty$ whenever M is not a torsion module: in fact, in that case M contains a copy of R, hence $\infty = L_v(R) \le L_v(M) \le \infty$. It follows that L_v is not significant outside the category of the torsion R-modules.

We recall now the usual definition of algebraic entropy induced by a length function, applying it to L_v (cf. [10]).

Let G be an R-module and denote by $fL(G)$ the family of its submodules N such that $L_v(N)$ is finite (we just say that N has finite length). If $\phi \in \mathrm{End}_R(G)$, for every positive integer n and every $F \in fL(G)$ we set

$$T_n(\phi, F) = F + \phi F + \phi^2 F + \dots + \phi^{n-1} F.$$

$T_n(\phi, F)$ is said to be the partial n-trajectory of F.

The submodule of G

$$T(\phi, F) = \sum_{n>0} T_n(\phi, F) = \sum_{n\ge 0} \phi^n F$$

will be called the ϕ-trajectory of F. The ϕ-trajectory of an element x is just the ϕ-trajectory of the cyclic submodule Rx, and, obviously, it coincides with the smallest ϕ-invariant submodule of G containing x. It is simply denoted by $T(\phi, x)$, and called the cyclic ϕ-trajectory generated by x.

Given the submodule F of finite length and the endomorphism ϕ of G, for each $n \ge 1$ we take the real number

$$H_n(\phi, F) = L_v(T_n(\phi, F)),$$

and define

$$H(\phi, F) = \lim_{n\to\infty} \frac{H_n(\phi, F)}{n}.$$

Since L_v is a length function, the above limit exists and is finite.

We define the algebraic entropy induced by L_v (*algebraic v-entropy*, for short) of an endomorphism ϕ of G as

$$\mathrm{ent}_v(\phi) = \sup_{F\in fL(G)} H(\phi, F).$$

Property (2) says that every finitely generated torsion R-module has finite length. The converse is not true when the valuation v is non-discrete. Indeed, under the present circumstances we have $v(P) = 0$, hence, by the definition, $L_v(R/P) = 0$, and therefore any R/P-vector space V has length 0, not just the finitely generated ones. Less obvious examples of R-modules not finitely generated but with finite length may be found in Example 2.2 of [10] (see also Example 4.5 of [8]).

Naturally, it should be desirable to compute the algebraic entropy taking not all the submodules of finite length, but just the finitely generated ones. This is possible by Proposition 4.2 of [8], proved for any ring R and any length function L of Mod R. We restate that result just for the case we are interested in.

Proposition 2.1. ([8]) *In the above notation, if G is a torsion R-module and $\phi \in \mathrm{End}_R(G)$, we have*

$$\mathrm{ent}_v(\phi) = \sup_{N\in\mathcal{F}(G)} H(\phi, N),$$

where $\mathcal{F}(G)$ denotes the set of the finitely generated submodules of G.

In what follows, we will always compute the entropy confining ourselves to finitely generated torsion submodules. Indeed, if M contains a torsion-free finitely generated submodule, then $\mathrm{ent}_v(\phi) = \infty$ for every $0 \ne \phi \in \mathrm{End}_R(G)$. Note also that $\mathrm{ent}_v(\phi) = 0$ for every $0 \ne \phi \in \mathrm{End}_R(N)$, when N is a finitely generated torsion R-module: indeed, in this case ϕ is annihilated by a monic polynomial with coefficients in R, and therefore $T(\phi, N)$ is finitely generated (cf. the proof of Proposition 4.1).

We end this section observing that ent_v satisfies the standard basic properties of every algebraic entropy induced by a length function (e.g., see [10]). We don't recall those properties, since we won't need them.

3 Computation of v-entropies.

The aim of this section is to get a good description of the cyclic ϕ-trajectories contained in the torsion R-module G, where $\phi \in \mathrm{End}_R(G)$.

To simplify the notation, in what follows, for assigned $\phi \in \mathrm{End}_R(G)$ and $x \in G$, we will denote by T_n the n-partial trajectory $T_n(\phi, x)$.

Proposition 3.1. *Let G be a torsion R-module, $\phi \in \mathrm{End}_R(G)$, $x \in G$. If the ϕ-trajectory $T(\phi, x)$ is not finitely generated, then* $\mathrm{gen}\, T_n = \mathrm{gen}\langle x, \phi x, \dots, \phi^{n-1}x\rangle = n$ *for every* $n > 0$.

Proof. Let us assume, for a contradiction, that $\mathrm{gen}\, T_m < m$ for some $m > 0$. Then $\sum_{i=0}^{m-1} a_i\phi^i x = 0$, for suitable $a_i \in R$ not all in P. From this relation it follows that there exist polynomials p_0, p_1, with p_0 monic, such that $p_0(\phi)x = \lambda p_1(\phi)x$, where $\lambda \in P$. Now take $k > 0$ large enough such that $\lambda^k x = 0$. This is possible, since R is archimedean. Then we get $p_0^k(\phi)x = p_1^k(\phi)(\lambda^k x) = 0$. We conclude that x is annihilated by $p_0^k(\phi)$, where p_0^k is a monic polynomial, of degree h, say, and so $T_n \subseteq T_{h+1}$ for every $n > 0$. It follows that $T(\phi, x) = T_{h+1}$ is finitely generated, impossible. □

The following two lemmas are crucial to get the next Proposition 4.4.

Lemma 3.2. *Let G be a torsion R-module, $\phi \in \mathrm{End}_R(G)$, $y \in G$. If $y = \lambda\sum_{i=1}^m a_i\phi^i y$, for some $m > 0$, $a_i \in R$ and $\lambda \in P$, then $y = 0$.*

Proof. We assume, for a contradiction, that $y \neq 0$, hence $0 \neq \mathrm{Ann}\, y \neq R$. Since R is archimedean, there exists $r \notin \mathrm{Ann}\, y$ such that $\lambda r y = 0$. Then we get

$$0 \neq ry = \lambda r\sum_{i=1}^{n} a_i\phi^i y = \sum_{i=1}^{n} a_i\phi^i(\lambda r y) = 0.$$

We reached a contradiction. □

Lemma 3.3. *Let G be a torsion R-module, $\phi \in \mathrm{End}_R(G)$, $y \in G$. If $\phi^k y = \sum_{0\le i<k} a_i\phi^i y + \lambda\sum_{j>k} b_j\phi^j y$, for some $a_i, b_j \in R$ (almost all zero) and $\lambda \in P$, then $\phi^k y = \sum_{0\le i<k} c_i\phi^i y$, for suitable $c_i \in R$.*

Proof. Firstly we prove that, for every $n \ge k$, we can write a relation

$$\phi^n y = \sum_{0\le i<k} a_{ni}\phi^i y + \lambda\sum_{j>k} b_{nj}\phi^j y,$$

for suitable $a_{ni}, b_{nj} \in R$, almost all zero. We make induction on n. The hypothesis yields the case $n = k$. Assume that the above relation is valid. Then we get

$$\begin{aligned}\phi^{n+1}y &= \phi(\sum_{0\le i<k-1} a_{ni}\phi^i y) + a_{nk}\phi^k y + \phi(\lambda\sum_{j>k} b_{nj}\phi^j y)\\ &= \sum_{0\le i<k-1} a_{ni}\phi^{i+1}y + a_{nk}(\sum_{0\le i<k} a_i\phi^i y + \lambda\sum_{j>k} b_j\phi^j y) + \lambda\sum_{j>k} b_{nj}\phi^{j+1}y\\ &= \sum_{0\le i<k} a_{n+1,i}\phi^i y + \lambda\sum_{j>k} b_{n+1,j}\phi^j y,\end{aligned}$$

for suitable $a_{n+1,i}, b_{n+1,j} \in R$, almost all zero.

Now we prove that, for every $m > 0$, we can write a relation

$$\phi^k y = \sum_{0\le i<k} c_{mi}\phi^i y + \lambda^m\sum_{n>k} d_{mn}\phi^n y,$$

for suitable $c_{mi}, d_{mn} \in R$, almost all zero. We make induction on m. The case $m = 1$ is covered by the hypothesis. Assume that the above relation is valid. Since $\phi^n y = \sum_{0\le i<k} a_{ni}\phi^i y + \lambda \sum_{j>k} b_{nj}\phi^j y$ for every $n > k$, we readily see that

$$\phi^k y = \sum_{0\le i<k} c_{m+1,i}\phi^i y + \lambda^{m+1}\sum_{n>k} d_{m+1,n}\phi^n y,$$

for suitable $c_{m+1,i}, d_{m+1,n} \in R$, almost all zero.

Since R is archimedean, there exists $t > 0$ such that $\lambda^t y = 0$. Under this circumstance, we get $\lambda^t \sum_{n>k} d_{tn}\phi^n y = \sum_{n>k} d_{tn}\phi^n(\lambda^t y) = 0$, whence $\phi^k y = \sum_{0\le i<k} c_{t,i}\phi^i y$, as required. □

Proposition 3.4. *Let G be a torsion R-module, $\phi \in \mathrm{End}_R(G)$, $x \in G$, and assume that $T(\phi, x)$ is not finitely generated. Then T_k is a pure submodule of T_n, for any $0 \le k < n$. In particular,*

$$T_0 = 0 < T_1 < \cdots < T_{n-1} < T_n$$

is a pure-composition series of T_n, with increasing annihilator sequence.

Proof. We make induction on k. Let us prove that $T_1 = Rx$ is pure in $T_n = \langle x, \phi x, \dots, \phi^{n-1}x\rangle$. We assume, for a contradiction, that Rx is not pure in T_n. Then there exists a relation $0 \ne ax = r\sum_{j\ge 1} a_j\phi^j x$ ($a, a_i, r \in R$), where $ax \notin rRx$, i.e., $r = a\lambda$ for some $\lambda \in P$. We set $y = ax$. Then we get $y = \lambda\sum_{j>1} a_i\phi^j y$, so $y = 0$ by Lemma 4.2, a contradiction.

Let the statement be true for $k - 1$, and assume, for a contradiction, that T_k is not pure in T_n. Then there exists a relation

$$0 \ne \sum_{i=0}^{k} b_i\phi^i x = r\sum_{j>k} a_j\phi^j x$$

where $\sum_{i=0}^k b_i\phi^i x \notin rT_k$. Note also that $\sum_{i=0}^k b_i\phi^i x \notin T_{k-1}$ (equivalently $b_k\phi^k x \notin T_{k-1}$), since T_{k-1} is pure in T_n, by induction. There exists a suitable $i \le k$ such that $v(r) > v(b_i)$. Let us observe that $v(b_k) < v(r)$, otherwise $b_k\phi^k x \in rT_k$, hence $\sum_{i<k} b_i\phi^i x \in rT_n \cap T_{k-1} = rT_{k-1}$, and so $\sum_{i=0}^k b_i\phi^i x \in rT_k$, impossible. It follows that $\sum_{i<k} b_i\phi^i x \in b_kT_n \cap T_{k-1} = b_kT_{k-1}$. In conclusion, we may assume that $c_i = b_i/b_k \in R$ for every $i < k$, and $ra_j/b_k = \lambda d_j$, where $d_j \in R$ and $\lambda \in P$, for every $j > k$.

Let us set $y = b_kx$. The above discussion shows that

$$\phi^k y = -\sum_{0\le i<k} c_i\phi^i y + \lambda\sum_{j>k} d_j\phi^j y,$$

hence $\phi^k y \in \langle y, \phi y, \dots, \phi^{k-1}y\rangle \le T_{k-1}$, in view of Lemma 4.3. But $\phi^k y = b_k\phi^k x \notin T_{k-1}$, a contradiction. We conclude that T_k is necessarily pure in T_n.

In particular, it follows that

$$T_0 = 0 < T_1 < \cdots < T_{n-1} < T_n$$

is a pure-composition series of T_n. It remains to show that this pure-composition series has increasing annihilator sequence. For $0 < k \le n$, let $A_k = \mathrm{Ann}(T_k/T_{k-1})$. Pick $r \in A_k$; then, from $r\phi^{k+1}x = \phi(r\phi^k x) \in \phi T_{k-1} \le T_k$, we get $r \in A_{k+1}$. We conclude that $A_k \subseteq A_{k+1}$ and therefore $A_1 = \mathrm{Ann}(x) \subseteq A_2 \subseteq \cdots \subseteq A_n$. □

We are in the position to compute the v-entropy of ϕ restricted to the cyclic ϕ-trajectory generated by $x \in G$. Naturally, we consider only the non-trivial case where $T(\phi, x)$ is not finitely generated.

Theorem 3.5. *Let G be a torsion R-module, $\phi \in \mathrm{End}_R(G)$, $x \in G$. For every $j > 0$, let $A_j = \mathrm{Ann}(T_j/T_{j-1})$. If $T = T(\phi, x)$ is not finitely generated, then $\mathrm{ent}_v(\phi|_T) = v(A)$, where $A = \bigcup_{j>0} A_j$.*

Proof. To simplify the symbols, in the proof we will denote $\phi|_T$ by ϕ. Since every finitely generated submodule of T is contained in some T_j, we have

$$\mathrm{ent}_v(\phi) = \sup_j\{H(\phi, T_j)\}$$

Note that $T_n(\phi, T_j) = T_{n+j-1}$, and recall that $v(A_1) \geq v(A_2) \geq \dots$ and $v(A) = \inf_k v(A_k)$.

Now take any integer $k > 0$. If $n + j - 1 > k$, by Proposition 4.4 and (2), we get

$$L_v(T_{n+j-1}) = \sum_{i=1}^{n+j-1} v(A_i) = \sum_{i=1}^{k-1} v(A_i) + \sum_{i=k}^{n+j-1} v(A_i) \leq \sum_{i=1}^{k-1} v(A_i) + (n + j - 1 - k)v(A_k),$$

hence

$$H(\phi, T_j) = \lim_{n\to\infty} \frac{L_v(T_{n+j-1})}{n} \leq v(A_k).$$

Since j and k were arbitrary, we get

$$\sup_j\{H(\phi, T_j)\} = \mathrm{ent}_v(\phi) \leq \inf_k v(A_k) = v(A).$$

Let us verify the reverse inequality $\mathrm{ent}_v(\phi) \geq v(A)$. In fact, $L_v(T_{n+j-1}) \geq (n + j - 1)v(A)$, whence

$$\lim_{n\to\infty} \frac{L_v(T_{n+j-1})}{n} \geq \lim_{n\to\infty} \frac{(n + j - 1)v(A)}{n} = v(A).$$

□

A result less precise but more general than Theorem 4.5 has some interest.

Proposition 3.6. *Let ϕ be an endomorphism of the torsion R-module G such that $G = T(\phi, F)$ for a suitable $F \in \mathcal{F}(G)$. Then* $\mathrm{ent}(\phi) \leq L_v(F)$. *In particular,* $\mathrm{ent}(\phi) < \infty$.

Proof. Since $G = T(\phi, F)$, every finitely generated submodule of G is contained in some $T_j = T_j(\phi, F)$. Observe that $T_n(\phi, T_j) = T_{n+j-1}$. Recall that $L_v(A+B) \leq L_v(A)+L_v(B)$ for all R-modules A, B, by the additivity property (1). It follows that $L_v(T_{n+j-1}) \leq (n + j - 1)L_v(F)$, since $L_v(\phi^k F) \leq L_v(F)$ for $k \geq 0$. Then

$$H(\phi, T_j) = \lim_{n\to\infty} \frac{L_v(T_{n+j-1})}{n} \leq \lim_{n\to\infty} \frac{(n + j - 1)L_v(F)}{n} = L_v(F).$$

The desired conclusion follows, since $j > 0$ was arbitrary. □

In our final result we get a special case of the Addition Theorem, namely, when the R-module G coincides with the cyclic trajectory $T(\phi, x)$, for some $\phi \in \mathrm{End}_R(G)$, $x \in G$, and the ϕ-invariant submodule coincides with aG, $a \in P$. We use a direct argument, based on Theorem 4.5, that is much simpler and shorter than the discussion leading to the proof of the general result in [8].

Proposition 3.7. *Let G be a torsion R-module, not finitely generated, such that $G = T(\phi, x)$ for suitable $\phi \in \mathrm{End}_R(G)$ and $x \in G$. Take any $a \in P$, and consider the fully invariant submodule $aG = T(\phi, ax)$ and the induced map $\bar{\phi} : G/aG \to G/aG$. Then we have*

$$\mathrm{ent}_v(\phi) = \mathrm{ent}_v(\phi|_{aG}) + \mathrm{ent}_v(\bar{\phi}) \qquad (\dagger)$$

Proof. For $n > 0$, let $T_n = \langle x, \phi x, \dots, \phi^{n-1}x\rangle$ and $A_n = \mathrm{Ann}(T_n/T_{n-1})$. We have seen in Theorem 4.5 that $\mathrm{ent}_v(\phi) = v(A)$, where $A = \bigcup_{j>0} A_j$. We firstly assume that aG is finitely generated. Then $\mathrm{ent}_v(\phi|_{aG}) = 0$ and there exists $k > 0$ such that $T_k \geq aG$. Consider the partial $\bar{\phi}$-trajectory $X_n = T_n(\bar{\phi}, T_k/aG) \leq G/aG$. Since $T_k \geq aG$, it follows that $T_j/T_k \cong (T_j/aG)/(T_k/aG)$ for every $j > k$, and therefore the last terms of the increasing annihilator sequence of X_n are $A_{k+1} \subseteq A_{k+2} \subseteq \dots \subseteq A_{k+n}$. Under the present circumstances, an argument similar to the proof of Theorem 4.5 shows that $\mathrm{ent}_v(\bar{\phi}) = v(A)$, hence (†) holds in this case.

Now we assume that $aG = T(\phi, ax)$ is not finitely generated; equivalently, $\mathrm{gen}(aT_j) = j$ for every $j > 0$. Note that aG not finitely generated implies that $a \notin A_j$, i.e., $aR \supset A_j$, for every $j > 0$. Since T_n is pure in T_{n+1}, a simple computation shows that aT_j has annihilator sequence $a^{-1}A_1 \subseteq a^{-1}A_2 \subseteq \dots \subseteq a^{-1}A_j$. Then Theorem 4.5, applied to $y = ax$ yields $\mathrm{ent}_v(\phi|_{aG}) = v(a^{-1}A) = v(A) - v(a)$. Thus, in order to verify (†), it remains to show that $\mathrm{ent}_v(\bar{\phi}) = v(a)$. Let us verify that

$$G/aG = T(\bar{\phi}, x + aG) = \bigoplus_{n\geq 0} \langle \phi^n x + aG\rangle \cong \bigoplus R/aR.$$

Assume, for a contradiction, that the sum is not direct. Then there exists $k > 0$ such that

$$b_k\phi^k x - \sum_{i=0}^{k-1} b_i\phi^i x = a\sum_{i=0}^{m} c_i\phi^i x \in aG,$$

where $b_k \notin aR$. If $m > k$, we get $ac_m \in A_m$, hence $ac_m\phi^m x = ac_m \sum_{i=0}^{m-1} d_i\phi^i x$, since T_{m-1} is pure in T_m. Thus we get a shorter relation

$$(b_k - ac_m d_k)\phi^k x - \sum_{i=0}^{k-1} b_i\phi^i x = a\sum_{i=0}^{m-1}(c_i + c_m d_i)\phi^i x \in aG,$$

where $b_k - ac_m d_k \notin aR$. Repeating the procedure, after a finite number of steps we get an equality of the form

$$e_k\phi^k x - \sum_{i=0}^{k-1} e_i\phi^i x = 0,$$

where $e_k \notin aR$. But this means that $e_k \in A_k \subset aR$, impossible. We reached a contradiction; then, necessarily, the sum is direct.

The desired conclusion follows, applying Theorem 4.5 to G/aG and $\bar{\phi}$. □

4 Computation of *v*-entropies.

The aim of this section is to get a good description of the cyclic ϕ-trajectories contained in the torsion R-module G, where $\phi \in \mathrm{End}_R(G)$.

To simplify the notation, in what follows, for assigned $\phi \in \mathrm{End}_R(G)$ and $x \in G$, we will denote by T_n the n-partial trajectory $T_n(\phi, x)$.

Proposition 4.1. *Let G be a torsion R-module, $\phi \in \mathrm{End}_R(G)$, $x \in G$. If the ϕ-trajectory $T(\phi, x)$ is not finitely generated, then* $\mathrm{gen}\, T_n = \mathrm{gen}\langle x, \phi x, \dots, \phi^{n-1}x\rangle = n$ *for every* $n > 0$.

Proof. Let us assume, for a contradiction, that $\mathrm{gen}\, T_m < m$ for some $m > 0$. Then $\sum_{i=0}^{m-1} a_i\phi^i x = 0$, for suitable $a_i \in R$ not all in P. From this relation it follows that there exist polynomials p_0, p_1, with p_0 monic, such that $p_0(\phi)x = \lambda p_1(\phi)x$, where $\lambda \in P$. Now take $k > 0$ large enough such that $\lambda^k x = 0$. This is possible, since R is archimedean. Then we get $p_0^k(\phi)x = p_1^k(\phi)(\lambda^k x) = 0$. We conclude that x is annihilated by $p_0^k(\phi)$, where p_0^k is a monic polynomial, of degree h, say, and so $T_n \subseteq T_{h+1}$ for every $n > 0$. It follows that $T(\phi, x) = T_{h+1}$ is finitely generated, impossible. □

The following two lemmas are crucial to get the next Proposition 4.4.

Lemma 4.2. *Let G be a torsion R-module, $\phi \in \mathrm{End}_R(G)$, $y \in G$. If $y = \lambda\sum_{i=1}^{m} a_i\phi^i y$, for some $m > 0$, $a_i \in R$ and $\lambda \in P$, then $y = 0$.*

Proof. We assume, for a contradiction, that $y \neq 0$, hence $0 \neq \mathrm{Ann}\, y \neq R$. Since R is archimedean, there exists $r \notin \mathrm{Ann}\, y$ such that $\lambda r y = 0$. Then we get

$$0 \neq ry = \lambda r\sum_{i=1}^{n} a_i\phi^i y = \sum_{i=1}^{n} a_i\phi^i(\lambda r y) = 0.$$

We reached a contradiction. □

Lemma 4.3. *Let G be a torsion R-module, $\phi \in \mathrm{End}_R(G)$, $y \in G$. If $\phi^k y = \sum_{0\le i<k} a_i\phi^i y + \lambda\sum_{j>k} b_j\phi^j y$, for some $a_i, b_j \in R$ (almost all zero) and $\lambda \in P$, then $\phi^k y = \sum_{0\le i<k} c_i\phi^i y$, for suitable $c_i \in R$.*

Proof. Firstly we prove that, for every $n \geq k$, we can write a relation

$$\phi^n y = \sum_{0 \leq i < k} a_{ni} \phi^i y + \lambda \sum_{j > k} b_{nj} \phi^j y,$$

for suitable $a_{ni}, b_{nj} \in R$, almost all zero. We make induction on n. The hypothesis yields the case $n = k$. Assume that the above relation is valid. Then we get

$$\begin{aligned} \phi^{n+1} y &= \phi(\sum_{0 \leq i < k-1} a_{ni} \phi^i y) + a_{nk} \phi^k y + \phi(\lambda \sum_{j>k} b_{nj} \phi^j y) \\ &= \sum_{0 \leq i < k-1} a_{ni} \phi^{i+1} y + a_{nk} (\sum_{0 \leq i < k} a_i \phi^i y + \lambda \sum_{j>k} b_j \phi^j y) + \lambda \sum_{j>k} b_{nj} \phi^{j+1} y \\ &= \sum_{0 \leq i < k} a_{n+1,i} \phi^i y + \lambda \sum_{j>k} b_{n+1,j} \phi^j y, \end{aligned}$$

for suitable $a_{n+1,i}, b_{n+1,j} \in R$, almost all zero.

Now we prove that, for every $m > 0$, we can write a relation

$$\phi^k y = \sum_{0 \leq i < k} c_{mi} \phi^i y + \lambda^m \sum_{n>k} d_{mn} \phi^n y,$$

for suitable $c_{mi}, d_{mn} \in R$, almost all zero. We make induction on m. The case $m = 1$ is covered by the hypothesis. Assume that the above relation is valid. Since $\phi^n y = \sum_{0 \leq i < k} a_{ni} \phi^i y + \lambda \sum_{j>k} b_{nj} \phi^j y$ for every $n > k$, we readily see that

$$\phi^k y = \sum_{0 \leq i < k} c_{m+1,i} \phi^i y + \lambda^{m+1} \sum_{n>k} d_{m+1,n} \phi^n y,$$

for suitable $c_{m+1,i}, d_{m+1,n} \in R$, almost all zero.

Since R is archimedean, there exists $t > 0$ such that $\lambda^t y = 0$. Under this circumstance, we get $\lambda^t \sum_{n>k} d_{tn} \phi^n y = \sum_{n>k} d_{tn} \phi^n (\lambda^t y) = 0$, whence $\phi^k y = \sum_{0 \leq i < k} c_{t,i} \phi^i y$, as required. □

Proposition 4.4. *Let G be a torsion R-module, $\phi \in \mathrm{End}_R(G)$, $x \in G$, and assume that $T(\phi, x)$ is not finitely generated. Then T_k is a pure submodule of T_n, for any $0 \leq k < n$. In particular,*

$$T_0 = 0 < T_1 < \cdots < T_{n-1} < T_n$$

is a pure-composition series of T_n, with increasing annihilator sequence.

Proof. We make induction on k. Let us prove that $T_1 = Rx$ is pure in $T_n = \langle x, \phi x, \ldots, \phi^{n-1} x \rangle$. We assume, for a contradiction, that Rx is not pure in T_n. Then there exists a relation $0 \neq ax = r \sum_{j \geq 1} a_j \phi^j x$ ($a, a_i, r \in R$), where $ax \notin rRx$, i.e., $r = a\lambda$ for some $\lambda \in P$. We set $y = ax$. Then we get $y = \lambda \sum_{j>1} a_i \phi^j y$, so $y = 0$ by Lemma 4.2, a contradiction.

Let the statement be true for $k - 1$, and assume, for a contradiction, that T_k is not pure in T_n. Then there exists a relation

$$0 \neq \sum_{i=0}^{k} b_i \phi^i x = r \sum_{j>k} a_j \phi^j x$$

where $\sum_{i=0}^{k} b_i \phi^i x \notin rT_k$. Note also that $\sum_{i=0}^{k} b_i \phi^i x \notin T_{k-1}$ (equivalently $b_k \phi^k x \notin T_{k-1}$), since T_{k-1} is pure in T_n, by induction. There exists a suitable $i \leq k$ such that $v(r) > v(b_i)$. Let us observe that $v(b_k) < v(r)$, otherwise $b_k \phi^k x \in rT_k$, hence $\sum_{i<k} b_i \phi^i x \in rT_n \cap T_{k-1} = rT_{k-1}$, and so $\sum_{i=0}^{k} b_i \phi^i x \in rT_k$, impossible. It follows that $\sum_{i<k} b_i \phi^i x \in b_k T_n \cap T_{k-1} = b_k T_{k-1}$. In conclusion, we may assume that $c_i = b_i / b_k \in R$ for every $i < k$, and $ra_j / b_k = \lambda d_j$, where $d_j \in R$ and $\lambda \in P$, for every $j > k$.

Let us set $y = b_k x$. The above discussion shows that

$$\phi^k y = - \sum_{0 \leq i < k} c_i \phi^i y + \lambda \sum_{j>k} d_j \phi^j y,$$

hence $\phi^k y \in \langle y, \phi y, \dots, \phi^{k-1} y\rangle \le T_{k-1}$, in view of Lemma 4.3. But $\phi^k y = b_k \phi^k x \notin T_{k-1}$, a contradiction. We conclude that T_k is necessarily pure in T_n.

In particular, it follows that

$$T_0 = 0 < T_1 < \dots < T_{n-1} < T_n$$

is a pure-composition series of T_n. It remains to show that this pure-composition series has increasing annihilator sequence. For $0 < k \le n$, let $A_k = \mathrm{Ann}(T_k/T_{k-1})$. Pick $r \in A_k$; then, from $r\phi^{k+1}x = \phi(r\phi^k x) \in \phi T_{k-1} \le T_k$, we get $r \in A_{k+1}$. We conclude that $A_k \subseteq A_{k+1}$ and therefore $A_1 = \mathrm{Ann}(x) \subseteq A_2 \subseteq \dots \subseteq A_n$. □

We are in the position to compute the v-entropy of ϕ restricted to the cyclic ϕ-trajectory generated by $x \in G$. Naturally, we consider only the non-trivial case where $T(\phi, x)$ is not finitely generated.

Theorem 4.5. *Let G be a torsion R-module, $\phi \in \mathrm{End}_R(G)$, $x \in G$. For every $j > 0$, let $A_j = \mathrm{Ann}(T_j/T_{j-1})$. If $T = T(\phi, x)$ is not finitely generated, then $\mathrm{ent}_v(\phi|_T) = v(A)$, where $A = \bigcup_{j>0} A_j$.*

Proof. To simplify the symbols, in the proof we will denote $\phi|_T$ by ϕ. Since every finitely generated submodule of T is contained in some T_j, we have

$$\mathrm{ent}_v(\phi) = \sup_j\{H(\phi, T_j)\}$$

Note that $T_n(\phi, T_j) = T_{n+j-1}$, and recall that $v(A_1) \ge v(A_2) \ge \dots$ and $v(A) = \inf_k v(A_k)$.

Now take any integer $k > 0$. If $n + j - 1 > k$, by Proposition 4.4 and (2), we get

$$\begin{aligned} L_v(T_{n+j-1}) &= \sum_{i=1}^{n+j-1} v(A_i) = \sum_{i=1}^{k-1} v(A_i) + \sum_{i=k}^{n+j-1} v(A_i) \\ &\le \sum_{i=1}^{k-1} v(A_i) + (n + j - 1 - k)v(A_k), \end{aligned}$$

hence

$$H(\phi, T_j) = \lim_{n\to\infty} \frac{L_v(T_{n+j-1})}{n} \le v(A_k).$$

Since j and k were arbitrary, we get

$$\sup_j\{H(\phi, T_j)\} = \mathrm{ent}_v(\phi) \le \inf_k v(A_k) = v(A).$$

Let us verify the reverse inequality $\mathrm{ent}_v(\phi) \ge v(A)$. In fact, $L_v(T_{n+j-1}) \ge (n + j - 1)v(A)$, whence

$$\lim_{n\to\infty} \frac{L_v(T_{n+j-1})}{n} \ge \lim_{n\to\infty} \frac{(n + j - 1)v(A)}{n} = v(A).$$

□

A result less precise but more general than Theorem 4.5 has some interest.

Proposition 4.6. *Let ϕ be an endomorphism of the torsion R-module G such that $G = T(\phi, F)$ for a suitable $F \in \mathcal{F}(G)$. Then $\mathrm{ent}(\phi) \le L_v(F)$. In particular, $\mathrm{ent}(\phi) < \infty$.*

Proof. Since $G = T(\phi, F)$, every finitely generated submodule of G is contained in some $T_j = T_j(\phi, F)$. Observe that $T_n(\phi, T_j) = T_{n+j-1}$. Recall that $L_v(A + B) \le L_v(A) + L_v(B)$ for all R-modules A, B, by the additivity property (1). It follows that $L_v(T_{n+j-1}) \le (n + j - 1)L_v(F)$, since $L_v(\phi^k F) \le L_v(F)$ for $k \ge 0$. Then

$$H(\phi, T_j) = \lim_{n\to\infty} \frac{L_v(T_{n+j-1})}{n} \le \lim_{n\to\infty} \frac{(n + j - 1)L_v(F)}{n} = L_v(F).$$

The desired conclusion follows, since $j > 0$ was arbitrary. □

In our final result we get a special case of the Addition Theorem, namely, when the R-module G coincides with the cyclic trajectory $T(\phi, x)$, for some $\phi \in \mathrm{End}_R(G)$, $x \in G$, and the ϕ-invariant submodule coincides with aG, $a \in P$. We use a direct argument, based on Theorem 4.5, that is much simpler and shorter than the discussion leading to the proof of the general result in [8].

Proposition 4.7. *Let G be a torsion R-module, not finitely generated, such that $G = T(\phi, x)$ for suitable $\phi \in \mathrm{End}_R(G)$ and $x \in G$. Take any $a \in P$, and consider the fully invariant submodule $aG = T(\phi, ax)$ and the induced map $\bar{\phi} : G/aG \to G/aG$. Then we have*

$$\mathrm{ent}_v(\phi) = \mathrm{ent}_v(\phi|_{aG}) + \mathrm{ent}_v(\bar{\phi}) \tag{†}$$

Proof. For $n > 0$, let $T_n = \langle x, \phi x, \dots, \phi^{n-1}x\rangle$ and $A_n = \mathrm{Ann}(T_n/T_{n-1})$. We have seen in Theorem 4.5 that $\mathrm{ent}_v(\phi) = v(A)$, where $A = \bigcup_{j>0} A_j$. We firstly assume that aG is finitely generated. Then $\mathrm{ent}_v(\phi|_{aG}) = 0$ and there exists $k > 0$ such that $T_k \geq aG$. Consider the partial $\bar{\phi}$-trajectory $X_n = T_n(\bar{\phi}, T_k/aG) \leq G/aG$. Since $T_k \geq aG$, it follows that $T_j/T_k \cong (T_j/aG)/(T_k/aG)$ for every $j > k$, and therefore the last terms of the increasing annihilator sequence of X_n are $A_{k+1} \subseteq A_{k+2} \subseteq \cdots \subseteq A_{k+n}$. Under the present circumstances, an argument similar to the proof of Theorem 4.5 shows that $\mathrm{ent}_v(\bar{\phi}) = v(A)$, hence (†) holds in this case.

Now we assume that $aG = T(\phi, ax)$ is not finitely generated; equivalently, $\mathrm{gen}(aT_j) = j$ for every $j > 0$. Note that aG not finitely generated implies that $a \notin A_j$, i.e., $aR \supset A_j$, for every $j > 0$. Since T_n is pure in T_{n+1}, a simple computation shows that aT_j has annihilator sequence $a^{-1}A_1 \subseteq a^{-1}A_2 \subseteq \cdots \subseteq a^{-1}A_j$. Then Theorem 4.5, applied to $y = ax$ yields $\mathrm{ent}_v(\phi|_{aG}) = v(a^{-1}A) = v(A) - v(a)$. Thus, in order to verify (†), it remains to show that $\mathrm{ent}_v(\bar{\phi}) = v(a)$. Let us verify that

$$G/aG = T(\bar{\phi}, x + aG) = \bigoplus_{n\geq 0}\langle \phi^n x + aG\rangle \cong \bigoplus R/aR.$$

Assume, for a contradiction, that the sum is not direct. Then there exists $k > 0$ such that

$$b_k\phi^k x - \sum_{i=0}^{k-1} b_i\phi^i x = a\sum_{i=0}^{m} c_i\phi^i x \in aG,$$

where $b_k \notin aR$. If $m > k$, we get $ac_m \in A_m$, hence $ac_m\phi^m x = ac_m \sum_{i=0}^{m-1} d_i\phi^i x$, since T_{m-1} is pure in T_m. Thus we get a shorter relation

$$(b_k - ac_m d_k)\phi^k x - \sum_{i=0}^{k-1} b_i\phi^i x = a\sum_{i=0}^{m-1}(c_i + c_m d_i)\phi^i x \in aG,$$

where $b_k - ac_m d_k \notin aR$. Repeating the procedure, after a finite number of steps we get an equality of the form

$$e_k\phi^k x - \sum_{i=0}^{k-1} e_i\phi^i x = 0,$$

where $e_k \notin aR$. But this means that $e_k \in A_k \subset aR$, impossible. We reached a contradiction; then, necessarily, the sum is direct.

The desired conclusion follows, applying Theorem 4.5 to G/aG and $\bar{\phi}$. □

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References

[1] R. L. Adler, A. G. Konheim, M. H. McAndrew, *Topological entropy*, Trans. Amer. Math. Soc. 114 (1965), 309–319.
[2] R. Bowen, *Entropy for group endomorphisms and homogeneous spaces*, Trans. Amer. Math. Soc. 153 (1971), 401–414.
[3] D. Dikranjan, A. Giordano Bruno, *Discrete dynamical systems in group theory*, Note Mat. 33 (2013), no. 1, 1–48.

[4] D. Dikranjan, B. Goldsmith, L. Salce, P. Zanardo, *Algebraic entropy for abelian groups*, Trans. Amer. Math. Soc. 361 (2009), no. 7, 3401–3434.

[5] L. Fuchs, L. Salce, *Modules over non-Noetherian domains*, Mathematical Surveys and Monographs, 84. American Mathematical Society, Providence, RI, 2001.

[6] D.G. Northcott, M. Reufel, *A generalization of the concept of length*, Quart. J. Math. Oxford (2) 16 (1965) 297–321.

[7] L. Salce, P. Vámos, S. Virili, *Length functions, multiplicities and algebraic entropy*, Forum Math. 25 (2013), no. 2, 255–282.

[8] L. Salce, S. Virili, *The addition theorem for algebraic entropies induced by non-discrete length functions*, to appear.

[9] L. Salce, P. Zanardo, *Finitely generated modules over valuation rings*, Comm. Algebra 12 (1984), no. 15-16, 1795–1812.

[10] L. Salce, P. Zanardo, *A general notion of algebraic entropy and the rank-entropy*, Forum Math. 21 (2009), no. 4, 561–587.

[11] P. Vámos, *Length functions on modules*, Ph.D. thesis, University of Sheffield, 1968.

[12] P. Vámos, *Additive functions and duality over Noetherian rings*, Quart. J. Math. Oxford Ser. (2) 19 (1968) 43–55.

[13] S. Virili, *Length functions of Grothendieck categories with applications to infinite group representations*, arXiv: 1410.8306.

[14] M. D. Weiss, *Algebraic and other entropies of group endomorphisms*, Math. Systems Theory, 8 (1974/75), no. 3, 243–248.

[15] P. Zanardo, *Multiplicative invariants and length functions over valuation domains*, J. Commut. Algebra 3 (2011), no. 4, 561–587.