Caratheodory Solutions and their Associated Graphs in Opinion Dynamics with Topological Interactions ⋆

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Abstract: Dynamical models of social influence may present discontinuous rules of interactions: discontinuities are unavoidable when interactions are of topological nature, i.e. when the dynamics is the outcome of interactions with a limited number of nearest neighbors. Here, we prove that classical solutions are not sufficient to describe the dynamics that is produced by such interactions, but one needs to use non-classical concepts of solutions instead. We first describe the time evolution of the interaction graph associated to Caratheodory solutions, whose properties depend on the dimension of the state space and on the number of considered neighbors. We then prove the existence of Caratheodory solutions for 2-nearest neighbors, via an algorithm defining an interaction graph.

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1. INTRODUCTION AND SUMMARY OF RESULTS

Researchers from many different fields have explored the behavior of large systems of active particles or agents. Examples include dynamics of opinions in social networks, animal groups, networked robots, pedestrian dynamics and language evolution. Their dynamics is written as an Ordinary Differential Equation (ODE in the following) in large dimension. One of the main phenomena of active particles is *self-organization* of the whole system, stemming from simple interaction rules at the particle level. Such interaction rules are often motivated by relationships among agents; thus, corresponding evolutions are referred to as *social dynamics*, see Aydoğdu et al. (2017); Proskurnikov and Tempo (2017, 2018).

The description of social dynamics may require ODEs with discontinuous vector fields, as we will show in the model studied in the present paper. Several concepts of solutions have been defined in mathematical analysis and in control theory, such as *classical, Caratheodory, Filippov, Krasovsky, Clarke-Ledyaev-Sontag-Subbotin and stratified solutions*. In this article, we will only focus on Caratheodory solutions, for which we recall the precise definition in Section 2.1 below. For a thorough discussion on these different concepts of solutions in social dynamics models, see e.g. Ceragioli et al. (2021a); Piccoli and Rossi (2021); Ceragioli et al. (2021b); Ceragioli and Frasca (2012, 2018b,a); Frasca et al. (2019).

We now informally describe the opinion dynamics model that we analyze in the present article, whose basic idea is that trust towards others has limitations. Some earlier work (by Hegselmann and Krause (2002) and followers) has assumed that an individual is influenced by others only if opinions are not too far from one another. Here, we describe the fact that one's confidence towards others is limited by describing the so-called *topological interactions*: we assume that an individual follows only a fixed number $\kappa \geq 1$ of neighbors, the ones whose opinions are the nearest to his own. Topological interactions can be motivated by cognitive limits of the individuals on the number of significant relationships with other individuals (see Ballerini et al. (2008); Dunbar (1992)). These limitations are particularly meaningful in the dynamics of contemporary society, where potential contacts and available information are virtually unlimited. The precise mathematical description of the model is postponed to Section 2.

The main results of this article are the following. First, we study the evolution of the interaction graph for Caratheodory solutions, i.e. the graph of the interacting neighbors at each instant of time. In Theorem 2, we will show that such graph is constant for positive times only for the case of $\kappa = 1$ when agents interact on the real line $\mathbb{R}$. Instead, the graph can evolve in time both when $\kappa > 1$ or where the state space for each agent is $\mathbb{R}^n$ with $n > 1$, as shown by relevant examples. The second main result is Theorem 4, stating that, in the case of $\kappa = 2$ neighbors, for any initial data there exists at least one Caratheodory solution. The presence of topological (thus,

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strongly discontinuous) interactions makes the problem quite hard, and we need a non-trivial algorithm to build the interaction graph.

The structure of the article is the following. In Section 2, we describe the model and show the need of non-classical solutions to make sense of discontinuous interactions. We then recall in Section 2.1 the definition of Caratheodory solutions and some known results about them. Section 3 presents the main results about the interaction graph: after its precise definition, we provide a couple of remarkable examples of solutions in which the interaction graph evolves in time. We finally prove Theorem 2. Section 4 contains our main result about existence of Caratheodory solutions, i.e. Theorem 4. Finally, Section 5 collects conclusions and future research directions.

2. THE TOPOLOGICAL INTERACTION MODEL AND ITS KNOWN PROPERTIES

Let us consider a set $V = \{1, \dots, N\}$ of N agents with states $x_i \in \mathbb{R}^n$ (e.g. position, opinion, speed). Each agent $i \in V$ interacts with other agents belonging to a subset of neighbors $N_i(x) \subseteq V$. The subset of neighbors $N_i(x)$ depends on the state and induces a graph $G(x) = (V, E)$ of interactions among the agents: V is the set of nodes and $ij \in E$ is a (directed) edge if $j \in N_i(x)$. We denote the set of edges by $E(x)$. The dynamics can be written as

$$\dot{x}_i = \sum_{j \in N_i(x)} a(|x_j - x_i|)(x_j - x_i), \qquad (1)$$

where the function $a : [0, +\infty[\to [0, +\infty[$ represents the strength of interactions among agents. It satisfies the following hypotheses from now on:

a is a non-decreasing C^1-function, with $a(r) > 0$.

The *topological* interaction model is obtained when agent i interacts only with a fixed number κ of neighbors, where $1 \leq \kappa \leq N-1$. More precisely, for every agent $i \in V$, his neighborhood $N_i(x)$ is defined in the following way: the elements of $V \setminus \{i\}$ are ordered by increasing values of $|x_j - x_i|$; then, the first κ elements of the list (i.e. those with smallest distance from i) form the set $N_i(x)$ of current neighbors of i. Should a tie between two or more agents arise, priority is given to agents with lower index.

This continuous-time topological interaction model was first pointed out in Aydoğdu et al. (2017), while several other models of opinion dynamics and collective motion have considered topological interactions in different forms: see Cristiani et al. (2011); Rossi and Frasca (2020) and references therein.

Observe the following key feature: the right hand side of (1) is a discontinuous function because of the possible changes in the neighbor sets. For this reason, one needs to carefully select a concept of solution to such discontinuous ODE. Here, we will only consider Caratheodory solutions, which are defined below in Section 2.1. For a thorough discussion on these different concepts of solutions in social dynamics models, see Ceragioli and Frasca (2018b); Piccoli and Rossi (2021); Ceragioli et al. (2021a).

Remark 1. (Metric bounded confidence models). A related family of *bounded confidence* models is given by *metric* interactions: the agents in $N_i(x)$ are all the agents within a given radius $R > 0$, i.e.

$$N_i(x) = \{j \in V \ \text{ s.t. } \ |x_j - x_i| < R\}.$$

Also in this case, the presence of the threshold implies that the ODE in (1) has discontinuous right hand side. Yet, metric interactions enjoy some nicer properties, compared to the topological ones, such as symmetry: if ij is an edge for $G(t)$, then ji is an edge too. See more details in Piccoli and Rossi (2021).

2.1 Caratheodory solutions

An autonomous ODE is written as:

$$\dot{x}(t) = g(x(t)) \qquad (2)$$

where $x \in \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^m$ is a measurable and locally bounded function (defined at every point). Many definitions of solutions for (2) are available, most of which coincide when g is sufficiently regular (e.g. locally Lipschitz). In this article we only consider the following.

Definition 1. Given the ODE (2) and $T > 0$, we define:

(1) A **classical solution** is a differentiable function $x : [0, T] \to \mathbb{R}^m$ that satisfies (2) at every time $t \in (0, T)$. At 0 and at T the equation must be satisfied with one-sided derivatives.
(2) A **Caratheodory solution** is an absolutely continuous function $x : [0, T] \to \mathbb{R}^m$ which satisfies (2) at almost every time $t \in [0, T]$. Equivalently, x is a solution in integral form:

$$x(t) = x(0) + \int_0^t g(x(s))\, ds.$$

It is clear that all classical solutions are Caratheodory too.

We will show in Examples 2-3 below that classical solutions to (1) may not exist. For this reason, in this paper we will concentrate on Caratheodory solutions to (1). We thus recall from Ceragioli et al. (2021a) two facts about them. The first one is contractivity of their support, which is a consequence of the fact that $a(r) > 0$, i.e. that interactions are always attractive.

Proposition 1. (Contractivity). Let $x(t)$ be a solution to (1). Then

$$co\left(\{x_1(T^1), \dots, x_N(T^1)\}\right) \supseteq co\left(\{x_1(T^2), \dots, x_N(T^2)\}\right),$$

for $0 \leq T^1 < T^2$, where the (closed) convex hull of A is

$$co(A) := \left\{ \sum_{i=1}^{\ell} \alpha_i x_i : \ell \in \mathbb{N}, \alpha_i \in [0,1], \sum_{i=1}^{\ell} \alpha_i = 1, x_i \in A \right\}.$$

The second fact is about the uniqueness of Caratheodory solutions from almost every initial data. We provide a counterexample to uniqueness from specific initial conditions in Example 1.

Theorem 1. (Uniqueness). Uniqueness of Caratheodory solutions for the topological bounded confidence model (1) does not hold for all initial data but does hold for almost every initial datum, i.e. the set of initial data in $\mathbb{R}^{nN}$ for which uniqueness does not hold has zero Lebesgue measure.

3. THE INTERACTION GRAPH

In this section, we define and study the graph associated to a Caratheodory solution to (1).

Definition 2. (Associated graph). Let $x(t)$ be a Caratheodory solution to (1). Then, the associated (directed) graph is $G(x(t)) := (V, E(t))$, where $E(t)$ is composed by edges $\{ij \text{ s.t. } i \in V, j \in N_i(x(t))\}$.

For all examples in this section, we set $a(r) \equiv 1$. As a consequence, the solutions of (1) are given by piecing together solutions of linear systems.

Observe that $N_i(x(t))$ is not uniquely determined by the initial data and is not even constant along trajectories, as the following example shows.

Example 1. (Non-uniqueness). Set $N = 4$, $n = 1$, $\kappa = 1$. Consider the initial condition $\overline{x} = (-1, 0, 1, 1)$. We now provide two Caratheodory solutions with such initial data. The first is given by choosing the edges of the interaction graph $E(t) = \{12, 23, 34, 43\}$ for $t > 0$: the solution is

$$x(t) = (1 - t\exp(-t) - 2\exp(-t), 1 - \exp(-t), 1, 1).$$

Note that $x(0) = \overline{x}$ and $x(t)$ satisfies (1) for all $t > 0$ but not for $t = 0$, hence it is not a classical solution. Also remark that $E(0) = \{12, 21, 34, 43\} \neq E(t)$ for $t > 0$.
The second solution is given by setting the edges of the interaction graph to be $\tilde{E}(t) = \{12, 23, 34, 43\}$ for all $t \geq 0$. In this case, the solution is

$$\tilde{x}(t) = \left(-\tfrac{1}{2} - \tfrac{1}{2}\exp(-2t), -\tfrac{1}{2} + \tfrac{1}{2}\exp(-2t), 1, 1\right).$$

Remark that this solution is (the unique) classical one.

Remark 2. (Initial data and interaction graphs). Example 1 also explains why the interaction graph is associated to a specific Caratheodory solution, and is not uniquely determined by the initial data.

In the previous example, the graph $G(t)$ is different at time $t = 0$ and for $t > 0$. Motivated by this and similar examples, we ask ourselves whether the associated graph of Caratheodory solutions of (1) is always constant for $t > 0$. We will prove in Theorem 2 that this fact only holds true when $\kappa = 1$ and $n = 1$. Instead, we now provide examples in which the associated graph is not constant when $\kappa > 1$ or $n > 1$.

3.1 Examples of non-constant interaction graphs

In this section, we provide two interesting examples of topological bounded confidence models for which the graph associated to a Caratheodory solution is not constant. For each example, we will provide a Caratheodory solution and prove that it is the unique Caratheodory solution for it. Since trajectories will be non-differentiable for some times, this will also ensure that classical solutions do not exist.

Example 2. ($\kappa = 1, n = 2$). In this example, we set $\kappa = 1$ and $n = 2$, i.e. agents evolve on the plane $\mathbb{R}^2$. Fix $L := \sqrt{7/8}$ and $N = 6$ agents in initial positions

$$x_1(0) := (0, 0), \quad x_2(0) := (1, 0), \quad x_3(0) := (1.9, 0),$$
$$x_4(0) := (0.5, L),\ x_5(0) := (-0.5, L),\ x_6(0) := (-1.4, L).$$

Since $|x_1(0) - x_4(0)| = |x_2(0) - x_4(0)| = |x_1(0) - x_5(0)| > 1$, for small times the graph $G(x(t)) = (V, E(t))$ has the following edges: $E(t) = \{12, 23, 32, 45, 56, 65\}$. By solving the associated linear system

$$\dot{x}_i = \textstyle\sum_{ij \in E(t)} (x_j - x_i), \tag{3}$$

there exists a time $T_1 > 0$ for which it holds

$$|x_1(T_1) - x_4(T_1)| = |x_1(T_1) - x_2(T_1)| = |x_4(T_1) - x_5(T_1)|$$
$$> |x_2(T_1) - x_3(T_1)| = |x_5(T_1) - x_6(T_1)|.$$

Then, there exists $T_2 > T_1$ for which $\{23, 32, 56, 65\} \subset E(t)$ for all $t \in (T_1, T_2)$. In principle, one needs to choose either $12 \in E(t)$ or $14 \in E(t)$ for $t \in (T_1, T_2)$, and similarly either $45 \in E(t)$ or $41 \in E(t)$. Direct computations of the solution of the associated linear system show that any of the possible choices forces

$$|x_1(t) - x_4(t)| < \min\{|x_1(t) - x_2(t)|, |x_4(t) - x_5(t)|\},$$

i.e. that the unique Caratheodory solution of (1) satisfies $E(t) = \{14, 41, 23, 32, 56, 65\}$ for all times $t > T_1$.
The trajectories are illustrated in Figure 1. The first component of x_1, x_2, x_3 is plotted as a function of time in Figure 2: the angle in the trajectory of x_1 corresponds to the change of the interaction graph.

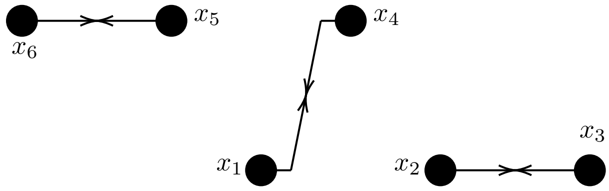

Fig. 1. Trajectories of Example 2.

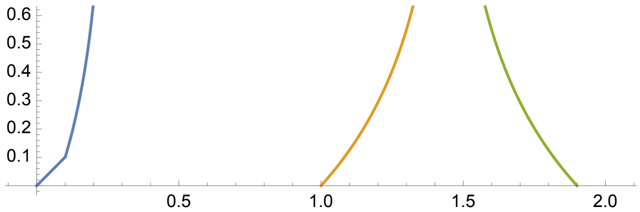

Fig. 2. Example 2: evolution of the first component of x_1, x_2, x_3.

Example 3. ($\kappa = 2, n = 1$). In this example, we set $\kappa = 2$ and agents evolve on the real line $\mathbb{R}$. Consider $N = 6$ agents with initial positions

$$x_1(0) = -11, \quad x_2(0) = -8, \quad x_3(0) = -3,$$
$$x_4(0) = 3, \quad x_5(0) = 8, \quad x_6(0) = 11.$$

By continuity, the interaction graph is constant on a given interval $[0, T_1)$, with edges $E(t) = \{12, 13, 21, 23, 32, 34, 43, 45, 54, 56, 64, 65\}$. By computing the solutions of the corresponding linear system (3), there exists a first time $T_1 > 0$ such that

$$|x_1(T_1) - x_3(T_1)| = |x_3(T_1) - x_4(T_1)|.$$

On a time interval (T_1, T_2), one certainly keeps

$$\{12, 13, 21, 23, 32, 45, 54, 56, 64, 65\} \in E(t)$$

and might eventually replace 34 with 31 and/or 43 with 45. In reality, for any of the possible choices, it holds

$$\max\{|x_1(t) - x_3(t)|, |x_4(t) - x_6(t)|\} < |x_3(t) - x_4(t)|.$$

Thus, the only admissible choice for (1) is to set $\{31, 45\} \in E(t)$ for $t \in (T_1, T_2)$, hence $E(t)$ is not constant along the trajectory. Afterwards, the graph is constant for $t > T_1$. The time evolution of the system is shown in Figure 3.

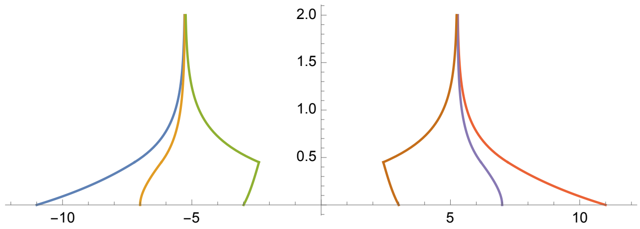

Fig. 3. Example 3: trajectories.

In both examples above, the Caratheodory solution is unique and is not always differentiable, thereby implying that no classical solution exists.

3.2 *The interaction graph for $\kappa = 1$ on the real line*

In this section, we prove the first main result of our article, i.e. that the interaction graph is constant for Caratheodory solutions of topological interaction models for $\kappa = 1$ in $\mathbb{R}$. Observe that the result is stated for a given Caratheodory solution: different solutions starting from the same initial datum may have different interaction graphs.

Theorem 2. (Graph is constant). Let $x(t)$ be a Caratheodory solution for (1) and $G(x(t))$ be the associated graph. If $\kappa = 1$ and the state space for agents is $\mathbb{R}$, then $G(x(t))$ is constant for $t > 0$.

PROOF. **Step 1.** We first prove that the ordering in $\mathbb{R}$ is preserved by Caratheodory solutions of (1) with $\kappa = 1$. Since the space is $\mathbb{R}$, one can rearrange indexes so that $x_i(0) \leq x_{i+1}(0)$ for all $i \in V \setminus \{N\}$. Moreover, rearrangement of indexes can preserve ordering in case of initial coinciding positions, i.e. if $x_i(0) = x_j(0)$ with $i < j$ before rearrangement, we can preserve it.

We now prove that inequalities are preserved along time. For simplicity of notation, we choose $i = 3$ and $N \geq 5$; the cases with $N \leq 4$ are identical. Consider the function $\phi(t) := x_4(t) - x_3(t)$, that is Lipschitz continuous with respect to time. From now on, only consider times where $x(t)$ is differentiable, hence $\phi(t)$ if differentiable too. We also drop dependence on time, for simplicity. For times where $\phi \geq 0$, one has two cases:

- either $|x_2 - x_3| \leq \phi$ and $\dot{x}_3 = a(|x_2 - x_3|)(x_2 - x_3) \leq 0$;
- or $|x_2 - x_3| \geq \phi$ and $\dot{x}_3 = a(\phi)\phi$.

In both cases, it holds $\dot{x}_3 \leq a(\phi)\phi$. By symmetry, one also has $\dot{x}_4 \geq -a(\phi)\phi$, then $\dot{\phi} = \dot{x}_4 - \dot{x}_3 \geq -2a(\phi)\phi$. Since solutions of $\dot{\phi} = -2a(\phi)\phi$ preserve the sign of the initial datum, then $\phi(0) > 0$ implies $\phi(t) > 0$, while $\phi(0) = 0$ implies $\phi(t) = 0$. Then, the order is preserved, particles cannot merge in finite time and particles with coinciding initial data keep being coinciding.

Step 2. We now prove the main statement: with no loss of generality, we choose the index $i = 3$ and prove that the unique edge of the form $3j$ is constant for $t > 0$. If $x_3(0) = x_2(0)$, then by Step 1 it holds $x_3(t) = x_2(t) = x_3(0)$. If $x_1(0) < x_3(0)$, then the edge is constantly 32, otherwise it is constantly 31. Similarly, if $x_2(0) < x_3(0) = x_4(0)$, then the edge is constantly 34.

Assume then $x_2(0) < x_3(0) < x_4(0)$ from now on. Define the function $f(t) := (x_4(t) - x_3(t)) - (x_3(t) - x_2(t))$, that is Lipschitz continuous with respect to time, and let t be a point of differentiability of $x(t)$. We have three cases, depending on the sign of $f(0)$.

If $f(0) > 0$, i.e. $|x_3(0) - x_2(0)| < |x_4(0) - x_3(0)|$, then necessarily $\dot{x}_3 = a(|x_3 - x_2|)(x_2 - x_3)$ as soon as $f(t) > 0$. To estimate $\dot{x}_2$, one has:

- either $|x_1 - x_2| \leq |x_2 - x_3|$ and $\dot{x}_2 = -a(|x_2 - x_1|)(x_2 - x_1) \geq -a(|x_3 - x_2|)(x_3 - x_2)$;
- or $|x_2 - x_3| \leq |x_1 - x_2|$ and $\dot{x}_2 = a(|x_3 - x_2|)(x_3 - x_2)$.

In both cases, it holds
$$\dot{x}_2 \geq -a(|x_3 - x_2|)(x_3 - x_2).$$
By using results of Step 1, one also has
$$\dot{x}_4 \geq -a(|x_4 - x_3|)(x_4 - x_3).$$
It then holds

$$\begin{aligned} \dot{f} \geq & -a(|x_4 - x_3|)(x_4 - x_3) - a(|x_3 - x_2|)(x_2 - x_3) \\ & -a(|x_3 - x_2|)(x_2 - x_3) - a(|x_3 - x_2|)(x_3 - x_2) \\ = & -\left(a(|x_4 - x_3|)(x_4 - x_3) - a(|x_3 - x_2|)(x_3 - x_2)\right). \end{aligned} \tag{4}$$

Introduce the the difference quotient of the function $a(r)r$, i.e. the following function of two variables:
$$\psi(r, h) := \frac{1}{h}(a(r + h)(r + h) - a(r)r).$$
Our hypotheses on $a(r)$ imply that $a(r)r$ is differentiable and increasing, then $\psi(r, h) \geq 0$ when restricted to $[0, +\infty) \times [0, +\infty)$. One can then rewrite (4) as
$$\dot{f}(t) \geq -\psi(x_3(t) - x_2(t), f(t))f(t),$$
whose solutions with $f(0) > 0$ preserve the sign for all times. This shows that $|x_3(0) - x_2(0)| < |x_4(0) - x_3(0)|$ implies $|x_3(t) - x_2(t)| < |x_4(t) - x_3(t)|$. If $x_2(0) > x_1(0)$, this implies that the edge 32 is constant, otherwise $x_2(0) = x_1(0)$ implies $x_2(t) = x_1(t)$ and the edge 31 is constant. The case $f(0) < 0$ is identical to the previous case, by reversing the sign.

We are left with the case $f(0) = 0$. Due to the study above, one necessarily has $f(t) \equiv 0$ on a suitable interval $[0, T]$, possibly with $T = 0, +\infty$, while $f(t)$ is either always strictly negative or strictly positive in $(T, +\infty)$. We now prove that $T = 0$, i.e. that $f(t)$ cannot be zero for positive times. By contradiction, assume that $T > 0$ and consider again the function $\phi(t) = x_4(t) - x_3(t)$, which is strictly positive due the hypothesis $x_4(0) > x_3(0)$ and to Step 1. Since $f(t) \equiv 0$ on $[0, T]$, it also holds $x_3(t) - x_2(t) = \phi(t)$. Again by considering only points $t \in [0, T]$ of differentiability for $x(t)$ and by dropping dependence on time, for each of the agents x_3, x_4, x_2, we have two possibilities:

- $\dot{x}_3 = a(\phi)\phi$ or $\dot{x}_3 = -a(\phi)\phi$;
- $\dot{x}_4 = -a(\phi)\phi$ or $\dot{x}_4 = a(|x_5 - x_4|)(x_5 - x_4) \geq 0$;
- $\dot{x}_2 = a(\phi)\phi$ or $\dot{x}_2 = a(|x_1 - x_2|)(x_1 - x_2) \leq 0$.

Since $f(t) \equiv 0$, then $\dot{f} = 0$, thus $\dot{x}_4 + \dot{x}_2 = 2\dot{x}_3$. The possibilities above are then reduced to the following:

- $\dot{x}_3 = a(\phi)\phi$, then $\dot{x}_4 = a(|x_5 - x_4|)(x_5 - x_4) = a(\phi)\phi$ and $\dot{x}_2 = a(\phi)\phi$;
- $\dot{x}_3 = -a(\phi)\phi$, then $\dot{x}_4 = -a(\phi)\phi$ and $\dot{x}_2 = a(|x_1 - x_2|)(x_1 - x_2) = -a(\phi)\phi$.

By hypotheses on $a(r)$, we have that $a(r)r$ is strictly increasing, hence injective: then, the first case reads as

$x_5 - x_4 = \phi$, while the second reads as $x_2 - x_1 = \phi$. Since ϕ is constant on $[0, T]$, the first case read as $\dot{x}_5 = \dot{x}_4$, while the second reads as $\dot{x}_1 = \dot{x}_2$. In the first case, one cannot have $\dot{x}_5 = a(|x_4 - x_5|)(x_4 - x_5) < 0$, hence one necessarily has $\dot{x}_5 = a(|x_6 - x_5|)(x_6 - x_5) = a(\phi)\phi$. This in turn implies $x_6 - x_5 = \phi$ and subsequently $\dot{x}_i = a(\phi)\phi$ for all $i \geq 2$. This holds also for the agent with highest index $i = N$, but this contradicts the fact that it necessarily holds

$$\dot{x}_N = a(|x_{N-1} - x_N|)(x_{N-1} - x_N) = -a(\phi)\phi < 0.$$

This implies that there exists no time $t \in [0, T]$ for which the first case is satisfied, then the second case holds for all $t \in [0, T]$ for which $x(t)$ is differentiable. But this second case is similar, as one has $\dot{x}_i = -a(\phi)\phi$ for all $i \leq 4$, which is in contradiction with the fact that for $i = 1$ it necessarily holds $\dot{x}_1 = a(|x_1 - x_2|)(x_2 - x_1) = a(\phi)\phi > 0$. □

We just proved that the interaction graph is constant. Since each component has a globally reachable node, we can draw the following important consequence.

Theorem 3. (Convergence). Let $x(t)$ be a Caratheodory solution to (1) and $\bar{G} = (V, \bar{E})$ the associated graph for $t > 0$. If $\kappa = 1$ and the state space for agents is $\mathbb{R}$, then $x(t)$ converges to some $x^\star$. Moreover, it holds $x_i^\star = x_j^\star$ if and only if there is a path from i to j or from j to i in $\bar{G}$.

A similar reasoning was employed in Ceragioli et al. (2021b) for a more restrictive definition of solutions. Theorem 3 was also proved by a Lyapunov argument in Ceragioli et al. (2021a).

4. EXISTENCE OF CARATHEODORY SOLUTIONS

In this section, we prove the second main result of this article, that is, the existence of Caratheodory solutions for the topological interaction models for $\kappa = 2$ neighbors. The case $\kappa = 1$ was already proved in Ceragioli et al. (2021a). The general case seems much harder to solve, as we will discuss later.

Theorem 4. (Existence). Let $\kappa = 2$. Then, for any initial condition, equation (1) admits a Caratheodory solution on $[0, +\infty)$.

Proof. We first fix notation. We denote by $\#B$ the cardinality of the set B. Given $\kappa \geq 1$, we denote by

$$\min^\kappa(B) := \min\{x \in \mathbb{R} \text{ s.t. } \#(B \cap (-\infty, x]) \geq \kappa\},$$

i.e. the minimal value ensuring that $B \cap (-\infty, x]$ contains at least κ elements. Similarly, given and indexed set $B = \{B_i\}_{i \in I}$. we denote by

$$\operatorname{argmin}^\kappa(B) := \{i \in I \text{ s.t. } B_i \leq \min^k(B)\},$$

i.e. the set of indexes of elements smaller than $\min^\kappa(B)$. It is clear that $\operatorname{argmin}^\kappa(B)$ contains at least κ elements. Moreover, it contains exactly κ elements in several relevant cases: e.g., when there is a single element $b \in B$ such that $b = \min^\kappa(B)$.

Since the case of several values $b \in B$ satisfying $b = \min^\kappa(B)$ is relevant for the rest of the discussion, we also define " strict" κ-minimum and κ-argmin as follows:

$$\operatorname{smin}^\kappa(B) := \max\{x \in \mathbb{R} \text{ s.t. } \#(B \cap (-\infty, x]) \leq \kappa\}$$
$$\operatorname{sargmin}^\kappa(B) := \{i \in I \text{ s.t. } B_i \leq \operatorname{smin}^k(B)\}.$$

It is clear that $\operatorname{sargmin}^\kappa(B)$ contains κ elements at most. Moreover, it contains exactly κ elements when it coincides with $\operatorname{argmin}^\kappa(B)$, e.g. in the case discussed above.

We now build a Caratheodory solution as follows. For each initial datum $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_N)$, we first define a directed graph $G = (V, E)$ for which there exists $T > 0$ and a curve defined on $[0, T]$ having G as connectivity graph on $(0, T)$. For each index $i \in V$ we choose exactly κ indexes, that we denote with $\Gamma(i)$, such that $ij \in E$ for $j \in \Gamma(i)$. This implies that

$$\dot{x}_i = \textstyle\sum_{j \in \Gamma(i)} a(|x_j - x_i|)(x_j - x_i). \tag{5}$$

for the whole time interval $(0, T)$. We then need to prove that the corresponding trajectory $(x_1(t), \ldots, x_N(t))$ is indeed a Caratheodory solution for (1).

Remark that one might be tempted to choose $\Gamma(i)$ to be equal to $N_i(\bar{x})$, that is, the set of nearest neighbors choosing the minimal index (in the lexicographic order) in case of ties. This solution cannot be effective, though, because problems arise exactly in case of ties. Hence, we will not choose $\Gamma(i)$ to be the set of nearest neighbors with minimal index at the initial time, but instead choose $\Gamma(i)$ to be the set of nearest neighbors for all $t \in (0, T)$. This delicate construction will be achieved by Algorithm 1, which takes as input an initial configuration $\bar{x}$ and produces as output the directed graph $G = (V, E)$, by iteratively adding edges into E through four Steps.

To simplify notation within the algorithm, we will first treat the specific case of coinciding particles separately. If there exist coinciding initial states $\bar{x}_i = \bar{x}_j$, one needs to treat any creation of edges from i as follows:

- if the edge ij is created, then create ji too;
- if an edge il with $l \neq j$ is created, then create jl too.

This rule ensures that we have $\dot{x}_i = \dot{x}_j$ in (5), hence the two particles keep coinciding all along the trajectory.

In writing the algorithm, we will make use of the following useful function, where k is an element of the set of indexes V, while J is a subset of it:

$$\begin{aligned}\psi_i(k, J) := \big(&\textstyle\sum_{l \in \Gamma(k)} a(|\bar{x}_l - \bar{x}_j|)(\bar{x}_l - \bar{x}_j) \\ &- \textstyle\sum_{l \in J} a(|\bar{x}_l - \bar{x}_i|)(\bar{x}_l - \bar{x}_i)\big) \cdot (\bar{x}_k - \bar{x}_i).\end{aligned} \tag{6}$$

It is easy to observe that $\psi_i(k, J)$ is the derivative of $|x_k - x_i|^2$ for $t = 0$, where neighbors of k are in $\Gamma(k)$, while the neighbors of i are in J.

It is easy to observe that the algorithm terminates: indeed, at each step the number of edges increases or keeps being constant. The only exception comes from Step 4, in which some steps provide smaller sets A_i instead of increasing the number of edges. This anyway leads to increase edges in future steps, as conditions for Step 2) or Step 3) are easier to be satisfied.

We now prove that the algorithm provides a graph $G = (V, E)$ such that the solution to (5) starting from $\bar{x}$ is a Caratheodory solution to (1) for a small interval of time $[0, T]$. In particular, we define

$$\begin{aligned}V^* := \{&(i, j, k) \text{ with } i \in \{1, \ldots, N\}, j \in \Gamma(i), k \notin \Gamma(i) \\ &\bar{x}_i \neq \bar{x}_j \neq \bar{x}_k \neq \bar{x}_i\},\end{aligned}$$

Algorithm 1: Graph construction

Step 1) **for** $i = 1, \dots, N$ **do**
  $\Gamma(i) \leftarrow \text{sargmin}^2(|\bar{x}_j - \bar{x}_i|)$;

Step 2) **while** *There exists i such that $\#\Gamma(i) < 2$ and all $j \in A_i := \text{argmin}^2(|\bar{x}_j - \bar{x}_i|) \setminus \Gamma(i)$ satisfy $\#\Gamma(j) = 2$* **do**
  if $\#\Gamma(i) = 1$ **then**
    Choose $j^* \in A_i$ as one element of $\text{argmin}_{j \in A_i} \psi_i(j, \Gamma(i) \cup \{j\})$;
    $\Gamma(i) \leftarrow \Gamma(i) \cup \{j^*\}$;
  else
    Choose $\{j_1^*, j_2^*\} \subset A_i$ as one pair realizing $\text{argmin}_{\{j_1,j_2\} \subset A_i} \max_{l=1,2} \psi_i(j_l, \{j_1, j_2\})$;
    $\Gamma(i) \leftarrow \Gamma(i) \cup \{j_1^*, j_2^*\}$;

Step 3) **if** *The set $B := \{i$ such that $\#\Gamma(i) = 1\}$ is nonempty* **then**
  Choose the subset $B' := \{i \in B$ such that $\text{argmin}^2_{j \in A_i}(|\bar{x}_j - \bar{x}_i|) = \max_{k \in B} \text{argmin}^2_{j \in A_k}(|\bar{x}_j - \bar{x}_k|)\}$;
  Choose one ordered pair $(i, j^*) \in B''$ with $B'' := \{(i, j) \in B' \times \{1, \dots, N\}$ with $j \in A_i \setminus \Gamma(i)\}$ that minimizes $\psi_i(j, \Gamma(i) \cup \{j\})$ on B'';
  $\Gamma(i) \leftarrow \Gamma(i) \cup \{j^*\}$;
  if $\Gamma(j^*) = 1$ **then**
    $\Gamma(j^*) \leftarrow \Gamma(j^*) \cup \{i\}$;
  Go to Step Step 2));

Step 4) **if** *There exists i with $\Gamma(i) = \emptyset$* **then**
  For each i, consider the set of pairs

$$B_i := \{\text{argmin}_{\{j_1,j_2\} \subset A_i} \max_{l=1,2} \psi_i(j_l, \{j_1, j_2\})\}.$$

  if *There exist i, j such that $\{j, k\} \in B_i$ and $\{i, l\} \in B_j$* **then**
    $\Gamma(i) \leftarrow \{j\}$ and $\Gamma(j) \leftarrow \{i\}$;
  else
    if *There exist i, j such that there exist neither $\{j, k\} \in B_i$ nor $\{i, l\} \in B_j$* **then**
      $A_i \leftarrow A_i \setminus \{j\}$ and $A_j \leftarrow A_j \setminus \{i\}$;
    else
      if *There exists i, j such that $\{j, k\} \in B_i$ and $\min_{l \in A_j} \max_{r=i,l} \psi_j(r, \{i, l\}) < \min_{\{l_1, l_2 \in A_j \setminus \{i\}\}} \psi_j(l_1, \{l_1, j_2\})$ when $\Gamma(i) = \{k\}$* **then**
        $\Gamma(i) \leftarrow \{j\}$;
      else
        $A_i \leftarrow A_i \setminus \{j\}$ and $A_j \leftarrow A_j \setminus \{i\}$;
  Go To Step Step 2));

and prove the following claim:

Claim A) For each $(i, j, k) \in V^*$ there exists a time $T_{ijk} > 0$ such that for all $t \in (0, T_{ijk})$ it holds

$$d_{ijk}(t) := |x_j(t) - x_i(t)|^2 - |x_k(t) - x_i(t)|^2 < 0.$$

The claim ensures that j keeps being one among the $\kappa = 2$ nearest neighbors of i for the whole time interval.

We prove Claim A in four steps, corresponding to the four Steps of the algorithm.

Step 1) Let $(i, j, k) \in V^*$ with $j \in \text{sargmin}^2(|\bar{x}_j - \bar{x}_i|)$. By Step 1, we set $j \in \Gamma(i)$. By definition, it then holds $d_{ijk}(0) < 0$. Thus, continuity of $d_{ijk}(t)$ ensures the existence of $T_{ijk} > 0$ satisfying Claim A. The case of $(i, j, k) \in V^*$ with $j \in \text{argmin}^2(|\bar{x}_j - \bar{x}_i|)$ and $k \notin \text{argmin}^2(|\bar{x}_k - \bar{x}_i|)$ is similar and can be treated by continuity as well.

We are now left to the (more complicated) case of ties. From now on, we assume $L := |x_j - x_i| = |x_k - x_i| > 0$, where strict positivity comes from the definition of V^*.

Step 2) If for some i, all possible neighbors $j \in A_i$ have a fixed dynamics, then it is sufficient to choose the neighbors of i as the ones ensuring $d_{ijk}(t) < 0$ for $t \in (0, T_{ijk})$. With this goal, it is sufficient to recall that

$$\psi_i(j, \Gamma(i) \cup J) - \psi_i(k, \Gamma(i) \cup J)$$

is the time derivative of d_{ijk} for $t = 0$ when one adds J as new neighbors of i. Thus, the two possibilities in Cycle 2 (in which one has to choose 1 or 2 neighbors, respectively) correspond to the fact that one chooses the neighbors ensuring that $\dot{d}_{ijk}$ is minimal among all possible choices. It is easy to observe that $\dot{d}_{ijk}(0) < 0$. By contradiction, if

$$\dot{d}_{ijk}(0) = \psi_i(j, J) - \psi_i(k, J) \geq 0,$$

replace the neighbor $j \in J$ with k, define $J' := \{k\} \cup J \setminus \{j\}$ and observe that it holds

$$\begin{aligned} &\psi_i(k, J') - \psi_i(j, J') = \psi_i(k, J) + (x_k - x_i) \cdot a(L)(x_j - x_i) \\ &-(x_k - x_i) \cdot a(L)(x_k - x_i) - \psi_i(j, J) \\ &-(x_j - x_i) \cdot a(L)(x_j - x_i) + (x_j - x_i) \cdot a(L)(x_k - x_i) = \\ &-(\psi_i(j, J) - \psi_i(k, J)) + 2a(L)((x_j - x_i) \cdot (x_k - x_i) - L^2). \end{aligned}$$

The first term is zero or negative, due to the hypothesis, while the second term is strictly negative, except for $L = 0$ or for $x_j = x_k$. Both cases are forbidden in V^*.

Step 3) is very similar to the previous step. Given i such that $\Gamma(i) = \{l\}$ and an edge ij is added, this means that the l neighbor was added at Cycle 1, i.e. $|x_i - x_l| < L$. We now prove $\dot{d}_{ijk}(0) < 0$ as follows:

- If $\#\Gamma(j) = \#\Gamma(k) = 2$ at this stage, follow the argument of Step 2.
- If $\#\Gamma(j) = 2 > \#\Gamma(k) = 1$, use the fact that neighbors of $\Gamma(k)$ are all at distance smaller or equal than L, otherwise $\Gamma(k) = 2$ due to the choice of the maximal distance in B'. Then, adding the edge ij decreases $\frac{d}{dt}|x_j - x_i|^2$ by a factor $a(L)L^2$, while adding any edge kr will increase $\frac{d}{dt}|x_k - x_i|^2$ by a factor that is strictly smaller than $a(L)L^2$. The only exception would be to choose $r = i$, but in this case one would have $\psi_i(k, \Gamma(i) \cup \{k\}) = \psi_i(j, \Gamma(i) \cup \{j\})$, thus

$$\begin{aligned} &\psi_k(i, \Gamma(k) \cup \{i\}) = \psi_k(i, \Gamma(k)) - a(L)L^2 = \\ &\psi_i(k, \Gamma(i) \cup \{k\}) - a(L)L^2 = \\ &\psi_i(j, \Gamma(i) \cup \{k\}) - a(L)L^2, \end{aligned}$$

  i.e. (i, j) is not a minimizer of $\psi_i(j, \Gamma(i) \cup \{j\})$ on B''.
- If $\#\Gamma(j) = 1$, the situation is similar to the previous one. One adds both edges ij and ji, to decrease $|x_i - x_i|^2$ by a factor $2a(L)L^2$, while any choice for $\Gamma(k)$ will increase it by a strictly smaller factor.

Step 4) Observe that the repetition of Steps 2 and 3 forces all agents to be in the following configuration when the Step 4 starts: they all satisfy $\#\Gamma(i) \in \{0, 2\}$, i.e. the case $\#\Gamma(i) = 1$ is no more present. In this case, all the different possibilities either provide links that ensure Claim A, or remove the links that cannot satisfy it. Then, going back to Step 2 and repeating the algorithm, we have that Claim A is proved.

We are now left to prove that there exists a Caratheodory solution to (1). Using Claim A, define $T := \min_{ijk} T_{ijk} > 0$. The claim, together with the discussion about coinciding agents, ensures that the solution to (5) is a Caratheodory solution to (1) on $[0, T]$. We now prove that we can always extend the solution to the time interval $[0, +\infty)$. Since the trajectory is compact, due to Proposition 1, then $x(T)$ is well-defined. By using the same algorithm starting from $x(T)$ and one can build a solution on some $[T, T_1]$ with $T_1 > T$, then on $[T_1, T_2]$ and so on. This implies that there exists a maximal interval I of definition of the solution. We claim that $I = [0, +\infty)$. By contradiction, if $I = [0, T^*)$ with $T^* < +\infty$, then $x(T^*)$ is well defined and I can be extended to $[0, T^*]$. Then, one uses the algorithm starting from $x(T^*)$ and extends the solution on $[0, T^* + \epsilon)$. Contradiction. ☐

Remark 3. (Larger κ). An algorithm that constructs Caratheodory solutions for the case $\kappa > 2$ seems much more complicated to devise, for two reasons:

- The combinatorics of cases seems increasing, as one can observe by comparing the algorithm for $\kappa = 2$ here with the one for $\kappa = 1$ given in Ceragioli et al. (2021a).
- In case of ties, when a first neighbor $j \in \Gamma(i)$ is found, adding one more neighbor k (which is required by $\kappa = 2$) cannot drastically change the dynamics of i, since j and k have the same weight. Instead, for $\kappa > 2$, the influence of the first neighbor can be overcome by the $\kappa - 1 > 1$ new neighbors.

5. CONCLUSIONS AND FUTURE DIRECTIONS

This paper has contributed some additional results to the study of continuous-time opinion dynamics with topological interactions. The resulting differential equations feature discontinuous right-hand sides, which requires appropriate notions of solutions. Furthermore, the interactions are inherently asymmetric, which makes the analysis harder.

In this paper, we have focused on Caratheodory solutions: building upon and extending the recent contributions by Ceragioli et al. (2021a,b), we have produced results about the existence of Caratheodory solutions, the properties of their associated graph of inter-agent interactions, and their convergence to equilibria when time goes to infinity.

In the light of our results, the most immediate open problem is proving the existence of Caratheodory solutions when the number of neighbors κ is greater than 2. Next, one should study their convergence properties: so far, convergence to equilibria has only been proved for $\kappa = 1$ by Ceragioli et al. (2021a).

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