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Closed normal subgroups of free pro-S-groups of finite rank

Andrea Lucchini

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1 Introduction

Let S be a finite simple group. A poly-S group is a finite group with all composition factors isomorphic to S. A profinite group is said to be a pro-S group if it is an inverse limit of poly-S groups. In [6] Jarden and Lubotzky determined some properties of pro-S groups and called for a systematic study of them. When S is the cyclic group of order p, the category of pro-S groups coincides with that of pro-p groups and has been intensively studied in the literature. Interesting results about pro-S groups when S is a non abelian simple group have been recently obtained by Fireman [4]. In this short note we solve an open question posed in Fireman's paper.

Let G be a pro-S group and let M be the intersection of all maximal open normal subgroups of G. It turns out that $G/M \cong S^{\alpha}$ for a suitable cardinal α, which is called the S-rank of G. In [10] Mel'nikov studied closed normal subgroups of infinitely generated free pro-S groups; in particular he proved that the free pro-S group $\hat{F}_{\mathfrak{m}}(S)$ of infinite rank $\mathfrak{m}$ has a closed normal subgroup with S-rank $\mathfrak{n}$ for each $\mathfrak{n} \leqslant \mathfrak{m}$. However Mel'nikov's methods are inapplicable in the finitely generated case and this led Fireman to propose the following open question.

Problem 1.1 ('Mel'nikov problem'). *Does each free pro-S group of finite rank have a closed normal subgroup of S-rank n for each $n \in \mathbb{N}$?*

Fireman [4, Proposition 4.3] proved that in order to solve this problem, one has to prove or disprove the following.

Conjecture 1. *Given $e \geqslant 2$, for every $n \in \mathbb{N}$ there exists an e-generated poly-S group with a normal subgroup isomorphic to S^n.*

In this paper we prove that the conjecture is true if $e \geqslant 3$ or $|S|$ is large enough.

Theorem 1.2. *Let S be a finite nonabelian simple group and let $l(S)$ be the smallest index of a proper subgroup of S.*

(i) *For every* $n \in \mathbb{N}$ *there exists a* 3*-generated poly-*S *group with a normal subgroup isomorphic to* S^n.

(ii) *If* $S^{l(S)}$ *is* 2*-generated, then for every* $n \in \mathbb{N}$ *there exists a* 2*-generated poly-*S *group with a normal subgroup isomorphic to* S^n.

As is well known (see for example [5]), S^n is 2-generated if and only if there are at least n orbits for the conjugation action of the automorphism group of S on the set of ordered pairs of elements of S that generate S. Hence $S^{l(S)}$ is 2-generated if and only if $P(S)|S| \geqslant l(S)|\mathrm{Out}\, S|$, where $P(S)$ denotes the probability of generating S by 2 randomly chosen elements. If $S = \mathrm{Alt}(6)$, then $P(S) = 53/90$, so

$$P(S)|S| = 212 \geqslant l(S)|\mathrm{Out}\, S| = 24.$$

If $S \neq \mathrm{Alt}(6)$, then $2|\mathrm{Out}\, S| \leqslant l(S)$ [1, Lemma 2.7] and $l(S)^2 \leqslant |S|$ [11, Proposition 3.9], hence the condition $P(S)|S| \geqslant l(S)|\mathrm{Out}(S)|$ is satisfied whenever $P(S) \geqslant 1/2$. It is proved in [7] that $P(S)$ approaches 1 as $|S|$ approaches ∞, so $S^{l(S)}$ is 2-generated if $|S|$ is large enough and we have the following.

Corollary 1.3. *If* $|S|$ *is large enough, then a free pro-*S *group of rank* 2 *has a closed normal subgroup of* S*-rank* n *for each* $n \in \mathbb{N}$.

Notice that if $N \cong S^m$ is a minimal normal subgroup of a poly-S group, then either $m = 1$ or $m \geqslant l(S)$. In particular if a 2-generated poly-S group G contains a normal subgroup N isomorphic to $S^{l(S)-1}$, then $S^{l(S)-1}$ is an epimorphic image of G and must be 2-generated. So the hypothesis "$S^{l(S)}$ is 2-generated" in Theorem 1.2 cannot be weakened too much.

2 Proof

We first recall some results useful for estimating the minimal number of generators $\mathrm{d}(G)$ of a finite group G.

Let L be a monolithic group, that is a group with a unique minimal normal subgroup N. For each positive integer k we let L^k be the k-fold direct power of L. The *crown-based power* of L of size k is the subgroup L_k of L^k defined by:

$$L_k = \{(l_1, \dots, l_k) \in L^k \mid l_1 \equiv \cdots \equiv l_k \bmod N\}.$$

Crown-based powers arise naturally when studying finite groups that need more generators than any proper quotient. A proof of the following theorem can be found in [2].

Theorem 2.1. *Let* m *be a natural number and let* G *be a finite group such that* $\mathrm{d}(G/N) \leqslant m$ *for every non-trivial normal subgroup* N, *but* $\mathrm{d}(G) > m$. *Then there exists a group* L *with a unique minimal normal subgroup* N *such that* $G \cong L_k$ *for some* k.

Let $\phi(d, G)$ denote the number of d-bases of G. In the case where the socle N of the monolithic group L is non-abelian, a bound on $\mathrm{d}(L_k)$ can be obtained using the following result [2].

Proposition 2.2. *Let L be a group with a unique minimal normal subgroup N such that N is non-abelian and let Γ denote the group of those automorphisms of L that act trivially on L/N. Assume $\mathrm{d}(L) \leqslant d$. Then $\mathrm{d}(L_k) \leqslant d$ if and only if $k \leqslant \frac{\phi(d,L)}{\phi(d,L/N)|\Gamma|}$.*

A family of monolithic groups will play an important role in our discussion. Let S be a finite non abelian simple group and let $l = l(S)$ be the smallest index of a proper subgroup of S. The group S has a transitive faithful permutation representation of degree l; the wreath product $S \wr S$ with respect to this permutation representation of S can be viewed as an imprimitive transitive permutation group of degree l^2 and, more generally, for each positive integer k, the k-iterated wreath product $S \wr \cdots \wr S$ has a transitive faithful permutation representation of degree l^k. We will denote this k-iterated wreath product by $L(S,k)$. Notice that $L(S,k)$ is a monolithic group, $\mathrm{soc}(L(S,k)) \cong S^{l^{k-1}}$ and if $k > 1$ then $L(S,k)/\mathrm{soc}(L(S,k)) \cong L(S,k-1)$. It follows from the classification of the finite simple groups that $\mathrm{d}(S) = 2$ and the main result of [9] implies that $\mathrm{d}(L(S,k)) = \mathrm{d}(L(S,k-1)) = \mathrm{d}(S) = 2$. The following holds.

Lemma 2.3. *Let $L = L(S,k)$.*

(i) *If $P(S)|S| \geqslant l(S)|\mathrm{Out}(S)|$, then $\mathrm{d}(L_t) \leqslant 2$ for each $t \leqslant l(S)$.*

(ii) *$\mathrm{d}(L_t) \leqslant 3$ for each $t \leqslant l(S)$.*

Proof. Let $l = l(S)$, $N = \mathrm{soc}\, L$ and

$$\omega = \frac{\phi(2,L)}{\phi(2,L/N)|\Gamma|} = \frac{P(L)|N|^2}{P(L/N)|\Gamma|}.$$

We have that $N \cong S^n$ with $n = l^{k-1}$.

Suppose $P(S)|S| \geqslant l(S)|\mathrm{Out}(S)|$. If $k = 1$, then $L = S$ and

$$\omega = \frac{P(S)|S|}{|\mathrm{Out}\, S|} \geqslant l.$$

Suppose $k > 1$. It is not difficult to prove (see for example the proof of [3, Lemma 1]) that $|\Gamma| \leqslant n|S^n|\,|\mathrm{Out}\, S|$. Moreover in [12] it is proved that

$$P(L) \geqslant P(S)\left(1 - \frac{16}{5}\frac{1}{2^l}\right).$$

Hence

$$\omega \geqslant \frac{P(L)|N|}{n|\mathrm{Out}\, S|} \geqslant \frac{P(S)}{|\mathrm{Out}\, S|}\left(1 - \frac{16}{5}\frac{1}{2^l}\right)\frac{|S|^n}{n} \geqslant l\left(1 - \frac{16}{5}\frac{1}{2^l}\right)\frac{|S|^{n-1}}{n}.$$

Since $n \geqslant l \geqslant 5$ and $|S| \geqslant 60$,

$$\left(1 - \frac{16}{5}\frac{1}{2^l}\right)\frac{|S|^{n-1}}{n} \geqslant \frac{9 \cdot 60^{n-1}}{10 \cdot n} \geqslant 1.$$

So $\omega \geqslant l$ in all the cases. Now, if $t \leqslant l$, then $t \leqslant \omega$, hence $\mathrm{d}(L_t) = 2$ by Proposition 2.2. This proves (1).

By [8, Lemma 1], $d(L_t) \leqslant 3$ if $t \leqslant |N|/n = |S|^n/n$. Since $|S| \geqslant 60$, we have that $|S|^n/n \geqslant |S| \geqslant l$, so (2) is also proved. $\square$

Proof of Theorem 1.2. Let $n \in \mathbb{N}$ and let $l = l(S)$ be the smallest index of a proper subgroup of S. We can write n in the form

$$n = a_0 + a_1 l + \cdots + a_r l^r$$

with $a_i \in \mathbb{N}$ and $0 \leqslant a_i < l$ for each $i \in \{0, \ldots, r\}$ and $a_r \neq 0$. Let $X = L(S, r)$. For each $i \leqslant r$, we define an action of X on the l^i-power $M_i = S^{l^i}$ as follows:

- $M_0 \cong S$ is centralized by X;
- if $i \neq 0$, then $L(S, i)$ is an epimorphic image of X, so X has a transitive permutation representation of degree l^i and acts on the direct power $M_i = S^{l^i}$ by permuting its coordinates.

The actions defined above can be used to define a diagonal action of G on M_i^n, for each $1 \leqslant i \leqslant r$ and $n \in \mathbb{N}$, so we may consider the semidirect product

$$G := (M_0^{a_0} \times \cdots \times M_r^{a_r}) \rtimes X.$$

Clearly G is an S-group with a normal subgroup $M_0^{a_0} \times \cdots \times M_r^{a_r}$ which is isomorphic to S^n. By Theorem 2.1, there exist a monolithic group L and an integer t such that L_t is an epimorphic image of G and $\mathrm{d}(G) = \mathrm{d}(L_t)$. By the way in which G has been constructed, the monolithic group L is either an epimorphic image of X, in which case $L \cong L(S, i)$ for some $i \leqslant r$, or $L \cong M_i \rtimes L(S, i) \cong L(S, i+1)$ for some $i \leqslant r$. Moreover a chief series of G contains exactly $a_i + 1$ chief factors isomorphic to S^{l^i} for each $i \in \{0, \ldots, r-1\}$ and a_r chief factors isomorphic S^{l^r}. This implies $t \leqslant l$ so, by Lemma 2.3, $\mathrm{d}(G) = \mathrm{d}(L_t) \leqslant 3$ and $\mathrm{d}(G) = \mathrm{d}(L_t) = 2$ if $P_S(2)|S| \geqslant l(S)|\mathrm{Out}(S)|$. $\square$

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Andrea Lucchini, Dipartimento di Matematica Pura ed Applicata, Via Trieste 63, 35121 Padova, Italy
E-mail: lucchini@math.unipd.it