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Optimal Extraction Under Endogenous Degradation Risk

Luca Grosset *,† , Maddalena Muttoni † and Elena Sartori †

Department of Mathematics "Tullio Levi-Civita", University of Padova, 35121 Padova, Italy; esartori@math.unipd.it (E.S.)
* Correspondence: luca.grosset@unipd.it
† These authors contributed equally to this work.

Abstract

We study the optimal extraction of a non-renewable resource under an endogenous risk of irreversible degradation. The extractor faces a stochastic switching time at which extraction costs permanently increase, with the hazard rate of this transition depending on the current extraction intensity. As a result, faster extraction not only accelerates depletion but also raises the probability of entering a high-cost regime. We formulate the problem as an optimal control model with a control-dependent hazard process and derive a deterministic equivalent representation. Although extraction before and after degradation is individually trivial, their coupling through the endogenous hazard generates a nonlinear control problem. We provide an explicit characterization of the optimal extraction policy and show that degradation risk fundamentally alters the optimal depletion path. In contrast to the deterministic benchmark, optimal extraction becomes smoother over time, as the decision maker trades off immediate profits against the expected increase in future costs. The analysis highlights how endogenous operational risk can discipline extraction incentives and offers new insights into the sustainable management of exhaustible resources under technological fragility.

Keywords: endogenous regime switching; hazard-rate models; optimal control; non-renewable resources

MSC: 91B76; 90C39; 49K15

1. Introduction

1.1. Background and Motivation

The economics of non-renewable resource extraction has long been grounded in the optimal control tradition inaugurated by Hotelling [1]. In such models, the central issue concerns how the intertemporal path of extraction balances current profits against future scarcity. While classical approaches focus on depletion, recent research emphasizes that technological or environmental degradation may modify extraction costs, thereby reshaping optimal trajectories. Two complementary strands of the literature motivate this study. The first, exemplified by Fossas and Corominas in [2], uses Pontryagin's Maximum Principle to describe the extraction path of a monopolistic supplier of a durable and exhaustible resource. Their analysis identifies bang–bang control structures and switching surfaces that govern extraction under convex cost and durability constraints. The second, exemplified by Cunha-e-Sá et al. in [3], explores how technological change and adoption of innovation influence extraction costs in competitive resource markets. Their results highlight the

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dynamic link between the size of available stocks, prices, and incentives to adopt cost-reducing technologies. However, both frameworks share a simplifying assumption: the cost structure evolves deterministically or through an exogenous shift. In many practical settings—such as mining, oil extraction, or aquifer exploitation—the transition from low to high extraction costs is a stochastic event triggered by the intensity of extraction itself. The faster the resource is extracted, the higher the probability that physical, geological, or ecological degradation will occur, causing irreversible increases in operational costs. This endogenous degradation mechanism introduces a feedback between the intensity of control and the risk of regime change, a feature largely unexplored in the literature.

1.2. Aim and Contribution

This paper extends the standard optimal control framework for non-renewable resources by endogenizing the stochastic degradation process. The model assumes that the probability of cost escalation follows a hazard function dependent on the current extraction rate. When degradation occurs, the extraction cost parameter increases permanently, capturing the loss of efficiency due to irreversible damage to the production environment. We reformulate the problem as a deterministic equivalent control problem using the survival probability of the low-cost regime as an auxiliary state variable. The resulting optimal control problem is nonlinear even if the two problems in Stage 1 and Stage 2 are static. The analysis reveals that optimal extraction paths differ markedly from the static benchmark. While the classical solution prescribes constant extraction, the introduction of degradation risk induces a more conservative strategy: the planner smooths extraction to prolong the low-cost phase and mitigate expected losses from cost escalation. The model thus bridges the deterministic frameworks in [2] and the stochastic, technology-oriented models in [3], unifying them within a dynamic framework.

1.3. Structure of the Paper

The remainder of the paper is structured as follows. Section 2 provides a brief overview of the literature on non-renewable resource extraction. Section 3 presents the baseline deterministic extraction model with linear costs and recalls its analytical solution. Section 4 sets up the stochastic switching-cost problem and shows how it can be reformulated as a nonlinear deterministic optimal control problem. In Section 5, we characterize the optimal control. In Section 6 we comment on the results obtained and analyze the trade-off between short-term profit and degradation risk. Finally, we conclude by outlining possible extensions, including how to relax some assumptions of the model.

2. Literature Review

The study of non-renewable resource extraction has a long intellectual lineage beginning with the seminal paper by Hotelling [1], who first formalized the intertemporal allocation of an exhaustible resource in a perfectly competitive market. His celebrated Hotelling rule, that the net price of a non-renewable resource must rise at the rate of interest along an efficient path, became the cornerstone of modern resource economics. Subsequent developments have extended this result to accommodate a variety of institutional, technological, and environmental complexities that affect the depletion dynamics.

2.1. Classical and Deterministic Approaches

Early contributions by Solow in [4] and by Dasgupta and Heal in [5], enriched the Hotelling framework by incorporating intertemporal welfare and capital accumulation, thus embedding the resource problem within growth theory. In these models, extraction costs are typically deterministic and convex, implying that the socially optimal extraction path equates marginal net benefits over time. Analytical tractability in such models often relies on quadratic or linear-quadratic specifications, which allow the use of dynamic programming and the derivation of explicit control laws. More recent research has emphasized the role of convex extraction costs and the optimal stopping conditions that emerge when resource exhaustion occurs within a finite horizon. These formulations have been particularly useful for analyzing technological change, market power, or uncertainty. In deterministic settings, extraction typically follows a monotonic decline, often linear in time, as in the canonical linear-quadratic model.

2.2. Market Structure and Durability

When market imperfections or durability of the resource are considered, the dynamics become richer. Levhari and Pindyck in [6] demonstrated that for durable exhaustible resources, Hotelling's rule does not necessarily hold under monopoly, because the monopolist's intertemporal trade-off is affected by the continuing presence of previously sold units in the market. Extending this insight, Fossas and Corominas in [2] analyzed the optimal extraction policy of a monopolistic supplier of a durable non-renewable resource within a continuous-time optimal control framework. Their model introduces an upper bound on the extraction rate and identifies bang–bang and singular controls governed by switching surfaces. This geometric characterization of the Hamiltonian system reveals that optimal policies consist of alternating phases of maximum extraction, inactivity, or constant-rate extraction at an equilibrium price. The contribution of Fossas and Corominas lies in showing how physical constraints and market durability generate endogenous regime switches even in deterministic environments. In contrast, the model developed in the present paper introduces regime switching not through market structure but through stochastic degradation, endogenously driven by the control variable itself. Hence, while both models involve multiple cost regimes, our framework transforms the switching event into a probabilistic process, linking it directly to the intensity of extraction. A competitive approach is described by Tur, Gromova, and Gromov in [7]. In this work, they study a differential game of non-renewable resource extraction under imperfect information about the initial resource stock. They introduce the value of information and derive optimal stock estimates that minimize worst-case losses when only bounds on the initial stock are known.

2.3. Technological Change and Cost Evolution

A parallel line of research investigates how technological innovation influences the extraction and depletion of non-renewable resources. Within this domain, Cunha-e-Sá, Balcão Reis, and Roseta-Palma in [3] formalized the decision problem of a competitive firm facing the option of adopting a cost-reducing technology when extraction costs are quadratic both in extraction rate and remaining stock. Their results show that the expected benefits from adoption increase with the size of the available stock and with the market price, contradicting the common intuition that innovation occurs only after significant depletion. By deriving analytical conditions for the boundary between the adoption and non-adoption regions, their model links technological progress to the state of resource stocks and market prices. This approach paved the way for considering endogenous technological dynamics in resource extraction problems. However, their framework assumes an exogenous arrival of innovation and a deterministic adoption cost, whereas the transition considered here is endogenously stochastic, determined by extraction decisions themselves. In this sense, our model replaces the discrete innovation event with a hazard-based degradation process that continuously evolves in response to the control path.

2.4. Uncertainty and Stochastic Switching Times

A powerful approach to describe uncertainty is to move to stochastic processes. For example, in [8] the authors study a natural resource extraction problem with exogenous

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regime-switching prices and an optimal stopping decision, modeling uncertainty through a Markov-switching geometric Brownian motion. The switch is described endogenously using deterministic optimal control with stochastic switching time.

Stochastic switching time in control problems refers to situations in which the system dynamics, payoffs, or constraints change at a random and generally irreversible time that is not chosen by the decision maker. Such problems naturally arise when agents face uncertainty about regime shifts, catastrophes, or structural changes that may occur during the planning horizon. A common modeling device is to describe the switching time through a hazard rate, which governs the instantaneous probability that the regime change occurs conditional on survival up to the current time. In economic applications, this framework captures the idea that optimal decisions must balance current performance against the risk of triggering or facing a future regime with different characteristics. In the two-stage approach of Kuhn and Wrzaczek in [9], stochastic switching time is interpreted as a source of rational risk: before the switch, agents optimally control the system while anticipating that a different control problem may become relevant after the random event. The expectation over the switching time links the two stages and implies that optimal pre-switch policies internalize both the likelihood and the consequences of the regime change. This typically leads to modified shadow prices and incentives already in the first stage, reflecting the option value of continuation under an alternative regime. Similarly, Polasky, de Zeeuw, and Wagener in [10] analyze optimal management under potential regime shifts, where the switching time represents an uncertain transition to a degraded or altered ecological state. Their results show that the mere possibility of a stochastic switch can substantially alter optimal control paths, often inducing more conservative behavior to reduce the probability or impact of the shift. Overall, stochastic switching time problems highlight how uncertainty about future regimes fundamentally reshapes optimal control strategies, even when the underlying dynamics within each regime are well understood.

A closely related but methodologically distinct strand of the literature studies resource extraction problems under uncertainty by modeling randomness in the effective horizon of the control problem. An important contribution in this direction is the work by López-Barrientos, Gromova, and Miroshnichenko [11], who analyze non-renewable resource extraction under random terminal times within a continuous-time optimal control and differential game framework. In their model, uncertainty affects the duration of the extraction process or the relevance of continuation payoffs, and is handled through probabilistic laws governing termination, such as Weibull or Chen distributions. Optimal policies are characterized using Dynamic Programming techniques and Hamilton–Jacobi–Bellman equations, leading to explicit feedback controls under suitable functional assumptions. While both approaches share the use of hazard-rate representations and the interpretation of uncertainty through survival probabilities, the present framework differs in a fundamental way. In [11] randomness affects the end of the game or the continuation of payoffs, whereas in our model uncertainty governs an endogenous and irreversible change in the cost structure, with the hazard rate depending explicitly on the control variable itself. This control-dependent hazard is the key novelty of our approach: extraction decisions not only influence instantaneous payoffs and future states, but also shape the probability law of the regime switch. As a result, the intertemporal trade-off faced by the decision maker arises from the endogenous feedback between control intensity and the likelihood of degradation, rather than from uncertainty about the terminal time alone. From a methodological perspective, we rely on Pontryagin's Maximum Principle (PMP) rather than Dynamic Programming because the deterministic equivalent formulation leads naturally to an open-loop control problem with smooth dynamics and a finite horizon. PMP allows for a transparent characterization of the interaction between the control and the hazard-driven auxiliary state, while preserving analytical tractability. In contrast, a Dynamic Programming approach would typically lead to a higher-dimensional Hamilton–Jacobi–Bellman equation, whose explicit solution would be difficult to obtain in the presence of a control-dependent hazard rate. In this sense, the two approaches can be seen as complementary: Dynamic Programming is particularly well suited to problems with exogenous random horizons and feedback controls, whereas PMP proves more convenient for isolating and characterizing the endogenous risk mechanism that lies at the core of the present contribution.

2.5. Positioning of the Present Work

The model proposed in this paper unifies elements from the above traditions: From Hotelling and subsequent deterministic models, it inherits the optimal control structure with quadratic costs and linear profits. From [2], it borrows the geometric intuition of regime switching and finite-horizon optimization under capacity constraints. From [3], it incorporates the idea that cost structures can evolve dynamically. From the stochastic switching time literature, it adopts the hazard-based treatment of random events affecting the control environment. The resulting model can thus be viewed as a bridge between deterministic and stochastic control frameworks for exhaustible resources. The analytical tractability is preserved, while introducing a realistic endogenous source of uncertainty. From a mathematical point of view, the model is interesting because, even though the two problems that make up Stage 1 and Stage 2 are static, the deterministic optimal control problem obtained by introducing the switching time is dynamic and nonlinear; nevertheless, it admits a closed-form solution that provides interesting insights.

3. Linear Extraction Costs

We consider the following optimal control problem:

$$\begin{aligned} &\max_{u(t)\in[0,U]} \quad \int_0^T (\pi u(t) - \kappa_1 u(t))dt \\ &\dot{x}(t) = -u(t) \\ &x(0) = \alpha > 0 \end{aligned} \tag{1}$$

where:

- $x(t) \geq 0$ is the state variable and represents the remaining stock of a non-renewable resource at time t;
- $u(t) \geq 0$ is the control variable and denotes the extraction rate of this resource at time t. It is bounded above by the maximum extraction rate, which is $U \in (0, +\infty)$;
- $\pi u(t)$ represents instantaneous linear revenue, with $\pi > 0$;
- $\kappa_1 u(t)$ is the linear extraction cost, with $\kappa_1 > 0$.

Here, as in [2], we have that extraction rate is bounded and the programming interval is finite. We make two strong assumptions on this model.

Assumption 1. *We assume that $\kappa_1 < \pi$ so that marginal extraction cost is lower than the marginal revenue. Hence, extraction is profitable at the margin. In the following we define $\mu := \pi - \kappa_1 > 0$; this is a key parameter in our discussion and it must be positive if we think that the extraction is profitable.*

Assumption 2. *We assume that $UT \leq \alpha$. Even if extraction occurs at the maximum feasible rate throughout the planning horizon, the resource stock is not exhausted. Equivalently, the planning horizon is not sufficiently long to allow full depletion of the resource.*

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A large part of the classical literature on optimal extraction of non-renewable resources models scarcity through an isoperimetric constraint on cumulative extraction, typically written as

$$\int_0^T u(t)\,dt \le \alpha,$$

where α denotes the initial stock of the resource. In these formulations, the associated multiplier admits a natural economic interpretation as a scarcity rent and plays a central role in determining the optimal extraction path [12]. In the present framework, such a constraint is not imposed explicitly, but is embedded in the stock dynamics $\dot{x}(t) = -u(t)$ together with the non-negativity constraint on the stock. Assumption 2 guarantees that even under extraction at the maximum feasible rate throughout the planning horizon, the stock is never exhausted. Consequently, the stock constraint remains slack along the optimal trajectory. From an economic standpoint, this implies that scarcity effects are intentionally abstracted from: the shadow value of the resource is identically zero, and extraction decisions are not influenced by intertemporal depletion considerations. Assumption 2 allows us to abstract from scarcity effects and focus exclusively on the role of endogenous degradation risk. When the stock constraint does not bind, the intertemporal link between current extraction and future payoffs arises exclusively through the effect of extraction intensity on the probability of a transition to a higher-cost regime. The model should therefore be interpreted as a benchmark within the classical extraction literature, designed to disentangle scarcity-driven incentives from those generated by endogenous risk.

Therefore, Assumption 2 plays a crucial conceptual role in the present framework. By ensuring that the initial stock is sufficiently large relative to the planning horizon and the maximal feasible extraction rate, the stock constraint never binds along the optimal path. As a consequence, the shadow value of the remaining resource is identically zero and classical scarcity rents, in the Hotelling sense [1,12], are intentionally abstracted from. In the absence of scarcity, the only intertemporal link between current and future payoffs arises through the hazard rate governing the transition to the high-cost regime. Nevertheless, this channel alone is sufficient to generate a genuinely dynamic and nonlinear optimal control problem, as current extraction decisions affect the expected future cost structure through their impact on the probability of regime switching. The resulting precautionary behavior—characterized by moderated extraction early in the horizon and accelerated extraction as the terminal date approaches—thus emerges independently of depletion considerations. Relaxing Assumption 2 would reintroduce an active stock constraint and a positive shadow value of the remaining resource, leading to a richer problem in which two distinct but interacting intertemporal forces coexist: scarcity rents associated with depletion and risk-related shadow costs induced by endogenous degradation. In such a setting, higher extraction would simultaneously reduce future availability and increase the likelihood of entering a high-cost regime, implying an explicit interaction between the costate variables associated with the stock and with the survival probability. Although this extension would generally preclude a closed-form solution and require numerical methods, the core economic trade-off highlighted in this paper is expected to remain robust. If anything, the presence of scarcity would tend to reinforce the incentive to moderate early extraction, as aggressive extraction would carry both a higher expected degradation cost and a higher opportunity cost in terms of foregone future rents. The model serves as a benchmark in which degradation risk can be studied independently of scarcity. By abstracting from the depletion effects, it clarifies how endogenous operational risk alone can discipline extraction paths. This perspective complements, rather than substitutes, the classical Hotelling framework and provides a transparent baseline for future extensions in which scarcity and endogenous degradation interact.

Under these strong assumptions, the optimal resource extraction policy is trivial: since marginal profits are positive and the stock constraint never binds, the optimal control requires extracting the resource at the maximum feasible rate at all times. We formalize this result in the following theorem.

Theorem 1. *The optimal control of (1) is $u^*(t) = U$, for all $t \in [0, T]$.*

Proof. We apply Pontryagin's Maximum Principle [13]. The Hamiltonian function is

$$H(x, u, p) = \mu u + p(-u) = (\mu - p)u.$$

The costate equation gives $\dot{p}(t) = 0$, hence $p(t)$ is constant. Due to Assumption 2, the terminal state is free, and the transversality condition requires $p(T) = 0$. Therefore, $p(t) = 0$ for all $t \in [0, T]$. Maximizing the Hamiltonian on $0 \le u \le U$ reduces to maximizing the linear function μu. Therefore, being $\kappa_1 < \pi$ from Assumption 1, the optimal control is $u^*(t) = U$. □

This result is trivial: since the instantaneous profit margin is positive $\mu = \pi - \kappa_1$, the necessary conditions confirm the intuitive bang–bang policy: extract as fast as possible when the margin is positive.

Remark 1. *We observe that this problem could be viewed as a static optimization problem, since the state variable $x_1(t)$ never reaches zero, due to Assumption 2. Nevertheless, we decide to present this dynamic formulation, as it offers greater flexibility for the generalizations described in the conclusion of this paper.*

Economically, the assumption of a constant linear extraction cost is overly restrictive when extraction occurs at the maximum feasible rate. A substantial body of the energy economics literature emphasizes that upstream extraction costs are inherently dynamic, evolving with physical depletion, geological complexity, and operational intensity. In particular, as documented by Aguilera in [14], production typically begins with low-cost, easily accessible reserves and progressively shifts to more technically challenging conditions, requiring higher capital expenditures, higher energy inputs and more complex recovery techniques. This process implies that marginal extraction costs tend to increase with cumulative extraction and may exhibit discrete upward shifts rather than smooth continuous increases, reflecting transitions across cost regimes. Empirical evidence from unconventional resources supports this mechanism. In the case of U.S. shale oil, recent industry-based estimates indicate that average breakeven prices have risen markedly as core acreage has been depleted and drilling activity has expanded into less productive zones. For 2025, breakeven costs are estimated at approximately \$70 per barrel, with projections suggesting further increases to \$90–95 USD per barrel in the next decade, driven by increased drilling intensity, decreased well productivity, and increased service costs [15]. These trends imply that maintaining high extraction rates becomes progressively more expensive over time. A similar pattern is observed in large conventional fields. Sustaining production in mature giant reservoirs such as Ghawar requires increasingly intensive reservoir management, including large-scale seawater injection and handling of rising water cuts. As the share of non-oil fluids increases, lifting, separation, and reinjection requirements mechanically increase per-barrel extraction and processing costs, illustrating how physical depletion can generate sharp increases in marginal costs even in fields traditionally considered low-cost [14]. Taken together, these considerations motivate modeling extraction costs as subject to a stochastic upward shift at an endogenous time τ. Before τ, extraction occurs under relatively stable conditions and can be approximated with a constant marginal cost. After τ, extraction enters a high-cost regime characterized by increasing marginal costs, which we capture parsimoniously through a convex cost function $u \mapsto \kappa_1 u + \frac{\kappa_2}{2} u^2$. This formulation reflects empirical evidence on cost escalation documented in the literature and provides an economically grounded justification for a regime-switching extraction-cost structure.

4. Quadratic Extraction Costs and Stochastic Switching Time

To account for the possibility that extraction conditions may deteriorate abruptly and unpredictably, we model the extraction process as consisting of two distinct regimes separated by a stochastic switching time τ. In the initial regime, extraction takes place under relatively stable technical and geological conditions, so that marginal extraction costs can be reasonably approximated by a constant level, and instantaneous profits are linear in the extraction rate. This specification is consistent with extraction from favorable reservoirs or core production areas, where operating conditions are relatively predictable.

However, as extraction proceeds, the producer may enter a high-cost regime due to factors such as depletion of high-quality reserves, declining productivity, increasing water cut, or the need for enhanced recovery techniques. Empirical evidence suggests that such transitions do not necessarily occur smoothly, but may arise suddenly as operating thresholds are crossed or new technical constraints become binding. Since the exact timing of this transition is uncertain, we represent the onset of the high-cost regime by a random switching time τ as illustrated in Figure 1. In the post-switch regime, a higher extraction intensity requires disproportionately larger inputs, reflecting the increasing marginal costs associated with drilling effort, energy use, and reservoir management. We capture this effect parsimoniously by adding a convex extraction cost, modeled as a quadratic term $\frac{\kappa_2}{2} u_2^2$. This specification penalizes aggressive extraction following the regime change and allows for optimal extraction rates inside. The continuity condition $x_1(\tau) = x_2(\tau, \tau)$ ensures the consistency of the resource stock between regimes, while the objective function is evaluated in expectation to account for the uncertainty in the timing of the regime change. All these considerations allow us to analyze the following optimal control problem with stochastic switching time.

$$\max_{u_1(t),u_2(s,t)\in[0,U]} \mathbb{E} \quad \left\{ \int_0^\tau \mu u_1(t)dt + \int_\tau^T \left(\mu u_2(\tau,t) - \frac{\kappa_2}{2} u_2^2(\tau,t)\right) dt \right\}$$

$$\begin{aligned}
\dot{x}_1(t) &= -u_1(t) \\
x_1(0) &= \alpha > 0 \\
x_1(\tau) &= x_2(\tau, \tau) \\
\dot{x}_2(\tau, t) &= -u_2(\tau, t)
\end{aligned}$$

where:

- $\kappa_2 u^2(t)/2$ is the extra quadratic extraction cost, with $\kappa_2 > 0$.

We assume that the switching time τ is an absolutely continuous random variable (technical assumption) and is characterized by

$$\mathbb{P}(\tau \le t + h \,|\, \tau > t) = \eta(u_1(t))h + o(h). \tag{2}$$

It is important to clarify that the function $\eta(u_1(t))$ does not represent a probability, but a hazard rate (or instantaneous switching intensity) governing the arrival of the degradation event. As such, $\eta(\cdot)$ is allowed to take any non–negative value and is not constrained to lie in the unit interval $[0, 1]$. The probability that the degradation occurs over a finite time interval is obtained only after integration through the survival function. Hence, while higher extraction intensities increase the instantaneous risk of switching, the associated probabilities remain well defined and bounded. This interpretation is standard in the literature on stochastic switching times and survival analysis (see, e.g., [16]).

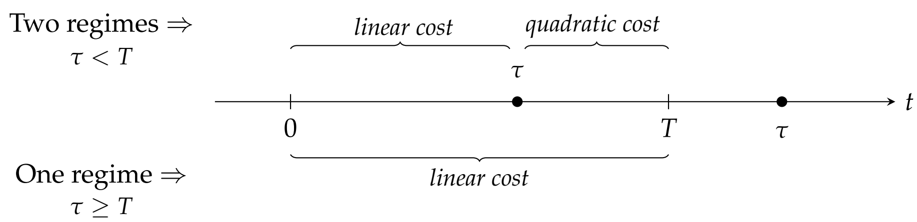

Figure 1. Graphical representation of the stochastic switching time.

Hence, the instantaneous probability of an irreversible degradation of the extraction site depends on the current extraction intensity. In particular, higher extraction rates increase the likelihood that the site undergoes a transition that substantially raises operating costs, for instance, through a change in the cost structure from linear to quadratic. The function $\eta(u_1)$ increases strictly $\eta'(u_1) > 0$ and captures the strength of the relationship between the extraction intensity and the probability of this transition. In the following, we assume that

Assumption 3. *$\eta(u_1) = \eta u_1^2$, hence the hazard rate is quadratic in the extraction rate, whereas $\eta > 0$.*

Let us now translate this problem into a deterministic optimal control problem. The distribution of random variable τ can be defined as follows:

$$z(t) = \mathbb{P}(\tau > t). \tag{3}$$

Starting from (2) we obtain

$$\mathbb{P}(\tau \leq t+h \,|\, \tau > t) = 1 - \mathbb{P}(\tau > t+h \,|\, \tau > t) = \frac{\mathbb{P}(\tau > t) - \mathbb{P}(\tau > t+h)}{\mathbb{P}(\tau > t)}.$$

Therefore, using (3) we get the following.

$$\frac{z(t) - z(t+h)}{z(t)} = \eta u_1^2(t)h + o(h).$$

When $h \to 0$ the previous equation becomes (using the regularity assumption of τ)

$$\dot{z}(t) = -\eta u_1^2(t) z(t).$$

If $z(0) = 1$, we immediately see that the survival probability is

$$z(t) = \mathbb{P}(\tau > t) = \exp\left(-\int_0^t \eta u_1^2(s)ds\right),$$

therefore

$$F_\tau(t) = \mathbb{P}(\tau \leq t) = 1 - \mathbb{P}(\tau > t) = 1 - \exp\left(-\int_0^t \eta u_1^2(s)ds\right).$$

Hence, the density function is

$$f_\tau(t) = \eta u_1^2(t) \exp\left(-\int_0^t \eta u_1^2(s)ds\right) = \eta u_1^2(t) z(t) = -\dot{z}(t).$$

Now, we want to find the expected value using the so called dynamic programming technique. We summarize all the results in the following Theorem. The idea is that we can solve the optimal control problem using Assumption 2: the final constraint on the state variable is always feasible.

Theorem 2. *Consider the second-stage optimal control problem*

$$\max_{u_2(\cdot)\in[0,U]} \int_t^T \Big(\mu u_2(s) - \frac{\kappa_2}{2}u_2^2(s)\Big)ds \tag{4}$$

subject to

$$\dot{x}_2(s) = -u_2(s), \qquad x_2(t) = x > 0.$$

Assume that Assumptions 1 and 2, and

$$\frac{\mu}{\kappa_2} < U \tag{5}$$

holds, then the optimal control is constant and given by

$$u_2^*(s) = \frac{\mu}{\kappa_2}, \qquad s \in [t, T],$$

and the associated value function is

$$V(t,x) = \frac{\mu^2}{2\kappa_2}(T-t),$$

which is independent of the initial stock x.

Proof. We apply Pontryagin's Maximum Principle [13]. The Hamiltonian associated with problem (4) is

$$H(x_2, u_2, p) = \mu u_2 - \frac{\kappa_2}{2}u_2^2 - pu_2,$$

where $p(\cdot)$ is the costate variable. Since the Hamiltonian does not depend explicitly on the state variable x_2, the costate equation reads $\dot{p}(s) = 0$, implying that $p(s)$ is constant on $[t, T]$. By Assumption 2 we have that $p(T) = 0$, Therefore, $p(s) = 0$ for all $s \in [t, T]$. Substituting $p(t) = 0$ into the Hamiltonian, the maximization problem reduces to

$$\max_{u_2\in[0,U]} \Big\{\mu u_2 - \frac{\kappa_2}{2}u_2^2\Big\}.$$

The first-order condition for an interior maximum is

$$\mu - \kappa_2 u_2 = 0,$$

which, using Assumption 1, gives

$$u_2^* = \frac{\mu}{\kappa_2}.$$

Since the Hamiltonian is strictly concave in u_2 and $\mu/\kappa_2 < U$ by assumption, this control is admissible and globally optimal. The optimal state trajectory is therefore

$$\dot{x}_2^*(s) = -\frac{\mu}{\kappa_2}, \qquad x_2^*(t) = x,$$

and the value of the objective function does not depend on x. Substituting the optimal control into the payoff functional, we obtain

$$V(t,x) = \int_t^T \left(\mu\frac{\mu}{\kappa_2} - \frac{\kappa_2}{2}\left(\frac{\mu}{\kappa_2}\right)^2\right)ds = \int_t^T \frac{\mu^2}{2\kappa_2}\,ds = \frac{\mu^2}{2\kappa_2}(T-t).$$

This concludes the proof. □

The content of this theorem suggests to introduce another assumption coming from (5).

Assumption 4. *We assume that, when the extraction cost rate is quadratic, the maximum extraction rate cannot be optimal. Hence, we assume that*

$$\frac{\mu}{\kappa_2} - U < 0.$$

This implies that if this inequality is not satisfied, our problem degenerates, as it would be considered optimal to extract at the maximum level in each stage.

Remark 2. *As in the previous optimal control problem, we note that, under Assumption 2, this problem can also be reformulated as a static optimization problem. Nevertheless, we choose to present it in a dynamic framework to facilitate the extension of the model to more general settings, as discussed in the conclusion. Although the proof follows directly from Pontryagin's Maximum Principle, it may still be useful when considering extensions of the model in which Assumption 2 is relaxed.*

The results shown in the previous propositions allow us to modify the model using a direct computation for the expected value with respect to the switching time random variable:

$$\max_{u_1(t)\in[0,U]} \quad \int_0^T \left[\int_0^s \mu u_1(t)dt + \frac{\mu^2}{2\kappa_2}(T-s)\right](-\dot{z}(s))ds + z(T)\int_0^T \mu u_1(t)dt$$

$$\begin{aligned}
&\dot{x}_1(t) = -u_1(t)\\
&x_1(0) = \alpha > 0\\
&\dot{z}(t) = -z(t)\eta u_1^2(t)\\
&z(0) = 1.
\end{aligned}$$

We now apply integration by parts to simplify the objective functional

$$\frac{\mu^2}{2\kappa_2}T + \int_0^T z(s)\left[\mu u_1(s) - \frac{\mu^2}{2\kappa_2}\right]ds.$$

Therefore, the optimal control problem becomes (the first addend is constant and does not enter the optimization process):

$$\max_{u_1(t)\in[0,U]} \quad \int_0^T z(s)\left[\mu u_1(s) - \frac{\mu^2}{2\kappa_2}\right]ds$$

$$\begin{aligned}
&\dot{x}_1(t) = -u_1(t)\\
&x_1(0) = \alpha > 0\\
&\dot{z}(t) = -z(t)\eta u_1^2(t)\\
&z(0) = 1.
\end{aligned} \tag{6}$$

5. Optimal Extraction with Switching Time

This section provides a complete characterization of the optimal control. We demonstrate that, even when beginning with two trivial static models, one is led to a problem of sufficient complexity that it does not admit an immediate analytical solution. The remainder of the paper is devoted to establishing that for this specific instance of the model (satisfying

Assumptions 1–4) it is possible to derive an analytically satisfactory characterization of the optimal control. Several related issues are left for future investigation and are described in Section 6.

We briefly comment on the well-posedness of problem (6) and on the existence and uniqueness of an optimal control.

The admissible control set is compact, and the state dynamics are governed by a system of ordinary differential equations that is globally Lipschitz in the state variables for any measurable control bounded in $[0, U]$. Moreover, the running payoff is continuous in all arguments and jointly concave in the control variable, while the state equations depend on the control in a smooth way. These properties ensure the existence of an optimal control by standard results in deterministic optimal control theory [17].

Uniqueness follows from the strict concavity of the Hamiltonian with respect to the control on the interior of the admissible set, which implies pointwise uniqueness of the maximizer almost everywhere. Since the state equations admit a unique solution for any admissible control, the resulting optimal trajectory is uniquely determined. In addition, the Hamiltonian maximization condition together with the adjoint system yields a necessary condition that is also sufficient for global optimality, due to the concavity of the Hamiltonian and the absence of state constraints binding along the optimal path. Hence, the candidate control characterized in Theorem 3 is not only locally optimal but globally optimal over the admissible class [17]. Although the control enters nonlinearly through the survival probability, the problem remains deterministic and finite-dimensional after the introduction of the auxiliary state variable. The regularity of the hazard rate and the boundedness of the control ensure that the survival probability remains strictly positive and continuously differentiable on $[0, T]$, guaranteeing that the optimal control problem is well defined over the entire planning horizon.

Theorem 3. *The optimal control of (6) is the function*

$$u_1^*(t) = \begin{cases} U & t \in [t^*, T] \\ \frac{\mu}{2\eta p_2(t)} & t \in [0, t^*], \end{cases}$$

where

$$t^* = \max\left\{0, T + \frac{1}{\eta U^2} \ln\left(1 + \frac{U}{2\left(\frac{\mu}{2\kappa_2} - U\right)}\right)\right\}$$

and

$$p_2(t) = \frac{\kappa_2}{2\eta}\left[1 + W_0\left(\left(\frac{\mu}{U\kappa_2} - 1\right)\exp\left(\frac{\mu}{U\kappa_2} - 1 - \frac{\eta\mu^2}{\kappa_2^2}(t^* - t)\right)\right)\right]$$

with W_0 the principal branch of the Lambert function.

Proof. We apply Pontryagin's Maximum Principle [13]. The state variables are $x_1(t)$ and $z(t)$, whose dynamics are given by

$$\dot{x}_1(t) = -u_1(t), \qquad x_1(0) = \alpha > 0,$$

$$\dot{z}(t) = -\eta u_1^2(t)\, z(t), \qquad z(0) = 1,$$

and control constraint $u_1(t) \in [0, U]$. The payoff is

$$\int_0^T z(t)\left(\mu u_1(t) - \frac{\mu^2}{2\kappa_2}\right) dt.$$

Since $z(0) = 1$ and $\dot{z}(t) = -\eta u_1^2(t) z(t)$, it follows that

$$z(t) = \exp\left(-\int_0^t \eta u_1^2(s)\,ds\right) \in (0,1], \qquad \forall t \in [0,T].$$

From now on, the proof is organized step by step.

1. Hamiltonian and adjoint equations. Let $(p_1(t), p_2(t))$ be the costate variables associated with $(x_1(t), z(t))$. The Hamiltonian is

$$H(x_1, z, u_1, p_1, p_2) = z\left(\mu u_1 - \frac{\mu^2}{2\kappa_2}\right) - p_1 u_1 - p_2\,\eta u_1^2 z.$$

Since H does not depend on x_1, the adjoint equation for p_1 is

$$\dot{p}_1(t) = -\partial_{x_1} H = 0,$$

hence p_1 is constant. The terminal state $x_1(T)$ is free (by Assumption 2 the nonnegativity constraint on x_1 never binds, and there is no terminal payoff), the transversality condition implies $p_1(T) = 0$, and therefore

$$p_1(t) \equiv 0 \qquad \text{for all } t \in [0,T].$$

For p_2, we have

$$\partial_z H = \mu u_1 - \frac{\mu^2}{2\kappa_2} - p_2 \eta u_1^2,$$

hence

$$\dot{p}_2(t) = -\partial_z H = -\left(\mu u_1(t) - \frac{\mu^2}{2\kappa_2}\right) + p_2(t)\eta u_1^2(t), \qquad p_2(T) = 0, \tag{7}$$

where $p_2(T) = 0$ follows because $z(T)$ is free and there is no terminal payoff.

2. Pointwise maximization. Using $p_1 \equiv 0$, the Hamiltonian maximization over $u_1 \in [0, U]$ is equivalent to maximizing

$$\Phi(u_1, t) := z(t)\left(\mu u_1 - \frac{\mu^2}{2\kappa_2}\right) - p_2(t)\eta u_1^2 z(t).$$

Since $z(t) > 0$, the first-order condition for an interior maximizer is

$$\partial_{u_1} \Phi(u_1, t) = z(t)(\mu - 2\eta p_2(t) u_1) = 0,$$

which gives the candidate

$$u_1^\circ(t) = \frac{\mu}{2\eta p_2(t)}.$$

Moreover,

$$\partial_{u_1}^2 \Phi(u_1, t) = -2\eta p_2(t) z(t).$$

Hence, whenever $p_2(t) > 0$ the map $u_1 \mapsto \Phi(u_1, t)$ is strictly concave and the maximizer is unique. Therefore the optimal control has the feedback form

$$u_1^*(t) = \min\left\{U, \frac{\mu}{2\eta p_2(t)}\right\}. \tag{8}$$

3. Behavior near T and the switching time. Since $p_2(T) = 0$, the unconstrained maximizer $\mu/(2\eta p_2(t))$ diverges as $t \uparrow T$, hence $u_1^*(t) = U$ for t sufficiently close to T. Substituting $u_1^*(t) = U$ into (7) yields, on such an interval,

$$\dot{p}_2(t) = \frac{\mu^2}{2\kappa_2} - \mu U + \eta U^2 p_2(t) = \mu\left(\frac{\mu}{2\kappa_2} - U\right) + \eta U^2 p_2(t).$$

By Assumption 4, $\mu/\kappa_2 < U$, hence $\mu/(2\kappa_2) - U < 0$ and in particular

$$\dot{p}_2(T) = \frac{\mu^2}{2\kappa_2} - \mu U < 0,$$

which implies $p_2(t) > 0$ for $t < T$ sufficiently close to T. Solving the linear ODE backward from $p_2(T) = 0$ gives

$$p_2(t) = \frac{\mu\left(U - \frac{\mu}{2\kappa_2}\right)}{\eta U^2}\left(1 - e^{-\eta U^2 (T-t)}\right), \qquad t \in [t^*, T], \tag{9}$$

for some (possibly zero) switching time $t^* \in [0, T]$ to be determined. The switch from the boundary arc $u_1^* = U$ to the interior arc occurs when

$$U = \frac{\mu}{2\eta p_2(t^*)} \quad \Longleftrightarrow \quad p_2(t^*) = \frac{\mu}{2\eta U}.$$

Using (9) and solving for t^* yields

$$t^* = T + \frac{1}{\eta U^2} \ln\left(1 + \frac{U}{2\left(\frac{\mu}{2\kappa_2} - U\right)}\right).$$

If this value is negative, then the constraint binds over the whole horizon and $u_1^*(t) \equiv U$. Hence we set

$$t^* = \max\left\{0,\ T + \frac{1}{\eta U^2} \ln\left(1 + \frac{U}{2\left(\frac{\mu}{2\kappa_2} - U\right)}\right)\right\}.$$

Note also that the control is continuous at t^* since $\mu/(2\eta p_2(t^*)) = U$.

4. Interior arc and Lambert-W_0 representation. In a left neighborhood of t^* we have $u_1^*(t) = \mu/(2\eta p_2(t))$ because $\dot{p}_2(t^*) < 0$. Substituting this into (7) yields the scalar autonomous ODE

$$\dot{p}_2(t) = \frac{\mu^2}{2\kappa_2} - \frac{\mu^2}{4\eta p_2(t)}, \qquad p_2(t^*) = \frac{\mu}{2\eta U}. \tag{10}$$

This ODE has equilibrium $p_2^{eq} = \kappa_2/(2\eta)$, and Assumption 4 implies $p_2^{eq} > \mu/(2\eta U)$, so the solution remains positive and decreases in $[0, t^*]$. Therefore, $u_1^*(t) = \mu/(2\eta p_2(t))$ in the whole $[0, t^*]$. Separating variables and integrating gives the implicit identity

$$\frac{2\eta}{\kappa_2} p_2(t) + \ln(\kappa_2 - 2\eta p_2(t)) = \frac{\mu}{U\kappa_2} + \ln\left(\kappa_2 - \frac{\mu}{U}\right) - \frac{\eta\mu^2}{\kappa_2^2}(t^* - t).$$

Let $y(t) := \kappa_2 - 2\eta p_2(t)$. Then the previous equation becomes

$$\ln y(t) - \frac{y(t)}{\kappa_2} = \frac{\mu}{U\kappa_2} + \ln\left(\kappa_2 - \frac{\mu}{U}\right) - \frac{\eta\mu^2}{\kappa_2^2}(t^* - t) - 1,$$

hence

$$y(t)e^{-y(t)/\kappa_2} = \left(\kappa_2 - \frac{\mu}{U}\right)\exp\left(\frac{\mu}{U\kappa_2} - 1 - \frac{\eta\mu^2}{\kappa_2^2}(t^* - t)\right).$$

Applying the principal branch W_0 of the Lambert function yields

$$-\frac{y(t)}{\kappa_2} = W_0\left(\left(\frac{\mu}{U\kappa_2} - 1\right)\exp\left(\frac{\mu}{U\kappa_2} - 1 - \frac{\eta\mu^2}{\kappa_2^2}(t^* - t)\right)\right),$$

and therefore

$$p_2(t) = \frac{\kappa_2}{2\eta}\left[1 + W_0\left(\left(\frac{\mu}{U\kappa_2} - 1\right)\exp\left(\frac{\mu}{U\kappa_2} - 1 - \frac{\eta\mu^2}{\kappa_2^2}(t^* - t)\right)\right)\right].$$

Combining this expression with (8) yields the claimed form of the optimal feedback control:

$$u_1^*(t) = \begin{cases} U, & t \in [t^*, T], \\ \dfrac{\mu}{2\eta p_2(t)}, & t \in [0, t^*]. \end{cases}$$

□

The analytical tractability of the model relies on a specific set of functional forms: linear revenues, a quadratic post-switch extraction cost, and a quadratic hazard rate linking extraction intensity to the probability of regime switching. These assumptions are instrumental in obtaining a closed-form solution and, in particular, in expressing the optimal pre-switch control and the switching time explicitly in terms of the Lambert W_0 function. It is therefore important to distinguish clearly between those features of the optimal policy that are structural and those that are artifacts of the quadratic specification.

The underlying economic mechanism does not rely on the quadratic specification itself. What drives the smoothing of extraction and the emergence of an interior phase followed by a terminal boundary phase is the interaction between an increasing hazard rate and an irreversible deterioration of extraction conditions. Higher extraction today increases current profits, but also raises the probability of entering a high-cost regime whose consequences are borne over the remaining horizon. This intertemporal trade-off generates a precautionary incentive to moderate extraction early on, when the option value of preserving the low-cost regime is high, and to accelerate extraction later as the terminal date approaches and the value of this option vanishes. These forces operate whenever the hazard rate is increasing and convex in extraction, and the post-switch problem features higher marginal costs, regardless of the exact curvature of these functions.

Assumption 3, which specifies a quadratic hazard rate $\eta(u_1) = \eta u_1^2$, plays a purely technical role in the analysis. Its purpose is to ensure analytical tractability and to allow for a closed–form characterization of the optimal pre–switch control and of the switching time t^*, which can be expressed explicitly in terms of the Lambert W_0 function. The quadratic specification is not essential for the underlying economic mechanism. If the hazard rate were replaced by a more general non–negative increasing and convex function of the extraction rate, the same qualitative trade–off would persist: higher extraction would increase current profits but also raise the instantaneous risk of entering the irreversible high–cost regime, thereby inducing precautionary incentives early in the planning horizon. What would be lost in the absence of Assumption 3 is not the structure of the optimal policy, but its explicit closed–form representation. In that case, the optimal control would still be characterized by Pontryagin's Maximum Principle, but the resulting system would generally require an implicit or numerical solution.

By contrast, the precise analytical shape of the optimal control, the explicit formula for the switching time, and the feedback representation in terms of the Lambert W_0 function are parametric outcomes tied to the quadratic specification. Under alternative functional forms—such as non-quadratic convex costs or more general increasing hazard rates (the same economic logic would apply, but the optimal policy would typically need to be characterized implicitly or computed numerically). In such cases, the threshold structure would persist in qualitative terms, while its exact location and the time profile of the control would depend on the specific functional assumptions.

The present framework should therefore be interpreted as a benchmark that isolates the core economic mechanism linking extraction intensity to endogenous degradation risk in a transparent and analytically tractable way. The quadratic specification allows this mechanism to be fully characterized in closed form and provides a useful reference point for more general models in which explicit solutions are no longer available. This explicit solvability helps clarify how endogenous risk alone can generate precautionary extraction behavior, even in the absence of scarcity.

Although the optimal feedback can be written explicitly in terms of the Lambert W_0 function, the qualitative behavior of the solution is more transparently conveyed by its numerical representation, which highlights the monotonicity of the interior arc and the emergence of a terminal boundary phase.

Therefore, to complement the analytical characterization, we briefly illustrate the optimal policy with a simple numerical example. We fix parameter values that satisfy Assumptions 1–4 and compute the optimal control using the closed-form expressions derived above. The resulting extraction path highlights the qualitative properties of the solution. In particular, extraction is initially internal and strictly below capacity, and increases smoothly over time. At the threshold t^*, the policy reaches the upper limit and remains in capacity until the end date. This numerical illustration confirms that endogenous degradation risk induces a precautionary smoothing of extraction relative to the deterministic benchmark. It also clarifies the role of the switching time and helps visualize the transition between the interior and boundary regimes implied by the analytical solution.

Figure 2 illustrates a numerical example of optimal pre-switch extraction control $u_1^*(t)$ on the planning horizon $[0, T]$, with parameters fixed at $T = 10$, $U = 5$, $\kappa_2 = 0.5$ and $\mu = 2$, while varying the parameter of hazard sensitivity η. The parameter η governs the instantaneous probability rate of switching to the high–cost regime and therefore measures how strongly the extraction intensity affects the risk of irreversible degradation. The figure reports the optimal control in stage 1 for several values of η, highlighting the analytic structure $u_1^*(t) = \min\{U, \mu/(2\eta p_2(t))\}$, which consists of an initial interior arc followed by a terminal boundary arc where the capacity constraint is bound. As η increases, the probability of switching to stage 2 increases for any given extraction level, making the expected cost of aggressive extraction greater. Hence, the optimal policy becomes more cautious, with lower extraction rates early in the planning horizon to reduce the probability of a costly regime shift.

This behavior contrasts sharply with the benchmark case in which the possibility of switching is ignored, where the optimal control would be constant and equal to the capacity bound $U = 5$ throughout the horizon. The saturation of the control constraint occurs only in the final portion of the planning interval, reflecting the fact that near the end of the programming interval the option value of preserving the low–cost regime vanishes: at that point, maintaining a conservative policy becomes more expensive than extracting at full capacity or tolerating a late switch to stage 2.

Figure 3 illustrates the time profile of the optimal extraction control when the parameters are fixed as in Figure 2 and the hazard sensitivity is set to $\eta = 0.025$. In this case, several possible switching times from stage 1 to stage 2 are simulated exogenously and are represented by vertical dashed lines, equidistantly spaced over the interval $[0, T]$. For each realized switching time, the control follows the optimal pre–switch feedback policy $u_1^*(t)$ up to the switching date and then instantaneously jumps to the optimal post–switch policy, which is constant and given by $u_2^*(t) = \mu/\kappa_2 = 4$ for the chosen parameter values. As the switching time increases, the control trajectory follows the interior arc of the stage 1 optimal policy for a longer portion of the horizon, reflecting a prolonged period in which the low–cost regime is preserved. Conversely, earlier switches truncate the interior arc and force the planner to adopt the constant stage 2 control sooner, thereby exposing the system earlier to higher marginal costs. The figure highlights how the realized timing of an irreversible regime change interacts with the optimal pre–switch policy: later switches allow the planner to fully exploit the cautious interior trajectory implied by the endogenous hazard, while earlier switches shorten this phase and reduce the scope for precautionary behavior.

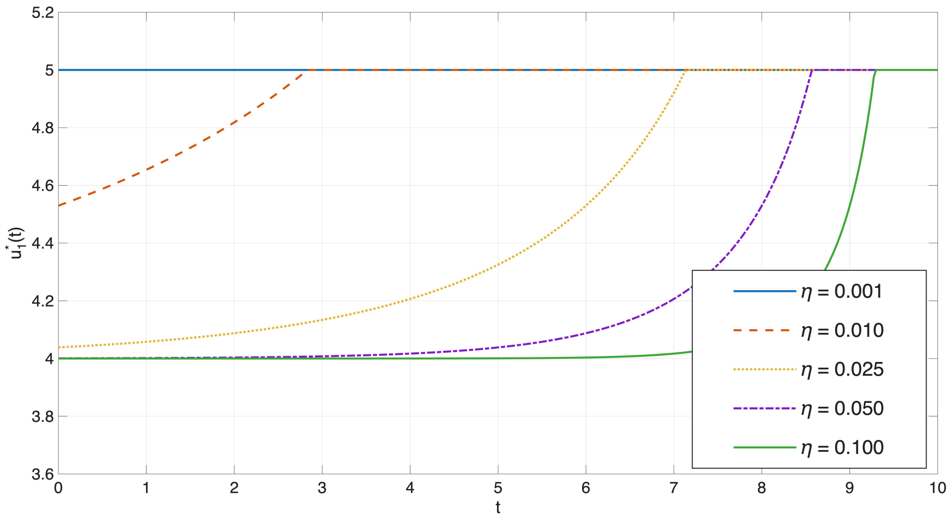

Figure 2. Numerical example of the optimal pre-switch extraction control $u_1^*(t)$.

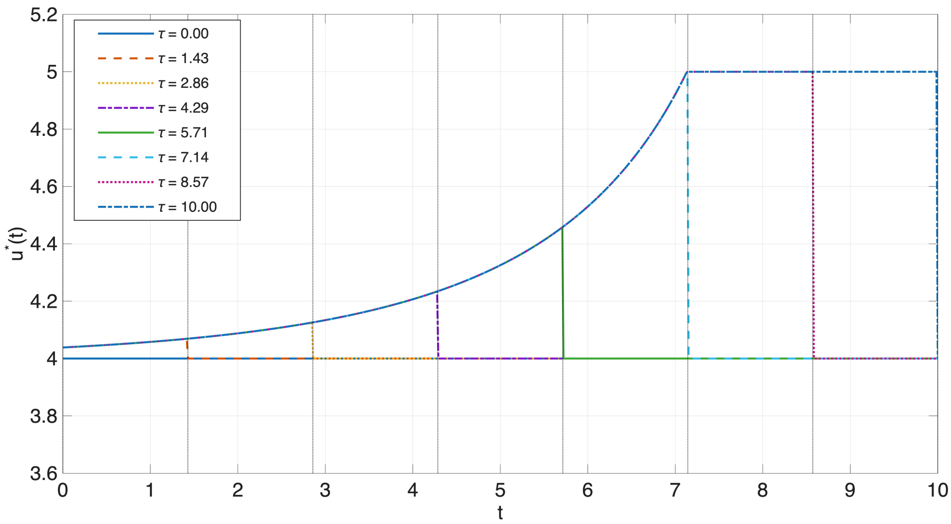

Figure 3. Sample paths of $u^*(t)$ with different switching times τ.

To further clarify the novelty of the framework, it is useful to contrast it with two related classes of models. In models with exogenous regime switching, the probability of a cost change is independent of the control, so that extraction affects expected payoffs but not the law of motion of uncertainty itself. In models with stock-dependent costs, dynamics arise from scarcity through the shadow value of the resource. By contrast, the present model generates nontrivial dynamics even in the absence of scarcity, because the control directly influences the evolution of the survival probability. This endogenous feedback introduces a distinct source of nonlinearity that is absent in both classes of models.

6. Comments and Conclusions

This paper has studied optimal extraction of a non-renewable resource when the extraction environment may deteriorate irreversibly at an endogenous and uncertain time. The baseline model with constant linear marginal cost delivers the familiar bang–bang outcome: under a positive margin and a non-binding stock constraint over the horizon, it is optimal to extract at capacity for the entire planning interval. We then depart from this deterministic setting by allowing extraction costs to switch from a low-cost linear regime to a high-cost regime with an additional quadratic term, where the switching time is stochastic and governed by a hazard rate that depends on the pre-switch extraction intensity.

The key modeling step was to translate the two-stage problem with random switching time into a deterministic optimal control problem by introducing the survival probability of the low-cost regime as an auxiliary state. Although each stage, taken in isolation, is essentially static under Assumption 2, the resulting deterministic equivalent problem is genuinely dynamic and nonlinear, because current extraction affects future payoffs through its effect on the likelihood of entering the high-cost regime. This wedge is precisely what is missing in standard deterministic formulations and in models where cost shifts arrive exogenously.

As just underlined, Assumption 2 plays a central role in the analytical tractability of the model and deserves an explicit interpretative discussion. By ensuring that the initial stock is sufficiently large relative to the maximal feasible extraction rate and the planning horizon, Assumption 2 guarantees that the stock constraint never binds along the optimal path. Economically, this eliminates scarcity effects and implies a zero shadow value of the resource throughout the problem. As a consequence, the continuation value after a regime switch is independent of the remaining stock, which is the key reason why the stochastic control problem can be reduced to a low-dimensional deterministic one. This simplification is not merely technical: it removes the intertemporal feedback from remaining reserves to current extraction incentives, allowing the optimal policy to be characterized without tracking a scarcity rent. Relaxing Assumption 2 would reintroduce scarcity effects and fundamentally alter the structure of the problem. When the stock constraint becomes potentially binding, the continuation value after the switch depends on the remaining stock, the shadow value of the resource becomes positive, and extraction decisions reflect an explicit trade-off between current returns and preserving future flexibility. In this case, the problem no longer admits a closed-form reduction and instead leads to a higher-dimensional control problem with endogenous shadow prices. We expect the main qualitative insights to carry over to more general specifications. In particular, the tension between higher extraction, increasing current profits while raising the likelihood of an adverse regime switch, and the role of uncertainty in shaping precautionary behavior continues to operate even when scarcity effects are present. Our results can therefore be interpreted as a benchmark that isolates these forces in the absence of scarcity, while highlighting the mechanisms through which relaxing Assumption 2 would enrich—rather than overturn—the economic trade-offs captured by the model.

The main analytical contribution is the closed-form characterization of the optimal pre-switch policy under Assumptions 1–4 and a quadratic hazard. The optimal extraction rule exhibits a threshold structure with a single switching time t^*: the extraction is interior and time-varying on $[0, t^*]$, and it reaches the capacity constraint U on $[t^*, T]$. Intuitively, early in the horizon, the planner moderates extraction to preserve the low-cost regime by keeping the endogenous hazard low; as the terminal date approaches, the value of preserving the low-cost regime diminishes and the optimal policy converges to an "extract fast" strategy. In this sense, the model generates a form of precautionary smoothing that is absent in the linear-cost benchmark, while still allowing for aggressive late-horizon extraction. The explicit solution in terms of the Lambert function W_0 clarifies how primitives shape behavior, higher hazard sensitivity η, strengthens early conservatism and tends to push t^* forward, while a higher post-switch curvature κ_2 increases the expected cost of switching and similarly makes delaying degradation more valuable. By contrast, a higher capacity U relaxes the physical constraint and, depending on the parameters, can enlarge the region in which the interior policy is optimal.

A natural question concerns the robustness of the qualitative structure of the optimal policy, characterized by early moderation and late acceleration, under alternative specifications of the degradation risk. Although our analysis focuses on a quadratic hazard rate depending on the current extraction flow, the underlying mechanism generating this pattern is more general. The incentive to moderate extraction early on arises from the fact that a higher current intensity raises the probability of entering an irreversible high-cost regime, the expected cost of which is borne over the remaining horizon. As long as the hazard rate increases with extraction, the same intertemporal trade-off persists. Lower early extraction reduces the risk of costly degradation, while future rewards still matter. Later in the horizon, the option value of preserving the low-cost regime diminishes, and accelerated extraction becomes optimal. Letting the hazard depend on cumulative extraction or remaining stock would reinforce this logic without changing the basic trade-off between early precaution and late extraction. Such extensions would generally rule out closed-form solutions and require numerical methods. Nevertheless, we expect the qualitative timing result to remain robust, with moderation early on and more aggressive extraction near the end of the programming interval.

Several extensions can broaden the scope of the framework and relax the strongest assumptions. First, relaxing Assumption 2 (so that depletion may bind) would reintroduce an active stock constraint and a non-trivial shadow value of the remaining resource, likely generating additional switching surfaces and potentially multiple regimes (interior extraction, capacity extraction, and eventual inactivity). In this respect, Theorems 1 and 2 provide a useful benchmark: they help clarify how relaxing Assumption 2 would affect the costate variable and how the pre-switch and post-switch problems would become more complex. Second, allowing the hazard rate to depend not only on u_1 but also on cumulative extraction or remaining stock, $\eta = \eta(u_1, x)$, would capture geological depletion and ecological fatigue more realistically and would sharpen the link between scarcity and the risk of degradation. Finally, extending the model to multiple switching events or to partial reversibility would allow the analysis of repeated shocks and recovery processes, which are empirically relevant.

Although these extensions are promising, their application will likely require numerical methods for their analysis. In this sense, the closed-form solution derived in this paper can serve as a useful benchmark and validation test for computational approaches.

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