Research Article

Fabio Paronetto*

Harnack inequality for parabolic equations with coefficients depending on time

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Abstract: We define a homogeneous De Giorgi class of order $p = 2$ that contains the solutions of evolution equations of the types $\rho(x, t)u_t + Au = 0$ and $(\rho(x, t)u)_t + Au = 0$, where $\rho > 0$ almost everywhere and A is a suitable elliptic operator. For functions belonging to this class, we prove a Harnack inequality. As a byproduct, one can obtain Hölder continuity for solutions of a subclass of the first equation.

Keywords: Parabolic equations, De Giorgi classes, Harnack inequality, weights

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1 Introduction

In this paper, we consider the two parabolic equations

$$\rho(x, t)\frac{\partial u}{\partial t} - \operatorname{div}(A(x, t, u, Du)) + B(x, t, u, Du) + C(x, t, u) = 0 \quad \text{in } \Omega \times (0, T), \tag{1.1}$$

$$\frac{\partial}{\partial t}(\rho(x, t)u) - \operatorname{div}(A(x, t, u, Du)) + B(x, t, u, Du) + C(x, t, u) = 0 \quad \text{in } \Omega \times (0, T), \tag{1.2}$$

where the coefficient ρ is in L^∞ and $\rho > 0$ almost everywhere, and therefore it can degenerate to zero. The functions

$$\begin{aligned} A &: \Omega \times (0, T) \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \\ B &: \Omega \times (0, T) \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}, \\ C &: \Omega \times (0, T) \times \mathbb{R} \to \mathbb{R} \end{aligned}$$

are Carathéodory functions with $A, B, C \in L^\infty(\Omega \times (0, T) \times \mathbb{R} \times \mathbb{R}^n)$ satisfying

$$\begin{aligned} (A(x, t, u, \xi), \xi) &\geqslant \lambda|\xi|^2, \\ |A(x, t, u, \xi)| &\leqslant \Lambda|\xi|, \\ |B(x, t, u, \xi)| &\leqslant M|\xi|, \\ |C(x, t, u)| &\leqslant N|u|, \end{aligned}$$

for some given positive constants λ, Λ, M, N and for every $u \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ and a.e. $(x, t) \in \Omega \times (0, T)$.

We show that functions belonging to a suitable De Giorgi class containing the solutions of equations (1.1) and (1.2) satisfy a Harnack inequality. In particular, we show the following result, but we refer to Theorem 5.1 and Theorem 5.2 for the precise statements.

***Corresponding author: Fabio Paronetto,** Dipartimento di Matematica “Tullio Levi Civita”, Università di Padova, via Trieste 63, 35121, Padova, Italy, e-mail: fabio.paronetto@unipd.it

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Theorem. *There is a constant $c > 0$ such that for every $(x_o, t_o) \in \Omega \times (0, T)$ and $r > 0$ such that*

$$B_{5r}(x_o) \times (t_o - (5r)^2 \mathsf{h}_*(x_o, t_o, 5r), t_o + (5r)^2 \mathsf{h}^*(x_o, t_o, 5r)) \subset \Omega \times (0, T)$$

and

$$B_r(x_o) \times (t_o, t_o + 5r^2 \mathsf{h}^*(x_o, t_o, r)) \subset \Omega \times (0, T),$$

if $u \geqslant 0$ is a solution of (1.1) *or* (1.2)*, then*

$$\sup_{B_r(x_o)} u(x, t_o - r^2 \mathsf{h}_*(x_o, t_o, r)) \leqslant c \inf_{B_r(x_o)} u(x, t_o + r^2 \mathsf{h}^*(x_o, t_o, r)),$$

where

$$\mathsf{h}^*(x_o, t_o, r) = \frac{1}{r^2 |B_r(x_o)|} \int_{t_o}^{t_o+r^2} \int_{B_r(x_o)} \rho(x, t)\, dx\, dt,$$
$$\mathsf{h}_*(x_o, t_o, r) = \frac{1}{r^2 |B_r(x_o)|} \int_{t_o-r^2}^{t_o} \int_{B_r(x_o)} \rho(x, t)\, dx\, dt.$$

Moreover, adapting to the parabolic case the classical simple argument due to Moser (see [18], but also, e.g., [12, Section 7.9]), one can show that solutions of (1.1) with $C = 0$ are locally Hölder continuous (see also Remark 3.3).

Equations like (1.1) and (1.2) arise as natural generalizations of the case $\rho = \rho(x)$ to the time dependent case $\rho = \rho(x, t)$. Applications can be found in some diffusion and in fluid flow problems (see [2, Chapter 3], in particular [2, Example 5.14]). In particular, this occurs in hydrology when dealing with transport of contaminants in unsaturated or variably saturated media (see [7]) or in density dependent flow in porous media (see, for instance, [17]).

There is a wide literature regarding the Harnack inequality for parabolic equations. Starting from the very first results due to Hadamard and Pini (see [16, 25]), the next important step due to Moser (see [19]) was to consider linear parabolic equations in divergence form. Due to the nature of this parabolic operator (in particular, its invariance with respect to the scaling $x \mapsto hx$, $t \mapsto h^2 t$), the natural Harnack inequality for a solution u is

$$\sup_{B_r(x_o)} u(x, t_o - r^2) \leqslant C \inf_{B_r(x_o)} u(x, t_o + r^2). \tag{1.3}$$

We also mention the papers [1, 26], where operators with linear growth, but possibly nonlinear, are considered and an inequality of the type (1.3) is derived.

Nevertheless, unlike in the elliptic case, for parabolic operators with different growth the problem is more delicate. In 1986, DiBenedetto showed that a Harnack inequality analogous to (1.3), i.e. substituting 2 with p, cannot hold for solutions of the equation

$$u_t - \operatorname{div}(|Du|^{p-2} Du) = 0, \quad p > 2,$$

even if this equation is invariant with respect to the scaling $x \mapsto hx$, $t \mapsto h^p t$ (see [8]).

A modified Harnack inequality for this equation was proved by DiBenedetto, Gianazza and Vespri only in 2008 (see [9]).

Arriving to parabolic equations with degenerate coefficients of the type

$$v(x) \frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \Big(a_{ij}(x, t) \frac{\partial u}{\partial x_j} \Big) = 0$$

with

$$w_1(x, t)|\xi|^2 \leqslant (a(x, t)\xi, \xi) \leqslant L w_2(x, t)|\xi|^2, \quad L \geqslant 1,$$

and $v, w_1, w_2 > 0$ almost everywhere, we recall the papers [4–6], where Chiarenza and Serapioni considered

$$v \equiv 1, \quad w_1 = w_2 = w$$

with $w \in L^1_{\mathrm{loc}}$, $w > 0$ a.e., satisfying a Muckenhoupt condition A_2 (a subclass of A_∞ defined below in Definition 2.1) in the variable x uniformly with respect to time, i.e. there is $C > 0$ such that

$$\frac{1}{|B|}\int_B w(x,t)\,dx \frac{1}{|B|}\int_B w^{-1}(x,t)\,dx \leqslant C \quad \text{for every ball } B \text{ and every } t,$$

or equations with

$$v(x) = w_1(x) = w_2(x)$$

satisfying the A_2 condition. In [13, 14], a more general situation is considered:

$$v = v(x), \quad w_1 = w_1(x,t), \quad w_2 = w_2(x,t),$$

with w_1 possibly different from w_2.

In the present paper, we focus our attention on the coefficient in front of u, taking for this reason $w_1 = w_2 \equiv 1$ and $v = \rho(x,t)$, $\rho > 0$, almost everywhere and bounded, so possibly degenerating only to zero (and not to $+\infty$).

We conclude saying that we use a technique due to DiBenedetto, Gianazza and Vespri, which adapts in some sense the technique of De Giorgi (to prove boundedness and regularity of the solutions) for the elliptic case to the parabolic case. This is mainly contained in [11].

2 Preliminaries

From now on, we will consider an open set $\Omega \subset \mathbb{R}^n$, $T > 0$ and a function

$$\rho \in L^\infty(\Omega \times (0,T)), \quad \rho > 0 \text{ almost everywhere.}$$

We define for each $t \in (0,T)$,

$$L^2(\Omega, \rho(t)) = \{u \in L^1_{\mathrm{loc}}(\Omega) \mid u\rho^{1/2}(t) \in L^2(\Omega)\}$$

and

$$L^2(\Omega \times (0,T), \rho) = \{u \in L^1_{\mathrm{loc}}(\Omega \times (0,T)) \mid u\rho^{1/2} \in L^2(\Omega \times (0,T))\}.$$

Moreover, we define for each $t \in (0,T)$,

$$H^1(\Omega, \rho(t)) := \{u \in L^2(\Omega, \rho(t)) \mid Du \in L^2(\Omega)\}.$$

By Du we will always denote the vector of derivatives of a function u with respect to the variables $x_1, \dots, x_n$, $(x_1, \dots, x_n) \in \Omega$, even if u depends also on t. Then we define

$$\mathcal{V} := \{u \in L^2(\Omega \times (0,T), \rho) \mid Du \in L^2(\Omega \times (0,T))\}$$

and denote by $\mathcal{V}'$ the dual space of $\mathcal{V}$. We also define

$$\mathcal{V}_{\mathrm{loc}} := \{u \in L^2_{\mathrm{loc}}(\Omega \times (0,T), \rho) \mid Du \in L^2_{\mathrm{loc}}(\Omega \times (0,T))\}.$$

In the following, we will sometimes write

$$\rho(E) \quad \text{instead of} \quad \iint_E \rho(x,t)\,dx\,dt, \qquad E \subset \Omega \times (0,T),$$

$$\rho(t)(A) \quad \text{instead of} \quad \int_A \rho(x,t)\,dx, \qquad A \subset \Omega.$$

In [20, 23], some existence results are proved (where ρ may also be zero and negative). As byproducts, we get the existence of solutions for the following equations (A, B and C introduced in the previous section):

$$\rho(x,t)\frac{\partial u}{\partial t} - \operatorname{div}(A(x,t,u,Du)) + B(x,t,u,Du) + C(x,t,u) = 0,$$
$$\frac{\partial}{\partial t}(\rho(x,t)u) - \operatorname{div}(A(x,t,u,Du)) + B(x,t,u,Du) + C(x,t,u) = 0.$$

About ρ we will need some assumptions; before listing them, we recall some definitions.

Definition 2.1. Given a $\omega \in L^\infty(\mathbb{R}^n)$, $\omega > 0$ a.e., $K > 0$ and $q > 2$, we say that

$$\omega \in B^1_{2,q}(K)$$

if

$$\frac{r}{\rho}\Big(\frac{\omega(B_r(\bar{x}))}{\omega(B_\rho(\bar{x}))}\Big)^{1/q}\Big(\frac{|B_r(\bar{x})|}{|B_\rho(\bar{x})|}\Big)^{-1/2} \leqslant K \tag{2.1}$$

for every pair of concentric balls B_r and B_ρ with $0 < r < \rho$.

We say that ω is *doubling* if for every $t > 0$ there is a positive constant $c_\omega(t)$ such that

$$\int_{tB} \omega(x)\,dx \leqslant c_\omega(t)\int_B \omega(x)\,dx, \tag{2.2}$$

where B is a generic ball $B_r(\bar{x})$ and tB denotes the ball $B_{tr}(\bar{x})$.

We say that ω belongs to the class $A_\infty(K,\varsigma)$ if

$$\frac{\omega(S)}{\omega(B)} \leqslant K\Big(\frac{|S|}{|B|}\Big)^\varsigma$$

for every ball $B \subset \mathbb{R}^n$ and every measurable set $S \subset B$.

We will denote by $A_\infty(K)$ the set $\bigcup_{\varsigma>0} A_\infty(K,\varsigma)$, and $A_\infty = \bigcup_{K>0} A_\infty(K)$.

Remark 2.2. If $\omega \in A_\infty$, then ω is doubling (see [10]).

Remark 2.3. If $\omega \in A_\infty(K)$, then there is $p \geqslant 1$ such that

$$\Big(\frac{|S|}{|B|}\Big)^p \leqslant K\frac{\omega(S)}{\omega(B)}$$

for every measurable $S \subset B$ and every B ball of $\mathbb{R}^n$ (see [10]).

About the function ρ, we will assume that there are some positive constants K_1, K_2, K_3, ς, σ and $L \geqslant 0$ such that the following conditions (H) hold:

$$\begin{cases} [0,T] \ni t \mapsto \displaystyle\int_\Omega v(x)w(x)\rho(x,t)\,dx & \text{is absolutely continuous,} \\ \Big|\displaystyle\iint_\Omega v(x)w(x)\rho(x,t)\,dx\Big| \leqslant L\|v\|_{H^1(\Omega)}\|w\|_{H^1(\Omega)} & \text{for every } v, w \in H^1(\Omega), \end{cases} \tag{H1}$$

$$\rho(\cdot,t) \in B^1_{2,q}(K_1) \quad \text{for almost every } t \in (0,T), \tag{H2}$$
$$\rho(\cdot,t) \in A_\infty(K_2,\varsigma) \quad \text{for almost every } t \in (0,T), \tag{H3}$$
$$\rho \in A_\infty(K_3,\sigma). \tag{H4}$$

Comments on the assumptions. (H1) This assumption is needed in [20, 23] to obtain the results about existence and uniqueness of the solutions of (1.1) and (1.2) with suitable boundary conditions.

Notice that this requirement is done for every $v, w \in H^1(\Omega)$, which turns out to be a dense subset of $H^1(\Omega, \rho(t))$ for each $t \in (0,T)$.

This assumption is clearly satisfied if

$$\rho, \frac{\partial\rho}{\partial t} \in L^\infty(\Omega \times (0,T))$$

since in this case we get

$$\Big|\frac{d}{dt}\int_\Omega v(x)w(x)\rho(x,t)\,dx\Big| = \Big|\iint_\Omega v(x)w(x)\frac{\partial\rho}{\partial t}(x,t)\,dx\Big| \leqslant \|\rho_t\|_\infty\|v\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)}.$$

But more interesting are the cases when $t \mapsto \rho(x,t)$ is not absolutely continuous; it may be only continuous (see an example in [22]) and even discontinuous for every $x \in \Omega$. The simplest example is the following: consider ($\Omega = (0,1)$, $T = 1$)

$$\rho : (0,1) \times (0,1) \to \mathbb{R} \quad \text{such that} \quad \rho(x,t) := \begin{cases} 2 & \text{for } x < t, \\ 1 & \text{for } x > t. \end{cases}$$

Then

$$\frac{d}{dt} \int_\Omega v(x) w(x) \rho(x,t)\, dx = \frac{d}{dt}\Big(2 \int_0^t v(x) w(x)\, dx + \int_t^1 v(x) w(x)\, dx \Big) = v(x) w(x).$$

Since $H^1(0,1) \subset C^0([0,1])$, one can estimate the left-hand side in the previous equality by the H^1-norm of v and w (in this example, $H^1(\Omega, \rho(t)) = H^1(\Omega)$ for every t).

Analogous examples may be considered in higher dimensions, for which we refer to [20, 23].

Assumptions (H2) and (H3) are needed for Theorem 2.8 to hold. Notice moreover that in fact they hold for *every*, and not for almost every, $t \in [0,T]$, thanks to assumption (H1).

Finally, assumption (H4) (more precisely, its consequence (C3) stated below) is needed in Lemma 4.6 and consequently for the expansion of positivity (Theorem 4.8).

Some consequences of the assumptions.

(C1) From assumption (H1), one can derive that for every ball $B \subset \Omega$ there exists a constant γ_B, depending only on B, such that

$$\sup_{t \in [0,T]} \int_B \rho(x,t)\, dx \leqslant \gamma_B \inf_{t \in [0,T]} \int_B \rho(x,t)\, dx.$$

By that, one also derives that

$$\sup_{t \in [0,T]} \Big[\int_B \rho(x,t)\, dx \Big]^{-1} \leqslant \gamma_B \inf_{t \in [0,T]} \Big[\int_B \rho(x,t)\, dx \Big]^{-1}.$$

This is actually a consequence of Lemma 2.4 stated below, which can be proved starting from (H1).

(C2) By Remark (2.2) and since, by (H3), $\rho(t) \in A_\infty(K_2)$ uniformly in t, we obtain that for every $\theta > 0$ there is a constant $c_\rho(\theta)$ (depending on K_2 and θ) such that

$$\int_{\theta B} \rho(x,t)\, dx \leqslant c_\rho(\theta) \int_B \rho(x,t)\, dx \quad \text{for every } t \in (0,T).$$

Similarly, by (H4) we also have that for every $\theta > 0$ there is a constant $C_\rho(\theta)$ such that

$$\int_{t_o - \theta\delta}^{t_o + \theta\delta} \int_{\theta B} \rho(x,t)\, dx\, dt \leqslant C_\rho(\theta) \int_{t_o - \delta}^{t_o + \delta} \int_B \rho(x,t)\, dx\, dt.$$

We will simply write c_ρ or C_ρ if $\theta = 2$.

(C3) Assumptions (H3) and (H4) imply that there is $p \geqslant 1$ such that

$$\Big(\frac{|S|}{|B|}\Big)^p \leqslant K_2 \frac{\rho(t)(S)}{\rho(t)(B)}, \qquad \Big(\frac{|\tilde S|}{|B \times I|}\Big)^p \leqslant K_3 \frac{\rho(\tilde S)}{\rho(B \times I)}$$

for almost every $t \in (0,T)$, every ball $B \subset \mathbb{R}^n$, every interval $I \subset \mathbb{R}$, every measurable set $S \subset B$, and every measurable set $\tilde S \subset B \times I$.

Lemma 2.4. *If* ρ *satisfies* (H1), *then*

$$[0,T] \ni t \mapsto \int_B \rho(x,t)\, dx \quad \textit{is continuous}$$

for every ball $B = B_r(x_o) \subset \Omega$.

Proof. Consider a function ζ such that

$$\zeta \equiv 1 \qquad \text{in } B_r(x_o),$$
$$\zeta \equiv 0 \qquad \text{outside of } B_R(x_o),$$
$$|D\zeta| \leqslant \frac{1}{R-r}.$$

Then, by the assumptions, the function

$$[0,T] \ni t \mapsto \int_{B_R} \zeta^n(x)\rho(x,t)\,dx$$

is continuous for every $n \in \mathbb{N}$, $n \geqslant 1$. Consider now $t, s \in [0,T]$ and estimate as follows:

$$\begin{aligned} |\rho(t)(B_r) - \rho(s)(B_r)| &= \Big| \int_{B_R} \zeta^n(x)\rho(x,t)\,dx - \int_{B_R\setminus B_r} \zeta^n(x)\rho(x,t)\,dx - \int_{B_R} \zeta^n(x)\rho(x,s)\,dx + \int_{B_R\setminus B_r} \zeta^n(x)\rho(x,s)\,dx \Big| \\ &\leqslant \Big| \int_{B_R} \zeta^n(x)\rho(x,t)\,dx - \int_{B_R} \zeta^n(x)\rho(x,s)\,dx \Big| + \Big| \int_{B_R\setminus B_r} \zeta^n(x)\rho(x,t)\,dx - \int_{B_R\setminus B_r} \zeta^n(x)\rho(x,s)\,dx \Big|. \end{aligned}$$

Now fix $\varepsilon > 0$. There is $\delta > 0$ such that if $|t-s| < \delta$, then

$$\Big| \int_{B_R} \zeta^n(x)\rho(x,t)\,dx - \int_{B_R} \zeta^n(x)\rho(x,s)\,dx \Big| < \frac{\varepsilon}{3}.$$

Moreover, we can also chose $n \in \mathbb{N}$ great enough in such a way that

$$\int_{B_R\setminus B_r} \zeta^n(x)\rho(x,t)\,dx < \frac{\varepsilon}{3} \quad \text{and} \quad \int_{B_R\setminus B_r} \zeta^n(x)\rho(x,s)\,dx < \frac{\varepsilon}{3}.$$

Then we conclude that

$$|\rho(t)(B_r) - \rho(s)(B_r)| < \varepsilon$$

for $|t-s| < \delta$. □

For the following result, see, e.g., [21, Lemma 2.14]. This lemma is needed to prove the main result (see in particular the fourth step).

Lemma 2.5. *Consider $x_o \in \Omega$ and $r > 0$ such that $B_{2r}(x_o) \subset \Omega$, $\sigma \in (0,r)$, $\omega \in B^1_{2,q}(K)$ for some $q > 2$, $\omega \in A_\infty$, and $\alpha, \beta > 0$. Consider an open and non-empty subset $\mathcal{B}$ of $B_r(x_o)$ such that $\mathcal{B}^\sigma = \{x \in \Omega \mid \operatorname{dist}(x,\mathcal{B}) < \sigma\}$ is a subset of $B_r(x_o)$. Then, for every $a > 0$ and $\varepsilon, \delta \in (0,1)$, there exists $\eta \in (0,1)$ such that for every $u \in H^1(\Omega,\omega)$ satisfying*

$$\int_{\mathcal{B}^\sigma} |Du|^2\,dx \leqslant \alpha \frac{|B_r(x_o)|}{r^2}$$

and

$$\omega(\{u > a\} \cap \mathcal{B}) \geqslant \beta\omega(B_r(x_o)),$$

there exists $x^* \in \mathcal{B}$ with $B_{\eta\rho}(x^*) \subset \mathcal{B}$ such that

$$\omega(\{u > \varepsilon a\} \cap B_{\eta r}(x^*)) > (1-\delta)\omega(B_{\eta r}(x^*)).$$

For the following result, see [3].

Theorem 2.6. *Consider $q > 2$, $r > 0$, $x_o \in \mathbb{R}^n$, and $\omega \in L^\infty(\mathbb{R}^n)$, $\omega > 0$ a.e., $\omega \in B^1_{2,q}(K)$ and doubling, i.e. satisfying* (2.1) *and* (2.2). *Then there is a constant Γ depending (only) on n, q, K, c_ω such that*

$$\Big[\frac{1}{\omega(B_r)} \int_{B_r} |u(x)|^q \omega(x)\,dx\Big]^{1/q} \leqslant \Gamma r \Big[\frac{1}{|B_r|} \int_{B_r} |Du(x)|^2\,dx\Big]^{1/2}$$

for every Lipschitz continuous function u defined in $B_r = B_r(x_0)$, with either support contained in $B_r(x_0)$ or with null mean value.

By Theorem 2.6 and one of its direct consequences (a result contained in [14, 15], but see also [21, Theorem 2.9]), one gets the following result.

Remark 2.7. The assumptions required in the result in [15] involve the ratio between two weights. In our case (in Theorem 2.8) this assumption is simply

$$(\rho(t))^{-1} \in A_\infty(\rho(t)), \quad \text{where } A_\infty(\rho(t)) \text{ is the } A_\infty \text{ class with respect to the measure } \rho(t)\,dx). \tag{2.3}$$

If ρ satisfies (H3), this request is satisfied.

Instead of (H3), since we suppose $\rho \in L^\infty$, another condition which implies (2.3) is that there is a constant $C > 0$ such that

$$\|\rho(t)\|_{L^\infty(B)} \leqslant C\frac{1}{|B|}\int_B \rho(x,t)\,dx = C\frac{\rho(t)(B)}{|B|}$$

for every ball B and for a.e. $t \in (0,T)$.

Theorem 2.8. *Consider $q > 2$ and $\rho \in L^\infty(\mathbb{R}^{n+1})$, $\rho > 0$ a.e., satisfying (H2) and (H3) for some $K_1, K_2 > 0$. Then there is $\kappa \in (1, \frac{q}{2})$ and Γ depending (only) on $n, q, K_1, K_2, \varsigma$ such that*

$$\begin{aligned}\int_{s_1}^{s_2} \frac{1}{\eta(t)(B_r)}\int_{B_r} |u|^{2\kappa}(x,t)\eta(x,t)\,dx\,dt \\ \leqslant \Gamma^2 r^2\Big(\max_{s_1\leqslant t\leqslant s_2}\frac{1}{\rho(t)(B_r)}\int_{B_r} u^2(x,t)\rho(x,t)\,dx\Big)^{\kappa-1}\frac{1}{|B_r|}\int_{s_1}^{s_2}\int_{B_r}|Du|^2(x,t)\,dx\,dt\end{aligned}$$

for every $B_r(x_o) \subset \mathbb{R}^n$, every $(s_1, s_2) \in (0,T)$, every Lipschitz continuous function u defined in $B_r \times (s_1, s_2)$, $u(\cdot, t)$ with either support contained in $B_r(x_o)$ or with null mean value, and where the inequality holds both with $\eta = \rho$ and $\eta \equiv 1$.

Proof. Here we do not present the proof, since it can be derived adapting easily that of [21, Theorem 2.9]. We only stress that (2.3) is needed, but this is a consequence of (H3) as observed in Remark 2.7. □

Remark 2.9. About the proof of the previous theorem, we want to consider the following question: whether $\eta = \rho$ requiring ρ to be doubling (together to $\rho(t) \in B^1_{2,q}(K_1)$ uniformly in t) is in fact sufficient to get the thesis of Theorem 2.8. The assumption $\rho(t) \in A_\infty(K_2)$ is needed just to get the thesis with $\eta \equiv 1$ (and, more general, with $\eta \neq \rho$).

In this regard, see [21, Remark 2.3 and Theorem 2.9].

Other results useful in the sequel are the following two lemmas, for which we refer to [21].

Lemma 2.10. *Consider ω as in Theorem 2.6, $k, l \in \mathbb{R}$ with $k < l$, and $p \in (1,2]$. Then*

$$(l-k)\omega(\{v<k\})\omega(\{v>l\}) \leqslant 2\Gamma r(\omega(B_r))^2\Big(\frac{1}{|B_r|}\int_{B_r\cap\{k<v<l\}}|Dv|^p\,dx\Big)^{1/p}$$

for every Lipschitz continuous function v defined in the ball B_r.

We conclude stating a standard lemma (see, for instance, [12, Lemma 7.1]) needed later.

Lemma 2.11. *Let $(y_h)_h$ be a sequence of positive real numbers such that*

$$y_{h+1} \leqslant c b^h y_h^{1+\alpha}$$

with $c, \alpha > 0$ and $b > 1$. If $y_0 \leqslant c^{-1/\alpha} b^{-1/\alpha^2}$, then

$$\lim_{h\to+\infty} y_h = 0.$$

3 DG classes

We are going now to introduce a De Giorgi class suited to the parabolic equations (1.1) and (1.2) of order 2; before, we need to we consider a suitable class of decay functions, namely the set

$$\mathcal{X}_c := \{\zeta \in \mathrm{Lip}(\Omega \times (0,T)) \mid \zeta(\cdot\,, t) \in \mathrm{Lip}_c(\Omega) \text{ for each } t \in (0,T),\ \zeta \geqslant 0,\ \zeta_t \geqslant 0\}.$$

By v_+ we will denote the positive part of a function v, and by v_- we will denote the negative part of a function v.

Definition 3.1. Given $\gamma > 0$, we say that a function $u \in \mathcal{V}_{\mathrm{loc}}$, such that

$$(0,T) \ni t \mapsto \int_\Omega u^2(x,t)\rho(x,t)\,dx \in C^0_{\mathrm{loc}}(0,T),$$

belongs to $DG(\Omega, T, \rho, \gamma)$ if u satisfies for every function $\zeta \in \mathcal{X}_c$, every $t_1, t_2 \in (0,T)$ and every $k \in \mathbb{R}$ the inequality

$$\begin{aligned}\int_\Omega (u-k)_+^2\zeta^2\rho(x,t_2)\,dx &+ \int_{t_1}^{t_2}\int_\Omega |D(u-k)_+|^2\zeta^2\,dx\,dt\\ &\leqslant \int_\Omega (u-k)_+^2\zeta^2\rho(x,t_1)\,dx + \gamma\Big[\int_{t_1}^{t_2}\int_\Omega (u-k)_+^2|D\zeta|^2\,dx\,dt + \int_{t_1}^{t_2}\int_\Omega (u-k)_+^2\zeta\zeta_t\rho\,dx\,dt\\ &\qquad + \int_{t_1}^{t_2}\int_\Omega (u-k)_+^2\zeta^2\,dx\,dt + k^2\int_{t_1}^{t_2}\int_{\{u(t)>k\}}(\zeta^2 + |D\zeta|^2)\,dx\,dt\Big].\end{aligned} \tag{3.1}$$

and if u satisfies an analogous inequality for every function $\zeta \in \mathcal{X}_c$, every $t_1, t_2 \in (0,T)$, and every $k \in \mathbb{R}$, with $(u-k)_-$ instead of $(u-k)_+$ and with

$$k^2\int_{t_1}^{t_2}\int_{\{u(t)<k\}}(\zeta^2 + |D\zeta|^2)\,dx\,dt$$

instead of the last term in inequality (3.1).

Remark 3.2 (Comments on Definition 3.1).
- The energy inequality (3.1) can be derived as done in [22] starting from both equations (1.1) and (1.2). Notice that in [22] only $k \geqslant 0$ is considered to derive (3.1) (and only $k \leqslant 0$ to derive the analogous one for $(u-k)_-$): indeed, these restrictions are due to the fact that in [22] a wider class was considered and can be removed in our case.
- A solution of (1.1) belongs to $\{v \in \mathcal{V} \mid \rho v' \in \mathcal{V}'\}$ and a solution of (1.2) belongs to $\{v \in \mathcal{V} \mid (\rho v)' \in \mathcal{V}'\}$, but in fact these two spaces, under assumption (H1), turn out to be the same space.
- A function belonging to $DG(\Omega, T, \rho, \gamma)$ is locally bounded (see, for instance, [24]).

Remark 3.3. The solutions of equation (1.1) with $C \equiv 0$ satisfy

$$\begin{aligned}\int_\Omega (u-k)_+^2\zeta^2\rho(x,t_2)\,dx &+ \int_{t_1}^{t_2}\int_\Omega |D(u-k)_+|^2\zeta^2\,dx\,dt\\ &\leqslant \int_\Omega (u-k)_+^2\zeta^2\rho(x,t_1)\,dx + \gamma\Big[\int_{t_1}^{t_2}\int_\Omega (u-k)_+^2|D\zeta|^2\,dx\,dt\\ &\qquad + \int_{t_1}^{t_2}\int_\Omega (u-k)_+^2\zeta\zeta_t\rho\,dx\,dt + \int_{t_1}^{t_2}\int_\Omega (u-k)_+^2\zeta^2\,dx\,dt\Big]\end{aligned} \tag{3.2}$$

and the analogous one for $(u-k)_-$. Notice that if u satisfies these inequalities, also $u + c$ satisfies them, where c is an arbitrary constant (in particular, this holds for solutions of (1.1) with $C \equiv 0$).

This is the key point to use the argument due to Moser, by which one can derive the local Hölder continuity from the Harnack inequality for functions satisfying (3.2) and the analogous estimate with $(u-k)_-$.

For fixed $(x_o, t_o) \in \Omega \times (0,T)$ and $r > 0$, we define the quantities

$$\mathsf{h}_*(x_o, t_o, r) := \frac{1}{r^2|B_r(x_o)|} \int_{t_o - r^2}^{t_o} \int_{B_r(x_o)} \rho(x,t)\,dx\,dt,$$

$$\mathsf{h}^*(x_o, t_o, r) := \frac{1}{r^2|B_r(x_o)|} \int_{t_o}^{t_o + r^2} \int_{B_r(x_o)} \rho(x,t)\,dx\,dt.$$

Notice that

$$\begin{cases} \mathsf{h}^*(x_o, t_o - r^2, r) = \mathsf{h}_*(x_o, t_o, r), \\ \quad\mathsf{h}^*(x_o, t_o, r) = \mathsf{h}_*(x_o, t_o + r^2, r). \end{cases} \tag{3.3}$$

Notice that for $0 < r < \tilde{r}$ one has

$$\mathsf{h}_*(x_o, t_o, r) \leqslant \Big(\frac{\tilde{r}}{r}\Big)^{n+2} \mathsf{h}_*(x_o, t_o, \tilde{r}), \quad \mathsf{h}^*(x_o, t_o, r) \leqslant \Big(\frac{\tilde{r}}{r}\Big)^{n+2} \mathsf{h}^*(x_o, t_o, \tilde{r}). \tag{3.4}$$

Once we fixed $(x_o, t_o) \in \Omega \times (0,T)$, for a generic $R > 0$ we will set

$$Q_R(x_o, t_o) := B_R(x_o) \times (t_o - R^2 \mathsf{h}_*(x_o, t_o, R), t_o),$$
$$Q^R(x_o, t_o) := B_R(x_o) \times (t_o, t_o + R^2 \mathsf{h}^*(x_o, t_o, R)).$$

Given $r \in (0, R]$ and $\theta > 0$ (possibly also greater than 1), we define

$$\begin{cases} Q_{r,\theta}(x_o, t_o) := B_r(x_o) \times (t_o - \theta r^2 \mathsf{h}_*(x_o, t_o, R), t_o), \\ Q^{r,\theta}(x_o, t_o) := B_r(x_o) \times (t_o, t_o + \theta r^2 \mathsf{h}^*(x_o, t_o, R)), \end{cases} \tag{3.5}$$

where h_* and h^* always refer to a fixed radius R bigger than or equal to r (only for $\theta = 1$ and $r = R$ the cylinder $Q_{r,\theta}$ coincides with Q_R). To lighten the notation, we will omit writing (x_o, t_o) if not strictly necessary and simply write

$$Q_R, \quad Q^R, \quad Q_{r,\theta}, \quad Q^{r,\theta}$$

if it is clear which point we are referring to.

4 Expansion of positivity

We recall that $u \in DG(\Omega, T, \rho, \gamma)$ is locally bounded, as mentioned in Remark 3.2.

Proposition 4.1. *Consider $(\bar{x}, \bar{t}) \in \Omega \times (0,T)$ and $R > 0$ such that*

$$Q_R = Q_R(\bar{x}, \bar{t}) \subset \Omega \times (0,T).$$

Let $\mathsf{h}_* = \mathsf{h}_*(\bar{x}, \bar{t}, R)$. Consider $r, \tilde{r}, \theta, \tilde{\theta}$ satisfying

$$0 < r < \tilde{r} \leqslant R, \quad 0 < \theta < \tilde{\theta} \qquad \text{such that} \qquad Q_{\tilde{r},\tilde{\theta}}(\bar{x}, \bar{t}) \subset \Omega \times (0,T).$$

Then, for every choice of $a, \sigma \in (0,1)$, every $u \in DG(\Omega, T, \rho, \gamma)$ and every μ_+, ω satisfying

$$\mu_+ \geqslant \sup_{Q_{\tilde{r},\tilde{\theta}}(\bar{x},\bar{t})} u, \quad \omega \geqslant \operatorname{osc}_{Q_{\tilde{r},\tilde{\theta}}(\bar{x},\bar{t})} u,$$

there is ν, depending (only) on γ, $1-a$, κ, $\gamma_{B_{\tilde{r}}(\bar{x})}$, Γ, $\tilde{r}^{-1}$, $\tilde{\theta}^{-1}$, $R/\tilde{r}$, $c_\rho(R/\tilde{r})$, $C_\rho(\tilde{\theta})$, $\tilde{r} - r$, $\tilde{\theta} - \theta$, and $\frac{\mu_+}{\sigma\omega}$, such that if

$$\frac{|\{(x,t) \in Q_{\tilde{r},\tilde{\theta}}(\bar{x},\bar{t}) \mid u(x,t) > \mu_+ - \sigma\omega\}|}{|Q_{\tilde{r},\tilde{\theta}}(\bar{x},\bar{t})|} + \frac{\rho(\{(x,t) \in Q_{\tilde{r},\tilde{\theta}}(\bar{x},\bar{t}) \mid u(x,t) > \mu_+ - \sigma\omega\})}{\rho(Q_{\tilde{r},\tilde{\theta}}(\bar{x},\bar{t}))} \leqslant \nu,$$

then

$$u(x,t) \leqslant \mu_+ - a\sigma\omega \quad \text{for a.e. } (x,t) \in Q_{r,\theta}(\bar{x},\bar{t}).$$

Remark 4.2. This is the only result (indeed, also Proposition 4.3) stated for each function belonging to $DG(\Omega, T, \rho, \gamma)$, while in all of the remainder we consider nonnegative functions.

In this way, the constant ν may depend on the oscillation of a specific function in the class $DG(\Omega, T, \rho, \gamma)$, but in fact it depends on the ratio

$$\frac{\mu_+}{\omega},$$

as highlighted in the statement. In particular, if one confines to consider a function $u \geqslant 0$ (as we will do) and choose

$$\mu_+ = \sup_{Q_{\tilde{r},\tilde{\theta}}(\bar{x},\bar{t})} u, \qquad \omega = \operatorname{osc}_{Q_{\tilde{r},\tilde{\theta}}(\bar{x},\bar{t})} u,$$

in fact ν does not depend anymore on μ_+ or on ω since $\frac{\mu_+}{\omega} \leqslant 1$ (see in particular the last lines of the following proof and also step 1 of the proof of Theorem 5.2).

Proof. Consider for $h \in \mathbb{N}$,

$$\begin{aligned}
r_h &= r + \frac{\tilde{r} - r}{2^h}, \\
\theta_h &= \theta + \frac{\tilde{\theta} - \theta}{4^h}, \\
Q_h &:= Q_{r_h,\theta_h}(\bar{x}, \bar{t}), \\
\sigma_h &= a\sigma + \frac{1-a}{2^h}\sigma, \\
k_h &= \mu_+ - \sigma_h \omega.
\end{aligned}$$

With these choices, we have that

$$\theta_h - \theta_{h+1} = 3\frac{\tilde{\theta} - \theta}{(\tilde{r} - r)^2}(r_h - r_{h+1})^2$$

and that

$$Q_{h+1} \subset Q_h.$$

Consider now a sequence of functions satisfying

$$\begin{aligned}
&\zeta_h \equiv 1 \quad \text{in } Q_{h+1}, \\
&\zeta_h \equiv 0 \quad \text{outside of } Q_h \quad \text{for } t \leqslant \bar{t}, \\
&0 \leqslant \zeta_h \leqslant 1, \\
&|D\zeta_h| \leqslant \frac{1}{r_h - r_{h+1}}
\end{aligned}$$

and

$$\begin{aligned}
0 \leqslant (\zeta_h)_t &\leqslant \frac{1}{(\theta_h r_h^2 - \theta_{h+1} r_{h+1}^2)\mathsf{h}_*} \\
&\leqslant \frac{1}{(\theta_h r^2 - \theta_{h+1} r^2)\mathsf{h}_*} \\
&= \frac{4^{h+1}}{3r^2\mathsf{h}_*}\frac{1}{\tilde{\theta} - \theta} \\
&= \frac{1}{3r^2\mathsf{h}_*(r_h - r_{h+1})^2}\frac{(\tilde{r} - r)^2}{\tilde{\theta} - \theta} \\
&< \frac{1}{r^2\mathsf{h}_*(r_h - r_{h+1})^2}\frac{(\tilde{r} - r)^2}{\tilde{\theta} - \theta},
\end{aligned}$$

and finally define

$$A_h = \{(x, t) \in Q_h \mid u(x, t) > k_h\}.$$

First, we have (κ is the value in Theorem 2.8 and γ_B is defined in (C1))

$$\begin{aligned}\frac{((1-a)\sigma\omega)^2}{2^{2h+2}}\frac{|A_{h+1}|}{|Q_{\tilde r,\tilde\theta}|} &= (k_{h+1}-k_h)^2\frac{|A_{h+1}|}{|Q_{\tilde r,\tilde\theta}|}\\ &\leqslant \frac{1}{|Q_{\tilde r,\tilde\theta}|}\iint_{A_{h+1}}(u-k_h)_+^2\,dx\,dt\\ &\leqslant \frac{1}{|Q_{\tilde r,\tilde\theta}|}\iint_{Q_{h+1}}(u-k_h)_+^2\,dx\,dt\\ &\leqslant \frac{1}{|Q_{\tilde r,\tilde\theta}|}\iint_{Q_h}(u-k_h)_+^2\zeta_h^2\,dx\,dt\\ &\leqslant \Big(\frac{|A_h|}{|Q_{\tilde r,\tilde\theta}|}\Big)^{\frac{\kappa-1}{\kappa}}\Big(\frac{1}{|Q_{\tilde r,\tilde\theta}|}\iint_{Q_h}(u-k_h)_+^{2\kappa}\zeta_h^{2\kappa}\,dx\,dt\Big)^{\frac{1}{\kappa}}\\ &= \Big(\frac{|A_h|}{|Q_{\tilde r,\tilde\theta}|}\Big)^{\frac{\kappa-1}{\kappa}}\Big(\frac{1}{\tilde\theta\tilde r^2\mathsf{h}_*}\Big)^{\frac{1}{\kappa}}\Big(\frac{1}{|B_{\tilde r}|}\iint_{Q_h}(u-k_h)_+^{2\kappa}\zeta_h^{2\kappa}\,dx\,dt\Big)^{\frac{1}{\kappa}}.\end{aligned}\tag{4.1}$$

Similarly,

$$\begin{aligned}&\frac{((1-a)\sigma\omega)^2}{2^{2h+2}}\rho(A_{h+1})\inf_{t\in(\bar t-\theta_{h+1}r_{h+1}^2\mathsf{h}_*,\bar t)}\frac{1}{\rho(t)(B_{\tilde r})}\\ &\quad= (k_{h+1}-k_h)^2\rho(A_{h+1})\inf_{t\in(\bar t-\theta_{h+1}r_{h+1}^2\mathsf{h}_*,\bar t)}\frac{1}{\rho(t)(B_{\tilde r})}\\ &\quad\leqslant \inf_{t\in(\bar t-\theta_{h+1}r_{h+1}^2\mathsf{h}_*,\bar t)}\frac{1}{\rho(t)(B_{\tilde r})}\iint_{A_{h+1}}(u-k_h)_+^2\rho\,dx\,dt\\ &\quad\leqslant \iint_{A_h}\frac{1}{\rho(t)(B_{\tilde r})}(u-k_h)_+^2\zeta_h^2\rho\,dx\,dt\\ &\quad\leqslant \Big(\iint_{A_h}\frac{1}{\rho(t)(B_{\tilde r})}\rho\,dx\,dt\Big)^{\frac{\kappa-1}{\kappa}}\Big(\iint_{Q_h}\frac{1}{\rho(t)(B_{\tilde r})}(u-k_h)_+^{2\kappa}\zeta_h^{2\kappa}\rho\,dx\,dt\Big)^{\frac{1}{\kappa}},\end{aligned}$$

by which

$$\frac{((1-a)\sigma\omega)^2}{2^{2h+2}}\frac{1}{\gamma_{B_{\tilde r}}}\frac{\rho(A_{h+1})}{\rho(Q_{\tilde r,\tilde\theta})}\leqslant \gamma_{B_{\tilde r}}^{\frac{\kappa-1}{\kappa}}\cdot\Big(\frac{\rho(A_h)}{\rho(Q_{\tilde r,\tilde\theta})}\Big)^{\frac{\kappa-1}{\kappa}}\Big(\frac{1}{\tilde\theta\tilde r^2\mathsf{h}_*}\Big)^{\frac{1}{\kappa}}\Big(\iint_{Q_h}\frac{1}{\rho(t)(B_{\tilde r})}(u-k_h)_+^{2\kappa}\zeta_h^{2\kappa}\rho\,dx\,dt\Big)^{\frac{1}{\kappa}}.\tag{4.2}$$

Now we use Theorem 2.8 to estimate the last factors in (4.1) and (4.2). In the following, η may denote the function ρ or the constant function 1. We get

$$\begin{aligned}&\Big(\iint_{Q_h}\frac{1}{\eta(t)(B_{\tilde r})}(u-k_h)_+^{2\kappa}\zeta_h^{2\kappa}\eta(x,t)\,dx\,dt\Big)^{\frac{1}{\kappa}}\\ &\quad\leqslant \Gamma^{\frac{2}{\kappa}}\tilde r^{\frac{2}{\kappa}}\Big(\max_{\bar t-\theta_h r_h^2\mathsf{h}_*\leqslant t\leqslant\bar t}\frac{1}{\rho(t)(B_{\tilde r})}\int_{B_{r_h}}(u-k_h)^2\zeta_h^2\rho(x,t)\,dx\Big)^{\frac{\kappa-1}{\kappa}}\Big(\frac{1}{|B_{\tilde r}|}\iint_{Q_h}|D((u-k_h)_+\zeta_h)|^2\,dx\,dt\Big)^{\frac{1}{\kappa}}\\ &\quad\leqslant \Gamma^{\frac{2}{\kappa}}\tilde r^{\frac{2}{\kappa}}\Big(\max_{\bar t-\theta_h r_h^2\mathsf{h}_*\leqslant t\leqslant\bar t}\frac{1}{\rho(t)(B_{\tilde r})}\Big)^{\frac{\kappa-1}{\kappa}}\Big(\frac{1}{|B_{\tilde r}|}\Big)^{\frac{1}{\kappa}}\\ &\qquad\cdot\Big[\max_{\bar t-\theta_h r_h^2\mathsf{h}_*\leqslant t\leqslant\bar t}\int_{B_{r_h}}(u-k_h)^2\zeta_h^2\rho(x,t)\,dx+2\iint_{Q_h}|D(u-k_h)_+|^2\zeta_h^2\,dx\,dt\\ &\qquad\quad+2\iint_{Q_h}|D\zeta_h|^2(u-k_h)_+^2\,dx\,dt\Big].\end{aligned}\tag{4.3}$$

Now since

$$(u - k_h)_+ \leqslant \sigma_h \omega, \tag{4.4}$$

we can estimate the last term in the right-hand side as

$$2 \iint_{Q_h} |D\zeta_h|^2 (u - k_h)_+^2 \, dx \, dt \leqslant 2 \frac{|A_h|}{(r_h - r_{h+1})^2} (\sigma_h \omega)^2. \tag{4.5}$$

Now we estimate the first two of the three addends of the last factor in the right-hand side using (3.1):

$$\begin{aligned}
&\max_{\bar t - \theta_h r_h^2 \mathsf{h}_* \leqslant t \leqslant \bar t} \int_{B_{r_h}} (u - k_h)^2 \zeta_h^2 \rho(x,t) \, dx + 2 \iint_{Q_h} |D(u - k_h)_+|^2 \zeta_h^2 \, dx \, dt \\
&\leqslant 2\gamma \Big[\iint_{Q_h} (u - k_h)_+^2 |D\zeta_h|^2 \, dx \, dt + \iint_{Q_h} (u - k_h)_+^2 \zeta_h (\zeta_h)_t \rho \, dx \, dt \\
&\qquad + \iint_{Q_h} (u - k_h)_+^2 \zeta_h^2 \, dx \, dt + k_h^2 \iint_{A_h} (\zeta_h^2 + |D\zeta_h|^2) \, dx \, dt \Big] \\
&\leqslant 2\gamma \Big[\frac{1}{(r_h - r_{h+1})^2} \iint_{Q_h} (u - k_h)_+^2 \, dx \, dt + \frac{1}{r^2 \mathsf{h}_* (r_h - r_{h+1})^2} \frac{(\tilde r - r)^2}{\tilde\theta - \theta} \iint_{Q_h} (u - k_h)_+^2 \rho \, dx \, dt \\
&\qquad + \iint_{Q_h} (u - k_h)_+^2 \, dx \, dt + k_h^2 \iint_{A_h} (\zeta_h^2 + |D\zeta_h|^2) \, dx \, dt \Big].
\end{aligned}$$

Now using again (4.4), we get

$$\begin{aligned}
&\max_{\bar t - \theta_h r_h^2 \mathsf{h}_* \leqslant t \leqslant \bar t} \int_{B_{r_h}} (u - k_h)^2 \zeta_h^2 \rho(x,t) \, dx + 2 \iint_{Q_h} |D(u - k_h)_+|^2 \zeta_h^2 \, dx \, dt \\
&\leqslant 2\gamma \Big[\frac{|A_h| (\sigma_h \omega)^2}{(r_h - r_{h+1})^2} + \frac{\rho(A_h) (\sigma_h \omega)^2}{r^2 \mathsf{h}_* (r_h - r_{h+1})^2} \frac{(\tilde r - r)^2}{\tilde\theta - \theta} \\
&\qquad + |A_h| (\sigma_h \omega)^2 + |A_h| k_h^2 \Big(1 + \frac{1}{(r_h - r_{h+1})^2} \Big) \Big].
\end{aligned} \tag{4.6}$$

By (3.4), we get

$$\begin{aligned}
\frac{1}{\mathsf{h}_*(\bar x, \bar t, R)} &\leqslant \Big(\frac{R}{\tilde r} \Big)^{n+2} \frac{1}{\mathsf{h}_*(\bar x, \bar t, \tilde r)} \\
&= \Big(\frac{R}{\tilde r} \Big)^{n+2} \frac{|Q_{\tilde r}|}{\rho(Q_{\tilde r})} \\
&= \Big(\frac{R}{\tilde r} \Big)^{n+2} \frac{1}{\tilde\theta} \frac{|Q_{\tilde r, \tilde\theta}|}{\rho(Q_{\tilde r})} \\
&\leqslant \Big(\frac{R}{\tilde r} \Big)^{n+2} \frac{C_\rho(\tilde\theta)}{\tilde\theta} \frac{|Q_{\tilde r, \tilde\theta}|}{\rho(Q_{\tilde r, \tilde\theta})},
\end{aligned}$$

where C_ρ is defined in Remark 2.9. Then, continuing estimating in (4.6), we finally get

$$\begin{aligned}
&\max_{\bar t - \theta_h r_h^2 \mathsf{h}_* \leqslant t \leqslant \bar t} \int_{B_{r_h}} (u - k_h)^2 \zeta_h^2 \rho(x,t) \, dx + 2 \iint_{Q_h} |D(u - k_h)_+|^2 \zeta_h^2 \, dx \, dt \\
&\leqslant \frac{2\gamma |Q_{\tilde r, \tilde\theta}|}{(r_h - r_{h+1})^2} [((\sigma_h \omega)^2 + k_h^2)((r_h - r_{h+1})^2 + 1)] \frac{|A_h|}{|Q_{\tilde r, \tilde\theta}|} \\
&\qquad + \Big(\frac{R}{\tilde r} \Big)^{n+2} \frac{C_\rho(\tilde\theta)}{\tilde\theta} \frac{2\gamma |Q_{\tilde r, \tilde\theta}|}{(r_h - r_{h+1})^2} \frac{(\sigma_h \omega)^2}{r^2} \frac{\rho(A_h)}{\rho(Q_{\tilde r, \tilde\theta})} \frac{(\tilde r - r)^2}{\tilde\theta - \theta}.
\end{aligned} \tag{4.7}$$

Now we call

$$z_h := \frac{|A_h|}{|Q_{\tilde r, \tilde\theta}|}, \quad x_h := \frac{\rho(A_h)}{\rho(Q_{\tilde r, \tilde\theta})}, \quad y_h := z_h + x_h.$$

Summing (4.1) and (4.2) and taking into account (4.3), (4.5) and (4.7), we get

$$y_{h+1} \leqslant \frac{1}{((1-a)\sigma\omega)^2}\big(1+\gamma_{B_{\tilde r}}^{1+\frac{\kappa-1}{\kappa}}\big)\Big(\frac{1}{\tilde\theta\tilde r^2 \mathsf{h}_*}\Big)^{\frac{1}{\kappa}} 16^{h+1}\cdot\Gamma^{\frac{2}{\kappa}}\tilde r^{\frac{2}{\kappa}}\Big(\max_{\bar t-\theta_h r_h^2\mathsf{h}_*\leqslant t\leqslant\bar t}\frac{1}{\rho(t)(B_{\tilde r})}\Big)^{\frac{\kappa-1}{\kappa}}\Big(\frac{1}{|B_{\tilde r}|}\Big)^{\frac{1}{\kappa}}$$
$$\cdot\frac{2\gamma|Q_{\tilde r,\tilde\theta}|}{(\tilde r-r)^2}\Big[((\sigma_h\omega)^2+k_h^2)\Big(\frac{(\tilde r-r)^2}{2^{2h+2}}+2\Big)+\Big(\frac{R}{\tilde r}\Big)^{n+2}\frac{C_\rho(\tilde\theta)}{\tilde\theta}\frac{(\sigma_h\omega)^2}{r^2}\frac{(\tilde r-r)^2}{\tilde\theta-\theta}\Big]y_h^{1+\frac{\kappa-1}{\kappa}}.$$

Since

$$\begin{aligned}\Big(\frac{1}{\tilde\theta\tilde r^2\mathsf{h}_*}\Big)^{\frac{1}{\kappa}}\tilde r^{\frac{2}{\kappa}}&\Big(\max_{\bar t-\theta_h r_h^2\mathsf{h}_*\leqslant t\leqslant\bar t}\frac{1}{\rho(t)(B_{\tilde r})}\Big)^{\frac{\kappa-1}{\kappa}}\Big(\frac{1}{|B_{\tilde r}|}\Big)^{\frac{1}{\kappa}}|Q_{\tilde r,\tilde\theta}|\\
&=\tilde r^{\frac{2}{\kappa}}\Big(\max_{\bar t-\theta_h r_h^2\mathsf{h}_*\leqslant t\leqslant\bar t}\frac{1}{\rho(t)(B_{\tilde r})}\Big)^{\frac{\kappa-1}{\kappa}}|Q_{\tilde r,\tilde\theta}|^{\frac{\kappa-1}{\kappa}}\\
&=\tilde r^{\frac{2}{\kappa}}\Big(\frac{|B_{\tilde r}|}{\min_{\bar t-\theta_h r_h^2\mathsf{h}_*\leqslant t\leqslant\bar t}\rho(t)(B_{\tilde r})}\Big)^{\frac{\kappa-1}{\kappa}}(\tilde\theta\tilde r^2\mathsf{h}_*)^{\frac{\kappa-1}{\kappa}}\\
&\leqslant\tilde r^{\frac{2}{\kappa}}\Big(\frac{|B_{\tilde r}|}{\min_{\bar t-\theta_h r_h^2\mathsf{h}_*\leqslant t\leqslant\bar t}\rho(t)(B_{\tilde r})}\Big)^{\frac{\kappa-1}{\kappa}}\Big(\tilde\theta\tilde r^2\frac{\max_{\bar t-R^2\leqslant t\leqslant\bar t}\rho(t)(B_R)}{|B_R|}\Big)^{\frac{\kappa-1}{\kappa}}\\
&\leqslant\tilde r^2\tilde\theta^{\frac{\kappa-1}{\kappa}}\Big(c_\rho\Big(\frac{R}{\tilde r}\Big)\Big)^{\frac{\kappa-1}{\kappa}}\gamma_{B_{\tilde r}}^{\frac{\kappa-1}{\kappa}},\end{aligned}$$

where $c_\rho(\theta)$ is defined in Remark 2.9, we finally get

$$y_{h+1}\leqslant \mathsf{c}16^h y_h^{1+\frac{\kappa-1}{\kappa}},$$

where

$$\mathsf{c}=32\frac{\gamma\gamma_{B_{\tilde r}}^{\frac{\kappa-1}{\kappa}}\Gamma^{\frac{2}{\kappa}}}{(1-a)^2}\big(1+\gamma_{B_{\tilde r}}^{1+\frac{\kappa-1}{\kappa}}\big)\frac{\tilde r^2}{(\tilde r-r)^2}\tilde\theta^{\frac{\kappa-1}{\kappa}}\Big(c_\rho\Big(\frac{R}{\tilde r}\Big)\Big)^{\frac{\kappa-1}{\kappa}}$$
$$\cdot\Big[\Big(1+\Big(\frac{\mu_+}{\sigma\omega}-a\Big)^2\Big)\frac{(\tilde r-r)^2+8}{4}+\Big(\frac{R}{\tilde r}\Big)^{n+2}\frac{C_\rho(\tilde\theta)}{\tilde\theta}\frac{1}{r^2}\frac{(\tilde r-r)^2}{\tilde\theta-\theta}\Big].$$

Then, by Lemma 2.11, we get the thesis provided that

$$y_0\leqslant\mathsf{c}^{-\frac{\kappa}{\kappa-1}}16^{-\frac{\kappa^2}{(\kappa-1)^2}}$$

and choosing

$$\nu=\mathsf{c}^{-\frac{\kappa}{\kappa-1}}16^{-\frac{\kappa^2}{(\kappa-1)^2}}.$$

The proof is finished. □

Similarly, one can prove the following proposition.

Proposition 4.3. *Consider* $(\bar x,\bar t)\in\Omega\times(0,T)$ *and* $R>0$ *such that*

$$Q_R=Q_R(\bar x,\bar t)\subset\Omega\times(0,T).$$

Let $\mathsf{h}_*=\mathsf{h}_*(\bar x,\bar t,R)$. *Consider* $\hat r,\tilde r,\hat\theta,\tilde\theta$ *satisfying*

$$0<\hat r<\tilde r\leqslant R,\quad 0<\hat\theta<\tilde\theta\qquad\textit{such that}\qquad Q_{\tilde r,\tilde\theta}(\bar x,\bar t)\subset\Omega\times(0,T).$$

Then, for every choice of $a,\sigma\in(0,1)$, *every* $u\in DG(\Omega,T,\rho,\gamma)$ *and every* μ_-,ω *satisfying*

$$\mu_-\leqslant\inf_{Q_{\tilde r,\tilde\theta}(\bar x,\bar t)}u,\quad \omega\geqslant\operatorname{osc}_{Q_{\tilde r,\tilde\theta}(\bar x,\bar t)}u,$$

there is ν, *depending (only) on* γ, $1-a$, κ, $\gamma_{B_{\tilde r}(\bar x)}$, Γ, $\tilde r^{-1}$, $\tilde\theta^{-1}$, $R/\tilde r$, $c_\rho(R/\tilde r)$, $C_\rho(\tilde\theta)$, $\tilde r-r$, $\tilde\theta-\theta$, *and* $\frac{\mu_-}{\sigma\omega}$, *such that if*

$$\frac{|\{(x,t)\in Q_{\tilde r,\tilde\theta}(\bar x,\bar t)\mid u(x,t)<\mu_-+\sigma\omega\}|}{|Q_{\tilde r,\tilde\theta}(\bar x,\bar t)|}+\frac{\rho(\{(x,t)\in Q_{\tilde r,\tilde\theta}(\bar x,\bar t)\mid u(x,t)<\mu_-+\sigma\omega\})}{\rho(Q_{\tilde r,\tilde\theta}(\bar x,\bar t))}\leqslant\nu,$$

then

$$u(x,t)\geqslant\mu_-+a\sigma\omega\quad\textit{for a.e. }(x,t)\in Q_{\hat r,\hat\theta}(\bar x,\bar t).$$

Remark 4.4. Fix $(\tilde{x}, \tilde{t}) \in \Omega \times (0, T)$ and $h > 0$ and set

$$D_r(h; \tilde{t})(\tilde{x}) := \{x \in B_r(\tilde{x}) \mid u(x, \tilde{t}) < h\}.$$

Observe that the condition $u(x, \tilde{t}) \geqslant h$ for every $x \in B_r(\tilde{x})$ implies $D_{4r}(h; \tilde{t})(\tilde{x}) \subset B_{4r}(\tilde{x}) \setminus B_r(\tilde{x})$, and thus

$$\rho(\tilde{t})(D_{4r}(h; \tilde{t})(\tilde{x})) \leqslant \Big(1 - \frac{1}{c_\rho^2}\Big)\rho(\tilde{t})(B_{4r}(\tilde{x})).$$

Lemma 4.5. *Consider $(\bar{y}, \bar{s})$ for which, given $R, r, \theta > 0$,*

$$\begin{aligned} B_{4r}(\bar{y}) &\subseteq B_R(\bar{y}), \\ Q^R(\bar{y}, \bar{s}) &= B_R(\bar{y}) \times (\bar{s}, \bar{s} + \mathsf{h}^*) \subset \Omega \times (0, T), \quad \mathsf{h}^* = \mathsf{h}^*(\bar{y}, \bar{s}, R), \\ Q^{4r,\theta}(\bar{y}, \bar{s}) &= B_{4r}(\bar{y}) \times (\bar{s}, \bar{s} + 16\theta r^2 \mathsf{h}^*) \subset \Omega \times (0, T). \end{aligned}$$

Then there exist $\vartheta \in (0, \theta)$, ϑ depending on $\rho(\bar{s})(B_{4r}(\bar{y}))$, and $\eta \in (0, 1)$ such that, given $h > 0$ and $u \geqslant 0$ in $DG(\Omega, T, \rho, \gamma)$ for which

$$u(x, \bar{s}) \geqslant h \quad \textit{a.e. in } B_r(\bar{y}),$$

it holds

$$\rho(t)(D_{4r}(\eta h; t)(\bar{y})) < \Big(1 - \frac{1}{2c_\rho^2}\Big)\rho(t)(B_{4r}(\bar{y}))$$

for every $t \in [\bar{s}, \bar{s} + \vartheta(4r)^2 \mathsf{h}^*(\bar{y}, \bar{s}, R)]$.

Proof. In the following, we set

$$D^{r,\theta}(k) = \{(x, t) \in Q^{r,\theta}(\bar{y}, \bar{s}) \mid u(x, t) < k\}.$$

Consider $\sigma \in (0, 1)$ and a function $\zeta \in \mathcal{X}_c$, $\zeta = \zeta(x)$, satisfying

$$\begin{aligned} &\zeta \equiv 1 && \text{in } B_{4r(1-\sigma)}(\bar{y}), \\ &\zeta \equiv 0 && \text{outside of } B_{4r}(\bar{y}), \\ &0 \leqslant \zeta \leqslant 1, \\ &|D\zeta| \leqslant \frac{1}{4r\sigma}. \end{aligned}$$

Consider $\vartheta > 0$ to be fixed below. We apply now the energy estimate (3.1) with this choice of ζ to the function $(u - h)_-$ and get

$$\begin{aligned} &\sup_{t \in (\bar{s}, \bar{s} + \vartheta(4r)^2 \mathsf{h}^*)} \int_{B_{4r(1-\sigma)}} (u - h)_-^2 \rho(x, t)\, dx \\ &\quad \leqslant \int_{B_{4r}} (u - h)_-^2 \rho(x, \bar{s})\, dx + \gamma\Big(\frac{1}{(4r\sigma)^2} + 1\Big)\Big[\int_{\bar{s}}^{\bar{s} + \vartheta(4r)^2 \mathsf{h}^*} \int_{B_{4r}} (u - h)_-^2\, dx\, dt + h^2 |D^{4r,\vartheta}(h)|\Big]. \end{aligned}$$

Now, using Remark 4.4 to estimate the first term on the right-hand side, and the fact that $u(x, \bar{s}) \geqslant 0$ implies $(u - h)_-(x, \bar{s}) \leqslant h$, to estimate the first term we get

$$\sup_{t \in (\bar{s}, \bar{s} + \vartheta(4r)^2 \mathsf{h}^*)} \int_{B_{4r(1-\sigma)}} (u - h)_-^2 \rho(x, t)\, dx \leqslant h^2 \Big(1 - \frac{1}{c_\rho^2}\Big)\rho(\bar{s})(B_{4r}(\bar{y})) + 2\gamma\Big(\frac{1}{(4r\sigma)^2} + 1\Big) h^2 |D^{4r,\vartheta}(h)|.$$

Writing, for some $\eta \in (0, 1)$ to be fixed and for $t \in (\bar{s}, \bar{s} + \vartheta(4r)^2 \mathsf{h}^*)$,

$$D_{4r}(\eta h; t)(\bar{y}) = D_{4r(1-\sigma)}(\eta h; t)(\bar{y}) \cup (\{x \in B_{4r}(\bar{y}) \setminus B_{4r(1-\sigma)}(\bar{y}) \mid u(x, t) < \eta\}),$$

we derive that

$$\rho(t)(D_{4r}(\eta h; t)(\bar{y})) \leqslant \rho(t)(D_{4r(1-\sigma)}(\eta h; t)(\bar{y})) + \rho(t)(B_{4r}(\bar{y}) \setminus B_{4r(1-\sigma)}(\bar{y})).$$

On the other hand,

$$\int_{B_{4r(1-\sigma)}} (u-h)_-^2 \rho(x,t)\,dx \geqslant \int_{D_{4r(1-\sigma)}(\eta h;t)(\bar y)} (u-h)_-^2 \rho(x,t)\,dx$$
$$\geqslant h^2(1-\eta)^2 \rho(t)(D_{4r(1-\sigma)}(\eta h;t)(\bar y)).$$

Finally, we obtain

$$\begin{aligned}\rho(t)(D_{4r}(\eta h;t)(\bar y)) &\leqslant \rho(t)(D_{4r(1-\sigma)}(\eta h;t)(\bar y)) + \rho(t)(B_{4r}(\bar y)\setminus B_{4r(1-\sigma)}(\bar y))\\ &\leqslant \frac{1}{h^2(1-\eta)^2}\int_{B_{4r(1-\sigma)}} (u-h)_-^2\rho(x,t)\,dx + \rho(t)(B_{4r}(\bar y)\setminus B_{4r(1-\sigma)}(\bar y))\\ &\leqslant \frac{1}{(1-\eta)^2}\Big[\Big(1-\frac{1}{c_\rho^2}\Big)\rho(\bar s)(B_{4r}(\bar y)) + 2\gamma\Big(\frac{1}{(4r\sigma)^2}+1\Big)|D^{4r,\vartheta}(h)|\Big]\\ &\qquad + \rho(t)(B_{4r}(\bar y)\setminus B_{4r(1-\sigma)}(\bar y))\end{aligned} \tag{4.8}$$

for every $t \in (\bar s, \bar s + \vartheta(4r)^2 \mathrm{h}^*)$. Now we argue by contradiction: if the thesis were not true, then for every $\vartheta, \eta \in (0,1)$ we would have $\tau \in (\bar s, \bar s + \vartheta(4r)^2\mathrm{h}^*)$ for which

$$\rho(\tau)(D_{4r}(\eta h;\tau)(\bar y)) \geqslant \Big(1-\frac{1}{2c_\rho^2}\Big)\rho(\tau)(B_{4r}(\bar y)).$$

By this and (4.8) (with $t=\tau$), we would get

$$\begin{aligned}&\Big(1-\frac{1}{2c_\rho^2}\Big)\rho(\tau)(B_{4r}(\bar y))\\ &\quad\leqslant \rho(\tau)(B_{4r}(\bar y)\setminus B_{4r(1-\sigma)}(\bar y)) + \frac{1}{(1-\eta)^2}\Big[\Big(1-\frac{1}{c_\rho^2}\Big)\rho(\bar s)(B_{4r}(\bar y)) + 2\gamma\Big(\frac{1}{(4r\sigma)^2}+1\Big)|D^{4r,\vartheta}(h)|\Big].\end{aligned}$$

Then we could choose ϑ depending on σ in such a way that

$$\lim_{\sigma\to 0^+}\frac{|D^{4r,\vartheta}(h)|}{\sigma^2} = 0.$$

By the uniform continuity (see Lemma 2.4) of the function $[0,T]\ni t\mapsto \rho(t)(B_{4r}(\bar y))$, we have that for every $\varepsilon>0$ there is $\delta>0$ such that if

$$\vartheta(4r)^2\mathrm{h}^*(\bar y,\bar s,R) < \delta,$$

then

$$|\rho(t_2)(B_{4r}(\bar y)) - \rho(t_1)(B_{4r}(\bar y))| < \varepsilon \quad \text{for every } t_1,t_2\in[\bar s,\bar s+\vartheta(4r)^2\mathrm{h}^*(\bar y,\bar s,R)].$$

Then, adding and subtracting on the left-hand side the term $\rho(\bar s)(B_{4r}(\bar y))$, letting σ go to zero and dividing by $\rho(\bar s)(B_{4r}(\bar y))$, we would get

$$\Big(1-\frac{1}{2c_\rho^2}\Big) \leqslant \frac{1}{(1-\eta)^2}\Big(1-\frac{1}{c_\rho^2}\Big) + \Big(1-\frac{1}{2c_\rho^2}\Big)\frac{\varepsilon}{\rho(\bar s)(B_{4r}(\bar y))}.$$

Since this inequality holds independently of the values of η and ε, we would have a contradiction. Notice that, having to choose $\varepsilon(\rho(\bar s)(B_{4r}(\bar y)))^{-1}$ small enough and δ depending on ε, the value of ϑ depends on $\rho(\bar s)(B_{4r}(\bar y))$. Indeed, we could choose η and ε satisfying

$$\frac{1}{(1-\eta)^2}\Big(1-\frac{1}{c_\rho^2}\Big) < \alpha\Big(1-\frac{1}{2c_\rho^2}\Big),$$
$$\Big(1-\frac{1}{2c_\rho^2}\Big)\frac{\varepsilon}{\rho(\bar s)(B_{4r}(\bar y))} < (1-\alpha)\Big(1-\frac{1}{2c_\rho^2}\Big)$$

for some $\alpha\in(0,1)$. These imply to choose

$$\alpha(1-\eta)^2 > \Big(1-\frac{1}{c_\rho^2}\Big)\Big(1-\frac{1}{2c_\rho^2}\Big)^{-1},$$
$$\varepsilon < (1-\alpha)\rho(\bar s)(B_{4r}(\bar y)).$$

Once one chose η and α satisfying the first inequality, one can choose ε, and consequently ϑ. □

Lemma 4.6. *Consider $r, \theta > 0$, $(\bar{y}, \bar{s})$ for which*

$$Q^R(\bar{y}, \bar{s}) = B_R(\bar{y}) \times (\bar{s}, \bar{s} + \mathsf{h}^*) \subset \Omega \times (0, T),$$

with $R = 5r$ and $\mathsf{h}^* = \mathsf{h}^*(\bar{y}, \bar{s}, 5r)$, and

$$Q^{4r,\theta}(\bar{y}, \bar{s}) = B_{4r}(\bar{y}) \times (\bar{s}, \bar{s} + \theta(4r)^2\mathsf{h}^*) \subset \Omega \times (0, T).$$

Consider $u \in DG(\Omega, T, \rho, \gamma)$, $u \geqslant 0$, and $h > 0$, such that

$$u(x, \bar{s}) \geqslant h \quad \text{a.e. in } B_r(\bar{y}).$$

Consider $\vartheta \in (0, \theta)$ as in Lemma 4.5 and suppose moreover that

$$B_{5r}(\bar{y}) \times (\bar{s} - \vartheta(4r)^2\mathsf{h}_* \bar{s} + \vartheta(4r)^2\mathsf{h}^*) \subset \Omega \times (0, T) \quad \text{with } \mathsf{h}_* = \mathsf{h}_*(\bar{y}, \bar{s}, 5r).$$

Then there is $p \geqslant 1$ and for every $\varepsilon > 0$ there exists

$$\eta_1 = \eta_1(\varepsilon, n, p, \gamma, \Gamma, c_\rho, \gamma_{B_{4r}(\bar{y})}, \gamma_{B_{5r}(\bar{y})}, \vartheta, r, \mathsf{h}_*/\mathsf{h}^*) \in (0, 1)$$

such that

$$\rho([u < \eta_1 h] \cap (B_{4r}(\bar{y}) \times (\bar{s}, \bar{s} + \vartheta(4r)^2\mathsf{h}^*))) < \varepsilon\rho(B_{4r}(\bar{y}) \times (\bar{s}, \bar{s} + \vartheta(4r)^2\mathsf{h}^*)),$$
$$|[u < \eta_1 h] \cap (B_{4r}(\bar{y}) \times (\bar{s}, \bar{s} + \vartheta(4r)^2\mathsf{h}^*))| < (K_3\varepsilon)^{1/p}|B_{4r}(\bar{y}) \times (\bar{s}, \bar{s} + \vartheta(4r)^2\mathsf{h}^*)|.$$

Remark 4.7. Notice that, observing the dependence of ϑ on $\rho(\bar{s})(B_{4r}(\bar{y}))$ in Lemma 4.5, one obtains that the smaller $\rho(\bar{s})(B_{4r}(\bar{y}))$ is, the smaller ϑ, and thus also η_1, is.

Proof. Consider $\eta > 0$, $m \in \mathbb{N}$ and, for u and h as in the assumptions, consider the function $(u - \eta h 2^{-m})_-$. Setting

$$A_r(h; s)(y) := \{x \in B_r(y) \mid u(x, s) > h\},$$
$$D_r(h; s)(y) := \{x \in B_r(y) \mid u(x, s) < h\},$$

by Lemma 2.10 (taking $k = \eta h/2^m$, $l = \eta h/2^{m-1}$, $p \in (1, 2)$ and $\omega = \rho(\sigma)$) for each $\sigma \in (\bar{s}, \bar{s} + \theta(4r)^2\mathsf{h}^*)$, we get

$$\begin{aligned}
&\int_{B_{4r}(\bar{y})} \Big(u - \frac{\eta h}{2^m}\Big)_- \rho(x, \sigma)\, dx \\
&\qquad \leqslant \frac{\eta h}{2^m} \rho(\sigma)(D_{4r}(\eta h 2^{-m}; \sigma)(\bar{y})) \\
&\qquad \leqslant \frac{2\Gamma r(\rho(\sigma)(B_{4r}))^2}{\rho(\sigma)(A_{4r}(\eta h 2^{-m+1}; \sigma)(\bar{y}))} \Big(\frac{1}{|B_{4r}|} \int_{D_{4r}(\eta h 2^{-m+1}; \sigma)(\bar{y}) \setminus D_{4r}(\eta h 2^{-m}; \sigma)(\bar{y})} |Du(x, \sigma)|^p\, dx\Big)^{\frac{1}{p}}.
\end{aligned} \tag{4.9}$$

Now, since (at least for $m \geqslant 2$)

$$A_{4r}(\eta h 2^{-m+1}; \sigma)(\bar{y}) \supseteq B_{4r}(\bar{y}) \setminus D_{4r}(\eta h; \sigma)(\bar{y}),$$

if we consider $\eta \in (0, 1)$ as in Lemma 4.5, we get

$$\rho(\sigma)(A_{4r}(\eta h 2^{-m+1}; \sigma)(\bar{y})) > \Big(1 - \frac{1}{2c_\rho^2}\Big)\rho(\sigma)(B_{4r}(\bar{y}))$$

for every $\sigma \in [\bar{s}, \bar{s} + \vartheta(4r)^2\mathsf{h}^*]$. Then, by that and (4.9), we derive

$$\int_{B_{4r}(\bar{y})} \Big(u - \frac{\eta h}{2^m}\Big)_- \rho(x, \sigma)\, dx < \frac{4\Gamma c_\rho^2 r \rho(\sigma)(B_{4r})}{2c_\rho^2 - 1} \Big(\frac{1}{|B_{4r}|} \int_{D_{4r}(\eta h 2^{-m+1}; \sigma)(\bar{y}) \setminus D_{4r}(\eta h 2^{-m}; \sigma)(\bar{y})} |Du(x, \sigma)|^p\, dx\Big)^{\frac{1}{p}}. \tag{4.10}$$

Now we define the sequences

$$a_m := \int_{\bar s}^{\bar s+\vartheta(4r)^2 h^*} \rho(\sigma)(D_{4r}(\eta h 2^{-m};\sigma)(\bar y))\,d\sigma = \rho\Big(Q^{4r,\vartheta}(\bar y,\bar s)\cap\Big\{u<\frac{\eta h}{2^m}\Big\}\Big),$$

$$b_m := \int_{\bar s}^{\bar s+\vartheta(4r)^2 h^*} |D_{4r}(\eta h 2^{-m};\sigma)(\bar y)|\,d\sigma = \Big|Q^{4r,\vartheta}(\bar y,\bar s)\cap\Big\{u<\frac{\eta h}{2^m}\Big\}\Big|.$$

Notice that, integrating the left-hand side in (4.10), we get

$$\int_{\bar s}^{\bar s+\vartheta(4r)^2 h^*}\int_{B_{4r}(\bar y)}\Big(u-\frac{\eta h}{2^m}\Big)_-\rho\,dx\,d\sigma \geqslant \frac{\eta h}{2^{m+1}}a_{m+1}. \tag{4.11}$$

While integrating the right-hand side and using (C1), we obtain

$$\begin{aligned}
&\int_{\bar s}^{\bar s+\vartheta(4r)^2 h^*}\rho(\sigma)(B_{4r})\Big(\frac{1}{|B_{4r}|}\int_{D_{4r}(\eta h2^{-m+1};\sigma)(\bar y)\setminus D_{4r}(\eta h2^{-m};\sigma)(\bar y)}|Du(x,\sigma)|^p\,dx\Big)^{\frac1p}d\sigma\\
&\leqslant \max_{\bar s\leqslant\sigma\leqslant\bar s+\vartheta(4r)^2h^*}\rho(\sigma)(B_{4r})\frac{(\vartheta(4r)^2h^*)^{\frac{p-1}{p}}}{|B_{4r}|^{\frac1p}}\Big(\int_{\bar s}^{\bar s+\vartheta(4r)^2 h^*}\int_{D_{4r}(\eta h2^{-m+1};\sigma)(\bar y)\setminus D_{4r}(\eta h2^{-m};\sigma)(\bar y)}|Du|^p\,dx\,d\sigma\Big)^{\frac1p}\\
&\leqslant \frac{\gamma_{B_{4r}}\rho(B_{4r}(\bar y)\times(\bar s,\bar s+\vartheta(4r)^2h^*))}{(\vartheta(4r)^2h^*|B_{4r}|)^{\frac1p}}\Big(\iint_{Q^{4r,\vartheta}}\Big|D\Big(u-\frac{\eta h}{2^m}\Big)_-\Big|^2dx\,d\sigma\Big)^{\frac12}(b_{m-1}-b_m)^{\frac{2-p}{2p}}.
\end{aligned}\tag{4.12}$$

We want to estimate

$$\iint_{Q^{4r,\vartheta}}|D(u-\frac{\eta h}{2^m})_-|^2\,dx\,d\sigma.$$

To do that consider a function $\zeta\in\mathcal{X}_c$ satisfying

$$\begin{aligned}
\zeta &\equiv 1 && \text{in } Q^{4r,\vartheta}=Q^{4r,\vartheta}(\bar y,\bar s)=B_{4r}(\bar y)\times(\bar s,\bar s+\vartheta(4r)^2h^*),\\
\zeta &\equiv 0 && \text{outside of } B_{5r}(\bar y)\times(\bar s-\vartheta(4r)^2h_*,\bar s+\vartheta(4r)^2h^*)\quad\text{for } t\leqslant\bar s+\vartheta(4r)^2h^*,\\
0 &\leqslant\zeta\leqslant 1,\\
|D\zeta| &\leqslant\frac1r,\\
0 &\leqslant\zeta_t\leqslant\frac{1}{\vartheta(4r)^2h_*}.
\end{aligned}$$

Then apply (3.1) to the function $(u-\eta h2^{-m})_-$ for some $\eta>0$, $m\in\mathbb{N}$ and ζ as above:

$$\begin{aligned}
&\int_{\bar s}^{\bar s+\vartheta(4r)^2h^*}\int_{B_{4r}}\Big|D\Big(u-\frac{\eta h}{2^m}\Big)_-\Big|^2(x,t)\,dx\,dt\\
&\leqslant\gamma\int_{\bar s-\vartheta(4r)^2h_*}^{\bar s+\vartheta(4r)^2h^*}\int_{B_{5r}}\Big(u-\frac{\eta h}{2^m}\Big)_-^2(|D\zeta|^2+\zeta\zeta_t\rho)\,dx\,dt\\
&\leqslant\frac{\gamma}{r^2}\Big[\int_{\bar s-\vartheta(4r)^2h_*}^{\bar s+\vartheta(4r)^2h^*}\int_{B_{5r}}\Big(u-\frac{\eta h}{2^m}\Big)_-^2dx\,dt+\frac{1}{\vartheta(4r)^2h_*}\int_{\bar s-\vartheta(4r)^2h_*}^{\bar s+\vartheta(4r)^2h^*}\int_{B_{5r}}\Big(u-\frac{\eta h}{2^m}\Big)_-^2\rho\,dx\,dt\Big]\\
&\leqslant\frac{\gamma}{r^2}\frac{\eta^2h^2}{2^{2m}}\Big[\vartheta(4r)^2(h_*+h^*)|B_{5r}|+\frac{1}{\vartheta(4r)^2h_*}\rho(B_{5r}(\bar y)\times(\bar s-\vartheta(4r)^2h_*,\bar s+\vartheta(4r)^2h^*))\Big]\\
&=\frac{\gamma}{r^2}\frac{\eta^2h^2}{2^{2m}}|B_{5r}|\Big[\vartheta(4r)^2(h_*+h^*)+\frac{25}{16}\frac{1}{\vartheta}\frac{\rho(B_{5r}(\bar y)\times(\bar s-\vartheta(4r)^2h_*,\bar s+\vartheta(4r)^2h^*))}{\rho(B_{5r}(\bar y)\times(\bar s-(5r)^2h_*,\bar s))}\Big].
\end{aligned}\tag{4.13}$$

From (4.10)–(4.13), we get

$$\frac{\eta h}{2^{m+1}} a_{m+1} < \frac{4\Gamma c_\rho^2 r}{2c_\rho^2 - 1} \frac{\gamma_{B_{4r}} \rho(B_{4r}(\bar y) \times (\bar s, \bar s + \vartheta(4r)^2 \mathsf{h}^*))}{(\vartheta(4r)^2 \mathsf{h}^* |B_{4r}|)^{\frac{1}{p}}} \frac{\gamma^{\frac12}}{r} \frac{\eta h}{2^m} |B_{5r}|^{\frac12} \cdot \Big[\vartheta(4r)^2(\mathsf{h}_* + \mathsf{h}^*) + \frac{25}{16}\frac{1}{\vartheta} \frac{\rho(B_{5r}(\bar y) \times (\bar s - \vartheta(4r)^2 \mathsf{h}_*, \bar s + \vartheta(4r)^2 \mathsf{h}^*))}{\rho(B_{5r}(\bar y) \times (\bar s - (5r)^2 \mathsf{h}_*, \bar s))}\Big]^{\frac12} (b_{m-1} - b_m)^{\frac{2-p}{2p}}.$$

Since

$$\frac{\rho(B_{5r}(\bar y) \times (\bar s - \vartheta(4r)^2 \mathsf{h}_*, \bar s + \vartheta(4r)^2 \mathsf{h}^*))}{\rho(B_{5r}(\bar y) \times (\bar s - (5r)^2 \mathsf{h}_*, \bar s))} \leqslant \frac{\vartheta(4r)^2 (\mathsf{h}_* + \mathsf{h}^*) \gamma_{B_{5r}} \min_{\bar s - \vartheta(4r)^2 \mathsf{h}_* \leqslant \sigma \leqslant \bar s + \vartheta(4r)^2 \mathsf{h}^*} \rho(\sigma)(B_{5r}(\bar y))}{(5r)^2 \mathsf{h}_* \min_{\bar s - (5r)^2 \mathsf{h}_* \leqslant \sigma \leqslant \bar s} \rho(\sigma)(B_{5r}(\bar y))} \leqslant \frac{16}{25} \frac{\vartheta(\mathsf{h}_* + \mathsf{h}^*)\gamma_{B_{5r}}}{\mathsf{h}_*}$$

and

$$\frac{|B_{5r}|^{\frac12}}{|B_{4r}|^{\frac1p}} \leqslant \frac{|B_{8r}|^{\frac12}}{|B_{4r}|^{\frac1p}} \leqslant \frac{\sqrt{2^n} |B_{4r}|^{\frac12}}{|B_{4r}|^{\frac1p}} = \frac{\sqrt{2^n}}{|B_{4r}|^{\frac{2-p}{2p}}},$$

we get

$$a_{m+1}^{\frac{2p}{2-p}} < \Big(\frac{8\Gamma c_\rho^2 \sqrt{\gamma} \gamma_{B_{4r}}}{2c_\rho^2 - 1}\Big)^{\frac{2p}{2-p}} \frac{(\rho(B_{4r}(\bar y) \times (\bar s, \bar s + \vartheta(4r)^2 \mathsf{h}^*)))^{\frac{2p}{2-p}}}{(\vartheta(4r)^2 \mathsf{h}^*)^{\frac{2}{2-p}}} \frac{(2^n)^{\frac{p}{2-p}}}{|B_{4r}|} \cdot \Big[\vartheta(4r)^2(\mathsf{h}_* + \mathsf{h}^*) + \frac{(\mathsf{h}_* + \mathsf{h}^*)\gamma_{B_{5r}}}{\mathsf{h}_*}\Big]^{\frac{p}{2-p}} (b_{m-1} - b_m).$$

First of all, notice that, by this and since $\{b_m\}_m$ is decreasing, one gets that the fact that

$$\sum_{m=1}^{+\infty} a_{m+1}^{\frac{2p}{2-p}}$$

converges, implies

$$\lim_{m \to +\infty} a_m = 0.$$

More precisely, we estimate the generic term a_m: setting the quantity

$$\mathsf{Q} := \Big(\frac{8\Gamma c_\rho^2 \sqrt{\gamma} \gamma_{B_{4r}}}{2c_\rho^2 - 1}\Big)^{\frac{2p}{2-p}} \frac{(\rho(B_{4r}(\bar y) \times (\bar s, \bar s + \vartheta(4r)^2 \mathsf{h}^*)))^{\frac{2p}{2-p}}}{(\vartheta(4r)^2 \mathsf{h}^*)^{\frac{2}{2-p}}} \frac{(2^n)^{\frac{p}{2-p}}}{|B_{4r}|} \cdot \Big[\vartheta(4r)^2(\mathsf{h}_* + \mathsf{h}^*) + \gamma_{B_{5r}} \frac{\mathsf{h}_* + \mathsf{h}^*}{\mathsf{h}_*}\Big]^{\frac{p}{2-p}},$$

summing till a generic $m_o \in \mathbb{N}$ and since $\{a_m\}_m$ is decreasing, we get

$$m_o a_{m_o+1}^{\frac{2p}{2-p}} \leqslant \sum_{m=1}^{m_o} a_{m+1}^{\frac{2p}{2-p}} < \mathsf{Q}(b_0 - b_{m_o}) < \mathsf{Q} b_0.$$

Since

$$b_0 = |Q^{4r,\vartheta}(\bar y, \bar s) \cap \{u < \eta h\}| < |Q^{4r,\vartheta}(\bar y, \bar s)| = \vartheta(4r)^2 \mathsf{h}^* |B_{4r}(\bar y)|$$

we get

$$m_o a_{m_o+1}^{\frac{2p}{2-p}} < \mathsf{Q} \vartheta(4r)^2 \mathsf{h}^* |B_{4r}(\bar y)|.$$

Finally,

$$a_{m_o+1} < \mathsf{C} \rho(B_{4r}(\bar y) \times (\bar s, \bar s + \vartheta(4r)^2 \mathsf{h}^*)),$$

where

$$\mathsf{C} := \Big(\frac{1}{m_o}\Big)^{\frac{2-p}{2p}} \Big(\frac{8\Gamma c_\rho^2 \sqrt{\gamma} \gamma_{B_{4r}}}{2c_\rho^2 - 1}\Big) \sqrt{\frac{\mathsf{h}_* + \mathsf{h}^*}{\mathsf{h}^*}} \sqrt{2^n \Big(1 + \frac{\gamma_{B_{5r}}}{\vartheta(4r)^2}\Big)}.$$

Now, once we fixed $\varepsilon > 0$, requiring that $\mathsf{C} \leqslant \varepsilon$, we can find m_o for which the first point of the thesis is true.

To get the second point, by (C3) and Remark 2.3, we also get that there is $p \geqslant 1$ such that

$$\begin{aligned}\frac{|[u<\eta_1 h]\cap(B_{4r}(\bar y)\times(\bar s,\bar s+\vartheta(4r)^2\mathsf{h}^*))|^p}{|B_{4r}(\bar y)\times(\bar s,\bar s+\vartheta(4r)^2\mathsf{h}^*)|^p}\\ &\leqslant K_3\frac{\rho([u<\eta_1 h]\cap(B_{4r}(\bar y)\times(\bar s,\bar s+\vartheta(4r)^2\mathsf{h}^*)))}{\rho(B_{4r}(\bar y)\times(\bar s,\bar s+\vartheta(4r)^2\mathsf{h}^*))}\\ &< K_3\varepsilon,\end{aligned}$$

by which

$$|[u<\eta_1 h]\cap(B_{4r}(\bar y)\times(\bar s,\bar s+\vartheta(4r)^2\mathsf{h}^*))|<(K_3\varepsilon)^{1/p}|B_{4r}(\bar y)\times(\bar s,\bar s+\vartheta(4r)^2\mathsf{h}^*)|,$$

as desired. □

Theorem 4.8. *Consider (x_o,t_o) and $r>0$ such that*

$$B_{5r}(x_o)\times(t_o-(5r)^2\mathsf{h}_*,t_o+(5r)^2\mathsf{h}^*)\subset\Omega\times(0,T),$$

where

$$\mathsf{h}_*=\mathsf{h}_*(x_o,t_o,5r),\qquad \mathsf{h}^*=\mathsf{h}^*(x_o,t_o,5r).$$

Let $\vartheta\in(0,1)$ be the value determined in Lemma 4.5 corresponding to $\theta=1$. Then, for every $\bar\vartheta\in(0,\vartheta)$, there is $\lambda\in(0,1)$ depending only on K_3, n, p, γ, Γ, c_ρ, $\gamma_{B_{4r}(x_o)}$, $\gamma_{B_{5r}(x_o)}$, ϑ, r, and $\mathsf{h}_*/\mathsf{h}^*$, such that, for every $h>0$ and $u\geqslant 0$ in $DG(\Omega,T,\rho,\gamma)$, if

$$u(\cdot,t_o)\geqslant h\quad \text{a.e. in } B_r(x_o),$$

then

$$u\geqslant\lambda h\quad \text{a.e. in } B_{2r}(x_o)\times\big[t_o+\bar\vartheta(5r)^2\mathsf{h}^*(x_o,t_o,5r),t_o+\vartheta(5r)^2\mathsf{h}^*(x_o,t_o,5r)\big].$$

Remark 4.9. As observed after Lemma 4.6 and since λ depends on the same quantities on which η_1 depends, we have that the smaller $\rho(t_o)(B_{4r}(x_o))$ is, the smaller ϑ and λ are.

Proof. In Proposition 4.3, consider $\bar x=x_o$, $\bar t=t_o+\vartheta(5r)^2\mathsf{h}^*(x_o,t_o,5r)$, $\mu_-=0$, $\sigma\omega=c$ (with c arbitrary, to be chosen later), $R=5r$, $\tilde r=4r$, $\hat r=2r$, and $\tilde\theta=\vartheta$ where ϑ is the value determined in Lemma 4.5 corresponding to $\theta=1$. Notice that

$$\begin{aligned}Q_{4r,\vartheta}(\bar x,\bar t)&=B_{4r}(\bar x)\times(\bar t-\vartheta(5r)^2\mathsf{h}^*(x_o,t_o,5r),\bar t)\\&=B_{4r}(x_o)\times(t_o,t_o+\vartheta(5r)^2\mathsf{h}^*(x_o,t_o,5r))\\&=Q^{4r,\vartheta}(x_o,t_o).\end{aligned}$$

Similarly, for any $\hat\vartheta\in(0,\vartheta)$,

$$\begin{aligned}Q_{2r,\hat\vartheta}(\bar x,\bar t)&=B_{2r}(\bar x)\times(\bar t-\hat\vartheta(5r)^2\mathsf{h}^*(x_o,t_o,5r),\bar t)\\&=B_{2r}(x_o)\times(t_o+(\vartheta-\hat\vartheta)(5r)^2\mathsf{h}^*(x_o,t_o,5r),t_o+\vartheta(5r)^2\mathsf{h}^*(x_o,t_o,5r)).\end{aligned}$$

Now, given $a\in(0,1)$ and for every $\hat\vartheta\in(0,\vartheta)$, we get the existence of $\bar\nu>0$ such that if

$$\frac{|\{(x,t)\in Q_{4r,\vartheta}(\bar x,\bar t)\mid u(x,t)<c\}|}{|Q_{4r,\vartheta}(\bar x,\bar t)|}+\frac{\rho(\{(x,t)\in Q_{4r,\vartheta}(\bar x,\bar t)\mid u(x,t)<c\})}{\rho(Q_{4r,\vartheta}(\bar x,\bar t))}\leqslant\bar\nu,$$

then

$$u(x,t)\geqslant ac\quad \text{for a.e. } (x,t)\in Q_{2r,\hat\vartheta}(\bar x,\bar t).$$

To conclude, consider

$$\bar\vartheta=\vartheta-\hat\vartheta,$$

which is arbitrary in $(0,\vartheta)$, and consider $\varepsilon>0$ such that

$$\varepsilon+(K_3\varepsilon)^{1/p}=\bar\nu.$$

Corresponding to this value of ε, we get the existence of η_1 such that Lemma 4.6, with $\bar y=\bar x=x_o$ and $\bar s=t_o$, holds. Choosing then $c=\eta_1 h$, we conclude taking $\lambda=a\eta_1$. Choosing, for instance, $a=\frac{1}{2}$, we drop the dependence of a, and then λ depends on the same constants on which η_1 depends and, since η_1 depends also on ε, consequently λ also depends on K_3, p and on the constants on which η_1 depends. □

5 The Harnack inequality

Theorem 5.1. *There exists* η *such that for every constant* $c \in (0,1]$, *every* $u \in DG(\Omega, T, \rho, \gamma)$, $u \geqslant 0$, *every* $(x_o, t_o) \in \Omega \times (0,T)$, *and* $r > 0$ *such that*

$$B_{5r}(x_o) \times (t_o - (5r)^2 \mathsf{h}_*(x_o, t_o, 5r), t_o + (5r)^2 \mathsf{h}^*(x_o, t_o, 5r) \subset \Omega \times (0,T)$$

and

$$B_r(x_o) \times (t_o, t_o + 5r^2 \mathsf{h}^*(x_o, t_o, r)) \subset \Omega \times (0,T),$$

one has that

$$u(x_o, t_o) \leqslant \eta \inf_{B_r(x_o)} u(x, t_o + cr^2 \mathsf{h}^*(x_o, t_o, r)),$$

where we recall that

$$\mathsf{h}^*(x_o, t_o, r) = \frac{1}{r^2 |B_r(x_o)|} \int_{t_o}^{t_o + r^2} \int_{B_r(x_o)} \rho(x,t)\, dx\, dt.$$

The constant η *depends (only) on* K_3, n, p, γ, Γ, c_ρ, y, r, $\rho(t_o)(B_r(x_o))$, *and* $\mathsf{h}^*(x_o, t_o, r)$, *where*

$$\mathsf{y} := \max\Big\{\mathsf{y}_{B_{4r}(x_o)}, \mathsf{y}_{B_{5r}(x_o)}, \sup\Big\{\mathsf{y}_{B_\rho(y)} \;\Big|\; \rho \in \Big(0, \frac{5r}{2}\Big), \ y \in (0,r)\Big\}\Big\}.$$

Similarly, one can prove the following theorem.

Theorem 5.2. *There exists* $\tilde{\eta}$ *such that for every constant* $c \in (0,1]$, *every* $u \in DG(\Omega, T, \rho, \gamma)$, $u \geqslant 0$, *every* $(x_o, t_o) \in \Omega \times (0,T)$, *and* $r > 0$ *such that*

$$B_{5r}(x_o) \times (t_o - (5r)^2 \mathsf{h}_*(x_o, t_o, 5r), t_o + (5r)^2 \mathsf{h}^*(x_o, t_o, 5r)) \subset \Omega \times (0,T)$$

and

$$B_r(x_o) \times (t_o - 3r^2 \mathsf{h}_*(x_o, t_o, r), t_o) \subset \Omega \times (0,T),$$

one has that

$$\tilde{\eta} \sup_{B_r(x_o)} u(x, t_o - cr^2 \mathsf{h}_*(x_o, t_o, r)) \leqslant u(x_o, t_o).$$

Proof of Theorem 5.1. For the sake of simplicity, we suppose that $t_o = 0$, which is always possible up to a translation.

We may write $u(x_o, 0) = br^{-\xi}$ for some $b, \xi > 0$ to be fixed later. Define the functions

$$\mathscr{M}(s) = \sup_{C_s(x_o,0)} u, \qquad \mathscr{N}(s) = b(r-s)^{-\xi}, \qquad s \in [0,r),$$

where, for a generic point $(\upsilon, \sigma) \in \Omega \times (0,T)$,

$$C_s(\upsilon, \sigma) = B_s(\upsilon) \times (\sigma - s^2 \mathsf{h}^*(x_o, 0, r), \sigma).$$

Let us denote by $s_o \in [0,r)$ the largest solution of $\mathscr{M}(s) = \mathscr{N}(s)$ (notice that 0 is a solution). Define

$$M := \mathscr{N}(s_o) = b(r - s_o)^{-\xi}.$$

We can find $(y_o, \tau_o) \in C_{s_o}(x_o, 0)$ such that

$$\frac{3}{4} M < \sup_{C_{\frac{r_o}{4}}(y_o,\tau_o)} u \leqslant M, \tag{5.1}$$

where $r_o = (r - s_o)/2$. In this way, we get

$$C_{r_o}(y_o, \tau_o) \subset C_{\frac{r+s_o}{2}}(x_o, 0),$$

and therefore

$$\sup_{C_{r_o}(y_o,\tau_o)} u \leqslant \sup_{C_{\frac{r+s_o}{2}}(x_o,0)} u = \mathscr{M}\Big(\frac{r+s_o}{2}\Big) < \mathscr{N}\Big(\frac{r+s_o}{2}\Big) = 2^\xi M.$$

We now proceed dividing the proof in seven steps.

Step 1. In this step, we want to show that there is $\bar{\nu} \in (0,1)$ such that

$$\rho\Big(\Big\{u > \frac{M}{2}\Big\} \cap C_{r_o/2}(y_o,\tau_o)\Big) > \bar{\nu}\rho(C_{r_o/2}(y_o,\tau_o)). \tag{5.2}$$

To prove that, we prove first that there is $\bar{\bar{\nu}} \in (0,1)$ such that

$$\frac{\rho(\{u > \frac{M}{2}\} \cap C_{r_o/2}(y_o,\tau_o))}{\rho(C_{r_o/2}(y_o,\tau_o))} + \frac{|\{u > \frac{M}{2}\} \cap C_{r_o/2}(y_o,\tau_o)|}{|C_{r_o/2}(y_o,\tau_o)|} > \bar{\bar{\nu}}. \tag{5.3}$$

Assume, by contradiction, that this is not true. Taking, in Proposition 4.1,

$$\begin{aligned}
(\bar{x},\bar{t}) &= (y_o,\tau_o),\\
r &= \frac{r_o}{4},\\
\tilde{r} &= \frac{r_o}{2},\\
R &= r,\\
\tilde{\theta}\mathsf{h}_*(y_o,\tau_o,r) &= \Big(\frac{r_o}{2}\Big)^2 \mathsf{h}^*(x_o,0,r),\\
\theta\mathsf{h}_*(y_o,\tau_o,r) &= \Big(\frac{r_o}{4}\Big)^2 \mathsf{h}^*(x_o,0,r),\\
\mu_+ &= \omega = 2^{\xi}M,\\
\sigma &= 1-2^{-\xi-1},\\
a &= \sigma^{-1}\Big(1-\frac{3}{2^{\xi+2}}\Big),
\end{aligned}$$

we obtain that

$$u \leqslant \frac{3M}{4} \quad \text{in } C_{\frac{r_o}{4}}(y_o,\tau_o),$$

which contradicts (5.1). Notice that, according to Proposition 4.1, $\bar{\bar{\nu}}$ depends on γ, ξ, M, κ, $\gamma_{B_{\frac{r_o}{2}}(y_o)}$, Γ, r_o^{-1}, r/r_o, $c_\rho(r/r_o)$, $\mathsf{h}_*(y_o,\tau_o,r)/\mathsf{h}^*(x_o,0,r)$, $C_\rho(\mathsf{h}_*(y_o,\tau_o,r)/\mathsf{h}^*(x_o,0,r))$, r_o, and $\mathsf{h}^*(x_o,0,r)$.

Now, by (5.3), we derive that at least one of the two addends is greater than $\bar{\bar{\nu}}/2$. If

$$\frac{\rho(\{u > \frac{M}{2}\} \cap C_{r_o/2}(y_o,\tau_o))}{\rho(C_{r_o/2}(y_o,\tau_o))} > \frac{\bar{\bar{\nu}}}{2}$$

and since $\rho \in A_\infty(K_3,\sigma)$, we get that

$$\frac{\rho(\{u > \frac{M}{2}\} \cap C_{r_o/2}(y_o,\tau_o))}{\rho(C_{r_o/2}(y_o,\tau_o))} \leqslant K_3\Big(\frac{|\{u > \frac{M}{2}\} \cap C_{r_o/2}(y_o,\tau_o)|}{|C_{r_o/2}(y_o,\tau_o)|}\Big)^{\sigma}.$$

Then

$$\frac{|\{u > \frac{M}{2}\} \cap C_{r_o/2}(y_o,\tau_o)|}{|C_{r_o/2}(y_o,\tau_o)|} > \Big(\frac{1}{K_3}\frac{\bar{\bar{\nu}}}{2}\Big)^{\frac{1}{\sigma}}.$$

In this case, we consider

$$\bar{\nu} := \min\Big\{\frac{\bar{\bar{\nu}}}{2}, \Big(\frac{1}{K_3}\frac{\bar{\bar{\nu}}}{2}\Big)^{\frac{1}{\sigma}}\Big\}.$$

If instead

$$\frac{|\{u > \frac{M}{2}\} \cap C_{r_o/2}(y_o,\tau_o)|}{|C_{r_o/2}(y_o,\tau_o)|} > \frac{\bar{\bar{\nu}}}{2},$$

by (C3) we get that

$$\Big(\frac{|\{u > \frac{M}{2}\} \cap C_{r_o/2}(y_o,\tau_o)|}{|C_{r_o/2}(y_o,\tau_o)|}\Big)^{p} \leqslant K_3\frac{\rho(\{u > \frac{M}{2}\} \cap C_{r_o/2}(y_o,\tau_o))}{\rho(C_{r_o/2}(y_o,\tau_o))}.$$

In this case, we consider

$$\bar{\nu} := \min\Big\{\frac{\bar{\bar{\nu}}}{2}, \Big(\frac{1}{K_3}\frac{\bar{\bar{\nu}}}{2}\Big)^{p}\Big\}.$$

Step 2. In this step, we want to prove that

$$\iint_{C_{r_o/2}(y_o,\tau_o)} |Du|^2\,dx\,dt \leqslant C(2^\xi M)^2 |B_{r_o}(y_o)|,$$

where

$$C = \gamma\Big((4 + r_o^2)\mathsf{h}^*(x_o,0,r) + \frac{4}{3}\frac{\rho(B_{r_o}(y_o)\times(\tau_o - r_o^2\mathsf{h}^*(x_o,0,r),\tau_o))}{|B_{r_o}(y_o)|r_o^2\mathsf{h}^*(x_o,0,r)}\Big).$$

Consider a function ζ defined in $\Omega\times(0,\tau_o]$ such that

$$\begin{aligned}
&\zeta \equiv 1 && \text{in } B_{\frac{r_o}{2}}(y_o)\times\Big(\tau_o - \Big(\frac{r_o}{2}\Big)^2\mathsf{h}^*(x_o,0,r),\tau_o\Big),\\
&\zeta \equiv 0 && \text{outside of } C_{r_o}(y_o,\tau_o) \quad \text{for } t\leqslant\tau_o,\\
&0\leqslant\zeta\leqslant 1,\\
&|D\zeta| \leqslant \frac{2}{r_o},\\
&0 \leqslant \zeta_t \leqslant \frac{4}{3}\frac{1}{r_o^2}\frac{1}{\mathsf{h}^*(x_o,0,r)}.
\end{aligned}$$

Using this function ζ in (3.1), taking $k = 0$ and since $u \leqslant 2^\xi M$ in $C_{r_o}(y_o,\tau_o)$, we get that

$$\begin{aligned}
&\int_{\tau_o - \frac{r_o^2}{4}\mathsf{h}^*(x_o,0,r)}^{\tau_o}\int_{B_{r_o/2}} |Du|^2\,dx\,dt\\
&\leqslant \gamma \int_{\tau_o - r_o^2\mathsf{h}^*(x_o,0,r)}^{\tau_o}\int_{B_{r_o}} u^2(|D\zeta|^2 + \zeta\zeta_t\rho + \zeta^2)\,dx\,dt\\
&\leqslant \gamma\Big[\frac{1}{r_o^2}(2^\xi M)^2\Big(4|B_{r_o}(y_o)|r_o^2\mathsf{h}^*(x_o,0,r) + \frac{4}{3}\frac{\rho(B_{r_o}(y_o)\times(\tau_o - r_o^2\mathsf{h}^*(x_o,0,r),\tau_o))}{\mathsf{h}^*(x_o,0,r)}\Big)\\
&\qquad + (2^\xi M)^2|B_{r_o}(y_o)|r_o^2\mathsf{h}^*(x_o,0,r)\Big]\\
&\leqslant \gamma(2^\xi M)^2|B_{r_o}(y_o)|\Big[(4 + r_o^2)\mathsf{h}^*(x_o,0,r) + \frac{4}{3}\frac{\rho(B_{r_o}(y_o)\times(\tau_o - r_o^2\mathsf{h}^*(x_o,0,r),\tau_o))}{|B_{r_o}(y_o)|r_o^2\mathsf{h}^*(x_o,0,r)}\Big].
\end{aligned}$$

Step 3. The goal of this step is to show the existence of

$$\bar t \in \Big[\tau_o - \frac{r_o^2}{4}\mathsf{h}^*(x_o,0,r),\tau_o\Big]$$

such that

$$\begin{cases}
\dfrac{\rho(\bar t)(\{x\in B_{\frac{r_o}{2}}(y_o) \mid u(x,\bar t) > \frac{M}{2}\})}{\rho(\bar t)(B_{\frac{r_o}{2}}(y_o))} > \dfrac{\bar\nu}{2\gamma_{B_{\frac{r_o}{2}}(y_o)}},\\
\displaystyle\int_{B_{r_o/2}(y_o)} |Du(x,\bar t)|^2\,dx \leqslant \alpha\frac{|B_{r_o/2}(y_o)|}{r_o^2/4}(2^\xi M)^2,
\end{cases} \tag{5.4}$$

where

$$\alpha = \frac{4\gamma 2^n \gamma_{B_{\frac{r_o}{2}}(y_o)}}{\bar\nu\mathsf{h}^*(x_o,0,r)}\Big[(4 + r_o^2)\mathsf{h}^*(x_o,0,r) + \frac{4}{3}\frac{\rho(B_{r_o}(y_o)\times(\tau_o - r_o^2\mathsf{h}^*(x_o,0,r),\tau_o))}{|B_{r_o}(y_o)|r_o^2\mathsf{h}^*(x_o,0,r)}\Big] \tag{5.5}$$

and $\bar\nu$ has been determined in step 1. To do that, we define the following sets:

$$\begin{aligned}
A(t) &= \Big\{x\in B_{\frac{r_o}{2}}(y_o) \;\Big|\; u(x,t) \geqslant \frac{M}{2}\Big\}\\
I &= \Big\{t\in(\tau_o - \tau_1,\tau_o] \;\Big|\; \rho(t)(A(t)) > \frac{\bar\nu}{2\gamma_{B_{\frac{r_o}{2}}(y_o)}}\rho(t)(B_{\frac{r_o}{2}}(y_o))\Big\},\\
J_\alpha &= \Big\{t\in(\tau_o - \tau_1,\tau_o] \;\Big|\; \int_{B_{r_o/2}(y_o)} |Du(x,t)|^2\,dx \leqslant \alpha\frac{|B_{r_o/2}(y_o)|}{r_o^2/4}(2^\xi M)^2\Big\},
\end{aligned}$$

with $\alpha > 0$ and where, for the sake of simplicity, we denote by τ_1 the quantity

$$\tau_1 := \frac{r_o^2}{4} \mathrm{h}^*(x_o, 0, r).$$

By (5.2), we get

$$\begin{aligned}
\bar{\nu} & \min_{t \in [\tau_o - \tau_1, \tau_o]} \rho(t)(B_{r_o/2}(y_o))\tau_1 \\
& < \int_{\tau_o - \tau_1}^{\tau_o} \rho(t)(A(t))\, dt \\
& = \int_I \rho(t)(A(t))\, dt + \int_{(\tau_o - \tau_1, \tau_o)\setminus I} \rho(t)(A(t))\, dt \\
& \leqslant \max_{t \in [\tau_o - \tau_1, \tau_o]} \rho(t)(B_{\frac{r_o}{2}}(y_o))|I| + \frac{\bar{\nu}}{2\gamma_{B_{\frac{r_o}{2}}(y_o)}} \max_{t \in [\tau_o - \tau_1, \tau_o]} \rho(t)(B_{\frac{r_o}{2}}(y_o))|(\tau_o - \tau_1, \tau_o) \setminus I| \\
& \leqslant \max_{t \in [\tau_o - \tau_1, \tau_o]} \rho(t)(B_{\frac{r_o}{2}}(y_o))\tau_1 \Big[\frac{|I|}{\tau_1} + \frac{\bar{\nu}}{2\gamma_{B_{\frac{r_o}{2}}(y_o)}} \Big] \\
& \leqslant \gamma_{B_{\frac{r_o}{2}}(y_o)} \min_{t \in [\tau_o - \tau_1, \tau_o]} \rho(t)(B_{\frac{r_o}{2}}(y_o))\tau_1 \Big[\frac{|I|}{\tau_1} + \frac{\bar{\nu}}{2\gamma_{B_{\frac{r_o}{2}}(y_o)}} \Big].
\end{aligned}$$

From this, using also (C1), we derive the following lower bound on I:

$$|I| > \frac{\bar{\nu}}{2\gamma_{B_{\frac{r_o}{2}}(y_o)}} \tau_1 = \frac{\bar{\nu}}{2\gamma_{B_{\frac{r_o}{2}}(y_o)}} \frac{r_o^2}{4} \mathrm{h}^*(x_o, 0, r).$$

On the other hand, by step 2 we get

$$\begin{aligned}
\alpha & \frac{|B_{r_o/2}(y_o)|}{r_o^2/4} (2^\xi M)^2 (\tau_1 - |J_\alpha|) \\
& \leqslant \int_{\tau_o - \tau_1}^{\tau_o} \int_{B_{r_o/2}(y_o)} |Du|^2 \, dx\, dt \\
& \leqslant \gamma (2^\xi M)^2 |B_{r_o}(y_o)| \Big((4 + r_o^2)\mathrm{h}^*(x_o, 0, r) + \frac{4}{3} \frac{\rho(B_{r_o}(y_o) \times (\tau_o - r_o^2 \mathrm{h}^*(x_o, 0, r), \tau_o))}{|B_{r_o}(y_o)| r_o^2 \mathrm{h}^*(x_o, 0, r)} \Big),
\end{aligned}$$

by which

$$|J_\alpha| \geqslant \frac{r_o^2}{4} \Big[\mathrm{h}^*(x_o, 0, r) - \frac{\gamma 2^n}{\alpha} \Big((4 + r_o^2)\mathrm{h}^*(x_o, 0, r) + \frac{4}{3} \frac{\rho(B_{r_o}(y_o) \times (\tau_o - r_o^2 \mathrm{h}^*(x_o, 0, r), \tau_o))}{|B_{r_o}(y_o)| r_o^2 \mathrm{h}^*(x_o, 0, r)} \Big) \Big].$$

Then, since

$$\begin{aligned}
|I \cap J_\alpha| & = |I| + |J_\alpha| - |I \cup J_\alpha| \\
& \geqslant \frac{\bar{\nu}}{2\gamma_{B_{\frac{r_o}{2}}(y_o)}} \frac{r_o^2}{4} \mathrm{h}^*(x_o, 0, r) \\
& \quad + \frac{r_o^2}{4} \Big[\mathrm{h}^*(x_o, 0, r) - \frac{\gamma 2^n}{\alpha} \Big((4 + r_o^2)\mathrm{h}^*(x_o, 0, r) + \frac{4}{3} \frac{\rho(B_{r_o}(y_o) \times (\tau_o - r_o^2 \mathrm{h}^*(x_o, 0, r), \tau_o))}{|B_{r_o}(y_o)| r_o^2 \mathrm{h}^*(x_o, 0, r)} \Big) \Big] \\
& \quad - \frac{r_o^2}{4} \mathrm{h}^*(x_o, 0, r) \\
& \geqslant \frac{\bar{\nu}}{2\gamma_{B_{\frac{r_o}{2}}(y_o)}} \frac{r_o^2}{4} \mathrm{h}^*(x_o, 0, r) - \frac{\gamma 2^n}{\alpha} \frac{r_o^2}{4} \Big((4 + r_o^2)\mathrm{h}^*(x_o, 0, r) + \frac{4}{3} \frac{\rho(B_{r_o}(y_o) \times (\tau_o - r_o^2 \mathrm{h}^*(x_o, 0, r), \tau_o))}{|B_{r_o}(y_o)| r_o^2 \mathrm{h}^*(x_o, 0, r)} \Big),
\end{aligned}$$

taking

$$\frac{1}{\alpha} = \frac{\bar{\nu}}{4\gamma_{B_{\frac{r_o}{2}}(y_o)}} \mathrm{h}^*(x_o, 0, r) \frac{1}{\gamma 2^n} \Big((4 + r_o^2)\mathrm{h}^*(x_o, 0, r) + \frac{4}{3} \frac{\rho(B_{r_o}(y_o) \times (\tau_o - r_o^2 \mathrm{h}^*(x_o, 0, r), \tau_o))}{|B_{r_o}(y_o)| r_o^2 \mathrm{h}^*(x_o, 0, r)} \Big)^{-1},$$

one gets that

$$|I \cap J_\alpha| = |I| + |J_\alpha| - |I \cup J_\alpha| \geqslant \frac{\bar\nu}{4\gamma_{B_{\frac{r_o}{2}}(y_o)}} \frac{r_o^2}{4} \mathsf{h}^*(x_o, 0, r).$$

Then we get the existence of

$$\bar t \in \Big[\tau_o - \frac{r_o^2}{4} \frac{\rho(B_r(x_o) \times (0, r^2))}{r^2 |B_r(x_o)|}, \tau_o\Big]$$

such that (5.4) holds.

Step 4. The goal of this step is to show that for every $\delta \in (0,1)$ there are $\eta \in (0,1)$ and $\hat x \in B_{r_o/2}(y_o)$ such that

$$B_{\eta \frac{r_o}{2}}(\hat x) \subset B_{\frac{r_o}{2}}(y_o)$$

and

$$\rho(\bar t)\Big(\Big\{u(\cdot, \bar t) \leqslant \frac{M}{4}\Big\} \cap B_{\eta\frac{r_o}{2}}(\hat x)\Big) \leqslant \delta \rho(\bar t)(B_{\eta \frac{r_o}{2}}(\hat x)). \tag{5.6}$$

To show this, it is sufficient to use step 3 and to apply Lemma 2.5 to the function u with $\omega = \rho(\bar t)$, $a = \frac{M}{2}$, $\varepsilon = \frac12$, $r = r_o/2$, $x_o = y_o$, $\mathcal{B} = B_{r_o/2}(y_o)$, and $\sigma = r_o/2$, so that

$$\mathcal{B}^\sigma = B_{r_o}(y_o), \quad \beta = \frac{\bar\nu}{2\gamma_{B_{\frac{r_o}{2}}(y_o)}},$$

and α is as defined in (5.5). Then we get (5.6).

Step 5. In this step, we want to show that an estimate like (5.6) holds also in a cylinder.

Precisely, we show that for every $\bar\delta \in (0,1)$ there is $\varepsilon \in (0,1)$, which will depend only on $\bar\delta$, and

$$\bar s = \Big(\frac{\varepsilon \eta r_o}{4}\Big)^2 \mathsf{h}^*\Big(\hat x, \bar t, \frac{\eta r_o}{4}\Big)$$

such that (with $\hat x, \bar t, \eta, r_o$ as in the previous step)

$$\rho\Big(\Big\{u \leqslant \frac{M}{8}\Big\} \cap (B_{\varepsilon\eta\frac{r_o}{4}}(\hat x) \times [\bar t, \bar t + \bar s])\Big) \leqslant \bar\delta \rho(B_{\varepsilon\eta\frac{r_o}{4}}(\hat x) \times [\bar t, \bar t + \bar s]). \tag{5.7}$$

Apply the energy estimate (3.1) to the function $(u - \frac{M}{4})_-$ in the interval $[\bar t, \bar t + s]$ and ball $B_{\eta r_o/2}(\hat x)$, where $\bar t$ is the one satisfying (5.4) and (5.6), and $s \leqslant \bar s$ with $\varepsilon > 0$ to be chosen.

As test function, consider a function $\zeta = \zeta(x)$ such that $\zeta = 1$ in $B_{\eta r_o/4}(\hat x)$, $\zeta = 0$ outside of $B_{\eta r_o/2}(\hat x)$, and $|D\zeta| \leqslant \frac{4}{\eta r_o}$. Using also (5.6) and integrating in $(\bar t, \bar t + s]$ with $s \in (0, \bar s]$, we get

$$\begin{aligned}
&\int_{B_{\eta r_o/4}(\hat x)} \Big(u - \frac{M}{4}\Big)_-^2 \rho(x,s)\,dx \\
&\quad\leqslant \int_{B_{\eta r_o/2}(\hat x)} \Big(u - \frac{M}{4}\Big)_-^2 \rho(x,\bar t)\,dx \\
&\qquad + \gamma\Big(\frac{16}{\eta^2 r_o^2} + 1\Big)\Big[\int_{\bar t}^{\bar t + \bar s} \int_{B_{\eta r_o/2}(\hat x)} \Big(u - \frac{M}{4}\Big)_-^2(x,t)\,dx\,dt + \frac{M^2}{16} \int_{\bar t}^{\bar t + \bar s} \int_{\{u(t) < \frac{M}{4}\}} dx\,dt\Big] \\
&\quad\leqslant \delta \frac{M^2}{16} \rho(\bar t)(B_{\eta\frac{r_o}{2}}(\hat x)) + \gamma\Big(\frac{16}{\eta^2 r_o^2} + 1\Big)\frac{M^2}{16} 2\bar s |B_{\eta r_o/2}(\hat x)| \\
&\quad= \frac{M^2}{16}\Big[\delta + 2\varepsilon^2 \gamma\Big(1 + \frac{\eta^2 r_o^2}{16}\Big)\mathsf{h}^*\Big(\hat x, \bar t, \frac{\eta r_o}{4}\Big)\Big]\rho(\bar t)(B_{\eta\frac{r_o}{2}}(\hat x)) \\
&\quad\leqslant \gamma_{B_{\eta\frac{r_o}{2}}(\hat x)} \frac{M^2}{16}\Big[\delta + 2\varepsilon^2 \gamma\Big(1 + \frac{\eta^2 r_o^2}{16}\Big)\mathsf{h}^*\Big(\hat x, \bar t, \frac{\eta r_o}{4}\Big)\Big]\rho(s)(B_{\eta\frac{r_o}{2}}(\hat x)),
\end{aligned}$$

where we recall that γ_B is defined in (C1). Now, after taking

$$B(s) = \Big\{x \in B_{\eta r_o/4}(\hat x) \;\Big|\; u(x,s) \leqslant \frac{M}{8}\Big\},$$

we have that

$$\int_{B_{\eta r_o/4}(\hat{x})} \Big(u - \frac{M}{4}\Big)_-^2 (x,s)\rho(x,s)\,dx \geqslant \int_{B(s)} \Big(u - \frac{M}{4}\Big)_-^2 (x,s)\rho(x,s)\,dx \geqslant \frac{M^2}{64}\rho(s)(B(s)).$$

Then, for every $s \in [\bar{t}, \bar{t} + \bar{s}]$,

$$\begin{aligned}\frac{M^2}{64}\rho(s)(B(s)) &\leqslant \gamma_{B_{\eta\frac{r_o}{2}}(\hat{x})} \frac{M^2}{16}\Big[delta + 2\varepsilon^2\gamma\Big(1 + \frac{\eta^2 r_o^2}{16}\Big)\mathsf{h}^*\Big(\hat{x}, \bar{t}, \frac{\eta r_o}{4}\Big)\Big]\rho(s)(B_{\eta\frac{r_o}{2}}(\hat{x})) \\ &\leqslant c_\rho \gamma_{B_{\eta\frac{r_o}{2}}(\hat{x})} \frac{M^2}{16}\Big[\delta + 2\varepsilon^2\gamma\Big(1 + \frac{\eta^2 r_o^2}{16}\Big)\mathsf{h}^*\Big(\hat{x}, \bar{t}, \frac{\eta r_o}{4}\Big)\Big]\rho(s)(B_{\eta\frac{r_o}{4}}(\hat{x})).\end{aligned}$$

Integrating in $[\bar{t}, \bar{t} + \bar{s}]$, we get

$$\begin{aligned}&\rho\Big(\Big\{u \leqslant \frac{M}{8}\Big\} \cap (Q^{\frac{\eta r_o}{4},\varepsilon^2}(\hat{x}, \bar{t}))\Big) \\ &\qquad \leqslant c_\rho \gamma_{B_{\eta\frac{r_o}{2}}(\hat{x})} \frac{M^2}{16}\Big[\delta + 2\varepsilon^2\gamma\Big(1 + \frac{\eta^2 r_o^2}{16}\Big)\mathsf{h}^*\Big(\hat{x}, \bar{t}, \frac{\eta r_o}{4}\Big)\Big]\rho(Q^{\frac{\eta r_o}{4},\varepsilon^2}(\hat{x}, \bar{t})),\end{aligned}$$

where $Q^{\frac{\eta r_o}{4},\varepsilon^2}(\hat{x}, \bar{t})$, according to the definition (3.5) with $R = \eta r_o/4$, is

$$B_{\eta\frac{r_o}{4}}(\hat{x}) \times [\bar{t}, \bar{t} + \bar{s}] = B_{\eta\frac{r_o}{4}}(\hat{x}) \times \Big[\bar{t}, \bar{t} + \Big(\varepsilon\frac{\eta r_o}{4}\Big)^2 \mathsf{h}^*\Big(\hat{x}, \bar{t}, \frac{\eta r_o}{4}\Big)\Big].$$

Now for any $\bar{\delta} \in (0,1)$, since δ (chosen in step 4) and ε are arbitrarily chosen in $(0,1)$, one can find δ and ε in such a way that

$$\bar{\delta} = c_\rho \gamma_{B_{\eta\frac{r_o}{2}}(\hat{x})} \frac{M^2}{16}\Big[\delta + 2\varepsilon^2\gamma\Big(1 + \frac{\eta^2 r_o^2}{16}\Big)\mathsf{h}^*\Big(\hat{x}, \bar{t}, \frac{\eta r_o}{4}\Big)\Big].$$

Step 6. We want to show in this step that there is an instant $\hat{s}$ such that

$$u(x, \hat{s}) \geqslant \frac{M}{16} \quad \text{for a.e. } x \in B_{\frac{\eta r_o}{8}}(\hat{x}). \tag{5.8}$$

Setting $\hat{s} := \bar{t} + \bar{s}$, we have that

$$\hat{s} := \bar{t} + \bar{s} = \bar{t} + (\varepsilon\eta r_o/4)^2 \mathsf{h}^*\Big(\hat{x}, \bar{t}, \frac{\eta r_o}{4}\Big).$$

Then, by (3.3),

$$\begin{aligned}Q^{\frac{\eta r_o}{4},\varepsilon^2}(\hat{x}, \bar{t}) &= B_{\eta\frac{r_o}{4}}(\hat{x}) \times [\bar{t}, \bar{t} + \bar{s}] \\ &= B_{\eta\frac{r_o}{4}}(\hat{x}) \times [\hat{s} - \bar{s}, \hat{s}] \\ &= B_{\eta\frac{r_o}{4}}(\hat{x}) \times \Big[\hat{s} - \varepsilon^2\Big(\frac{\eta r_o}{4}\Big)^2 \mathsf{h}^*\Big(\hat{x}, \bar{t}, \frac{\eta r_o}{4}\Big), \hat{s}\Big] \\ &= B_{\eta\frac{r_o}{4}}(\hat{x}) \times \Big[\hat{s} - \varepsilon^2\Big(\frac{\eta r_o}{4}\Big)^2 \mathsf{h}_*\Big(\hat{x}, \bar{t} + \Big(\frac{\eta r_o}{4}\Big)^2, \frac{\eta r_o}{4}\Big), \hat{s}\Big].\end{aligned}$$

Writing

$$\mathsf{h}_*\Big(\hat{x}, \bar{t} + \Big(\frac{\eta r_o}{4}\Big)^2, \frac{\eta r_o}{4}\Big) = \frac{\mathsf{h}_*(\hat{x}, \bar{t} + (\frac{\eta r_o}{4})^2, \frac{\eta r_o}{4})}{\mathsf{h}_*(\hat{x}, \hat{s}, \frac{\eta r_o}{4})} \mathsf{h}_*\Big(\hat{x}, \hat{s}, \frac{\eta r_o}{4}\Big)$$

and setting

$$\alpha = \varepsilon^2 \frac{\mathsf{h}_*(\hat{x}, \bar{t} + (\frac{\eta r_o}{4})^2, \frac{\eta r_o}{4})}{\mathsf{h}_*(\hat{x}, \hat{s}, \frac{\eta r_o}{4})},$$

we get that (see also (3.5) for the definition of these cylinders)

$$Q^{\frac{\eta r_o}{4},\varepsilon^2}(\hat{x}, \bar{t}) = Q_{\frac{\eta r_o}{4},\alpha}(\hat{x}, \hat{s}). \tag{5.9}$$

Notice that, since ε is arbitrarily chosen, we can always suppose that

$$Q_{\frac{\eta r_o}{4},\alpha}(\hat{x}, \hat{s}) \subset C_{s_o}(x_o, 0),$$

so that

$$\operatorname{osc}_{Q_{\frac{\eta r_o}{4},\alpha}(\hat{x},\hat{s})} u \leqslant 2^{\xi} M.$$

Now we come back to the previous step and observe that, since $\rho \in A_\infty(K_3, \sigma)$, we have (see (C3))

$$\Big(\frac{|\{u \leqslant \frac{M}{8}\} \cap (Q^{\frac{\eta r_o}{2},\varepsilon^2}(\hat{x},\bar{t}))|}{|Q^{\frac{\eta r_o}{2},\varepsilon^2}(\hat{x},\bar{t})|}\Big)^p \leqslant K_3 \frac{\rho(\{u \leqslant \frac{M}{8}\} \cap (Q^{\frac{\eta r_o}{2},\varepsilon^2}(\hat{x},\bar{t})))}{\rho(Q^{\frac{\eta r_o}{2},\varepsilon^2}(\hat{x},\bar{t}))}.$$

Consequently, by (5.7),

$$\frac{|\{u \leqslant \frac{M}{8}\} \cap (Q^{\frac{\eta r_o}{2},\varepsilon^2}(\hat{x},\bar{t}))|}{|Q^{\frac{\eta r_o}{2},\varepsilon^2}(\hat{x},\bar{t})|} \leqslant (K_3\bar{\delta})^{\frac{1}{p}}.$$

Moreover, one can choose $\bar{\delta}$ in such a way that $(K_3\bar{\delta})^{1/p} < 1$. In this way, we get (using (5.9))

$$\frac{|\{u \leqslant \frac{M}{8}\} \cap (Q_{\frac{\eta r_o}{4},\alpha}(\hat{x},\hat{s}))|}{|Q_{\frac{\eta r_o}{4},\alpha}(\hat{x},\hat{s})|} + \frac{\rho(\{u \leqslant \frac{M}{8}\} \cap (Q_{\frac{\eta r_o}{4},\alpha}(\hat{x},\hat{s})))}{\rho(Q_{\frac{\eta r_o}{4},\alpha}(\hat{x},\hat{s}))} \leqslant (K_3\bar{\delta})^{\frac{1}{p}} + \bar{\delta}.$$

Now we use Proposition 4.3 in the cylinder $Q_{\frac{\eta r_o}{4},\alpha}(\hat{x},\hat{s})$ defined in (5.9) with

$$\begin{cases} \hat{r} = \dfrac{\eta r_o}{8}, \\ \tilde{r} = R = \dfrac{\eta r_o}{4}, \\ \tilde{\theta} = \alpha, \\ \hat{\theta} = \dfrac{\alpha}{2}, \\ \mu_- = 0, \\ a = \dfrac{1}{2}, \\ \sigma = \dfrac{1}{2^{\xi}8}, \\ \omega = 2^{\xi} M. \end{cases} \tag{5.10}$$

By Proposition 4.3, we can choose $\bar{\delta}$ such that the value ν of Proposition 4.3 satisfies $\nu = \bar{\delta} + (K_3\bar{\delta})^{1/p}$. With the choices (5.10) and this choice of $\bar{\delta}$, we derive that

$$u(x,t) \geqslant \frac{M}{16} \quad \text{for a.e. } (x,t) \in Q_{\frac{\eta r_o}{8},\frac{\alpha}{2}}(\hat{x},\hat{s}),$$

and in particular (5.8).

Step 7. Denoting by $\hat{r}$ the quantity $\eta r_o/8$, by Theorem 4.8 and by (5.8), we get the existence of $\vartheta \in (0,1)$ such that for every $\bar{\vartheta} \in (0,\vartheta)$ there is $\lambda > 0$ such that

$$u \geqslant \lambda \frac{M}{16} \quad \text{a.e. in } B_{2\hat{r}}(\hat{x}) \times [\hat{s} + \bar{\vartheta}(5\hat{r})^2 \mathsf{h}^*(\hat{x},\hat{s},5\hat{r}), \hat{s} + \vartheta(5\hat{r})^2 \mathsf{h}^*(\hat{x},\hat{s},5\hat{r})].$$

Applying again Theorem 4.8 in $B_{2\hat{r}}(\hat{x}) \times \{t\}$ for every

$$t \in [\hat{s} + \bar{\vartheta}(5\hat{r})^2 \mathsf{h}^*(\hat{x},\hat{s},5\hat{r}), \hat{s} + \vartheta(5\hat{r})^2 \mathsf{h}^*(\hat{x},\hat{s},5\hat{r})],$$

we obtain

$$u \geqslant \lambda^2 \frac{M}{16} \quad \text{a.e. in } B_{4\hat{r}}(\hat{x}) \times \hat{J}_2,$$

where

$$\hat{J}_2 = \bigcup_{\omega \in [\bar{\vartheta},\vartheta]} [\hat{s} + \omega(5\hat{r})^2 \mathsf{h}^*(\hat{x},\hat{s},5\hat{r}) + \bar{\vartheta}(10\hat{r})^2 \mathsf{h}^*(\hat{x},\hat{s} + \omega(5\hat{r})^2 \mathsf{h}^*(\hat{x},\hat{s},5\hat{r}),10\hat{r}),$$
$$\hat{s} + \omega(5\hat{r})^2 \mathsf{h}^*(\hat{x},\hat{s},5\hat{r}) + \vartheta(10\hat{r})^2 \mathsf{h}^*(\hat{x},\hat{s} + \omega(5\hat{r})^2 \mathsf{h}^*(\hat{x},\hat{s},5\hat{r}),10\hat{r})].$$

In particular,

$$u \geqslant \lambda^2 \frac{M}{16} \quad \text{a.e. in } B_{4\hat{r}}(\hat{x}) \times \hat{I}_2,$$

where

$$\hat{I}_2 = \big[\hat{s} + \bar{\vartheta}(5\hat{r})^2 \mathrm{h}^*(\hat{x}, \hat{s}, 5\hat{r}) + \bar{\vartheta}(10\hat{r})^2 \mathrm{h}^*(\hat{x}, \hat{s} + \omega(5\hat{r})^2 \mathrm{h}^*(\hat{x}, \hat{s}, 5\hat{r}), 10\hat{r}), \\ \hat{s} + \vartheta(5\hat{r})^2 \mathrm{h}^*(\hat{x}, \hat{s}, 5\hat{r}) + \vartheta(10\hat{r})^2 \mathrm{h}^*(\hat{x}, \hat{s} + \omega(5\hat{r})^2 \mathrm{h}^*(\hat{x}, \hat{s}, 5\hat{r}), 10\hat{r})\big].$$

Iterating this argument after m steps, one reaches

$$u \geqslant \lambda^m \frac{M}{16} \quad \text{a.e. in } B_{2^m\hat{r}}(\hat{x}) \times [\hat{s}_m, \hat{t}_m],$$

where

$$\begin{aligned}
\hat{s}_0 &= \hat{s}, \\
\hat{s}_1 &= \hat{s} + \bar{\vartheta}(5\hat{r})^2 \mathrm{h}^*(\hat{x}, \hat{s}, 5\hat{r}) = \hat{s}_0 + \bar{\vartheta}\tau_1, \\
\hat{s}_2 &= \hat{s} + \bar{\vartheta}(5\hat{r})^2 \mathrm{h}^*(\hat{x}, \hat{s}, 5\hat{r}) + \bar{\vartheta}(10\hat{r})^2 \mathrm{h}^*(\hat{x}, \hat{s} + \bar{\vartheta}(5\hat{r})^2 \mathrm{h}^*(\hat{x}, \hat{s}, 5\hat{r}), 10\hat{r}) = \hat{s}_0 + \bar{\vartheta}\tau_1 + \bar{\vartheta}\tau_2 = \hat{s}_1 + \bar{\vartheta}\tau_2, \\
\hat{s}_3 &= \hat{s}_2 + \bar{\vartheta}(20\hat{r})^2 \mathrm{h}^*(\hat{x}, \hat{s}_2, 20\hat{r}) = \hat{s}_2 + \bar{\vartheta}\tau_3, \\
&\ \vdots \\
\hat{s}_m &= \hat{s}_{m-1} + \bar{\vartheta}\tau_m = \hat{s} + \bar{\vartheta}(\tau_1 + \tau_2 + \ldots + \tau_m), \\
\tau_1 &= (5\hat{r})^2 \mathrm{h}^*(\hat{x}, \hat{s}, 5\hat{r}), \\
\tau_2 &= (10\hat{r})^2 \mathrm{h}^*(\hat{x}, \hat{s} + \bar{\vartheta}(5\hat{r})^2 \mathrm{h}^*(\hat{x}, \hat{s}, 5\hat{r}), 10\hat{r}), \\
&\ \vdots \\
\tau_m &= (52^{m-1}\hat{r})^2 \mathrm{h}^*(\hat{x}, \hat{s}_{m-1}, 52^{m-1}\hat{r}), \\
\hat{t}_k &= \hat{s}_{k-1} + \vartheta(\tau_1 + \tau_2 + \ldots + \tau_k), \qquad k = 1, \ldots m.
\end{aligned}$$

Notice that, by definition, for k between 1 and m,

$$\begin{aligned}
\hat{s}_k &= \hat{s} + 25\bar{\vartheta} \sum_{j=1}^{k} \frac{4^{j-1}\hat{r}^2}{(\tau_j - \tau_{j-1})|B_{5\cdot 2^{j-1}\hat{r}}(\hat{x})|} \int_{\tau_{j-1}}^{\tau_j} \int_{B_{5\cdot 2^{j-1}\hat{r}}(\hat{x})} \rho(x, t)\, dx\, dt = \hat{s} + 25\bar{\vartheta}\hat{r}^2 \sum_{j=1}^{k} 4^{j-1} \fint\!\!\fint_{C_j} \rho(x, t)\, dx\, dt, \\
\hat{t}_k &= \hat{s} + 25\vartheta\hat{r}^2 \sum_{j=1}^{k} 4^{j-1} \fint\!\!\fint_{C_j} \rho(x, t)\, dx\, dt,
\end{aligned}$$

where

$$C_j := B_{5\cdot 2^{j-1}\hat{r}}(\hat{x}) \times [\tau_{j-1}, \tau_j].$$

We can choose m in such a way that

$$2r \leqslant 2^m\hat{r} \leqslant 4r. \tag{5.11}$$

Here the estimate $2r \leqslant 2^m\hat{r}$ ensures that $B_{2^m\hat{r}}(\hat{x}) \supset B_r(x_o)$ and that $B_{2^m\hat{r}}(\hat{x}) \subset B_{5r}(x_o)$. Now, according to the choice of r_o made in (5.1), we have that

$$\hat{r} = \frac{\eta r_o}{8} = \frac{\eta r_o}{8} = \frac{\eta(r - s_o)}{16}.$$

Recalling that

$$M = b(r - s_o)^{-\xi},$$

we have (now using $2^m\hat{r} \leqslant 4r$)

$$u \geqslant \lambda^m \frac{M}{16} = \frac{\lambda^m}{16} b(r - s_o)^{-\xi} = \frac{\lambda^m}{16} b\Big(\frac{\eta}{16\hat{r}}\Big)^{\xi} \geqslant (2^{\xi}\lambda)^m b\eta^{\xi} 2^{-6\xi-4} r^{-\xi}$$

a.e. in $B_{2^m\hat{r}}(\hat{x}) \times [\hat{s}_m, \hat{t}_m]$. First of all, we get rid of the dependence of m, still to be chosen, in the right-hand side simply by choosing a value of the parameter ξ in such a way that

$$2^{\xi}\lambda = 1,$$

that is, $\xi = \log_2 \lambda^{-1}$ (remember that $\lambda \in (0,1)$); consequently, one has fixed also the value of b. By this choice, we have, in particular,

$$u \geqslant b\eta^{\xi} 2^{-6\xi-4} r^{-\xi} = c_o u(x_o, 0) \quad \text{a.e. in } B_r(x_o) \times [\hat{s}_m, \hat{t}_m],$$

where

$$c_o := \eta^{\xi} 2^{-6\xi-4} \quad \text{and} \quad u(x_o, 0) = b\eta^{\xi} 2^{-6\xi-4} r^{-\xi}$$

for the choice made at the beginning of the proof. Now to conclude, we have to choose $\hat{s}_m$ and $\hat{t}_m$ in such a way that

$$cr^2 \mathsf{h}^*(x_o, 0, r) \in [\hat{s}_m, \hat{t}_m],$$

where $c \in (0,1]$ is arbitrarily chosen. Set, for simplicity,

$$\rho_j := \iint_{C_j} \rho(x,t)\,dx\,dt$$

and observe that ($\hat{s} < 0$)

$$\begin{aligned}
\hat{s}_m &= \hat{s} + 25\bar{\vartheta}\hat{r}^2 \sum_{j=1}^{m} 4^{j-1}\rho_j \\
&< 25\bar{\vartheta}\hat{r}^2 \sum_{j=1}^{m} 4^{j-1}\rho_j \\
&\leqslant 25\bar{\vartheta}\hat{r}^2 \frac{4^m - 1}{3} \max_{j=0,\dots,m-1} \rho_j \\
&< 25\bar{\vartheta}\hat{r}^2 \frac{4^m}{3} \max_{j=0,\dots,m-1} \rho_j \\
&\leqslant 25\bar{\vartheta} \frac{4r}{3} \max_{j=0,\dots,m-1} \rho_j,
\end{aligned}$$

where in the last inequality we have used (5.11). To guarantee that $\hat{s}_m \leqslant cr^2\mathsf{h}^*(x_o, 0, r)$, it is then sufficient to require that

$$25\bar{\vartheta}\frac{16r^2}{3} \max_{j=0,\dots,m-1} \rho_j \leqslant cr^2\mathsf{h}^*(x_o, 0, r)$$

that is,

$$\bar{\vartheta} \leqslant \frac{3}{400} \frac{c\mathsf{h}^*(x_o, 0, r)}{\max_{j=0,\dots,m-1} \rho_j}.$$

It remains in some sense to force $\hat{t}_m \geqslant cr^2\mathsf{h}^*(x_o, 0, r)$. If this is true, we choose

$$\eta = c_o^{-1}$$

and we conclude. Otherwise, since $\hat{s}_m < cr^2\mathsf{h}^*(x_o, 0, r)$, we consider $\tilde{t} > \hat{s}_m$ such that

$$u(x, \tilde{t}) \geqslant c_o u(x_o, 0) \quad \text{a.e. in } B_r(x_o).$$

We can suppose, taking $\bar{\vartheta}$ smaller than the choice made above if necessary, that

$$\tilde{t} + \bar{\vartheta}r^2\mathsf{h}^*(x_o, 0, r) \leqslant cr^2\mathsf{h}^*(x_o, 0, r).$$

By Theorem 4.8, we then get that

$$u(x,t) \geqslant \lambda c_o u(x_o, 0) \quad \text{a.e. in } B_{2r}(x_o) \times [\tilde{t} + \bar{\vartheta}(5r)^2\mathsf{h}^*(x_o, \tilde{t}, 5r), \tilde{t} + \vartheta(5r)^2\mathsf{h}^*(x_o, \tilde{t}, 5r)]. \tag{5.12}$$

Notice that

$$\gamma_{B_{5r}(x_o)}^{-1} \mathsf{h}^*(x_o, 0, 5r) \leqslant \mathsf{h}^*(x_o, \tilde{t}, 5r) \leqslant \gamma_{B_{5r}(x_o)} \mathsf{h}^*(x_o, 0, 5r).$$

Then in particular, taking, if necessary, $\bar{\vartheta}$ smaller than before and anyway $\bar{\vartheta} < \vartheta \gamma_{B_{5r}(x_o)}^{-2}$, we have

$$u(x, t) \geqslant \lambda c_o u(x_o, 0)$$

a.e. in

$$B_r(x_o) \times [\tilde{t} + \bar{\vartheta}\gamma_{B_{5r}(x_o)}(5r)^2 \mathsf{h}^*(x_o, 0, 5r), \tilde{t} + \vartheta\gamma_{B_{5r}(x_o)}^{-1}(5r)^2 \mathsf{h}^*(x_o, 0, 5r)].$$

Now if

$$\tilde{t} + \vartheta\gamma_{B_{5r}(x_o)}^{-1}(5r)^2 \mathsf{h}^*(x_o, 0, 5r) > cr^2 \mathsf{h}^*(x_o, 0, r),$$

we can conclude taking

$$\eta = (\lambda c_o)^{-1};$$

otherwise, we go on. Restarting from (5.12) restricted to $B_r(x_o)$ and since $\lambda \in (0, 1)$, we get that

$$u(x, t) \geqslant \lambda^2 c_o u(x_o, 0)$$

a.e. in

$$\begin{aligned} B_r(x_o) \times [&\tilde{t} + \bar{\vartheta}(5r)^2 \mathsf{h}^*(x_o, \tilde{t}, 5r) + \bar{\vartheta}(5r)^2 \mathsf{h}^*(x_o, \tilde{t} + \bar{\vartheta}(5r)^2 \mathsf{h}^*(x_o, \tilde{t}, 5r), 5r), \\ &\tilde{t} + \vartheta(5r)^2 \mathsf{h}^*(x_o, \tilde{t}, 5r) + \vartheta(5r)^2 \mathsf{h}^*(x_o, \tilde{t} + \bar{\vartheta}(5r)^2 \mathsf{h}^*(x_o, \tilde{t}, 5r), 5r)]. \end{aligned}$$

With the additional constraint

$$\bar{\vartheta} \mathsf{h}^*(x_o, \tilde{t}, 5r) + \bar{\vartheta} \mathsf{h}^*(x_o, \tilde{t} + \bar{\vartheta}(5r)^2 \mathsf{h}^*(x_o, \tilde{t}, 5r), 5r) < \vartheta \mathsf{h}^*(x_o, \tilde{t}, 5r),$$

we obtain

$$u(x, t) \geqslant \lambda^2 c_o u(x_o, 0)$$

a.e. in

$$B_r(x_o) \times [\tilde{t} + \bar{\vartheta}\tau^*(x_o, \tilde{t}, 5r), \tilde{t} + \vartheta(5r)^2 \mathsf{h}^*(x_o, \tilde{t}, 5r) + \vartheta(5r)^2 \mathsf{h}^*(x_o, \tilde{t} + \bar{\vartheta}(5r)^2 \mathsf{h}^*(x_o, \tilde{t}, 5r), 5r)].$$

Now if

$$\tilde{t} + \vartheta \mathsf{h}^*(x_o, \tilde{t}, 5r) + \vartheta \mathsf{h}^*(x_o, \tilde{t} + \bar{\vartheta}(5r)^2 \mathsf{h}^*(x_o, \tilde{t}, 5r), 5r) > c\mathsf{h}^*(x_o, 0, r),$$

one concludes choosing

$$\eta = (\lambda^2 c_o)^{-1}.$$

Iterating this procedure, if necessary, k times, we get that

$$u(x, t) \geqslant \lambda^k c_o u(x_o, 0) \quad \text{a.e. in} \quad B_r(x_o) \times [\tilde{t} + \bar{\vartheta}(5r)^2 \mathsf{h}^*(x_o, \tilde{t}, 5r), \tilde{t} + \vartheta t_k^*],$$

where

$$\begin{aligned} t_0^* &:= 0, \\ t_1^* &:= (5r)^2 \mathsf{h}^*(x_o, \tilde{t}, 5r), \\ t_2^* &:= (5r)^2 \mathsf{h}^*(x_o, \tilde{t}, 5r) + (5r)^2 \mathsf{h}^*(x_o, \tilde{t} + \bar{\vartheta} t_1^*, 5r), \\ &\ \vdots \\ t_k^* &:= \sum_{j=0}^{k-1} (5r)^2 \mathsf{h}^*\Big(x_o, \tilde{t} + \bar{\vartheta} \sum_{i=0}^{j} t_i^*, 5r\Big). \end{aligned}$$

At this point, we need

$$\tilde{t} + \vartheta t_k^* > cr^2 \mathsf{h}^*(x_o, 0, r).$$

Since (by the definition of $C_s(\upsilon, \sigma)$ at the beginning of the proof)

$$\tilde{t} > -r^2 \mathrm{h}^*(x_o, 0, r),$$

the previous request is satisfied if

$$\vartheta t_k^* > (c+1) r^2 \mathrm{h}^*(x_o, 0, r).$$

Moreover, we want $\tilde{t} + \vartheta t_k^*$ to remain in the domain, i.e.

$$\tilde{t} + \vartheta t_k^* < 4r^2 \mathrm{h}^*(x_o, 0, r),$$

which is true if $\vartheta t_k^* < 5r^2 \mathrm{h}^*(x_o, 0, r)$. Since $c \in (0, 1]$, we then require that

$$2\mathrm{h}^*(x_o, 0, r) < \vartheta \frac{t_k^*}{r^2} < 5\mathrm{h}^*(x_o, 0, r).$$

Since, if necessary, ϑ can be taken smaller, one can always find $k \in \mathbb{N}^*$ such that the previous inequalities are satisfied. Notice that k depends (only) on ϑ, $\bar{\vartheta}$ and on the mean value of ρ in the cylinder $B_r(x_o) \times (t_o, t_o + r^2)$, and not on the function u. Now we conclude taking $\eta = (\lambda^k c_o)^{-1}$. Notice that increasing k does not increase η, and therefore η is independent of k. □

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