BEC in Nonextensive Statistical Mechanics

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Abstract

We discuss the Bose-Einstein condensation (BEC) for an ideal gas of bosons in the framework of Tsallis's nonextensive statistical mechanics. We study the corrections to the standard BEC formulas due to a weak nonextensivity of the system. In particular, we consider three cases in the D-dimensional space: the homogeneous gas, the gas in a harmonic trap and the relativistic homogenous gas. The results show that small deviations from the extensive Bose statistics produce remarkably large changes in the BEC transition temperature.

PACS numbers: 05.30-d; 03.75.Fi

A decade ago, Tsallis introduced a nonextensive statistical mechanics (NSM) to describe systems for which the additivity property of entropy does not hold.[1] The NSM can describe systems for which long-range microscopic memory, fractal space-time constraints or long-range interactions affect the thermalization process.[2] The NSM is characterized by a parameter q such that $(q-1)$ is a measure of the lack of extensivity: in the limit $q \to 1$ one recovers the familiar statistical mechanics but for $q \neq 1$ one obtains generalized Boltzmann, Fermi and Bose distributions.[3] In the last few years the NSM has been applied in different contexts like solar neutrinos,[4] high energy nuclear collisions[5] and the cosmic microwave background radiation.[6] In such cases it has been found that a small deviation from standard statistics is sufficient for eliminating the discrepancy between theoretical calculations and experimental data.

Recently, there has been a renewed theoretical interest on Bose-Einstein condensation (BEC) (for a review see Ref. 7), motivated by the experimental achievement of BEC with trapped weakly-interacting alkali-metal atoms.[8] In this paper we analyze the consequences of weak nonextensivity on BEC for an ideal Bose gas. From the generalized Bose-Einstein distribution we derive the BEC transition temperature, the condensed fraction and the energy per particle in three different cases: the homogeneous gas, the gas in a harmonic trap and the relativistic homogenous gas. All the calculations are performed by assuming a D-dimensional space.

For a quantum gas of identical bosons in the grand canonical ensemble, the NSM predicts that the average number of particles with energy ϵ is given by

$$\langle n(\epsilon) \rangle_q = \frac{1}{[1+\beta(q-1)(\epsilon-\mu)]^{1/(q-1)}-1} \, , \tag{1}$$

where μ is the chemical potential and $\beta = 1/(kT)$ with k the Boltzmann constant and T the temperature.[2] This generalized distribution follows from the minimization of the Tsallis's generalized entropy under the dilute gas assumption, namely the different single-particle states of the systems are regarded as independent. Thus, this is not an exact formula but it has been shown to be extremely accurate, in particular near $q = 1$.[9] When $q < 1$ the generalized distribution has an upper cut-off: $(\epsilon - \mu) \leq kT/(1-q)$. In the limit $q \to 1$ the generalized distribution becomes the standard Bose-Einstein distribution. For $q > 1$ there is no cut-off and the (power-law) decay is slower than exponential. Because of the unphysical cut-off for $q < 1$, in this paper we discuss only the case $q \geq 1$.

We want study the effects of weak nonextensivity on the BEC properties. We assume that $(q-1) < 1$ and by performing a Taylor expansion of the generalized Bose distributions in the parameter $(q-1)$, at first order we obtain

$$\langle n(\epsilon)\rangle_q = \frac{1}{e^{\beta(\epsilon-\mu)}-1} + \frac{1}{2}(q-1)\frac{\beta^2(\epsilon-\mu)^2 e^{\beta(\epsilon-\mu)}}{(e^{\beta(\epsilon-\mu)}-1)^2} \ . \tag{2}$$

This is the weak nonextensivity correction to the standard Bose-Einstein distribution and the starting point for our calculations.

The total number of particle for our system of non-interacting bosons reads

$$N = \int_0^\infty d\epsilon \ \rho(\epsilon) \ \langle n(\epsilon)\rangle_q \ , \tag{3}$$

where $\rho(\epsilon)$ is the density of states. It can be obtained from the formula

$$\rho(\epsilon) = \int \frac{d^D\mathbf{q}d^D\mathbf{p}}{(2\pi\hbar)^D}\delta(\epsilon - H(\mathbf{p},\mathbf{q})) \ , \tag{4}$$

where $H(\mathbf{p},\mathbf{q})$ is the classical single-particle Hamiltonian of the system in a D-dimensional space. It is easy to show that for a homogenous gas the density of states in a D-dimensional box of volume V is given by

$$\rho(\epsilon) = \frac{V}{\Gamma(D/2)}\left(\frac{m}{2\pi\hbar^2}\right)^{D/2} \epsilon^{(D-2)/2} \ , \tag{5}$$

where m is the mass of the particle. Instead, for a gas in a harmonic trap one finds

$$\rho(\epsilon) = \frac{\epsilon^{D-1}}{(\hbar\bar{\omega})^D\Gamma(D)} \ , \tag{6}$$

where $\bar{\omega}$ is the geometric average of the trap frequencies. $\Gamma(x)$ is the factorial function.

At the BEC transition temperature T_q, the chemical potential μ is zero and at $\mu = 0$ the number of particles N can be analytically determined from Eq. (2) and (3). By inverting the function $N = N(T_q)$ one finds the transition temperature. It is given by

$$kT_q = \left(\frac{2\pi\hbar^2}{m}\right)\frac{(N/V)^{2/D}}{\zeta(D/2)^{2/D}}\left[1 + \frac{1}{2}(q-1)\frac{\Gamma(D/2+2)\zeta(D/2+1)}{\Gamma(D/2)\zeta(D/2)}\right]^{-2/D} \tag{7}$$

for the homogenous gas, and by

$$kT_q = \frac{\hbar\bar{\omega}}{\zeta(D)^{1/D}} N^{1/D} \left[1 + \frac{1}{2}(q-1)\frac{\Gamma(D+2)\zeta(D+1)}{\Gamma(D)\zeta(D)} \right]^{-1/D} \tag{8}$$

for a gas in a harmonic trap. $\zeta(x)$ is the Riemann ζ-function. Obviously, for $q = 1$ one recovers standard BEC formulas. Moreover one observes that for $D = 2$ there is no BEC in the homogenous gas because $\zeta(1) = \infty$. Instead, BEC is possible with $D = 2$ in a harmonic trap. Note that the inclusion of an attractive interaction can modify the stability of the Bose condensate. A discussion of the the role of dimensionality in the stability of a weakly-interacting condensate can be found in Ref. 10.

An inspection of Eq. (7) and (8) shows that the critical temperature T_q grows by increasing the nonextensive parameter q and the space dimension D. It is important to stress that such effect is quite strong. For example, with $q = 1.1$ and $D = 3$ we have that the relative difference $(T_q - T_1)/T_1$ is 6.32% for the homogenous gas and and 15.48% for the gas in a harmonic trap.

Below T_q, a macroscopic number N_0 of particle occupies the single-particle ground-state of the system. It follows that Eq. (3) gives the number $N - N_0$ of non-condensed particles and the condensed fraction is $N_0/N = 1-(T/T_q)^{D/2}$ for the homogenous gas and $N_0/N = 1 - (T/T_q)^D$ for the gas in harmonic trap. For the sake of completeness, we calculate also the energy

$$E = \int_0^\infty d\epsilon \; \epsilon \; \rho(\epsilon) \; \langle n(\epsilon) \rangle_q \; . \tag{9}$$

From the energy one can easily obtain the specific heat and the other thermodynamical quantities. We find

$$\frac{E}{KT} = V \left(\frac{kT}{2\pi\hbar^2} \right)^{D/2} \frac{D}{2} \zeta(D/2+1) \left[1 + \frac{1}{2}(q-1)\frac{\Gamma(D/2+3)\zeta(D/2+2)}{\Gamma(D/2+1)\zeta(D/2+1)} \right] \tag{10}$$

for the homogenous gas, and by

$$\frac{E}{KT} = \left(\frac{kT}{\hbar\bar{\omega}} \right)^D D\zeta(D+1) \left[1 + \frac{1}{2}(q-1)\frac{\Gamma(D+3)\zeta(D+2)}{\Gamma(D+1)\zeta(D+1)} \right] \tag{11}$$

for a gas in a harmonic trap. Note that our formulas of the energy can be easily generalized above the critical temperature T_q by substituting the Riemann function $\zeta(D)$ with the polylogarithm function $Li_D(z) = \sum_{k=1}^{\infty} z^k/k^D$, that depends on the fugacity $z = e^{\beta\mu}$.

In the case of a relativistic gas, the total number of particles is not conserved because of the production of antiparticles, which becomes relevant when kT is comparable with mc^2. The conserved quantity is the difference between the number N of particles and the number $\bar{N}$ of antiparticles, i.e. the net conserved *charge*

$$Q = N - \bar{N} = \int d\epsilon \; \rho(\epsilon) \left[\langle n(\epsilon) \rangle_q - \langle \bar{n}(\epsilon) \rangle_q \right] \; , \tag{12}$$

where $\langle \bar{n}(\epsilon) \rangle_q$ is obtained from $\langle n(\epsilon) \rangle_q$ with the substitution $\mu \to -\mu$. Thus the chemical potential μ describes both bosons and antibosons: the sign of μ indicates whether particles outnumber antiparticles or vice. Moreover, because both $\langle n(\epsilon) \rangle_q$ and $\langle \bar{n}(\epsilon) \rangle_q$ must be positive definite, it follows that $|\mu| \leq mc^2$.[11]

As well known, the classical single-particle Hamiltonian of a relativistic ideal gas is $H = \sqrt{p^2c^2 + m^2c^4}$ and the density of states reads

$$\rho(\epsilon) = \frac{V 2\pi^{D/2}}{(2\pi\hbar c)^D \Gamma(D/2)} \epsilon (\epsilon^2 - m^2c^4)^{(D-2)/2} \; . \tag{13}$$

It is interesting to observe that in the ultrarelativistic limit, the density of states is $\rho(\epsilon) = (V 2\pi^{D/2})/((2\pi\hbar c)^D \Gamma(D/2)) \epsilon^{(D-1)}$ and it has the same power law of the non-relativistic gas in a harmonic potential. The critical temperature T_q at which BEC occurs corresponds to $|\mu| = mc^2$. In the ultrarelativistic region $kT \gg mc^2$ one can expand Q at first order in μ and then obtains

$$kT_q = \left(\frac{(2\pi\hbar c)^D \Gamma(D/2)}{4\pi^{D/2}\Gamma(D)\zeta(D-1)} \frac{|Q|/V}{mc^2} \right)^{1/(D-1)} \times$$
$$\times \left[1 + \frac{1}{2}(q-1) \frac{(D-1)\Gamma(D+1)\zeta(D)}{\Gamma(D)\zeta(D-1)} \right]^{-1/(D-1)} \; . \tag{14}$$

Note that, as in the non-relativistic case, for a homogenous gas there is BEC only for $D > 2$. Also for the relativistic gas the critical temperature T_q is a growing function of the nonextensive parameter q (for $q \geq 1$) and of the space dimension D. By using the previously introduced values $q = 1.1$ and $D = 3$ we find $(T_q - T_1)/T_1 = 6.83\%$. Finally, we obtain that below T_q the condensed fraction reads $Q_0/Q = 1 - (T/T_q)^{(D-1)}$.

In conclusion, we have analyzed the consequences of Tsallis's nonextensive statistical mechanics on BEC. We have studied three non-interacting systems with a generic spatial dimension: the homogeneous gas, the gas in a harmonic trap and the relativistic homogenous gas. The calculations show that a very small deviation from the extensive Bose statistics produces remarkable changes in the BEC transition temperature. This result may have important consequences, for instance in the formation of Quark-Gluon Plasma[12] and in the thermodynamics of the Higgs field in the early Universe.[13] We observe that the inter-particle interaction can strongly modify the BEC transition temperature and the condensate properties: one of our future projects will be the study of nonextensive statistical mechanics for interacting systems.

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