

Nontriviality of rings of integral-valued polynomials

Giulio Peruginelli* Nicholas J. Werner†

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Abstract

Let S be a subset of $\overline{\mathbb{Z}}$, the ring of all algebraic integers. A polynomial $f \in \mathbb{Q}[X]$ is said to be integral-valued on S if $f(s) \in \overline{\mathbb{Z}}$ for all $s \in S$. The set $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ of all integral-valued polynomials on S forms a subring of $\mathbb{Q}[X]$ containing $\mathbb{Z}[X]$. We say that $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is trivial if $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}}) = \mathbb{Z}[X]$, and nontrivial otherwise. We give a collection of necessary and sufficient conditions on S in order $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ to be nontrivial. Our characterizations involve, variously, topological conditions on S with respect to fixed extensions of the p -adic valuations to $\overline{\mathbb{Q}}$; pseudo-monotone sequences contained in S ; ramification indices and residue field degrees; and the polynomial closure of S in $\overline{\mathbb{Z}}$.

Keywords: integer-valued polynomial, algebraic integers, pseudo-monotone sequences, polynomial closure.

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Conflict of Interest

The authors declare that they have no conflicts of interest.

1 Introduction and Motivation

Let $\overline{\mathbb{Q}}$ be a fixed algebraic closure of the field of rational numbers, and let $\overline{\mathbb{Z}}$ be the integral closure of \mathbb{Z} in $\overline{\mathbb{Q}}$, which is the ring of all algebraic integers. For a subset $S \subseteq \overline{\mathbb{Z}}$, a polynomial $f \in \mathbb{Q}[X]$ is said to be *integral-valued* if $f(s) \in \overline{\mathbb{Z}}$ for all $s \in S$. When $S \subseteq \mathbb{Z}$, the definition reduces to the classical notion of integer-valued polynomials, i.e., $f(S) \subseteq \mathbb{Z}$. We adopt the general term ‘integral-valued’ for polynomials with rational coefficients to underline the fact that the evaluation occurs at elements which are *integral* over \mathbb{Z} . Sets of integral-valued polynomials often form rings with interesting properties, and these rings have been of interest in recent years [7, 13, 15, 18, 21, 27].

*Department of Mathematics “Tullio Levi-Civita” University of Padova, Via Trieste, 63 35121 Padova, Italy. gperugin@math.unipd.it

†Department of Mathematics, Computer and Information Science, State University of New York at Old Westbury, Old Westbury, NY 11560, USA. wernern@oldwestbury.edu

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For each positive integer n , let \mathcal{A}_n be the set of algebraic integers of degree at most n . In [15] (see also [21] and [27]) the following ring of integral-valued polynomials was introduced in order to provide an example of a Prüfer domain strictly contained between $\mathbb{Z}[X]$ and the classical ring of integer-valued polynomials $\text{Int}(\mathbb{Z}) = \{f \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}$:

$$\text{Int}_{\mathbb{Q}}(\mathcal{A}_n) := \{f \in \mathbb{Q}[X] \mid f(\mathcal{A}_n) \subseteq \mathcal{A}_n\}.$$

Note that for $n = 1$, we have $\text{Int}(\mathbb{Z})$. Clearly, $\mathcal{A}_n \subsetneq \mathcal{A}_{n+1}$ for each n . If α is an algebraic number of degree at most n and $f \in \mathbb{Q}[X]$, then the degree of $f(\alpha)$ is also bounded by n . As we will later show in Lemma 4.2, there exist polynomials in $\text{Int}_{\mathbb{Q}}(\mathcal{A}_{n+1})$ that are not in $\text{Int}_{\mathbb{Q}}(\mathcal{A}_n)$. Thus, we have the following chain of strict inclusions (see Lemma 4.2):

$$\dots \subsetneq \text{Int}_{\mathbb{Q}}(\mathcal{A}_{n+1}) \subsetneq \text{Int}_{\mathbb{Q}}(\mathcal{A}_n) \subsetneq \text{Int}_{\mathbb{Q}}(\mathcal{A}_{n-1}) \subsetneq \dots \subsetneq \text{Int}_{\mathbb{Q}}(\mathcal{A}_1) = \text{Int}(\mathbb{Z}). \quad (1.1)$$

By [15, Theorem 3.9], for each n the ring $\text{Int}_{\mathbb{Q}}(\mathcal{A}_n)$ is a Prüfer domain. In particular, this means that the polynomial ring $\mathbb{Z}[X]$ is strictly contained in $\text{Int}_{\mathbb{Q}}(\mathcal{A}_n)$ for each $n \geq 1$. Thus, for $n \geq 2$, these rings provide examples of Prüfer domains strictly contained between $\mathbb{Z}[X]$ and $\text{Int}(\mathbb{Z})$.

More generally, for $S \subseteq \overline{\mathbb{Z}}$ one may then consider the ring

$$\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}}) := \{f \in \mathbb{Q}[X] \mid f(S) \subseteq \overline{\mathbb{Z}}\}.$$

Note that $\text{Int}_{\mathbb{Q}}(\mathcal{A}_n) = \text{Int}_{\mathbb{Q}}(\mathcal{A}_n, \overline{\mathbb{Z}})$. It is stated in [15, p. 2482] that $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ lies properly between $\mathbb{Z}[X]$ and $\text{Int}(\mathbb{Z})$ for every subset S of $\overline{\mathbb{Z}}$ properly containing \mathbb{Z} . The containment $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}}) \subsetneq \text{Int}(\mathbb{Z})$ holds whenever $\mathbb{Z} \subsetneq S$ (see [13]), but we demonstrate below in Example 1.3 that $\mathbb{Z}[X]$ may equal $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$. Hence, the aforementioned statement from [15, p. 2482] is false.

Definition 1.2. For any $\alpha \in \overline{\mathbb{Z}}$, let $O_{\mathbb{Q}(\alpha)}$ be the ring of integers of $\mathbb{Q}(\alpha)$. A subset $S \subseteq \overline{\mathbb{Z}}$ is said to have *unbounded degree* if the set $\{[\mathbb{Q}(\alpha) : \mathbb{Q}] \mid \alpha \in S\}$ is unbounded. For $\alpha \in \overline{\mathbb{Z}}$, the *index* of α is $\iota_{\alpha} := [O_{\mathbb{Q}(\alpha)} : \mathbb{Z}[\alpha]]$.

Example 1.3. Let $S \subseteq \overline{\mathbb{Z}}$ be such that S contains a sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ of unbounded degree and $\iota_{\alpha_n} = 1$ for all n . We will show that $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}}) = \mathbb{Z}[X]$.

Certainly, $\mathbb{Z}[X] \subseteq \text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$. Let $f(X) \in \text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ of degree d , and write $f(X) = \sum_{i=0}^d a_i X^i$, where each $a_i \in \mathbb{Q}$. Take an element $\alpha \in S$ of degree $n > d$, and consider $f(\alpha)$. By assumption $f(\alpha)$ is integral over \mathbb{Z} and so it belongs to $O_{\mathbb{Q}(\alpha)} = \mathbb{Z}[\alpha]$. Thus,

$$f(\alpha) = a_0 + a_1\alpha + \dots + a_d\alpha^d \in \mathbb{Z}[\alpha].$$

As a \mathbb{Z} -module, $\mathbb{Z}[\alpha]$ is a free with basis given by $1, \alpha, \dots, \alpha^{n-1}$, so we also have

$$f(\alpha) = b_0 + b_1\alpha + \dots + b_{n-1}\alpha^{n-1}$$

for some integers b_0, \dots, b_{n-1} that are uniquely determined by $f(\alpha)$. Since $n > d$ and every element of the \mathbb{Q} -vector space $\mathbb{Q}(\alpha)$ can be written uniquely as a \mathbb{Q} -linear combination of $1, \alpha, \dots, \alpha^{n-1}$, each coefficients a_i must be in \mathbb{Z} . Thus, $f(X) \in \mathbb{Z}[X]$ and consequently $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}}) = \mathbb{Z}[X]$.

Thankfully, the fallacious claim from [15] was merely an expository remark, and does not affect the results of that paper.

In particular, Example 1.3 applies when $S = \overline{\mathbb{Z}}$ or when $S = \{\zeta_n\}_{n \in \mathbb{N}}$, where ζ_n is a primitive n^{th} root of unity. Moreover, since $\text{Int}_{\mathbb{Q}}(\overline{\mathbb{Z}}, \overline{\mathbb{Z}}) = \bigcap_{n \in \mathbb{N}} \text{Int}_{\mathbb{Q}}(\mathcal{A}_n)$, Example 1.3 demonstrates that the intersection of all the rings in (1.1) is equal to $\mathbb{Z}[X]$, which answers a question posed by David Dobbs to the first author in 2014.

The goal of this paper is to characterize those subsets $S \subseteq \overline{\mathbb{Z}}$ such that $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}}) \neq \mathbb{Z}[X]$. In pursuit of this problem, we will deal with more general rings of integer-valued polynomials, and describe when these rings are trivial, in the sense that they are equal to ordinary rings of polynomials. We refer to the papers [26, 29] for studies on related problems.

Definition 1.4. Let D be an integral domain, let L be a field containing D , and let F be a subfield of L . For each subset $S \subseteq D$, we define the following ring of integer-valued polynomials:

$$\text{Int}_F(S, D) := \{f \in F[X] \mid f(S) \subseteq D\}.$$

In the literature, when F is the fraction field of D , the above ring is denoted by $\text{Int}(S, D)$. Because our work will often involve changing the field of coefficients of these polynomials, we will always include the subscript F in $\text{Int}_F(S, D)$ for the sake of clarity, even in the case where F is the fraction field of D .

It is clear that $(D \cap F)[X] \subseteq \text{Int}_F(S, D)$. We say that $\text{Int}_F(S, D)$ is *trivial* if $(D \cap F)[X] = \text{Int}_F(S, D)$, and *nontrivial* otherwise.

We seek to describe those subsets $S \subseteq \overline{\mathbb{Z}}$ for which $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is nontrivial. If S is of bounded degree n , then $S \subseteq \mathcal{A}_n$, and we have $\mathbb{Z}[X] \subsetneq \text{Int}_{\mathbb{Q}}(\mathcal{A}_n) \subseteq \text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$. Thus, the sets of interest in this problem are all of unbounded degree. Example 1.3 shows that if S has unbounded degree and $\iota_s = 1$ for all $s \in S$, then $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is trivial. However, if we relax the condition on the indices ι_s , then $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ could be trivial or nontrivial.

Example 1.5. Let $S = 2\overline{\mathbb{Z}} = \{2\alpha \mid \alpha \in \overline{\mathbb{Z}}\}$. Then, S has unbounded degree and not all of the indices ι_s are equal to 1. For this S , $X/2 \in \text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$, and hence $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is nontrivial. It is also possible to construct a set S that has unbounded degree and with not all ι_s equal to 1, but for which $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is trivial; see Example 4.11.

Example 1.6. Fix a prime p . For each $k \in \mathbb{N}$ let $a_k = 1 - \frac{1}{2^k}$, and take $S = \{p^{a_k}\}_{k \in \mathbb{N}} = \{p^{1/2}, p^{3/4}, p^{7/8}, \dots\}$. Then, S has unbounded degree, but $f(X) = X^2/p \in \text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ because $f(p^{a_k}) = p^{a_k - 1}$ for all $k \geq 2$.

This example can be generalized. Once again fix $p \in \mathbb{P}$, but also fix a positive integer $n \geq 2$. For each $k \in \mathbb{N}$, let $b_k = \frac{1}{n-1}(1 - \frac{1}{n^k})$. Take $S = \{p^{b_k}\}_{k \in \mathbb{N}}$. Then, $f(X) = X^n/p \in \text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$, because $f(p^{b_1}) = 1$ and $f(p^{b_k}) = p^{b_k - 1}$ for all $k \geq 2$.

In this paper, we provide both local and global characterizations of subsets $S \subseteq \overline{\mathbb{Z}}$ for which $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is nontrivial. We begin our study by considering integer-valued polynomial rings over subsets of valuation domains. Let V be a valuation domain with fraction field K , and suppose $S \subseteq V$. In Section 2, we review the concept of pseudo-monotone sequences (as defined in [6]), and we use these sequences to provide necessary and sufficient conditions on S in order for $\text{Int}_K(S, V)$ to be nontrivial (Theorem 2.7). In Section 3, we return to the case where $S \subseteq \overline{\mathbb{Z}}$ and relate $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ to rings of integer-valued polynomials over valuation domains in p -adic fields. In this way, Theorem 2.7 can be applied to $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ (see Theorem 3.2 and Corollary 3.3), and provides several characterizations of when $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is nontrivial.

In Section 4, we examine global conditions that imply that $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is trivial or nontrivial. We also consider how the degrees and indices of the algebraic integers in S can influence $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$, present variations on Examples 1.3 and 1.6. Section 5 examines similar questions with regard to ramification indices and residue field degrees in number fields generated by elements of S . We produce a new descending chain of Prüfer domains between $\mathbb{Z}[X]$ and $\text{Int}(\mathbb{Z})$ by considering for each $n \in \mathbb{N}$ the compositum $\mathbb{Q}^{(n)} = \mathbb{Q}(\mathcal{A}_n)$ of all the number fields of degree bounded by n and the ring $\text{Int}_{\mathbb{Q}}(O_{\mathbb{Q}^{(n)}})$ of polynomials integral-valued over the ring of integers $O_{\mathbb{Q}^{(n)}}$ of this infinite algebraic extension (which is a non-Noetherian almost Dedekind domain with finite residue fields). Moreover, we review some classical examples due to Gilmer [12, Example 14] and Chabert [5, Example 6.2] of an infinite algebraic extension K of \mathbb{Q} such that the integral closure D in K of $\mathbb{Z}_{(p)}$ for some prime $p \in \mathbb{Z}$ is an almost Dedekind domain with finite residue fields such that either the set of the residue field degrees (ramification indexes, respectively) of all the prime ideals of D over $\mathbb{Z}_{(p)}$ are unbounded and $\text{Int}_{\mathbb{Q}}(D)$ is trivial. By means of the double boundedness condition of Loper [14], in Theorem 5.18 we prove that if D is the integral closure of $\mathbb{Z}_{(p)}$ in some algebraic extension of \mathbb{Q} , then $\text{Int}_{\mathbb{Q}}(D)$ is nontrivial if and only if $\text{Int}_{\mathbb{Q}}(D)$ is Prüfer.

The final section of the paper considers a broad generalization of Example 1.5, describes the polynomial closure in $\overline{\mathbb{Z}}$ of a subset S of the ring of all algebraic integers, and relates this concept to the nontriviality of $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$.

2 Nontriviality over Valuation Domains

Throughout, V is a valuation domain with valuation v , value group Γ_v , maximal ideal M , and fraction field K . We begin by recalling the definitions of two kinds of pseudo-monotone sequences and related objects, which were introduced in [6] in order to study the polynomial closure of subsets of a rank one valuation domain and studied further in [22, 24]. These definitions originate from the notion of pseudo-convergent sequence given by Ostrowski in [20], and we will not use this kind of sequence in this paper. After proving some basic properties of such sequences, we use them to characterize when $\text{Int}_K(S, V)$ is nontrivial (see Theorem 2.7).

Definition 2.1. Let Λ be an index set which we assume to be infinite and well-ordered, and let $E = \{s_i\}_{i \in \Lambda}$ be a sequence in K . Then, E is said to be

- (i) *pseudo-divergent* if $v(s_i - s_j) > v(s_j - s_k)$ for all $i < j < k \in \Lambda$;
- (ii) *pseudo-stationary* if $v(s_i - s_j) = v(s_k - s_\ell) \in \Gamma_v$ for all $i \neq j \in \Lambda, k \neq \ell \in \Lambda$.

A sequence that satisfies either of these two properties is said to be *pseudo-monotone*.

Definition 2.2. Let $E = \{s_i\}_{i \in \Lambda}$ be a pseudo-monotone sequence in K . We define the *gauge* of E as the following sequence $\{\delta_i\}_{i \in \Lambda}$ of Γ_v :

- (i) if E is pseudo-divergent, for each $i \in \Lambda$ we set $\delta_i = v(s_i - s_j)$, with $j \in \Lambda$ and $j < i$;
- (ii) if E is pseudo-stationary, we let $\delta = v(s_i - s_j)$ for any $i, j \in \Lambda, i \neq j$.

Definition 2.3. Each pseudo-monotone sequence $E = \{s_i\}_{i \in \Lambda}$ in K has an associated *breadth ideal*, which we denote by $\text{Br}(E)$. The definition of $\text{Br}(E)$ depends on which one of the two properties from Definition 2.1 that E satisfies.

- (i) If E is pseudo-divergent, then $\text{Br}(E) = \{x \in K \mid v(x) > \delta_i \text{ for some } i \in \Lambda\}$;
- (ii) If E is pseudo-stationary, then $\text{Br}(E) = \{x \in K \mid v(x) \geq \delta\}$.

For a pseudo-divergent sequence E , it is possible that $\text{Br}(E) = K$ (this occurs when the gauge of E is coinital in Γ_v). In all the other cases, $\text{Br}(E)$ is a fractional V -ideal. When V has rank one and $E = \{s_n\}_{n \in \mathbb{N}}$ is a pseudo-monotone sequence, the *breadth* $\delta_E \in \mathbb{R} \cup \{\infty\}$ of E is $\delta_E := \lim_{n \rightarrow \infty} v(s_{n+1} - s_n)$. In this case, δ_i can be replaced with δ_E in each of the above definitions for $\text{Br}(E)$.

Note that if E is pseudo-stationary and $\delta = v(c)$ for some $c \in K$, then $\text{Br}(E) = cV$ is a principal (fractional) ideal. Also, if $E \subset V$ is any pseudo-monotone sequence, then $\text{Br}(E) \subseteq V$, and $\text{Br}(E) \subseteq M$ if E is pseudo-divergent.

Finally, we recall that, given $a \in K$ and $\delta \in \Gamma_v$, the ball of center a and radius δ is $B(a, \delta) = \{x \in K \mid v(x - a) \geq \delta\}$. For each $b \in B(a, \delta)$ it is well known that $B(a, \delta) = B(b, \delta)$.

Lemma 2.4. *Let $S \subseteq V$. Assume there exist a finite subset $T \subseteq S$ and $b \in M$ such that for all $s \in S$, there exists $t \in T$ such that $v(s - t) \geq v(b)$.*

- (1) *If S contains a pseudo-divergent sequence E , then $\text{Br}(E) \subseteq bM \subsetneq M$.*
- (2) *If S contains a pseudo-stationary sequence E , then $\text{Br}(E) \subseteq bV \subsetneq V$.*

Note that the assumption in this lemma is equivalent to saying that S is covered by a finite union of balls $B(t, \delta)$, where $\delta > 0$ is fixed and each $t \in T$.

Proof. Let $\delta = v(b) > 0$. Note that if either one of the two conditions is satisfied, then S is infinite. In particular, $b \neq 0$, because otherwise S would be finite. Since T is finite, there exists $t \in T$ such that $B(t, \delta)$ contains an infinite subsequence $E' = \{s_i\}_{i \in \Lambda'}$ of E . If E is pseudo-divergent (respectively, pseudo-stationary), then E' is also pseudo-divergent (resp., pseudo-stationary), and $\text{Br}(E') = \text{Br}(E)$. For any distinct $i, j \in \Lambda'$, we have $v(s_i - s_j) = v((s_i - t) + (t - s_j)) \geq \delta$.

When E' is pseudo-divergent, the gauge of E' is strictly decreasing, and the above inequality implies that $v(s_i - s_j) > \delta$. It follows that $\text{Br}(E') \subseteq bM \subsetneq M$. If instead E' is pseudo-stationary, then $v(s_i - s_j)$ is constant for all $i, j \in \Lambda'$. In this case, we have $v(s_i - s_j) \geq \delta > 0$, which means that $\text{Br}(E') \subseteq bV \subsetneq V$. \square

For a value group Γ_v associated to the valuation v , the divisible hull of Γ_v is $\mathbb{Q}\Gamma_v := \Gamma_v \otimes_{\mathbb{Z}} \mathbb{Q}$. Note that if L/K is an algebraic extension and w is an extension of v to L , then $w(\alpha) \in \mathbb{Q}\Gamma_v$ for all $\alpha \in L$.

Lemma 2.5. *Let $S \subseteq V$. The following are equivalent.*

- (1) *There exist a finite subset $T \subseteq S$ and $\lambda \in \mathbb{Q}\Gamma_v \cup \{\infty\}$ such that $\lambda > 0$ and for all $s \in S$, there exists $t \in T$ such that $v(s - t) \geq \lambda$.*
- (2) *There exist a finite subset $T \subseteq S$ and $\delta \in \Gamma_v \cup \{\infty\}$ such that $\delta > 0$ and for all $s \in S$, there exists $t \in T$ such that $v(s - t) \geq \delta$.*
- (3) *There exists $b \in M$ such that S/bV is finite.*

Proof. Note that $\lambda = \infty$ or $\delta = \infty$ in (1) or (2) corresponds to $b = 0$ in (3). In this case, the set S is finite. So, we assume throughout that $\lambda \neq \infty$ and $\delta \neq \infty$.

(1) \Rightarrow (2) Assume that (1) holds. Let $\Gamma' = \{\gamma \in \Gamma_v \mid \gamma \geq \lambda\}$. Suppose first that Γ' contains a minimum element γ' . Then, for all $x \in V$, $v(x) \geq \lambda$ if and only if $v(x) \geq \gamma'$, so we may take $\delta = \gamma'$ and (2) holds. So, assume that Γ' has no minimum element. In particular, this means that $\lambda \notin \Gamma_v$.

Since $\lambda \in \mathbb{Q}\Gamma_v \setminus \Gamma_v$, there exists $\gamma \in \Gamma_v$ and $m \in \mathbb{Z}$, $m \geq 2$, such that $\lambda = \gamma/m$. For each $1 \leq k \leq m-1$, let

$$I_k = \{\varepsilon \in \Gamma' \mid k\lambda < \varepsilon \leq (k+1)\lambda\}.$$

We claim that there exists $k \in \{1, \dots, m-1\}$ and $\varepsilon, \varepsilon' \in \Gamma'$ such that

$$k\lambda < \varepsilon < \varepsilon' \leq (k+1)\lambda. \quad (2.6)$$

To see this, let $\delta_1 = \gamma = m\lambda \in \Gamma'$. Since Γ' has no minimum value in Γ_v , there exist $\delta_2, \dots, \delta_m \in \Gamma'$ such that $\delta_1 > \delta_2 > \dots > \delta_m > \lambda$. Since there are $m-1$ sets I_1, \dots, I_{m-1} , some I_k must contain at least two distinct elements $\delta_i, \delta_j \in \{\delta_1, \dots, \delta_m\}$ such that $\delta_i > \delta_j$. For this k , taking $\varepsilon = \delta_j$ and $\varepsilon' = \delta_i$ satisfies (2.6). Moreover, (2.6) implies that $0 < \varepsilon' - \varepsilon < \lambda$. Thus, we may take $\delta = \varepsilon' - \varepsilon \in \Gamma_v$, and then for $x \in V$, $v(x) \geq \lambda$ implies that $v(x) \geq \delta > 0$. Hence, (2) holds.

(2) \Rightarrow (3) Assume that (2) holds. Choose $b \in M$, such that $v(b) = \delta$. Let $T = \{t_1, \dots, t_m\}$. We claim that $S/bV = \{t_i + bV \mid 1 \leq i \leq m\}$. Indeed, given $s \in S$, there exists i such that $v(s - t_i) \geq \delta = v(b)$. So, $s - t_i \in bV$ and hence $s + bV = t_i + bV$.

(3) \Rightarrow (1) Assume that (3) holds. Let $t_1, \dots, t_m \in S$ be such that $S/bV = \{t_i + bV \mid 1 \leq i \leq m\}$. Take $T = \{t_1, \dots, t_m\}$ and $\lambda = v(b) > 0$. Then, for any $s \in S$, there exists i such that $s + bV = t_i + bV$, and for this i we have $v(s - t_i) \geq v(b) = \lambda$. \square

Theorem 2.7. *Let $S \subseteq V$. The following are equivalent.*

- (1) $\text{Int}_K(S, V)$ is nontrivial.
- (2) There exist a finite subset $T \subseteq S$ and $\lambda \in \mathbb{Q}\Gamma_v \cup \{\infty\}$ such that $\lambda > 0$ and for all $s \in S$, there exists $t \in T$ such that $v(s - t) \geq \lambda$.
- (3) There exist a finite subset $T \subseteq S$ and $\delta \in \Gamma_v \cup \{\infty\}$ such that $\delta > 0$ and for each $s \in S$, there exists $t \in T$ such that $v(s - t) \geq \delta$.
- (4) There exists $b \in M$ such that S/bV is finite.
- (5) S contains neither a pseudo-divergent sequence E with $\text{Br}(E) = M$, nor a pseudo-stationary sequence E with $\text{Br}(E) = V$.

Proof. The theorem is trivial if S is finite (which is equivalent to having $\lambda = \infty$ in (2), $\delta = \infty$ in (3), or $b = 0$ in (4)), so we will assume throughout that S is infinite.

(1) \Rightarrow (2) Assume $V[X] \not\subseteq \text{Int}_K(S, V)$. Among all polynomials in $\text{Int}_K(S, V) \setminus V[X]$, choose $f(X)$ of minimal degree m . Then, $m \geq 1$. Write $f(X) = g(X)/d$, where $g(X) \in V[X]$ and $d \in M$. Let a be the leading coefficient of g .

We claim that $v(a) < v(d)$. If $v(a) \geq v(d)$, then $aX^m/d \in V[X]$, and $f(X) - aX^m/d$ would be either an element of $V[X]$, or an element of $\text{Int}_K(S, V) \setminus V[X]$ of degree less than m . We reach a contradiction in either case. So, $v(a) < v(d)$.

Next, let L/K be the splitting field of $g(X)$, and let w be an extension of v to L . In L , we have $g(X) = a(X - \alpha_1) \cdots (X - \alpha_m)$ for some $\alpha_1, \dots, \alpha_m \in L$. Let $\lambda = (v(d) - v(a))/m \in \mathbb{Q}\Gamma_v$, which is greater than 0. Then, for all $s \in S$, there exists $1 \leq i \leq m$ such that $w(s - \alpha_i) \geq \lambda$. Indeed, if this is not the case, then for some $s \in S$,

$$v(f(s)) = w(f(s)) < v(a) + m\lambda - v(d) = 0,$$

and $f \notin \text{Int}_K(S, V)$.

For each $1 \leq i \leq m$, let $B_w(\alpha_i, \lambda) = \{x \in L \mid w(x - \alpha_i) \geq \lambda\}$ and let $B_i = B_w(\alpha_i, \lambda) \cap S$. Whenever B_i has nonempty intersection with S , we may choose $t_i \in B_i \cap S$. Take $T = \{t_i \mid 1 \leq i \leq m \text{ and } B_i \cap S \neq \emptyset\}$. Note that $T \neq \emptyset$ because we showed above that $S \subseteq \bigcup_{i=1}^m B_w(\alpha_i, \lambda)$. Given $s \in S$, find i such that $w(s - \alpha_i) \geq \lambda$. Then, since $w(t_i - \alpha_i) \geq \lambda$, we have $v(s - t_i) = w(s - t_i) \geq \lambda$. Thus, (2) holds.

(2) \Leftrightarrow (3) \Leftrightarrow (4) This is the content of Lemma 2.5.

(4) \Rightarrow (5) This follows from Lemma 2.4.

(5) \Rightarrow (1) We prove the contrapositive. Assume that $V[X] = \text{Int}_K(S, V)$. Then, for all $m \geq 1$ and all $s_1, \dots, s_m \in S$, and all $d \in M$, the polynomial $(X - s_1) \cdots (X - s_m)/d$ is not in $\text{Int}_K(S, V)$. Consequently,

$$\begin{aligned} &\text{for all } s_1, \dots, s_m \in S, \text{ and all } \gamma \in \Gamma_v \text{ with } \gamma > 0, \\ &\text{there exists } t \in S \text{ such that } v(\prod_{i=1}^m (t - s_i)) < \gamma. \end{aligned} \tag{2.8}$$

Suppose first that v is discrete. By (2.8), given any $s_1, \dots, s_m \in S$, there exists $s_{m+1} \in S$ such that $v(\prod_{i=1}^m (s_{m+1} - s_i)) = 0$. So, we can construct a sequence $E = \{s_1, s_2, \dots\}$ such that $v(s_i - s_j) = 0$ for all $i \neq j$. This sequence is pseudo-stationary with $\text{Br}(E) = V$.

For the remainder of the proof, we will assume that v is not discrete. If S contains a pseudo-stationary sequence E with $\text{Br}(E) = V$, then we are done. So, we will further assume that this is not the case. We will demonstrate how to construct a pseudo-divergent sequence $E \subseteq S$ with $\text{Br}(E) = M$.

Choose $s_1 \in S$. For each $n \geq 1$, choose—if possible— $s_{n+1} \in S$ such that $v(\prod_{i=1}^n (s_{n+1} - s_i)) = 0$. We cannot do this indefinitely, since then $\{s_n\}_{n \in \mathbb{N}}$ would be pseudo-stationary with breadth ideal equal to V . So, there exists $N \geq 1$ such that

$$v(s_i - s_j) = 0 \text{ for all } 1 \leq i < j \leq N, \text{ and } v(\prod_{i=1}^N (s - s_i)) > 0 \text{ for all } s \in S. \tag{2.9}$$

In other words, for each $s \in S$, there exists a unique $i \in \{1, \dots, N\}$ such that $v(s - s_i) > 0$ and for every $j \in \{1, \dots, N\}$, $j \neq i$, we have $v(s - s_j) = 0$.

Let $\delta_1 \in \Gamma_v$, $\delta_1 > 0$. By (2.8) and (2.9), there exists $s_{N+1} \in S$ such that $0 < v(\prod_{i=1}^N (s_{N+1} - s_i)) < \delta_1$. By the previous paragraph, we have $v(s_{N+1} - s_{i_1}) > 0$ for a unique $i_1 \in \{1, \dots, N\}$ and $v(s_{N+1} - s_j) = 0$ for every $j \in \{1, \dots, N\} \setminus \{i_1\}$. In particular, $0 < v(s_{N+1} - s_{i_1}) < \delta_1$. We choose now $\delta_2 \in \Gamma_v$ such that $0 < \delta_2 < v(s_{N+1} - s_{i_1})$ and correspondingly, by (2.8) and (2.9), we can find $s_{N+2} \in S$ such that $0 < v(\prod_{i=1}^{N+1} (s_{N+2} - s_i)) < \delta_2$. As before, there exists a unique $i_2 \in \{1, \dots, N\}$ such that $v(s_{N+2} - s_{i_2}) > 0$ and $v(s_{N+2} - s_j) = 0$ for every $j \in \{1, \dots, N\} \setminus \{i_2\}$.

At this point, we have

$$0 < v(s_{N+2} - s_{i_2}) < \delta_2 < v(s_{N+1} - s_{i_1}) < \delta_1.$$

If we continue in this way, we can produce a sequence $E = \{s_{N+k}\}_{k \in \mathbb{N}} \subseteq S$ and a sequence $\{\delta_k\}_{k \in \mathbb{N}} \subseteq \{\gamma \in \Gamma_v \mid \gamma > 0\}$ such that $\{\delta_k\}_{k \in \mathbb{N}}$ decreases to 0 and for each $k, \ell \in \mathbb{N}$, $k < \ell$ we have

$$0 < v(s_{N+\ell} - s_{i_\ell}) < \delta_\ell < v(s_{N+k} - s_{i_k}) < \delta_k \quad (2.10)$$

where $i_k, i_\ell \in \{1, \dots, N\}$. For each $1 \leq i \leq N$, define

$$E_i := \{s_{N+k} \mid k \in \mathbb{N} \text{ and } v(s_{N+k} - s_i) > 0\}.$$

Note that if $s_{N+k} \in E_i$, then $0 < v(s_{N+k} - s_i) < \delta_k$ and $v(s_{N+k} - s_j) = 0$ for every $j \in \{1, \dots, N\} \setminus \{i\}$.

Now, at least one of the sets E_i , $i = 1, \dots, N$, must be infinite. Without loss of generality, assume that E_1 is infinite. We will prove that E_1 is pseudo-divergent with $\text{Br}(E_1) = M$. Write $E_1 = \{s_{k_0}, s_{k_1}, s_{k_2}, \dots\}$, where $s_{k_0} = s_1$ and $k_1 < k_2 < \dots$. By (2.10), for $k_j > k_i$ we have

$$0 < v(s_{k_j} - s_1) < \delta_{k_j} < v(s_{k_i} - s_1) < \delta_{k_i}$$

which shows that E_1 is pseudo-divergent with pseudo-limit s_1 (see for example [6, Remark 4.7]). Moreover, $\text{Br}(E_1) = M$ because the sequence $\{\delta_k\}_{k \in \mathbb{N}}$ decreases to 0. \square

3 Local Case

In this section, we show how Theorem 2.7 can be used to decide whether or not $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is trivial. Let \mathbb{P} be the set of all prime numbers. For $p \in \mathbb{P}$, v_p is the standard p -adic valuation, $\mathbb{Z}_{(p)}$ is the localization of \mathbb{Z} at $p\mathbb{Z}$, \mathbb{Z}_p denotes the ring of p -adic integers, \mathbb{Q}_p is the field of p -adic numbers, and \mathbf{F}_p is the finite field with p elements. We let $\overline{\mathbb{Q}_p}$ be an algebraic closure of \mathbb{Q}_p , and let $\overline{\mathbb{Z}_p}$ be the absolute integral closure of \mathbb{Z}_p . Similarly, $\overline{\mathbb{Z}_{(p)}}$ denotes the integral closure of $\mathbb{Z}_{(p)}$ in $\overline{\mathbb{Q}}$. Note that $\overline{\mathbb{Z}_p}$ is a rank one non-discrete valuation domain with residue field equal to an algebraic closure of \mathbf{F}_p .

Definition 3.1. Let $p \in \mathbb{P}$ and let $S \subseteq \overline{\mathbb{Z}_{(p)}}$. Let $\mathcal{P}(S) \subseteq \mathbb{Z}_{(p)}[X]$ be set of minimal polynomials over \mathbb{Q} of all the elements of S . We define $\Sigma_p(S)$ to be the set of roots in $\overline{\mathbb{Z}_p}$ of the polynomials in $\mathcal{P}(S)$.

Theorem 3.2. *Let $S \subseteq \overline{\mathbb{Z}}$. The following are equivalent.*

- (1) $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is nontrivial.
- (2) There exists $p \in \mathbb{P}$ such that $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}_{(p)}})$ is nontrivial.
- (3) There exists $p \in \mathbb{P}$ such that $\text{Int}_{\overline{\mathbb{Q}_p}}(\Sigma_p(S), \overline{\mathbb{Z}_p})$ is nontrivial.
- (4) There exists $p \in \mathbb{P}$ such that $\text{Int}_{\overline{\mathbb{Q}_p}}(\Sigma_p(S), \overline{\mathbb{Z}_p})$ is nontrivial.

The proof of Theorem 3.2 is quite long and requires numerous intermediate results. To keep things organized, we will prove the equivalences in the theorem one at a time. Before doing this, we mention the connection between Theorem 3.2 and our work from Section 2. Since $\overline{\mathbb{Z}_p}$ is a valuation domain with fraction field $\overline{\mathbb{Q}_p}$, combining Theorem 2.7 and Theorem 3.2 yields the following corollary.

Corollary 3.3. *Let $S \subseteq \overline{\mathbb{Z}}$. Then, $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is nontrivial if and only if there exists $p \in \mathbb{P}$ such that $\Sigma_p(S) \subseteq \overline{\mathbb{Z}_p}$ satisfies one of the equivalent conditions of Theorem 2.7.*

Equivalence of 3.2(1) and 3.2(2)

First, we show that our rings of integral-valued polynomials are well-behaved with respect to localization at primes of \mathbb{Z} .

Lemma 3.4. *Let $p \in \mathbb{P}$. Then, $\overline{\mathbb{Z}}_{(p)} = \overline{\mathbb{Z}}_{(p)}$. Moreover, $(\mathcal{A}_n)_{(p)} = \{\frac{\alpha}{t} \mid \alpha \in \mathcal{A}_n, t \in \mathbb{Z} \setminus p\mathbb{Z}\}$, and so $\bigcap_{p \in \mathbb{P}} (\mathcal{A}_n)_{(p)} = \mathcal{A}_n$.*

Proof. This follows from the standard principle (see e.g. [1, Proposition 5.12]) that integral closure and localization commute with one another. \square

Since $\overline{\mathbb{Z}} = \bigcap_{p \in \mathbb{P}} \overline{\mathbb{Z}}_{(p)}$, we clearly have

$$\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}}) = \bigcap_{p \in \mathbb{P}} \text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}}_{(p)}). \quad (3.5)$$

The next proposition shows that for each $p \in \mathbb{P}$, $\text{Int}(S, \overline{\mathbb{Z}}_{(p)})$ is the localization of $\text{Int}(S, \overline{\mathbb{Z}})$ at the multiplicative set $\mathbb{Z} \setminus p\mathbb{Z}$.

Proposition 3.6. *Let $S \subseteq \overline{\mathbb{Z}}$ and let $p \in \mathbb{P}$. Then, $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})_{(p)} = \text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}}_{(p)})$.*

Proof. First, let $f \in \text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ and $n \in \mathbb{Z} \setminus p\mathbb{Z}$. Then, for each $s \in S$, we have that $\frac{f(s)}{n}$ is integral over $\mathbb{Z}_{(p)}$ by Lemma 3.4. Thus, $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})_{(p)} \subseteq \text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}}_{(p)})$.

The proof of the other containment is similar to the proof of [8, (7)]. Let $f \in \text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}}_{(p)})$. There exists $d \in \mathbb{Z}, d \neq 0$ such that $df(X) \in \mathbb{Z}[X]$. Suppose that $d = p^a t$, for some $a \geq 0$ and $t \in \mathbb{Z}$ not divisible by p . Let $g(X) = tf(X)$. Then $g \in \text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}}_{(p)})$. Moreover, for each prime $q \neq p$, $g \in \mathbb{Z}_{(q)}[X] \subseteq \text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}}_{(q)})$. Now, we have

$$g \in \bigcap_{p \in \mathbb{P}} \text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}}_{(p)}) = \text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$$

so that $f = \frac{g}{t}$ is in $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})_{(p)}$ because t is invertible in $\mathbb{Z}_{(p)}$. \square

Proposition 3.7. *Let $S \subseteq \overline{\mathbb{Z}}$. Then $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is trivial if and only if, for each $p \in \mathbb{P}$, $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}}_{(p)})$ is trivial.*

Proof. The ‘if’ direction follows by (3.5) and the fact that $\mathbb{Z}[X] = \bigcap_{p \in \mathbb{P}} \mathbb{Z}_{(p)}[X]$. Conversely, suppose that $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}}) = \mathbb{Z}[X]$. Localizing at any $p \in \mathbb{P}$, by Proposition 3.6 we get $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})_{(p)} = \mathbb{Z}_{(p)}[X] = \text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}}_{(p)})$. \square

Equivalence of 3.2(2) and 3.2(3)

Proposition 3.8. *Let $p \in \mathbb{P}$ and let $S \subseteq \overline{\mathbb{Z}}_{(p)}$. Then, $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}}_{(p)}) = \text{Int}_{\mathbb{Q}}(\Sigma_p(S), \overline{\mathbb{Z}}_p)$.*

Proof. First, assume $f \in \text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}}_{(p)})$ and let $\alpha \in \Sigma_p(S)$. Then, by definition of the set $\Sigma_p(S)$, there exists $\beta \in S$ such that α and β have the same minimal polynomial over \mathbb{Q} . In particular, $\alpha = \sigma(\beta')$, where $\beta' \in \overline{\mathbb{Q}}$ is a conjugate of β over \mathbb{Q} and $\sigma : \mathbb{Q}(\beta') \rightarrow \mathbb{Q}_p(\alpha)$ is a \mathbb{Q} -embedding (which corresponds to some prime ideal P of $O_{\mathbb{Q}(\beta')}$ above p). By assumption, $f(\beta)$ is integral over $\mathbb{Z}_{(p)}$; but, f is also integral-valued on β' , even if β' might not lie in S . Hence, $f(\alpha) = \sigma(f(\beta')) \in O_{\mathbb{Q}_p(\alpha)} \subseteq \overline{\mathbb{Z}}_p$.

Conversely, suppose $f \in \text{Int}_{\mathbb{Q}}(\Sigma_p(S), \overline{\mathbb{Z}_p})$ and let $\beta \in S$. For each \mathbb{Q} -embedding $\sigma : \mathbb{Q}(\beta) \rightarrow \overline{\mathbb{Q}_p}$, we have $\sigma(\beta) \in \Sigma_p(S)$. Thus, $f(\sigma(\beta)) \in O_{\mathbb{Q}_p(\sigma(\beta))}$. Denote the localization of $O_{\mathbb{Q}(\beta)}$ at a prime P by $O_{\mathbb{Q}(\beta), P}$. Then,

$$f(\beta) \in \bigcap_{P|p} O_{\mathbb{Q}(\beta), P} = O_{\mathbb{Q}(\beta), (p)}.$$

Since $O_{\mathbb{Q}(\beta), (p)} \subseteq \overline{\mathbb{Z}_p}$, we obtain the desired result. \square

Proposition 3.9. *Let $p \in \mathbb{P}$ and $S \subseteq \overline{\mathbb{Z}_p}$. Then $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}_p})$ is nontrivial if and only if $\text{Int}_{\mathbb{Q}_p}(\Sigma_p(S), \overline{\mathbb{Z}_p})$ is nontrivial.*

Proof. Note that by Proposition 3.8, we have $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}_p}) = \text{Int}_{\mathbb{Q}}(\Sigma_p(S), \overline{\mathbb{Z}_p})$ so that

$$\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}_p}) = \text{Int}_{\mathbb{Q}_p}(\Sigma_p(S), \overline{\mathbb{Z}_p}) \cap \mathbb{Q}[X].$$

Now, $\overline{\mathbb{Z}_p} \cap \mathbb{Q}_p = \mathbb{Z}_p$ and $\mathbb{Z}_p \cap \mathbb{Q} = \mathbb{Z}_{(p)}$. So, if $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}_p})$ is nontrivial, then $\text{Int}_{\mathbb{Q}_p}(\Sigma_p(S), \overline{\mathbb{Z}_p})$ is nontrivial.

Conversely, suppose that there exists $f \in \text{Int}_{\mathbb{Q}_p}(\Sigma_p(S), \overline{\mathbb{Z}_p}) \setminus \mathbb{Z}_p[X]$. Since $\mathbb{Q}_p = \mathbb{Z}_p[\frac{1}{p}]$, we may write

$$f(X) = \frac{F(X)}{p^n} = \frac{\sum_{i=0}^d \alpha_i X^i}{p^n}$$

where $F(X) = \sum_i \alpha_i X^i \in \mathbb{Z}_p[X]$ and $n \geq 1$. For each $i = 0, \dots, d$, choose $a_i \in \mathbb{Z}_{(p)}$ such that $v_p(a_i - \alpha_i) \geq n$. We consider the polynomial

$$g(X) = \frac{G(X)}{p^n} = \frac{\sum_{i=0}^d a_i X^i}{p^n}.$$

By construction, $g - f \in \mathbb{Z}_p[X]$, so $g \in \mathbb{Q}[X] \setminus \mathbb{Z}_{(p)}[X]$. We will show that $g \in \text{Int}_{\mathbb{Q}}(\Sigma_p(S), \overline{\mathbb{Z}_p})$. Let $s \in \Sigma_p(S)$. Then,

$$g(s) = \frac{G(s) - F(s) + F(s)}{p^n} = \frac{G(s) - F(s)}{p^n} + f(s)$$

For each $0 \leq i \leq d$, we have $v_p(a_i - \alpha_i) + iv(s) \geq n$, so $\frac{G(s) - F(s)}{p^n}$ has non-negative valuation. Thus, $g(s) \in \overline{\mathbb{Z}_p}$. \square

Equivalence of 3.2(3) and 3.2(4)

For a prime p , we denote by G_p the absolute Galois group of \mathbb{Q}_p , that is, $G_p = \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$. For a subset S of $\overline{\mathbb{Z}_p}$ we set

$$G_p(S) = \{\sigma(s) \mid s \in S, \sigma \in G_p\},$$

which is a subset of $\overline{\mathbb{Z}_p}$ consisting of the union of all the conjugates of the elements in S . Note that we have the equality $\text{Int}_{\mathbb{Q}_p}(S, \overline{\mathbb{Z}_p}) = \text{Int}_{\mathbb{Q}_p}(G_p(S), \overline{\mathbb{Z}_p})$.

Proposition 3.10. *Let $p \in \mathbb{P}$ and let $S \subseteq \overline{\mathbb{Z}_p}$. The following are equivalent.*

- (1) $\text{Int}_{\mathbb{Q}_p}(S, \overline{\mathbb{Z}_p})$ is nontrivial.
- (2) $\text{Int}_{\overline{\mathbb{Q}_p}}(G_p(S), \overline{\mathbb{Z}_p})$ is nontrivial.

(3) $\text{Int}_{\overline{\mathbb{Q}_p}}(S, \overline{\mathbb{Z}_p})$ is nontrivial.

Proof. (1) \Leftrightarrow (2) By [23, Lemma 2.20], the integral closure of $\text{Int}_{\mathbb{Q}_p}(S, \overline{\mathbb{Z}_p}) = \text{Int}_{\mathbb{Q}_p}(G_p(S), \overline{\mathbb{Z}_p})$ in $\overline{\mathbb{Q}_p}(X)$ is equal to $\text{Int}_{\overline{\mathbb{Q}_p}}(G_p(S), \overline{\mathbb{Z}_p})$. In particular, $\text{Int}_{\mathbb{Q}_p}(S, \overline{\mathbb{Z}_p})$ strictly contains $\mathbb{Z}_p[X]$ if and only if $\text{Int}_{\overline{\mathbb{Q}_p}}(G_p(S), \overline{\mathbb{Z}_p})$ strictly contains $\overline{\mathbb{Z}_p}[X]$.

(2) \Rightarrow (3) Since $S \subseteq G_p(S)$, we have $\text{Int}_{\overline{\mathbb{Q}_p}}(G_p(S), \overline{\mathbb{Z}_p}) \subseteq \text{Int}_{\overline{\mathbb{Q}_p}}(S, \overline{\mathbb{Z}_p})$. So, (2) implies (3).

(3) \Rightarrow (2) Assume $\text{Int}_{\overline{\mathbb{Q}_p}}(S, \overline{\mathbb{Z}_p})$ is nontrivial. By Theorem 2.7, there exists a finite set $T \subseteq S$ and $\delta > 0$ such that for each $s \in S$, there exists $t \in T$ with $v(s - t) \geq \delta$. Note that since T is finite, so is the set $G_p(T)$ of all images of T under G_p . Let $\sigma(s) \in G_p(S)$, where $\sigma \in G_p$ and $s \in S$. Then, there exists $t \in T$ with $v(s - t) \geq \delta$. Since $\overline{\mathbb{Q}_p}$ is Henselian, we have $v(\sigma(\alpha)) = v(\alpha)$ for all $\alpha \in \overline{\mathbb{Q}_p}$. So,

$$v(\sigma(s) - \sigma(t)) = v(\sigma(s - t)) = v(s - t) \geq \delta.$$

Thus, condition (2) of Theorem 2.7 holds for $G_p(S)$, and we conclude that $\text{Int}_{\overline{\mathbb{Q}_p}}(G_p(S), \overline{\mathbb{Z}_p})$ is nontrivial. \square

4 Global Conditions and Examples

Here, we examine global conditions that can be used to decide when $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is nontrivial. Our first theorem of this type relates $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ to some distinguished polynomials that lie in the rings $\text{Int}_{\mathbb{Q}}(\mathcal{A}_n)$.

Definition 4.1. For each $p \in \mathbb{P}$ and each positive integer $n \geq 1$, let

$$\Psi_{p,n}(X) = (X^{p^n} - X)(X^{p^{n-1}} - X) \cdots (X^p - X).$$

Recall that when $\alpha \in \overline{\mathbb{Z}}$, the index of α is $\iota_\alpha = [O_{\mathbb{Q}(\alpha)} : \mathbb{Z}[\alpha]]$. The lemma below summarizes the basic relationships among $\Psi_{p,n}$, ι_α , and integral-valued polynomials.

Lemma 4.2. Let $n \in \mathbb{N}$ and $p \in \mathbb{P}$.

- (1) ([3, Theorem 3]) Modulo p , $\Psi_{p,n}$ is the monic least common multiple of all polynomials in $\mathbb{F}_p[X]$ of degree at most n .
- (2) For each prime p , $\Psi_{p,n}/p \in \text{Int}_{\mathbb{Q}}(\mathcal{A}_n) \setminus \text{Int}_{\mathbb{Q}}(\mathcal{A}_{n+1})$.
- (3) Let $\alpha \in \overline{\mathbb{Z}}$ and let $f \in \text{Int}_{\mathbb{Q}}(\{\alpha\}, \overline{\mathbb{Z}})$. If $\deg f < [\mathbb{Q}(\alpha) : \mathbb{Q}]$, then $\iota_\alpha f \in \mathbb{Z}[X]$. In particular, $f \in \mathbb{Z}_{(q)}[X]$ for each prime q not dividing ι_α .

Proof. For (2), if $\alpha \in \mathcal{A}_n$ with minimal polynomial $m_\alpha(X)$, then $m_\alpha \mid \Psi_{p,n}$ modulo p . Hence, $\Psi_{p,n}(\alpha) \in pO_{\mathbb{Q}(\alpha)}$. To show that $\Psi_{p,n}/p \notin \text{Int}_{\mathbb{Q}}(\mathcal{A}_{n+1})$, let $f \in \mathbb{F}_p[X]$ be monic and irreducible of degree $n+1$. Then, $f \nmid \Psi_{p,n} \pmod{p}$. Next, let $F \in \mathbb{Z}[X]$ be monic, irreducible, and such that $F \equiv f \pmod{p}$ (the existence of F follows from Perron's Criterion for irreducibility [28, Theorem 2.2.5]). Let $\beta \in \overline{\mathbb{Z}}$ be a root of F , and let $O = O_{\mathbb{Q}(\beta)}$; note that $\beta \in \mathcal{A}_{n+1}$. Suppose that $\Psi_{p,n}(\beta)/p \in \overline{\mathbb{Z}}$. Then, $\Psi_{p,n}(\beta) \in pO$, and hence $\Psi_{p,n}(\beta) \equiv 0 \pmod{p}$. Since the image of β in O/pO is a root of f , this means that $f \mid \Psi_{p,n} \pmod{p}$, which is a contradiction.

For (3), let $N = [\mathbb{Q}(\alpha) : \mathbb{Q}]$ and assume that $f(X) = \sum_{i=0}^d a_i X^i$, where each $a_i \in \mathbb{Q}$ and $d < N$. Then, $f(\alpha) \in \overline{\mathbb{Z}} \cap \mathbb{Q}(\alpha) = O_{\mathbb{Q}(\alpha)}$. Now, $\iota_\alpha O_{\mathbb{Q}(\alpha)} \subseteq \mathbb{Z}[\alpha]$, so $\sum_{i=0}^d \iota_\alpha a_i \alpha^i \in \mathbb{Z}[\alpha]$. Since $\mathbb{Z}[\alpha]$ is a free \mathbb{Z} -module with basis $1, \alpha, \dots, \alpha^{N-1}$ and $d < N$, each $\iota_\alpha a_i \in \mathbb{Z}$, as required. \square

In the examples later in this section, we will sometimes use extensions of p -adic valuations to $\overline{\mathbb{Q}}$ in order to prove that $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is trivial. The next two lemmas demonstrate how to do this.

Lemma 4.3. *Let $S \subseteq \overline{\mathbb{Z}}$.*

- (1) *Let $\sigma \in G = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and U be a valuation domain of $\overline{\mathbb{Q}}$ extending $\mathbb{Z}_{(p)}$ for some $p \in \mathbb{P}$. Then $\text{Int}_{\mathbb{Q}}(S, U) = \text{Int}_{\mathbb{Q}}(\sigma(S), \sigma(U))$. In particular, if S is G -invariant (i.e., $\sigma(S) = S, \forall \sigma \in G$) and U, U' are two valuation domains extending $\mathbb{Z}_{(p)}$ then $\text{Int}_{\mathbb{Q}}(S, U) = \text{Int}_{\mathbb{Q}}(S, U')$ and so $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}_{(p)}}) = \text{Int}_{\mathbb{Q}}(S, U)$ for any valuation domain U extending $\mathbb{Z}_{(p)}$.*
- (2) *Let $p \in \mathbb{P}$. If there exists an extension u of v_p to $\overline{\mathbb{Q}}$ with associated valuation domain U such that $\text{Int}_{\overline{\mathbb{Q}}}(S, U)$ is trivial, then $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}_{(p)}}) = \mathbb{Z}_{(p)}[X]$.*
- (3) *If, for every $p \in \mathbb{P}$, there exists an extension u_p of v_p to $\overline{\mathbb{Q}}$ with associated valuation domain U_p such that $\text{Int}_{\overline{\mathbb{Q}}}(S, U_p)$ is trivial, then $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is trivial.*

Proof. (1) Let $f \in \text{Int}_{\mathbb{Q}}(S, U)$. Since $f \in \mathbb{Q}[X]$, $\sigma(f(s)) = f(\sigma(s)) \in \sigma(U)$ for each $s \in S$, so $f \in \text{Int}_{\mathbb{Q}}(\sigma(S), \sigma(U))$. The other containment is proved in the same way by applying σ^{-1} to the previous relation. Now, suppose that S is G -invariant. The valuation domains U, U' are conjugate by some $\sigma \in G$, so $U' = \sigma(U)$ and $\text{Int}_{\mathbb{Q}}(S, U) = \text{Int}_{\mathbb{Q}}(S, U')$. For the final claim, note that $\overline{\mathbb{Z}_{(p)}} = \bigcap_{u|v_p} U$, where the intersection is taken over all extensions u of v_p to $\overline{\mathbb{Q}}$, and U is the valuation domain corresponding to u . We have,

$$\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}_{(p)}}) = \bigcap_{u|v_p} \text{Int}_{\mathbb{Q}}(S, U) \quad (4.4)$$

and so if S is G -invariant, we have $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}_{(p)}}) = \text{Int}_{\mathbb{Q}}(S, U)$ for each u extending v_p to $\overline{\mathbb{Q}}$.

(2) Assume the desired u and U exist. Then, $\text{Int}_{\overline{\mathbb{Q}}}(S, U) = U[X]$, so

$$\text{Int}_{\mathbb{Q}}(S, U) = \text{Int}_{\overline{\mathbb{Q}}}(S, U) \cap \mathbb{Q}[X] = \mathbb{Z}_{(p)}[X].$$

By (4.4), in order for $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}_{(p)}})$ to be trivial it is sufficient that $\text{Int}_{\mathbb{Q}}(S, U) = \mathbb{Z}_{(p)}[X]$ for only one extension u of v_p . As shown above, this holds for u , so we conclude that $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}_{(p)}}) = \mathbb{Z}_{(p)}[X]$.

(3) By (2), we have $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}_{(p)}}) = \mathbb{Z}_{(p)}[X]$ for each p , and then $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}}) = \mathbb{Z}[X]$ by (3.5). \square

Remark 4.5. We stress that the condition appearing in item (2) of Lemma 4.3 is sufficient but not necessary, if S is not G -invariant ((1) of the same Lemma). We will see in Example 5.9 (due to Gilmer, [12, Example 14]) and in Example 5.14 (due to Chabert, [5, Example 6.2]) that there are subsets $S \subseteq \overline{\mathbb{Z}_{(p)}}$ such that $\text{Int}_{\mathbb{Q}}(S, U)$ is non trivial for each extension U of $\mathbb{Z}_{(p)}$ to $\overline{\mathbb{Q}}$ but their intersection $\bigcap_{U|\mathbb{Z}_{(p)}} \text{Int}_{\mathbb{Q}}(S, U) = \text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}_{(p)}})$ (4.4) is trivial (see Propositions 5.13 and 5.15).

Lemma 4.6. *Let $S = \{s_i\}_{i \in \mathbb{N}} \subseteq \overline{\mathbb{Z}}$. Assume that $p \in \mathbb{P}$ and u is an extension of v_p to $\overline{\mathbb{Q}}$ such that one of the following two conditions holds:*

- (i) $\{u(s_i)\}_{i \in \mathbb{N}}$ is a strictly decreasing sequence with limit 0.
- (ii) $u(s_i - s_j) = 0$ for all distinct $i, j \in \mathbb{N}$.

Then, $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}_{(p)}}) = \mathbb{Z}_{(p)}[X]$. Moreover, if for every $p \in \mathbb{P}$ there exists an extension u of v_p satisfying either (i) or (ii), then $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is trivial.

Proof. Let $U \subseteq \overline{\mathbb{Q}}$ be the valuation domain associated to u , and let M be the maximal ideal of U . If (i) holds, then with respect to u , S is a pseudo-divergent sequence with breadth ideal M ; and if (ii) holds, then S is pseudo-stationary with respect to u , and the breadth ideal is U . In either case, $\text{Int}_{\overline{\mathbb{Q}}}(S, U)$ is trivial by Theorem 2.7. By Lemma 4.3, $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}_{(p)}}) = \mathbb{Z}_{(p)}[X]$, and if this is true for every prime p , then $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is trivial by (3.5). \square

Next, we will explore examples of unbounded sets $S \subseteq \overline{\mathbb{Z}}$ such that $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is trivial. Later in Sections 5 and 6, we construct unbounded sets for which $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is nontrivial. As demonstrated in Example 1.3, the indices of algebraic integers can be used to give a sufficient condition for $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ to be trivial. We now restate that example, and give a slight variation that does not require each index ι_s to be 1.

Proposition 4.7. *Let $S \subseteq \overline{\mathbb{Z}}$.*

- (1) *Assume that S contains a sequence $\{s_i\}_{i \in \mathbb{N}} \subseteq S$ of unbounded degree and such that $\iota_{s_i} = 1$ for all $i \in \mathbb{N}$. Then, $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is trivial.*
- (2) *Assume that for each integer $n \geq 1$, there exists a finite subset $\{s_1, \dots, s_m\} \subseteq S$ such that $[\mathbb{Q}(s_i) : \mathbb{Q}] > n$ for each $1 \leq i \leq m$ and $\gcd(\iota_{s_1}, \dots, \iota_{s_m}) = 1$. Then, $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is trivial.*

Proof. Part (1) is Example 1.3. For (2), let $f \in \text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ of degree d and let s_1, \dots, s_m be the elements of S with the stated properties. For each i , we have $[\mathbb{Q}(s_i) : \mathbb{Q}] > d$, so $\iota_{s_i} f \in \mathbb{Z}[X]$ by Lemma 4.2(3). The condition $\gcd(\iota_{s_1}, \dots, \iota_{s_m}) = 1$ then implies that $f \in \mathbb{Z}[X]$. \square

In Example 4.11, we will show that the conditions on indices in Proposition 4.7 are not necessary for $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ to be trivial. This example requires Dedekind's Index Theorem, which describes the primes p such that $p \mid \iota_{\alpha}$. For this theorem, we will follow the treatment given in [10]. An equivalent statement is available in [9, Theorem 6.1.4].

Theorem 4.8. (*Dedekind Index Theorem*) *Let α be an algebraic integer with minimal polynomial $f \in \mathbb{Z}[X]$. For a prime p , use a bar to denote reduction mod p . Factor f mod p as*

$$\overline{f} = \overline{\pi_1}^{e_1} \cdots \overline{\pi_k}^{e_k},$$

where each $\overline{\pi_i} \in \mathbb{F}_p[X]$ is monic and irreducible, and each $e_i \geq 1$. For each i , let π_i be a monic lift of $\overline{\pi_i}$ to $\mathbb{Z}[X]$ and let $F \in \mathbb{Z}[X]$ be such that

$$f = \pi_1^{e_1} \cdots \pi_k^{e_k} + pF.$$

Then, $p \mid \iota_{\alpha}$ if and only if $\overline{\pi_j} \mid \overline{f}$ in $\mathbb{F}_p[X]$ for some $1 \leq j \leq k$ such that $e_j \geq 2$.

Definition 4.9. Let $c, n \in \mathbb{Z}$ be such that $c \geq 2$ and $n \geq 2$. We define

$$f_{c,n}(X) := X^n + c^3 X^{n-1} + c^2$$

and $\iota_{c,n} := \iota_{\alpha}$, where α is a root of $f_{c,n}$.

Lemma 4.10. *Let $c, n \in \mathbb{Z}$ be such that $c \geq 2$ and $n \geq 2$.*

- (1) *The polynomial $f_{c,n}$ is irreducible over \mathbb{Q} .*

- (2) Let $\alpha \in \overline{\mathbb{Q}}$ be a root of $f_{c,n}$ and u an extension of the p -adic valuation v_p to $\overline{\mathbb{Q}}$. If p is a prime and $p \mid c$, then $u(\alpha) = 2v_p(c)/n$.
- (3) If p is a prime and $p \mid c$, then $p \mid \iota_{c,n}$.
- (4) If q is a prime and $q \nmid c$ but $q \mid n$, then $q \nmid \iota_{c,n}$.
- (5) If $m \in \mathbb{N}$ and q is a prime such that $q \nmid c$, then there exists a prime $\ell > m$ such that $q \nmid \iota_{c,\ell}$.

Proof. (1) This follows from Perron's Criterion for irreducibility [28, Theorem 2.2.5].

(2) We have $\alpha^n = -c^2(c\alpha^{n-1} + 1)$. If p divides c , then $u(\alpha^n) = v_p(c^2)$, and so $u(\alpha) = 2v_p(c)/n$.

(3) Assume $p \mid c$, and use a bar to denote reduction mod p . Then, $\overline{f_{c,n}}(X) = X^n$, and $f_{c,n}(X) = X^n + pF_{c,n}(X)$, where $F_{c,n}(X) = \frac{c^3}{p}X^{n-1} + \frac{c^2}{p}$. Note that $F_{c,n} \in \mathbb{Z}[x]$ and $\overline{F_{c,n}} = 0$. By Theorem 4.8, $p \mid \iota_{p,n}$.

(4) Assume $q \nmid c$ and $q \mid n$. When reduced mod q , the derivative of $f_{c,n}(X)$ equals $(n-1)c^3X^{n-2}$, which is either constant or has 0 as its only root. Since $q \nmid f_{c,n}(0)$, we see that $f_{c,n} \bmod q$ is separable. Hence, when applying the Dedekind Index Theorem, each exponent e_j will equal 1. We conclude that $q \nmid \iota_{c,n}$.

(5) We have $f'_{c,n}(X) = X^{n-2}((n-1)c^3 + nX)$. By the Dirichlet's theorem on arithmetic progressions, there exists a prime ℓ of the form $qk + 1$, $\ell > m$; in particular, $v_q(\ell - 1) > 0$ and so, modulo q , the only root of $f'_{c,\ell}(X)$ is 0 (note that $v_q(\ell) = 0$) which is not a root of $f_{c,\ell}$ modulo q because we are assuming $v_q(c) = 0$. Hence, $f_{c,\ell}$ is separable modulo q so that $q \nmid \iota_{c,\ell}$ by Theorem 4.8. \square

Example 4.11. It is possible that the indices of the elements of an unbounded sequence $S = \{s_i\}_{i \in \mathbb{N}} \subseteq \overline{\mathbb{Z}}$ can all share a common prime factor, yet the corresponding ring of integral-valued polynomials $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is trivial.

Fix a prime p , and let $q_1 < q_2 < \dots$ be an ordering of all the primes not equal to p . For each $i \in \mathbb{N}$, let f_{p,q_i} be as in Definition 4.9, and let $s_i \in \overline{\mathbb{Q}}$ be a root of f_{p,q_i} . Take $S = \{s_i\}_{i \in \mathbb{N}}$. Then, $\{[\mathbb{Q}(s_i) : \mathbb{Q}]\}_{i \in \mathbb{N}}$ is unbounded, and by Lemma 4.10 each ι_{s_i} is divisible by p and not divisible by q_i . We will show that $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is trivial.

By Lemma 4.10(2), $u_p(s_i) = 2/q_i$ for each i , so $\{u_p(s_i)\}_{i \in \mathbb{N}}$ is a strictly decreasing sequence with limit 0. By Lemma 4.6, $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}_{(p)}}) = \mathbb{Z}_{(p)}[X]$.

Let now q be a prime different from p . Given $f \in \text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$, by Lemma 4.10(5), there exists a prime q_i such that $[\mathbb{Q}(s_i) : \mathbb{Q}] = q_i > \deg(f)$ and $q \nmid \iota_{s_i}$. By Lemma 4.2(3), $\iota_{s_i} \cdot f \in \mathbb{Z}[X]$ so that f belongs to $\mathbb{Z}_{(q)}[X]$. This argument shows that $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}_{(q)}}) = \mathbb{Z}_{(q)}[X]$ for every prime $q \neq p$.

By (3.5), we conclude that $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}}) = \mathbb{Z}[X]$.

In the following examples, for each $p \in \mathbb{P}$, we fix an extension u_p of v_p to $\overline{\mathbb{Q}}$ with corresponding valuation domain U_p having maximal ideal \overline{M}_p . For each positive integer n , let ζ_n be a primitive n^{th} root of unity. Sequences of these algebraic integers can lead to interesting examples of sets S for which $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is trivial. In the next three examples, we will construct a sequence $S \subseteq \overline{\mathbb{Z}}$ that is pseudo-divergent with $\text{Br}(S) = \overline{M}_p$ with respect to u_p for a single prime p , but pseudo-stationary with $\text{Br}(S) = U_q$ at all other primes q ; a sequence S that is pseudo-stationary with $\text{Br}(S) = \overline{M}_p$ with respect to all primes; and a sequence S which is eventually pseudo-divergent with $\text{Br}(S) = \overline{M}_p$ with respect to each prime.

Example 4.12. Let $p \in \mathbb{P}$ and take $S = \{\zeta_{p^k}\}_{k \in \mathbb{N}}$. Then, $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is trivial by Proposition 4.7(1). We claim that S is a pseudo-divergent sequence with $\text{Br}(S) = M_p$ with respect to u_p and it is pseudo-stationary with $\text{Br}(S) = U_q$ with respect to u_q for every prime $q \neq p$.

Whenever $j < k$, we have $\zeta_{p^k} - \zeta_{p^j} = \zeta_{p^k}(1 - \zeta_{p^m}^m)$, where $m = p^{k-j} - 1$. Since m and p^k are coprime, $\zeta_{p^m}^m$ is also a primitive p^k -th root of unity. So,

$$u_q(\zeta_{p^k} - \zeta_{p^j}) = u_q(1 - \zeta_{p^m}^m) = u_q(1 - \zeta_{p^k}).$$

We recall that the prime p is totally ramified in $\mathbb{Q}(\zeta_{p^k})$ for every $k \in \mathbb{N}$ (see for example [16, Theorem 26]), and $u_p(1 - \zeta_{p^k}) = (p^{k-1}(p-1))^{-1}$ by [30, Chapter 2, p. 9]. Moreover, if q is a prime different from p then $u_q(1 - \zeta_{p^k}) = 0$. These calculations prove the claims about S .

Example 4.13. Let $S = \{\zeta_p\}_{p \in \mathbb{P}}$. Once again, $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is trivial by Proposition 4.7. We will show that S is pseudo-stationary with $\text{Br}(E) = U_p$ with respect to u_p for every prime p .

Given primes $p < q$, we have $\zeta_q - \zeta_p = \zeta_q(1 - \zeta_{pq}^{q-p})$, and ζ_{pq}^{q-p} is a primitive pq -th root of unity because $q-p$ is coprime to pq . So, for any prime l ,

$$u_l(\zeta_q - \zeta_p) = u_l(1 - \zeta_{pq}^{q-p}) = u_l(1 - \zeta_{pq}).$$

By [30, Proposition 2.8], $1 - \zeta_{pq}$ is a unit of $\mathbb{Z}[\zeta_{pq}]$. Hence, $u_l(\zeta_q - \zeta_p) = 0$, and so for every prime l , S is pseudo-stationary with respect to u_l with breadth ideal U_l .

Note that in both Example 4.12 and Example 4.13, one could use Lemma 4.6 instead of Proposition 4.7 to conclude that $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is trivial.

It is not possible for a sequence $S = \{s_k\}_{k \in \mathbb{N}} \subseteq \overline{\mathbb{Z}}$ to be pseudo-divergent with respect to all primes. If that were the case, then for all $i \neq j$, the difference $s_i - s_j$ would have finite positive value at each integral prime, which is impossible. However, we can construct a sequence S that is *eventually* pseudo-divergent with respect to each prime.

Example 4.14. Let $\mathbb{P} = \{p_1, p_2, \dots\}$. Define $s_1 = p_1$, $s_2 = (p_1 p_2)^{1/2}$, $s_3 = (p_1 p_2 p_3)^{1/3}$, and in general $s_k = (p_1 \cdots p_k)^{1/k}$ for each $k \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$,

$$u_{p_n}(s_k) = \begin{cases} 0, & k < n \\ \frac{1}{k}, & k \geq n. \end{cases}$$

Thus, for every prime p , the sequence $\{u_p(s_k)\}_{k \in \mathbb{N}}$ eventually strictly decreases to 0. Hence, S is eventually pseudo-divergent with breadth ideal equal to \overline{M}_p with respect to every prime p . As in the prior two examples, $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is trivial.

5 Ramification Indices and Residue Field Degrees

We first show that having bounds on both the ramification indices and residue field degrees of elements of S is sufficient for $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ to be nontrivial. Similar conditions appear in [14] in a theorem which classifies the almost Dedekind domains D with finite residue fields such that $\text{Int}(D)$ is a Prüfer domain.

Lemma 5.1. *Let $S \subseteq \overline{\mathbb{Z}}$. Assume that there exists $p \in \mathbb{P}$ such that S has both bounded ramification indices and bounded residue field degrees at p . That is, assume that there exist $e_0, f_0 \in \mathbb{N}$ such that for all $s \in S$ and every prime P_s of $O_{\mathbb{Q}(s)}$ above p , we have $e(P_s|p) \leq e_0$ and $f(P_s|p) \leq f_0$. Then, both $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ and $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}_{(p)}})$ are nontrivial.*

Proof. Fix $s \in S$, and let P_s be a prime of $O_{\mathbb{Q}(s)}$ above p . Let $q = p^{f_0!}$. Then, $O_{\mathbb{Q}(s)}/P_s$ is a subfield of \mathbb{F}_q , so $X^q - X$ maps $O_{\mathbb{Q}(s)}$ into P_s under evaluation for every $s \in S$. Since all ramification indices over p are bounded above by e_0 , it follows that $(X^q - X)^{e_0}$ sends $O_{\mathbb{Q}(s)}$ into $pO_{\mathbb{Q}(s)}$. Because the values of q and e_0 are independent of s , this is true for all elements of S . Thus, the polynomial $((X^q - X)^{e_0})/p$ is in $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}}) \subseteq \text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}(p)})$, and both rings are nontrivial. \square

The hypothesis of Lemma 5.1 is clearly met when S is of bounded degree. However, there exist sets S of unbounded degree for which Lemma 5.1 can be applied.

Example 5.2. Let $\mathbb{Q}^{(n)} = \mathbb{Q}(\mathcal{A}_n)$ be the compositum in $\overline{\mathbb{Q}}$ of all number fields of degree bounded by n and let $O_{\mathbb{Q}^{(n)}}$ be the ring of integers of $\mathbb{Q}^{(n)}$. It is known that there exists a global bound on both ramification indexes and residue field degrees of valuations of $\mathbb{Q}^{(n)}$. That is, there exists $N \in \mathbb{N}$ such that if u_p is a valuation of $\mathbb{Q}^{(n)}$ extending some v_p , $p \in \mathbb{P}$, then $e(u_p|v_p) \leq N$ and $f(u_p|v_p) \leq N$ [2, Proposition 4.5.3, p. 118]; we stress that N is independent from p . So, any subset $S \subseteq O_{\mathbb{Q}^{(n)}}$ meets the conditions of Lemma 5.1, and hence $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is nontrivial. Furthermore, subsets of $O_{\mathbb{Q}^{(n)}}$ may have unbounded degree. For an explicit example, let $\mathbb{P} = \{p_k\}_{k \in \mathbb{N}}$. For each $k \in \mathbb{N}$, let $s_k = \sum_{i=1}^k \sqrt{p_i}$, and take $S = \{s_k\}_{k \in \mathbb{N}}$. Then, $S \subseteq O_{\mathbb{Q}^{(2)}}$ and for each k , $[\mathbb{Q}(s_k) : \mathbb{Q}] = 2^k$.

As mentioned earlier, the double-boundedness condition of Lemma 5.1 was used in [14] to classify the almost Dedekind domains D with finite residue fields such that $\text{Int}(D)$ is Prüfer. We do not know whether this condition guarantees that $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is Prüfer. However, we can prove that this is the case when $S \subseteq O_{\mathbb{Q}^{(n)}}$.

Proposition 5.3. *Let $n \in \mathbb{N}$. Define $\mathbb{Q}^{(n)}$ and $O_{\mathbb{Q}^{(n)}}$ as in Example 5.2, and let $S \subseteq O_{\mathbb{Q}^{(n)}}$. Then, $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is a Prüfer domain.*

Proof. For readability, let $K = \mathbb{Q}^{(n)}$ and $O = O_{\mathbb{Q}^{(n)}}$. As usual, we let $\text{Int}(O) = \{f \in K[X] \mid f(O) \subseteq O\}$ and $\text{Int}_{\mathbb{Q}}(O) = \text{Int}(O) \cap \mathbb{Q}[X]$. Then,

$$\text{Int}_{\mathbb{Q}}(O) = \text{Int}_{\mathbb{Q}}(O, \overline{\mathbb{Z}}) \subseteq \text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}}).$$

Since overrings of Prüfer domains are Prüfer, it suffices to prove that $\text{Int}_{\mathbb{Q}}(O)$ is Prüfer. By the properties of ramification indexes and residue field degree mentioned in Example 5.2, it follows that O is an almost Dedekind domain with finite residue fields satisfying the double-boundedness condition of [14, Theorem 2.5]. Hence, $\text{Int}(O)$ is Prüfer. By [23, Lemma 2.20], $\text{Int}(O)$ is integral over $\text{Int}_{\mathbb{Q}}(O)$, so $\text{Int}_{\mathbb{Q}}(O)$ is Prüfer by [11, Theorem 22.4]. \square

Remark 5.4. For each $n \in \mathbb{N}$, we have $\text{Int}_{\mathbb{Q}}(O_{\mathbb{Q}^{(n)}}) \subseteq \text{Int}_{\mathbb{Q}}(\mathcal{A}_n)$. As in (1.1), we have a descending chain of Prüfer domains:

$$\dots \subseteq \text{Int}_{\mathbb{Q}}(O_{\mathbb{Q}^{(n+1)}}) \subseteq \text{Int}_{\mathbb{Q}}(O_{\mathbb{Q}^{(n)}}) \subseteq \dots \subseteq \text{Int}_{\mathbb{Q}}(O_{\mathbb{Q}^{(1)}}) = \text{Int}(\mathbb{Z}).$$

It is not known whether each containment in this train is strict. Regardless, as was shown in the Introduction for the rings $\text{Int}_{\mathbb{Q}}(\mathcal{A}_n)$, $n \in \mathbb{N}$, we have $\bigcap_{n \in \mathbb{N}} \text{Int}_{\mathbb{Q}}(O_{\mathbb{Q}^{(n)}}) = \text{Int}_{\mathbb{Q}}(\overline{\mathbb{Z}}) = \mathbb{Z}[X]$.

While bounds on both ramification indices and residue field degrees are sufficient to conclude that $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is nontrivial by Lemma 5.1, neither condition is necessary for nontriviality. The remainder of this section is devoted to examples that illustrate this. We also consider how to interpret the presence of either unbounded residue field degrees (respectively, ramification degrees) or in terms of pseudo-stationary (respectively, pseudo-divergent) sequences in suitable images of S in $\overline{\mathbb{Z}_p}$.

Example 5.5. Let $S \subseteq \overline{\mathbb{Z}}$ and $p \in \mathbb{P}$. Let I be the set of all ramification indices $e(P_s|p)$, where s runs through S and P_s runs through the prime ideals of $O_{\mathbb{Q}(s)}$ above p . We give two examples to show that if I is unbounded, then $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ may or may not be trivial.

First, let S be as in Example 1.6. The associated set of ramification indices is $I = \{2^k \mid k \in \mathbb{N}\}$, which is unbounded. However, $X^2/p \in \text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$, so the ring of integral-valued polynomials is nontrivial. By contrast, if S is the set defined in Example 4.14, then S still exhibits unbounded ramification indices, but $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is trivial.

As for the situation of ramification indices, we can show that the bound of the residue field degrees is not necessary for the $\text{Int}_{\mathbb{Q}}$ -ring to be nontrivial.

Example 5.6. For a fixed prime $p \in \mathbb{P}$, we consider the m^{th} roots of unity in $\overline{\mathbb{Z}}$ with $(p, m) = 1$; we recall that p is not ramified in $\mathbb{Q}(\zeta_m)$ [16, Corollary of Theorem 26]. The set of residue field degrees of the set of prime ideals of the finite extensions generated by the elements of the set $S = \{p \cdot \zeta_m \mid m \in \mathbb{N}, (p, m) = 1\}$ is unbounded but $u_p(p \cdot \zeta_m) > 0$ for each of the relevant m 's. Clearly, $\frac{X}{p} \in \text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$. The important condition here is that the residue set S/\overline{P} , for each prime ideal $\overline{P} \subset \overline{\mathbb{Z}}$ above p , is finite.

Notation 5.7. For $p \in \mathbb{P}$, we denote by \mathcal{P}_p the set of prime ideals \overline{P} of $\overline{\mathbb{Z}}$ lying above p .

Lemma 5.8. Let $S \subseteq \overline{\mathbb{Z}_{(p)}}$ for some $p \in \mathbb{P}$. Suppose there exists $n \in \mathbb{N}$ such that either one of the following conditions is satisfied:

- (i) there exists some $\overline{P} \in \mathcal{P}_p$ such that $\#S/\overline{P} > n$.
- (ii) there exist some $\overline{P} \in \mathcal{P}_p$ and $s \in S$ such that $0 < v_{\overline{P}}(s) < \frac{1}{n}$.

Then, given $f \in \text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}_{(p)}}) \setminus \mathbb{Z}_{(p)}[X]$, we have $\deg(f) > n$. In particular, if for every $n \in \mathbb{N}$ either one of these conditions hold, then $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}_{(p)}}) = \mathbb{Z}_{(p)}[X]$.

Proof. Let $f \in \text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}_{(p)}}) \setminus \mathbb{Z}_{(p)}[X]$ be of degree $d \leq n$; without loss of generality, we may assume that $f = \frac{g}{p}$ for some monic $g \in \mathbb{Z}_{(p)}[X]$ (see the arguments of the proof of Proposition 6.4 for the monic assumption; we may multiply f by a suitable power of p in order to get p at the denominator).

(i) By assumption, there exists a prime ideal $\overline{P} \in \mathcal{P}_p$ such that $\#S/\overline{P} > n$, that is, there exist $s_0, \dots, s_n \in S$ such that $u_{\overline{P}}(s_i - s_j) = 0$ for each $0 \leq i < j \leq n$. In particular, $\overline{s}_i = s_i \pmod{\overline{P}}$ are $n+1$ distinct elements of $\overline{\mathbb{F}_p}$. Then $g(s_i) \equiv 0 \pmod{\overline{P}}$, for $i = 0, \dots, n$, that is, $\overline{g}(\overline{s}_i) = 0$ in $\overline{\mathbb{F}_p}$, where $\overline{g} \in \overline{\mathbb{F}_p}[X]$ is the reduction modulo \overline{P} of g . It follows that $\overline{g} \in \overline{\mathbb{F}_p}[X]$ is a polynomial of degree $\leq n$ which has $n+1$ distinct roots in $\overline{\mathbb{F}_p}$, a contradiction.

(ii) By assumption, there exists a prime ideal $\overline{P} \in \mathcal{P}_p$ and $s \in S$ such that $v_{\overline{P}}(s) < \frac{1}{n}$. Let $g(X) = \prod_{i=1}^d (X - \alpha_i)$ over $\overline{\mathbb{Q}}$, where each $\alpha_i \in \mathcal{A}_n$. Fix $i \in \{1, \dots, n\}$. If $v_{\overline{P}}(\alpha_i)$ is not 0 then it is equal to $\frac{a}{n} > 0$, for some $a \in \mathbb{N}$ (not necessarily coprime with n) since $e(\overline{P} \cap O_{\mathbb{Q}(\alpha_i)} | p) \leq [\mathbb{Q}(\alpha_i) : \mathbb{Q}] \leq n$. If this second case occurs, then $0 < v_{\overline{P}}(s) < \frac{1}{n} \leq v_{\overline{P}}(\alpha_i)$. In either case we have:

$$v_{\overline{P}}(s - \alpha_i) = \min\{v_{\overline{P}}(\alpha_i), v_{\overline{P}}(s)\} = \begin{cases} 0, & \text{if } v_{\overline{P}}(\alpha_i) = 0 \\ v_{\overline{P}}(s), & \text{otherwise.} \end{cases}$$

Hence, $v_{\overline{P}}(g(s)) \leq \sum_{i=1}^n v_{\overline{P}}(s) \leq nv_{\overline{P}}(s) < 1 = v_{\overline{P}}(p)$ which contradicts the fact that $f(s) \in \overline{\mathbb{Z}_{(p)}} \subset \overline{\mathbb{Z}_{\overline{P}}}$. \square

Example 5.9. For an explicit example of a subset S of $\overline{\mathbb{Z}}$ for which S/\overline{P} is finite for each maximal ideal $\overline{P} \in \mathcal{P}_p$ for some prime $p \in \mathbb{Z}$ but $\{\#S/\overline{P} \mid \overline{P} \in \mathcal{P}_p\}$ is unbounded, let S be equal to the almost Dedekind domain D with finite residue fields of [12, Example 14] (see also [4, Example VI.4.18]). In that example, D is the integral closure of $\mathbb{Z}_{(p)}$ in a suitable infinite algebraic extension of \mathbb{Q} constructed as the union of a tower of finite algebraic extensions, so by Lemma 5.8 (see also [4, Lemma VI.4.2]) applied to that example shows that $\text{Int}_{\mathbb{Q}}(D, \overline{\mathbb{Z}}) = \text{Int}_{\mathbb{Q}}(D) = \mathbb{Z}_{(p)}[X]$.

Note that in Example 5.9 there are neither pseudo-stationary sequence nor pseudo-divergent sequences in D with respect to every possible extension of v_p to $\overline{\mathbb{Q}}$ because D is locally a DVR with finite residue field. Nonetheless, $\text{Int}_{\mathbb{Q}}(D)$ is trivial. The reason for this is that if we work with all of the \mathbb{Q} -embeddings of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}_p}$, and consider the union of all of the images of D under these embeddings, then we obtain the same $\text{Int}_{\mathbb{Q}}$ -ring (Lemma 5.11) and this union contains a pseudo-stationary sequence (see Proposition 5.13).

Definition 5.10. For each $\overline{P} \in \mathcal{P}_p$, let $u_{\overline{P}}$ be the valuation associated to $\overline{\mathbb{Z}_{\overline{P}}}$. By [19, Chapter 2, §8, Theorem 8.1], for each $\overline{P} \in \mathcal{P}_p$, the extension $u_{\overline{P}}$ of v_p to $\overline{\mathbb{Q}}$ is equal to $v_p \circ \tau_{\overline{P}}$, for some \mathbb{Q} -embedding $\tau_{\overline{P}} : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$, where v_p is the unique extension to $\overline{\mathbb{Q}_p}$ of the p -adic valuation. Given $S \subseteq \overline{\mathbb{Z}}$, we define the following subset of $\overline{\mathbb{Z}_p}$:

$$\mathcal{S}_p = \bigcup_{\overline{P} \in \mathcal{P}_p} \tau_{\overline{P}}(S).$$

The following lemma is analogous to Proposition 3.8.

Lemma 5.11. *Let $p \in \mathbb{P}$, $S \subseteq \overline{\mathbb{Z}_{(p)}}$ and $\overline{P} \in \mathcal{P}_p$. Then, we have*

$$\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}_{\overline{P}}}) = \text{Int}_{\mathbb{Q}}(\tau_{\overline{P}}(S), \overline{\mathbb{Z}_p}).$$

In particular,

$$\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}_{(p)}}) = \text{Int}_{\mathbb{Q}}(\mathcal{S}_p, \overline{\mathbb{Z}_p}).$$

Proof. (\subseteq) Let $f \in \text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}_{\overline{P}}})$ and $\tau_{\overline{P}}(s) \in \tau_{\overline{P}}(S)$, for some $s \in S$. Then $f(s) \in \overline{\mathbb{Z}_{\overline{P}}}$, so $u_{\overline{P}}(f(s)) \geq 0$. Hence,

$$v_p \circ \tau_{\overline{P}}(f(s)) = v_p(f(\tau_{\overline{P}}(s))) \geq 0$$

so that $f(\tau_{\overline{P}}(s)) \in \overline{\mathbb{Z}_p}$. It follows that $f \in \text{Int}_{\mathbb{Q}}(\tau_{\overline{P}}(S), \overline{\mathbb{Z}_p})$.

(\supseteq) Let $f \in \text{Int}_{\mathbb{Q}}(\tau_{\overline{P}}(S), \overline{\mathbb{Z}_p})$ and $s \in S$. By definition we have $f(\tau_{\overline{P}}(s)) \in \overline{\mathbb{Z}_p}$. Hence, applying $\tau_{\overline{P}}^{-1}$ to the previous relation we get $f(s) \in \overline{\mathbb{Z}_{\overline{P}}}$, and so $f \in \text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}_{\overline{P}}})$.

The final claim follows from the facts that $\overline{\mathbb{Z}_{(p)}} = \bigcap_{\overline{P} \in \mathcal{P}_p} \overline{\mathbb{Z}_{\overline{P}}}$ and $\bigcap_{\overline{P} \in \mathcal{P}_p} \text{Int}_{\mathbb{Q}}(\tau_{\overline{P}}(S), \overline{\mathbb{Z}_p}) = \text{Int}_{\mathbb{Q}}(\mathcal{S}_p, \overline{\mathbb{Z}_p})$. \square

Remark 5.12. Recall the set $\Sigma_p(S)$ that was introduced in Definition 3.1. One can show that $\Sigma_p(S)$ is equal to the $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ -closure of \mathcal{S}_p . Consequently, many of the results in Section 3 can be stated in terms of \mathcal{S}_p rather than $\Sigma_p(S)$. In particular, the following equivalences hold:

$$\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}_{(p)}}) \text{ is nontrivial} \stackrel{\text{Prop 3.9}}{\iff} \text{Int}_{\mathbb{Q}}(\mathcal{S}_p, \overline{\mathbb{Z}_p}) \text{ is nontrivial} \stackrel{\text{Prop 3.10}}{\iff} \text{Int}_{\overline{\mathbb{Q}_p}}(\mathcal{S}_p, \overline{\mathbb{Z}_p}) \text{ is nontrivial}$$

The proof of each equivalence is identical to that of the cited statement. One can then apply Theorem 2.7 to the ring $\text{Int}_{\overline{\mathbb{Q}_p}}(\mathcal{S}_p, \overline{\mathbb{Z}_p})$ to establish whether it is nontrivial or not.

Proposition 5.13. *Let $S \subseteq \overline{\mathbb{Z}}_{(p)}$ and let \overline{M}_p be the maximal ideal of $\overline{\mathbb{Z}}_p$. The following are equivalent.*

- (1) $\{\#S/\overline{P} \mid \overline{P} \in \mathcal{P}_p\}$ is unbounded.
- (2) $\mathcal{S}_p/\overline{M}_p$ is infinite.
- (3) \mathcal{S}_p contains a pseudo-stationary sequence E (with respect to v_p , the unique valuation of $\overline{\mathbb{Z}}_p$) such that $\text{Br}(E) = \overline{\mathbb{Z}}_p$.

Furthermore, if any of these conditions holds, then $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}}_{(p)})$ is trivial.

Proof. (1) \Rightarrow (2). Suppose that $\{\#S/\overline{P} \mid \overline{P} \in \mathcal{P}_p\}$ is unbounded, but $\#\mathcal{S}_p/\overline{M}_p = n$ for some $n \in \mathbb{N}$. By assumption, there exists some $\overline{P} \in \mathcal{P}_p$ such that $\#S/\overline{P} > n$. Thus, there exist $s_0, \dots, s_n \in S$ such that $u_{\overline{P}}(s_i - s_j) = 0$ for each $0 \leq i < j \leq n$. Applying the \mathbb{Q} -embedding $\tau_{\overline{P}}$ to each s_i , we obtain $n + 1$ elements $\tau_{\overline{P}}(s_0), \dots, \tau_{\overline{P}}(s_n) \in \mathcal{S}_p$ such that

$$v_p(\tau_{\overline{P}}(s_i) - \tau_{\overline{P}}(s_j)) = v_p \circ \tau_{\overline{P}}(s_i - s_j) = u_{\overline{P}}(s_i - s_j) = 0$$

for each $0 \leq i < j \leq n$. This is impossible, since some two of the $\tau_{\overline{P}}(s_i)$ must lie in the same residue class modulo \overline{M}_p . We conclude that $\mathcal{S}_p/\overline{M}_p$ is infinite.

(2) \Rightarrow (1). Assume that $\mathcal{S}_p/\overline{M}_p$ is infinite. Clearly, we have the identities

$$\frac{\mathcal{S}_p}{\overline{M}_p} = \frac{\bigcup_{\overline{P} \in \mathcal{P}_p} \tau_{\overline{P}}(S)}{\overline{M}_p} = \bigcup_{\overline{P} \in \mathcal{P}_p} \frac{\tau_{\overline{P}}(S)}{\overline{M}_p}.$$

Moreover, for each $\overline{P} \in \mathcal{P}_p$, $\tau_{\overline{P}}$ establishes a bijection between S/\overline{P} and $\tau_{\overline{P}}(S)/\overline{M}_p$. If $\tau_{\overline{P}}(S)/\overline{M}_p$ is infinite for some $\overline{P} \in \mathcal{P}_p$, then considering the pullback via $\tau_{\overline{P}}^{-1}$ we have that S/\overline{P} is infinite. If instead $\tau_{\overline{P}}(S)/\overline{M}_p$ is finite for each $\overline{P} \in \mathcal{P}_p$, since $\mathcal{S}_p/\overline{M}_p$ is infinite, this means that there exists a sequence $\{\overline{P}_n\}_{n \in \mathbb{N}} \subseteq \mathcal{P}_p$ such that $\#\tau_{\overline{P}_n}(S)/\overline{M}_p = d_n < \infty$ is diverging to infinity. This amounts to say via the previous one-to-one correspondence that $\#S/\overline{P}_n = d_n$ is diverging to infinity, which means that $\{\#S/\overline{P} \mid \overline{P} \in \mathcal{P}_p\}$ is unbounded.

(2) \Leftrightarrow (3) If $\mathcal{S}_p/\overline{M}_p$ is infinite, then we can choose a sequence $E = \{t_i\}_{i \in \mathbb{N}} \subseteq \mathcal{S}_p$ such that each t_i lies in a distinct residue class modulo \overline{M}_p . We then have $v_p(t_i - t_j) = 0$ whenever $i \neq j$, so E is pseudo-stationary and $\text{Br}(E) = \overline{\mathbb{Z}}_p$. Conversely, if $\mathcal{S}_p/\overline{M}_p$ is finite, then no such pseudo-stationary sequence can exist in S .

For the last claim, apply Theorem 2.7 and the equivalences noted in Remark 5.12. \square

We conjecture that there is an analogous result linking unbounded ramification indices of S at p to pseudo-divergent sequences in \mathcal{S}_p . However, determining the correct conditions for such a relation to hold in general has proven to be difficult. Nevertheless, we can provide one example showing that such a connection is possible, at least in the case where S is a ring.

Example 5.14. Following [5, Example 6.2] (see also [4, Example VI.4.17]), one may construct an almost Dedekind domain D with the following properties:

- D has finite residue fields.
- D is the integral closure of $\mathbb{Z}_{(p)}$ in a (suitably chosen) infinite algebraic extension K of \mathbb{Q} .

- The set $\{e(\overline{P} \cap D \mid p) \mid \overline{P} \in \mathcal{P}_p\}$ is unbounded.

For this domain D , every valuation overring is a DVR, and every residue field of D is equal to \mathbb{F}_p . These conditions guarantee that the ramification indices of D are bounded. Furthermore, D cannot contain any pseudo-divergent sequence with respect to any of its maximal prime ideals $\overline{P} \cap D$, because $D_{\overline{P} \cap D}$ is a DVR. We will show that $\mathcal{D}'_p := \bigcup_{\overline{P} \in \mathcal{P}_p} \tau_{\overline{P}}(D_{\overline{P} \cap D}) \subseteq \overline{\mathbb{Z}}_p$ contains a pseudo-divergent sequence and by an argument similar to Lemma 5.11, the ring $\text{Int}_{\mathbb{Q}}(D, \overline{\mathbb{Z}}_p)$ must equal $\mathbb{Z}_{(p)}[X]$.

Proposition 5.15. *Let $D \subseteq \overline{\mathbb{Z}}_{(p)}$ be a ring containing $\mathbb{Z}_{(p)}$ and define \mathcal{D}'_p as above. The following are equivalent.*

- (1) $\{e(P \mid p) \mid p \in P \subset D\}$ is unbounded.
- (2) For each $n \in \mathbb{N}$, there exist $P_n \subset D$, with $p \in P_n$ and $s_n \in D_{P_n}$ such that $0 < v_{P_n}(s_n) < \frac{1}{n}$.
- (3) There exists a pseudo-divergent sequence $E \subset \mathcal{D}'_p$ with $\text{Br}(E) = \overline{M}_p$.

Moreover, if any one of these conditions holds, then $\text{Int}_{\mathbb{Q}}(D) = \mathbb{Z}_{(p)}[X]$, and thus is trivial.

Proof. (1) \Rightarrow (2) If for some prime ideal P of D containing p we have $e(P \mid p) = \infty$, then it is easy to see that for each $n \in \mathbb{N}$, there exists $s_n \in D_P$ with $0 < v_P(s_n) < \frac{1}{n}$. Suppose instead that exists a sequence $\{P_n\}_{n \in \mathbb{N}}$ of prime ideals of D containing p such that $e(P_n \mid p) = e_n > n$ for each $n \in \mathbb{N}$. We claim that there exists $s_n \in D_{P_n}$ such that $v_{P_n}(s_n) = \frac{1}{e_n} < \frac{1}{n}$ which is precisely condition (2). Indeed, for each $n \in \mathbb{N}$ there exists $\alpha_n \in D$ such that $v_{P_n}(\alpha_n) = \frac{a_n}{e_n}$, for some $a_n \in \mathbb{Z}$ coprime with e_n . Now, if $aa_n + be_n = 1$ for some $a, b \in \mathbb{Z}$, then $s_n = \alpha_n^a \cdot p^b$ is an element of D_{P_n} with the desired value.

(2) \Rightarrow (3) Suppose (2) holds. Then for each $n \in \mathbb{N}$ we have $0 < v_{P_n}(s_n) = v_p(\tau_{P_n}(s_n)) = v_p(\tilde{s}_n) < \frac{1}{n}$, where $\tilde{s}_n \in \mathcal{D}'_p$. We remark that P_n could be the same prime ideal for infinitely many n 's; this corresponds to the case $e(P \mid p) = \infty$. Suppose that for some $n \in \mathbb{N}$, we have $\tilde{s}_1, \dots, \tilde{s}_n$ so that $0 < v_p(\tilde{s}_{i+1}) < v_p(\tilde{s}_i)$ for all $i = 1, \dots, n-1$. If we take $m \in \mathbb{N}$ so that $0 < \frac{1}{m} \leq v_p(\tilde{s}_n)$, then by assumption there exists $\tilde{s}_{n+1} \in \mathcal{D}'_p$ such that $0 < v_p(\tilde{s}_{n+1}) < \frac{1}{m} \leq v_p(\tilde{s}_n)$. In this way we can extract a subsequence E from $\{\tilde{s}_n\}_{n \in \mathbb{N}} \subset \mathcal{D}'_p$ which is pseudo-divergent (i.e., $0 < v_p(\tilde{s}_{n+1}) < v_p(\tilde{s}_n)$ for each $n \in \mathbb{N}$) and has $\text{Br}(E) = \overline{M}_p$.

(3) \Rightarrow (1) Assume that (3) holds. Then, there exists a pseudo-divergent sequence $E = \{\tilde{s}_n\}_{n \in \mathbb{N}} \subset \mathcal{D}'_p$ with $\text{Br}(E) = \overline{M}_p$. Hence, $v_p(\tilde{s}_n)$ strictly decreases to 0 and, up to taking a subsequence, we may assume that $0 < v_p(\tilde{s}_n) < \frac{1}{n}$. Since $\tilde{s}_n = \tau_{P_n}(s_n)$ for some $P_n \subset D$, $p \in D$ and $s_n \in D_{P_n}$, we have $0 < v_{P_n}(s_n) = v_p(\tau_{P_n}(s_n)) < \frac{1}{n}$ which implies that $n < e(P_n \mid p)$. Thus $\{e(P_n \mid p)\}_{n \in \mathbb{N}}$ is unbounded. Note that, as in the proof of (2) \Rightarrow (3), if P_n is the same prime ideal P for infinitely many n 's, then $e(P \mid p) = \infty$.

The last claim follows from Lemma 5.8. □

We end this section by showing that when S is an integrally closed subring of $\overline{\mathbb{Z}}_{(p)}$ containing $\mathbb{Z}_{(p)}$, then the conditions of Lemma 5.1 are both necessary and sufficient for $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ to be nontrivial.

Notation 5.16. For an algebraic extension K of \mathbb{Q} and a prime $p \in \mathbb{P}$, let D be the integral closure of $\mathbb{Z}_{(p)}$ in K (note that $D = O_{K, (p)}$, the localization of the integral closure O_K of \mathbb{Z} in K with respect to p). We set

$$E_{D,p} := \{e(P|p) \mid P \subset D\} \quad \text{and} \quad F_{D,p} := \{f(P|p) \mid P \subset D\}.$$

Lemma 5.17. *Let K be an infinite algebraic extension of \mathbb{Q} and let K' be the Galois closure of K over \mathbb{Q} . Let $p \in \mathbb{P}$ and let D and D' be the integral closures of $\mathbb{Z}_{(p)}$ in K and K' , respectively. If $E_{D,p}$ and $F_{D,p}$ are bounded, then $E_{D',p}$ and $F_{D',p}$ are bounded.*

Proof. Let $\sigma \in G = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Note that $E_{\sigma(D),p} = E_{D,p}$ and $F_{\sigma(D),p} = F_{D,p}$. Since \mathbb{Q}_p has only finitely many extensions of bounded degree, there exists a finite extension L of \mathbb{Q}_p such that, for every $\sigma \in G$, L contains the completion of $\sigma(K)$ with respect to any prime ideal of $\sigma(D)$. In particular, there are only finitely many such completions.

Note that K' is equal to the compositum of the fields $\sigma(K)$, $\sigma \in G$. Let Q be a prime ideal of D' . We claim that the completion $\widehat{K'}$ of K' with respect to Q is equal to the compositum of the completions $\widehat{\sigma(K)}$ of $\sigma(K)$ with respect to $Q \cap \sigma(D)$ (which by the above are finite in number). Indeed, $\widehat{K'}$ contains $\widehat{\sigma(K)}$ for every $\sigma \in G$ and so $\widehat{K'}$ contains their compositum; conversely, since $\widehat{K'}$ is the completion of K' , it is contained in the compositum of the (finitely many) fields $\widehat{\sigma(K)}$, $\sigma \in G$, because the latter is complete and contains K' .

Now, if L' is the Galois closure of L over \mathbb{Q}_p , then it is clear that $\widehat{K'}$ is contained in L' . Since this holds for every prime ideal Q of D' (and L' is independent from the choice of the prime ideal Q), it follows that $E_{D',p}$ and $F_{D',p}$ are bounded. \square

Theorem 5.18. *Let D be an integrally closed subring of $\overline{\mathbb{Z}_{(p)}}$ containing $\mathbb{Z}_{(p)}$. The following conditions are equivalent.*

- (1) $\text{Int}_{\mathbb{Q}}(D)$ is Prüfer.
- (2) $\text{Int}_{\mathbb{Q}}(D)$ is nontrivial (i.e., $\mathbb{Z}_{(p)}[X] \subsetneq \text{Int}_{\mathbb{Q}}(D)$).
- (3) The sets $F_{D,p}$ and $E_{D,p}$ are bounded.

Proof. Clearly, (1) implies (2).

(2) \Rightarrow (3) If F_p is unbounded then by Proposition 5.13 and Remark 5.12, $\text{Int}_{\mathbb{Q}}(D)$ is trivial. Similarly, if E_p is unbounded the same conclusion holds by Proposition 5.15.

(3) \Rightarrow (1) Let K' be the Galois closure of K and let D' be the integral closure of D in K' . By Lemma 5.17, $E_{D',p}$ and $F_{D',p}$ are bounded. Therefore, $\text{Int}_{K'}(D')$ is Prüfer by [14, Theorem 2.5]. We consider now the set $G(D) = \bigcup_{\sigma \in G} \sigma(D) \subset D'$. Since $\text{Int}_{K'}(D') \subseteq \text{Int}_{K'}(G(D), D')$, the latter ring is Prüfer, too. By [23, Lemma 2.20], $\text{Int}_{K'}(G(D), D')$ is the integral closure in $K'(X)$ of $\text{Int}_{\mathbb{Q}}(D) = \text{Int}_{\mathbb{Q}}(G(D), D')$. It follows by [11, Theorem 22.4] that $\text{Int}_{\mathbb{Q}}(D) = \text{Int}_{K'}(G(D), D') \cap \mathbb{Q}(X)$ is Prüfer. \square

Corollary 5.19. *Let $D \subseteq \overline{\mathbb{Z}}$ be an integrally closed subring.*

- (1) $\text{Int}_{\mathbb{Q}}(D)$ is nontrivial if and only if there exists some $p \in \mathbb{P}$ such that both $E_{D,p}$ and $E_{F,p}$ are bounded.
- (2) $\text{Int}_{\mathbb{Q}}(D)$ is Prüfer if and only if for each $p \in \mathbb{P}$ the sets $E_{D,p}$ and $E_{D,p}$ are bounded.

Proof. (1) This follows by Proposition 3.7 and Theorem 5.18.

(2) It is easy to show that $\text{Int}_{\mathbb{Q}}(D)$ is Prüfer if and only if for each $p \in \mathbb{P}$, the localization $\text{Int}_{\mathbb{Q}}(D)_{(p)}$ is Prüfer. By Proposition 3.6, the latter ring is equal to $\text{Int}_{\mathbb{Q}}(D, \overline{\mathbb{Z}_{(p)}})$, and since $\overline{\mathbb{Z}_{(p)}} \cap K = D_{(p)}$, we have $\text{Int}_{\mathbb{Q}}(D, \overline{\mathbb{Z}_{(p)}}) = \text{Int}_{\mathbb{Q}}(D, D_{(p)})$. Finally, $\text{Int}_{\mathbb{Q}}(D, D_{(p)}) = \text{Int}_{\mathbb{Q}}(D_{(p)})$ by [4, Corollary I.2.6]. The conclusion then follows by Theorem 5.18. \square

6 Polynomial closure of subsets of algebraic integers

Recall that by Example 1.5, when $S = 2\bar{\mathbb{Z}} = \{2\alpha \mid \alpha \in \bar{\mathbb{Z}}\}$, the ring $\text{Int}_{\mathbb{Q}}(S, \bar{\mathbb{Z}})$ is nontrivial because it contains $X/2$. This example admits a rather strong generalization.

Definition 6.1. Let $f \in \mathbb{Z}[X]$ be nonconstant and monic. Let $d \in \mathbb{Z}, d \geq 1$. For each $\alpha \in \bar{\mathbb{Z}}$, let $\mathcal{Z}(f(X) - d\alpha) \subseteq \bar{\mathbb{Z}}$ be the set of all the roots of $f(X) - d\alpha$. Define $\mathcal{S}(f, d)$ to be the following subset of $\bar{\mathbb{Z}}$:

$$\mathcal{S}(f, d) := \bigcup_{\alpha \in \bar{\mathbb{Z}}} \mathcal{Z}(f(X) - d\alpha).$$

Note that if $d = 1$, then $\mathcal{S}(f, 1) = \bar{\mathbb{Z}}$.

With this notation, $2\bar{\mathbb{Z}} = \mathcal{S}(X, 2)$. The sets $\mathcal{S}(f, d)$ provide many examples of subsets of $\bar{\mathbb{Z}}$ for which the associated integral-valued polynomial ring is nontrivial.

Example 6.2. Let $f \in \mathbb{Z}[X]$ be nonconstant and monic, and let $d \geq 2$. Then, $\mathcal{S}(f, d)$ has unbounded degree, and $\text{Int}_{\mathbb{Q}}(\mathcal{S}(f, d), \bar{\mathbb{Z}})$ contains $f(X)/d$, hence is nontrivial.

There is more to say about the relationship between the sets $\mathcal{S}(f, d)$ and $\text{Int}_{\mathbb{Q}}(S, \bar{\mathbb{Z}})$.

Definition 6.3. Given a subset $S \subseteq \bar{\mathbb{Z}}$, the *polynomial closure* of S in $\bar{\mathbb{Z}}$ is the largest subset $T \subseteq \bar{\mathbb{Z}}$ containing S such that $\text{Int}_{\mathbb{Q}}(S, \bar{\mathbb{Z}}) = \text{Int}_{\mathbb{Q}}(T, \bar{\mathbb{Z}})$. We say that S is *polynomially closed in $\bar{\mathbb{Z}}$* if $S = T$. In the literature [6, 25], the polynomial closure of S is often denoted by \bar{S} . We will not employ this notation in order to avoid confusion with integral closures or algebraic closures.

In Theorem 6.5 below, we prove that each polynomially closed subset of $\bar{\mathbb{Z}}$ is equal to an intersection of $\mathcal{S}(f, d)$ sets. In particular, this means that if $\text{Int}_{\mathbb{Q}}(S, \bar{\mathbb{Z}})$ is nontrivial, then $S \subseteq \mathcal{S}(f, d)$ for some f and some $d \geq 2$.

We recall now the definitions of characteristic ideals and regular bases, as discussed in [4, Section II.1]. For each integer $n \geq 0$, let $\mathfrak{I}_n(S)$ be the n^{th} characteristic ideal of $\text{Int}_{\mathbb{Q}}(S, \bar{\mathbb{Z}})$ of degree n , which consists of 0 and all the leading coefficients of polynomials in $\text{Int}_{\mathbb{Q}}(S, \bar{\mathbb{Z}})$ of degree n . For each $n \geq 0$, $\mathfrak{I}_n(S)$ is a principal fractional ideal of \mathbb{Z} that contains \mathbb{Z} . Let R be a ring such that $\mathbb{Z}[X] \subseteq R \subseteq \mathbb{Q}[X]$. Then, a regular basis for R is a \mathbb{Z} -module basis $\{b_n\}_{n \geq 0}$ for R such that each b_n has degree n . This happens precisely when $\mathfrak{I}_n(S)$ is principal, generated by the leading coefficient of b_n , for each $n \in \mathbb{N}$ [4, Theorem II.1.4].

Proposition 6.4. *Let $S \subseteq \bar{\mathbb{Z}}$. Then, $\text{Int}_{\mathbb{Q}}(S, \bar{\mathbb{Z}})$ has a regular basis of the form $\{f_n(X)/d_n\}_{n \geq 0}$ such that for each $n \geq 0$, $f_n \in \mathbb{Z}[X]$ is monic and d_n is a positive integer.*

Proof. Note that in the case $\text{Int}_{\mathbb{Q}}(S, \bar{\mathbb{Z}})$ is trivial, a regular basis is given by $\{X^n\}_{n \in \mathbb{N}}$.

For each $n \geq 0$, the characteristic ideal $\mathfrak{I}_n(S)$ is a principal fractional ideal of \mathbb{Z} . Fix n , and assume that $\mathfrak{I}_n(S)$ is generated by $\frac{a}{b} \in \mathbb{Q}$, where a and b are coprime integers and $b > 0$. Let $x, y \in \mathbb{Z}$ such that $ax + by = 1$. Since $\mathbb{Z} \subseteq \mathfrak{I}_n(S)$, we have $\frac{1}{b} = x(\frac{a}{b}) + y \in \mathfrak{I}_n(S)$. Thus, we may assume without loss of generality that each $\mathfrak{I}_n(S)$ is generated by $\frac{1}{d_n}$, where $d_n \in \mathbb{Z}$ and $d_n \geq 1$.

Now, by [4, Proposition II.1.4], $\text{Int}_{\mathbb{Q}}(S, \bar{\mathbb{Z}})$ has a regular basis $\{g_n\}_{n \geq 0} \subseteq \mathbb{Q}[X]$ such that for each n , g_n has degree n and the leading coefficient of g_n generates $\mathfrak{I}_n(S)$. Moreover, by [4, Proposition II.1.7], each coefficient of g_n is an element of $\mathfrak{I}_n(S)$. In light of the previous paragraph, we see that there are integers c_0, \dots, c_{n-1} such that

$$g_n(X) = \frac{1}{d_n} X^n + \sum_{i=0}^{n-1} \frac{c_i}{d_n} X^i.$$

Taking $f_n(X) = X^n + \sum_{i=0}^{n-1} c_i X^i$, we achieve the desired regular basis. \square

Theorem 6.5. *Let $S \subseteq \overline{\mathbb{Z}}$. Let $\{f_n(X)/d_n\}_{n \geq 0}$ be a regular basis for $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ such that each $f_n \in \mathbb{Z}[X]$ is monic and each d_n is a positive integer. Let $\mathcal{F} = \{f_n(X)/d_n \mid d_n \geq 2\}$.*

(1) $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is nontrivial if and only if $\mathcal{F} \neq \emptyset$.

(2) The polynomial closure of S in $\overline{\mathbb{Z}}$ is $\bigcap_{f(X)/d \in \mathcal{F}} \mathcal{S}(f, d)$.

Proof. Part (1) is clear. For (2), let T be the polynomial closure of S in $\overline{\mathbb{Z}}$. If \mathcal{F} is empty, then $T = \overline{\mathbb{Z}}$ because $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}}) = \mathbb{Z}[X] = \text{Int}_{\mathbb{Q}}(\overline{\mathbb{Z}}, \overline{\mathbb{Z}})$, and an intersection of an empty collection of subsets of $\overline{\mathbb{Z}}$ is also equal to $\overline{\mathbb{Z}}$. So, assume that $\mathcal{F} \neq \emptyset$. Let $\alpha \in T$, and let $f(X)/d \in \mathcal{F}$. Then, $f(X)/d \in \text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}}) = \text{Int}_{\mathbb{Q}}(T, \overline{\mathbb{Z}})$, so $f(\alpha)/d \in \overline{\mathbb{Z}}$ and $\alpha \in \mathcal{S}(f, d)$. Thus,

$$T \subseteq \bigcap_{f(X)/d \in \mathcal{F}} \mathcal{S}(f, d).$$

Conversely, let $\beta \in \bigcap_{f(X)/d \in \mathcal{F}} \mathcal{S}(f, d)$, let $g \in \text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$, and let $m = \deg g$. Since $\{f_n(X)/d_n\}_{n \geq 0}$ is a regular basis for $\text{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$, there exist $c_0, \dots, c_m \in \mathbb{Z}$ such that

$$g(X) = \sum_{i=0}^m \frac{c_i f_i(X)}{d_i}.$$

For each $0 \leq i \leq m$, if $d_i = 1$ then clearly $f_i(\beta)/d_i \in \overline{\mathbb{Z}}$. If $d_i \geq 2$, then $f_i(X)/d_i \in \mathcal{F}$, so $\beta \in \mathcal{S}(f_i, d_i)$, which means that $f_i(\beta)/d_i \in \overline{\mathbb{Z}}$. It follows that $g(\beta) \in \overline{\mathbb{Z}}$. Thus, $\beta \in T$, and therefore $T = \bigcap_{f(X)/d \in \mathcal{F}} \mathcal{S}(f, d)$. \square

We close the paper by using a theorem of McQuillan to prove that $\mathcal{A}_1 = \mathbb{Z}$ is polynomially closed in $\overline{\mathbb{Z}}$. It is an open problem to determine whether \mathcal{A}_n is polynomially closed in $\overline{\mathbb{Z}}$ when $n \geq 2$, although we suspect that this is the case.

Theorem 6.6. [17, Theorem] *Let $\alpha \in \overline{\mathbb{Z}}$ and let $\text{Int}(\mathbb{Z})(\alpha) = \{f(\alpha) \mid f \in \text{Int}(\mathbb{Z})\}$. Then,*

$$\text{Int}(\mathbb{Z})(\alpha) = \bigcap_{P \in \mathcal{S}_1} O_{\mathbb{Q}(\alpha), P}$$

where the intersection is over the family \mathcal{S}_1 of prime ideals P of $O_{\mathbb{Q}(\alpha)}$ which are totally split over \mathbb{Z} , that is $e(P \mid P \cap \mathbb{Z})f(P \mid P \cap \mathbb{Z}) = 1$.

Proposition 6.7. \mathbb{Z} is polynomially closed in $\overline{\mathbb{Z}}$.

Proof. Given $\alpha \in \overline{\mathbb{Z}} \setminus \mathbb{Z}$, define $\text{Int}(\mathbb{Z})(\alpha)$ as in Theorem 6.6. Since not all the prime ideals of $O_{\mathbb{Q}(\alpha)}$ are totally split over \mathbb{Z} , by Theorem 6.6 $\text{Int}(\mathbb{Z})(\alpha)$ is a proper overring of $O_{\mathbb{Q}(\alpha)}$. Thus, there exists $f \in \text{Int}(\mathbb{Z})$ such that $f(\alpha) \notin O_{\mathbb{Q}(\alpha)}$. This is equivalent to having $f(\alpha) \notin \overline{\mathbb{Z}}$, so we conclude that α is not in the polynomial closure of \mathbb{Z} in $\overline{\mathbb{Z}}$. \square

Conjecture 6.8. For each $n \geq 1$, \mathcal{A}_n is polynomially closed in $\overline{\mathbb{Z}}$.

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