F-ANALYTIC B-PAIRS

by

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Abstract. — In this note, we define the notion of F-analytic B-pairs and we prove that its category is equivalent to the one of F-analytic (φ_q, Γ_K)-modules.

Introduction

Let p be a prime and let K be a finite extension of $\mathbf{Q}_p$. One of the main tools to study p-adic representations of $\mathcal{G}_K = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$ is to operate a "dévissage" of the extension $\overline{\mathbf{Q}}_p/K$ through an intermediate extension K_∞/K which contains most of the ramification of $\overline{\mathbf{Q}}_p/K$ but such that K_∞/K is nice enough (for example when K_∞/K is an infinite almost totally ramified p-adic Lie extension).

In some sense, the simplest extension one can choose for K_∞/K is the cyclotomic extension of K. Using the theory of fields of norms [**21**] attached to the cyclotomic extension of K, Fontaine has constructed [**13**] a theory of cyclotomic (φ, Γ_K)-modules, which are finite dimensional vector spaces defined on a local field $\mathbf{B}_K$ which is of dimension 2, and endowed with semilinear actions of a Frobenius φ and of $\Gamma_K = \mathrm{Gal}(K(\mu_{p^\infty})/K)$ that commute one to another. Moreover, Fontaine has constructed a functor $V \mapsto D(V)$ which is an equivalence of categories between p-adic representations of $\mathcal{G}_K$ and étale (φ, Γ_K)-modules (which means that φ is of slope 0). The main theorem of [**6**] show that these (φ, Γ_K)-modules are overconvergent and it allows us to relate the cyclotomic (φ, Γ_K)-modules with classical p-adic Hodge theory, using the fact that the resulting overconvergent (φ, Γ_K)-modules give rise to what we still call (φ, Γ_K)-modules but defined on the cyclotomic Robba ring $\mathbf{B}^\dagger_{\mathrm{rig},K}$.

The construction of the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$ [**10**] relies heavily on this construction, and in particular on the computations made by Colmez in the trianguline case [**9**].

In order to extend this correspondence to $\mathrm{GL}_2(F)$, it seems necessary to replace the theory of cyclotomic (φ, Γ_K)-modules by Lubin-Tate (φ_q, Γ_K)-modules, where $F \subset K$ and

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K_∞/K is generated by the torsion points of a Lubin-Tate group attached to a uniformizer of F. Specializing Fontaine's constructions, Kisin and Ren have shown that we can attach to each representation of $\mathcal{G}_K$ a Lubin-Tate (φ_q, Γ_K)-module $D(V)$ over a 2-dimensional local field $\mathbf{B}_K$ (which is not the same as in the cyclotomic case) and such that $V \mapsto D(V)$ gives rise to an equivalence of categories when the image is restricted to the subcategory of étale objects.

However, unlike in the cyclotomic case, the resulting Lubin-Tate (φ_q, Γ_K)-modules are usually not overconvergent. The main theorem of [**2**] shows that F-analytic (φ_q, Γ_K)-modules are always overconvergent. The generalization of trianguline representations in the $\mathbf{Q}_p$-cyclotomic case to F-analytic representations has been studied in [**15**] (and Kisin and Ren mainly studied F-analytic crystalline representations in [**16**]).

A generalization of trianguline representations in the cyclotomic case for $\mathcal{G}_K$ has been done by Nakamura in [**18**] using the language of Berger's B-pairs [**1**] (and their natural extension to E-representations which are called $E - B-$pairs in [**18**]) but as noted in the introduction of [**15**], this language does not appear well suited to deal with Lubin-Tate objects.

In [**2**, Rem. 10.3] Berger notes that his results and methods should extend to prove that there is an equivalence of categories between F-analytic (φ_q, Γ_K)-modules and F-analytic B-pairs, and it is this result this note is meant to prove.

In the cyclotomic case, it is often useful to switch between cyclotomic (φ, Γ_K)-modules and B-pairs, some properties being easier to prove using one of the categories instead of the other, and it so should be in the Lubin-Tate case, using the following:

Theorem 0.1. — *There is an equivalence of categories between F-analytic B-pairs and F-analytic (φ_q, Γ_K)-modules.*

In particular, a recent result of Porat [**19**, Thm. 6.8] shows that for F-analytic 2-dimensional representations of $\mathcal{G}_F$, V is trianguline in the cyclotomic sense if and only if it is trianguline in the sense of [**15**]. His theorem actually extends to F-analytic representations of arbitrary dimension as a straightforward consequence of our theorem 0.1:

Theorem 0.2. — *Let V be an F-analytic representation of $\mathcal{G}_K$. Then V is trianguline in the cyclotomic sense if and only if it is trianguline in the sense of* [**15**].

As stated above, the usual language of B-pairs is not well suited to deal with Lubin-Tate objects. Ding has constructed in [**11**] a variant of Berger's B-pairs with a Lubin-Tate flavour. For any embedding $\sigma : F \to \overline{\mathbf{Q}}_p$, and for any B-pair D, Ding constructs what he calls a B_σ-pair D_σ, such that $D \mapsto D_\sigma$ is an equivalence of categories between B-pairs and B_σ-pairs. In the F-analytic case, we construct a functor $D \mapsto W(D)$ from the category of F-analytic (φ_q, Γ_K)-modules to the category of F-analytic B_{id}-pairs and which is the natural Lubin-Tate analogue of the constructions of Berger [**1**]. In particular, the following ensues from theorem 0.1 but the correspondence between objects is easier to see:

Theorem 0.3. — *The functor $D \mapsto W(D)$, from the category of F-analytic (φ_q, Γ_K)-modules to the category of F-analytic B_{id}-pairs is an equivalence of categories.*

Structure of the note

The first three sections of this note are meant to recall the setting, notations and few properties of Lubin-Tate extensions, (φ_q, Γ_K)-modules and locally analytic vectors from [**2**] that are needed for the rest of this note. In particular, these are pretty much the same as [**2**, §1, 2 and 3]. Section 4 explains the notion of F-analyticity in the case of F-representations and (φ_q, Γ_K)-modules. In section 5, we recall the notion of (B, E)-pairs, define F-analyticity for (B, E)-pairs and prove the main theorem of this note, and how to derive from it theorem 0.2 which is the generalization of Porat's result. In section 6 we explain how to replace the category of F-analytic B-pairs by the one of F-analytic B_{id}-pairs. The last section is a quick summary of the rings that appear throughout this paper.

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1. Lubin-Tate extensions

Let F be a finite extension of $\mathbf{Q}_p$, let $\mathcal{O}_F, \pi$ and k_F denote respectively its ring of integers, a uniformizer of $\mathcal{O}_F$ and its residue field. Let $h \geq 1$ be such that $|k_F| = q = p^h$. We let $F_0 = W(k_F)[1/p]$, the maximal unramified extension of $\mathbf{Q}_p$ inside F, and we let e to be the ramification index of F. Let Σ be the set of embeddings of F in $\overline{\mathbf{Q}}_p$, and let σ be the absolute Frobenius on F_0. For $\tau \in \Sigma$, there exists a unique $n(\tau) \in \{0, \ldots, h-1\}$ such that $\overline{\tau} = \overline{\sigma}^{n(\tau)}$ on k_F. We also let E to be a field of coefficients which is a finite Galois extension of $\mathbf{Q}_p$ containing F (hence F^{Gal}), and write Σ_E for $\mathrm{Gal}(E/\mathbf{Q}_p)$. We let $\Sigma_0 = \Sigma \setminus \{\mathrm{id}\}$.

Let S be a formal $\mathcal{O}_F$-module Lubin-Tate group law attached to π, such that the endomorphism of multiplication by π is given by the power series $[\pi](T) = T^q + \pi T$. For $a \in \mathcal{O}_F$, we will denote $[a](T)$ the power series giving the endomorphism of multiplication by a for S. Let F_n be the field generated by F and the points of π^n-torsion, that is the roots of $[\pi^n](T)$. Let $F_\infty = \bigcup_{n \geq 1} F_n$, $\Gamma_F = \mathrm{Gal}(F_\infty/F)$ and $H_F = \mathrm{Gal}(\overline{\mathbf{Q}}_p/F_\infty)$. Let χ_π be the attached Lubin-Tate character. Note that there exists an unramified character $\eta : \mathcal{G}_F \to \mathbf{Z}_p^\times$ such that $N_{F/\mathbf{Q}_p}(\chi_\pi) = \eta\chi_{\mathrm{cycl}}$, where χ_{cycl} is the cyclotomic character.

If K is a finite extension of F, we write $K_n = KF_n$ and $K_\infty = KF_\infty$. We let $\Gamma_K = \mathrm{Gal}(K_\infty/K)$ and $H_K = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K_\infty)$. We let $K_\infty^\eta = \overline{\mathbf{Q}}_p^{\ker \eta\chi_{\mathrm{cycl}}}$, so that $K_\infty^\eta \subset K_\infty$ and that $\eta\chi_{\mathrm{cycl}}$ identifies $\mathrm{Gal}(K_\infty^\eta/K)$ with an open subgroup of $\mathbf{Z}_p^\times$.

Now let $\Gamma_n = \mathrm{Gal}(K_\infty/K_n)$ so that $\Gamma_n = \{g \in \Gamma_K \text{ such that } \chi_\pi(g) \in 1 + \pi^n\mathcal{O}_F\}$. Let $u_0 = 0$ and for each $n \geq 1$, chose $u_n \in \overline{\mathbf{Q}}_p$ such that $[\pi](u_n) = u_{n-1}$, with $u_1 \neq 0$. We have $v_p(u_n) = 1/q^{n-1}(q-1)e$ for $n \geq 1$ and $F_n = F(u_n)$. We also let $Q_n(T)$ be the minimal polynomial of u_n over F, so that $Q_0(T) = T$, $Q_1(T) = [\pi](T)/T$ and

$Q_{n+1}(T) = Q_n([\pi](T))$ if $n \geq 1$. Let $\log_{\mathrm{LT}}(T) = T + O(\deg \geq 2) \in F[\![T]\!]$ denote the Lubin-Tate logarithm map, which converges on the open unit disk and satisfies $\log_{\mathrm{LT}}([a](T)) = a \cdot \log_{\mathrm{LT}}(T)$ if $a \in \mathcal{O}_F$. Note that we have $\log_{\mathrm{LT}}(T) = T \cdot \prod_{k \geq 1} Q_k(T)/\pi$. We also let $\exp_{\mathrm{LT}}(T)$ denote the inverse of $\log_{\mathrm{LT}}(T)$.

Let $\mathcal{O}_{\mathbf{C}_p}^\flat = \{(x_0, x_1, \ldots)$, with $x_n \in \mathcal{O}_{\mathbf{C}_p}/\pi$ and such that $x_{n+1}^q = x_n$ for all $n \geq 0\}$. This ring is endowed with the valuation $v_{\mathbf{E}}(\cdot)$ defined by $v_{\mathbf{E}}(x) = \lim_{n \to +\infty} q^n v_p(\widehat{x}_n)$ where $\widehat{x}_n \in \mathcal{O}_{\mathbf{C}_p}$ lifts x_n. The ring $\mathcal{O}_{\mathbf{C}_p}^\flat$ is complete for $v_{\mathbf{E}}(\cdot)$. If the $\{u_n\}_{n \geq 0}$ are as above, then $\overline{u} = (\overline{u}_0, \overline{u}_1, \ldots) \in \mathcal{O}_{\mathbf{C}_p}^\flat$ and $v_{\mathbf{E}}(\overline{u}) = q/(q-1)e$. Let $\mathbf{C}_p^\flat$ be the fraction field of $\mathcal{O}_{\mathbf{C}_p}^\flat$.

Let $W_F(\cdot) = \mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} W(\cdot)$ be the F-Witt vectors. Let $\widetilde{\mathbf{A}}^+ = \mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} W(\mathcal{O}_{\mathbf{C}_p}^\flat)$, $\widetilde{\mathbf{A}} = \mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} W(\mathbf{C}_p^\flat)$ and let $\widetilde{\mathbf{B}}^+ = \widetilde{\mathbf{A}}^+[1/\pi]$ and $\widetilde{\mathbf{B}} = \widetilde{\mathbf{A}}[1/\pi]$. These rings are preserved by the Frobenius map $\varphi_q = \mathrm{id} \otimes \varphi^h$. By [**7**, §9.2], there exists $u \in \widetilde{\mathbf{A}}^+$, whose image in $\mathcal{O}_{\mathbf{C}_p}^\flat$ is $\overline{u}$, and such that $\varphi_q(u) = [\pi](u)$ and $g(u) = [\chi_\pi(g)](u)$ if $g \in \Gamma_F$.

Every element of $\widetilde{\mathbf{B}}^+[1/[\overline{u}]]$ can be written uniquely as a sum $\sum_{k \gg -\infty} \pi^k [x_k]$ where $\{x_k\}_{k \in \mathbf{Z}}$ is a bounded sequence of $\mathbf{C}_p^\flat$. For $r \geq 0$, we define a valuation $V(\cdot, r)$ on $\widetilde{\mathbf{B}}^+[1/[\overline{u}]]$ by

$$V(x, r) = \inf_{k \in \mathbf{Z}} \left(\frac{k}{e} + \frac{p-1}{pr} v_{\mathbf{E}}(x_k) \right) \text{ if } x = \sum_{k \gg -\infty} \pi^k [x_k].$$

If I is a closed subinterval of $[0; +\infty[$, then let $V(x, I) = \inf_{r \in I} V(x, r)$. We define $\widetilde{\mathbf{B}}^I$ to be the completion of $\widetilde{\mathbf{B}}^+[1/[\overline{u}]]$ for the valuation $V(\cdot, I)$ if $0 \notin I$. If $I = [0; r]$, then let $\widetilde{\mathbf{B}}^I$ be the completion of $\widetilde{\mathbf{B}}^+$ for $V(\cdot, I)$.

For $\rho > 0$, let $\rho' = \rho \cdot e \cdot p/(p-1) \cdot (q-1)/q$ as in [**2**, §3]. We have $V(u^i, r) = i/r'$ for $i \in \mathbf{Z}$ if $r > 1$ (see [**2**, §3]).

Let I be either a subinterval of $]1; +\infty[$ or such that $0 \in I$, and let $f(Y) = \sum_{k \in \mathbf{Z}} a_k Y^k$ be a power series with $a_k \in F$ and such that $v_p(a_k) + k/\rho' \to +\infty$ when $|k| \to +\infty$ for all $\rho \in I$. The series $f(u)$ converges in $\widetilde{\mathbf{B}}^I$ and we let $\mathbf{B}_F^I$ denote the set of $f(u)$ where $f(Y)$ is as above. It is a subring of $\widetilde{\mathbf{B}}_F^I = (\widetilde{\mathbf{B}}^I)^{H_F}$, which is stable under the action of Γ_F. The Frobenius map gives rise to a map $\varphi_q : \mathbf{B}_F^I \to \mathbf{B}_F^{qI}$. If $m \geq 0$, then we have $\varphi_q^{-m}(\mathbf{B}_F^{q^m I}) \subset \widetilde{\mathbf{B}}_F^I$ and we let $\mathbf{B}_{F,m}^I = \varphi_q^{-m}(\mathbf{B}_F^{q^m I})$.

We will write $\mathbf{B}_{\mathrm{rig},F}^{\dagger,r}$ for $\mathbf{B}_F^{[r;+\infty[}$. Let $\mathbf{B}_F^{\dagger,r}$ denote the set of $f(u) \in \mathbf{B}_{\mathrm{rig},F}^{\dagger,r}$ such that the sequence $\{a_k\}_{k \in \mathbf{Z}}$ is bounded. Let $\mathbf{B}_F^\dagger = \cup_{r \gg 0} \mathbf{B}_F^{\dagger,r}$. Its residue field $\mathbf{E}_F$ is isomorphic to $\mathbf{F}_q(\!(\overline{u})\!)$. If K is a finite extension of F then by the theory of the field of norms (see [**21**]), there corresponds to K/F a separable extension $\mathbf{E}_K/\mathbf{E}_F$, of degree $[K_\infty : F_\infty]$. Since $\mathbf{B}_F^\dagger$ is a Henselian field, there exists a finite unramified extension $\mathbf{B}_K^\dagger/\mathbf{B}_F^\dagger$ of degree $f = [K_\infty : F_\infty]$ whose residue field is $\mathbf{E}_K$ (see §2 and §3 of [**17**]). There exist therefore $r(K) > 0$ and elements $x_1, \ldots, x_f$ in $\mathbf{B}_K^{\dagger, r(K)}$ such that $\mathbf{B}_K^{\dagger,s} = \oplus_{i=1}^f \mathbf{B}_F^{\dagger,s} \cdot x_i$ for all $s \geq r(K)$. Note that the rings $\mathbf{B}_K^\dagger$ are actually contained inside $\widetilde{\mathbf{B}}$. We also let $\mathbf{B}_K$ to be the p-adic completion of $\mathbf{B}_K^\dagger$ inside $\widetilde{\mathbf{B}}$, and $\mathbf{A}_K$ its ring of integers for the p-adic topology (note that we could have defined $\mathbf{A}_F$ as the p-adic completion of $\mathcal{O}_F[\![u]\!][1/u]$ inside $\widetilde{\mathbf{A}}$, put $\mathbf{B}_F = \mathbf{A}_F[1/\pi]$ and used the same argument as in the beginning of [**8**, §6.1] to define $\mathbf{B}_K$). Let $\mathbf{B}$ be the p-adic completion of $\bigcup_{K/F} \mathbf{B}_K$ inside $\widetilde{\mathbf{B}}$.

Let $\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$ denote the Fréchet completion of $\mathbf{B}^{\dagger,r}_K$ for the valuations $\{V(\cdot,[r;s])\}_{s\geq r}$. Let $\mathbf{B}^{\dagger,r}_{\mathrm{rig},K,m} = \varphi_q^{-m}(\mathbf{B}^{\dagger,q^m r}_{\mathrm{rig},K})$ and $\mathbf{B}^{\dagger,r}_{\mathrm{rig},K,\infty} = \cup_{m\geq 0}\mathbf{B}^{\dagger,r}_{\mathrm{rig},K,m}$. Let $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$ denote the Fréchet completion of $\widetilde{\mathbf{B}}^+[1/[\overline{u}]]$ for the valuations $\{V(\cdot,[r;s])\}_{s\geq r}$. Let $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} = \cup_{r\gg 0}\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$, $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig},K} = (\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}})^{H_K}$ and $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig},K} = (\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}})^{H_K}$.

Recall that K^η_∞/K is the extension of K attached to $\eta\chi_{\mathrm{cycl}}$. Let $\Gamma'_K = \mathrm{Gal}(K^\eta_\infty/K)$. Let $\mathbf{B}^{\dagger}_{K,\eta}$, $\mathbf{B}^{I}_{K,\eta}$ and $\mathbf{B}^{\dagger}_{\mathrm{rig},K,\eta}$ be as in [**2**, §8]. By the same arguments as in [**2**, §8], there is an equivalence of categories between étale (φ,Γ'_K)-modules over $E\otimes_{\mathbf{Q}_p}\mathbf{B}^{\dagger}_{\mathrm{rig},K,\eta}$ (it is also true over $E\otimes_{\mathbf{Q}_p}\mathbf{B}^{\dagger}_{K,\eta}$) and E-representations of $\mathcal{G}_K$. We will also denote by $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig},\eta}$ the ring $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ in the specific case of $F=\mathbf{Q}_p$, so that $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} = F\otimes_{F_0}\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig},\eta}$. Note that the ring $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig},\eta}$ does actually not depend on η but we use this notation for convenience.

A (φ_q,Γ_K)-module over $\mathbf{B}_K$ is a $\mathbf{B}_K$-vector space $\mathbf{D}$ of finite dimension d, along with a semilinear Frobenius map φ_q and a commuting continuous and semilinear action of Γ_K. We say that $\mathbf{D}$ is étale if there exists a basis of $\mathbf{D}$ in which $\mathrm{Mat}(\varphi)$ belongs to $\mathrm{GL}_d(\mathbf{A}_K)$. By specializing the constructions of [**13**], Kisin and Ren prove the following theorem [**16**, Thm. 1.6].

Theorem 1.1. — *The functors $V\mapsto(\mathbf{B}\otimes_F V)^{H_K}$ and $\mathbf{D}\mapsto(\mathbf{B}\otimes_{\mathbf{B}_K}\mathbf{D})^{\varphi_q=1}$ give rise to mutually inverse equivalences of categories between the category of F-linear representations of $\mathcal{G}_K$ and the category of étale (φ_q,Γ_K)-modules over $\mathbf{B}_K$.*

We say that a (φ_q,Γ_K)-module $\mathbf{D}$ is overconvergent if there exists a basis of $\mathbf{D}$ in which the matrices of φ_q and of all $g\in\Gamma_K$ have entries in $\mathbf{B}^{\dagger}_K$. This basis generates a $\mathbf{B}^{\dagger}_K$-vector space $\mathbf{D}^{\dagger}$ which is canonically attached to $\mathbf{D}$. Theorem 1.1 extends more generally to an equivalence of categories between the category of E-linear representations of $\mathcal{G}_K$ and the category of étale (φ_q,Γ_K)-modules over $E\otimes_F\mathbf{B}_K$.

2. Locally, pro-analytic and F-analytic vectors

In this section, we recall the theory of locally analytic vectors of Schneider and Teitelbaum [**20**] but here we follow the constructions of Emerton [**12**] as in [**2**]. We also define the notion of F-analytic vectors relative to the Galois group of a Lubin-Tate extension, following the definitions of [**2**]. We will use the following multi-index notations: if $\mathbf{c}=(c_1,\dots,c_d)$ and $\mathbf{k}=(k_1,\dots,k_d)\in\mathbf{N}^d$ (here $\mathbf{N}=\mathbf{Z}^{\geq 0}$), then we let $\mathbf{c}^{\mathbf{k}} = c_1^{k_1}\cdot\ldots\cdot c_d^{k_d}$.

Let G be a p-adic Lie group, and let W be a $\mathbf{Q}_p$-Banach representation of G. Let H be an open subgroup of G such that there exists coordinates $c_1,\cdots,c_d : H\to\mathbf{Z}_p$ giving rise to an analytic bijection $\mathbf{c}: H\to\mathbf{Z}_p^d$. We say that $w\in W$ is an H-analytic vector if there exists a sequence $\{w_{\mathbf{k}}\}_{\mathbf{k}\in\mathbf{N}^d}$ such that $w_{\mathbf{k}}\to 0$ in W and such that $g(w)=\sum_{\mathbf{k}\in\mathbf{N}^d}\mathbf{c}(g)^{\mathbf{k}}w_{\mathbf{k}}$ for all $g\in H$. We let $W^{H\text{-an}}$ be the space of H-analytic vectors. This space injects into $\mathcal{C}^{\mathrm{an}}(H,W)$, the space of all analytic functions $f: H\to W$. Note that $\mathcal{C}^{\mathrm{an}}(H,W)$ is a Banach space equipped with its usual Banach norm, so that we can endow $W^{H\text{-an}}$ with the induced norm, that we will denote by $||\cdot||_H$. With this definition, we have $||w||_H=\sup_{\mathbf{k}\in\mathbf{N}^d}||w_{\mathbf{k}}||$ and $(W^{H\text{-an}},||\cdot||_H)$ is a Banach space.

We say that a vector w of W is locally analytic if there exists an open subgroup H as above such that $w\in W^{H\text{-an}}$. Let W^{la} be the space of such vectors, so that $W^{\mathrm{la}} =$

$\bigcup_H W^{H\text{-an}}$, where H runs through a sequence of open subgroups of G. The space W^{la} is naturally endowed with the inductive limit topology, so that it is an LB space. Note that in the Lubin-Tate setting, we have $W^{\mathrm{la}} = \bigcup_{n\in\mathbf{N}} W^{\Gamma_n-\mathrm{an}}$.

Let W be a Fréchet space whose topology is defined by a sequence $\{p_i\}_{i\geq 1}$ of seminorms. Let W_i be the Hausdorff completion of W at p_i, so that $W = \varprojlim_{i\geq 1} W_i$. The space W^{la} can be defined but as stated in [**2**], this space is too small in general for what we are interested in, and so we make the following definition, following [**2**, Def. 2.3]:

Definition 2.1. — If $W = \varprojlim_{i\geq 1} W_i$ is a Fréchet representation of G, then we say that a vector $w \in W$ is pro-analytic if its image $\pi_i(w)$ in W_i is locally analytic for all i. We let W^{pa} denote the set of all pro-analytic vectors of W.

We extend the definition of W^{la} and W^{pa} for LB and LF spaces respectively.

Proposition 2.2. — *Let G be a p-adic Lie group, let B be a Banach G-ring and let W be a free B-module of finite rank, equipped with a compatible G-action. If the B-module W has a basis $w_1, \dots, w_d$ in which $g \mapsto \mathrm{Mat}(g)$ is a globally analytic function $G \to \mathrm{GL}_d(B) \subset M_d(B)$, then*

1. *$W^{H\text{-an}} = \bigoplus_{j=1}^d B^{H\text{-an}} \cdot w_j$ if H is a subgroup of G;*
2. *$W^{\mathrm{la}} = \bigoplus_{j=1}^d B^{\mathrm{la}} \cdot w_j$.*

Let G be a p-adic Lie group, let B be a Fréchet G-ring and let W be a free B-module of finite rank, equipped with a compatible G-action. If the B-module W has a basis $w_1, \dots, w_d$ in which $g \mapsto \mathrm{Mat}(g)$ is a pro-analytic function $G \to \mathrm{GL}_d(B) \subset M_d(B)$, then

$$W^{\mathrm{pa}} = \bigoplus_{j=1}^{d} B^{\mathrm{pa}} \cdot w_j.$$

Proof. — The part for Banach rings is proven in [**4**, Prop. 2.3] and the one for Fréchet rings is proven in [**2**, Prop. 2.4]. □

The map $\ell : g \mapsto \log_p \chi_\pi(g)$ gives an F-analytic isomorphism between Γ_n and $\pi^n\mathcal{O}_F$ for $n \gg 0$. If W is an F-linear Banach representation of Γ_K and $n \gg 0$, then we say, following [**2**], that an element $w \in W$ is F-analytic on Γ_n if there exists a sequence $\{w_k\}_{k\geq 0}$ of elements of W with $\pi^{nk}w_k \to 0$ such that $g(w) = \sum_{k\geq 0} \ell(g)^k w_k$ for all $g \in \Gamma_n$. Let $W^{\Gamma_n\text{-an},F\text{-la}}$ denote the space of such elements. Let $W^{F\text{-la}} = \bigcup_{n\geq 1} W^{\Gamma_n\text{-an},F\text{-la}}$.

Lemma 2.3. — *We have $W^{\Gamma_n\text{-an},F\text{-la}} = W^{\Gamma_n\text{-an}} \cap W^{F\text{-la}}$.*

Proof. — See [**2**, Lemm. 2.5]. □

If $\tau \in \Sigma$, we let ∇_τ denote the derivative in the direction τ, which belongs to $E \otimes_{\mathbf{Q}_p} \mathrm{Lie}(\Gamma_F)$. It can be defined as follows: the E-vector space $\mathrm{Hom}_{\mathbf{Q}_p}(F, E)$ is generated by the elements of Σ. If W is an E-linear Banach representation of Γ_K and if $w \in W^{\mathrm{la}}$ and $g \in \Gamma_K$, then there exists elements $\{\nabla_\tau\}_{\tau\in\Sigma}$ of $F^{\mathrm{Gal}} \otimes_{\mathbf{Q}_p} \mathrm{Lie}(\Gamma_F)$ such that we can write

$$\log g(w) = \sum_{\tau\in\Sigma} \tau(\ell(g)) \cdot \nabla_\tau(w).$$

With the same notation, there exist $m \gg 0$ and elements $\{w_{\mathbf{k}}\}_{\mathbf{k}\in\mathbf{N}^\Sigma}$ such that if $g \in \Gamma_m$, then $g(w) = \sum_{\mathbf{k}\in\mathbf{N}^\Sigma} \ell(g)^{\mathbf{k}} w_{\mathbf{k}}$, where $\ell(g)^{\mathbf{k}} = \prod_{\tau\in\Sigma} \tau \circ \ell(g)^{k_\tau}$. We have $\nabla_\tau(w) = w_{\mathbf{1}_\tau}$ where $\mathbf{1}_\tau$ is the Σ-tuple whose entries are 0 except the τ-th one which is 1. If $\mathbf{k} \in \mathbf{N}^\Sigma$, and if we set $\nabla^{\mathbf{k}}(w) = \prod_{\tau\in\Sigma} \nabla_\tau^{k_\tau}(w)$, then $w_{\mathbf{k}} = \nabla^{\mathbf{k}}(w)/\mathbf{k}!$.

Remark 2.4. — If $w \in W^{\mathrm{la}}$, then $w \in W^{F\text{-la}}$ if and only if $\nabla_\tau(w) = 0$ for all $\tau \in \Sigma \setminus \{\mathrm{id}\}$.

We have the following structure result for locally and pro-analytic vectors in the rings $\widetilde{\mathbf{B}}^I$:

Theorem 2.5. — *Let $I = [r_\ell; r_k]$ with $\ell \leq k$, let K be a finite extension of F, and let $m \geq 0$ be such that t_π and t_π/Q_k belong to $(\widetilde{\mathbf{B}}^I_F)^{\Gamma_{m+k}\text{-an},F\text{-la}}$.*

1. $(\widetilde{\mathbf{B}}^I_F)^{\Gamma_{m+k}\text{-an},F\text{-la}} \subset \mathbf{B}^I_{F,m}$;
2. $(\widetilde{\mathbf{B}}^I_K)^{F\text{-la}} = \mathbf{B}^I_{K,\infty}$;
3. $(\widetilde{\mathbf{B}}^{\dagger,r_\ell}_{\mathrm{rig},K})^{F\text{-pa}} = \mathbf{B}^{\dagger,r_\ell}_{\mathrm{rig},K,\infty}$.

Proof. — This is [**2**, Thm. 4.4]. □

3. F-analyticity

We say, following [**2**, §7] that an F-linear representation V of $\mathcal{G}_K$ is F-analytic if $\mathbf{C}_p \otimes^\tau_F V$ is the trivial $\mathbf{C}_p$-semilinear representation of $\mathcal{G}_K$ for all embeddings $\tau \neq \mathrm{id} \in \Sigma$.

The following lemma shows that the condition for an E-representation to be F-analytic depends only on the restriction of the elements of Σ_E to F.

Lemma 3.1. — *If V is an E-representation of $\mathcal{G}_K$, then the following are equivalent:*

1. *V seen as an F-representation is F-analytic;*
2. *$\mathbf{C}_p \otimes^g_E V$ is the trivial $\mathbf{C}_p$-semilinear representation of $\mathcal{G}_K$ for all $g \in \mathrm{Gal}(E/\mathbf{Q}_p)$ such that $g|_F \neq \mathrm{id}$.*

Proof. — See [**2**, Lemm. 7.2]. □

Definition 3.2. — If $\mathbf{D}^\dagger_{\mathrm{rig}}$ is a (φ_q, Γ_K)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$, and if $g \in \Gamma_K$ is close enough to 1, then the series $\log(g) = \log(1 + (g-1))$ gives rise to a differential operator $\nabla_g : \mathbf{D}^\dagger_{\mathrm{rig}} \to \mathbf{D}^\dagger_{\mathrm{rig}}$. The map $\mathrm{Lie}\,\Gamma_K \to \mathrm{End}(\mathbf{D}^\dagger_{\mathrm{rig}})$ arising from $v \mapsto \nabla_{\exp(v)}$ is $\mathbf{Q}_p$-linear, and we say, following [**16**, §2.1], [**15**, §1.3] and [**2**, §7], that $\mathbf{D}^\dagger_{\mathrm{rig}}$ is F-analytic if this map is F-linear. This is the same as asking the elements of $\mathbf{D}^\dagger_{\mathrm{rig}}$ to be pro-F-analytic vectors for the action of Γ_K.

Given $\tau \in \Sigma$ and $f(Y) = \sum_{k\in\mathbf{Z}} a_k Y^k$ with $a_k \in F$, let $f^\tau(Y) = \sum_{k\in\mathbf{Z}} \tau(a_k) Y^k$. For $\tau \in \Sigma$, let $\widetilde{n}(\tau)$ be the lift of $n(\tau) \in \mathbf{Z}/h\mathbf{Z}$ belonging to $\{0, \ldots, h-1\}$. Recall that E is a finite extension of F that contains F^{Gal} and that if $\tau \in \Sigma$, then we have $\nabla_\tau \in E \otimes_F \mathrm{Lie}(\Gamma_F)$. The field E is a field of coefficients, so that $\mathcal{G}_K$ acts E-linearly below.

Let $t_\pi = \log_{\mathrm{LT}}(u) \in \mathbf{B}^+_{\mathrm{rig},K}$. Note that we actually have $t_\pi \in \mathbf{B}^+_{\mathrm{rig},F}$, and that $\varphi_q(t_\pi) = \pi t_\pi$ and $g(t_\pi) = \chi_\pi(g) t_\pi$ if $g \in \mathcal{G}_F$. Let $y_\tau = (\tau \otimes \varphi^{\widetilde{n}(\tau)})(u) \in \mathcal{O}_E \otimes_{\mathcal{O}_F} \widetilde{\mathbf{A}}^+$. We have

$g(y_\tau) = [\chi_\pi(g)]^\tau(y_\tau)$ and $\varphi_q(y_\tau) = [\pi]^\tau(y_\tau) = \tau(\pi)y_\tau + y_\tau^q$. Let $t_\tau = (\tau \otimes \varphi^{\widetilde{n(\tau)}})(t_\pi) = \log_{\mathrm{LT}}^\tau(y_\tau)$, let $Q_n = Q_n(u)$ and $Q_n^\tau = Q_n^\tau(y_\tau)$, so that $t_\tau = y_\tau \prod_{n\geq 1} Q_n^\tau/\pi$.

We have $\nabla_\tau(y_\tau) = t_\tau \cdot v_\tau$ where $v_\tau = (\partial(T \oplus_{\mathrm{LT}} U)/\partial U)^\tau(y_\tau, 0)$ is a unit (see §2.1 of [**16**]). Let $\partial_\tau = t_\tau^{-1} v_\tau^{-1} \nabla_\tau$ so that $\partial_\tau(y_\tau) = 1$. If $\tau, \upsilon \in \Sigma$, then $\partial_\tau \circ \partial_\upsilon = \partial_\upsilon \circ \partial_\tau$, and $\partial_\tau(y_\upsilon) = 0$ if $\tau \neq \upsilon$.

Lemma 3.3. — *We have* $\partial_\tau((E \otimes_F \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K})^{\mathrm{pa}}) \subset (E \otimes_F \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K})^{\mathrm{pa}}$.

Proof. — See [**2**, Lemm. 5.2]. □

Proposition 3.4. — *Let M be a (φ_q, Γ_K)-module over $E \otimes_F (\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K})^{\mathrm{pa}}$. Let*

$$\mathrm{Sol}(M) = \{x \in M \text{ such that } \nabla_\tau(x) = 0 \text{ for all } \tau \in \Sigma_0\}.$$

If for all $\tau \in \Sigma_0$, $\nabla_\tau(M) \subset t_\tau \cdot M$, then there exists a unique (φ_q, Γ_K)-module $\mathbf{D}^\dagger_{\mathrm{rig}}$ over $E \otimes_F \mathbf{B}^\dagger_{\mathrm{rig},K}$ such that $\mathrm{Sol}(M) = (E \otimes_F (\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K})^{F\text{-pa}}) \otimes_{E\otimes_F \mathbf{B}^\dagger_{\mathrm{rig},K}} \mathbf{D}^\dagger_{\mathrm{rig}}$ and such that $M = (E \otimes_F (\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K})^{\mathrm{pa}}) \otimes_{E\otimes_F \mathbf{B}^\dagger_{\mathrm{rig},K}} \mathbf{D}^\dagger_{\mathrm{rig}}$, and $\mathbf{D}^\dagger_{\mathrm{rig}}$ is an F-analytic (φ_q, Γ_K)-module.

Moreover, if $\mathbf{D}$ is a (φ_q, Γ_K)-module over $E \otimes_F \mathbf{B}^\dagger_{\mathrm{rig},K}$, and if $M = (E \otimes_F \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K}) \otimes_{E\otimes_F \mathbf{B}^\dagger_{\mathrm{rig},K}} \mathbf{D}$, then $\mathbf{D}$ is F-analytic if and only if for all $\tau \in \Sigma_0$, $\nabla_\tau(M^{\mathrm{pa}}) \subset t_\tau \cdot M^{\mathrm{pa}}$, and in this case we have $\mathbf{D} = \mathbf{D}^\dagger_{\mathrm{rig}}$.

Proof. — We first prove the first part of the theorem. Let M be a (φ_q, Γ_K)-module over $E \otimes_F (\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K})^{\mathrm{pa}}$. Theorem 6.1 of [**2**] shows that

$$\mathrm{Sol}(M) = \{x \in M \text{ such that } \nabla_\tau(x) = 0 \text{ for all } \tau \in \Sigma_0\}$$

is a free $E \otimes_F (\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K})^{F\text{-pa}}$-module of rank d such that

$$(E \otimes_F \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K}) \otimes_{E\otimes_F (\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K})^{F\text{-pa}})^{F\text{-pa}}} \mathrm{Sol}(M) = (E \otimes_F \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K}) \otimes_E \mathbf{D}.$$

By (3) of theorem 2.5, we have $(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K})^{F\text{-pa}} = \mathbf{B}^\dagger_{\mathrm{rig},K,\infty} = \bigcup_{n\geq 0} \mathbf{B}^\dagger_{\mathrm{rig},K,n}$. Since Γ_K is topologically of finite type, there exist $n \geq 0$, and a basis $s_1, \ldots, s_d$ of $\mathrm{Sol}(M)$ such that $\mathrm{Mat}(\varphi_q) \in \mathrm{GL}_d(E \otimes_F \mathbf{B}^\dagger_{\mathrm{rig},K,n})$ and $\mathrm{Mat}(g) \in \mathrm{GL}_d(E \otimes_F \mathbf{B}^\dagger_{\mathrm{rig},K,n})$ for all $g \in \Gamma_K$. If $\mathbf{D}^\dagger_{\mathrm{rig}} = \oplus_{i=1}^d (E \otimes_F \mathbf{B}^\dagger_{\mathrm{rig},K}) \cdot \varphi_q^n(s_i)$, then $\mathbf{D}^\dagger_{\mathrm{rig}}$ is a (φ_q, Γ_K)-module over $E \otimes_F \mathbf{B}^\dagger_{\mathrm{rig},K}$ such that $\mathrm{Sol}(M) = (E \otimes_F (\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K})^{F\text{-pa}}) \otimes_{E\otimes_F \mathbf{B}^\dagger_{\mathrm{rig},K}} \mathbf{D}^\dagger_{\mathrm{rig}}$.

The module $\mathbf{D}^\dagger_{\mathrm{rig}}$ is uniquely determined by this condition: if there are two such modules and if X denotes the change of basis matrix and P_1, P_2 denote the matrices of φ_q, then $X \in \mathrm{GL}_d(E \otimes_F \mathbf{B}^\dagger_{\mathrm{rig},K,n})$ for $n \gg 0$, and the equation $X = P_2^{-1}\varphi(X)P_1$ implies that $X \in \mathrm{GL}_d(E \otimes_F \mathbf{B}^\dagger_{\mathrm{rig},K})$.

Since $\mathrm{Sol}(M)$ is a free $E \otimes_F (\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K})^{F\text{-pa}}$-module, $\mathbf{D}^\dagger_{\mathrm{rig}}$ is also free of the same rank.

Now, let $\mathbf{D}$ be a (φ_q, Γ_K)-module over $E \otimes_F \mathbf{B}^\dagger_{\mathrm{rig},K}$, such that $M = (E \otimes_F \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K})^{\mathrm{pa}} \otimes_{E\otimes_F \mathbf{B}^\dagger_{\mathrm{rig},K}} \mathbf{D}$ is such that for all $\tau \in \Sigma_0$, $\nabla_\tau(M) \subset t_\tau \cdot M$. We then have $\mathbf{D} \subset \mathrm{Sol}(M)$ so that $\mathbf{D}$ is F-analytic by the above. If $\mathbf{D}$ is an F-analytic (φ_q, Γ_K)-module over $E \otimes_F \mathbf{B}^\dagger_{\mathrm{rig},K}$, then we have $\nabla_\tau(x) = 0$ for all $x \in \mathbf{D}$ by remark 2.4 and so $\nabla_\tau(M) \subset t_\tau \cdot M$ for $M = (E \otimes_F \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K})^{\mathrm{pa}} \otimes_{E\otimes_F \mathbf{B}^\dagger_{\mathrm{rig},K}} \mathbf{D}$ by lemma 3.3.

We have

$$M = (E \otimes_F \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig},K})^{\mathrm{pa}} \otimes_{E \otimes_F \mathbf{B}^{\dagger}_{\mathrm{rig},K}} \mathbf{D} = (E \otimes_F \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig},K})^{\mathrm{pa}} \otimes_{E \otimes_F \mathbf{B}^{\dagger}_{\mathrm{rig},K}} \mathbf{D}^{\dagger}_{\mathrm{rig}},$$

and by taking the F-analytic elements, since both $\mathbf{D}$ and $\mathbf{D}^{\dagger}_{\mathrm{rig}}$ are F-analytic, we get that

$$M^{F-\mathrm{pa}} = (E \otimes_F \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig},K})^{F-\mathrm{pa}} \otimes_{E \otimes_F \mathbf{B}^{\dagger,F-\mathrm{pa}}_{\mathrm{rig},K}} \mathbf{D} = (E \otimes_F \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig},K})^{F-\mathrm{pa}} \otimes_{E \otimes_F \mathbf{B}^{\dagger,F-\mathrm{pa}}_{\mathrm{rig},K}} \mathbf{D}^{\dagger}_{\mathrm{rig}}.$$

As above, if X denotes the base change matrix between $\mathbf{D}$ and $\mathbf{D}^{\dagger}_{\mathrm{rig}}$, we obtain that $X \in \mathrm{GL}_d(E \otimes_F \mathbf{B}^{\dagger}_{\mathrm{rig},K})$ so that $\mathbf{D} = \mathbf{D}^{\dagger}_{\mathrm{rig}}$. □

4. (B, E)-pairs

Let $\mathbf{B}^+_{\mathrm{dR}}, \mathbf{B}_{\mathrm{dR}}, \mathbf{B}^+_{\mathrm{cris}}$ and $\mathbf{B}_{\mathrm{cris}}$ be the usual Fontaine's rings of p-adic periods, defined for example in [**14**]. These rings come equipped with an action of $\mathcal{G}_{\mathbf{Q}_p}$, and the rings $\mathbf{B}^+_{\mathrm{cris}}$ and $\mathbf{B}_{\mathrm{cris}}$ are endowed with an injective Frobenius φ. We let $\mathbf{B}_e = (\mathbf{B}_{\mathrm{cris}})^{\varphi=1}$. Berger defined in [**1**] the notion of B-pairs, that is pairs $W = (W_e, W^+_{dR})$, where W_e is a free $\mathbf{B}_e$-module of finite rank, equipped with a semilinear continuous action of $\mathcal{G}_K$ and where W^+_{dR} is a $\mathcal{G}_K$-stable $\mathbf{B}^+_{\mathrm{dR}}$-lattice inside $W_{dR} = \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} W_e$. To a p-adic representation V, one can attach the B-pair $W(V) = (\mathbf{B}_e \otimes_{\mathbf{Q}_p} V, \mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)$, and the functor $V \mapsto W(V)$ is fully faithful since $\mathbf{B}_e \cap \mathbf{B}^+_{\mathrm{dR}} = \mathbf{Q}_p$. Recall that t is the usual t in p-adic Hodge theory (note that t corresponds to the element t_p for $F = \mathbf{Q}_p$) and that $\mathbf{B}^+_{\mathrm{dR}}/t\mathbf{B}^+_{\mathrm{dR}} = \mathbf{C}_p$.

Berger showed [**1**, Thm. 2.2.7] how to attach to any B-pair a cyclotomic (φ, Γ)-module $D(W)$ on the (cyclotomic) Robba ring, and that this functor induces an equivalence of categories.

Let E be a field of coefficients as previously. Let $\mathbf{B}_{e,E} = E \otimes_{\mathbf{Q}_p} \mathbf{B}_e$, $\mathbf{B}^+_{\mathrm{dR},E} = E \otimes_{\mathbf{Q}_p} \mathbf{B}^+_{\mathrm{dR}}$ and $\mathbf{B}_{\mathrm{dR},E} = E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}$, where $\mathcal{G}_{\mathbf{Q}_p}$ acts E-linearly on E. A (B, E)-pair is a pair $W = (W_e, W^+_{dR})$, where W_e is a free $\mathbf{B}_{e,E}$-module of finite rank, equipped with a semilinear continuous action of $\mathcal{G}_K$ and where W^+_{dR} is a $\mathcal{G}_K$-stable $\mathbf{B}^+_{\mathrm{dR},E}$-lattice inside $W_{dR} = \mathbf{B}_{\mathrm{dR},E} \otimes_{\mathbf{B}_{e,E}} W_e$. To an E representation V, one can attach the (B, E)-pair $W(V) = (\mathbf{B}_e \otimes_{\mathbf{Q}_p} V, \mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)$, and this functor is once again fully faithful. Theorem 2.2.7 of [**1**] has been extended by Nakamura [**18**, Thm. 1.36] for (B, E)-pairs and cyclotomic E-(φ, Γ)-modules, that is (φ, Γ)-modules over the cyclotomic Robba ring tensored by E over $\mathbf{Q}_p$.

Let F, E be as in §1. Note that we have an isomorphism $E \otimes_{\mathbf{Q}_p} F \simeq \prod_{\tau \in \Sigma} E$, given by $a \otimes b \mapsto (a\tau(b))_{\tau \in \Sigma}$. Since $F \subset \mathbf{B}^+_{\mathrm{dR}}$, we have natural isomorphisms

$$E \otimes_{\mathbf{Q}_p} \mathbf{B}^+_{\mathrm{dR}} \simeq (E \otimes_{\mathbf{Q}_p} F) \otimes_F \mathbf{B}^+_{\mathrm{dR}} \simeq (\prod_{\tau \in \Sigma} E) \otimes_F \mathbf{B}^+_{\mathrm{dR}} \simeq \prod_{\tau \in \Sigma} \mathbf{B}^+_{\mathrm{dR},\tau}$$

where $\mathbf{B}^+_{\mathrm{dR},\tau} = E \otimes^{\tau}_F \mathbf{B}^+_{\mathrm{dR}}$, and

$$E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}} \simeq \prod_{\tau \in \Sigma} \mathbf{B}_{\mathrm{dR},\tau}$$

where $\mathbf{B}_{\mathrm{dR},\tau} = E \otimes^{\tau}_F \mathbf{B}_{\mathrm{dR}}$.

We thus get decompositions $W^+_{dR} \simeq \prod_{\tau \in \Sigma} W^+_{dR,\tau}$ and $W_{dR} \simeq \prod_{\tau \in \Sigma} W_{dR,\tau}$.

We say that a (B,E)-pair is F-analytic if for all $\tau \in \Sigma_0$, $W^+_{dR,\tau}/tW^+_{dR,\tau}$ is the trivial $\mathbf{C}_p$-semilinear representation of $\mathcal{G}_K$. The following lemma shows that this definition is compatible with the one of F-analytic representation:

Lemma 4.1. — *Let V be an E-representation of $\mathcal{G}_K$. Then V is F-analytic if and only if the (B,E)-pair $W(V) = (W_e, W^+_{\mathrm{dR}}) = (\mathbf{B}_e \otimes_{\mathbf{Q}_p} V, \mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)$ is F-analytic.*

Proof. — We have $\mathbf{B}^+_{\mathrm{dR}}/t\mathbf{B}^+_{\mathrm{dR}} = \mathbf{C}_p$, so that $W^+_{dR}/tW^+_{dR} = \mathbf{C}_p \otimes_{\mathbf{Q}_p} V \simeq \prod_{\tau\in\Sigma}(\mathbf{C}_p \otimes^\tau_F V)$, and $W^+_{dR,\tau}/tW^+_{dR,\tau} = \mathbf{C}_p \otimes^\tau_F V$, and so the equivalence is clear. □

Lemma 4.2. — *We have $\mathbf{B}_{e,E} = E \otimes_F (\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t])^{\varphi_q=1}$.*

Proof. — First, recall that $\mathbf{B}_e = (\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},\eta}[1/t])^{\varphi=1}$ (this is [**1**, Lemm. 1.1.7]). Since φ_q is F-linear, we have $(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t])^{\varphi_q=1} = (F \otimes_{F_0} \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},\eta}[1/t])^{\varphi_q=1} = F \otimes_{F_0} (\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},\eta}[1/t])^{\varphi^h=1}$. Now since $\mathrm{Gal}(F_0/\mathbf{Q}_p)$ acts F_0-semi-linearly on $(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},\eta}[1/t])^{\varphi^h=1}$ by φ, Speiser's lemma implies that $(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},\eta}[1/t])^{\varphi^h=1} = F_0 \otimes_{\mathbf{Q}_p} \mathbf{B}_e$. Thus, we get that

$$\mathbf{B}_{e,E} = E \otimes_{\mathbf{Q}_p} \mathbf{B}_e = E \otimes_F F \otimes_{F_0} (F_0 \otimes \mathbf{B}_e)$$

and what we just did implies that

$$\mathbf{B}_{e,E} = E \otimes_F (\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t])^{\varphi_q=1}.$$

□

Lemma 4.3. —

1. *The t-adic valuation of the τ'-component of the image of t_τ by the map $\widetilde{\mathbf{B}}^+_{\mathrm{rig}} \to F \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}} = \prod_{\tau'\in\Sigma} \mathbf{B}_{\mathrm{dR}}$ given by $x \mapsto \{(\tau' \otimes \varphi^{n(\tau')})(x)\}_{\tau'\in\Sigma}$ is 1 if $\tau' = \tau^{-1}$ and 0 otherwise.*
2. *There exists $u \in (F \otimes \widehat{\mathbf{Q}_p}^{\mathrm{unr}})^\times$ such that $\prod_{\tau\in\Sigma} t_\tau = u \cdot t$ in $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}$.*

Proof. — These are items 2 and 3 of [**5**, Prop. 2.4], using $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ instead of $F \otimes_{F_0} \mathbf{B}^+_{\mathrm{cris}}$. □

Lemma 4.2 allows us to see $E \otimes_F \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t]$ as a $\mathbf{B}_{e,E}$-module.

Let $\Omega = \{(\tau,n) \in \mathrm{Gal}(E/\mathbf{Q}_p) \times \mathbf{Z}$ such that $n(\tau|_F) \equiv n \bmod h\}$. For $n \geq 0$, let $r_n = p^{n-1}(p-1)$, and for $r > 0$, let $n(r)$ be the least integer n such that $r_n \geq r$. For $r \geq 0$, we let $\Omega_r = \{(\tau,n) \in \Omega$ such that $n \geq n(r)\}$. For $g = (\tau,n) \in \Omega$, we let $\tau(g) = \tau$ and $n(g) = n$. If $\min(I) \geq r$ and if $g \in \Omega_r$, we have a map $\iota_g : E \otimes_F \widetilde{\mathbf{B}}^I \to E \otimes_F^{\tau(g)_{|F}} \mathbf{B}^+_{\mathrm{dR}} = \mathbf{B}_{\mathrm{dR},\tau(g)_{|F}}$, defined in [**2**, §5] and given by $x \mapsto (g^{-1} \otimes (g|_F^{-1} \otimes \varphi^{-n(g)}))(x)$.

Lemma 4.4. — *Let W be a (B,E)-pair of rank d, and let*

$$\widetilde{D}^r(W) = \left\{y \in (E \otimes_F \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}[1/t]) \otimes_{\mathbf{B}_{e,E}} W_e \text{ such that } \iota_g(y) \in W^+_{dR,\tau(g)_{|F}} \text{ for all } g \in \Omega_r\right\}.$$

Then:

1. *$\widetilde{D}^r(W)$ is a free $E \otimes_F \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$-module of rank d;*
2. *$\widetilde{D}^r(W)[1/t] = (E \otimes_F \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}[1/t]) \otimes_{\mathbf{B}_{e,E}} W_e$.*

Proof. — This is [**1**, Lemm. 2.2.1] tensored by E. □

If W is a (B,E)-pair, we let $\widetilde{D}(W) = (E \otimes_F \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}) \otimes_{E\otimes_F \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}} \widetilde{D}^r(W)$, and if I is a subinterval of $[r;+\infty[$, we let $\widetilde{D}^I(W) = (E \otimes_F \widetilde{\mathbf{B}}^I) \otimes_{E\otimes_F \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}} \widetilde{D}^r(W)$. By the same argument as in [**1**, Lemm. 2.2.2], this does not depend on the choice of $r \in I$.

Proposition 4.5. — *If W is a (B,E)-pair of rank d, then there exists a unique (φ_q, Γ'_K)-module $\mathbf{D}_\eta(W)$ over $E \otimes_{F_0} \mathbf{B}^{\dagger}_{\mathrm{rig},K,\eta}$ such that $(E \otimes_{F_0} \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}) \otimes_{E\otimes_{F_0}\mathbf{B}^{\dagger}_{\mathrm{rig},K,\eta}} \mathbf{D}_\eta(W) = \widetilde{D}(W)$.*

Proof. — This is [**1**, Prop. 2.2.5] up to a tensor product, and using the twisted cyclotomic case instead of the classical one, but again by using [**2**, §8], it does not change the arguments of the proof. □

For $r \geq 0$ such that $\mathbf{D}_\eta(W)$ and all its structures are defined over $E \otimes_{F_0} \mathbf{B}^{\dagger,r}_{\mathrm{rig},K,\eta}$, we let $\mathbf{D}^r_\eta(W)$ be the associated $(E \otimes_{F_0} \mathbf{B}^{\dagger,r}_{\mathrm{rig},K,\eta})$-module so that $\mathbf{D}_\eta(W) = (E \otimes_F \mathbf{B}^{\dagger}_{\mathrm{rig},K,\eta}) \otimes_{E\otimes_{F_0}\mathbf{B}^{\dagger,r}_{\mathrm{rig},K,\eta}} \mathbf{D}^r_\eta(W)$. For $I = [r;s]$, we let $\mathbf{D}^I_\eta = (E \otimes_{F_0} \mathbf{B}^I_{K,\eta}) \otimes_{E\otimes_{F_0}\mathbf{B}^{\dagger,r}_{\mathrm{rig},K,\eta}} \mathbf{D}^r_\eta(W)$. Let $\widetilde{D}^I_K(W) = (\widetilde{D}^I(W))^{H_K}$ and $\widetilde{D}_K(W) = \widetilde{D}(W)^{H_K}$, so that $\widetilde{D}^I_K(W) = (E \otimes_F \widetilde{\mathbf{B}}^I_K) \otimes_{E\otimes_{F_0}\mathbf{B}^I_{K,\eta}} \mathbf{D}^I_\eta(W)$ and $\widetilde{D}_K(W) = (E \otimes_F \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig},K}) \otimes_{E\otimes_{F_0}\mathbf{B}^{\dagger}_{\mathrm{rig},K,\eta}} \mathbf{D}_\eta(W)$ (since $\mathbf{D}_\eta(W)$ is invariant under H_K).

Proposition 4.6. — *We have*

1. *$\widetilde{D}^I_K(W)^{\mathrm{la}} = (E \otimes_F \widetilde{\mathbf{B}}^I_K)^{\mathrm{la}} \otimes_{E\otimes_F \mathbf{B}^I_{K,\eta}} \mathbf{D}^I_\eta(W)$;*
2. *$\widetilde{D}_K(W)^{\mathrm{pa}} = (E \otimes_F \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig},K})^{\mathrm{pa}} \otimes_{E\otimes_F \mathbf{B}^{\dagger}_{\mathrm{rig},K,\eta}} \mathbf{D}_\eta(W)$.*

Proof. — The same proof as [**16**, §2.1] shows that the elements of $\mathbf{D}^I_\eta(W)$ are locally analytic vectors, and the result now follows from proposition 2.2. □

Theorem 4.7. — *If W is an F-analytic (B,E)-pair of rank d, then there exists a unique F-analytic (φ_q, Γ_K)-module $D(W)$ over $E \otimes_F \mathbf{B}^{\dagger}_{\mathrm{rig},K}$ such that*

$$(E \otimes_F \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}) \otimes_{E\otimes_F \mathbf{B}^{\dagger}_{\mathrm{rig},K}} D(W) = \widetilde{D}(W).$$

Proof. — Let W be an F-analytic (B,E)-pair of rank d, and let $\widetilde{D}_K(W)$ be as above. Let $r \geq 0$ and let $y \in (\widetilde{D}^r_K(W))^{\mathrm{pa}}$. Let $\tau \in \Sigma \setminus \{\mathrm{id}\}$ and let

$$\Omega_{\tau,r} = \{g \in \Omega \text{ such that } n(g) \geq n(r) \text{ and } \tau(g) = \tau\}.$$

Let $g \in \Omega_{\tau,r}$. We have $\iota_g(y) \in W^+_{dR,\tau}$. Write x_g for the image of $\iota_g(y)$ in $W^+_{dR,\tau}/tW^+_{dR,\tau}$. Since the filtration on $W_{dR,\tau}$ is Galois stable, we get that x_g is invariant under H_K (since $\iota_g(y)$ is), and is a locally analytic vector of $(W^+_{dR,\tau}/tW^+_{dR,\tau})^{H_K}$ using the fact that $y \in (\widetilde{D}_K(W)^r)^{\mathrm{pa}}$. Note that $\nabla_{\mathrm{id}} = 0$ on $((W^+_{dR,\tau}/tW^+_{dR,\tau})^{H_K})^{\mathrm{la}}$ since W is F-analytic and by [**2**, Prop. 2.10]. This shows that $\nabla_{\mathrm{id}}(x_g) = 0$ and so $\nabla_{\mathrm{id}}(\iota_g(y)) = 0 \mod t_\pi$ (recall that t and t_π both generate the kernel of θ in $\mathbf{B}^+_{\mathrm{dR}}$ by lemma 4.3). Using the fact that $\iota_g \circ \nabla_\tau = \nabla_{\mathrm{id}} \circ \iota_g$, this implies that $t_\pi | \iota_g \circ \nabla_\tau(y)$ in $W^+_{dR,\tau}$. By lemma 4.3, this proves that $\iota_g((Q^\tau_n)^{-1} \cdot \nabla_\tau(y)) \in W^+_{dR,\tau}$ for $n = n(g)$. By definition of $\widetilde{D}^r(W)$, this proves that $\nabla_\tau(y) \in Q^\tau_n \cdot \widetilde{D}^r(W)$ for all $n \geq n(r)$, and so ∇_τ is divisible by $\prod_{n=n(r)}^{+\infty} Q^\tau_n$ in $\widetilde{D}^r(W)$ (the argument for the divisibility by an infinite product is the same as the one given in the proof of [**2**, Lemm. 10.2]), hence by t_τ.

In particular, for all $\tau \in \Sigma_0$, we have $\nabla_\tau(\widetilde{D}^r(W)^{\mathrm{pa}}) \subset t_\tau \cdot \widetilde{D}^r(W)^{\mathrm{pa}}$. By proposition 3.4, there exists a unique (φ_q,Γ_K)-module $\mathbf{D}^\dagger_{\mathrm{rig}}$ over $E \otimes_F \mathbf{B}^\dagger_{\mathrm{rig},K}$ such that $(E \otimes_F \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}) \otimes_{E\otimes_F \mathbf{B}^\dagger_{\mathrm{rig},K}} \mathbf{D}^\dagger_{\mathrm{rig}} = \widetilde{D}(W)$, which is what we wanted. □

Proposition 4.8. — *If $\mathbf{D}$ is a φ_q-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$, then there exists $r(\mathbf{D}) \geq r(K)$ such that, for all $r \geq r(\mathbf{D})$, there exists a unique sub $\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$-module $\mathbf{D}_r$ of $\mathbf{D}$ such that:*

1. *$\mathbf{D} = \mathbf{B}^\dagger_{\mathrm{rig},K} \otimes_{\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}} \mathbf{D}_r$;*
2. *the $\mathbf{B}^{\dagger,qr}_{\mathrm{rig},K}$-module $\mathbf{B}^{\dagger,qr}_{\mathrm{rig},K} \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K},r} \mathbf{D}_r$ has a basis contained inside $\varphi_q(\mathbf{D})$. Moreover, if $\mathbf{D}$ is a (φ_q,Γ_K)-module, one has $g(\mathbf{D}_r) = \mathbf{D}_r$ for all $g \in \Gamma_K$.*

Proof. — This is exactly the same proof as [**1**, Thm. I.3.3] but using Lubin-Tate (φ_q,Γ_K)-modules instead of cyclotomic ones, and tensoring by E over F. □

Proposition 4.9. — *If $\mathbf{D}$ is a (φ_q,Γ_K)-module over $E \otimes_F \mathbf{B}^\dagger_{\mathrm{rig},K}$, free of rank d, then*

1. *$W_e(\mathbf{D}) = (E \otimes_F \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K}[1/t] \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} \mathbf{D})^{\varphi_q=1}$ is a free $\mathbf{B}_{e,E}$-module of rank d which is $\mathcal{G}_K$-stable;*
2. *$W^+_{dR} = \prod_{\tau\in\Sigma} \left((E \otimes_F \mathbf{B}^+_{\mathrm{dR}}) \otimes^{\iota_g}_{E\otimes_F \mathbf{B}^{\dagger,r_{n(g)}}_{\mathrm{rig},K}} \mathbf{D}^{r_{n(g)}}\right)_{g\in\Omega_{r,\tau}}$ does not depend on $n(g) \gg 0$ and is a free $E \otimes_{\mathbf{Q}_p} \mathbf{B}^+_{\mathrm{dR}} = (\mathbf{B}^+_{\mathrm{dR}\tau})_{\tau\in\Sigma}$-module of rank d and $\mathcal{G}_K$-stable.*
3. *$W(\mathbf{D}) = (W_e(\mathbf{D}), W^+_{dR}(\mathbf{D}))$ is a (B,E)-pair. Moreover, if $\mathbf{D}$ is F-analytic, then so is $W(D)$.*

Proof. — The proof of items 1 and 2 is the same as [**1**, Prop. 2.2.6]. Assume now that $\mathbf{D}$ is F-analytic, and let us prove that $W(\mathbf{D})$ is F-analytic. Let $\tau \in \Sigma \setminus \{\mathrm{id}\}$.

By item 2, we have $W^+_{dR,\tau} = (E \otimes_F \mathbf{B}^+_{\mathrm{dR}}) \otimes^{\iota_g}_{E\otimes_F \mathbf{B}^{\dagger,r_{n(g)}}_{\mathrm{rig},K}} \mathbf{D}^{r_{n(g)}}$ for some $g \in \Omega_{r,\tau}$. We can find a basis $e_1,\ldots,e_d$ of $\mathbf{D}^{r_{n(g)}}$ over $E \otimes_F \mathbf{B}^{\dagger,r_{n(g)}}_{\mathrm{rig},K}$ such that the image of the basis $\iota_g(e_1),\ldots,\iota_g(e_d)$ of $W^+_{dR,\tau}$ over $E\otimes_F \mathbf{B}^+_{\mathrm{dR}}$ modulo t_π is a basis of the $E\otimes_F \mathbf{C}_p$-representation $W^+_{dR,\tau}/tW^+_{dR,\tau}$.

Since the e_i are pro-analytic vectors of $\mathbf{D}^{r_{n(g)}}$ for the action of Γ_K, the same argument as in the proof of theorem 4.7 shows that their image in $W^+_{dR,\tau}/tW^+_{dR,\tau}$ are invariant under H_K and locally analytic vectors of $(W^+_{dR,\tau}/tW^+_{dR,\tau})^{H_K}$. Since

$$\nabla_\tau\left((E \otimes_F \widetilde{\mathbf{B}}^{\dagger,r_{n(g)}}_{\mathrm{rig},K})^{\mathrm{pa}} \otimes_{E\otimes_F \mathbf{B}^{\dagger,r_{n(g)}}_{\mathrm{rig},K}} \mathbf{D}^{r_{n(g)}}\right) \subset t_\tau \cdot \left((E \otimes_F \widetilde{\mathbf{B}}^{\dagger,r_{n(g)}}_{\mathrm{rig},K})^{\mathrm{pa}} \otimes_{E\otimes_F \mathbf{B}^{\dagger,r_{n(g)}}_{\mathrm{rig},K}} \mathbf{D}^{r_{n(g)}}\right)$$

by lemma 2.4 and since

$$W^+_{dR,\tau} = (E \otimes_F \mathbf{B}^+_{\mathrm{dR}}) \otimes^{\iota_g}_{E\otimes_F \mathbf{B}^{\dagger,r_{n(g)}}_{\mathrm{rig},K}} ((E \otimes_F \widetilde{\mathbf{B}}^{\dagger,r_{n(g)}}_{\mathrm{rig},K})^{\mathrm{pa}} \otimes_{E\otimes_F \mathbf{B}^{\dagger,r_{n(g)}}_{\mathrm{rig},K}} \mathbf{D}^{r_{n(g)}})$$

we get that $\nabla_{\mathrm{id}}(e_i) = 0 \bmod t_\pi$ for all i since $\iota_g \circ \nabla_\tau = \nabla_{\mathrm{id}} \circ \iota_g$ and since $\iota_g(t_\tau) = t_\pi$.

This implies that $\nabla_{\mathrm{id}} = 0$ on $(W^+_{dR,\tau}/tW^+_{dR,\tau})^{H_K,\mathrm{la}}$ so that $(W^+_{dR,\tau}/tW^+_{dR,\tau})$ is $\mathbf{C}_p$-admissible as an $E \otimes_F \mathbf{C}_p$ representation of $\mathcal{G}_K$, using the discussion following [**4**, Thm. 4.11]. □

Theorem 4.10. — *The two functors $W \mapsto D(W)$ and $\mathbf{D} \mapsto W(\mathbf{D})$ are inverse one to another and induce an equivalence of categories between the category of F-analytic (B,E)-pairs and the category of F-analytic (φ_q,Γ_K)-modules.*

Proof. — Let $W = (W_e, W^+_{dR})$ be an F-analytic (B,E)-pair and let $\mathbf{D} = D(W)$. By definition of $W(\mathbf{D})$, we have

$$(E \otimes_F \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t]) \otimes_{\mathbf{B}_{e,E}} W_e(\mathbf{D}) = (E \otimes_F \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t]) \otimes_{E\otimes_F \mathbf{B}^\dagger_{\mathrm{rig},K}} \mathbf{D}$$

and by definition of $D(W)$, we have

$$(E \otimes_F \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t]) \otimes_{\mathbf{B}_{e,E}} W_e = (E \otimes_F \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t]) \otimes_{E\otimes_F \mathbf{B}^\dagger_{\mathrm{rig},K}} \mathbf{D}$$

so that, taking the invariants by φ_q, we get that $W_e \simeq W(\mathbf{D})$ as $\mathbf{B}_{e,E}$-representations.

Let $\tau \in \Sigma$. By definition of $W^+_{dR,\tau}(\mathbf{D})$, we have $W^+_{dR,\tau}(\mathbf{D}) = (E \otimes_F \mathbf{B}^+_{\mathrm{dR}}) \otimes^{\iota_g} \mathbf{D}^{r_{n(g)}}$ for some $g \in \Omega_{r,\tau}$ with r big enough, and hence

$$W^+_{dR,\tau}(\mathbf{D}) = (E \otimes_F \mathbf{B}^+_{\mathrm{dR}}) \otimes^{\iota_g} \widetilde{D}^{r_{n(g)}}$$

where $\widetilde{D}^r = \widetilde{D}^r(W) = (E \otimes_F \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}) \otimes_{E\otimes_F \mathbf{B}^{\dagger,r}_{\mathrm{rig},K}} \mathbf{D}^r$ by proposition 4.5. Recall that

$$\widetilde{D}^r(W) = \left\{ y \in (E \otimes_F \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}[1/t]) \otimes_{\mathbf{B}_{e,E}} W_e \text{ such that } \iota_g(y) \in W^+_{dR,\tau(g)_{|F}} \text{ for all } g \in \Omega_r \right\},$$

so that, after tensoring by $E \otimes_F \mathbf{B}^+_{\mathrm{dR}}$ over ι_g, we get $W^+_{dR,\tau}(\mathbf{D}(W)) = W^+_{dR,\tau}$.

Let $\mathbf{D}$ be an F-analytic (φ_q,Γ_K)-module and let $W = W(\mathbf{D})$ and $\widetilde{D} = (E \otimes_F \mathbf{B}^\dagger_{\mathrm{rig}}) \otimes_{E\otimes_F \mathbf{B}^\dagger_{\mathrm{rig},K}} \mathbf{D}$. The same reasoning as above shows that

$$(E \otimes_F \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t]) \otimes_{E\otimes_F \mathbf{B}^\dagger_{\mathrm{rig},K}} \mathbf{D} = (E \otimes_F \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t]) \otimes_{E\otimes_F \mathbf{B}^\dagger_{\mathrm{rig},K}} \mathbf{D}(W(\mathbf{D}))$$

and that

$$(E \otimes_F \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t]) \otimes_{E\otimes_F \mathbf{B}^\dagger_{\mathrm{rig},K}} \widetilde{D} = (E \otimes_F \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t]) \otimes_{E\otimes_F \mathbf{B}^\dagger_{\mathrm{rig},K}} \widetilde{D}(W(\mathbf{D})).$$

If M is a (φ_q,Γ_K)-module over $E \otimes_F \mathbf{B}^\dagger_{\mathrm{rig}}$, note that we can recover M inside $M[1/t]$ by

$$M = \left\{ x \in M[1/t] \text{ such that } \iota_g(x) \in (E \otimes_F \mathbf{B}^+_{\mathrm{dR}}) \otimes^{\iota_g}_{E\otimes_F \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}} M \text{ for all } g \text{ with } n(g) \gg 0 \right\}.$$

In particular, since

$$(E \otimes_F \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t]) \otimes_{E\otimes_F \mathbf{B}^\dagger_{\mathrm{rig},K}} \widetilde{D} = (E \otimes_F \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t]) \otimes_{E\otimes_F \mathbf{B}^\dagger_{\mathrm{rig},K}} \widetilde{D}(W(\mathbf{D})),$$

this shows that

$$\widetilde{D} = \widetilde{D}(W(\mathbf{D})).$$

Since $\mathbf{D}$ is F-analytic, we have $\nabla_\tau((\widetilde{D}_K)^{\mathrm{pa}}) \subset t_\tau \cdot (\widetilde{D}_K)^{\mathrm{pa}})$ for all $\tau \in \Sigma \backslash \{\mathrm{id}\}$ by proposition 3.4, hence there exists, still by proposition 3.4, a unique F-analytic (φ_q, Γ_K)-module $\mathbf{D}^\dagger_{\mathrm{rig}}$ over $E \otimes_F \mathbf{B}^\dagger_{\mathrm{rig},K}$ such that

$$\mathrm{Sol}(\widetilde{D}_K^{\mathrm{pa}}) = (E \otimes_F (\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K})^{F\text{-pa}}) \otimes_{E \otimes_F \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K}} \mathbf{D}^\dagger_{\mathrm{rig}}$$

and such that

$$\widetilde{D} = (E \otimes_F \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K}) \otimes_{E \otimes_F \mathbf{B}^\dagger_{\mathrm{rig},K}} \mathbf{D}^\dagger_{\mathrm{rig}}$$

In particular, we have $\mathbf{D} = \mathbf{D}(W(\mathbf{D})) = \mathbf{D}^\dagger_{\mathrm{rig}}$, which concludes the proof. □

We now explain how to use this result to generalize Porat's result [**19**, Thm. 6.8]. Recall that an E-representation V is said to be split trianguline if its corresponding cyclotomic (φ, Γ)-module $\mathbf{D}^\dagger_{\mathrm{cycl}}(V)$ over the Robba ring $E \otimes \mathbf{B}^\dagger_{\mathrm{rig},K,\eta}$ (here we take η to be the trivial unramified character of $\mathcal{G}_K$) is a successive extension of (φ, Γ)-modules of rank 1. Note that this is the same as asking that $D = \mathbf{D}^\dagger_{\mathrm{cycl}}(V)$ is equipped with a strictly increasing filtration $\mathrm{Fil}_0(D) = \{0\} \subset \mathrm{Fil}_1(D) \subset \cdots \subset \mathrm{Fil}_d(D) = D$ of cyclotomic (φ, Γ)-modules over $E \otimes \mathbf{B}^\dagger_{\mathrm{rig},K,\eta}$ which are direct summands of D as $E \otimes \mathbf{B}^\dagger_{\mathrm{rig},K,\eta}$-modules, where $d = \dim_E(V)$.

Recall (see the beginning of §3 of [**3**]) that it is equivalent to ask the (B, E)-pair $W(V)$ attached to V to be a successive extension of (B, E)-pairs of rank 1.

An E-representation V is said to be trianguline if there exists an extension E' of E such that $E' \otimes_E V$ is split trianguline.

An F-analytic E-representation V of $\mathcal{G}_K$ is said to be split Lubin-Tate trianguline if its (φ_q, Γ_K)-module over $E \otimes \mathbf{B}^\dagger_{\mathrm{rig},K}$ is a successive extension of (φ_q, Γ_K)-modules of rank 1, and to be Lubin-Tate trianguline if there exists E'/E a finite extension such that $E' \otimes_E V$ is Lubin-Tate trianguline.

Theorem 4.11. — *Let V be an F-analytic representation of $\mathcal{G}_K$. Then V is trianguline in the cyclotomic sense if and only if it is Lubin-Tate trianguline.*

Proof. — First note that it suffices to prove the result for split trianguline representations. Now let V be an F-analytic representation of $\mathcal{G}_K$. Assume that it is trianguline in the cyclotomic sense. Then its corresponding (B, E)-pair $W(V)$ is a successive extension of (B, E)-pairs of rank 1. There exists therefore a triangulation of the (B, E)-pair $W(V)$, that is a filtration

$$0 = W_0 \subset W_1 \subset \cdot \subset W_d = W(V)$$

by sub-(B, E)-pairs such that W_i is saturated in W_{i+1} and the quotient W_{i+1}/W_i is a rank 1 (B, E)-pair.

Since V is F-analytic, so is $W(V)$ by lemma 4.1, and thus so are the W_i. By theorem 4.10, for any i, $D_i := D(W_i)$ is an F-analytic Lubin-Tate (φ_q, Γ_K)-module over $E \otimes \mathbf{B}^\dagger_{\mathrm{rig},K}$, and we have

$$0 = D_0 \subset D_1 \subset \cdot \subset D_d = D(W(V)) = \mathbf{D}^\dagger_{\mathrm{rig}}(V).$$

Moreover, because W_i is saturated in W_{i+1} and the quotient W_{i+1}/W_i is a rank 1 F-analytic (B, E)-pair, we get that D_i is saturated in D_{i+1} and that the quotient is a rank

1 F-analytic Lubin-Tate (φ_q, Γ_K)-module, so that V is split trianguline in the Lubin-Tate sense.

For the converse, assume that $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ is a successive extension of rank 1 F-analytic Lubin-Tate (φ_q, Γ_K)-modules. Then we have a triangulation

$$0 = D_0 \subset D_1 \subset \cdot \subset D_d = D(W(V)) = \mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$$

where D_i is saturated in D_{i+1} and the quotient is a rank 1 F-analytic Lubin-Tate (φ_q, Γ_K)-module. By theorem 4.10, if $W_i = W(D_i)$ then

$$0 = W_0 \subset W_1 \subset \cdot \subset W_d = W(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) = W(V)$$

is a triangulation of $W(V)$ such that W_i is saturated in W_{i+1} and the quotient W_{i+1}/W_i is a rank 1 (B, E)-pair and thus V is split trianguline in the usual sense. □

5. A simpler equivalence in the F-analytic case

Let $\mathbf{B}^{\mathrm{LT}}_{e,F} = (\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}[1/t_\pi])^{\varphi_q=1} = (\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}[1/t_\pi])^{\varphi_q=1}$. Following [**11**], we make the following definition:

Definition 5.1. — 1. Let $\sigma \in \Sigma_E$ be any embedding. A B_σ-pair is the data of a couple $W_\sigma = (W^{\mathrm{LT}}_{\sigma,E}, W^{+}_{\mathrm{dR},\sigma})$ where $W^{\mathrm{LT}}_{\sigma,E}$ is a finite free $E \otimes^{\sigma}_{F} \mathbf{B}^{\mathrm{LT}}_{e,F}$-module equipped with a semi-linear $\mathcal{G}_K$ action and $W^{+}_{\mathrm{dR},\sigma}$ is a $\mathcal{G}_K$-invariant $\mathbf{B}^{+}_{\mathrm{dR},\sigma}$-lattice in $W_{\mathrm{dR},\sigma} := W^{\mathrm{LT}}_{\sigma,E} \otimes_{E\otimes^{\sigma}_{F}\mathbf{B}^{\mathrm{LT}}_{e,F}} \mathbf{B}_{\mathrm{dR},\sigma}$.

2. For two B_σ-pairs W_σ, W'_σ, a morphism $f : W_\sigma \longrightarrow W'_\sigma$ is a $\mathcal{G}_K$-invariant $E \otimes^{\sigma}_{F} \mathbf{B}^{\mathrm{LT}}_{e,F}$-linear map $f^{\mathrm{LT}}_{\sigma,E} : W^{\mathrm{LT}}_{\sigma,E} \longrightarrow (W'_{\sigma,E})^{\mathrm{LT}}$ such that the induced $\mathbf{B}_{\mathrm{dR},\sigma}$-linear map $f_{\mathrm{dR},\sigma} := f^{\mathrm{LT}}_{\sigma,E} \otimes \mathrm{id} : W_{\mathrm{dR},\sigma} \longrightarrow W'_{\mathrm{dR},\sigma}$ sends $W^{+}_{\mathrm{dR},\sigma}$ to $(W')^{+}_{\mathrm{dR},\sigma}$.

Let $W = (W_e, W^{+}_{\mathrm{dR}})$ be a (B, E)-pair. Let $W^{\mathrm{LT}}_{\sigma,E} =$

$$\left\{ w \in W_e : \tau(w) \in W^{+}_{\mathrm{dR},\sigma\circ\tau^{-1}} \text{ for all } \tau \in \mathrm{Gal}(E/\mathbf{Q}_p), \tau_{|F} \neq \mathrm{id} \right\}.$$

By [**11**, Lemm. 1.3], this is an $E \otimes^{\sigma}_{F} \mathbf{B}^{\mathrm{LT}}_{e,F}$-module. Proposition 3.7 of [**11**] shows that for $\sigma \in \Sigma_E$, the functor $F_\sigma : \{(B, E) - \text{pairs}\} \longrightarrow \{B_\sigma - \text{pairs}\}$ given by $W = (W_e, W^{+}_{\mathrm{dR}}) \mapsto W_\sigma = (W^{\mathrm{LT}}_{\sigma,E}, W^{+}_{\mathrm{dR},\sigma})$ induces an equivalence of categories.

For $\sigma \in \Sigma_E$, let G_σ denote the inverse functor of F_σ defined by Ding in [**11**, Lemm. 3.8]. We say that a B_{id}-pair W is F-analytic if for all $\sigma \in \Sigma_E$ such that $\sigma_{|F} \neq \mathrm{id}_F$, then $W^{+}_{\mathrm{dR},\sigma}/tW^{+}_{\mathrm{dR},\sigma}$ is the trivial $\mathbf{C}_p$-representation of $\mathcal{G}_K$, where $W^{+}_{\mathrm{dR},\sigma}$ is the second component of the B_σ-pair $F_\sigma \circ G_{\mathrm{id}}(W)$. By [**11**, Lemm. 3.9], this is the same as asking that the corresponding (B, E)-pair $G_{\mathrm{id}}(W)$ is F-analytic.

Proposition 5.2. — *If $\mathbf{D}$ is a (φ_q, Γ_K)-module over $E \otimes_F \mathbf{B}^{\dagger}_{\mathrm{rig},K}$, free of rank d, then*

1. $W^{\mathrm{LT}}_{\mathrm{id},E}(\mathbf{D}) = (E \otimes_F \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig},K}[1/t_\pi] \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},K}} \mathbf{D})^{\varphi_q=1}$ *is a free $E \otimes^{\sigma}_{F} \mathbf{B}^{\mathrm{LT}}_{e,F}$-module of rank d which is $\mathcal{G}_K$-stable;*
2. $W^{+}_{\mathrm{dR},\mathrm{id}} = \left((E \otimes_F \mathbf{B}^{+}_{\mathrm{dR}}) \otimes^{\iota_g}_{E\otimes_F \mathbf{B}^{\dagger,r_{n(g)}}_{\mathrm{rig},K}} \mathbf{D}^{r_{n(g)}} \right)_{g\in\Omega_{\mathrm{id},r}}$ *does not depend on $n(g) \gg 0$ and is a free $\mathbf{B}^{+}_{\mathrm{dR},\mathrm{id}}$-module of rank d which is $\mathcal{G}_K$-stable.*

3. $W(\mathbf{D})^{\mathrm{LT}} = (W_{\mathrm{id},E}^{\mathrm{LT}}(\mathbf{D}), W_{\mathrm{dR,id}}^{+}(\mathbf{D}))$ *is a* B_{id}*-pair. Moreover, if* $\mathbf{D}$ *is* F*-analytic, then so is* $W(\mathbf{D})$.

Proof. — The proof of items 1, 2 and 3 is the same as in 4.9. The part on F-analyticity now follows from the remark above and the fact that the B_{id}-pair $W(\mathbf{D})$ we just constructed is exactly $F_{\mathrm{id}}(W')$ where W' is the (B,E)-pair attached to $\mathbf{D}$ constructed in proposition 4.9. □

Lemma 5.3. — *Let* $\widetilde{D}^r(W)^{\mathrm{LT}} =$

$$\left\{y \in (E \otimes_F \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}[1/t_\pi]) \otimes_{E \otimes_F \mathbf{B}_{e,F}^{\mathrm{LT}}} W_{\mathrm{id},E}^{\mathrm{LT}}, \iota_g(y) \in W_{\mathrm{dR,id}}^{+}, g \in \Omega_r, \tau(g) = \mathrm{id}\right\}.$$

Then:

1. $\widetilde{D}^r(W)^{\mathrm{LT}}$ *is a free* $E \otimes_F \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}$*-module of rank* d*;*
2. $\widetilde{D}^r(W)^{\mathrm{LT}}[1/t_\pi] = E \otimes_F \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}[1/t_\pi] \otimes_{E \otimes_F \mathbf{B}_{e,F}^{\mathrm{LT}}} W_{\mathrm{id},E}^{\mathrm{LT}}$.

Proof. — This is the same proof as in lemma 4.4 but here we do not need to keep track of all the embeddings. □

We know that there are enough pro-analytic vectors inside $\widetilde{D}(W)^{\mathrm{LT}}$, just because we already know by the constructions of §4 that it contains the F-analytic (φ_q, Γ_K)-module $D(W')$ attached to $W' = G_{\mathrm{id}}(W)$ of theorem 4.7. We can now recover it by taking the pro-analytic vectors of $\widetilde{D}(W)^{\mathrm{LT}}$ and taking the module $\mathbf{D}_{\mathrm{rig}}^{\dagger}(\widetilde{D}(W)^{\mathrm{LT}})$ given by proposition 3.4. In particular, the following is a straightforward consequence of our previous constructions:

Theorem 5.4. — *The functors* $\mathbf{D} \mapsto W(\mathbf{D})^{\mathrm{LT}}$ *and* $W_{\mathrm{id}} \mapsto \mathbf{D}_{\mathrm{rig}}^{\dagger}(\widetilde{D}(W)^{\mathrm{LT}})$ *are inverse of each other an give rise to an equivalence of categories between the category of* F*-analytic* (φ_q, Γ_K)*-modules and the category of* F*-analytic* B_{id}*-pairs.*

6. Quick summary of the rings

While most of the rings mentioned in this paper should be well known to the experts, we give here a description or an interpretation of those rings in order for the reader to have a better intuition of what they are.

Recall that F_0 is a finite unramified extension of $\mathbf{Q}_p$, F/F_0 is a finite totally ramified extension and K/F is a finite extension. We also let F_∞/F denote the Lubin-Tate extension of F attached to a uniformizer π of F. We let K' denote the maximal unramified extension of F inside KF_∞.

For $I = [r,s]$ a compact subinterval of $[0,+\infty[$ such that $0 \in I$ or $I \subset [1,+\infty[$, we let $C(I)$ denote the annulus

$$\left\{z \in \mathbf{C}_p, p^{-1/r'} \leq |z|_p \leq p^{-1/s'}\right\}$$

where if $\rho \geq 0$, then $\rho' = \rho \cdot e \cdot p/(p-1) \cdot (q-1)/q$, and we admit that $p^{-1/r'} = 0$ if $r = 0$.

For $I = [r,+\infty[$, we let $C(I)$ denote the annulus

$$\left\{z \in \mathbf{C}_p, p^{-1/r'} \leq |z|_p < 1\right\}$$

Let X be a variable. Then we have the following description of the rings $\mathbf{B}_K^I$, $\mathbf{B}_K^{\dagger,r}$ and $\mathbf{B}_{K,\mathrm{rig}}^{\dagger,r}$:

$$\mathbf{B}_K^I = \{\text{Laurent series } f(X) \text{ with coefficients in } K', \text{ which converges on } C(I)\}$$
$$\mathbf{B}_K^{\dagger,r} = \{\text{Laurent series } f(X) \text{ with coefficients in } K', \text{which converges on } C([r,+\infty[) \text{ and is bounded}\}$$
$$\mathbf{B}_{\mathrm{rig},K}^{\dagger,r} = \{\text{Laurent series } f(X) \text{ with coefficients in } K', \text{ which converges on } C(I)\}.$$

If $F = K$, then $\mathrm{Gal}(\overline{\mathbf{Q}}_p/F)$ acts on these rings by $g(X) = [\chi_\pi(g)](X)$ and we have maps $\varphi_q : \mathbf{B}_K^I \longrightarrow \mathbf{B}_K^{qI}, \mathbf{B}_K^{\dagger,r} \longrightarrow \mathbf{B}_K^{\dagger,qr}, \mathbf{B}_{\mathrm{rig},K}^{\dagger,r} \longrightarrow \mathbf{B}_{\mathrm{rig},K}^{\dagger,qr}$ defined by $X \mapsto [\pi](X)$.

When $F \neq K$, there is still a way to define actions of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$ and φ_q, but they are usually no longer explicit.

The elements of $\widetilde{\mathbf{B}}^I$, $\widetilde{\mathbf{B}}^{\dagger,r}$ and $\widetilde{\mathbf{B}}_{\mathrm{rig},K}^{\dagger,r}$ cannot be directly interpreted as functions on some annulus, but one should think of them as limits of algebraic functions. With that in mind, $\widetilde{\mathbf{B}}^I$ is the ring of limits of algebraic functions on $C(I)$, $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}$ is the ring of limits of algebraic functions on $C([r,+\infty[)$, and $\widetilde{\mathbf{B}}^{\dagger,r}$ is the subring of $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}$ consisting of bounded elements.

The rings $\widetilde{\mathbf{B}}^I$, $\widetilde{\mathbf{B}}^{\dagger,r}$ and $\widetilde{\mathbf{B}}_{\mathrm{rig},K}^{\dagger,r}$ come equipped with an action of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/F)$, and with maps $\varphi_q : \widetilde{\mathbf{B}}^I \longrightarrow \widetilde{\mathbf{B}}^{qI}, \widetilde{\mathbf{B}}^{\dagger,r} \longrightarrow \widetilde{\mathbf{B}}^{\dagger,qr}, \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r} \longrightarrow \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,qr}$, which coincides with the actions defined above on $\mathbf{B}_K^I$, $\mathbf{B}_K^{\dagger,r}$ and $\mathbf{B}_{\mathrm{rig},K}^{\dagger,r}$.

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