





Dual Seminorms, Ergodic Coefficients and Semicontraction Theory

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Abstract—Dynamical systems that are contracting on a subspace are said to be semicontracting. Semicontraction theory is a useful tool in the study of consensus algorithms and dynamical flow systems such as Markov chains. To develop a comprehensive theory of semicontracting systems, we investigate seminorms on vector spaces and define two canonical notions: Projection and distance seminorms. We show that the well-known ℓ_p ergodic coefficients are induced matrix seminorms and play a central role in stability problems. In particular, we formulate a duality theorem that explains why the Markov–Dobrushin coefficient is the rate of contraction for both averaging and conservation flows in discrete time. Moreover, we obtain parallel results for induced matrix logarithmic seminorms. Finally, we propose comprehensive theorems for strong semicontractivity of linear and nonlinear time-varying dynamical systems with invariance and conservation properties both in discrete and continuous time.

Index Terms—Duality, ergodic coefficients, induced matrix seminorm, logarithmic norm, semicontraction theory.

I. INTRODUCTION

A. Problem Description and Motivation

BEFORE Stefan Banach proved his famous contraction principle in 1922 [1], Andrey Markov started in 1906 [2] the study of stochastic processes. As documented by Eugene Seneta [3], Markov established a key contraction inequality and a corresponding contraction factor now known with the name of *ergodic coefficient* of a Markov chain. This article aims to provide a modern semicontraction theory approach to explain and generalize ergodic coefficients.

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To be concrete, let the matrix A be row-stochastic and consider the discrete-time dynamical systems

$$x(k+1) = Ax(k), \quad (1a)$$

$$\pi(k+1) = A^T \pi(k). \quad (1b)$$

These systems, and the continuous time counterpart defined by the Laplacian matrix, are perhaps the simplest examples of general averaging-based dynamics (e.g., robotic coordination and distributed optimization) and dynamical flow systems (e.g., compartmental and traffic systems). Important generalizations include systems that satisfy invariance properties (generalizing row-stochasticity) or conservation properties (generalizing column-stochasticity); in all these (linear and nonlinear) cases, the system is at most marginally stable. Markov and later scientists essentially showed that, under a certain connectivity assumption, maps of the form $\pi \mapsto A^T \pi$ are contraction maps with respect to the total variation distance on the simplex. To be specific, define the total variation distance on the simplex by $d_{TV}(\pi, \sigma) = \frac{1}{2} \sum_i |\pi_i - \sigma_i|$. Then any two solutions $\pi(k), \sigma(k)$ to (1b) satisfy

$$d_{TV}(\pi(k) - \sigma(k)) \leq \tau_1(A)^k d_{TV}(\pi(0) - \sigma(0)) \quad (2)$$

where $\tau_1(A)$ is the so-called *Markov–Dobrushin ergodic coefficient* defined by

$$\tau_1(A) := \max_{\|z\|_1=1, z^T \mathbf{1}_n=0} \|A^T z\|_1. \quad (3)$$

In short, when $\tau_1(A) < 1$, existence, uniqueness and global exponential stability of an equilibrium π^* in the simplex for system (1b) are ensured.

Now comes a remarkable similarity. If one defines the seminorm $\|x\|_{\text{dist},\infty} = \frac{1}{2}(\max_i \{x_i\} - \min_j \{x_j\})$, the following fact is also known [4, Th. 1.1] about averaging systems of the form (1a):

$$\|x(k)\|_{\text{dist},\infty} \leq \tau_1(A)^k \|x(0)\|_{\text{dist},\infty}. \quad (4)$$

Despite the extensive research in this field, numerous known related facts remain somehow mysterious and numerous related mathematical questions remain open. For example, why is the same ergodic coefficient τ_1 relevant for the contraction properties of both dynamical flow systems and averaging systems? How does one generalize the bounds (2) and (4) to ergodic coefficients τ_p defined with respect to arbitrary ℓ_p norms (instead of the ℓ_1 norm)? What are the canonical Lyapunov functions for

both systems (1a) and (1b), whose discrete-time variation along the flow is described by $\tau_p(A)$? How does one define ergodic coefficients for continuous-time systems? Is there a unifying contraction theoretic framework that applies to time-varying and nonlinear systems with generalized invariance or conservation properties?

The interest in non-Euclidean norms (e.g., ℓ_1 , ℓ_∞ , and polyhedral norms) is motivated by classes of network systems, such as biological transcriptional systems [5], Hopfield neural networks [6], chemical reaction networks [7], traffic networks [8], vehicle platoons [9], and coupled oscillators [10].

B. Contributions

This article provides a comprehensive answer to all the open research questions outlined above.

In order to define Lyapunov functions for averaging, flow systems and their generalizations to nonlinear dynamical systems with invariant subspaces, we study seminorms, induced matrix seminorms for discrete-time systems and logarithmic seminorms for continuous-time systems. A key contribution of this article is to explain precisely in what sense ergodic coefficients are *induced matrix seminorms* and, when less than unity, contraction factors for discrete-time systems. This equality is the fundamental reason why ergodic coefficients play a critical role in robust stability theory for discrete-time dynamical systems with invariance properties. It is surprising that induced norms are widely studied in the matrix theory literature, but induced seminorms much less (e.g., see [11]).

After characterizing various seminorms' properties, we define two canonical sets of seminorms, namely, *distance* and *projection seminorms*, and establish remarkable duality properties between the two. Our first result generalizes and strengthens the so-called *Markov contraction inequality* as a duality result between the aforementioned seminorms. Our duality result precisely explains why the induced matrix seminorms for both A and A^T are identical, when computed with respect to dual seminorms. Particular emphasis is given to the case of *consensus seminorms*, i.e., seminorms whose kernel is the consensus space (i.e., seminorms that are positive definite about the consensus space). Consensus seminorms appear naturally in averaging algorithms and surprisingly in systems with conservation property (such as Markov chains and dynamical flow systems).

It is an elementary algebraic observation that the total variation distance on the simplex arises from the restriction of the ℓ_1 projection consensus seminorm.

We then leverage all these notions to provide a general nonlinear semicontraction theory, grounded in two key theorems both for continuous and discrete time-varying dynamical systems. The semicontraction theory we develop is tailored to systems with invariance or conservation properties. More in detail, when the system's Jacobian leaves invariant either the seminorm kernel (invariance property) or its orthogonal complement (conservation property), there is a well defined notion of *perpendicular dynamics* which is strictly contracting. For both systems, in the linear time varying case, we show how canonical Lyapunov functions (some of which partly known in

the literature) naturally arise from seminorms. For the nonlinear case, our first key theorem establishes conditions and features of strong semicontracting continuous time, time-varying systems that enjoy the invariance property. The theorem extends [12, Th. 13] through the formulation of a cascade decomposition and by establishing a strong contractivity property on the orthogonal complement to the seminorm kernel. The second key theorem is entirely novel and pertains semicontraction conditions for continuous time, time-varying, dynamical systems that enjoy the conservation property. A discrete-time version of these two theorems is also provided.

C. Literature Review

Interest in contractivity of dynamical systems via matrix measures can be traced back to Demidovič [13] and Krasovskiĭ [14]. Logarithmic norms have been exploited in control theory later on by Desoer and Vidyasagar in [15] and applied in the study of contraction theory for dynamical systems for the first time by Lohmiller and Slotine [16]. In the context of control theory, this literature inspired many generalizations of contraction theory such as partial contraction [17], weak- and semicontraction [12], horizontal contraction on Riemannian and Finsler manifolds [18], [19], etc.

In particular, partial contraction refers to convergence of systems trajectories to a specific behavior, or a manifold [20], see also [21] for a survey on this theory. While partial contraction establishes convergence to a manifold, semicontraction ensures contractivity on the subspace perpendicular to the kernel of the seminorm. For a characterization of partial contraction in the ℓ_2 -norm for the study of synchronization in networked systems, see [17]. The notion of partial contraction is closely related to the one of semicontraction and weak contraction proposed and investigated in [12]. Semicontraction theory relies on a relaxed concept of matrix measure, known as matrix semimeasure. For this reason, contractivity of a dynamical system is only ensured on a certain subspace and the distance between trajectories is allowed to increase along certain directions.

A relevant behavior, to which semicontraction theory applies, is the one of consensus for dynamical systems. Strictly related to consensus, when it comes to stochastic systems, is the concept of (weak) ergodicity [22]. The concept of weak ergodicity was first formalized in 1931 by Kolmogorov [23], who stated that a sequence of stochastic matrices is *weakly ergodic* if the rows of the matrix product tend to become identical as the number of factors increases. The study of ergodicity coefficients is traced back to the pioneering work of Markov [2], in 1906, in which a first expression of ergodicity coefficients was provided in the context of the weak law of large numbers. Subsequent works from Doeblin [24] and Dobrushin [25] provided conditions for weak ergodicity. The key results in this research area were extended and then reviewed by Seneta in the 80s, see, e.g., [26]. A survey of ergodicity coefficients was given by Ipsen and Selee [27]. A historical discussion was given by Hartfiel [4, Ch. 1], and a recent treatment on their connection with spectral graph theory was given by Marsli and Hall [28]. A characterization of "convergability" [29], namely the convergence of a product

of an infinite number of stochastic matrices, was studied by Liu et al. in [29], where a different approach, based on optimally deflated matrices, was proposed. Despite the evident relation between ergodicity coefficients, contraction factors and induced matrix seminorms, especially in the context of stochastic and averaging systems [30], to the best of our knowledge none in the past has shed full light on their connections (see [31] for some preliminary work in this direction). This manuscript aims to bridge the existing gap in the scientific literature between semicontraction and ergodicity of dynamical systems.

D. Article Organization

The rest of this article is organized as follows. Section II presents notation and preliminary results. Section III introduces the projection and distance seminorms and establishes their duality relationship. Section IV pertains with induced matrix seminorms and induced matrix logarithmic-seminorms. In Section V semicontraction theory is applied to dynamical systems. Finally, Section VII concludes the article.

All theorems in this manuscript are new. Lemmas and Corollaries are either new or simple derivations from known results. Due to page constraints some proofs are omitted and reported in the extended technical report [32].

II. NOTATION AND PRELIMINARIES

A. Notation

The set $\mathbb{R}_{\geq 0}$ is the set of nonnegative real numbers. Let $I_n \in \mathbb{R}^{n \times n}$ denote the identity matrix of size n . Let $\mathbf{1}_n$ and $\mathbf{0}_n$ denote the n dimensional column vectors whose entries are all equal to 1 and 0, respectively. Let \mathbf{e}_i denote the i th vector of the canonical basis in \mathbb{R}^n . For a matrix $A \in \mathbb{R}^{n \times n}$, let A^T denote its transpose, $[A]_{i,j}$ its (i, j) th entry. The matrix A is *nonnegative* if all its entries are nonnegative, it is *row stochastic* if it is nonnegative and $A\mathbf{1}_n = \mathbf{1}_n$, it is *column stochastic* if A^T is row stochastic.

Given $A \in \mathbb{R}^{n \times n}$, a vector subspace $\mathcal{K} \subseteq \mathbb{R}^n$ is *A-invariant* if $A\mathcal{K} \subseteq \mathcal{K}$. The symbol $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ denotes the standard inner product on \mathbb{R}^n . We let Π_{\perp} denote the orthogonal projection matrix onto \mathcal{K}^{\perp} , where the symbol \mathcal{K}^{\perp} denotes the orthogonal complement of \mathcal{K} . Note that $\Pi_{\perp} = \Pi_{\perp}^T$, and if $\mathcal{K} = \text{span}\{\mathbf{1}_n\}$, then $\Pi_{\perp} = I_n - \mathbf{1}_n\mathbf{1}_n^T/n =: \Pi_n$. Given $x \in \mathbb{R}^n$, the perpendicular and parallel components of x to \mathcal{K} are denoted by $x_{\perp} = \Pi_{\perp}x$ and $x_{\parallel} = (I_n - \Pi_{\perp})x$, respectively. Define the *n-simplex* as $\Delta_n = \{v \in \mathbb{R}_{\geq 0}^n \mid \mathbf{1}_n^T v = 1\}$ and the *sign function*, $\text{sign} : \mathbb{R} \rightarrow \{-1, 0, 1\}$, as $\text{sign}(x) = \frac{x}{|x|}$ if $x \neq 0$, and $\text{sign}(0) = 0$. Let $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ denote the ceiling and floor functions, respectively. Given two matrices $A, B \in \mathbb{R}^{n \times n}$, we use the notation $A \preceq B$ to indicate that $A - B$ is a negative semidefinite matrix.

A directed, weighted graph is a triple [33], $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, \dots, n\}$ is the set of vertices, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of arcs and \mathcal{A} is the adjacency matrix. An arc (i, j) belongs to \mathcal{G} if and only if $[A]_{ij} \neq 0$. Two nodes $i, j \in \mathcal{V}$ are *weakly adjacent* if either $(i, j) \in \mathcal{E}$ or $(j, i) \in \mathcal{E}$.

Given a real vector space V , the *dual space* V^* is the vector space of linear maps from V into \mathbb{R} . If $V = \mathbb{R}^n$, then V^* is the

vector space of row vectors in \mathbb{R}^n . In this case, it is typical to make a slight abuse of notation and assume $V^* = \mathbb{R}^n$.

B. Basic Concepts

We start with some basic useful concepts. For $x \in \mathbb{R}^n$ and $p \in \mathbb{N}$, the ℓ_p -norm of x is as follows:

$$\|x\|_p \triangleq \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$$

while the ℓ_{∞} -norm is as follows:

$$\|x\|_{\infty} = \lim_{p \rightarrow \infty} \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} = \max_i |x_i|.$$

For $A \in \mathbb{R}^{n \times m}$ and $p \in [1, \infty]$, the ℓ_p -induced norm of A is as follows:

$$\|A\|_p = \max_{\substack{x \in \mathbb{R}^m \\ \|x\|_p \leq 1}} \|Ax\|_p.$$

Definition 1 (Seminorms): A function $\|\cdot\| : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ is a *seminorm* on \mathbb{R}^n if it satisfies the following properties for all $x, y \in \mathbb{R}^n$ and $a \in \mathbb{R}$:

$$(\text{homogeneity}) : \|\|ax\|\| = |a| \|\|x\|\|, \text{ and}$$

$$(\text{subadditivity}) : \|\|x + y\|\| \leq \|\|x\|\| + \|\|y\|\|.$$

The *kernel* of a seminorm is the vector space

$$\mathcal{K} \triangleq \ker(\|\cdot\|) = \{x \in \mathbb{R}^n : \|x\| = 0\}.$$

Note that, $\|x\| = 0$ does not imply $x = \mathbf{0}_n$.

From now onward, for a seminorm $\|\cdot\|$ on \mathbb{R}^n with kernel \mathcal{K} we will use the symbol $\|\cdot\|^{\mathcal{K}}$.

Lemma 2 (Seminorms of orthogonal projections): Let $\|\cdot\|^{\mathcal{K}}$ be a seminorm on \mathbb{R}^n with kernel \mathcal{K} , and let Π_{\perp} be the orthogonal projection matrix onto \mathcal{K}^{\perp} . For all $x \in \mathbb{R}^n$, $\|x\|^{\mathcal{K}} = \|\|\Pi_{\perp}x\|\|^{\mathcal{K}}$.

Proof: The result is a direct consequence of the reverse triangle inequality and the sub additivity property of seminorms applied to the orthogonal decomposition $x = x_{\perp} + x_{\parallel}$, with $x_{\parallel} \in \mathcal{K}$. \square

Definition 3 (Induced seminorm [12]): Given a seminorm $\|\cdot\|^{\mathcal{K}} : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ with kernel \mathcal{K} , the induced seminorm on $\mathbb{R}^{n \times n}$ is as follows:

$$\|A\|^{\mathcal{K}} \triangleq \max_{\substack{\|x\|^{\mathcal{K}} \leq 1 \\ x \perp \mathcal{K}}} \|Ax\|^{\mathcal{K}}.$$

Definition 4 (Matrix logarithmic seminorms [34]): Given a seminorm $\|\cdot\|^{\mathcal{K}} : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ with kernel \mathcal{K} , the induced matrix logarithmic seminorm on $\mathbb{R}^{n \times n}$ is as follows:

$$\mu^{\mathcal{K}}(A) \triangleq \lim_{h \rightarrow 0^+} \frac{\|\|I_n + hA\|\|^{\mathcal{K}} - 1}{h}.$$

Definition 5 (Generalized ℓ_p ergodicity coefficient [35]): Given $p \in [1, \infty]$ and a vector subspace $\mathcal{K} \subset \mathbb{R}^m$, the *generalized ℓ_p ergodicity coefficient* $\tau_p : \mathcal{K} \times \mathbb{R}^{m \times n} \rightarrow \mathbb{R}_{\geq 0}$ is defined

by

$$\tau_p(\mathcal{K}, A) := \max_{\substack{\|z\|_p=1 \\ z \perp \mathcal{K}}} \|A^T z\|_p. \quad (5)$$

Remark 6 (Markov-Dobrushin ergodic coefficient): For the case $p = 1$, $\mathcal{K} = \text{span}\{\mathbf{1}_n\}$, the coefficient in (5) is known [27], [28], [36] as the *Markov-Dobrushin ergodic coefficient* and simply denoted by $\tau_1(A)$.

The ergodicity coefficient (5) can be interpreted as the induced norm of the operator defined on the real (normed) linear space \mathcal{K}^\perp by $x \rightarrow A^T x$ [37].

Lemma 7 (ℓ_2 -Norm LMI characterization [34]): Given any $A \in \mathbb{R}^{n \times n}$

$$\|A\|_2 = \min\{b \in \mathbb{R}_{\geq 0} \mid A^T A \preceq b^2 I_n\}.$$

III. SEMINORMS AND DUALITY

A. Projection and Distance Seminorms

In the following we provide the definition of projection and distance seminorms. These two seminorms serve as Lyapunov functions for certain classes of systems and will play a fundamental role in the duality result.

Definition 8 (Projection and distance seminorms): Let $\mathcal{K} \subset \mathbb{R}^n$ be a vector space and $\Pi_\perp \in \mathbb{R}^{n \times n}$ be the orthogonal projection matrix onto \mathcal{K}^\perp . For each $p \in [1, \infty]$, define the ℓ_p -projection seminorm with respect to \mathcal{K} by

$$\|x\|_{\text{proj},p}^{\mathcal{K}} \triangleq \|\Pi_\perp x\|_p \quad (6)$$

and the ℓ_p -distance seminorm with respect to \mathcal{K} by

$$\|x\|_{\text{dist},p}^{\mathcal{K}} \triangleq \text{dist}_p(x, \mathcal{K}) = \min_{u \in \mathcal{K}} \|x - u\|_p. \quad (7)$$

Note that the optimization problem (7) is well posed since the norm function is convex.

Lemma 9 (Basic properties): For each $p \in [1, \infty]$

- i) $\ker(\|\cdot\|_{\text{proj},p}^{\mathcal{K}}) = \ker(\|\cdot\|_{\text{dist},p}^{\mathcal{K}}) = \mathcal{K}$.
- ii) $\|x\|_{\text{dist},p}^{\mathcal{K}} \leq \min\{\|x\|_p, \|x\|_{\text{proj},p}^{\mathcal{K}}\}$ for all $x \in \mathbb{R}^n$.

Proof: Statement (i) is obvious from (6) and (7). Next, we compute

$$\min_{u \in \mathcal{K}} \|x - u\|_p \leq \|x - \mathbf{0}_n\|_p = \|x\|_p,$$

$$\min_{u \in \mathcal{K}} \|x - u\|_p \leq \|x - (I_n - \Pi_\perp)x\|_p = \|x\|_{\text{proj},p}^{\mathcal{K}}.$$

This completes the proof of statement (ii). \square

It is not true in general that $\|x\|_{\text{proj},p}^{\mathcal{K}} \leq \|x\|_p$.

Example 10 (Seminorms for consensus and stationary distribution): When $\mathcal{K} = \text{span}\{\mathbf{1}_n\}$ and $p \in \{1, 2, \infty\}$, explicit formulas for the ℓ_p -projection and distance seminorms are either easily derivable or available in the literature [29], [34], [38]. For each $x \in \mathbb{R}^n$, with the shorthand $x_{\text{avg}} = \frac{1}{n} \mathbf{1}_n^T x$,

$$\|x\|_{\text{proj},1}^{\mathcal{K}} = \sum_{i=1}^n |x_i - x_{\text{avg}}|,$$

$$\|x\|_{\text{proj},2}^{\mathcal{K}} = \left(\frac{1}{n} \sum_{i,j} (x_i - x_j)^2 \right)^{1/2},$$

$$\|x\|_{\text{proj},\infty}^{\mathcal{K}} = \max_i |x_i - x_{\text{avg}}|.$$

Next, sort the entries of x according to $x_{(1)} \geq x_{(2)} \geq \dots \geq x_{(n)}$. With this shorthand

$$\|x\|_{\text{dist},1}^{\mathcal{K}} = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} x_{(i)} - \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^n x_{(i)},$$

$$\|x\|_{\text{dist},2}^{\mathcal{K}} = \left(\frac{1}{n} \sum_{i,j} (x_i - x_j)^2 \right)^{1/2},$$

$$\|x\|_{\text{dist},\infty}^{\mathcal{K}} = \frac{1}{2} (x_{(1)} - x_{(n)}) = \frac{1}{2} \left(\max_i \{x_i\} - \min_j \{x_j\} \right).$$

Fig. 1 illustrates the unit disks for these seminorms on \mathbb{R}^3 .

Example 11 (Total variation distance): The total variation [39, Sec. 4.1] is a metric on the simplex Δ_n defined by

$$d_{\text{TV}}(x, y) \triangleq \frac{1}{2} \sum_{i=1}^n |x_i - y_i|.$$

Given any two vectors $x, y \in \Delta_n$, a simple derivation shows

$$d_{\text{TV}}(x, y) = \frac{1}{2} \|x - y\|_{\text{proj},1}^{\mathcal{K}}$$

where $\|\cdot\|_{\text{proj},1}^{\mathcal{K}}$ is the ℓ_1 -projection seminorm with respect to the kernel $\mathcal{K} = \text{span}\{\mathbf{1}_n\}$.

B. Duality

In this section we establish a useful duality relationship between projection and distance seminorms. We start with the notion of dual seminorm.

Definition 12 (Dual seminorm): Let $\|\cdot\|_{\mathcal{K}}$ be a seminorm on a real vector space $V \subseteq \mathbb{R}^n$ with kernel $\mathcal{K} \subset V$. The dual seminorm is the function $\|\cdot\|_{\star}^{\mathcal{K}} : V^* \rightarrow \mathbb{R}$ defined by

$$\|x\|_{\star}^{\mathcal{K}} \triangleq \max_{\substack{\|y\|_{\mathcal{K}} \leq 1 \\ y \perp \mathcal{K}}} \langle x, y \rangle.$$

We omit the proof of the following natural result.

Lemma 13 (Well-posedness of dual seminorms [32]): Let $\|\cdot\|_{\mathcal{K}}$ be a seminorm on a real vector space V with kernel \mathcal{K} . Then, the dual seminorm $\|\cdot\|_{\star}^{\mathcal{K}}$ is a seminorm on V^* .

When $V = \mathbb{R}^n$, we make the usual identification $(\mathbb{R}^n)^* = \mathbb{R}^n$. In this case, the kernel of the dual seminorm is identical to the kernel of the primal seminorm.

Next, we present an important generalization to arbitrary ℓ_p/ℓ_q norms of the Markov contraction inequality from [38, Lemma 2.3].

Lemma 14 (Markov contraction inequality): Let $p, q \in [1, \infty]$ satisfy $p^{-1} + q^{-1} = 1$ (with the convention $1/\infty = 0$) and consider a vector space $\mathcal{K} \subset \mathbb{R}^n$. For all $x, y \in \mathbb{R}^n$

$$x^T \Pi_\perp y \leq \|x\|_{\text{proj},p}^{\mathcal{K}} \|y\|_{\text{dist},q}^{\mathcal{K}}.$$

Proof: For each $u \in \mathcal{K}$ satisfying $\|y - u\|_q = \|y\|_{\text{dist},q}^{\mathcal{K}}$

$$x^T \Pi_\perp y = x^T \Pi_\perp (y - u) \stackrel{\text{(H\"older's ineq)}}{\leq} \|\Pi_\perp x\|_p \|y - u\|_q.$$

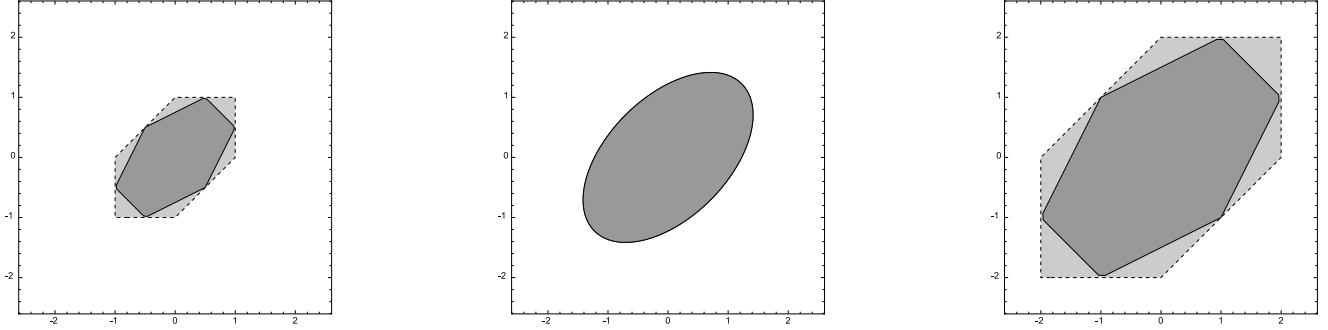


Fig. 1. 2-D sections of 3-D unit disks of projection (solid contours) and distance (dashed contours) consensus seminorms. We plot the section corresponding to $(x_1, x_2, x_3 = 0)$ for $p = 1$ (left), $p = 2$ (center), and $p = \infty$ (right).

The result follows from minimizing with respect to u . \square

Remark 15 (Markov contraction and Hölder's inequalities):

For the inner product of vectors perpendicular to a subspace, the Markov contraction inequality provides a tighter bound than the Hölder's inequality $x^T y \leq \|x\|_p \|y\|_q$. In fact, as a consequence of Lemma 9 (ii)

$$x^T \Pi_{\perp} y \leq \|x\|_{\text{proj}, p}^{\mathcal{K}} \|y\|_{\text{dist}, q}^{\mathcal{K}} \leq \|x\|_{\text{proj}, p}^{\mathcal{K}} \|y\|_{\text{proj}, q}^{\mathcal{K}}.$$

Next, we recall that, for unconstrained vectors, the Hölder's inequality provides a tight bound in the sense that, for all $x \in \mathbb{R}^n$, there exists $y \in \mathbb{R}^n$ such that $x^T y = \|x\|_p \|y\|_q$, with p and q as in Lemma 14. We now show this tightness result also for the Markov contraction inequality, thereby establishing the duality relationship between the projection and the distance seminorms.

Theorem 16 (Duality of distance and projection seminorms):

Let $p, q \in [1, \infty]$ satisfy that $p^{-1} + q^{-1} = 1$ (with the convention $1/\infty = 0$) and let $\mathcal{K} \subset \mathbb{R}^n$ be a vector subspace. Then, $\|\cdot\|_{\text{dist}, p}^{\mathcal{K}}$ and $\|\cdot\|_{\text{proj}, q}^{\mathcal{K}}$, with kernel \mathcal{K} , are dual seminorms

$$\|\cdot\|_{\text{dist}, p}^{\mathcal{K}} = \left(\|\cdot\|_{\text{proj}, q}^{\mathcal{K}} \right)_{\star} \quad (8)$$

$$\|\cdot\|_{\text{proj}, q}^{\mathcal{K}} = \left(\|\cdot\|_{\text{dist}, p}^{\mathcal{K}} \right)_{\star}. \quad (9)$$

Proof: To prove (8), consider two cases. First, if $x \in \mathcal{K}$, then $\|x\|_{\text{dist}, p}^{\mathcal{K}} = 0$. On the other hand, $y \in \mathcal{K}^{\perp}$ implies $y^T x = 0$. So, both sides of (8) are zero. Second, if $x \notin \mathcal{K}$, by Lemma 37 in Appendix I, there exists $\psi_p(x) \in \mathcal{K}^{\perp}$ with $\|\psi_p(x)\|_{\text{proj}, q}^{\mathcal{K}} = 1$ such that

$$\|x\|_{\text{dist}, p}^{\mathcal{K}} = \psi_p(x)^T x \leq \max_{\substack{\|y\|_{\text{proj}, q}^{\mathcal{K}} \leq 1 \\ y \in \mathcal{K}^{\perp}}} y^T x = \left(\|\cdot\|_{\text{proj}, q}^{\mathcal{K}} \right)_{\star}.$$

To prove the opposite inequality, choose any $y \in \mathbb{R}^n$ such that $\|y\|_{\text{proj}, q}^{\mathcal{K}} \leq 1$ and $y \in \mathcal{K}^{\perp}$. Then $\|y\|_q = \|y\|_{\text{proj}, q}^{\mathcal{K}} \leq 1$, so by Lemma 14

$$y^T x \leq \|y\|_q \|x\|_{\text{dist}, p}^{\mathcal{K}} \leq \|x\|_{\text{dist}, p}^{\mathcal{K}}.$$

To prove equality (9) we notice, as in the previous first case, that if $x \in \mathcal{K}$, then $\|x\|_{\text{proj}, q}^{\mathcal{K}} = 0$, while $y \in \mathcal{K}^{\perp}$ implies that $y^T x = 0$, so both sides of (9) are zero. Otherwise, in the second case, if $x \notin \mathcal{K}$, by Lemma 38, there exists $\zeta_q(x) \in \mathcal{K}^{\perp}$ with

$\|\zeta_q(x)\|_{\text{dist}, p}^{\mathcal{K}} \leq 1$ such that

$$\|x\|_{\text{proj}, q}^{\mathcal{K}} = \zeta_q(x)^T x \leq \max_{\substack{\|y\|_{\text{dist}, p}^{\mathcal{K}} \leq 1 \\ y \in \mathcal{K}^{\perp}}} y^T x = \left(\|\cdot\|_{\text{dist}, p}^{\mathcal{K}} \right)_{\star}.$$

To prove the opposite inequality, choose any $y \in \mathbb{R}^n$ such that $\|y\|_{\text{dist}, p}^{\mathcal{K}} \leq 1$ and $y \in \mathcal{K}^{\perp}$. Lemma 14 implies

$$x^T y = x^T \Pi_{\perp} y \leq \|x\|_{\text{proj}, q}^{\mathcal{K}} \|y\|_{\text{dist}, q}^{\mathcal{K}} \leq \|x\|_{\text{proj}, q}^{\mathcal{K}}.$$

This concludes the proof. \square

IV. INDUCED MATRIX SEMINORMS AND LOGARITHMIC SEMINORMS

A. Induced Matrix Seminorms

In the following we list some basic properties related to induced matrix seminorms.

Lemma 17 (Properties of induced matrix seminorms): Let $\|\cdot\|_{\mathcal{K}}$ be a seminorm on \mathbb{R}^n with kernel \mathcal{K} . For any $A, B \in \mathbb{R}^{n \times n}$

i) $\|Ax\|_{\mathcal{K}} \leq \|A\|_{\mathcal{K}} \|x\|_{\mathcal{K}}$ for all $x \in \mathcal{K}^{\perp}$.

If $A\mathcal{K} \subseteq \mathcal{K}$, then

ii) $\|A\|_{\mathcal{K}} = \max_{\|x\|_{\mathcal{K}} \leq 1} \|Ax\|_{\mathcal{K}}$,

iii) $\|Ax\|_{\mathcal{K}} \leq \|A\|_{\mathcal{K}} \|x\|_{\mathcal{K}}$, and

iv) $\|AB\|_{\mathcal{K}} \leq \|A\|_{\mathcal{K}} \|B\|_{\mathcal{K}}$.

Proof: Property (i) was proven in [40]. To prove property (ii), decompose any vector $x \in \mathbb{R}^n$ as $x = x_{\perp} + x_{\parallel}$, with $x_{\perp} \in \mathcal{K}^{\perp}$ and $x_{\parallel} \in \mathcal{K}$, and notice that

$$\begin{aligned} \max_{\|x\|_{\mathcal{K}} \leq 1} \|Ax\|_{\mathcal{K}} &= \max_{\|x\|_{\mathcal{K}} \leq 1} \|A(x_{\perp} + x_{\parallel})\|_{\mathcal{K}} \\ &= \max_{\|x_{\perp}\|_{\mathcal{K}} \leq 1} \|Ax_{\perp}\|_{\mathcal{K}} = \max_{\|y\|_{\mathcal{K}} \leq 1} \|Ay\|_{\mathcal{K}} = \|A\|_{\mathcal{K}} \end{aligned}$$

where the second equality is based on Lemma 2 and exploits the fact that $A\mathcal{K} \subseteq \mathcal{K}$. To prove property (iii) we notice that, by adopting the same decomposition as before

$$\begin{aligned} \|Ax\|_{\mathcal{K}} &= \|Ax_{\perp}\|_{\mathcal{K}} = \left\| A \left(x_{\perp} / \|x_{\perp}\|_{\mathcal{K}} \right) \right\|_{\mathcal{K}} \|x_{\perp}\|_{\mathcal{K}} \\ &\leq \|A\|_{\mathcal{K}} \|x_{\perp}\|_{\mathcal{K}} = \|A\|_{\mathcal{K}} \|x\|_{\mathcal{K}} \end{aligned}$$

where the first equality is based on Lemma 2 and exploits the fact that $AK \subseteq \mathcal{K}$, while the inequality derives from Definition 3. Property (iv) can be found in [40]. \square

Based on Theorem 16 we are now in the position to provide one of the main results of this manuscript.

For a matrix $A \in \mathbb{R}^{k \times n}$, and for $p, q \in [1, \infty]$, with $p^{-1} + q^{-1} = 1$, it holds that

$$\|A\|_p = \|A^\top\|_q \quad (10)$$

where $\|\cdot\|_p : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ and $\|\cdot\|_q : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ are dual norms.

The following theorem represents a generalization of the duality relationship between induced matrix norms (10) to seminorms.

Theorem 18 (Duality of induced matrix seminorms): Let $p, q \in [1, \infty]$ such that $p^{-1} + q^{-1} = 1$. For any matrix $A \in \mathbb{R}^{n \times n}$, and any vector space $\mathcal{K} \subseteq \mathbb{R}^n$

$$\| \|A^\top\| \|_{\text{proj}, q}^{\mathcal{K}} = \| \|A\| \|_{\text{dist}, p}^{\mathcal{K}}. \quad (11)$$

Additionally, if $AK \subseteq \mathcal{K}$, then

$$\tau_q(\mathcal{K}, A) = \| \|A^\top\| \|_{\text{proj}, q}^{\mathcal{K}} = \| \|A\| \|_{\text{dist}, p}^{\mathcal{K}}. \quad (12)$$

Proof: (11) is a direct consequence of Theorem 16

$$\begin{aligned} \| \|A^\top\| \|_{\text{proj}, q}^{\mathcal{K}} &= \max_{\substack{\|x\|_{\text{proj}, q}^{\mathcal{K}} \leq 1 \\ x \perp \mathcal{K}}} \| \|A^\top x\| \|_{\text{proj}, q}^{\mathcal{K}} \\ &\stackrel{(9)}{=} \max_{\substack{\|x\|_{\text{proj}, q}^{\mathcal{K}} \leq 1 \\ x \perp \mathcal{K}}} \max_{\substack{\|y\|_{\text{dist}, p}^{\mathcal{K}} \leq 1 \\ y \perp \mathcal{K}}} y^\top A^\top x \\ &= \max_{\substack{\|y\|_{\text{dist}, p}^{\mathcal{K}} \leq 1 \\ y \perp \mathcal{K}}} \max_{\substack{\|x\|_{\text{proj}, q}^{\mathcal{K}} \leq 1 \\ x \perp \mathcal{K}}} x^\top A y \\ &\stackrel{(8)}{=} \max_{\substack{\|y\|_{\text{dist}, p}^{\mathcal{K}} \leq 1 \\ y \perp \mathcal{K}}} \| \|A y\| \|_{\text{dist}, p}^{\mathcal{K}} = \| \|A\| \|_{\text{dist}, p}^{\mathcal{K}}. \end{aligned}$$

To prove (12) note that

$$\begin{aligned} \| \|A^\top\| \|_{\text{proj}, q}^{\mathcal{K}} &= \max_{\substack{\|x\|_{\text{proj}, q}^{\mathcal{K}} \leq 1 \\ x \perp \mathcal{K}}} \| \|A^\top x\| \|_{\text{proj}, q}^{\mathcal{K}} \\ &= \max_{\substack{\|\Pi_\perp x\|_q \leq 1 \\ x \perp \mathcal{K}}} \|\Pi_\perp A^\top x\|_q = \max_{\substack{\|x\|_q \leq 1 \\ x \perp \mathcal{K}}} \|A^\top x\|_q = \tau_q(\mathcal{K}, A) \end{aligned}$$

where the second-last equality follows from the fact that $A^\top \mathcal{K}^\perp \subseteq \mathcal{K}^\perp$ and, since $x \in \mathcal{K}^\perp$, $x = \Pi_\perp x$. \square

In the following we provide some explicit expressions for the distance seminorm of row-stochastic matrices for the case in which the kernel of the seminorms is the consensus subspace. The explicit expressions can be derived by the ones available in the literature for ergodicity coefficients [27], [41]. The expression for the projection seminorm of column stochastic matrices can be easily derived by the duality result from Theorem 18 and hence, omitted.

Corollary 19 (Formulas for induced matrix seminorms): Consider the consensus distance seminorm. Let $A \in \mathbb{R}^{n \times n}$.

Assume that the entries of each column $j \in \{1, 2, \dots, n\}$ are sorted so that $a_{(1),j} \geq a_{(2),j} \geq \dots \geq a_{(n),j}$.

If A is row-stochastic, then

$$\| \|A\| \|_{\text{dist}, 1}^{\mathcal{K}} = \max_j \left\{ \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} a_{(i),j} - \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^n a_{(i),j} \right\}, \quad (13)$$

$$\| \|A\| \|_{\text{dist}, 2}^{\mathcal{K}} = \|\Pi_n A\|_2 = \min \{ b \geq 0 : A^\top \Pi_n A \preceq b^2 \Pi_n \}, \quad (14)$$

$$\begin{aligned} \| \|A\| \|_{\text{dist}, \infty}^{\mathcal{K}} &= \frac{1}{2} \max_{i \neq j} \sum_{k=1}^n |a_{ik} - a_{jk}| \\ &= 1 - \min_{i \neq j} \sum_{k=1}^n \min \{ a_{ik}, a_{jk} \}. \end{aligned} \quad (15)$$

Proof: The formulas (13), (15), and the first equality in (14) follow from the equivalence $\| \|A\| \|_{\text{dist}, p}^{\mathcal{K}} = \tau_q(\mathbb{1}_n, A)$ in Theorem 18 and by applying the explicit expressions for $\tau_q(\mathbb{1}_n, A)$ provided in Theorem 3.7, Corollary 3.9, Theorem 4.2, and Theorem 6.19 from [27].

The second equality in (14) follows from Lemma 7, since

$$\begin{aligned} \|\Pi_n A\|_2 &= \min_{b \in \mathbb{R}} \{ (\Pi_n A)^\top (\Pi_n A) \preceq b^2 I_n \} \\ &= \min_{b \in \mathbb{R}} \{ A^\top \Pi_n A \preceq b^2 I_n \}. \end{aligned}$$

Since $b^2 \Pi_n \preceq b^2 I_n$, it is clear that $A^\top \Pi_n A \preceq b^2 \Pi_n$ implies $A^\top \Pi_n A \preceq b^2 I_n$. Conversely, assume $A^\top \Pi_n A \preceq b^2 I_n$, so that $v^\top A^\top \Pi_n A v \leq b^2 v^\top v$ for all $v \in \mathbb{R}^n$. Then, for any $u \in \mathbb{R}^n$, we can decompose $u = u_\perp + u_\parallel$, with $u_\perp \in \text{span}\{\mathbb{1}_n\}^\perp$ and $u_\parallel \in \text{span}\{\mathbb{1}_n\}$. Since A is row stochastic

$$u^\top A^\top \Pi_n A u = u_\perp^\top A^\top \Pi_n A u_\perp \leq b^2 u_\perp^\top u_\perp = b^2 u^\top \Pi_n u$$

and thus $A^\top \Pi_n A \preceq b^2 \Pi_n$. This way we have proved that $A^\top \Pi_n A \preceq b^2 I_n$ if and only if $A^\top \Pi_n A \preceq b^2 \Pi_n$. In turn, this implies

$$\|\Pi_n A\|_2 = \min_{b \in \mathbb{R}} \{ A^\top \Pi_n A \preceq b^2 \Pi_n \}. \quad \square$$

Explicit expressions for induced matrix seminorms for column stochastic matrices are provided in Eq. (17)-(19) in [32].

B. Induced Matrix Logarithmic Seminorms

We now present a duality result for induced matrix logarithmic seminorms which is parallel to the one in Theorem 18.

Theorem 20 (Dual logarithmic seminorms): Let $p, q \in [1, \infty]$ be such that $p^{-1} + q^{-1} = 1$. For any matrix $M \in \mathbb{R}^{n \times n}$, and any kernel \mathcal{K}

$$\mu_{\text{dist}, p}^{\mathcal{K}}(M) = \mu_{\text{proj}, q}^{\mathcal{K}}(M^\top).$$

Proof: The equality directly follows from the duality of distance and projection induced matrix seminorms. See [32]. \square

We derive now explicit formulas for ℓ_p -distance logarithmic seminorm of (minus) Laplacian matrices, for $p \in \{1, 2, \infty\}$.

Theorem 21 (Explicit formulas for distance logarithmic seminorms): Consider the consensus distance and projection seminorms. Let $L \in \mathbb{R}^{n \times n}$ be the Laplacian matrix corresponding to an adjacency matrix $A \in \mathbb{R}^{n \times n}$ without self-loops, and let $d_{\text{out}} = A \mathbb{1}_n$. For each $i \in \{1, 2, \dots, n\}$, sort the off-diagonal

entries of Ae_j according to

$$a_{(1),j} \geq a_{(2),j} \geq \cdots \geq a_{(n-1),j}.$$

Then

$$\mu_{\text{dist},1}^{\mathcal{K}}(-L) = -\min_j \left\{ [d_{\text{out}}]_j - \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor - 1} a_{(i),j} + \sum_{i=\lceil \frac{n}{2} \rceil}^{n-1} a_{(i),j} \right\},$$

$$\mu_{\text{dist},2}^{\mathcal{K}}(-L) = \min_{b \in \mathbb{R}} \{ b : \Pi_n L + L^T \Pi_n \succeq -2b \Pi_n \},$$

$$\mu_{\text{dist},\infty}^{\mathcal{K}}(-L) = -\min_{i \neq j} \left\{ a_{ij} + a_{ji} + \sum_{k \neq i,j} \min\{a_{ik}, a_{jk}\} \right\}.$$

Proof: Given $h > 0$, set $S_h = I_n - hL$. Observe that S_h is row-stochastic for every $h > 0$, and its entries are as follows:

$$[S_h]_{ij} = \begin{cases} 1 - h[d_{\text{out}}]_i, & i = j, \\ ha_{ij}, & i \neq j. \end{cases}$$

Also, $\mu^{\mathcal{K}}(-L) = \lim_{h \rightarrow 0^+} h^{-1} (\| \| S_h \| \|_{\mathcal{K}} - 1)$ for any seminorm $\| \| \cdot \| \|_{\mathcal{K}}$.

Case $\| \| \cdot \| \|_{\text{dist},1}^{\mathcal{K}}$: For each $j \in \{1, 2, \dots, n\}$, sort the entries of $S_h e_j$ as follows:

$$(S_h)_{(1),j} \geq (S_h)_{(2),j} \geq \cdots \geq (S_h)_{(n),j}.$$

Assume h is so small that $(S_h)_{(1),j} = (S_h)_{j,j}$. Then by (13)

$$\| \| S_h \| \|_{\text{dist},1}^{\mathcal{K}} = \max_j \left\{ \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} s_{(i),j} - \sum_{i=\lceil \frac{n}{2} \rceil + 1}^n s_{(i),j} \right\}$$

$$= 1 + h \max_j \left\{ -[d_{\text{out}}]_j + \sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} a_{(i),j} - \sum_{i=\lceil \frac{n}{2} \rceil + 1}^n a_{(i),j} \right\}.$$

Substituting into (4) yields the formula for $\mu_{\text{dist},1}^{\mathcal{K}}(-L)$, since the order of the off-diagonal elements of Ae_j is identical to the order of the off-diagonal elements of $S_h e_j$ for all $h > 0$.

Case $\| \| \cdot \| \|_{\text{dist},2}^{\mathcal{K}}$: By (14)

$$\begin{aligned} \| \| S_h \| \|_{\text{dist},2}^{\mathcal{K}} &= \min_{b \geq 0} \{ b : S_h^T \Pi_n S_h \preceq b^2 \Pi_n \} \\ &= \min \{ b \geq 0 : (I_n - hL)^T \Pi_n (I_n - hL) \preceq b^2 \Pi_n \} \\ &= \min \{ b \geq 0 : h^2 L^T \Pi_n L - h \Pi_n L - h L^T \Pi_n \\ &\quad \preceq (b^2 - 1) \Pi_n \}. \end{aligned}$$

Therefore, $\frac{\| \| S_h \| \|_{\text{dist},2}^{\mathcal{K}}}{h}$ is equal to

$$\min \left\{ \frac{b-1}{h} : b \geq 0, h^2 L^T \Pi_n L - h \Pi_n L - h L^T \Pi_n \preceq (b^2 - 1) \Pi_n \right\}.$$

Let $\bar{b} = h^{-1}(b-1)$, so that $b \geq 0$, if and only if $\bar{b} \geq -h^{-1}$, and $(b^2 - 1) = h\bar{b}(2 + h\bar{b})$. Performing this change of variables

$$\mu_{\text{dist},2}^{\mathcal{K}}(L) = \lim_{h \rightarrow 0^+} \frac{\| \| S_h \| \|_{\text{dist},2}^{\mathcal{K}} - 1}{h}$$

$$\begin{aligned} &= \lim_{h \rightarrow 0^+} \min \{ \bar{b} \geq -h^{-1} : hL^T \Pi_n L - \Pi_n L - L^T \Pi_n \\ &\quad \preceq \bar{b}(2 + h\bar{b}) \Pi_n \} \\ &= \min \{ \bar{b} \geq 0 : -\Pi_n L - L^T \Pi_n \preceq 2\bar{b} \Pi_n \} \end{aligned}$$

which is equivalent to the formula for $\mu_{\text{dist},2}^{\mathcal{K}}(-L)$.

Case $\| \| \cdot \| \|_{\text{dist},\infty}^{\mathcal{K}}$: Assume h is sufficiently small that $1 - h[d_{\text{out}}]_i > ha_{ji}$ for all i, j . Applying (15),

$$\begin{aligned} \| \| S_h \| \|_{\text{dist},\infty}^{\mathcal{K}} &= 1 - \min_{i \neq j} \left\{ \min \{ 1 - h[d_{\text{out}}]_i, ha_{ji} \} \right. \\ &\quad \left. + \min \{ 1 - h[d_{\text{out}}]_i, ha_{ij} \} \right. \\ &\quad \left. + h \sum_{k \neq i,j} \min \{ a_{ik}, a_{jk} \} \right\} \\ &= 1 - h \min_{i \neq j} \left\{ a_{ij} + a_{ji} + \sum_{k \neq i,j} \min \{ a_{ik}, a_{jk} \} \right\}. \end{aligned}$$

Substituting into (4) yields the formula for $\mu_{\text{dist},\infty}^{\mathcal{K}}(-L)$. \square

Explicit formulas for projection logarithmic seminorms are provided in Corollary 24 in [32].

We also notice that, when $L = L^T$, one can also show that $\mu_{\text{dist},2}^{\mathcal{K}}(-L) = -\lambda_2(L)$, i.e., minus the second smallest eigenvalue of L (e.g., see [33, Exercise 6.3]). Explicit expressions for the ℓ_p -projection logarithmic seminorm of Laplacian matrices, for $p \in \{1, 2, \infty\}$, are derived by duality and hence, omitted.

V. SEMICONTRACTING DYNAMICAL SYSTEMS

We exploit now the duality result of induced matrix seminorms and induced matrix logarithmic seminorms for the study of strong semicontractivity of dynamical systems. We also provide some theoretical results that formalize semicontractivity conditions for linear and nonlinear dynamical systems both in discrete and continuous time.

Given a vector subspace $\mathcal{K} \subset \mathbb{R}^n$ and a vector field $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, the *perpendicular vector field* $f_{\perp} : \mathbb{R}^n \rightarrow \mathcal{K}^{\perp}$ and the *parallel vector field* $f_{\parallel} : \mathbb{R}^n \rightarrow \mathcal{K}$ are denoted for all $x \in \mathbb{R}^n$ by $f_{\perp}(x) = \Pi_{\perp} f(x)$ and $f_{\parallel}(x) = (I_n - \Pi_{\perp})f(x)$, respectively. Given a seminorm $\| \| \cdot \| \|_{\mathcal{K}} : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$, with kernel \mathcal{K} , the domain restriction of $\| \| \cdot \| \|_{\mathcal{K}}$ to \mathcal{K}^{\perp} , will be denoted by $\| \| \cdot \| \|_{\perp} : \mathcal{K}^{\perp} \rightarrow \mathbb{R}_{\geq 0}$.

Definition 22 (Invariant sets): Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$. A subspace $V \subset \mathbb{R}^n$ is *f-invariant* on a domain $C \subseteq \mathbb{R}^n$ if $f(x+v) - f(x) \in V$ for all $x \in C$ and $v \in V$.

Lemma 23 (Differential characterization of invariance): Given a continuously differentiable map $f : C \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$, let $Df(x)$ denote its Jacobian. A subspace $V \subset \mathbb{R}^n$ is *f-invariant*, if and only if $Df(x)V \subseteq V$ for all $x \in C$.

Proof: If V is *f-invariant*, then $f(x+hv) - f(x) \in V$ for all $x \in C$, $v \in V$, and $h \in \mathbb{R}$, which implies that

$$Df(x)v = \lim_{h \rightarrow 0} \frac{f(x+hv) - f(x)}{h} \in V.$$

Thus, $Df(x)V \subseteq V$ for all $x \in C$. To prove the converse, assume $Df(x)V \subseteq V$; then for all $v \in V$

$$f(x+v) - f(x) = \int_0^1 Df(x+\alpha v)v d\alpha \in V.$$

□

A. Discrete Time Semicontraction

Let us consider the discrete time, time-varying, nonlinear dynamics

$$x(k+1) = f(k, x(k)) \quad (16)$$

with $k \in \mathbb{Z}_{\geq 0}$, $x \in \mathbb{R}^n$. We assume f to be continuously differentiable in the second argument. In the following we give a generalized definition of strongly semicontracting discrete-time system with respect to the one in [12]. The generalization applies to systems with arbitrary contraction step.

Definition 24 (Semicontracting discrete-time systems): Let $\|\cdot\|^\mathcal{K}$ be a seminorm on \mathbb{R}^n with kernel \mathcal{K} . If there exists $m \in \mathbb{N}_{>0}$, $\rho < 1$ and a domain $C \subseteq \mathbb{R}^n$ for which the time-varying vector field $f: \mathbb{Z}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is such that¹

$$\|D(f^m(k, x))\|^\mathcal{K} \leq \rho \quad (17)$$

for all $k \in \mathbb{Z}_{\geq 0}$ and $x \in C$, then the vector field is *strongly semicontracting* on C with rate $\sqrt[m]{\rho}$.

The next Lemma provides sufficient conditions for two important classes of discrete-time systems to be strongly semicontracting.

Lemma 25 (Strong semicontractivity of discrete-time affine systems): Given a subspace $\mathcal{K} \subset \mathbb{R}^n$ and $p, q \in [1, \infty]$ with $p^{-1} + q^{-1} = 1$, consider a sequence of matrices $\{A(k)\}_{k \in \mathbb{Z}_{\geq 0}} \subset \mathbb{R}^{n \times n}$ satisfying

$$A(k)\mathcal{K} \subseteq \mathcal{K} \quad \text{for all } k \in \mathbb{Z}_{\geq 0}, \quad (\text{invariance})$$

$$\rho \triangleq \sup_{k \in \mathbb{Z}_{\geq 0}} \tau_p(\mathcal{K}, A(k)) < 1. \quad (\text{semicontractivity})$$

Then

i) the system

$$x(k+1) = A(k)x(k) + b(k), \quad b(k) \in \mathbb{R}^n \quad (18)$$

is strongly semicontracting with rate ρ in the ℓ_q -distance seminorm with kernel \mathcal{K} . Moreover

$$\|x(k) - y(k)\|_{\text{dist},q}^\mathcal{K} \leq \rho^k \|x(0) - y(0)\|_{\text{dist},q}^\mathcal{K}.$$

ii) The system

$$x(k+1) = A^\top(k)x(k) + b(k), \quad b(k) \in \mathbb{R}^n \quad (19)$$

is strongly semicontracting with rate ρ in the ℓ_p -projection seminorm with kernel \mathcal{K} . Moreover, for any $x(0), y(0)$ satisfying $x(0) - y(0) \in \mathcal{K}^\perp$

$$\|x(k) - y(k)\|_{\text{proj},p}^\mathcal{K} \leq \rho^k \|x(0) - y(0)\|_{\text{proj},p}^\mathcal{K}.$$

¹ f^m is the m -th iterate of f defined recursively by $f^m(k, x) = f(f^{m-1}(k, x))$, with f^0 the identity operator.

Proof: The proof of part (i) follows from (12) in Theorem 18, and from the conditional submultiplicative property (iii) Lemma 17

$$\begin{aligned} & \|x(k+1) - y(k+1)\|_{\text{dist},q}^\mathcal{K} \\ & \leq \|A(k)\|_{\text{dist},q}^\mathcal{K} \|x(k) - y(k)\|_{\text{dist},q}^\mathcal{K} \\ & = \tau_p(\mathcal{K}, A(k)) \|x(k) - y(k)\|_{\text{dist},q}^\mathcal{K}. \end{aligned} \quad (20)$$

The proof of part (ii) follows from Lemma 17 part (i) since $x(k) - y(k) \in \mathcal{K}^\perp$, $\forall k \in \mathbb{Z}_{\geq 0}$, as a consequence of the invariance assumption (invariance) and therefore,

$$\begin{aligned} & \|x(k+1) - y(k+1)\|_{\text{proj},p}^\mathcal{K} \\ & \leq \|A^\top(k)\|_{\text{proj},p}^\mathcal{K} \|x(k) - y(k)\|_{\text{proj},p}^\mathcal{K} \\ & = \tau_p(\mathcal{K}, A(k)) \|x(k) - y(k)\|_{\text{proj},p}^\mathcal{K}. \end{aligned} \quad (21)$$

□

Remark 26 (Averaging and flow systems): When the subspace \mathcal{K} is the consensus subspace, the matrices $\{A(k)\}_{k=0}^\infty$ are row-stochastic and the term $b(k) \equiv 0_n \forall k \in \mathbb{Z}_{\geq 0}$, the systems (18) and (19) are the standard averaging (1a) and flow systems (1b) in the Introduction and the bounds (20) and (21) are precisely the bounds (2) and (4) stated in the Introduction.

The following theorem focuses on strong semicontractivity of discrete-time dynamical systems that enjoy the invariance property of the kernel of the seminorm.

Theorem 27 (Discrete time semicontracting dynamics with invariance property): Consider a system as in (16). Let $\mathcal{K} \subset \mathbb{R}^n$ be an f -invariant subspace, and suppose that f is strongly semicontracting with rate $\rho < 1$, with respect to a seminorm $\|\cdot\|^\mathcal{K}$ on \mathbb{R}^n with kernel \mathcal{K} . Then

i) the system admits the cascade decomposition

$$x_{\parallel}(k+1) = f_{\parallel}(k, x_{\parallel}(k) + x_{\perp}(k)), \quad (22)$$

$$x_{\perp}(k+1) = f_{\perp}(k, x_{\perp}(k)); \quad (23)$$

ii) the perpendicular dynamics (23) are strongly contracting on \mathcal{K}^\perp with rate ρ , with respect to $\|\cdot\|_{\perp} : \mathcal{K}^\perp \rightarrow \mathbb{R}_{\geq 0}$;

iii) for any two trajectories $x(k), y(k)$ of (16)

$$\|x(k) - y(k)\|^\mathcal{K} \leq \rho^k \|x(0) - y(0)\|^\mathcal{K}$$

for all $k \in \mathbb{Z}_{\geq 0}$.

Proof: Regarding part (i), the cascade decomposition (22)–(23) follows from the observation that

$$\begin{aligned} x_{\perp}(k+1) &= \Pi_{\perp} f(k, x_{\parallel}(k) + x_{\perp}(k)) \\ &= \Pi_{\perp} f(k, x_{\perp}(k)) = f_{\perp}(k, x_{\perp}(k)) \end{aligned}$$

where the second equality is due to the f -invariance of \mathcal{K} . Part (ii) follows from

$$\begin{aligned} \max_{\substack{y \perp \mathcal{K} \\ k \geq 0}} \|Df_{\perp}(k, y)\|^\mathcal{K} &= \max_{\substack{y \perp \mathcal{K} \\ k \geq 0}} \|\Pi_{\perp} Df(k, y)\|^\mathcal{K} \\ &= \max_{\substack{y \perp \mathcal{K} \\ k \geq 0}} \|Df(k, y)\|^\mathcal{K} \leq \max_{\substack{x \in \mathbb{R}^n \\ k \geq 0}} \|Df(k, x)\|^\mathcal{K} \leq \rho \end{aligned}$$

where the second equality follows from the fact that for a generic matrix A , $\|A\|^{\mathcal{K}} = \|\Pi_{\perp} A\|^{\mathcal{K}}$. Part (iii) is a direct consequence of (ii). \square

The following theorem focuses on strong semicontractivity of discrete-time dynamical systems that enjoy the invariance property of the orthogonal complement of the kernel of the seminorm.

Theorem 28 (Discrete time semicontracting dynamics with conservation property): Consider a system as in (16). Let $\mathcal{K} \subset \mathbb{R}^n$ be such that \mathcal{K}^{\perp} is an f -invariant subspace. Let $f : \mathbb{Z}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be strongly semicontracting with rate $\rho < 1$ with respect to a seminorm $\|\cdot\|^{\mathcal{K}}$ on \mathbb{R}^n with kernel \mathcal{K} . Then

i) the system admits the cascade decomposition

$$x_{\parallel}(k+1) = f_{\parallel}(k, x_{\parallel}(k)), \quad (24)$$

$$x_{\perp}(k+1) = f_{\perp}(k, x_{\parallel}(k) + x_{\perp}(k)); \quad (25)$$

ii) for each $x_{\parallel} \in \mathcal{K}$, the vector field $x_{\perp} \mapsto f_{\perp}(k, x_{\parallel} + x_{\perp})$ is strongly contracting with rate ρ , with respect to $\|\cdot\|_{\perp} : \mathcal{K}^{\perp} \rightarrow \mathbb{R}_{\geq 0}$;

iii) if the map $x_{\perp} \mapsto f_{\perp}(k, x_{\parallel} + x_{\perp})$ is Lipschitz² with constant $\ell \in \mathbb{R}$ with respect to some metric $d_{\mathcal{K}}$ on \mathcal{K} , then for any two trajectories $x(k), y(k)$ of (16), satisfying $x(0) - y(0) \in \mathcal{K}^{\perp}$

$$\begin{aligned} & \| \|x(k+1) - y(k+1)\| \|^{\mathcal{K}} \\ & \leq \rho \| \|x(k) - y(k)\| \|^{\mathcal{K}} + \ell d_{\mathcal{K}}(x_{\parallel}(k), y_{\parallel}(k)) \end{aligned}$$

for all $k \in \mathbb{Z}_{\geq 0}$.

Proof: Regarding part (i), the cascade decomposition (24) and (25) follows from the observation that

$$\begin{aligned} x_{\parallel}(k+1) &= (I_n - \Pi_{\perp})f(k, x_{\parallel}(k) + x_{\perp}(k)) \\ &= (I_n - \Pi_{\perp})f(k, x_{\parallel}(k)) = f_{\parallel}(k, x_{\parallel}(k)) \end{aligned}$$

where the second equality is due to the f -invariance of \mathcal{K}^{\perp} . To prove (ii), fix $x_{\parallel} \in \mathcal{K}$, and pick any $x_{\perp}, y_{\perp} \in \mathcal{K}^{\perp}$. Then

$$\begin{aligned} & \| f_{\perp}(k, x_{\parallel} + x_{\perp}(k)) - f_{\perp}(k, x_{\parallel} + y_{\perp}(k)) \|_{\perp} \\ &= \| \|f_{\perp}(k, x_{\parallel} + x_{\perp}(k)) - f_{\perp}(k, x_{\parallel} + y_{\perp}(k))\| \|^{\mathcal{K}} \\ &\leq \rho \| \|x_{\perp}(k) - y_{\perp}(k)\| \|^{\mathcal{K}} = \rho \| \|x_{\perp}(k) - y_{\perp}(k)\|_{\perp}. \end{aligned}$$

To prove (iii), let $x_{\parallel}, y_{\parallel} \in \mathcal{K}$ and $x_{\perp}, y_{\perp} \in \mathcal{K}^{\perp}$. Then

$$\begin{aligned} & \| \|f(k, x_{\parallel}(k) + x_{\perp}(k)) - f(k, y_{\parallel}(k) + y_{\perp}(k))\| \|^{\mathcal{K}} \\ &\leq \| \|f(k, x_{\parallel}(k) + x_{\perp}(k)) - f(k, y_{\parallel}(k) + x_{\perp}(k))\| \|^{\mathcal{K}} \\ &\quad + \| \|f(k, y_{\parallel}(k) + x_{\perp}(k)) - f(k, y_{\parallel}(k) + y_{\perp}(k))\| \|^{\mathcal{K}} \\ &\leq \ell d_{\mathcal{K}}(x_{\parallel}(k), y_{\parallel}(k)) + \rho \| \|x(k) - y(k)\| \|^{\mathcal{K}} \end{aligned}$$

where the first inequality is due to the subadditivity property and the second one follows from point (ii) and the invariance of \mathcal{K}^{\perp} . \square

²That is, for all $x_{\parallel}, y_{\parallel} \in \mathcal{K}$, $z_{\perp} \in \mathcal{K}^{\perp}$, and $k \in \mathbb{Z}_{\geq 0}$, we have $\| \|f_{\perp}(k, x_{\parallel} + z_{\perp}) - f_{\perp}(k, y_{\parallel} + z_{\perp})\|_{\perp} \| \leq \ell d_{\mathcal{K}}(x_{\parallel}, y_{\parallel})$ (see [34, Definition 1.5]).

B. Continuous Time Semicontraction

Let us consider the continuous time, time-varying, nonlinear dynamics

$$\dot{x}(t) = f(t, x(t)) \quad (26)$$

with $t \in \mathbb{R}_{\geq 0}, x \in \mathbb{R}^n$. We assume f to be continuously differentiable in the second argument.

Definition 29 (Semicontracting continuous time systems): Let $\|\cdot\|^{\mathcal{K}}$ be a seminorm on \mathbb{R}^n with kernel \mathcal{K} . The time-varying vector field $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is *strongly infinitesimally semicontracting* with rate $c > 0$ on a domain $C \subseteq \mathbb{R}^n$ if $\forall t \in \mathbb{R}_{\geq 0}$ and $x \in C$

$$\mu^{\mathcal{K}}(Df(t, x)) \leq -c.$$

Lemma 30 provides sufficient conditions for two fundamental continuous time dynamical systems to be strongly infinitesimally semicontracting.

Lemma 30 (Strong semicontractivity of continuous-time affine systems): Given a subspace $\mathcal{K} \subset \mathbb{R}^n$ and $p, q \in [1, \infty]$ with $p^{-1} + q^{-1} = 1$, consider a sequence of matrices $\{A(t)\}_{t \in \mathbb{R}_{\geq 0}} \subset \mathbb{R}^{n \times n}$ satisfying

$$A(t)\mathcal{K} \subseteq \mathcal{K} \quad \text{for all } t \in \mathbb{R}_{\geq 0}, \quad (\text{invariance})$$

$$c \triangleq - \sup_{t \in \mathbb{R}_{\geq 0}} \mu_{\text{dist}, p}^{\mathcal{K}}(A(t)) > 0. \quad (\text{semicontractivity})$$

Then

i) the system

$$\dot{x}(t) = A(t)x(t) + b(t), \quad b(t) \in \mathbb{R}^n$$

is strongly infinitesimally semicontracting with rate c in the ℓ_p -distance seminorm with kernel \mathcal{K} , moreover

$$\| \|x(t) - y(t)\| \|^{\mathcal{K}_{\text{dist}, p}} \leq e^{-ct} \| \|x(0) - y(0)\| \|^{\mathcal{K}_{\text{dist}, p}}, \quad \forall t$$

ii) the system

$$\dot{x}(t) = A^{\top}(t)x(t) + b(t), \quad b(t) \in \mathbb{R}^n \quad (27)$$

is strongly infinitesimally semicontracting with rate c in the ℓ_q -projection seminorm with kernel \mathcal{K} , moreover, for any $x(0), y(0)$ satisfying $x(0) - y(0) \in \mathcal{K}^{\perp}$

$$\| \|x(t) - y(t)\| \|^{\mathcal{K}_{\text{proj}, q}} \leq e^{-ct} \| \|x(0) - y(0)\| \|^{\mathcal{K}_{\text{proj}, q}}, \quad \forall t. \quad (28)$$

Proof: The proof of part (i) follows from Theorem 13, part (i) in [12]. To prove part (ii) we follow a similar reasoning as in Theorem 11 from [12]. In fact, for all $x(0), y(0)$ such that $x(0) - y(0) \in \mathcal{K}^{\perp}$, since the solutions $t \mapsto x(t)$ of (27) are differentiable, by defining $z(t) \triangleq x(t) - y(t)$, for small h , one can write

$$z(t+h) = z(t) + h(A^{\top}(t)z(t)) + o(h) = \Pi_{\perp}(z(t+h))$$

since $z(t) \in \mathcal{K}^{\perp}$ and $A^{\top}\mathcal{K}^{\perp} \subseteq \mathcal{K}^{\perp}$ by hypothesis. Therefore, by Lemma 2 and Lemma 17 part (i)

$$\begin{aligned} & \frac{\| \|z(t+h)\| \|^{\mathcal{K}} - \| \|z(t)\| \|^{\mathcal{K}}}{h} \\ & \leq \frac{\| \|I_n + hA^{\top}(t)\| \|^{\mathcal{K}} - 1}{h} \| \|z(t)\| \|^{\mathcal{K}} + \frac{o(h)}{h}. \end{aligned}$$

Taking the limit as $h \rightarrow 0^+$, one gets $\frac{d}{dt} |||z(t)|||^\mathcal{K} \leq \mu^\mathcal{K}(A^\top(t)) |||z(t)|||^\mathcal{K}$. Finally, from the Grönwall comparison inequality (e.g., see [34, Exercise 2.1])

$$\begin{aligned} & |||x(t) - y(t)|||^\mathcal{K} \\ & \leq \exp\left(\int_0^t \mu^\mathcal{K}(A^\top(\tau)) d\tau\right) |||x(0) - y(0)|||^\mathcal{K}. \end{aligned}$$

Equation (28) follows from the fact that $\mu_{\text{dist},p}^\mathcal{K}(A(t)) = \mu_{\text{proj},q}^\mathcal{K}(A^\top(t)) \leq -c$ for all t . \square

The following theorem focuses on strong infinitesimal semicontractivity of continuous-time dynamical systems that enjoy the invariance property of the kernel of the seminorm. This theorem extends Theorem 13 from [12] through the formulation of a cascade decomposition and by establishing a strong contractivity property on the orthogonal complement to the seminorm kernel.

Theorem 31 (Continuous time semicontracting dynamics with invariance property, partially from [12]): Consider a system as in (26). Let $\mathcal{K} \subset \mathbb{R}^n$ be an f -invariant subspace and suppose that f is strongly infinitesimally semicontracting with rate $c > 0$, with respect to a seminorm $||| \cdot |||^\mathcal{K}$ in \mathbb{R}^n with kernel \mathcal{K} . Then

i) The system admits the cascade decomposition

$$\dot{x}_\parallel(t) = f_\parallel(t, x_\parallel(t) + x_\perp(t)), \quad (29)$$

$$\dot{x}_\perp(t) = f_\perp(t, x_\perp(t)). \quad (30)$$

ii) The perpendicular dynamics (30) are strongly infinitesimally contracting on \mathcal{K}^\perp with rate c , with respect to $\|\cdot\|_\perp : \mathcal{K}^\perp \rightarrow \mathbb{R}_{\geq 0}$.

iii) For any two trajectories $x(t), y(t)$ of (26)

$$|||x(t) - y(t)|||^\mathcal{K} \leq e^{-ct} |||x(0) - y(0)|||^\mathcal{K}$$

for all $t \in \mathbb{R}_{\geq 0}$.

Proof: Regarding part (i), the cascade decomposition is obtained by following the same reasoning as in Theorem 27. Part (ii) follows from

$$\begin{aligned} \mu^\mathcal{K}(Df_\perp(t, y)) &= \mu^\mathcal{K}(\Pi_\perp Df(t, y)) \\ &\leq \mu^\mathcal{K}(Df(t, x)) \leq -c \end{aligned}$$

where the first equality follows from the fact that for a generic matrix A , $\mu^\mathcal{K}(A) = \mu^\mathcal{K}(\Pi_\perp A)$. Part (iii) is a direct consequence of part (ii). \square

The following theorem focuses on strong semicontractivity of continuous-time dynamical systems that enjoy the invariance property of the orthogonal complement of the kernel of the seminorm.

Theorem 32 (Continuous time semicontracting dynamics with conservation property): Consider a system as in (26). Let $\mathcal{K} \subset \mathbb{R}^n$ be such that \mathcal{K}^\perp is an f -invariant subspace. Let $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be strongly infinitesimally semicontracting with rate $c > 0$ with respect to a seminorm $||| \cdot |||^\mathcal{K}$ on \mathbb{R}^n with kernel \mathcal{K} . Then

i) The system admits the cascade decomposition

$$\dot{x}_\parallel(t) = f_\parallel(t, x_\parallel(t)), \quad (31)$$

$$\dot{x}_\perp(t) = f_\perp(t, x_\parallel(t) + x_\perp(t)). \quad (32)$$

ii) For each $x_\parallel \in \mathcal{K}$, the vector field $x_\perp \mapsto f_\perp(t, x_\parallel + x_\perp)$ is strongly infinitesimally contracting with rate c , with respect to $\|\cdot\|_\perp : \mathcal{K}^\perp \rightarrow \mathbb{R}_{\geq 0}$.

iii) If the map $x_\parallel \mapsto f_\perp(t, x_\parallel + x_\perp)$ is Lipschitz³ continuous with constant $\ell \in \mathbb{R}$ with respect to some metric $d_\mathcal{K}$ on \mathcal{K} , then for any two trajectories $x(t), y(t)$ of (26), satisfying $x(0) - y(0) \in \mathcal{K}^\perp$

$$\begin{aligned} & D^+ |||x(t) - y(t)|||^\mathcal{K} \\ & \leq -c |||x(t) - y(t)|||^\mathcal{K} + \ell d_\mathcal{K}(x_\parallel(t), y_\parallel(t)) \end{aligned}$$

for all $t \in \mathbb{R}_{\geq 0}$, where $D^+(\cdot)$ indicates the upper right Dini derivative [34, Sec. 2.1].

Proof: The proof follows the same arguments as Theorem 28 for discrete-time systems. \square

Remark 33 (Seminorms as Lyapunov functions): Lemmas 25 and 30 and Theorems 27, 28, 31, and 32 all show that the seminorm of the difference between any two trajectories serves as an (incremental) Lyapunov function to prove incremental stability for linear systems, and practical stability and input-to-state stability for nonlinear systems both in continuous and discrete time.

VI. GRAPH THEORETICAL CONDITIONS FOR SEMICONTRACTIVITY

We now provide graph theoretical conditions for the systems (1) and their continuous time counterpart to be semicontracting with respect to ℓ_p distance and projection seminorms, for $p \in \{1, 2, \infty\}$. For the discrete-time case, the following conditions are topological abstractions of algebraic conditions in [27], [29]. Lemma 35 is novel.

Lemma 34 (Topological conditions for discrete-time averaging systems): The averaging system (1a) $x(k+1) = Ax(k)$ with A row stochastic is strongly semicontracting

i) in the ℓ_1 distance consensus seminorm if A is doubly stochastic and $\mathcal{G}(A)$ is strongly connected and aperiodic;

ii) in the ℓ_2 distance consensus seminorm if A is doubly stochastic and $\mathcal{G}(A)$ is weakly connected with self loops at each node;

iii) in the ℓ_∞ distance consensus seminorm if $\mathcal{G}(A)$ has self loops at each node and a globally reachable node.

Proof: Conditions in claim (i) ensure, in particular, that there exists $m \in \mathbb{N}$ such that A^m has at least $\lfloor \frac{n}{2} \rfloor + 1$ nonzero entries in each column so the expression in (13) takes value less than one. Consequently, the system is strongly semicontracting according to condition (17) in Definition 24.

Claim (ii) directly follows from Lemma 40 and Theorem 8 in [29]. Finally, according to Corollary 4.5 in [33], conditions in claim (iii) ensure that there exists $m \in \mathbb{N}$ such that A^m (has a column with all nonzero entries and hence) is scrambling. Consequently, according to Corollary 3.9 in [27] and condition (17) in Definition 24 the system is strongly semicontracting. \square

³That is, for all $x_\parallel, y_\parallel \in \mathcal{K}$, $z_\perp \in \mathcal{K}^\perp$, and $t \in \mathbb{R}_{\geq 0}$, we have $\|f_\perp(t, x_\parallel + z_\perp) - f_\perp(t, y_\parallel + z_\perp)\|_\perp \leq \ell d_\mathcal{K}(x_\parallel, y_\parallel)$.

Strong semicontractivity of Markov chains in the ℓ_p projection seminorms, $p \in \{1, 2, \infty\}$, can be derived by duality.

Lemma 35 (Topological conditions for continuous-time averaging): The averaging system $\dot{x} = -Lx$ with L the Laplacian of a graph with adjacency matrix A and without self-loops, is strongly infinitesimally semicontracting

- i) in the ℓ_1 distance consensus seminorm if A is doubly stochastic and every node has at least $\lfloor \frac{n}{2} \rfloor$ in-neighbors;
- ii) in the ℓ_2 distance consensus seminorm if A is doubly stochastic and $\mathcal{G}(A)$ is weakly connected;
- iii) in the ℓ_∞ distance consensus seminorm if every two nodes are either (weakly) adjacent or have a common out-neighbor.

Proof: To prove (i), note that $\mu_{\text{dist},1}^{\mathcal{K}}(-L) < 0$, if and only if

$$\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor - 1} a_{(i),j} - \sum_{j=\lfloor \frac{n}{2} \rfloor}^{n-1} a_{(i),j} < 1, \quad \forall i$$

that for A doubly stochastic is fulfilled if and only if each node has at least $\lfloor \frac{n}{2} \rfloor$ in-neighbors.

To prove (ii) note that for A doubly stochastic $L\Pi_n = \Pi_n L$ and hence, the formula for $\mu_{\text{dist},2}^{\mathcal{K}}(-L)$ in Theorem 21 reads as follows:

$$\mu_{\text{dist},2}^{\mathcal{K}}(-L) = \min_b \left\{ b : \frac{L + L^T}{2} + bI_n \succeq 0 \text{ on } \mathcal{K}^\perp \right\}.$$

The minimum is obtained for $b = -\lambda_2(\frac{L+L^T}{2})$ so that, for $\mathcal{G}(A)$ weakly connected, $\mu_{\text{dist},2}^{\mathcal{K}}(-L) < 0$.

To prove (iii) note that $\mu_{\text{dist},\infty}^{\mathcal{K}}(-L) < 0$, if and only if

$$a_{ij} + a_{ji} + \sum_{k \neq i,j} \min\{a_{ik}, a_{jk}\} > 0 \quad \forall i \neq j$$

that for nonnegative adjacency matrices is true, if and only if (i, j) is an edge or, (j, i) is an edge, or (i, k) and (j, k) are an edge for some third node k . \square

VII. CONCLUSION

We have studied seminorms on vector spaces and induced matrix seminorms for discrete- and continuous-time dynamical systems. We have shown how the natural distance and projection seminorms are dual and how the long-studied ℓ_p ergodic coefficients of a row-stochastic matrix are precisely induced matrix seminorms. We have provided a comprehensive treatment of semicontraction for discrete- and continuous-time systems with invariance or conservation properties. Future research directions include the application of semicontraction theory to systems with symmetries, such as robotic vehicles ($SE(3)$ symmetry) and coupled oscillators (torus symmetry), as well as systems with invariance properties, such as population games and evolutionary dynamics (whose state space is the simplex). See some recent progress in [42]. A long-term elusive task is the definition of an ergodic coefficient that is strictly less than unity for row-stochastic matrices satisfying weak connectivity properties.

APPENDIX I SEMINORM COEFFICIENTS

Here, we recall some useful properties of standard p -norms. In the following, for a differentiable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, we denote by $\nabla(f)$ its gradient.

Lemma 36 (Properties of differentiable p -norms [43]): Let p and $q \in (1, \infty)$, with $p^{-1} + q^{-1} = 1$, then $\|\cdot\|_p$ has the following properties:

- i) $\|x\|_p$ is differentiable on \mathbb{R}^n ;
- ii) $\|x\|_p = x^T \nabla(\|x\|_p)$ for all $x \in \mathbb{R}^n$;
- iii) $\|\nabla(\|x\|_p)\|_q = 1$ for all $x \neq 0_n$.

Proof: See the final remark and equation (18) from [43]. \square

Based on Lemma 36, we establish a novel and useful characterization of the distance and projection seminorms.

Lemma 37 (Coefficients for distance seminorms): Let $p, q \in [1, \infty]$ be such that $p^{-1} + q^{-1} = 1$ and let $\mathcal{K} \subset \mathbb{R}^n$ be a vector subspace. There exists a distance coefficient map $\psi_p : \mathbb{R}^n \rightarrow \mathcal{K}^\perp$ such that, for all $x \in \mathbb{R}^n$,

- i) $\psi_p(x) = 0_n$ if $x \in \mathcal{K}$ and $\|\psi_p(x)\|_{\text{proj},q}^{\mathcal{K}} = 1$ otherwise;
- ii) $\|x\|_{\text{dist},p}^{\mathcal{K}} = \psi_p(x)^T x$.

Proof: Let $V \in \mathbb{R}^{n \times k}$ be a matrix whose columns are a basis for \mathcal{K} , so that we can write

$$\|x\|_{\text{dist},p}^{\mathcal{K}} = \min_{\alpha \in \mathbb{R}^k} \|x - V\alpha\|_p.$$

At the optimum α^* , 0_k is a subgradient of $\|x - V\alpha^*\|_p$

$$0_n \in \partial\|x - V\alpha^*\|_p = -V^T G_p(x - V\alpha^*)$$

where G_p is the subdifferential $G_p = \partial\|\cdot\|_p \subset \mathbb{R}^n$. Consequently, there exists a vector $\psi_p(x) \in G_p(x - V\alpha^*)$ such that $\psi_p(x) \in \ker(V^T) = \mathcal{K}^\perp$. Note that $\|\psi_p(x)\|_{\text{proj},q}^{\mathcal{K}} = \|\psi_p(x)\|_q$, that $\psi_p(x)^T x = \psi_p(x)^T (x - V\alpha^*)$, and that $\|x\|_{\text{dist},p}^{\mathcal{K}} = \|x - V\alpha^*\|_p$, so we need only to show for each $p \in [1, \infty]$ that $\|\psi_p(x)\|_q = 1$ and that $\psi_p(x)^T (x - V\alpha^*) = \|x - V\alpha^*\|_p$.

Case $p = 1$: Using the standard formula for the subgradient of the absolute value function [44], $\psi_1(x) \in G_1(x - V\alpha^*)$ implies that

$$(\psi_1(x))_i = \begin{cases} \text{sgn}((x - V\alpha^*)_i) & (x - V\alpha^*)_i \neq 0 \\ -1 \text{ or } +1, & (x - V\alpha^*)_i = 0. \end{cases}, \quad \forall i$$

If $x \notin \mathcal{K}$, then $x - V\alpha^* \neq 0_n$, so $\|\psi_1(x)\|_\infty = 1$. Furthermore

$$\begin{aligned} (x - V\alpha^*)^T \psi_1(x) &= \sum_{i:(x-V\alpha^*)_i \neq 0} (x - V\alpha^*)_i (\psi_1(x))_i \\ &= \sum_{i:(x-V\alpha^*)_i \neq 0} (x - V\alpha^*)_i \text{sgn}((x - V\alpha^*)_i) \\ &= \|x - V\alpha^*\|. \end{aligned}$$

Case $p \in (1, \infty)$: If $p \in (1, \infty)$, then $\|\cdot\|_p$ is differentiable, so $G_p(z) = \nabla\|z\|_p$ for all $z \in \mathbb{R}^n$, and thus $\psi_p(x) = \nabla\|x - V\alpha^*\|_p$ (where the gradient is taken with respect to $x - V\alpha^*$). If $x \notin \mathcal{K}^\perp$, then $x - V\alpha^* \neq 0_n$, so $\|\psi_p(x)\|_q = 1$ due to

Lemma 36. A further consequence of this lemma is that

$$(x - V\alpha^*)^\top \psi_p(x) = \|x - V\alpha^*\|_p = \|x\|_{\text{dist},p}^{\mathcal{K}}.$$

Case $p = \infty$: Let $\mathcal{I} \subseteq \{1, 2, \dots, n\}$ be the set of indices such that $\|x - V\alpha^*\|_\infty = |x - V\alpha^*|_i$. Using a standard formula for the subdifferential of a pointwise maximum [44], $\psi_\infty(x) \in G_\infty(x - V\alpha^*)$ implies that

$$\psi_\infty(x) \in \text{conv} \bigcup_{i \in \mathcal{I}} \partial |x - V\alpha^*|_i$$

where conv denotes the convex hull, and the subdifferential of each absolute value is with respect to its argument. Therefore, there exist $g_i \in \partial |x - V\alpha^*|_i$ for each $i \in \mathcal{I}$, as well as convex weights λ_i , such that

$$\psi_\infty(x) = \sum_{i \in \mathcal{I}} \lambda_i g_i.$$

For each g_i , we have $[g_i]_j = 0$ for $j \neq i$, since $|z|_i$ only depends on z_i for any $z \in \mathbb{R}^n$. Furthermore, if $x \notin \mathcal{K}$, then $x - V\alpha^* \neq \mathbf{0}_n$, so $|x - V\alpha^*|_i > 0$ for all $i \in \mathcal{I}$, which implies that $[g_i]_i = \text{sgn}(x - V\alpha^*)_i$. Together, these two observations imply that

$$\|\psi_\infty(x)\|_1 = \sum_{i \in \mathcal{I}} \lambda_i \|g_i\|_1 = \sum_{i \in \mathcal{I}} \lambda_i = 1.$$

Finally

$$\begin{aligned} (x - V\alpha^*)^\top \psi_\infty(x) &= \sum_{i \in \mathcal{I}} (x - V\alpha^*)_i \sum_{j \in \mathcal{I}} \lambda_j [g_j]_i \\ &= \sum_{i \in \mathcal{I}} \lambda_i (x - V\alpha^*)_i \text{sgn}(x - V\alpha^*)_i = \|x - V\alpha^*\|_\infty. \end{aligned}$$

□

Lemma 38 (Coefficients for projection seminorms): Let $p, q \in [1, \infty]$ be such that $p^{-1} + q^{-1} = 1$ and $\mathcal{K} \subset \mathbb{R}^n$ be a vector subspace. There exists a projection coefficient map $\zeta_p : \mathbb{R}^n \rightarrow \mathcal{K}^\perp$ such that, for all $x \in \mathbb{R}^n$,

- i) $\zeta_p(x) = \mathbf{0}_n$ if $x \in \mathcal{K}$ and $\|x\|_{\text{dist},q}^{\mathcal{K}} \leq 1$ otherwise;
- ii) $\|x\|_{\text{proj},p}^{\mathcal{K}} = \zeta_p(x)^\top x$.

Proof: Let $x \in \mathbb{R}^n$ and define $x_\perp = \Pi_\perp x$.

Case $p = 1$: Let $\zeta_1(x) = \Pi_\perp \text{sgn}(x_\perp)$. By Lemmas 2 and 9 (ii)

$$\| \zeta_1(x) \|_{\text{dist},\infty}^{\mathcal{K}} = \| \text{sgn}(x_\perp) \|_{\text{dist},\infty}^{\mathcal{K}} \leq \| \text{sgn}(x_\perp) \|_\infty \leq 1$$

where $x \in \mathcal{K}$ implies that $\text{sgn}(x_\perp) = \mathbf{0}_n$. Furthermore,

$$\zeta_1(x)^\top x = \text{sgn}(x_\perp)^\top \Pi_\perp x = \|x_\perp\|_1 = \|x\|_{\text{proj},1}^{\mathcal{K}}.$$

Case $p \in (1, \infty)$: Let $\zeta_p(x) = \Pi_\perp \nabla(\|x_\perp\|_p)$. By Lemmas 2, 9 (ii), and 36 (iii), if $x \notin \mathcal{K}$, then

$$\| \zeta_p(x) \|_{\text{dist},q}^{\mathcal{K}} = \| \nabla(\|x_\perp\|_p) \|_{\text{dist},q}^{\mathcal{K}} \leq \| \nabla(\|x_\perp\|_p) \|_q = 1.$$

But if $x \in \mathcal{K}$, then $x_\perp = \mathbf{0}_n$, so $\zeta_p(x) = \mathbf{0}_n$. Furthermore, as a consequence of Lemma 36 (ii)

$$\zeta_p(x)^\top x = (\nabla(\|x_\perp\|_p))^\top \Pi_\perp x = \|x_\perp\|_p = \|x\|_{\text{proj},p}^{\mathcal{K}}.$$

Case $p = \infty$: Let $i \in \{1, 2, \dots, n\}$ be such that $\|x_\perp\|_\infty = |x_\perp|_i$, and let $\zeta_\infty(x) = \text{sgn}(x_\perp)_i \Pi_\perp \mathbf{e}_i$. By Lemmas 2 and 9 (ii) $\| \zeta_\infty(x) \|_{\text{dist},1}^{\mathcal{K}} = \| \text{sgn}(x_\perp)_i \mathbf{e}_i \|_{\text{dist},1}^{\mathcal{K}} \leq \| \text{sgn}(x_\perp)_i \mathbf{e}_i \|_1 \leq 1$ where $x \in \mathcal{K}$ implies that $\text{sgn}(x_\perp) = \mathbf{0}_n$. Furthermore

$$\zeta_\infty(x)^\top x = \text{sgn}(x_\perp)_i \mathbf{e}_i^\top x_\perp = \|x_\perp\|_\infty = \|x\|_{\text{proj},\infty}^{\mathcal{K}}.$$

□

APPENDIX II OPTIMAL DEFLATION

We present here a comparative analysis between induced seminorms and the notion of optimal deflation given in [29].

Definition 39 (p -optimal deflation [29]): For each $p \in [1, \infty]$, the p -optimal deflation of a matrix $A \in \mathbb{R}^{k \times n}$ is

$$|A|_p \triangleq \min_{v \in \mathbb{R}^n} \|A - \mathbf{1}_n v^\top\|_p. \quad (33)$$

Lemma 40 (Bounds on matrix seminorms): Given a row-stochastic matrix $A \in \mathbb{R}^{n \times n}$, for each $p \in [1, \infty]$

$$\| \|A\|_{\text{dist},p}^{\mathcal{K}} \leq |A|_p \leq \|A\|_p.$$

Proof: We first establish that $|A|_p \geq \| \|A\|_{\text{dist},p}^{\mathcal{K}}$. By the max–min inequality [44, Sec. 5.4.1]

$$\begin{aligned} |A|_p &= \min_{v \in \mathbb{R}^n} \max_{\|w\|_p \leq 1} \|(A - \mathbf{1}_n v^\top)w\|_p \\ &\geq \max_{\|w\|_p \leq 1} \min_{v \in \mathbb{R}^n} \|(Aw) - (v^\top w)\mathbf{1}_n\|_p \\ &\geq \max_{\|w\|_p \leq 1} \|Aw\|_{\text{dist},p}^{\mathcal{K}}. \end{aligned}$$

Let $w \in \mathbb{R}^n$ satisfy $\|w\|_{\text{dist},p}^{\mathcal{K}} \leq 1$ so that $\|w - \alpha \mathbf{1}_n\|_p \leq 1$ for some $\alpha \in \mathbb{R}$. Let $u = w - \alpha \mathbf{1}_n$, and observe that $\|u\|_p \leq 1$, and that $\| \|Au\|_{\text{dist},p}^{\mathcal{K}} = \| \|Aw\|_{\text{dist},p}^{\mathcal{K}}$, since A is row-stochastic and $\| \cdot \|_{\text{dist},p}^{\mathcal{K}}$ is invariant with respect to perturbations in $\text{span}\{\mathbf{1}_n\}$. Therefore

$$\begin{aligned} \max_{\|w\|_p \leq 1} \| \|Aw\|_{\text{dist},p}^{\mathcal{K}} &\geq \max_{\|w\|_{\text{dist},p}^{\mathcal{K}} \leq 1} \| \|Aw\|_{\text{dist},p}^{\mathcal{K}} \\ &\geq \max_{\substack{\|w\|_{\text{dist},p}^{\mathcal{K}} \leq 1 \\ \mathbf{1}_n^\top w = 0}} \| \|Aw\|_{\text{dist},p}^{\mathcal{K}} = \| \|A\|_{\text{dist},p}^{\mathcal{K}}. \end{aligned}$$

The inequality $|A|_p \leq \|A\|_p$ is obtained at $v = \mathbf{0}_n$ in (33). □

Here, a conjecture on the equivalence between ℓ_p distance seminorm and p -optimal deflation as in Definition 39 is proposed.

Conjecture 41 (Optimal deflation and distance seminorm): For each $p \in [1, \infty]$ and row-stochastic matrix $A \in \mathbb{R}^{n \times n}$

$$|A|_p = \| \|A\|_{\text{dist},p}^{\mathcal{K}}.$$

Here are some reasons in support of this conjecture, as follows.

- i) Expressions given in [45] for $p \in \{1, 2, \infty\}$ of $x \in \mathbb{R}^n$ and of $A \in \mathbb{R}^{n \times n}$ row stochastic, of $|x|_p$ and $|A|_p$ coincide with the ones of $\|x\|_{\text{dist},p}^{\mathcal{K}}$ and $\| \|A\|_{\text{dist},p}^{\mathcal{K}}$, respectively.

- ii) If the envelope theorem [46, Th. 1.F.1] could be applied⁴ to the projection seminorm, with kernel $\mathcal{K} = \text{span}\{\mathbf{1}_n\}$, it would lead to the orthogonality constraint with respect to \mathcal{K} and consequently to the equivalence between $\|A\|_p$ and $\|A^T\|_{\text{proj},q}^{\mathcal{K}}$.

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⁴It requires to prove the continuous differentiability of v in (33) as a function of the inner optimization vector related to the induced matrix norm.



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