

# ON THE BEST CONSTANT IN THE NONLOCAL ISOPERIMETRIC INEQUALITY OF ALMGREN AND LIEB

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ABSTRACT. In 1989 Almgren and Lieb proved a rearrangement inequality for the Sobolev spaces of fractional order  $W^{s,p}$ . The case  $p = 2$  of their result implies the nonlocal isoperimetric inequality

$$\frac{P_s(E)}{|E|^{\frac{N-2s}{N}}} \geq \frac{P_s(B_1)}{|B_1|^{\frac{N-2s}{N}}}, \quad 0 < s < 1/2,$$

where  $P_s$  indicates the fractional  $s$ -perimeter, and  $B_1$  is the unit ball in  $\mathbb{R}^N$ . In this note we explicitly compute the best constant, and show that for any  $0 < s < 1/2$ , one has

$$\frac{P_s(B_1)}{|B_1|^{\frac{N-2s}{N}}} = \frac{N\pi^{\frac{N}{2}+s}\Gamma(1-2s)}{s\Gamma(\frac{N}{2}+1)\frac{2s}{N}\Gamma(1-s)\Gamma(\frac{N+2-2s}{2})}.$$

## 1. A SIMPLE PROOF OF THE COMPUTATION OF THE BEST CONSTANT

In their 1989 paper [2, Theorem 9.2 (i)], Almgren and Lieb proved that, if  $f \in W^{s,p}$ , for  $0 < s < 1$  and  $1 \leq p < \infty$ , then also  $f^* \in W^{s,p}$  and

$$(1.1) \quad \|f^*\|_{W^{s,p}} \leq \|f\|_{W^{s,p}},$$

where  $f^*$  denotes the non-increasing rearrangement of  $|f|$ . Here, for  $1 \leq p < \infty$  and  $s > 0$  we have denoted by  $W^{s,p}$  the Banach space of functions  $f \in L^p$  with finite Aronszajn-Gagliardo-Slobedetzky seminorm,

$$(1.2) \quad [f]_{p,s} = \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x) - f(y)|^p}{|x - y|^{N+ps}} dx dy \right)^{1/p},$$

see e.g. [1] or also [11] (throughout this note we assume  $N \geq 2$ ). Notice that if  $\delta_\lambda f(x) = f(\lambda x)$ , with  $\lambda > 0$ , then  $[\delta_\lambda f]_{p,s}^p = \lambda^{-N+ps} [f]_{p,s}^p$ . Consider now the nonlocal perimeter of a set,

$$(1.3) \quad P_s(E) = [\mathbf{1}_E]_{2,s}^2 = [\mathbf{1}_E]_{1,2s}.$$

This notion has appeared in the works of Bourgain, Brezis and Mironescu [4], [5], [6], of Maz'ya [16], and of Caffarelli, Roquejoffre and Savin [7]. These latter authors have begun the study

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*Key words and phrases.* Fractional perimeter, isoperimetric inequality, optimal constant.

The author was supported in part by a Progetto SID (Investimento Strategico di Dipartimento) "Non-local operators in geometry and in free boundary problems, and their connection with the applied sciences", University of Padova, 2017.

of the Plateau problem with respect to a family of fractional perimeters. By the above noted scaling property, we have

$$(1.4) \quad \frac{P_s(\delta_\lambda E)}{|\delta_\lambda E|^{(N-2s)/N}} = \frac{P_s(E)}{|E|^{(N-2s)/N}}.$$

This observation suggests that the fractional perimeter should satisfy the following isoperimetric inequality: given  $0 < s < 1/2$ , there exists a constant  $i(N, s) > 0$  such that for any measurable set  $E \subset \mathbb{R}^N$ , such that  $|E| < \infty$ , one has

$$(1.5) \quad \frac{P_s(E)}{|E|^{(N-2s)/N}} \geq i(N, s).$$

(we note that when  $s \geq 1/2$  any non-empty open set has infinite  $s$ -perimeter, see e.g. the proof of Proposition 1.1 below). In fact, it is well-known (see e.g. [13], [14] and [12]) that (1.5) is contained in the Almgren-Lieb inequality (1.1), since the latter, combined with the observation (1.4), implies

$$(1.6) \quad \frac{P_s(E)}{|E|^{\frac{N-2s}{N}}} \geq \frac{P_s(B_1)}{|B_1|^{\frac{N-2s}{N}}},$$

where  $B_1 \subset \mathbb{R}^N$  is the unit ball. The important case of equality in (1.6) is contained in the works [13] and [14].

In this note we present a simple proof of the following explicit expression of the best constant in the right-hand side of (1.6). In connection with our result the reader should see the remarks at the end of this note.

**Proposition 1.1.** *For any  $0 < s < 1/2$ , one has*

$$(1.7) \quad \frac{P_s(B_1)}{|B_1|^{\frac{N-2s}{N}}} = \frac{N\pi^{\frac{N}{2}+s}\Gamma(1-2s)}{s\Gamma(\frac{N}{2}+1)\frac{2s}{N}\Gamma(1-s)\Gamma(\frac{N+2-2s}{2})}.$$

It is worth noting that the exact limiting behavior of the isoperimetric quotient  $\frac{P_s(B_1)}{|B_1|^{\frac{N-2s}{N}}}$  at the poles  $s = \frac{1}{2}$ , or  $s = 0$ , is captured by the factor  $\frac{\Gamma(1-2s)}{s}$  (recall that  $\Gamma(z)$  has a simple pole at  $z = 0$  with residue 1). Hereafter, we indicate with  $\sigma_{N-1} = \frac{2\pi^{\frac{N}{2}}}{\Gamma(N/2)}$  the  $(N-1)$ -dimensional volume of the unit sphere  $\mathbb{S}^{N-1} \subset \mathbb{R}^N$ , and with  $\omega_N = \sigma_{N-1}/N$  the  $N$ -dimensional volume of the unit ball. One has from (1.7)

$$(1.8) \quad \lim_{s \rightarrow 0^+} s \frac{P_s(B_1)}{|B_1|^{\frac{N-2s}{N}}} = \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} = \sigma_{N-1}, \quad \lim_{s \rightarrow \frac{1}{2}^-} (1-2s)P_s(B_1) = \frac{2N\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}+1)\frac{1}{N}\Gamma(\frac{N+1}{2})}|B_1|^{\frac{N-1}{N}}.$$

Both limit relations in (1.8) are special cases of well-known results. In fact, the case  $p = 2$  of [18, Theor. 3] gives for  $f \in W^{s,2}$ ,

$$\lim_{s \rightarrow 0^+} s \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x) - f(y)|^2}{|x - y|^{N+2s}} dx dy = \sigma_{N-1} \|f\|_{L^2(\mathbb{R}^N)}^2.$$

Taking  $f = \mathbf{1}_{B_1}$  in such result, we obtain the first relation in (1.8). On the other hand, we recall that, in answer to a question posed in [4], J. Dávila in [8, Theor. 1] extended to any dimension their limiting formula for  $N = 1$ , and proved

$$(1.9) \quad \lim_{s \nearrow 1/2} (1-2s)P_s(E) = \left( \int_{\mathbb{S}^{N-1}} |\langle e_N, \omega \rangle| d\sigma(\omega) \right) P(E),$$

where  $e_N = (0, \dots, 0, 1)$ . Since one easily recognises that  $\int_{\mathbb{S}^{N-1}} |\langle e_N, \omega \rangle| = \frac{2\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N+1}{2})}$ , it is clear that taking  $E = B_1$  in (1.9) we obtain the latter relation in (1.8).

*Proof of Proposition 1.1.* Using Plancherel theorem (we adopt the definition of Fourier transform  $\hat{f}(\xi) = \int_{\mathbb{R}^N} e^{-2\pi i \langle \xi, x \rangle} f(x) dx$ , which gives  $\|f\|_2 = \|\hat{f}\|_2$ ), we easily obtain for an arbitrary function  $f \in L^2$

$$[f]_{2,s}^2 = 2 \int_{\mathbb{R}^N} |\hat{f}(\xi)|^2 \int_{\mathbb{R}^N} \frac{1 - \cos(2\pi \langle h, \xi \rangle)}{|h|^{N+2s}} dh d\xi.$$

Now, a simple computation gives

$$\int_{\mathbb{R}^N} \frac{1 - \cos(2\pi \langle h, \xi \rangle)}{|h|^{N+2s}} dh = (2\pi|\xi|)^{2s} \int_{\mathbb{R}^N} \frac{1 - \cos(h_N)}{|h|^{N+2s}} dh = \frac{\pi^{\frac{N}{2}} \Gamma(1-s)}{s 2^{2s} \Gamma(\frac{N+2s}{2})} (2\pi|\xi|)^{2s},$$

where in the last equality we have used the well-known identity

$$\int_{\mathbb{R}^N} \frac{1 - \cos(h_N)}{|h|^{N+2s}} dh = \frac{\pi^{\frac{N}{2}} \Gamma(1-s)}{s 2^{2s} \Gamma(\frac{N+2s}{2})},$$

see e.g. [13, Lemma 3.1]. We conclude that the fractional perimeter of the unit ball is given by

$$(1.10) \quad P_s(B_1) = [\mathbf{1}_{B_1}]_{2,s}^2 = \frac{2\pi^{\frac{N}{2}+2s} \Gamma(1-s)}{s \Gamma(\frac{N+2s}{2})} \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{\mathbf{1}}_{B_1}(\xi)|^2 d\xi.$$

In what follows, we denote by  $J_\nu(z)$  the Bessel function of the first kind and order  $\nu$ . Using Bochner's formula  $\hat{u}(\xi) = 2\pi|\xi|^{-\frac{N}{2}+1} \int_0^\infty r^{\frac{N}{2}} f(r) J_{\frac{N}{2}-1}(2\pi|\xi|r) dr$  for the Fourier transform of a spherically symmetric function  $u(x) = f(|x|)$ , see [3, Theorem 40 p.69], in combination with the identity  $\int_0^1 x^{\nu+1} J_\nu(ax) dx = a^{-1} J_{\nu+1}(a)$ ,  $\Re \nu > -1$ , see [15, 6.561, 5., p.683], we have

$$(1.11) \quad \hat{\mathbf{1}}_{B_1}(\xi) = 2\pi|\xi|^{-\frac{N}{2}+1} \int_0^1 r^{\frac{N}{2}} J_{\frac{N}{2}-1}(2\pi|\xi|r) dr = |\xi|^{-\frac{N}{2}} J_{\frac{N}{2}}(2\pi|\xi|).$$

Since the asymptotic behaviour of  $J_\nu$  is given by  $J_\nu(z) \cong \frac{2^{-\nu}}{\Gamma(\nu+1)} z^\nu$ , as  $z \rightarrow 0$ ,  $J_\nu(z) = O(z^{-1/2})$ , as  $z \rightarrow +\infty$ , we see that  $|\xi|^s \hat{\mathbf{1}}_{B_1}(\xi) \in L^2(\mathbb{R}^N)$  if and only if  $s < 1/2$  (notice that this shows that a ball has infinite  $s$ -perimeter if  $1/2 \leq s < 1$ ). For  $0 < s < 1/2$  we thus find

$$(1.12) \quad \int_{\mathbb{R}^N} |\xi|^{2s-N} |J_{\frac{N}{2}}(2\pi|\xi|)|^2 d\xi = \sigma_{N-1} \int_0^\infty r^{-(1-2s)} |J_{\frac{N}{2}}(2\pi r)|^2 dr.$$

The latter integral can be computed explicitly using a special case of the beautiful, classical formula of Weber-Schafheitlin from 1880/1888: let  $\Re(\nu + \mu + 1) > \Re\lambda > 0$ ,  $\alpha > 0$ , then

$$(1.13) \quad \int_0^\infty r^{-\lambda} J_\nu(\alpha r) J_\mu(\alpha r) dr = \frac{\alpha^{\lambda-1} \Gamma(\lambda) \Gamma(\frac{\nu+\mu-\lambda+1}{2})}{2^\lambda \Gamma(\frac{\mu-\nu+\lambda+1}{2}) \Gamma(\frac{\nu+\mu+\lambda+1}{2}) \Gamma(\frac{\nu-\mu+\lambda+1}{2})},$$

see 6.574, 2. on p. 692 in [15], but for a proof see 13.4 on p. 398 in [22], or the original papers of Sonine [21, pp. 51-52] and Schafheitlin [20]. Taking  $\nu = \mu = N/2$ ,  $\lambda = 1 - 2s > 0$ , and  $\alpha = 2\pi$  in (1.13), we thus find

$$\int_0^\infty r^{-(1-2s)} |J_{\frac{N}{2}}(2\pi r)|^2 dr = \frac{\Gamma(1-2s) \Gamma(\frac{N+2s}{2})}{2\pi^{2s} \Gamma(1-s)^2 \Gamma(\frac{N+2-2s}{2})}.$$

Combining this observation with (1.10), (1.12), we obtain

$$\begin{aligned} \frac{P_s(B_1)}{|B_1|^{\frac{N-2s}{N}}} &= \frac{2\pi^{\frac{N}{2}+2s} \Gamma(1-s)}{s \Gamma(\frac{N+2s}{2})} \omega_N^{\frac{2s-N}{N}} \sigma_{N-1} \frac{\Gamma(1-2s) \Gamma(\frac{N+2s}{2})}{2\pi^{2s} \Gamma(1-s)^2 \Gamma(\frac{N+2-2s}{2})} \\ &= \frac{N\pi^{\frac{N}{2}+s} \Gamma(1-2s)}{s \Gamma(\frac{N}{2}+1)^{\frac{2s}{N}} \Gamma(1-s) \Gamma(\frac{N+2-2s}{2})}, \end{aligned}$$

which is the desired conclusion (1.7). □

In closing, the following two remarks are in order. First, in (4.2) and (1.4) of their 2008 work [14], Frank and Seiringer had already shown that

$$(1.14) \quad \frac{P_s(B_1)}{|B_1|^{\frac{N-2s}{N}}} = \frac{N\pi^s}{(N-2s) \Gamma(\frac{N}{2}+1)^{\frac{2s}{N}}} C_{N,s,1},$$

where (keeping in mind that their  $s$  corresponds to our  $2s$ ) they defined

$$(1.15) \quad C_{N,s,1} = 2\sigma_{N-2} \int_0^1 r^{-(1-2s)} (1-r^{N-2s}) \int_{-1}^1 \frac{(1-t^2)^{\frac{N-3}{2}}}{(1-2rt+r^2)^{\frac{N+2s}{2}}} dt dr.$$

The authors provide the explicit values of  $C_{N,s,1}$  only for  $N = 1$  or  $3$ , but the integral in the right-hand side of (1.15) does not seem to be easily computable, in general. We note that our formula (1.7) does exactly that.

Secondly, after a preliminary version of this note was completed, R. Frank has kindly informed us that the explicit value in our formula (1.7) can also be obtained by combining Proposition 2.3 in the work [12] with a result in Samko's book [19] which is itself cited in [12]. In a subsequent conversation, A. Figalli has kindly told us that, although an expression of the best constant is not explicitly written in their work, one can extract it from the following chain of results (which for the reader's sake we have outlined in detail, also keeping in mind that their  $s$  corresponds to our  $2s$ ):

1) The first key step is formula (2.11) in Proposition 2.3 in [12] which states

$$(1.16) \quad P_s(B_1) = \frac{\sigma_{N-1}}{2s(N-2s)} \lambda_1^s.$$

Here,  $\lambda_1^s$  indicates the first eigenvalue of the following operator in formula (2.7) in [12] :

$$\mathcal{J}_s u = \frac{2^{1-2s} \pi^{\frac{N-1}{2}} \Gamma(\frac{1-2s}{2})}{(1+2s) \Gamma(\frac{N+2s}{2})} \mathcal{D}^{1+s} u,$$

where  $\mathcal{D}^{1+s}$  is the hypersingular operator on  $\mathbb{S}^{N-1}$  defined by

$$\mathcal{D}^{1+s} u(x) = \frac{2s2^s \Gamma(\frac{N+2s}{2})}{\pi^{\frac{N-1}{2}} \Gamma(\frac{1-2s}{2})} \text{P. V.} \int_{\mathbb{S}^{N-1}} \frac{u(x) - u(y)}{|x - y|^{N+2s}} d\sigma(y).$$

2) Denoting by  $\lambda_1^*(s)$  the first eigenvalue of the operator  $\mathcal{D}^{1+s}$ , then by the above definition of  $\mathcal{J}_s$  one has that

$$\lambda_1^s = \frac{2^{1-2s} \pi^{\frac{N-1}{2}} \Gamma(\frac{1-2s}{2})}{(1+2s) \Gamma(\frac{N+2s}{2})} \lambda_1^*(s).$$

3) Finally,  $\lambda_1^*(s)$  is contained in Lemma 6.26 in [19]. The latter gives (see also (2.4) in [12])

$$\lambda_1^*(s) = \frac{\Gamma(\frac{N+2+2s}{2})}{\Gamma(\frac{N-2s}{2})} - \frac{\Gamma(\frac{N+2s}{2})}{\Gamma(\frac{N-2-2s}{2})}.$$

However, it should be noted that the indirect proof outlined in (1)-(3) is not self-contained and relies on several auxiliary results. For instance, the proof of (1.16) above, i.e., (2.11) in [12, Prop. 2.3], uses various special calculations involving the operator  $\mathcal{J}_s$  and, per se, is at least as long as the whole proof of Proposition 1.1. More importantly, (1)-(3) involve facts from harmonic analysis on the sphere  $\mathbb{S}^{N-1}$  which our simple proof of (1.7) avoid altogether. For instance, it rests on Lemma 6.26 from [19] which is not self-contained since its proof hinges on the Funk-Hecke formula for spherical harmonics (see [19, Theor. 1.7]), and on the expression, in terms of various special integrals involving Gegenbauer polynomials, of the coefficients in the Fourier-Laplace series of a function on the sphere.

As a final comment, we note that since in Proposition 1.1 we explicitly compute  $P_s(B_1)$ , our result provides an alternative direct computation of the above mentioned number  $\lambda_1^s$  in (1.16).

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