(Machine) Learning Supergravity Vacua

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This dissertation is submitted for the degree of
Doctor of Philosophy

March 2022

Non al denaro non all'amore né al cielo . . .

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Chapter 1

Introduction

1.1 Is the Standard Model truly a good model?

The course of theoretical particle physics has been very successful in the last decades. The Standard Model, based on Quantum Mechanics and Special Relativity principles, proved to be faithfully up to the scale proven at LHC, namely order ten Tev. It has received different confirmations and proofs from various experiments, and thus it is an excellent model that describes gauge interactions so far. Fermions come in three families of leptons that differ only by masses. The standard model has been confirmed numerous times and also provides a good amount of predictions. On the other hand, it has several drawbacks. There are a number of free parameters, such as coupling constants, mixing angles, and masses in the model, which have to be measured from experiments and there is no known principle to fix them. There is an even larger degree of arbitrariness in the choice of the Gauge group, namely $SU(3) \times SU(2) \times U(1)$, which corresponds to the strong, weak, and hypercharge interactions, respectively. This is only one single choice of field theory out of the infinite class of possible local quantum field theories. Another issue that arises, studying the Standard Model of particle physics, regards naturalness, or in other words, the problem of fine tuning. Indeed, some parameters (Higgs mass and θ term) necessitate fine-tuning at tree level in order to obtain the correct experimental results, in light of the great quantum corrections these quantities receive. The list of problems is not yet finished, huge complications arise when trying to couple the Standard Model with gravity. Indeed, approaching the quantisation of gravity around a flat background results in infinities in the related Feynman diagrams. The latter cannot be tamed by the usual renormalisation techniques [1]. This feature is in common with all the other theories in the framework of local quantum field theories, it is not possible to quantise General Relativity with the usual well-known method of QFT. Fine tuning is related also to this last issue, indeed without new interactions beyond the Standard

Model, there would not be any fine tuning problem. One last, pending puzzle is the one regarding the Cosmological Constant. The Standard Model predicts its values to be various (order 70) orders of magnitude higher than the observed value [2]. Other small problems also arise when considering the lack of particles to serve as dark-matter candidates or the lack of a first-order phase transition needed for baryogenesys.
On the other hand, a truly fundamental theory should be able to include gravity at a quantum level and possibly solve the previous puzzles. There have been many attempts to address some of these problems over the years.
Supersymmetric extensions of the Standard Model have been considered to solve the fine tuning issue regarding the SM parameters, but they do not solve any of the fundamental problems related to the shortcomings of quantum field theories, in particular supersymmetric extensions of gravity, supergravity theories, do not appear to be perturbatively renormalizable either (even though there have been extensive studies that show divergences in $N = 8$ supergravity do not show up before five loops, and current consensus is that the first counter-term appears at seven loops, though it is still possible that its coefficient is vanishing [3]). This may be due to the fact that space-time itself should lose its continuous nature at Planck length scale and its geometry be no more smooth. For example, an option could be changing the smoothness of the spacetime itself at small Planckian distance scales and discretising in some manner. Another possibility is non-commutative geometry, which assumes that the spacetime coordinates do not commute and instead obey an uncertainty principle, leads to a description of quantum gravity, these formulations have been ruled out by experiments at Fermilab [4], where possible variations of photon speed with energy have been tested up to energy scales beyond the Planck scale. Similarly, the approach of loop quantum gravity leads to quantisation of space-time at very small distances. Still, this approach lacks a unifying framework between gravity and the other sectors (matters), it fails to reproduce General Relativity in some limit and its perturbative expansion, and it also fails to describe how divergences [1] are cured by the quantisation procedure and all the elementary tests about black holes. In addition, their formulation is not yet at a level where it is possible to perform tests on the theory. An interesting approach is the one proposed by Weinberg on the asymptotic safety of gravity [5], which prescribes the existence of a safe UV fixed point for General Relativity. At the time of writing this thesis, there were no clues about the existence of such a fixed point.
String theory, on the other hand, starts from a different perspective. Particles are not anymore fundamental objects and not even space-time is; instead 1-dimensional objects called strings become the new fundamental constituent of the theory. These objects have an intrinsic length ℓ_s, so they cannot be used to prove physics at scales smaller than this length. Basically, there

is no meaning to the geometry of space-time below this scale. In addition, fields, particles, and space-time are emergent phenomena coming out of strings excitations. This provides a unifying framework for gravity and the other forces.
The theory possesses many useful and relevant features. In order to be unitary, preserve local Lorentz invariance and be anomaly-free, the bosonic string theory must live in 26 space-time dimensions, if we add supersymmetric partners and write down a supersymmetric theory the number of dimensions falls down to 10. The perturbative expansion of superstring theory is finite [6, 7], and explicit computations have been performed up to 2 loops order [8–16]. Implying that individual loops are free of UV divergences. On the other hand, the convergence of the perturbation series and the completeness of the theory is still under debate. Obviously, the theory contains both open and closed strings, the latter give rise to the degrees of freedom of gravity, while the former provide the degrees of freedom of gauge theories. In addition, it was soon shown that the various string theories were all linked and that are different realisations of an eleven-dimensional theory, called M-Theory (or the theory of everything). The number of dimensions is related to the fact that 11 is the maximal space-time dimension where it is possible to have a supergravity theory without higher spin fields (with linear realisation of supersymmetry and with a single time-like direction).

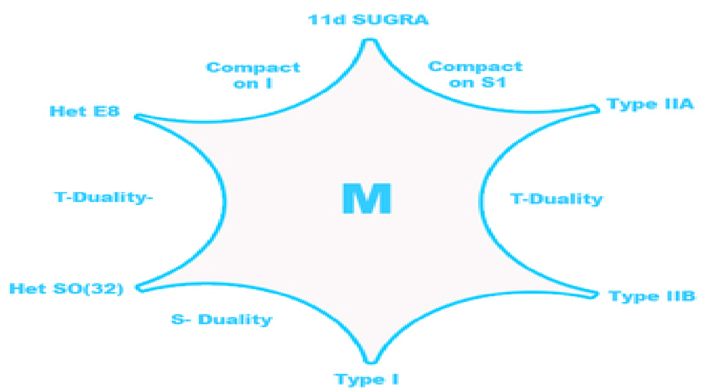


Fig. 1.1 Relations among various String Theories and M-Theory

From the first years, after it was discovered that String Theory encapsulated both gauge theories and gravity, people soon realised that there were too many space-time dimensions with respect to the ones we can sense in our universe. This brought back to the stage the old concept of dimensional reduction and compactification. The idea first came to Theodor Kaluza in 1919 and then was perfected by Klein in 1926. They started with a

pure gravitational theory in five dimensions, namely the Hilbert–Einstein term, with a 15-components metric, and assuming 1 space-like dimension is curled upon itself, forming a circle, they deduced, using the so-called cylinder conditions (fields do not depend on this fifth coordinate) a four-dimensional theory which contained the usual General Relativity term plus Maxwell theory and a scalar. However, this reduction presented many problems when applied to string theory. Most of the problems arose because, by using Kaluza–Klein reduction (KK reduction) on a torus in supergravity, it was not possible to have theories with broken supersymmetry in lower dimensions. This was a great issue, because in our world we do not observe supersymmetry at the energy scales we have tested so far, so supersymmetry should be a broken symmetry in the current phase of our universe. Another problem was the presence of a huge amount of massless scalar fields, parameterizing the geometry of the internal space-time (the compactification manifold). Being massless, they may be the mediator of long-range forces not observed in nature; therefore, they need to be stabilised (gain mass). In order to solve these problems, in 1979 Scherk and Schwarz identified a new method of dimensional reduction called generalised reduction. In Appendix A there is a detailed description of the KK and Scherk and Schwarz reductions. The latter used internal and spacetime simmetries to give the fields a particular dependence on the compactified coordinates, this would result in mass terms for the various field, and therefore solving the problem of moduli stabilisation (at least partly). In addition, if we make explicit the dependence on the compactified coordinates for the gravitino(s) and not for the graviton, we would also be able to break supersymmetry in lower dimensions.
In the late 90ies, people realised that string theory wasn't only a theory of strings, but it encompassed higher-dimensional objects where the ends of open strings could move, namely D-branes. These objects can be charged under some gauge groups. Other higher-dimensional objects were also introduced later, such as O-planes. These concepts made possible the development of other ways to solve the moduli stabilisation problems and create a potential for those scalar fields. Soon, people realised that internal manifolds could support 'fluxes'

$$\int_{\Sigma} \mathcal{F}^p = C_{\Sigma},$$

where $\mathcal{F}^p$ is the field strength of a q-form (which are present in type IIA, IIB, and 11 dimensional supergravity) and C_{Σ} are constants. Fluxes with all indices in the internal manifold do not break Lorentz invariance and so are allowed, in addition, fluxes along the external directions that are proportional to the Lorentz-invariant tensor are also allowed. They give rise to a potential as well, just as the electric and magnetic fields stored in a volume give rise to a non-vanishing Hamiltonian and, therefore, to a non-vanishing potential.

In this thesis, we will be interested in compactifications that preserve supersymmetry. It is also possible to obtain perfectly reasonable models with supersymmetry broken at the string scale, but usually these constructions suffer from instabilities. From the point of view of the low-energy theory in lower dimensions (except N = 1), all these possible reductions are encapsulated in a unique tensor, the Embedding Tensor. Namely, it contains all the possible deformations of the theory, Fluxes, Twists of the internal Geometries, non-Geometrical fluxes, etc.

1.2 One tensor to rule them all: the Embedding Tensor

Studying the string theory low-energy effective theory in 10-dimensions or the low-energy effective theory of M-Theory, one always ends up with a supergravity theory. Either type IIA or IIB supergravity or 11-dimensional supergravity, which is unique and corresponds to the highest-dimensional supergravity theory. Indeed, it is not possible to build supergravities in more than 11 dimensions (with linear realisation of supersymmetry, a single time direction, and without particles with spin greater than 2). By compactifying these theories on some internal manifold, it is possible to obtain supergravity theories in lower dimensions, usually with extended supersymmetry. All the possible deformations that one can imagine to apply to the compactification manifold, when analysed from the point of view of the lower-dimensional supergravity, are encapsulated in one single supersymmetric deformation, the gauging.
Gauging is the only known deformation of extended supergravities so far [1], it comprises possible p-form fluxes as well as geometric fluxes that act by twisting the internal compactification manifold. It also includes some parameters with no clear higher-dimensional origin. The gauging procedure, among its effects, also has the creation of a scalar potential, is completely encoded in a single tensor, called the embedding tensor. The general idea is best illustrated in Fig.1.2. N-dimensional ungauged (maximal) supergravities are obtained from 11D/IIA/IIB supergrivities compactified on an n-torus. Among their peculiarities, there are large global symmetry groups and abelian gauge group, e.g. the maximal four-dimensional theory [17] has a global $E_{7(7)}$ and a local $U(1)^{28}$ group. Matter fields are uncharged under the local symmetry group, which gives them the name. They only have a maximally symmetric Minkowski ground state due to the absence of a scalar potential, therefore, all fields are massless. On the other hand, as shown in Fig. 1.2, there are other less trivial ways to reduce the number of dimensions, for example, with an internal manifold with more structures

[1]This is strictly speaking incorrect, as there exist also massive deformations in some cases (see e.g. the Romans' mass in D=10, N=2A).

than the torus. Following the diagonal arrow in the image, thus turning on possible p-form fluxes, torsion, or non-geometric fluxes, one ends up with more complicated effective field theories. Now, non-Abelian gauge symmetries are present in the theory, and matter fields

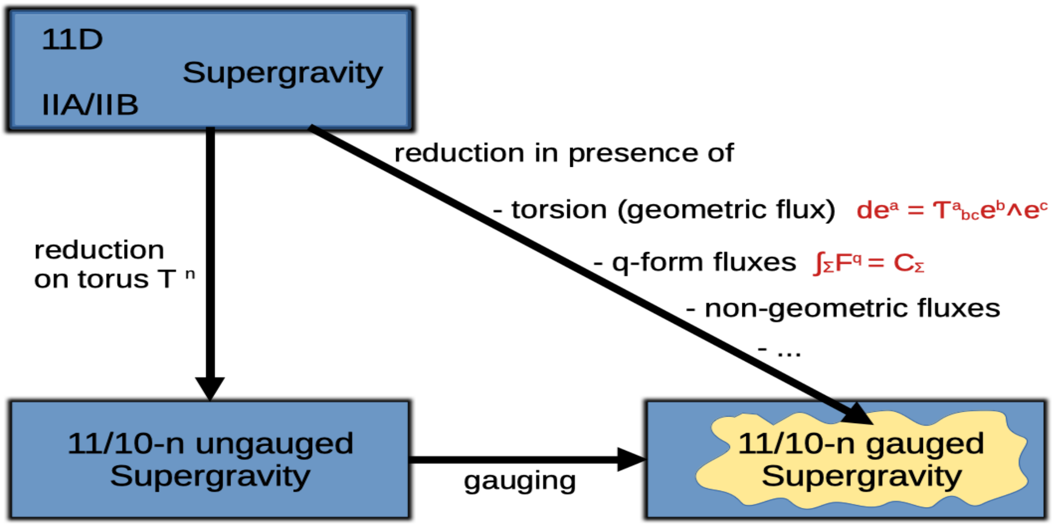


Fig. 1.2 General behaviour of gauged supergravities under compactification[2]

are charged under those. Consequently, these theories are called gauged supergravities in contrast to the ungauged ones. They also come with a scalar potential, resulting from a more complicated internal geometry, making them more interesting. As already explained before, a scalar potential can give rise to masses for the scalar particles (and other fields in the model), thus stabilising the moduli or providing mechanisms for spontaneous supersymmetry breaking. These are highly desired phenomenological properties. The scalar potential can also provide a non-vanishing cosmological constant. This has triggered the attention towards gauged supergravity, in addition, the supergravity regime of the bulk theory in the AdS/CFT conjecture is described by a gauged supergravity [18], [19], [20]. Indeed, the scalar potential can support a negative cosmological constant and thus an AdS ground state. The corresponding gauge groups of these models are usually compact and contain, in general, the R-symmetry groups of the boundary theories. The most prominent example [18], [19] is given by the five-dimensional maximal SO(6) gauged supergravity, which describes type IIB supergravity truncated on $AdS_5 \times S^5$. The scalar potential encodes non-trivial information about the four-dimensional SYM boundary theory, such as holographic RG flows and the anomalous conformal dimensions of operators.

From a pure low-dimensional point of view, one first obtains the ungauged theory by com-

[2]Note that known compactifications only cover a small part of the set of all possible gauged supergravities in lower dimension

pactifing on an n-torus and then, following the horizonatal line in Fig. 1.2, gauges the theory. Gauging proceeds by choosing a subgroup G_0 of the global symmetry group G of the ungauged theory and make it a local gauge group coupling it to the vector fields of the theory, recall that previously the vector fields supported an abelian gauge group. The first case where this happened was the SO(8) gauge theory in four space-time dimensions [21], which describes the compactification on an S^7 internal manifold of eleven-dimensional supergravity. $SO(8)$ is therefore embedded in the global symmetry group of 4 dimensional N = 8 supergravity, which is $E_{7(7)}$. The embeddings of the possible gauge groups are all described by one single tensor, which, unsurprisingly, takes the name of embedding tensor. Its features can be completely described group theoretically and the Lagrangian of gauged supergravities is completely parametrised by it. Indeed, the scalar potential, the mass terms for the various fields, and the new terms in the supersymmetry transformations are all given in terms of the embedding tensor. In addition, if we look at the higher-dimensional origin of the theory, the embedding tensor represents a compact way of encoding the presence of fluxes (p-form, geometrical and non-geometrical, etc.) in the lower-dimensional effective theory. In fact, while, for example, it was "easy" to construct geometrically N=8, SO(8) gauged supergravity, the power of the embedding tensor allowed Dall'Agata, Inverso and Triggiante to prove that this theory is only a point in an infinite family of theories [22].
In this thesis, we studied the vacua of maximal supergravity theories in 5 and 7 spacetime dimensions, both with analytical and numerical techniques. We have been able to discover new vacua and develop new tools that can be used for the research of vacua in other theories, with different space-time dimensions or different amounts of supersymmetry.

Chapter 2

Gauged Supergravities

2.1 Ungauged Supergravities

Before dealing with gauged supergravity theories, it is a wise idea to review what ungauged supergravities are and how to deal with them. In order to discuss vacua of gauged supergravities, we do not need to deep down in the vast topic of ungauged theories, we will only need to discuss the bosonic sector. The discussion is based on [23]
These theories contain always the metric $g_{\mu\nu}$, vector fields A^M_μ, each of them with an abelian redundancy and scalar fields ϕ^i. In addition, there can also be anti-symmetric q-forms $B^J_{\nu_1...\nu_q}$, these are generalisations of vector fields, each of them carry a gauge redundancy as well. The Lagrangian is given by

$$e^{-1}\mathcal{L}_{bos} = -\frac{1}{2}R - \frac{1}{2}G_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j - \frac{1}{4}\mathcal{M}_{MN}F^M_{\mu\nu}F^{\mu\nu N} + ..., \tag{2.1}$$

where $e = \sqrt{|detg_{\mu\nu}|}$, and the abelian field strengths are given in terms of the vector potentials by $F^M_{\mu\nu} = \partial_\mu A^M_\nu - \partial_\nu A^M_\mu$. We omitted the kinetic terms for the q-forms and also the topological terms. In this thesis we are gonna deal only with maximal supergravities, the reason is that the field content is completely fixed by supersymmetry and there is no freedom of choice, in addition Lagrangian and couplings are also restricted for the same reason.
The global symmetry group determines the Lagrangian, q-forms transform linearly under the global symmetry group G, while scalar fields transform non-linearly. The metric is invariant under G-transformations.

D	G_{max}	K_{max}
9	GL(2)	SO(2)
8	SL(2)× SL(3)	SO(2)× SO(3)
7	SL(5)	SO(5)
6	SO(5,5)	SO(5)× SO(5)
5	$E_{6(6)}$	USp(8)
4	$E_{7(7)}$	SU(8)
3	$E_{8(8)}$	SO(16)

Table 2.1 Global symmetry groups G with their maximal compact subgroup K in maximal supergravity in all dimensions

2.1.1 Scalar Fields

Scalar fields in extended supergravity theories (at least for all N$\geq$ 3 theories) are coordinates of a coset space G/K sigma-model [24]. G is the global symmetry group, in table 2.1 the groups for maximal supergravities in various dimensions are shown with their maximal compact subgroups which appear in the coset space G/K. In Appendix A it is shown how these coset spaces arise. It is possible to describe the scalar fields as a G-valued matrix **V**, but considering equivalent two group elements whenever they are multiplied on the right by an element in K. So the effective number of scalars is equal to the number of parameters in the coset space. The Lagrangian, therefore, has to remain invariant under any gauge transformation of the form

$$\mathbf{V}(x) \to \mathbf{V}(x)(k(x))^{-1}. \tag{2.2}$$

Obviously, it also has to stay invariant for any global transformation given by

$$\mathbf{V}(x) \to g\mathbf{V}(x), \tag{2.3}$$

with $k(x) \in K$ and $g \in G$. To write a Lagrangian that is invariant under gauge transformations, we also need a gauge field Q_μ, which lies in the Lie algebra of K and that transforms under gauge transformations 2.2 as

$$Q_\mu \to kQ_\mu k^{-1} - \partial_\mu k k^{-1}. \tag{2.4}$$

Thanks to it, it is possible to construct the covariant derivative

$$D_\mu \mathbf{V}(x) = \partial_\mu \mathbf{V}(x) - \mathbf{V}(x) Q_\mu . \tag{2.5}$$

This transforms just like the vielbein $\mathbf{V}(x)$, so that

$$\mathbf{V}^{-1} D_\mu \mathbf{V} \to k \mathbf{V}^{-1} D_\mu \mathbf{V} k^{-1}.$$

On the other hand, $\mathbf{V}^{-1} D_\mu \mathbf{V}$ is invariant under global transformations 2.3. A Lagrangian invariant under both symmetries is given by

$$\mathcal{L} = -\frac{1}{2} e Tr \left(\mathbf{V}^{-1} D_\mu \mathbf{V} \right)^2 . \tag{2.6}$$

In this Lagrangian, Q_μ appears without derivatives, and varying it keeping $\mathbf{V}$ fixed gives its equations of motions:

$$\delta \mathcal{L} = e Tr \delta Q^\mu ((\mathbf{V})^{-1} \partial_\mu \mathbf{V} - Q_\mu).$$

Since δQ_μ, analogously to Q_μ, is in the Lie algebra of K, then $P_\mu \equiv \mathbf{V}^{-1} \partial_\mu \mathbf{V} - Q_\mu$ lies in the orthogonal complement. Equivalently, we can define the left-invariant current as

$$J_\mu = \mathbf{V}^{-1} \partial \mathbf{V} \in \mathfrak{g} \equiv \text{Lie G}, \tag{2.7}$$

lying in the Lie algebra of G, and J_μ can be decomposed as

$$J_\mu = Q_\mu + P_\mu , \qquad \qquad Q_\mu \in \mathfrak{k}, \;\; P_\mu \in \mathfrak{p}, \tag{2.8}$$

where $\mathfrak{k} \equiv$ Lie K and $\mathfrak{p}$ is its complement within $\mathfrak{g}$ and they are orthogonal with respect to the Cartan–Killing form. Thus, it is obvious that P_μ can be rewritten as

$$P_\mu = \mathbf{V}^{-1} D_\mu \mathbf{V},$$

and, consequently, the Lagrangian takes the form

$$\mathcal{L} = -\frac{1}{2} Tr P_\mu P^\mu . \tag{2.9}$$

Since Q_μ behaves as a gauge field under K, it will act as a connection in the covariant derivatives for the fermionic fields, which transform in some linear representation under the local K symmetry:

$$D_{[\mu} \psi^i_{\nu]} \equiv \partial_{[\mu} \psi^i_{\nu]} - \frac{1}{4} \omega^{ab}_{[\mu} \gamma_{ab} \psi^i_{\nu]} - (Q_{[\mu})_k{}^i \psi^k_{\nu]} . \tag{2.10}$$

Another important feature of P_μ is that it can be used to construct K-invariant interactions terms for fermions. Usually, global $\mathfrak{g}$ transformations can be expanded as $\Lambda = \Lambda^\alpha t_\alpha$ into a basis of generators t_α. Note also that Q_μ is not a truly propagating gauge field, it is a composite field, whose presence is due to only taking care of any redundancy that arises in the parametrisation of the coset space. The vielbein $\mathbf{V}$ can be used instead to describe the couplings between bosons and fermions. Explicitly computing the transformation of the scalar matrix, one gets

$$\delta \mathbf{V}_R{}^{\underline{S}} = \Lambda^\beta (t_\beta)_R{}^N \mathbf{V}_N{}^{\underline{S}} - \mathbf{V}_R{}^{\underline{N}} k_{\underline{N}}{}^{\underline{S}}, \tag{2.11}$$

where the underlined indices $\underline{S}$, $\underline{N}$ refer to transformations under K, while the generators for the group G are represented by $(t_\beta)_R{}^N$. An example of interaction term, built from the scalar matrix, is $F^S \mathbf{V}_S{}^{\underline{R}} (\overline{\psi}\psi)_{\underline{R}}$, where $F^S_{\mu\nu}$ is the field strength for some vector. In the next section, the transformation of the gauge fields under G is given, providing a demonstration that this term is also invariant under global symmetries.

2.1.2 Vectors and Anti-symmetric q-forms

Vectors and q-forms in ungauged supergravity transform linearly under the global symmetry group G. Taking into consideration only the vectors, their transformation under the global symmetry group is

$$\delta A^R_\mu = -\Lambda^\beta (t_\beta)_S{}^R A^S_\mu, \tag{2.12}$$

where, as usual, $(t_\beta)_S{}^R$ are the generators for $\mathfrak{g}$. This transformation is valid only for $D > 4$, indeed, in $D = 4$ space-time dimensions one needs to take into consideration that vector fields are dual to vector fields. Similar transformations exist for higher-rank q-forms. Supersymmetry defines the field content in the various dimensions, therefore, it also gives which q-forms are present and in what representation of the global symmetry group G they transform. In table 2.2 the q-forms and their representations in each dimension are sketched. Physical scalars are determined by eliminating (dim K) scalars from the Adjoint representation of G. Underlined representations are subject to the self-duality relation concerning the $(D/2-1)$-forms in D-dimensions. Only half of these $(D/2-1)$-forms are propagating d.o.f. and appear in the Lagrangian.

To write down a kinetic term for the vectors, we need to introduce a manifestly K-invariant object. Indeed, the scalar fields can also be well described by a symmetric positive-definite scalar matrix, $\mathcal{M}$ given by

$$\mathcal{M} \equiv \mathbf{V}\Delta\mathbf{V}^T, \tag{2.13}$$

D	G	0	1	2	3
9	GL(2)	$\mathbf{1}^0+\mathbf{3}^0-1$	$\mathbf{1}^{-4}+\mathbf{2}^{+3}$	$\mathbf{2}^{-1}$	$\mathbf{1}^{+2}$
8	SL(2)$\times$ SL(3)	$(\mathbf{3}-1,\mathbf{1},\mathbf{8}-3)$	$(\mathbf{2},\mathbf{3}')$	$(\mathbf{1},\mathbf{3})$	$(\underline{\mathbf{2}},\mathbf{1})$
7	SL(5)	$\mathbf{25}-15$	$\mathbf{10}'$	$\mathbf{5}$	
6	SO(5,5)	$\mathbf{45}-30$	$\mathbf{16_c}$	$\underline{\mathbf{10}}$	
5	$E_{6(6)}$	$\mathbf{78}-36$	$\mathbf{27}'$		
4	$E_{7(7)}$	$\mathbf{133}-63$	$\underline{\mathbf{56}}$		
3	$E_{8(8)}$	$\mathbf{248}-120$			

Table 2.2 q-form field content in ungauged maximal supergravity and their G-representations

where Δ is a constant K-invariant positive definite matrix. On the other hand, $\mathcal{M}$ transforms under G as:

$$\delta\mathcal{M} = \Lambda\mathcal{M} + \mathcal{M}\Lambda^T, \tag{2.14}$$

where we suppressed the indices. In terms of $\mathcal{M}$ the scalar kinetic term becomes

$$\mathcal{L}_{scalars} = \frac{1}{8}Tr(\partial_\mu\mathcal{M}\partial^\mu\mathcal{M}^{-1}). \tag{2.15}$$

Now, thanks to this K-invariant object, it is possible to write down an invariant action for the vector fields as well, namely

$$\mathcal{L}_{vectors} = -\frac{1}{4}\mathcal{M}_{RS}F^R_{\mu\nu}F^{\mu\nu S}, \tag{2.16}$$

where $F^R_{\mu\nu} = \partial_\mu A^R_\nu - \partial_\nu A^R_\mu$ is the abelian field strength tensor. The action is manifestly G-invariant, but it is only valid in any spacetime dimension $D > 4$. In 4 space-time dimensions, vectors are dual to vectors and therefore there are complications arising due to that. In this thesis we will not deal with 4-spacetime dimensions explicitly, therefore these complications will not bother us anymore. Kinetic terms for higher-rank q-forms with $q < (D-1)/2$, follow analogously

$$\mathcal{L}_{q-forms} = -\frac{1}{2(q+1)!}\mathcal{M}_{IJ}F^I_{\nu_1\dots\nu_{q+1}}F^{\nu_1\dots\nu_{q+1}J}. \tag{2.17}$$

Dualities are very important in string theory and their low-energy effective theories, supergravities. There are some on-shell duality relations concerning q-forms that are very relevant in the construction of supergravities. Indeed, in D spacetime dimensions, on-shell q-forms and $(D-q-2)$-forms are dual, due to the fact that they transform in the same representation under the little group $SO(D-2)$. To lowest order, from eq. 2.17 the equations of motions for a q-form B^I are

$$\partial^\mu(\mathcal{M}_{IJ}F^J_{\mu\nu_1\ldots\nu_q}) = 0, \tag{2.18}$$

while the Bianchi identity for the field strength is given by

$$\partial_{[\nu_1}F^I_{\nu_2\ldots\nu_{q+2}]} = 0. \tag{2.19}$$

It is possible to write them in terms of dual field strength, defined by

$$G_{\mu_1\ldots\mu_{D-q-1}I} \equiv \frac{e}{(p+1)!}\varepsilon_{\mu_1\ldots\mu_{D-q-1}\nu_1\ldots\nu_{q+1}}\mathcal{M}_{IJ}F^{\nu_1\ldots\nu_{q+1}J}. \tag{2.20}$$

Consequently, equations of motions and the Bianchi Identity take, respectively, the new form

$$\partial_{[\mu_1}G_{\mu_2\ldots\mu_{D-q}]I} = 0 \qquad\qquad \partial^\mu(\mathcal{M}^{IJ}G_{\mu\nu_1\ldots\nu_{D-q-2}J}) = 0. \tag{2.21}$$

Thus, equations of motions and Bianchi identities exchange among each other, and therefore it is possible to define the dual $(D-q-2)$-forms C_I, given by

$$G_{\mu_1\ldots\mu_{D-q-1}J} \equiv (D-q-1)\partial_{[\mu_1}C_{\mu_2\ldots\mu_{D-q-1}]I}. \tag{2.22}$$

The duality extends to higher order in the fields in the E.O.M and Bianchi identities, thus providing a mechanism for exchange of q forms into $(D-q-2)$ forms. It follows trivially then, that there are many off-shell versions of an ungauged supergravity Lagrangian, which become on-shell equivalent using the duality relations just derived. Not always is it possible to dualise all the q-forms, given that some topological terms will force the presence of some gauge fields in the action. In any case, a Lagrangian with all the q-forms dualised to the lowest possible degree always exists, and gauged supergravities arise as deformations of this version of the theory. The latter is also the only theory where the G-action is a manifest symmetry. Now we have all the ingredients needed to deal with gauged extended supergravities.

2.2 The Gauging Procedure

The idea behind the gauging mechanism for supergravity theories is quite simple, it only consists in choosing a subgroup G_0 of G and making it local. This will consist in deforming the ungauged supergravity and will introduce new couplings and terms in the Lagrangian. In this chapter, the embedding tensor will be extensively explained. We will mostly deal with maximal supergravities in various dimensions.

2.2.1 The One Tensor

The global symmetry group G defines and organises the field content of ungauged supergravities. Bosonic fields transform as

$$\begin{aligned} \delta \mathbf{V} &= \Lambda^{\beta} t_{\beta} \mathbf{V}, \\ \delta A_{\mu}^{S} &= -\Lambda^{\beta} (t_{\beta})_{R}{}^{S} A_{\mu}^{R}, \end{aligned}$$

where Λ^{β} are constant parameters and $\beta = 1, ..., dimG$. We already noted that ungauged supergravities possess a $U(1)^{n_v}$ gauge symmetry, where n_v is the number of vectors in the theory and none of the matter fields is charged under it.

$$\delta_{U(1)} A_{\mu}^{R} = \partial_{\mu} \Lambda^{M}(x). \tag{2.23}$$

The U(1)-symmetry also acts on the higher-rank q-forms as an abelian tensor gauge symmetry. To select a subgroup $G_0 \subset G$, we pick a subset of generators of the Lie algebra of G, $\mathfrak{g}$. These new generators are denoted by X_M throughout this thesis, and they enter in the covariant derivatives as

$$\partial_{\mu} \to D_{\mu} \equiv \partial_{\mu} - g A_{\mu}^{M} X_{M}, \tag{2.24}$$

where g is a new coupling constant representing the gauge coupling. The question arises naturally, how one selects the generators of this new gauge group? The answer is the embedding tensor. Indeed, given the generators of $\mathfrak{g}$, one gets

$$X_M \equiv \Theta_M{}^{\beta} t_{\beta} \in \mathfrak{g}. \tag{2.25}$$

The embedding tensor, $\Theta_M{}^{\alpha}$, provides the linear combinations of generators of $\mathfrak{g}$ that become the new local gauge symmetry generators. It is a ($n_v \times$ dim G) matrix, and consequently the rank of Θ provides the dimension of the gauge group. If Θ is treated as a spurionic object that transforms under G according to its indices structure, then it preserves the G covariancy of the theory. Claiming that the theory is invariant under gauge symmetries implies the following transformations for the bosonic fields:

$$\delta \mathbf{V} = g\Lambda^R X_R \mathbf{V}, \tag{2.26}$$

$$\delta A_\mu^S = \partial_\mu \Lambda^S + g A_\mu^R X_{RQ}{}^S \Lambda^Q = D_\mu \Lambda^S, \tag{2.27}$$

where Λ depends on the coordinates and we have defined $X_{RQ}{}^S \equiv \Theta_R{}^\beta (t_\beta)_Q{}^S$. What it has been just written is not consistent in all the cases, indeed the Θ-tensor must satisfy some consistency constraints. The constraints are of two types: a linear and a quadratic constraint. Once these constraints are solved one would obtain a consistent gauge symmetry and consequently a consistent gauged Lagrangian. The quadratic constraint arises from the demand that the Θ-tensor stays invariant under the action of the local symmetry generators. From the form of its indices (one in the fundamental and one in the adjoint representation of the group G), one can immediately see that, in general, Θ is not invariant under G. In D = 3, the vectors transform in the adjoint representation, and therefore it is an exception. In any case, requiring, on the other hand, invariance under the action of G_0 leads to the quadratic constraint:

$$0 \stackrel{!}{=} \delta_R \Theta_S{}^\alpha \equiv \Theta_R{}^\beta \delta_\beta \Theta_S{}^\alpha = \Theta_R{}^\beta (t_\beta)_S{}^Q \Theta_Q{}^\alpha + \Theta_R{}^\beta f_{\beta\gamma}{}^\alpha \Theta_S{}^\gamma, \tag{2.28}$$

where we first noted that the action of G_0 is defined by the projection under Θ and then we used the fact that generators in the adjoint representation are given in terms of the structure constants $(t_\alpha)_\beta{}^\gamma = -f_{\alpha\beta}{}^\gamma$. It is possible to contract this equation with a generator t_α achieving

$$[X_R, X_S] = -X_{RS}{}^Q X_Q, \qquad X_{RS}{}^Q = \Theta_R{}^\beta (t_\beta)_S{}^Q. \tag{2.29}$$

So basically, this equation tells us that the gauge invariance of the embedding tensor implies the closure of the algebra for the generators 2.25. On the other hand, we can see that eq. 2.28 implies more than just the closure of an algebra. By symmetrising eq. 2.29 one obtains that what at first sight may seem as structure constants for the gauge algebra $\mathfrak{g}_0$ behave like those only when contracted with X_Q (in general $X_{(RS)}{}^Q$ does not vanish).
The linear constraint, instead, arises as a supersymmetry consistency condition. Sometimes, however, it is possible to obtain the linear constraint from the (deformed) tensor gauge algebra. The embedding tensor, considering its index structure, lives in the tensor product of $\mathcal{R}_v$ and $\mathcal{R}_{adj}$, and this tensor product can be expanded as a direct sum of several representations

$$\mathcal{R}_v \otimes \mathcal{R}_{adj} = \mathcal{R}_{v*} \oplus \ldots, \tag{2.30}$$

D	G	$\mathcal{R}_{adj}$	$\mathcal{R}_v$	$\Theta_M{}^\alpha$
9	GL(2)	$\mathbf{1}^0+\mathbf{3}^0$	$\mathbf{1}^{-4}+\mathbf{2}+3$	$\mathbf{2}^{-3}+\mathbf{3}^{+4}$
8	SL(2)× SL(3)	$(\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{8})$	$(\mathbf{2},\mathbf{3}')$	$(\mathbf{2},\mathbf{3})+(\mathbf{2},\mathbf{6}')$
7	SL(5)	$\mathbf{24}$	$\mathbf{10}'$	$\mathbf{15}+\mathbf{40}'$
6	SO(5,5)	$\mathbf{45}$	$\mathbf{16_c}$	$\mathbf{144_c}$
5	$E_{6(6)}$	$\mathbf{78}$	$\mathbf{27}'$	$\mathbf{351}'$
4	$E_{7(7)}$	$\mathbf{133}$	$\mathbf{56}$	$\mathbf{912}$
3	$E_{8(8)}$	$\mathbf{248}$	$\mathbf{248}$	$\mathbf{1}+\mathbf{3875}$

Table 2.3 Representations of the Embedding Tensor

then the linear constraint acts to remove some of the representations on the right-hand side of eq. 2.30. For what concerns maximal supergravities, the explicit representations where the embedding tensor lies are given in table 2.3. In this thesis, we will work only with maximal supergravities in 5 and 7 dimensions. Therefore, the relevant representations are $\mathbf{351}'$ for what concerns the five dimensions and $\mathbf{15}+\mathbf{40}'$ for what concerns the seven dimensions.

By solving these consistency constraints, it is possible to classify all possible gaugings. The quadratic constraint, unlike the linear representation one, is very difficult to solve, and it is not know any closed form for it in any dimension. Therefore, counting in-equivalent gaugings is still an unsolved problem.

2.2.2 The new tensor gauge algebra

Minimal couplings, introduced in the previous section are not the only deformation that has to be done in order to describe the new non-abelian nature of the gauge group we introduced. Field strengths for the vector fields have to be modified accordingly. Once this is done, consistency of gauge interactions will require a modification of the gauge algebra for the higher-rank q-forms and thus lead to an intertwining between q-forms and (q+1)-forms. One is naturally led to guess the following form for the vector field strength:

$$\mathcal{F}^S_{\mu\nu} = \partial_\mu A^S_\nu - \partial_\nu A^M_\mu + g X_{[RQ]}{}^S A^R_\mu A^Q_\nu, \tag{2.31}$$

but eventually this will fail. Indeed, as we already pointed out, the gauge algebra does not behave properly as it should.

$$[X_R, X_S] = -X_{RS}{}^Q X_Q, \tag{2.32}$$

with the "structure constants" given by

$$X_{RS}{}^{Q} \equiv \Theta_{R}{}^{\alpha}(t_{\alpha})_{S}{}^{Q} \equiv X_{[RS]}{}^{Q} + Z^{Q}{}_{RS}, \tag{2.33}$$

where the tensor Z is symmetric in its lower indices, and therefore there is a non-vanishing symmetric contribute for the structure constants. In particular, it is possible to show that maximal theories with irreducible theta tensor have no gauging with vanishing Z. On the other hand, the commutator is completely anti-symmetric in its indices, therefore it is clear that

$$Z^{Q}{}_{RS}X_{Q} = 0. \tag{2.34}$$

This follows directly from the quadratic constraint. One can think that this issue can be solved simply by claiming that the true structure constants are simply given by $X_{[RS]}{}^{Q}$, but a little more thought shows that this is also wrong. Indeed, the latter fail to satisfy the Jacobi identity:

$$X_{[MN]}{}^{P}X_{[QP]}{}^{R} + X_{[QM]}{}^{P}X_{[NP]}{}^{R} + X_{[NQ]}{}^{P}X_{[MP]}{}^{R} = -Z^{R}{}_{P[Q}X_{MN]}{}^{P}, \tag{2.35}$$

but it will be satisfied when contracted with X_R due to the quadratic constraint. The lack of closure of the Jacobi identity implies a non-covariance of the vector field strength 2.31. Indeed, under the new gauge transformations, it behaves as

$$\delta\mathcal{F}^{R}_{\mu\nu} = -g\Lambda^{Q}X_{QS}{}^{R}\mathcal{F}^{S}_{\mu\nu} + 2gZ^{R}{}_{QS}(\Lambda^{Q}\mathcal{F}^{S}_{\mu\nu} - A^{Q}_{[\mu}\delta A^{Q}_{\nu]}). \tag{2.36}$$

The second term is not what one would expect for the transformation of the field strength under gauge transformations. In any case, it immediately appears at first sight that due to the presence of the Z-tensor in the second term, the latter will vanish when contracted with X_R. Therefore, the product $\mathcal{F}^{M}_{\mu\nu}X_M$, which is what compare, for example, in the commutator of covariant derivatives, is a covariant object:

$$[D_{\mu}, D_{\nu}] = -g\mathcal{F}^{R}_{\mu\nu}X_{R}.$$

However, it is not possible to use $\mathcal{F}^{M}_{\mu\nu}X_M$ in the Lagrangian, because a possible kinetic term of the form $Tr[\mathcal{F}^{R}_{\mu\nu}X_R\mathcal{F}^{\mu\nu S}X_S]$ is not a smooth deformation of the ungauged kinetic term. Similar troubles arise when one tries to deform the kinetic terms of higher-rank q-forms. The true problem, that is hidden behind this, is that we are trying to build a G-covariant formalism, thus we are writing everything in terms of n_v generators, while only a subset of them will gauge the group G_0. The Theta-tensor has not maximal rank on a general basis and, therefore, some of the X_M will depend on others. In general, the vector fields split into two different types

$$A^N_\mu \to \begin{cases} A^n_\mu \to & \text{transforming in the adjoint of } G_0 \\ A^i_\mu \to & \text{transforming in some other reps. of } G_0 \end{cases} \tag{2.37}$$

In the case where the A^i_μ transform in some non-trivial representation of G_0 it is impossible to construct a consistent gauge theory. $Z^m{}_{RS}$ will vanish while $Z^i{}_{RS}$ will not. One may try to dualise these fields to solve this problem. This is what was done in the maximal SO(6) gauging, which involves only 15 of the 27 vector fields present in the theory [25]. The remaining has been dualised to 2-forms which during the gauging procedure become massive self-dual two forms. However, this procedure, while avoiding the non-covariance of the field-strength tensor, did not preserve explicit G-covariance. Thus, with this method, one first chooses the gauge group and consequently how the degrees of freedom are organised, and then one can construct the gauged Lagrangian. In addition, this procedure has been tried only in 5 and 7 space-time dimensions [25], [26] and with the highest rank gauge groups, thus it is not clear how to export the method to smaller groups and other spacetime dimensions. One, on the other hand, would like to have a general G-covariant Lagrangian without the need of first specifying the gauge group. This is what has been achieved with the covariant construction carried out by using the embedding tensor.
Indeed, recalling that the non covariant terms in the vector field strength gauge transformations, eq. 2.36, contain the tensor $Z^R{}_{QS}$, one can define ([27], [28]) a full covariant field strength as

$$\mathcal{H}^R_{\mu\nu} = \mathcal{F}^R_{\mu\nu} + gZ^R{}_{QS}B^{QS}_{\mu\nu}. \tag{2.38}$$

Where, $B^{QS}_{\mu\nu} = B^{(QS)}_{[\mu\nu]}$ are a two-form tensor fields, which will absorb the non-covariant terms in 2.36. The covariance of $\mathcal{H}^R_{\mu\nu}$ is ensured by the following gauge transformations

$$\delta A^R_\mu = D_\mu \Lambda^R - gZ^R{}_{QS}\Xi^{QS}_\mu, \tag{2.39}$$

$$\delta B^{RS}_{\mu\nu} = 2D_{[\mu}\Xi^{RS}_{\nu]} - 2\Lambda^{(R}\mathcal{H}^{S)}_{\mu\nu} + 2A^{(R}_{[\mu}\delta A^{S)}_{\nu]}, \tag{2.40}$$

where Ξ^{RS}_μ are the tensor gauge transformation parameters for the 2-forms. Therefore, it is possible to spot here a Stückelberg coupling between the vectors and the 2-forms, which arise specifically due to the presence of a non-vanishing Z-tensor. Here, it may seem that some totally new 2-forms have been added to the theory, but this is not the case, they cannot be added because the number of degrees of freedom is regulated and balanced by supersymmetry. Therefore, these 2-forms are provided by the ones already present in the

ungauged theory. Taking into account their index structure, one can see that 2-forms transform into some representation contained in the symmetric product of two vector representations $(\mathcal{R}_v \otimes \mathcal{R}_v)_{sym}$. This very simple fact, constraints the $Z^R{}_{SQ}$-tensor which should project, by means of its lower indices, only on those representations filled by the 2-forms of the theory. Being Z expressed in terms of the embedding tensor Θ, this projection will lead to the linear constraints. Here we can see how supersymmetry, which dictates the field content of the theory and their representation, implies the linear constraint.
For instance, it is possible to show how this works in maximal supergravity in four dimensions [29], where the Z-tensor can be expressed as:

$$\begin{aligned} Z^Q{}_{RS} = X_{(RS)}{}^Q &= \frac{1}{2}\Theta_R{}^\beta (t_\beta)_S{}^Q + \frac{1}{2}\Theta_S{}^\beta (t_\beta)_R{}^Q \\ &= -\frac{1}{2}\Theta^{Q\beta}(t_\beta)_{RS} + \frac{3}{2}X_{(RST)}\Omega^{QT}, \end{aligned}$$

where the indices are raised and lowered using the symplectic matrix $\Omega_{PQ} = \Omega^{PQ}$ with the north-west south-east conventions. Note that for D=4 dimensions, G-generators are embedded into the symplectic group (this is related to the vector-vector duality in 4 spacetime dimensions), therefore $(t_\beta)_{[RS]} = 0$. Once this form of the Z-tensor is inserted into the definition of the new vector field-strength tensor 2.38, it implies that the embedding tensor must satisfy the linear constraint $X_{(RST)} = 0$. Indeed, the 2-forms in the definition of the field strength appear always under the projection

$$Z^Q{}_{RS}B^{RS}_{\mu\nu} \equiv -\frac{1}{2}\Theta^{Q\beta}B_{\mu\nu\beta}, \quad \text{with} \quad B_{\mu\nu\beta} = B^{RS}_{\mu\nu}(t_\beta)_{RS}. \tag{2.41}$$

Consequently, two-forms in D=4 dimensions carry indices in the adjoint representation of G. This is indeed what one would expect from 2-forms in D=4, because 2-forms are on-shell dual to scalar field isometries which transform in the adjoint representation of G. Thus, we derived the linear constraint from bosonic considerations. If $X_{(QRS)} \neq 0$ would imply to include more 2-forms in the gauging procedure against supersymmetry, which fixes the field content of the theory.
It is important to know that the gauge variation of the 2-forms, given in 2.40 is exact only after a projection with the Z-tensor.
The previous mechanism, which induces Stückelberg types interactions between vector fields and 2-forms, is extended to include higher-rank q-forms if they are present. In this case, the

gauge transformations take the form

$$\begin{aligned}
\delta \mathbf{V} &= g\Theta\Lambda\mathbf{V}, \\
\delta A_\mu &= D_\mu\Lambda - g\Theta\Xi_\mu, \\
\delta B_{\mu\nu} &= 2D_{[\mu}\Xi_{\nu]} + ... - g\Theta\Phi_{\mu\nu}, \\
\delta C_{\mu\nu\rho} &= 3D_{[\mu}\Phi_{\nu\rho]} + ... - g\Theta\Sigma_{\mu\nu\rho}, \\
etc.
\end{aligned} \tag{2.42}$$

Therefore, the intertwining also extends between the (q-1) forms and the q forms, with the same Stückelberg mechanism. This also gives us a way to analyse the representations under which the field content of the theory transforms, which, for maximal supergravities, agree with what is expected once the embedding tensor is taken to lie in the representations given in Table 2.3.

2.2.3 Summary

In this chapter we reviewed the constructions of the ungauged supergravity theories, above all their bosonic sector, made by scalars, vectors, and higher-rank q-forms and organised by the global symmetry group G. The group G is obtained through torus-compactification from higher-dimensional supergravities. We also discussed the gauging procedure and how it is possible to have a completely G-covariant mechanism to accomplish it. The mechanism necessitates the use of a fundamental object, the embedding tensor, in terms of which it is possible to entirely construct the deformations of the ungauged Lagrangian. The embedding tensor can be defined group-theoretically and has to satisfy two constraints. The linear constraint determines the representation content of the embedding tensor and follows from supersymmetry. We also showed how one can deduce the linear constraint by simple considerations regarding the consistency of the tensor algebra on the lowest-rank tensor fields with the field content of the theory. The quadratic constraint instead arises by the requirement that the embedding tensor itself is invariant under a gauge transformation and it implies the existence of a gauge algebra for the new generators. Once these two constraints are solved, we obtain the most general, consistent, completely G-covariant formulation of gauged supergravity.

We also showed how, to have a covariant theory, one has to introduce a new field strength for the vectors (as well as for higher-rank q-forms) which will cause a Stückelberg intertwining between (q-1)-forms and q-forms. We are now ready to apply this method to specific cases.

Chapter 3

The Fifth and Seventh Dimension

3.1 The interesting 5 and 7

Five and seven are interesting numbers for physicists; apart from being the number of human senses and a happy number, respectively, they possess some peculiar characteristics once studied as the number of space-time dimensions. Indeed, this thesis will focus exactly on these dimensions for their properties. Conformal Field Theory (CFT) will be crucial in the following discussion. Indeed, the five dimensions have been the testbed for the AdS/CFT conjecture since its very beginning [18], [19], [20]. This conjecture relates a gravity theory (in particular, it has been formulated and extensively used within the framework of String Theory) living on an anti-de Sitter (AdS) background in D-dimensions to a Conformal Field Theory, living at the boundary of this spacetime, therefore on a (D-1) dimensional manifold. Conformal Field Theories are fundamental in High-Energy Physics because every Quantum Field Theory (QFT) can be thought of as a deformation starting from a related CFT, where to reach a QFT we add marginal and relevant couplings to the theory. So, by studying theory with an AdS background in 5-dimensions and using the AdS/CFT conjecture, it will be possible to obtain useful information about the Conformal Field Theories living on a 4-dimensional background and therefore about the possible QFT deformations. The AdS/CFT conjecture has been extensively analysed in five dimensions, and several examples exist here, so it is possible to compare our results with previous ones in the literature. Similar reasons led to the study of the seven dimensions. It has been proven that 6 is the maximum number of spacetime dimensions supporting the existence of a Conformal Field Theory (Super-Conformal actually) [30], so always by means of the AdS/CFT conjecture it is possible to study the possible CFTs in 6 dimensions by scrutinising the possible AdS backgrounds in 7 dimensions. In addition, being supergravity type IIA/IIB living in 10 dimensions and the low-energy limit of M-Theory living in 11 dimensions, 7 is not far from these numbers to be a good testbed for some of the recent Swampland conjectures such as the de Sitter(dS) conjecture [31], which claims that it is very difficult (impossible) to have de Sitter Vacua in String Theory. In the following sections, we will first review the 5-dimensional gauged maximal supergravities [32], and then we will proceed to the analysis of the 7-dimensional case [33].

3.2 The Maximal D=5 Supergravities

Let us start by analysing the embedding tensor group theoretically. In the ungauged version, the vector fields transform in the $\overline{\mathbf{27}}$ representation of $\mathrm{E}_{6(6)}$, which is the global symmetry group of the Lagrangian. We will label the 78 generators by $(t_\alpha)_M{}^N$ and therefore the variation of $A_\mu{}^P$ is given by $\delta A_\mu{}^R = -\Lambda^\beta (t_\beta)_Q{}^R A_\mu{}^Q$. Then, as usual, the gauge generators are given by $X_R = \Theta_R{}^\beta t_\beta$, where β and R run, respectively, from 1 to 78 and from 1 to 27, therefore the theta-tensor lives in the $\mathbf{27}\times\mathbf{78}$ representation of $E_{6(6)}$. Now we need to identify what are the constraints the embedding tensor has to satisfy. The new generators must define a Lie algebra

$$[X_P, X_Q] = f_{PQ}{}^R X_R, \tag{3.1}$$

where the structure constants are not yet known. The previous relation can also be written as

$$\Theta_P{}^\alpha \Theta_Q{}^\beta f_{\alpha\beta}{}^\gamma = f_{PQ}{}^R \Theta_R{}^\gamma, \tag{3.2}$$

where $f_{\alpha\beta}{}^\gamma$ are the $\mathrm{E}_{6(6)}$ structure constants: $[t_\alpha, t_\beta] = f_{\alpha\beta}{}^\gamma t_\gamma$. As already shown, the new structure constants $f_{PQ}{}^R$ must satisfy the Jacobi identity projected by the Theta tensor:

$$f_{[PQ}{}^R f_{S]R}{}^T \Theta_T{}^\beta = 0. \tag{3.3}$$

The gauge fields involved in the gauging (which are a subset of the 27 vector fields) must transform in the adjoint representation of the gauge group, as expected from gauge connections, but the gauge field charges coincide with X_R in the $\overline{\mathbf{27}}$ representation. Consequently, the new generators $(X_P)_Q{}^R$ have a decomposition consisting of the adjoint representation of the gauge group plus other terms that vanish once contracted with the embedding tensor:

$$(X_P)_Q{}^R \Theta_R{}^\alpha \equiv \Theta_P{}^\beta t_{\beta Q}{}^R \Theta_R{}^\alpha = -f_{PQ}{}^R \Theta_R{}^\alpha. \tag{3.4}$$

This is the same as 3.2 in the $\overline{\mathbf{27}}$ representation, and together they state the gauge invariance of the embedding tensor. Indeed, by putting them together, we can obtain the quadratic constraint

$$C_{PQ}{}^{\alpha} \equiv f_{\beta\gamma}{}^{\alpha}\Theta_P{}^{\beta}\Theta_Q{}^{\gamma} + t_{\beta Q}{}^{R}\Theta_P{}^{\beta}\Theta_R{}^{\alpha} = 0. \tag{3.5}$$

This is equivalent to 2.28. Instead, the linear constraint states that the theta tensor is limited to the **351** representation of $E_{6(6)}$:

$$\mathbf{27} \times \mathbf{78} = \mathbf{27} + \mathbf{351} + \overline{\mathbf{1728}}. \tag{3.6}$$

This implies that

$$t_{\beta P}{}^{Q}\Theta_Q{}^{\beta} = 0, \qquad (t_\beta t^\alpha)_P{}^{Q}\Theta_Q{}^{\beta} = -\frac{2}{3}\Theta_P{}^{\alpha} \tag{3.7}$$

where the Greek indices are raised with the inverse of the $E_{6(6)}$-invariant metric $\eta_{\alpha\beta} = tr(t_\alpha t_\beta)$. From these relations, it is possible to project the quadratic constraint as

$$t_{\beta P}{}^{Q}C_{RQ}{}^{\beta} = 0, \qquad (t_\beta t^\alpha)_P{}^{Q}C_{RQ}{}^{\beta} = -\frac{2}{3}C_{PR}{}^{\alpha}, \qquad t_{\beta P}{}^{Q}C_{QR}{}^{\beta} = t_{\beta R}{}^{Q}C_{QP}{}^{\beta}. \tag{3.8}$$

The previous projections simply imply that the quadratic constraint belongs to the representations in $\mathbf{27} \times \mathbf{351}$. But, obviously, the product of two embedding tensors lies in the symmetric product of two 351 representations. Explicitly illustrating both these representation products, one gets

$$\begin{aligned} (\mathbf{351} \times \mathbf{351})_s &= \overline{\mathbf{27}} + \mathbf{1728} + \mathbf{351}' + \mathbf{7722} + \mathbf{17550} + \mathbf{34398}, \\ \mathbf{27} \times \mathbf{351} &= \overline{\mathbf{27}} + \mathbf{1728} + \overline{\mathbf{351}} + \mathbf{7371}, \end{aligned} \tag{3.9}$$

Consequently, $C_{PQ}{}^{\alpha}$ lies in the $\overline{\mathbf{27}} + \mathbf{1728}$ representation. $X_{PQ}{}^{R}$ belongs to the same $E_{6(6)}$ representation of the theta tensor, because the $\mathrm{E}_{6(6)}$ generators are invariant tensors under $\mathrm{E}_{6(6)}$ transformations.
In general, it is also true that the product of three **27** representations has a singlet, labelled by d_{PQR} that is completely symmetric in all three indices. This is also valid for the product of three $\overline{\mathbf{27}}$ representations, so there exists also an invariant d^{PQR} tensor. These facts imply that

$$X_{PQ}{}^{Q} = 0 = X_{QP}{}^{Q} \qquad X_{P(Q}{}^{R}d_{ST)R} = 0 = X_{PQ}{}^{(R}d^{ST)Q}, \tag{3.10}$$

The first equation kills the 27 representation inside $X_{PQ}{}^{R}$ and is equivalent to the first of 3.7, the second instead, just states the invariance of the d-tensor. It is possible, in principle, to decompose the new generators $X_{PQ}{}^{R}$ into their symmetric and anti-symmetric parts, $X_{PQ}{}^{R} = X_{(PQ)}{}^{R} + X_{[PQ]}{}^{R}$, but they cannot transform into two irreducible representations because X lies in a unique representation, namely **351**. This can be checked by analysing the product of a $\overline{\mathbf{27}}$ representation with the symmetric and antisymmetric product of two **27** representations,

$$\begin{aligned}\overline{\mathbf{27}} \times (\mathbf{27} \times \mathbf{27})_s &= \overline{\mathbf{27}} \times (\overline{\mathbf{27}} + \mathbf{351}') = \mathbf{351} + 2(\mathbf{27}) + \overline{\mathbf{351}'} + \overline{\mathbf{1728}} + \overline{\mathbf{7722}}, \\ \overline{\mathbf{27}} \times (\mathbf{27} \times \mathbf{27})_a &= \overline{\mathbf{27}} \times \overline{\mathbf{351}} = \mathbf{351} + \mathbf{27} + \overline{\mathbf{1728}} + \overline{\mathbf{7371}}.\end{aligned} \tag{3.11}$$

We can see from the last relation that it is possible to build two contractions of the ${X_{PQ}}^R$ tensor with the d-tensor, these will produce some tensor of the form Z^{PQ} which from the index structure belongs to the $\overline{\mathbf{27}} \times \overline{\mathbf{27}}$ representation and must be antisymmetric in its indices in order to contain the **351** representation. In addition, obviously, the two contractions must produce the same tensor.

$$\begin{aligned}{X_{PQ}}^R d^{SPQ} &= Z^{RS}, \\ 2{X_{PQ}}^R d^{PST} d^{QUV} d_{SUR} &= Z^{TV}.\end{aligned} \tag{3.12}$$

This is in accordance with the second equation of 3.10, which guarantees that Z (as given by the first relation) is antisymmetric because the symmetric part vanishes. The coefficients agree perfectly as long as

$$d_{PQR} d^{SQR} = \delta_P^S. \tag{3.13}$$

In addition, another useful relation can be identified once one recalls that the product of four **27** representations gives one **27** representation:

$$d_{S(MN} d_{PQ)T} d^{STR} = \frac{2}{15} \delta^R_{(M} d_{NPQ)}. \tag{3.14}$$

Due to the latter, the inverse of 3.12 can be deduced:

$${X_{(PQ)}}^R = d_{PQS} Z^{RS}, \qquad {X_{[PQ]}}^R = 10 d_{PST} d_{QUV} d^{RSU} Z^{TV}. \tag{3.15}$$

Some other formulations of the quadratic constraint can be deduced. Indeed, among them, there is the form of the quadratic constraint that we are going to use. Taking into account the tensor $Z^{PQ} {\Theta_Q}^\alpha$, this transforms into $\overline{\mathbf{27}} \times \mathbf{78} = \overline{\mathbf{27}} + \overline{\mathbf{351}} + \mathbf{1728}$, which is equivalent to the square of the theta tensor and therefore must also be contained in the representations in the first line of 3.9. But the only representations they have in common are $\overline{\mathbf{27}}$ and **1728**, which vanish because of the Quadratic Constraint, so

$$Z^{PQ} {\Theta_Q}^\alpha = 0 \implies Z^{PQ} X_Q = 0. \tag{3.16}$$

Similarly, considering the second relation in 3.11 and the tensor $X_{PQ}{}^{[R}Z^{S]Q}$, one sees that the latter transforms into the $\overline{\mathbf{351}}+\overline{\mathbf{27}}+\mathbf{1728}+\mathbf{7371}$ representation. By comparing always with the square of the embedding tensor (symmetric), one ends up with the same representations put to 0 by the quadratic constraint, thus

$$X_{PQ}{}^{[R}Z^{S]Q}=0. \tag{3.17}$$

What we just obtained are equivalent versions of the quadratic constraint:

$$\begin{aligned} X_{MP}{}^{R}X_{NR}{}^{Q}-X_{NP}{}^{R}X_{MR}{}^{Q}+X_{MN}{}^{R}X_{RP}{}^{Q}=0, \\ Z^{PQ}X_{Q}=0, \\ X_{PQ}{}^{[R}Z^{S]Q}=0. \end{aligned} \tag{3.18}$$

3.2.1 The Scalar Sector

In order to discuss the vacua of the theory, which is of uttermost interest to us, we need to introduce the T-tensor. The scalars live on a coset manifold, which for the case of D=5 dimensions is given by $\mathrm{E}_{6(6)}/\mathrm{USp}(8)$, so it is possible to parameterize them as $\mathbf{V}(x)\in\mathrm{E}_{6(6)}$ in the fundamental **27** representation. The coset representative transforms from the right under a global $\mathrm{E}_{6(6)}$ transformation and from the left under local $\mathrm{USp}(8)$. The T-tensor is nothing more than a dressed embedding tensor, namely, a theta tensor contracted with the scalar coset representative:

$$T_{\underline{PQ}}{}^{\underline{R}}[\Theta,\phi]=\mathbf{V}^{-1P}_{\underline{P}}\mathbf{V}^{-1Q}_{\underline{Q}}\mathbf{V}_{R}{}^{\underline{R}}X_{PQ}{}^{R}. \tag{3.19}$$

Here, the underlined indices transform under USp(8). The theta tensor is treated as a spurionic object that transforms under G as $\Theta_{P}{}^{\beta}t_{\beta}\to g_{P}{}^{Q}\Theta_{Q}{}^{\beta}(gt_{\beta}g^{-1})$, where g is a global $\mathrm{E}_{6(6)}$ transformation. Once one picks the gauging and fixes the Θ-tensor, the duality invariance is broken. The constraints on the Θ-tensor transfer to a set of equivalent constraints on the T-tensor. Indeed, it is possible to express every variation of $\mathbf{V}$ as a (field dependent when needed) $\mathrm{E}_{6(6)}$ transformation acting from the right, e.g. considering a global $\mathrm{E}_{6(6)}$ transformation

$$\mathbf{V}\to\mathbf{V}'=g\mathbf{V}=\mathbf{V}\xi^{-1} \qquad \text{with} \quad \xi^{-1}=\mathbf{V}^{-1}g\mathbf{V}\in\mathrm{E}_{6(6)}. \tag{3.20}$$

Thus, every possible transformation, even supersymmetry, can be parametrised in this form and consequently have an effect on the T-tensor, described by

$$T_{\underline{PQ}}{}^{\underline{R}} \to T'_{\underline{PQ}}{}^{\underline{R}} = \xi_{\underline{P}}{}^{\underline{S}}\xi_{\underline{Q}}{}^{\underline{T}}(\xi^{-1})_{\underline{U}}{}^{\underline{R}}T_{\underline{ST}}{}^{\underline{U}}. \tag{3.21}$$

Therefore, the T-tensor varies under any transformation with a (possibly field-dependent) $\mathrm{E}_{6(6)}$ transformation. The maximal compact subgroup of $\mathrm{E}_{6(6)}$, namely USp(8), corresponds to the R-symmetry, and therefore the fermionic fields must transform under it. The gravitinos lie in the **8** representation, whereas fermions are in the **48** representation. The product of two **8** representations contains a singlet of the symplettic group, namely the skew-symmetric $\Omega^{AB} = (\Omega_{AB})^*$ with $A, B, .. = 1, ..., 8$, which satisfies the relation $\Omega^{AC}\Omega_{CB} = -\delta^A_B$. Therefore, once the fermionic content is introduced, it is more natural to switch to a notation in which the vector indices P, Q, R are replaced by a couple of indices $[AB]$, $A, B = 1, ..., 8$, which are antisymmetric and symplectically traceless. For example, the **27** representation is written in terms of a pseudo-real tensor x_{AB}.

$$x^P \equiv x^{AB} = (x_{AB})^* = \Omega^{AC}\Omega^{BD}x_{CD}, \qquad \Omega^{BC}x_{BC} = 0. \tag{3.22}$$

Therefore, raising and lowering indices are obtained through complex conjugation. An infinitesimal $\mathrm{E}_{6(6)}$ transformation is written as

$$\begin{aligned} \delta x_{AB} &= -2\Lambda_{[A}{}^{C}x_{B]C} + \Sigma_{ABCD}x^{CD}, \\ \delta y^{AB} &= 2\Lambda_{C}{}^{[A}y^{B]C} - \Sigma^{ABCD}y_{CD}. \end{aligned} \tag{3.23}$$

By doing so, we ensure that $x_{AB}y^{AB}$ is $\mathrm{E}_{6(6)}$ invariant. Under USp(8) the adjoint representation of $\mathrm{E}_{6(6)}$ decomposes as $\mathbf{78} \to \mathbf{36} + \mathbf{42}$, which is equivalent to the split in 3.23, where $\Lambda_A{}^B$ represents an USp(8) transformation and Σ^{ABCD} the remaining transformations (non-compact) within $\mathrm{E}_{6(6)}$. This implies some properties concerning the parameters of the $\mathrm{E}_{6(6)}$ transformations:

$$\begin{aligned} &\Lambda_A{}^B \equiv (\Lambda^A{}_B)^* = -\Lambda_B{}^A, \quad \Lambda_{[A}{}^C\Omega_{B]C} = 0, \quad \Lambda_A{}^A = 0, \quad \Sigma_{ABCD} = \Sigma_{[ABCD]}, \\ &\Sigma_{ABCD} \equiv (\Sigma^{ABCD})^* = \Omega_{AE}\Omega_{BF}\Omega_{CG}\Omega_{DH}\Sigma^{EFGH}, \quad \Omega^{AB}\Sigma_{ABCD} = 0. \end{aligned} \tag{3.24}$$

Furthermore, since Σ^{ABCD} lies in the irreducible **42** representation of USp(8), it must also be true that $\Omega_{[AB}\Sigma_{CDEF]} = 0$ (this should, in principle, lie in the **28** representation). Thanks to the latter equation, one obtains that $\Omega_{AB}y^{BC}\Omega_{CD}y^{DE}\Omega_{EF}y^{FA}$ remains unchanged under any $\mathrm{E}_{6(6)}$ transformation, therefore, it is the equivalent of the d_{PQR} tensor. The notation that we use will mix the vectorial $E_{6(6)}$ notation with the symplectic one just presented. Indeed, local USp(8) indices $\underline{P}, \underline{Q}, etc.$ will be denoted by symplectic traceless, antisymmetric pairs [ij]. The latter are also the indices labelling the fermionic fields. The coset representative then becomes $\mathbf{V}_P{}^{ij}$, with $\mathbf{V}_P{}^{ij}\Omega_{ij} = 0$. In addition, coset representatives are pseudoreal, meaning $\mathbf{V}_{Pij} \equiv (\mathbf{V}_P{}^{ij})^* = \Omega_{ki}\Omega_{lj}\mathbf{V}_P{}^{kl}$. The inverse of the coset representative $\mathbf{V}_P{}^{ij}$ is given by $\mathbf{V}_{ij}{}^P$ and the following relations are valid:

$$\begin{aligned} \mathbf{V}_P{}^{ij}\mathbf{V}_{ij}{}^Q &= \delta_P^Q, \\ \mathbf{V}_{ij}{}^P\mathbf{V}_P{}^{kl} &= \delta_{ij}^{kl} - \frac{1}{8}\Omega_{ij}\Omega^{kl}. \end{aligned} \tag{3.25}$$

As has already been pointed out, any variation of the coset representative is given in terms of a right multiplication by a possibly field-dependent $E_{6(6)}$ transformation. With the new notation, this is given by

$$\Delta\mathbf{V}_P{}^{ij} = \mathbf{V}_P{}^{kl}(2\delta_k^{[i}\mathcal{Q}_l^{j]} + \mathcal{P}^{ijpq}\Omega_{pk}\Omega_{ql}) \tag{3.26}$$

where, again, $\mathcal{Q}$ lies in the **36** representation of USp(8) and $\mathcal{P}$ in the **42** and satisfies the previous conditions 3.24. It is possible to invert 3.26 and obtain an expression for $\mathcal{Q}_i{}^j$ and $\mathcal{P}^{ijkl}$:

$$\begin{aligned} \mathcal{Q}_i{}^j &= \frac{1}{3}\mathbf{V}_{il}{}^P\Delta\mathbf{V}_P{}^{jl}, \\ \mathcal{P}^{ijkl} &= \mathbf{V}_{pq}{}^P\Delta\mathbf{V}_P{}^{[ij}\Omega^{k|p|}\Omega^{l]q}. \end{aligned} \tag{3.27}$$

These expressions are valid for any Δ, even for space-time derivatives and gauge transformations. When Δ is given by the gauge-covariant derivative $D_\mu = \partial_\mu - gA_\mu{}^P X_P$, the previous relations define the USp(8) composite connection $\mathcal{Q}_{\mu i}{}^j$ for USp(8), while $\mathcal{P}_\mu{}^{ijkl}$ represents the USp(8)-covariant tensor, which will enter the scalar kinetic term as it has been extensively explained in the second chapter:

$$\begin{aligned} \mathcal{Q}_{\mu i}{}^j &= \frac{1}{3}\mathbf{V}_{il}{}^P\partial_\mu\mathbf{V}_P{}^{jl} - gA_\mu{}^P Q_{Pi}{}^j, \\ \mathcal{P}_\mu{}^{ijkl} &= \mathbf{V}_{mn}{}^P\partial_{mu}\mathbf{V}_P{}^{[ij}\Omega^{k|m|}\Omega^{l]n} - gA_\mu{}^P\mathcal{P}_P{}^{ijkl}. \end{aligned} \tag{3.28}$$

Having defined:

$$\begin{aligned} \mathcal{Q}_{Pi}{}^j &= \frac{1}{3}\mathbf{V}_{il}{}^Q X_{PQ}{}^R\mathbf{V}_R{}^{jl}, \\ \mathcal{P}_P^{ijkl} &= \mathbf{V}_{mn}{}^Q X_{PQ}{}^R\mathbf{V}_R{}^{[ij}\Omega^{k|m|}\Omega^{l]n}. \end{aligned} \tag{3.29}$$

Thanks to these relations, we see that the T-tensor can be decomposed under USp(8), as

$$\begin{aligned} T^i{}_{jkl} &= \mathcal{Q}_{Pj}{}^i \mathbf{V}_{kl}{}^P, \\ T^{ijkl}{}_{mn} &= \mathcal{P}_P{}^{ijkl} \mathbf{V}_{mn}{}^P. \end{aligned} \tag{3.30}$$

It is also obvious that $\Omega^{k[i} T^{j]}{}_{kmn} = \Omega_{k[i} T^k{}_{j]mn} = 0$. In fact, by reversing the relation 3.19, one obtains the form of the generators in terms of the T-tensor:

$$X_{PQ}{}^R = \mathbf{V}_P{}^{kl} \mathbf{V}_Q{}^{mn} \mathbf{V}_{ij}{}^R [2\delta^i_m T^j{}_{nkl} + T^{ijpq}{}_{kl} \Omega_{pm} \Omega_{qn}]. \tag{3.31}$$

Now, let us discuss the formulation of the consistency constraints in terms of the T-tensor. The linear constraint forced the embedding tensor to live in the **351** representation of $E_{6(6)}$. Once the branchings of this representation under USp(8) are computed, we will obtain $\mathbf{36} + \mathbf{315}$. The T-tensor must, consequently, lie in these representations of USp(8). Indeed:

$$\begin{aligned} T^i{}_{jkl} &: \mathbf{36} \times \mathbf{27} = \mathbf{36} + \mathbf{315} + \mathbf{27} + \mathbf{594}, \\ T^{ijkl}{}_{mn} &: \mathbf{42} \times \mathbf{27} = \mathbf{315} + \mathbf{27} + \mathbf{792}. \end{aligned} \tag{3.32}$$

We can give a description of these 2 representations, **36** and **315** of USp(8),by means of two pseudoreal, symplectic traceless, tensors A_1^{ij} and $A_2{}^{i,jkl}$ with the properties $A_1^{[ij]} = 0$, $A_2{}^{i,jkl} = A_2{}^{i,[jkl]}$ and $A_2{}^{[i,jkl]} = 0$. Thus, the previous branching decomposition of the T-tensor representations explicitly gives:

$$\begin{aligned} T^{klmn}{}_{ij} &= 4A_2{}^{q,[klm} \delta^{n]}{}_{[i} \Omega_{j]q} + 3A_2{}^{p,q[kl} \Omega^{mn]} \Omega_{p[i} \Omega_{j]q}, \\ T_i^{jkl} &= -\Omega_{im} A_2{}^{(m,j)kl} - \Omega_{im} (\Omega^{m[k} A_1^{l]j} + \Omega^{j[k} A_1^{l]m} + \frac{1}{4} \Omega^{kl} A_1^{mj}), \end{aligned} \tag{3.33}$$

Consider now a variation of the coset representative, which, as usual, can be described as an $E_{6(6)}$ transformation from the right, this implies that once the transformation acts on the T-tensor, it gives back something proportional to the T-tensor again. Being the USp(8) transformations straightforward (and the Lagrangian invariant under those), we concentrate only on the remaining variations, namely variations of the type

$$\delta \mathbf{V}_P{}^{ij} = -\mathbf{V}_P{}^{kl} \Omega_{km} \Omega_{ln} \Sigma^{ijmn}. \tag{3.34}$$

This kind of variation determines the variations of the composite connection $\mathcal{Q}$, the USp(8) covariant tensor $\mathcal{P}$, and the T-tensor, just by using their definitions:

$$\begin{aligned}
\delta \mathcal{Q}_{\mu i}{}^{j} &= -\frac{1}{3}\mathcal{P}_{\mu iklm}\Sigma^{jklm} + \frac{1}{3}\Sigma_{iklm}\mathcal{P}_{\mu}{}^{jklm}, \\
\delta \mathcal{P}_{\mu}{}^{ijkl} &= -D_{\mu}\Sigma^{ijkl}, \\
\delta T^{i}{}_{jmn} &= \frac{1}{3}\Sigma_{jpqr}T^{ipqr}{}_{mn} - \frac{1}{3}\Omega^{iv}\Omega_{jw}\Sigma_{vpqr}T^{wpqr}{}_{mn} + \Sigma_{mnpq}\Omega^{pr}\Omega^{qs}T^{i}{}_{jrs}, \\
\delta T^{ijkl}{}_{mn} &= -4T^{[i}{}_{pmn}\Sigma^{jkl]p} + \Sigma_{mnpq}\Omega^{pr}\Omega^{qs}T^{ijkl}{}_{rs}.
\end{aligned} \tag{3.35}$$

Now, applying these variations to 3.33 and making use of $\Omega_{[ij}\Sigma_{klmn]} = 0$ for any Σ in the **42** representation of USp(8), one obtains the variations for the tensors A_1 and A_2.

$$\begin{aligned}
\delta A_{1}{}^{ij} =& \frac{4}{9}\Omega^{p(i}\Sigma^{k)klm}A_{2p,klm}, \\
\delta A_{2}{}^{i,jkl} =& \frac{3}{2}\left(\Omega^{mi}\Sigma^{jkln} + \Omega^{m[j}\Sigma^{kl]in}\right)A_{1mn} \\
&- \left(\Omega^{i[j}\Omega^{k|m|}\Sigma^{l]npq} - 3\Omega^{ni}\Omega^{m[j}\Sigma^{kl]pq} - \frac{1}{6}\Omega^{im}\Omega^{[kl}\Sigma^{j]npq}\right. \\
&\left. + \frac{1}{6}\Omega^{m[j}\Omega^{kl]}\Sigma^{inpq}\right)A_{2m,npq}.
\end{aligned} \tag{3.36}$$

The previous relations are useful for obtaining the equation of motion for the scalars on a fixed Lorentz-invariant background. From these variations, we can also show that

$$Z^{ij,kl} \equiv \Omega^{[i[k}A_{1}{}^{l]j]} + A_{2}{}^{[i,j]kl}, \tag{3.37}$$

which is an antisymmetric and symplectic traceless tensor in the indices couples [ij] and [kl], transforms as

$$\delta Z^{ij,kl} = -\Sigma^{ijmn}\Omega_{mp}\Omega_{nq}Z^{pq,kl} - \Sigma^{klmn}\Omega_{mp}\Omega_{nq}Z^{ij,pq}, \tag{3.38}$$

implying that this is nothing more than the dressed version of the Z^{PQ} tensor that we have already introduced before.

$$Z^{ij,kl} = \frac{1}{\sqrt{5}}\mathbf{V}_{P}{}^{ij}\mathbf{V}_{Q}{}^{kl}Z^{PQ}. \tag{3.39}$$

Here, we used the definition of Z^{RS} in terms of $X_{PQ}{}^{R}$ and d^{SPQ}, with X expressed in terms of the T-tensor and the following definitions for the invariant three-rank tensor

$$\begin{aligned} d_{PQR} &= \frac{2}{\sqrt{5}}\mathbf{V}_P{}^{ij}\mathbf{V}_Q{}^{kl}\mathbf{V}_R{}^{mn}\Omega_{jk}\Omega_{lm}\Omega_{ni}, \\ d^{PQR} &= \frac{2}{\sqrt{5}}\mathbf{V}_{ij}{}^{P}\mathbf{V}_{kl}{}^{Q}\mathbf{V}_{mn}{}^{R}\Omega^{jk}\Omega^{lm}\Omega^{ni}. \end{aligned} \tag{3.40}$$

These latter tensors have the expected form for an invariant USp(8) tensor, as we have already shown. Now, having defined all the useful quantities, we can express the quadratic constraint in an USp(8) covariant form, namely, choosing the second relation in 3.18 as the form of the quadratic constraint to use, we get

$$T^{i}{}_{jkl}Z^{kl,mn} = 0 = T^{ijkl}{}_{mn}Z^{mn,pq}. \tag{3.41}$$

This is the formulation of the quadratic constraint we will be using to study the vacua of the theory, once the linear constraint has been imposed, so working with the tensor A_1 and A_2 in the **36** and **315** representation, respectively.

3.2.2 The D=5 Lagrangian

For completeness, we add some information about the fermionic field content and the other bosonic fields present in the theory before writing down the Lagrangian.
The gravitinos $\psi_\mu{}^{i}$ lie in the **8** representation of USp(8), and the spinors χ^{ijk}, on the other hand, transform in the **48** representation. The covariant field strength tensor, instead, is given by

$$\mathcal{H}_{\mu\nu}{}^{ij} = \mathcal{H}_{\mu\nu}{}^{P}\mathbf{V}_P{}^{ij} = \left(\mathcal{F}_{\mu\nu}{}^{P} + gZ^{PQ}B_{\mu\nu Q}\right)\mathbf{V}_P{}^{ij}, \tag{3.42}$$

where $B_{\mu\nu Q}$ are the two forms that ensure the gauge covariance of the transformations. This is the classic St"uckelberg interaction that we expect from gauged supergravities. The supersymmetry transformations are given by

$$
\begin{aligned}
\delta_S e_\mu{}^a =& \frac{1}{2}\overline{\varepsilon}_i\gamma^a\psi_\mu{}^i,\\
\delta_S \mathbf{V}_P{}^{ij} =& i\mathbf{V}_P{}^{kl}\left[4\Omega_{p[k}\overline{\chi}_{lmn]}\varepsilon^p + 3\Omega_{[kl}\overline{\chi}_{mn]p}\varepsilon^p\right]\Omega^{mi}\Omega^{nj},\\
\delta_S A_\mu{}^P =& 2\left[i\Omega^{ik}\overline{\varepsilon}_k\psi_\mu{}^j + \overline{\varepsilon}_k\gamma_\mu\chi^{ijk}\right]\mathbf{V}_{ij}{}^P,\\
\delta_S B_{\mu\nu P} =& \frac{4}{\sqrt{5}}\mathbf{V}_P{}^{ij}\left[2\overline{\psi}_{[\mu i}\gamma_{\nu]}\varepsilon^k\Omega_{jk} - i\overline{\chi}_{ijk}\gamma_{\mu\nu}\varepsilon^k\right] + 2d_{PQR}A_{[\mu}{}^Q\delta_S A_{\nu]}{}^P,\\
\delta_S\psi_\mu{}^i =& \left(\partial_\mu\delta^i_j - \mathcal{Q}_\mu{}_j{}^i - \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\delta^i_j\right)\varepsilon^j\\
&+ i\left[\frac{1}{12}\left(\gamma_{\mu\nu\rho}\mathcal{H}^{\nu\rho ij} - 4\gamma^\nu\mathcal{H}_{\mu\nu}{}^{ij}\right) - g\gamma_\mu A_1{}^{ij}\right]\Omega_{jk}\varepsilon^k,\\
\delta_S\chi^{ijk} =& \frac{1}{2}i\gamma^\mu\mathcal{P}_\mu{}^{ijkl}\Omega_{lm}\varepsilon^m - \frac{3}{16}\gamma^{\mu\nu}\left[\mathcal{H}_{\mu\nu}{}^{ij}\varepsilon^{k]} - \frac{1}{3}\Omega^{[ij}\mathcal{H}_{\mu\nu}{}^{k]m}\Omega_{mn}\varepsilon^n\right]\\
&+ gA_2{}^{l,ijk}\Omega_{lm}\varepsilon^m,
\end{aligned}
\tag{3.43}
$$

The Lagrangian is then:

$$
\begin{aligned}
e^{-1}\mathcal{L} =& -\frac{1}{2}R - \frac{1}{2}\overline{\psi}_{\mu i}\gamma^{\mu\nu\rho}D_\nu\psi_\rho{}^i - \frac{1}{16}\mathcal{H}_{\mu\nu}{}^{ij}\mathcal{H}^{\mu\nu kl}\Omega_{ik}\Omega_{jl} - \frac{2}{3}\overline{\chi}_{ijk}\not{D}\chi^{ijk} - \frac{1}{12}\left|\mathcal{P}_\mu{}^{ijkl}\right|^2\\
&+ \frac{2}{3}i\mathcal{P}_\mu{}^{ijkl}\overline{\chi}_{ijk}\gamma^\nu\gamma^\mu\psi_\nu{}^m\Omega_{lm} + \mathcal{H}^{\rho\sigma ij}\left[\frac{1}{8}i\overline{\psi}_{\mu i}\gamma^{[\mu}\gamma_{\rho\sigma}\gamma^{\nu]}\psi_\nu{}^k\Omega_{kj} - \frac{1}{4}\overline{\chi}_{ijk}\gamma^\mu\gamma_{\rho\sigma}\psi_\mu{}^k\right.\\
&\left. - \frac{1}{2}i\overline{\chi}_{ikl}\gamma_{\rho\sigma}\chi^{mkl}\Omega_{mj}\right] + \frac{\sqrt{5}}{64e}i\varepsilon^{\mu\nu\rho\sigma\tau}\left\{gZ^{MN}B_{\mu\nu M}\left[D_\rho B_{\sigma\tau N} + \right.\right.\\
&\left. + 4d_{NPQ}A_\rho{}^P\left(\partial_\sigma A_\tau{}^Q + \frac{1}{3}gX_{[RS]}{}^QA_\sigma{}^RA_\tau{}^S\right)\right] - \frac{8}{3}d_{MNP}\left[A_\mu{}^M\partial_\nu A_\rho{}^N\partial_\sigma A_\tau{}^P\right.\\
&\left.\left. + \frac{3}{4}gX_{[QR]}{}^MA_\mu{}^NA_\nu^QA_\rho{}^R\left(\partial_\sigma A_\tau{}^P + \frac{1}{5}gX_{[ST]}{}^PA_\sigma{}^SA_\tau{}^T\right)\right]\right\} - \frac{3}{2}igA_1{}^{ik}\Omega_{kj}\overline{\psi}_{\mu i}\gamma^{\mu\nu}\psi_\nu{}^j\\
&- \frac{4}{3}g\Omega_{ml}A_2{}^{l,ijk}\overline{\chi}_{ijk}\gamma^\mu\psi_\mu{}^m + 2gi\Omega_{kp}\Omega_{lq}\left[-4A_2{}^{i,jpq} + A_1{}^{[i[p}\Omega^{q]j]}\right]\overline{\chi}_{ijm}\chi^{klm}\\
&+ g^2\left[3\left|A_1{}^{ij}\right|^2 - \frac{1}{3}\left|A_2{}^{i,jkl}\right|^2\right],
\end{aligned}
\tag{3.44}
$$

where the covariant derivatives of the fermionic fields are given by

$$\begin{aligned} D_\mu \psi_\nu{}^i &= \partial_\mu \psi_\nu{}^i - \mathcal{Q}_{\mu j}{}^i \psi_\nu{}^j - \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\psi_\nu{}^i, \\ D_\mu \chi^{ijk} &= \partial_\mu \chi^{ijk} - 3\mathcal{Q}_{\mu l}{}^{[i}\chi^{jk]l} - \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\chi^{ijk}. \end{aligned} \tag{3.45}$$

This is all we need to know about the D=5 gauged supergravity theories; further information about the scalar potential and the mass matrices will be provided in the following chapters, when dealing with the vacua.

3.3 The Maximal D=7 Supergravities

The global symmetry group in seven spacetime dimensions is $E_{4(4)}$ = SL(5), therefore, there are 24 generators $t^P{}_Q$ labelled with indices $P,Q = 1,\ldots,5$ satisfying $t^P{}_P = 0$. The generators obey an algebra

$$[t^P{}_Q, t^R{}_S] = \delta^R_Q t^P{}_S - \delta^P_S t^R{}_Q. \tag{3.46}$$

The ungauged formulation of the theory possesses vector fields with abelian redundancies in the $\overline{\mathbf{10}}$ representation of SL(5), namely $A^{PQ}_\mu = A^{[PQ]}_\mu$, so a G-transformation can be written $\delta A^{PQ}_\mu = 2\Lambda_R{}^{[P}A_\mu{}^{Q]R}$. The theory also possesses two forms $B_{\mu\nu P}$ that lie in the $\mathbf{5}$ representation of SL(5). The new generators for the gauge group G_0 are identified by $X_{MN} = X_{[MN]}$, and, of course, there may be as many gauge generators as vector fields in the theory. The generators are given by

$$X_{PQ} = \Theta_{PQ,R}{}^S t^R{}_S. \tag{3.47}$$

Therefore, the gauge-covariant derivatives act as

$$D_\mu = \nabla_\mu - gA^{PQ}_\mu \Theta_{PQ,R}{}^S t^R{}_S, \tag{3.48}$$

To find the consistency constraints, we first identify the representations contained in the most general embedding tensor:

$$\mathbf{10} \otimes \mathbf{24} = \mathbf{10} + \mathbf{15} + \overline{\mathbf{40}} + \mathbf{175}. \tag{3.49}$$

Consistency with supersymmetry implies that only the $\mathbf{15}$ and $\overline{\mathbf{40}}$ representations are present. This means that the embedding tensor can be decomposed into a symmetric matrix $Y_{PQ} = Y_{(PQ)}$ and a tensor $Z^{PQ,R} = Z^{[PQ],R}$ such that $Z^{[PQ,R]} = 0$:

$$\Theta_{PQ,R}{}^S = \delta^S_{[P}Y_{Q]R} - 2\varepsilon_{PQRTU}Z^{TU,S}. \tag{3.50}$$

The gauge group generators in the **5** representation take the form

$$(X_{PQ})_R{}^S = \Theta_{PQ,R}{}^S = \delta^S_{[P}Y_{Q]R} - 2\varepsilon_{PQRTU}Z^{TU,S}. \tag{3.51}$$

Where, the generators that satisfy 3.46 are chosen such that $(t^P{}_Q)_I{}^J = \delta^P_I\delta^J_Q - \frac{1}{5}\delta^J_I\delta^P_Q$, these indeed satisfy $t^P{}_P = 0$. The gauge generators in the **10** representation are given by $(X_{PQ})_{RS}{}^{TU} = 2(X_{PQ})_{[R}{}^{[T}\delta^{U]}_{S]}$, in addition note that

$$(X_{PQ})_{RS}{}^{TU} + (X_{RS})_{PQ}{}^{TU} = 2Z^{TU,W}d_{W,[PQ][RS]}. \tag{3.52}$$

The d-tensor just defined is the invariant tensor of SL(5), explicitly defined as $\mathrm{d}_{W,[PQ][RS]} = \varepsilon_{WPQRS}$. As usual, by demanding the closure of the gauge algebra or the invariance of the embedding tensor under gauge transformations, one attains the quadratic constraint:

$$(X_{MN})_{PQ}{}^{TU}\Theta_{TU,R}{}^S + (X_{MN})_R{}^T\Theta_{PQ,T}{}^S - (X_{MN})_T{}^S\Theta_{PQ,R}{}^T = 0. \tag{3.53}$$

Using the previous relations, it is possible to express this constraint in terms of the Y and Z tensors:

$$Y_{MQ}Z^{QN,P} + 2\varepsilon_{MRSTU}Z^{RS,N}Z^{TU,P} = 0. \tag{3.54}$$

The quadratic constraint has some irreducible projections on the $\overline{\mathbf{5}}$, $\overline{\mathbf{45}}$ and $\overline{\mathbf{70}}$ representations of SL(5), giving rise to

$$Z^{PQ,R}Y_{RS} = 0, \qquad Z^{PQ,R}X_{PQ} = 0, \tag{3.55}$$

with X_{PQ} that can lie in any representation. Another particular form of the quadratic constraints, equivalent to the invariance of the Θ-tensor and showing the closure of the algebra, is given by

$$[X_{PQ}, X_{RS}] = -(X_{PQ})_{RS}{}^{TU}X_{TU}. \tag{3.56}$$

The seven-dimensional theory possesses some q-forms in its field content, in particular there are the vector fields A^{PQ}_μ, the 2-forms $B_{\mu\nu P}$ transforming in the **5** representation of SL(5) and the 3-forms $S^P_{\mu\nu\rho}$ belonging to the $\overline{\mathbf{5}}$ of SL(5). The Θ tensor will project only on the components of these fields that are involved in the gaugings. The vector field strength, given by

$$\mathcal{F}^{PQ}_{\mu\nu} = 2\partial_{[\mu}A^{PQ}_{\nu]} + g(X_{RS})_{TU}{}^{PQ}A^{RS}_{[\mu}A^{TU}_{\nu]}, \tag{3.57}$$

does not behave as expected under gauge transformations, as was explained in the general description of the gauging procedure. Therefore, we introduce the covariant field strength tensor

$$\mathcal{H}^{PQ}_{\mu\nu} = \mathcal{F}^{PQ}_{\mu\nu} + gZ^{PQ,R}B_{\mu\nu R}, \tag{3.58}$$

creating the usual Stückelberg type coupling between 1 and 2-forms. In an analogous manner, one introduces the covariant field strength for the 2-forms

$$\mathcal{H}_{\mu\nu\rho M} = 3D_{[\mu}B_{\nu\rho]M} + 6\varepsilon_{MNPQR}A^{NP}_{[\mu}\left(\partial_\nu A^{QR}_{\rho]} + \frac{2}{3}gX_{ST,U}{}^{Q}A^{RU}_{\nu}A^{ST}_{\rho]}\right) + gY_{MN}S^{N}_{\mu\nu\rho}. \tag{3.59}$$

3.3.1 The Scalar Sector

The scalars, in seven dimensions, live on the coset manifold SL(5)/SO(5), where SO(5) $\sim$ USp(4) is the R symmetry of the theory. Therefore, the scalars are described by a matrix $\mathbf{V} \in$ SL(5), which transforms from the left under rigid SL(5) transformations and from the right under local SO(5) transformations. We will be using indices a,b,... = 1,...,4 for the **4** representation of USp(4). As usual, we introduce the antisymmetric, invariant, symplectic tensor Ω_{ab}, satisfying $(\Omega_{ab})^* = \Omega^{ab}$ and $\Omega_{ab}\Omega^{cb} = \delta^c_a$. Some useful USp(4) representations are given by:

$$\begin{array}{lllll}
\mathbf{1}: & V_{\mathbf{1}} & & & \\
\mathbf{5}: & V^{ab}_{\mathbf{5}} = V^{[ab]}_{\mathbf{5}}, & \Omega_{ab}V^{ab}_{\mathbf{5}} = 0, & & \\
\mathbf{10}: & V^{ab}_{\mathbf{10}} = V^{(ab)}_{\mathbf{10}}, & & & \\
\mathbf{14}: & V^{ab}_{\mathbf{14}\,cd} = V^{[ab]}_{\mathbf{14}\;[cd]}, & V^{ab}_{\mathbf{14}\,cb} = 0, & \Omega_{ab}V^{ab}_{\mathbf{14}\,cd} = 0 = \Omega^{cd}V^{ab}_{\mathbf{14}\,cd}, & \\
\mathbf{35}: & V^{ab}_{\mathbf{35}\,cd} = V^{[ab]}_{\mathbf{35}\;(cd)}, & V^{ab}_{\mathbf{35}\,cb} = 0, & \Omega_{ab}V^{ab}_{\mathbf{35}\,cd} = 0. &
\end{array} \tag{3.60}$$

In addition, objects that transform in these representations are pseudoreal, which means that

$$(V_{\mathbf{1}})^* = V_{\mathbf{1}}, \quad V_{\mathbf{5}ab} = (V^{ab}_{\mathbf{5}})^* = \Omega_{ac}\Omega_{bd}V^{cd}_{\mathbf{5}}, \quad V_{\mathbf{14}ab}{}^{cd} = (V^{ab}_{\mathbf{14}\,cd})^* = \Omega_{ae}\Omega_{bf}\Omega^{cg}\Omega^{dh}V^{ef}_{\mathbf{14}\,gh}, \tag{3.61}$$

so that we can raise and lower indices with complex conjugation or with the Ω tensor. SL(5) branches under its USp(4) subgroup as $\mathbf{24} \to \mathbf{10} + \mathbf{14}$, representing the compact and non-compact part of the group. SL(5) vector indices P,Q,R,... are substituted by the index pairs [ab] of USp(4). Then an element in the algebra of SL(5): $L = L_P{}^Q t^P{}_Q$ decomposes as

$$L_{ab}{}^{cd} = 2\Lambda_{[a}{}^{[c}\delta^{d]}_{b]} + \Sigma^{cd}{}_{ab}. \tag{3.62}$$

where Λ and Σ have the following properties:

$$\Lambda_{[a}{}^{c}\Omega_{b]c} = 0, \qquad \Sigma^{ab}{}_{cb} = 0, \qquad \Sigma^{ab}{}_{cd}\Omega^{cd} = 0 = \Omega_{ab}\Sigma^{ab}{}_{cd}, \tag{3.63}$$

in order to transform in the **10** and **14** representations, respectively. Consequently, the coset representative has the following index structure $\mathbf{V}_P{}^{ab} = \mathbf{V}_P{}^{[ab]}$, such that $\mathbf{V}_P{}^{ab}\Omega_{ab} = 0$. Its inverse is given by the matrix $\mathbf{V}_{ab}{}^{P}$ such that:

$$\mathbf{V}_P{}^{ab}\mathbf{V}_{ab}{}^{Q} = \delta_P^Q, \qquad \mathbf{V}_{ab}{}^{P}\mathbf{V}_P{}^{cd} = \delta_{ab}^{cd} - \frac{1}{4}\Omega_{ab}\Omega^{cd}. \tag{3.64}$$

Just as before, any variation of $\mathbf{V}$ can be equivalently described by a right multiplication with an element of the algebra of SL(5) (possibly dependent on the coordinates):

$$\delta\mathbf{V}_P{}^{ab} = \mathbf{V}_P{}^{cd}L_{cd}{}^{ab}(x) = \mathbf{V}_P{}^{cd}\Sigma^{ab}{}_{cd}(x) - 2\mathbf{V}_P{}^{c[a}\Lambda_c{}^{b]}(x). \tag{3.65}$$

The last piece is just an USp(4) transformation, which leaves the Lagrangian invariant. Therefore, we will only be concerned with variations of the type $\delta_\Sigma\mathbf{V}_P{}^{ab} = \mathbf{V}_P{}^{cd}\Sigma^{ab}{}_{cd}(x)$, which correspond to variations along the coset space SL(5)/SO(5). The USp(4) composite gauge field Q_μ and the P_μ tensor that enters the kinetic term for the scalars in the Lagrangian are given by

$$\mathbf{V}_{ab}{}^{M}(\partial_\mu\mathbf{V}_M{}^{cd} - gA_\mu^{PQ}X_{PQ,M}{}^{N}\mathbf{N}^{cd}) \equiv P_{\mu ab}{}^{cd} + 2Q_{\mu[a}{}^{[c}\delta_{b]}^{d]}. \tag{3.66}$$

The T-tensor is defined as the dressed version of the embedding tensor, explicitly

$$\begin{aligned} T_{(ef)[ab]}{}^{[cd]} \equiv& \sqrt{2}\mathbf{V}^{M}{}_{eg}\mathbf{V}^{N}{}_{fh}\Omega^{gh}\mathbf{V}^{P}{}_{ab}\Theta_{MN,P}{}^{Q}\mathbf{V}_Q{}^{cd} \\ =& \sqrt{2}\Omega^{h[c}\delta^{d]}_{(e}\mathbf{V}^{M}{}_{f)h}\mathbf{V}^{N}{}_{ab}Y_{MN} - 2\sqrt{2}\varepsilon_{MNPQR}Z^{PQ,S}\mathbf{V}^{M}{}_{eg}\mathbf{V}^{N}{}_{fh}\mathbf{V}^{R}{}_{ab}\mathbf{V}_S{}^{cd}\Omega^{gh}. \end{aligned} \tag{3.67}$$

Given that Y_{MN} and $Z^{MN,P}$ of Θ belong to the **15** and $\overline{\mathbf{40}}$ of SL(5), respectively, once they are decomposed under USp(4) one gets

$$\mathbf{15} + \overline{\mathbf{40}} \rightarrow (\mathbf{1} + \mathbf{14}) + (\mathbf{5} + \mathbf{35}). \tag{3.68}$$

Therefore, the T-tensor can be expressed in terms of these USp(4) representations, which we will label B, $B^{[ab]}{}_{[cd]}$, $C_{[ab]}$ and $C^{[ab]}{}_{(cd)}$, respectively:

$$\begin{aligned}
T_{(ef)ab}{}^{cd} =&\frac{1}{2}B\Omega_{a(e}\delta^{[c}_{f)}\delta^{d]}_{b} - \frac{1}{2}B\Omega_{b(e}\delta^{[c}_{f)}\delta^{d]}_{a} + \delta^{[c}_{(e}\Omega_{f)g}B^{d]g}{}_{ab} \\
&+ \frac{1}{2}C_{a(e}\delta^{[c}_{f)}\delta^{d]}_{b} - \frac{1}{2}C_{b(e}\delta^{[c}_{f)}\delta^{d]}_{a} - \frac{1}{8}\Omega^{cd}C_{a(e}\Omega_{f)b} + \frac{1}{8}\Omega^{cd}C_{b(e}\Omega_{f)a} \\
&+ \frac{1}{4}\Omega_{ab}C_{g(e}\delta^{[c}_{f)}\Omega^{d]g} + \frac{1}{2}\Omega_{e[a}C^{cd}{}_{b]f} + \frac{1}{2}\Omega_{f[a}C^{cd}{}_{b]e} + \frac{1}{4}\Omega_{ab}C^{cd}{}_{ef}.
\end{aligned} \tag{3.69}$$

An explicit USp(4) parameterisation can also be achieved for the tensor Y_{PQ} and $Z^{PQ,R}$:

$$Y_{PQ} = \mathbf{V}_P{}^{ab}\mathbf{V}_Q{}^{cd}Y_{ab,cd}, \qquad Z^{PQ,R} = \sqrt{2}\mathbf{V}_{ab}{}^{P}\mathbf{V}_{cd}{}^{Q}\mathbf{V}_{ef}{}^{R}\Omega^{bd}Z^{(ac)[ef]}, \tag{3.70}$$

where

$$\begin{aligned}
Y_{ab,cd} =&\frac{1}{\sqrt{2}}\left[(\Omega_{ac}\Omega_{bd} - \frac{1}{4}\Omega_{ab}\Omega_{cd})B + \Omega_{ae}\Omega_{bf}B^{[ef]}{}_{[cd]}\right], \\
Z^{(ab)[cd]} =&\frac{1}{16}\Omega^{a[c}C^{d]b} + \frac{1}{16}\Omega^{b[c}C^{d]a} - \frac{1}{8}\Omega^{ae}\Omega^{bf}C^{cd}{}_{ef}.
\end{aligned} \tag{3.71}$$

To compute the equation of motion for the scalars in a Lorentz-invariant background, we also need the variations of the scalar fields, now represented by the tensors B, $B^{ab}{}_{cd}$, C^{ab} and $C^{ab}{}_{cd}$. These are given by

$$\begin{aligned}
\delta_\Sigma B =& -\frac{2}{5}\Sigma^{ab}{}_{cd}B^{cd}{}_{ab}, \\
\delta_\Sigma B^{ab}{}_{cd} =& -2B\Sigma^{ab}{}_{cd} - \Sigma^{ab}{}_{gh}B^{gh}{}_{cd} - \Sigma^{gh}{}_{cd}B^{ab}{}_{gh} + \frac{2}{5}\left(\delta^{ab}_{cd} - \frac{1}{4}\Omega^{ab}\Omega_{cd}\right)\Sigma^{ef}{}_{gh}B^{gh}{}_{ef}, \\
\delta_\Sigma C^{ab} =& \frac{1}{2}\Sigma^{ab}{}_{cd}C^{cd} + 2\Omega^{e[a}\Sigma^{b]f}{}_{cd}C^{cd}{}_{ef}, \\
\delta_\Sigma C^{ab}{}_{cd} =& 4\Omega^{g[a}\Sigma^{b]h}{}_{g(c}C_{d)h} + \Omega^{g[a}\delta^{b]}_{(c}\Sigma^{kh}{}_{d)g}C_{kh} + \Omega^{gh}\delta^{[a}_{(c}\Sigma^{b]h}{}_{d)g}C_{kh} + \Sigma^{ab}{}_{gh}C^{gh}{}_{cd} \\
&+ \Sigma^{k[a}{}_{gh}\delta^{b]}_{(c}C^{gh}{}_{d)k} + 4\Sigma^{km}{}_{l(c}\Omega_{d)k}\Omega^{n[a}C^{b]l}{}_{mn} - \delta^{[a}_{(c}\Omega_{d)k}\Omega^{b]n}\Sigma^{km}{}_{lg}C^{gl}{}_{mn}.
\end{aligned} \tag{3.72}$$

The quadratic constraint, in the formulation 3.55, can also be expressed in terms of irreducible representations of USp(4), obtaining the following:

$$Z^{(ab)[ef]}\left[\Omega_{ce}\Omega_{df}B + \Omega_{eg}\Omega_{fh}B^{[gh]}{}_{[cd]}\right] = 0, \qquad Z^{(ab)[cd]}T_{(ab)ef}{}^{gh} = 0. \tag{3.73}$$

We can now write down, for completeness, the supersymmetry transformation and the Lagrangian.

3.3.2 The D = 7 Lagrangian

The fermionic field content of the lgrangian is given by the gravitinos ${\psi_\mu}^a$, transforming in the **4** (the fundamental) representation of the R-symmetry group USp(4) and the spinors χ^{abc} belonging to the **16** representation of Usp(4). This means that the spinors satisfy the following relations

$$\chi^{abc} = \chi^{[ab]c}, \qquad \Omega_{ab}\chi^{abc} = 0, \qquad \chi^{[abc]} = 0. \tag{3.74}$$

The fermions of this theory are symplectic Majorana, namely, they behave as

$$\overline{\chi}^T = \Omega_{ad}\Omega_{be}\Omega_{cf}C\chi^{def}, \qquad \overline{\psi}^T_{\mu a} = \Omega_{ab}C\psi^b_\mu, \tag{3.75}$$

where C is the charge conjugation matrix with the property $C = C^T = -C^{-1} = -C^\dagger$. The local supersymmetry transformations are

$$\begin{aligned}
\delta_S {e_\mu}^P =& \frac{1}{2}\overline{\varepsilon}_a\Gamma^P\psi^a_\mu, \\
\Delta_S A^{PQ}_\mu =& -{\mathbf{V}_{ab}}^{[P}{\mathbf{V}_{cd}}^{Q]}\Omega^{bd}\left(\frac{1}{2}\Omega^{ae}\overline{\varepsilon}_e\psi^c_\mu + \frac{1}{4}\overline{\varepsilon}_e\Gamma_\mu\chi^{eac}\right), \\
\Delta_S B_{\mu\nu P} =& {\mathbf{V}_P}^{ab}\left(-\Omega_{ac}\overline{\varepsilon}_b\Gamma_{[\mu}\psi^c_{\nu]} + \frac{1}{8}\Omega_{ac}\Omega_{bd}\overline{\varepsilon}_e\Gamma_{\mu\nu}\chi^{cde}\right), \\
\Delta_S S^P_{\mu\nu\rho} =& {\mathbf{V}_{ab}}^P\left(-\frac{3}{8}\Omega^{ac}\overline{\varepsilon}_c\Gamma_{[\mu\nu}\psi^b_{\rho]} - \frac{1}{32}\overline{\varepsilon}_e\Gamma_{\mu\nu\rho}\chi^{abe}\right), \\
\delta_S\psi^a_\mu =& D_\mu\varepsilon^a - \frac{1}{5\sqrt{2}}\mathcal{H}^{(ab)}_{\nu\rho}\Omega_{bc}\left({\Gamma^{\nu\rho}}_\mu + 8\Gamma^\nu\delta^\rho_\mu\right)\varepsilon^c \\
& - \frac{1}{15}\mathcal{H}_{\nu\rho\Lambda[bc]}\Omega^{ab}\left({\Gamma^{\nu\rho\lambda}}_\mu + \frac{9}{2}\Gamma^{\nu\rho}\delta^\lambda_\mu\right)\varepsilon^c - g\Gamma_\mu A^{ab}_1\Omega_{bc}\varepsilon^c, \\
\delta_S\chi^{abc} =& 2\Omega^{cd}{P_{\mu de}}^{ab}\Gamma^\mu\varepsilon^e - \sqrt{2}\left(\mathcal{H}^{c[a}_{\mu\nu}\Gamma^{\mu\nu}\varepsilon^{b]} - \frac{1}{5}(\Omega^{ab}\delta^c_g - \Omega^{c[a}\delta^{b]}_g)\Omega_{de}\mathcal{H}^{gd}_{\mu\nu}\Gamma^{\mu\nu}\varepsilon^e\right) \\
& - \frac{1}{6}\left(\Omega^{ad}\Omega^{be}\mathcal{H}_{\mu\nu\rho[de]}\Gamma^{\mu\nu\rho}\varepsilon^c - \frac{1}{5}(\Omega^{ab}\Omega^{cf} + 4\Omega^{c[a}\Omega^{b]f})\mathcal{H}_{\mu\nu\rho[fe]}\Gamma^{\mu\nu\rho}\varepsilon^e\right) \\
& + gA^{d,abc}_2\Omega_{de}\varepsilon^e, \\
\delta_S{\mathbf{V}_P}^{ab} =& \frac{1}{4}{\mathbf{V}_P}^{cd}\left(\Omega_{e[c}\overline{\varepsilon}_{d]}\chi^{abe} + \frac{1}{4}\Omega_{cd}\overline{\varepsilon}_e\chi^{abe} + \Omega_{ce}\Omega_{df}\overline{\varepsilon}_g\chi^{ef[a}\Omega^{b]g} + \frac{1}{4}\Omega_{ce}\Omega_{df}\Omega^{ab}\overline{\varepsilon}_g\chi^{efg}\right).
\end{aligned} \tag{3.76}$$

The variation for the q-forms is given in terms of the so-called covariant variations, which are defined by

$$\begin{aligned}
\Delta A_\mu^{PQ} \equiv & \delta A_\mu^{PQ}, \\
\Delta B_{\mu\nu P} \equiv & \delta B_{\mu\nu P} - 2\varepsilon_{PQRST} A_{[\mu}^{QR} \delta A_{\nu]}^{ST}, \\
Y_{MN}\Delta S_{\mu\nu\rho}^{N} \equiv & Y_{MN}\left(\delta S_{\mu\nu\rho}^{N} - 3B_{[\mu\nu P}\delta A_{\rho]}^{PN} + 2\varepsilon_{PQRST} A_{[\mu}^{NP} A_\nu^{QR} \delta A_{\rho]}^{ST}\right).
\end{aligned} \tag{3.77}$$

The latter are valid for any transformation, and in particular are useful for showing how the covariant field strengths transform covariantly. In addition, we have introduced the fermion shifts in the fermionic supersymmetry transformations. Those are defined by

$$A_1^{ab} \equiv -\frac{1}{4\sqrt{2}}\left(\frac{1}{4}B\Omega^{ab} + \frac{1}{5}C^{ab}\right), \tag{3.78}$$

$$A_2^{d,abc} \equiv \frac{1}{2\sqrt{2}}\left[\Omega^{ec}\Omega^{fd}(C^{ab}{}_{ef} - B^{ab}{}_{ef}) + \frac{1}{4}(C^{ab}\Omega^{cd} + \frac{1}{5}\Omega^{ab}C^{cd} + \frac{4}{5}\Omega^{c[a}C^{b]d})\right], \tag{3.79}$$

It will be useful when we determine the masses of the fermionic fields and the residual supersymmetry on the vacuum. We are now in a position to describe the full Lagrangian, it is divided into two pieces

$$\mathcal{L}_{7D} = \mathcal{L}_0 + \mathcal{L}_{VT}, \tag{3.80}$$

where $\mathcal{L}_{VT}$ is the topological vector tensor Lagrangian that is needed to have a gauge-invariant Lagrangian. When the coupling constant $g \to 0$, the topological Lagrangian $\mathcal{L}_{VT}$ reduces to the SL(5) invariant Chern-Simons term of the ungauged theory.

$$
\begin{aligned}
e^{-1}\mathcal{L} = &-\frac{1}{2}R - \Omega_{ac}\Omega_{bd}\mathcal{H}^{ab}_{\mu\nu}\mathcal{H}^{cd\mu\nu} - \frac{1}{6}\Omega^{ac}\Omega^{bd}\mathcal{H}_{\mu\nu\rho ab}\mathcal{H}^{\mu\nu\rho}_{cd} - \frac{1}{2}P_{\mu ab}{}^{cd}P^{\mu}_{cd}{}^{ab} \\
&-\frac{1}{2}\overline{\psi}_{\mu a}\Gamma^{\mu\nu\rho}D_\nu\psi^a_\rho - \frac{1}{8}\overline{\chi}_{abc}\not{D}\chi^{abc} - \frac{1}{2}P_{\mu ab}{}^{cd}\Omega_{ce}\overline{\psi}_{\nu d}\Gamma^\mu\Gamma^\nu\chi^{abe} \\
&+\frac{\sqrt{2}}{4}\mathcal{H}^{ab}_{\mu\nu}\left(-\overline{\psi}^\rho_a\Gamma_{[\rho}\Gamma^{\mu\nu}\Gamma_{\lambda]}\psi^{\lambda c}\Omega_{cb} + \overline{\psi}_{\rho c}\Gamma^{\mu\nu}\Gamma^\rho\chi^{cde}\Omega_{ad}\Omega_{be} + \frac{1}{2}\overline{\chi}_{acd}\Gamma^{\mu\nu}\chi^{edc}\Omega_{eb}\right) \\
&+\frac{1}{12}\mathcal{H}_{ab\mu\nu\rho}\left(-\Omega^{ac}\overline{\psi}^\lambda_c\Gamma_{[\lambda}\Gamma^{\mu\nu\rho}\Gamma_{\sigma]}\psi^{\sigma b} + \frac{1}{2}\overline{\psi}_{\lambda c}\Gamma^{\mu\nu\rho}\Gamma^\lambda\chi^{abc} + \frac{1}{4}\Omega^{ae}\overline{\chi}_{cde}\Gamma^{\mu\nu\rho}\chi^{cdb}\right) \\
&-\frac{5}{2}gA_1^{ab}\Omega_{bc}\overline{\psi}_{\mu a}\Gamma^{\mu\nu}\psi^c_\nu + \frac{1}{4}gA_2^{d,abc}\Omega_{de}\overline{\chi}_{abc}\Gamma^\mu\psi^e_\mu \\
&+\frac{g}{4\sqrt{2}}\left(\frac{3}{32}\delta^b_d\delta^c_e B + \frac{1}{8}\delta^b_d\Omega_{ef}C^{fc} + B^{bc}{}_{de} - C^{bc}{}_{de}\right)\overline{\chi}_{abc}\chi^{ade} \\
&+\frac{g^2}{128}\left(15B^2 + 2C^{ab}C_{ab} - 2B^{ab}{}_{cd}B^{cd}{}_{ab} - 2C^{[ab]}{}_{(cd)}C_{[ab]}{}^{(cd)}\right)
\end{aligned}
\tag{3.81}
$$

The topological Lagrangian is given by:

$$
\begin{aligned}
e^{-1}\mathcal{L}_{VT} = &-\frac{1}{9}\varepsilon^{\mu\nu\rho\lambda\sigma\tau\kappa}\times \\
&\times\Bigg[gY_{MN}S^M_{\mu\nu\rho}\Bigg(D_\lambda S^N_{\sigma\tau\kappa} + \frac{3}{2}gZ^{NP,Q}B_{\lambda\sigma P}B_{\tau\kappa Q} + 3\mathcal{F}^{NP}_{\lambda\sigma}B_{\tau\kappa P} \\
&+4\varepsilon_{PQRST}A^{NP}_\lambda A^{QR}_\sigma\partial_\tau A^{ST}_\kappa + g\varepsilon_{PQRWX}X_{ST,UV}{}^{WX}A^{NP}_\lambda A^{QR}_\sigma A^{ST}_\tau A^{UV}_\kappa\Bigg) \\
&+3gZ^{MN,P}(D_\mu B_{\nu\rho M})B_{\lambda\sigma N}B_{\tau\kappa P} - \frac{9}{2}\mathcal{F}^{MN}_{\mu\nu}B_{\rho\lambda M}D_\sigma B_{\tau\kappa N} \\
&+18\varepsilon_{MNPQR}\mathcal{F}^{MV}_{\mu\nu}A^{NP}_\rho\left(\partial_\lambda A^{QR}_\sigma + \frac{2}{3}gX_{ST,U}{}^QA^{RU}_\lambda A^{ST}_\sigma\right)B_{\tau\kappa V} \\
&+9g\varepsilon_{MNPQR}Z^{MV,W}A^{NP}_\mu\left(\partial_\nu A^{QR}_\rho + \frac{2}{3}gX_{ST,U}{}^QA^{RU}_\nu A^{ST}_\rho\right)B_{\lambda\sigma V}B_{\tau\kappa W} \\
&+\frac{36}{5}\varepsilon_{MPQTU}\varepsilon_{NRSVW}A^{MN}_\mu A^{PQ}_\nu A^{RS}_\rho(\partial_\lambda A_{\sigma}{}^{TU})(\partial_\tau A^{VW}_\kappa) \\
&+8g\varepsilon_{MPQRS}\varepsilon_{NTUZA}X_{VW,XY}{}^{ZA}A^{MN}_\mu A^{PQ}_\nu A^{TU}_\rho A^{VW}_\lambda A^{XY}_\sigma\partial_\tau A^{RS}_\kappa \\
&-\frac{4}{7}g^2\varepsilon_{MPQBC}\varepsilon_{NVWDE}X_{RS,TU}{}^{BC}X_{XY,ZA}{}^{DE}A^{MN}_\mu A^{PQ}_\nu A^{RS}_\rho A^{TU}_\lambda A^{VW}_\sigma A^{XY}_\tau A^{ZA}_\kappa\Bigg].
\end{aligned}
\tag{3.82}
$$

The topological Lagrangian provides a linear (in the derivatives) kinetic term for the three-form fields $S^P_{\mu\nu\rho}$, they appear contracted by the Y-tensor, and this will also be relevant when computing the mass spectrum for those. After this brief introduction to the five- and seven-dimensional gauged supergravities, we can start looking for the vacua of these theories.

3.4 Summary

In this section, we provided the relevant features for the D = 5, 7-dimensional maximal gauged supergravities, comprising the Θ and T tensor representations, the fermion shifts and the Lagrangian, in some detail. We also provided explicit forms of the quadratic constraints we use in the following computations; more information about these theories can be found in [32] and [33] for 5 and 7 dimensions, respectively.

Chapter 4

Residual Gauge Symmetry and Analytical Results

4.1 Old and new vacua of 5D maximal supergravity

Charting and analysing vacua of supergravity theories is a fundamental task to find which models can be related to string theory as well as to understand supersymmetry breaking, the possible mechanisms to generate critical points with a positive value of the cosmological constant and which supergravities lead to Anti-de Sitter (AdS) vacua with an interesting holographic dual. Among all possible theories, the maximally supersymmetric ones stand out for their fixed matter content and the limited number of possible deformations. For these reasons there has been an active interest in their gaugings and in the analysis of the resulting scalar potentials to understand their critical points, with a special emphasis on the theories obtained by reducing string or M-theory on spheres, which give models with vacua dual to maximally supersymmetric Conformal Field Theories (CFT).

The main challenges one faces when dealing with this problem are associated with the very complicated structure of the scalar potential, a function of 70 or 42 scalars in the maximal theory in 4 and 5 dimensions, respectively, which also depends on a large number of parameters (912 and 351, respectively) that fix the structure of the gaugings and therefore of the full Lagrangian, according to the rules specified in [32, 17]. Clearly such a large space of parameters makes the search for critical points complicated and attempts at a general classification extremely difficult. However, there has been some interesting progress in the last few years that expanded a lot our knowledge of this particular aspect of maximal supergravity theories.

There are mainly three techniques that have been used so far to find and analyse critical points of (maximal) supergravity theories. The first one relies on using symmetries to consistently truncate a particular theory to a subset of fields containing a limited number of scalars and then extremising the resulting simplified potential. Pioneered in [34, 35], this technique allowed for the first and only analytical results for the maximal theories from the 1980s until recent years. For what concerns maximal supergravity in 5 dimensions, this technique allowed the discovery of 5 vacua [25, 36–38] of the SO(6) and SO(3,3) gauged models in addition to the maximally symmetric one in [39], although often only partial results were available on the spectrum about these vacua. More recently, a new numerical approach, based on Machine Learning software libraries was developed and employed in a series of papers [40–47] where many new vacua of the maximal supergravities in 4 and 5 dimensions had been found. This also allowed to find precise information about the spectrum of scalar fluctuations, residual gauge groups, and residual supersymmetry. In particular, 27 new AdS vacua were found in the SO(6) maximal supergravity in 5 dimensions, with a detailed analysis in [45, 46]. While these approaches are very interesting and gave promising results, so far they have only been used to produce critical points for a fixed scalar potential, which results from a single specific gauging within the large infinite family of possible deformations. This leaves open the possibility that other vacua with the same residual symmetries appear in different gaugings. The approach we are going to show instead uses the power of the embedding tensor formalism in a way that allows for the search of critical points independently of the choice of gauging. This approach was pioneered in a very different context in [48]and was used in the context of maximal 4-dimensional supergravity in [49, 22, 50–54], as well as in half-maximal supergravity in four and three dimensions [55, 56]. In addition to the power of investigating in a single sweep all deformations of maximal supergravity, this approach has so far produced analytic results for the critical points and their full spectrum, also providing information on the gauging, the residual gauge symmetry and supersymmetry of the vacua. Moreover, for Minkowski vacua this led to understanding the moduli space of these theories [53] as well as their uplift to string theory [57]. Finally, since the vacua are obtained without specifying the gauging first, this means that we can exhaustively classify vacua with a given residual symmetry for all possible consistent gaugings.

In this thesis, we show this last technique by investigating critical points of maximal gauged supergravities in 5 and 7 dimensions with residual U(2) and U(1) symmetry, respectively. We recover all previously known vacua and find four new ones in five dimensions, with different gauge groups and cosmological constants. We also provide analytical results for their full mass spectra, thus completing partial results for old vacua as well as fully analysing new ones. We did not find new AdS vacua, so that the only such vacua with U(2) symmetry are those appearing in the maximal supergravity with SO(6) gauge group, but we have new Minkowski and de Sitter vacua. A particularly interesting result is that two of the vacua appear in the same theory with SO*(6) =SU(3,1) gauge group, providing the first example in 5 dimensions where a single gauging of a maximal supergravity theory produces vacua in different classes of the cosmological constant, one having a positive cosmological constant and the other a vanishing cosmological constant and residual supersymmetry[1].

In what follows, we will discuss in some detail our technique in section 4.1.2 and then proceed with a detailed analysis of the U(2) invariant sector in section 4.1.3. We tried to summarise all our results in tables that could be easily consulted and used for future reference.

4.1.1 Some Useful Notation for D=5

We have already established in the previous chapter the Lagrangian and supersymmetry transformations for the maximal supergravity theories in five space-time dimensions, we also briefly reviewed the bosonic sector, paying particular attention to the scalar field content. In order to facilitate reading, we collected in this section the main formulas and properties of the tensors relevant for our work. The centre of our analysis will be the scalar potential and the mass matrices. While everything can be defined in terms of the fermion shifts, for the scalar masses we preferred to use a convenient expression, which is valid only at the selected point of the scalar manifold we use as a basis for our analysis. As we shall see, we are not going to lose generality by this assumption.

Following a well-known general rule of gauged supergravity theories, the scalar potential is the square of the fermion shifts:

$$V = 3A^{ij}A_{ij} - \frac{1}{3}A^{i,jkl}A_{i,jkl}. \tag{4.1}$$

We are looking for maximally-symmetric vacua, where all fields are vanishing except for the scalar fields, which could have a constant vacuum expectation value and for the metric, which either describes a de Sitter, Minkowski or anti-de Sitter spacetime. The scalar equations of motion are solved by the critical point condition

$$U_{ijkl} - \frac{3}{2}\Omega_{[ij}U_{kl]pq}\Omega^{pq} + \frac{1}{8}\left(U_{mnpq}\Omega^{mn}\Omega^{pq}\right)\Omega_{[ij}\Omega_{kl]} = 0\,, \tag{4.2}$$

where the tensor U_{ijkl} is

$$U_{ijkl} = \frac{4}{3}A_1{}^{mq}A_{2m,[ijk}\Omega_{l]q} + 2A_2{}^{m,npq}A_{2n,m[ij}\Omega_{|p|k}\Omega_{l]q}\,. \tag{4.3}$$

[1]There is a similar instance in maximal supergravity in 3 dimensions [58]

Once we find a critical point, we derive the masses of the various fields by computing the eigenvalues of the respective mass matrices. For what concerns the gravitini ψ^i_μ, the mass matrix is directly proportional to the shift matrix A_{ij}.

$$\mathcal{M}^{(3/2)}_{ij} = \frac{3}{2} A_{ij}. \tag{4.4}$$

The masses of the other fermions χ^{ijk} are then fixed by the eigenvalues of (indices ijk and pqr are fully anti-symmetrised).

$$\mathcal{M}^{(1/2)}_{ijk,pqr} = 8A_{[i,j[pq}\Omega_{r]k]} + 2A_{[i[p}\Omega_{qj}\Omega_{k]r]} - \frac{10}{3} A_{l,ijk} A^{lm} (A_{m,stu} A^{n,stu})^{-1} A_{n,pqr}. \tag{4.5}$$

This mass matrix is the result of subtracting from the Lagrangian mass the appropriate term to remove the goldstinos from the spectrum for susy-breaking vacua. It is understood that in the case of a degenerate matrix $A_{m,stu}A^{n,stu}$, we only compute the inverse for its non-degenerate part, as this is the part related to the goldstino directions, which in the original Lagrangian mix the gravitinos and the spin-1/2 fields. The proof that such an additional term correctly produces $\mathcal{M}^{1/2}_{ijk,pqr} A^{s,pqr} = 0$ follows once one takes into account the equations of motion (4.2) and uses repeatedly the quadratic constraints (3.41). In particular, the matrix we are inverting is related to the shift of the gravitinos by means of the quadratic identity known as supersymmetric Ward identity

$$\frac{1}{3} A_{j,stu} A^{i,stu} = \frac{1}{8} \delta^i_j V + 3A^{ip} A_{pj}, \tag{4.6}$$

which also tells us that the expression is explicitly dependent on the value of the cosmological constant at the vacuum. This expression generalises previous similar formulae for maximal theories in 4 dimensions, which were obtained in particular instances where the cosmological constant was vanishing [59] or when the squared shifts had already been diagonalized [54]. A simple way to understand this expression can also be obtained by comparing it with the analogous expression for $N = 1$ supergravity presented in [60].

In addition, the masses of the bosonic degrees of freedom can be expressed in terms of the same tensors. The vector mass matrix is

$$\mathcal{M}_{(v)M}{}^N = \frac{1}{3} \mathbf{V}^{ij}_M T^{mnpq}{}_{ij} T_{mnpq}{}^{kl} \mathbf{V}^N_{kl}, \tag{4.7}$$

while the squared masses of the tensor fields follow from the eigenvalues of the matrix

$$\mathcal{M}_{(t)M}{}^N = \mathbf{V}^{ij}_M Z_{ijmn} Z^{mnkl} \mathbf{V}^N_{kl}. \tag{4.8}$$

These mass matrices are clearly redundant because the sum of vector and tensor fields present in the theory is fixed, given that the tensor fields appear by dualization of the vector fields. This means that both $M_{(v)}$ and $M_{(t)}$ are degenerate and contain zeros in the directions where the fields have been dualised.

All the above expressions have general validity and should be evaluated at the critical points that satisfy (4.2). For the scalar fields, on the other hand, following [49] we provide an expression that is valid only when the critical point is the base point of the manifold, i.e. when all scalars are vanishing. While this could seem a restriction, as we will explain in the next section, it allows us to obtain the full spectrum for any critical point in any arbitrary gauging. This is given in terms of the embedding tensor, the $\mathfrak{e}_{6(6)}$ generators, the $\mathfrak{e}_{6(6)}$ structure constants $f_{\alpha\beta}{}^{\gamma}$ and the $\mathfrak{e}_{6(6)}$ Cartan–Killing metric $\eta_{\alpha\beta}$:

$$\begin{aligned}\mathcal{M}_{\alpha}{}^{\beta} = & \frac{16}{5}\Big(\Theta_M{}^{\sigma}(t_\alpha t^\beta)_M{}^N\Theta_N{}^{\gamma}(\delta^{\gamma}_{\sigma}+5\,\eta_{\sigma\gamma})+\Theta_M{}^{\sigma}(t_\alpha)_M{}^N\Theta_N{}^{\gamma}f^{\beta}{}_{\gamma}{}^{\sigma} \\ & +\Theta_M{}^{\sigma}(t^\beta)_M{}^N\Theta_N{}^{\gamma}f_{\alpha\gamma}{}^{\sigma}+\Theta_M{}^{\sigma}\Theta_M{}^{\gamma}f_{\alpha\gamma}{}^{\delta}f^{\beta}{}_{\delta}{}^{\sigma}\Big)\,.\end{aligned} \tag{4.9}$$

The matrix is non-zero only in the non-compact directions, i.e. along the generators $t_\alpha \in \mathfrak{e}_{6(6)} \setminus \mathfrak{usp}(8)$. Moreover, all goldstone fields appear with a zero eigenvalue.

4.1.2 Extrema of the scalar potential

The procedure used to find and analyse the scalar potential has been developed in the case of maximal supergravities in [49], developing an old idea presented in a very different context [48]. The main point is that the scalar potential is a function of the scalar fields through the coset representatives $\mathbf{V}_M^{ij}$ and the embedding tensor $\Theta_M{}^{\alpha}$

$$V(\phi) = V\left(\mathbf{V}(\phi),\Theta\right). \tag{4.10}$$

As explained above, the vacua of the theory follow as solutions of the minimisation condition (4.2). This is generally a rather complicated expression of the scalar fields (at best ratios of polynomials and exponentials of the scalar fields). This is the reason why the task of finding solutions to such complicated systems of equations has always been very challenging, and researchers usually focus on restricted sets of scalar fields in order to simplify the task, which in any case is often performed only numerically.

The alternative proposed in [49] maps the problem to a coupled set of second- and first-order algebraic conditions on the gauging parameters. This is possible because the scalar manifold is homogeneous, and therefore each point on the manifold can be mapped to any other by an $E_{6(6)}$ transformation and at the same time the scalar potential is *invariant* under the simultaneous action of these transformations on both the coset representatives and on the embedding tensor. This implies that we can always map any critical point of the scalar potential to the "origin" at $\phi = 0$. At such a point, the scalar potential is a simple quadratic function of the embedding tensor

$$V = \frac{2}{15}\,\Theta_M{}^{\alpha}\Theta_M{}^{\beta}\left(\delta_{\alpha\beta} + 5\eta_{\alpha\beta}\right) \tag{4.11}$$

and the minimisation conditions become quadratic conditions on the embedding tensor, which should be solved together with the quadratic constraints (3.41). The result is that rather than fixing the gauge group and then performing a scan of all possible critical points of the scalar potential and then scanning among all possible gaugings, one can simply solve a set of quadratic conditions on the embedding tensor and then read the resulting values of Θ that fix at the same time the gauge group, the value of the cosmological constant and the masses at the critical point. Clearly any choice of point on the scalar manifold is equivalent, but choosing $\phi = 0$ has the advantage that it is a fixed point under the action of the maximal compact subgroup of the isometries, namely USp(8), and therefore we can consider modifications of the embedding tensor related only to the non-compact transformations, so that there is a one-to-one correspondence between the parameters in Θ related to the scalar fields and the independent directions on the scalar manifold, as shown in fig. 4.1. We advise the reader to consult [49] for more details. As we mentioned in the introduction, all of our results are fully analytic. The reason we are able to produce such results is related to the procedure we used to solve the quadratic conditions that come from the minimisation of the scalar potential and from the quadratic constraints. Although, in fact, we reduced our problem to a set of quadratic equations, we still generically have a very large number of parameters and quadratic equations. This implies that one does not always see a straightforward analytic solution, because the equations are coupled, and they could become very high in order in terms of a single variable.

We used mainly two techniques. The first one is based on a simplification of the set of quadratic equations by employing a choice of a more convenient Gröbner basis for the polynomial generating the same solutions. This has been done with the aid of the computer algebra system for polynomial computations `SINGULAR` [61]. Unfortunately, when the number of variables is very large, this can be extremely costly in time and therefore one has to resort to a different way of reducing the set of equations. We found a very effective procedure by borrowing an algorithm developed in the context of cryptography, where the solution of quadratic equations on finite fields is a common problem. In particular, we

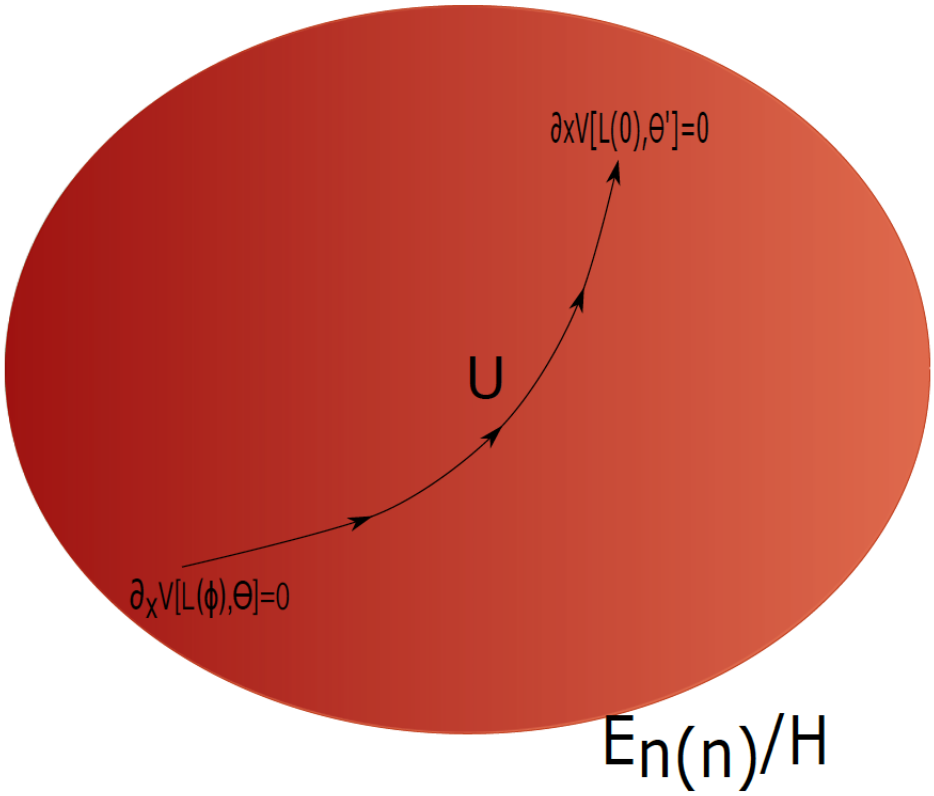


Fig. 4.1 $\mathrm{E}_{n(n)}$ transformations (U) mapping any point to the origin of homogenous space $\mathrm{E}_{n(n)}/H$

used the so-called XL algorithm [62], or extended linearisation. The idea is rather simple. Rather than solving directly the given set of quadratic equations, one produces sets of linear equations in the monomials appearing in the equations and in all equations obtained by multiplying the original set of equations by the variables and by their products up to a fixed order. This produces sets of linear equations that can be solved rapidly, and, once interpreted in terms of the original variables, they may reduce to equations in a single variable or in simpler sets of polynomial equations (like equality between different monomials). This allows to fix and eliminate some of the variables from the problem and then face a simpler set of equations, which could be solved directly or further simplified by another iteration of the same procedure, or by a more convenient choice of Gröbner basis. More information can be found in Appendix D.

4.1.3 Vacua with residual U(2) symmetry

In this work we decided to scan gauged maximal supergravity in 5 dimensions for vacua with a residual U(2) symmetry. Asking for a residual U(2) invariance of the vacuum (with respect to a gauged or global symmetry) imposes restrictions on the allowed coefficients of the embedding tensor and consequently of the fermion shift tensors, which should be singlets with respect to this residual symmetry. To perform a full analysis, we therefore looked at all the inequivalent embeddings of SU(2) in USp(8) and then singled out all possible inequivalent charge assignments for the remaining U(1)s, if any. We then performed the branching of the **36** and **315** representations of USp(8) specifying the fermion shifts with respect to the chosen embedding and classified all inequivalent cases. When the commutant of the residual symmetry group in USp(8) was non-trivial we used the commuting symmetries to further reduce the number of inequivalent variables by removing those that could be generated by the action of the commutant. Once the non-vanishing components of the fermion shifts had been identified, we then proceeded to solve the set of quadratic algebraic conditions coming from the scalar equations of motion (4.2) and the quadratic constraints (3.41) and then collected all solutions, which may still be related by duality transformations. Finally, we analysed their properties and computed their mass spectrum, as we shall discuss momentarily. In the summary tables, we collected all inequivalent vacua and reported the most general mass spectra for each of them. Unfortunately, two of the branchings still present a very large number of singlets (≥ 48) and even combining all the techniques mentioned above we have not been able to fully scan and solve their equations for all the allowed parameters, though for all solutions we recovered the same vacua we found in other branchings.

As a first step, we list the branchings we analysed by the inequivalent decompositions of the **8**-dimensional representation of USp(8) under SU(2) and then give one of the branching routes leading to this decomposition. For each case, we also give a table with the subcases based on possible different choices of the U(1) factor, when present. We also list the number of singlets in the fermion shifts, which are going to be the variables to be fixed by the quadratic conditions in order to find vacua.

Branchings

We find 13 different branchings of the fundamental representation of USp(8) under SU(2), which we therefore analyse separately. The labels on the various factors are self-explanatory: we use letters from the beginning of the alphabet to keep track of the various factors in the decompositions and we use *S* and *diag* to specify the symmetric and diagonal embedding of the group.

Case 1: $\mathbf{8} \to \mathbf{8}$.

The branching path is

$$\mathrm{USp}(8) \to \mathrm{SU}(2)_S. \tag{4.12}$$

This case leaves no singlets to discuss, so no vacua are possible for this choice.

Case 2: $\mathbf{8} \to \mathbf{6} + \mathbf{2}$.

The branching path is

$$\mathrm{USp}(8) \to \mathrm{SU}(2)_A \times \mathrm{USp}(6) \to \mathrm{SU}(2)_A \times \mathrm{SU}(2)_S \to \mathrm{SU}(2)_{diag} \tag{4.13}$$

There is only one singlet in $A_{i,jkl}$. Clearly, this is never going to give any vacuum because the critical point condition (4.2) would fix this parameter to zero.

Case 3: $\mathbf{8} \to \mathbf{6} + \mathbf{1} + \mathbf{1}$.

The branching path is

$$\mathrm{USp}(8) \to \mathrm{SU}(2)_A \times \mathrm{USp}(6) \to 1_A \times \mathrm{SU}(2)_S \tag{4.14}$$

There are 3 singlets in A_{ij} and no singlets in $A_{i,jkl}$.

Case 4: $\mathbf{8} \to \mathbf{4} + \mathbf{4}$.

The branching path is

$$\mathrm{USp}(8) \to \mathrm{USp}(4)^2 \to \mathrm{USp}(4)_{diag} \to SU(2)_S \tag{4.15}$$

We have one singlet in A_{ij} and three singlets in $A_{i,jkl}$.

Case 5: $\mathbf{8} \to \mathbf{4} + \mathbf{2} + \mathbf{2}$.

The branching path is

$$\mathrm{USp}(8) \to \mathrm{USp}(4)_A \times \mathrm{USp}(4)_B \to [\mathrm{SU}(2) \times \mathrm{U}(1)]_A \times \mathrm{SU}(2)_S \to \mathrm{SU}(2)_{diag} \tag{4.16}$$

We find just one singlet in A_{ij} and 8 in $A_{i,jkl}$.

Case 6: $\mathbf{8} \to \mathbf{4} + \mathbf{2} + \mathbf{1} + \mathbf{1}$.

The branching path is

$$\mathrm{USp}(8) \to \mathrm{USp}(4) \times \mathrm{USp}(4) \to [\mathrm{SU}(2)_A \times \mathrm{SU}(2)_B] \times \mathrm{SU}(2)_S \to \mathrm{SU}(2)_{B+S} \tag{4.17}$$

There are 3 singlets in A_{ij} and 5 singlets in $A_{i,jkl}$.

Case 7: $\mathbf{8} \to \mathbf{4} + 4 \cdot \mathbf{1}$.

The branching path is

$$\begin{aligned} \mathrm{USp}(8) &\to \mathrm{USp}(4)_A \times \mathrm{USp}(4)_B \to [\mathrm{SU}(2)_S]_A \times [\mathrm{SU}(2) \times \mathrm{U}(1)]_B \\ &\to \mathrm{SU}(2)_S \times \mathrm{U}(1)_A \times \mathrm{U}(1)_B \end{aligned} \tag{4.18}$$

In this case, we have two inequivalent choices of U(1) $\subset$ U(1)$_A\times$ U(1)$_B$, which we list in the following table.

#	**8** decomposition	charges choice	**36** singlets	**315** singlets
7	$\mathbf{8} \to \mathbf{4}_{00} + \mathbf{1}_{\pm 1 \pm 1}$	(q_A, q_B)		
7a	$\mathbf{8} \to \mathbf{4}_0 + 2 \cdot \mathbf{1}_{\pm 1}$	q_A	4	7
7b	$\mathbf{8} \to \mathbf{4}_0 + \mathbf{1}_{\pm 1} + 2 \cdot \mathbf{1}_0$	$\frac{q_A + q_B}{2}$	4	5

Table 4.1 Branchings for case 7.

Case 8: $\mathbf{8} \to 2 \cdot \mathbf{3} + \mathbf{2}$.

The branching path is

$$\begin{aligned} \mathrm{USp}(8) &\to \mathrm{SU}(2)_A \times \mathrm{USp}(6) \to \mathrm{SU}(2)_A \times [\mathrm{SU}(3) \times \mathrm{U}(1)]_B \\ &\to \mathrm{SU}(2)_A \times \mathrm{SO}(3)_B \to \mathrm{SU}(2)_{diag} \end{aligned} \tag{4.19}$$

The decomposition contains 3 singlets for A_{ij} and 6 singlets for $A_{i,jkl}$.

Case 9: $\mathbf{8} \to 2 \cdot \mathbf{3} + 2 \cdot \mathbf{1}$.

The branching path is

$$\mathrm{USp}(8) \to \mathrm{SU}(4) \times \mathrm{U}(1)_B \to \mathrm{SU}(3) \times \mathrm{U}(1)_A \times \mathrm{U}(1)_B \to \mathrm{SU}(2)_s \times \mathrm{U}(1)_A \times \mathrm{U}(1)_B. \tag{4.20}$$

This case already has 21 SU(2) singlets overall, therefore, we distinguish various subcases according to the choices of a U(1) factor, which we report in Table 4.2.

#	**8** decomposition	charges choice	**36** singlets	**315** singlets
9	$\mathbf{8} \to \mathbf{3}_{11} + \mathbf{3}_{-1-1} + \mathbf{1}_{-31} + \mathbf{1}_{3-1}$	(q_A, q_B)	2	1
9a	$\mathbf{8} \to \mathbf{3}_{\pm 1} + \mathbf{1}_{\pm 3}$	q_A	2	3
9b	$\mathbf{8} \to \mathbf{3}_{\pm 1} + \mathbf{1}_{\pm 1}$	q_B, $\frac{q_A+q_B}{2}$	2	5
9c	$\mathbf{8} \to 2 \cdot \mathbf{3}_0 + \mathbf{1}_{\pm 1}$	$\frac{q_A-q_B}{4}$	4	3
9d	$\mathbf{8} \to \mathbf{3}_{\pm 1} + 2 \cdot \mathbf{1}_0$	$\frac{q_A+3q_B}{4}$	4	1

Table 4.2 Branchings for case 9.

Case 10: $\mathbf{8} \to 4 \cdot \mathbf{2}$.

The branching path is

$$\mathrm{USp}(8) \to \mathrm{SU}(2)_A \times \mathrm{SU}(2)_B \times \mathrm{SU}(2)_C \to \mathrm{SU}(2)_C \times \mathrm{U}(1)_A \times \mathrm{U}(1)_B \tag{4.21}$$

This case has 51 singlets of SU(2) and therefore we classify various subcases according to a remaining U(2) symmetry. We collect all different branchings in Table 4.3.

#	**8** decomposition	charges choice	**36** singlets	**315** singlets
10	$\mathbf{8} \to \mathbf{2}_{\pm 1 \pm 1}$	(q_A, q_B)	2	3
10a	$\mathbf{8} \to 2 \cdot [\mathbf{2}_{\pm 1}]$	q_A	4	15
10b	$\mathbf{8} \to 2 \cdot [\mathbf{2}_0] + 2_{\pm 1}$	$\frac{q_A+q_B}{2}$	2	11
10c	$\mathbf{8} \to [\mathbf{2}_{\pm 3}] + [\mathbf{2}_{\pm 1}]$	$2q_A + q_B$	2	7

Table 4.3 Branchings for case 10.

#	**8** decomposition	charges choice	**36** singlets	**315** singlets
11	$\mathbf{8} \to \mathbf{2}_{11} + \mathbf{2}_{00} + \mathbf{2}_{-1-1} + \mathbf{1}_{2-1} + \mathbf{1}_{-21}$	(q_A, q_B)	2	5
11a	$\mathbf{8} \to \mathbf{2}_{\pm 1} + \mathbf{2}_0 + 1_{\pm 2}$	q_A	2	5
11b	$\mathbf{8} \to \mathbf{2}_{\pm 1} + \mathbf{2}_0 + 1_{\pm 1}$	q_B	2	7
11c	$\mathbf{8} \to \mathbf{2}_{\pm 2} + \mathbf{2}_0 + 1_{\pm 1}$	$q_A + q_B$	2	7
11d	$\mathbf{8} \to 3 \cdot \mathbf{2}_0 + 1_{\pm 1}$	$\frac{q_A - q_B}{3}$	4	23
11e	$\mathbf{8} \to \mathbf{2}_{\pm 1} + \mathbf{2}_0 + 2 \cdot 1_0$	$\frac{q_A + 2q_B}{3}$	4	7

Table 4.4 Branchings for case 11.

Case 11: $\mathbf{8} \to 3 \cdot \mathbf{2} + 2 \cdot \mathbf{1}$.
The branching path is

$$\begin{aligned} \mathrm{USp}(8) &\to \mathrm{SU}(2) \times \mathrm{USp}(6) \to \mathrm{SU}(2) \times [\mathrm{SU}(3) \times \mathrm{U}(1)] \\ &\to \mathrm{SU}(2) \times [\mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{U}(1)] \to \mathrm{SU}(2)_{diag} \times \mathrm{U}(1)_A \times \mathrm{U}(1)_B \end{aligned} \tag{4.22}$$

This case has 39 singlets of SU(2) and therefore we classify various subcases according to a remaining U(2) symmetry. The results are collected in Table 4.4.

Case 12: $\mathbf{8} \to 2 \cdot \mathbf{2} + 4 \cdot \mathbf{1}$.
The branching path is

$$\begin{aligned} \mathrm{USp}(8) &\to \mathrm{USp}(4)_A \times \mathrm{USp}(4)_B \to [\mathrm{SU}(2) \times \mathrm{U}(1)]_A \times [\mathrm{SU}(2) \times \mathrm{SU}(2)]_B \\ &\to \mathrm{SU}(2)_A \times \mathrm{U}(1)_A \times \mathrm{U}(1)_B \times \mathrm{U}(1)_C \end{aligned} \tag{4.23}$$

This case has 64 singlets of SU(2) and therefore we classify various subcases according to a remaining U(2) symmetry, which we list in Table 4.5.

#	**8** decomposition	charges choice	**36** singlets	**315** singlets
12	$\mathbf{8} \to \mathbf{2}_{\pm 100} + \mathbf{1}_{0\pm 10} + \mathbf{1}_{00\pm 1}$	(q_A, q_B, q_C)	3	5
12a	$\mathbf{8} \to \mathbf{2}_{\pm 1} + 4 \cdot \mathbf{1}_0$	q_A	11	21
12b	$\mathbf{8} \to 2 \cdot \mathbf{2}_0 + \mathbf{1}_{\pm 1} + 2 \cdot \mathbf{1}_0$	q_B	5	19
12c	$\mathbf{8} \to \mathbf{2}_{\pm 1} + 2 \cdot \mathbf{1}_{\pm 1}$	$q_A + q_B + q_C$	5	19
12d	$\mathbf{8} \to \mathbf{2}_{\pm 1} + \mathbf{1}_{\pm 1} + 2 \cdot \mathbf{1}_0$	$q_A + q_B$	5	9
12e	$\mathbf{8} \to 2 \cdot [\mathbf{2}_0 + \mathbf{1}_{\pm 1}]$	$q_B + q_C$	5	27
12f	$\mathbf{8} \to \mathbf{2}_{\pm 1} + \mathbf{1}_{\pm 2} + 2 \cdot \mathbf{1}_0$	$q_A + 2q_B$	5	15

Table 4.5 Branchings for case 12.

Case 13: $\mathbf{8} \to \mathbf{2} + 6 \cdot \mathbf{1}$.

The branching path is

$$\begin{aligned} &\mathrm{USp}(8) \to \mathrm{SU}(2) \times \mathrm{USp}(6) \to \mathrm{SU}(2) \times [\mathrm{SU}(2)_A \times \mathrm{USp}(4)] \\ &\quad \to \mathrm{SU}(2) \times [\mathrm{SU}(2)_A \times \mathrm{SU}(2)_B \times \mathrm{SU}(2)_C] \to \mathrm{SU}(2) \times \mathrm{U}(1)_A \times \mathrm{U}(1)_B \times \mathrm{U}(1)_C \end{aligned} \tag{4.24}$$

This case has 124 singlets of SU(2) and therefore we classify various subcases according to a remaining U(2) symmetry. Here, we list only the cases that are inequivalent *and* produce different sets of singlets. Note that cases 13e and 13f have only a subset of the singlets present in the other cases, so it is enough to solve cases 13a–13d. Case 13c has only Minkowski vacua: (M3), (M0) with $m_4 = 0$ and (M2). The results are presented in Table 4.6.

The branchings 13a and 13d present more than 48 singlets, and this hampered the simplification of the problem with any of the techniques used in this work in a reasonable amount of time. Anyway, all solutions we have been able to find for these branchings were already present in one of the other branchings.

#	**8** decomposition	charges choice	**36** singlets	**315** singlets
13	$\mathbf{8} \to \mathbf{2}_{000} + \mathbf{1}_{\pm 100} + \mathbf{1}_{0\pm 10} + \mathbf{1}_{00\pm 1}$	(q_A, q_B, q_C)	3	9
13a*	$\mathbf{8} \to \mathbf{2}_0 + \mathbf{1}_{\pm 1} + 4 \cdot \mathbf{1}_0$	q_A	11	37
13b	$\mathbf{8} \to \mathbf{2}_0 + 2 \cdot [\mathbf{1}_0 + \mathbf{1}_{\pm 1}]$	$q_A + q_B$	7	27
13c	$\mathbf{8} \to \mathbf{2}_0 + \mathbf{1}_{\pm 2} + \mathbf{1}_{\pm 1} + 2 \cdot \mathbf{1}_0$	$2q_A + q_B$	5	17
13d*	$\mathbf{8} \to \mathbf{2}_0 + 3 \cdot \mathbf{1}_{\pm 1}$	$q_A + q_B + q_C$	9	47
13e	$\mathbf{8} \to \mathbf{2}_0 + 2 \cdot \mathbf{1}_{\pm 1} + \mathbf{1}_{\pm 2}$	$q_A + q_B + 2q_C$	3	9
13f	$\mathbf{8} \to \mathbf{2}_0 + \mathbf{1}_{\pm 1} + 2 \cdot \mathbf{1}_{\pm 2}$	$q_A + 2q_B + 2q_C$	3	19

Table 4.6 Branchings for case 13.

Vacua

The search for vacua has been carried out by solving the sets of quadratic equations for the singlets in the tables above. Once we found solutions, we checked for each candidate vacuum the rank of the embedding tensor, the signature of the resulting Cartan–Killing matrix and the full mass spectrum. In general, we found 5 different Anti-de Sitter vacua, 5 Minkowski vacua, and 2 de Sitter vacua. The vacua with negative cosmological constant are all pertaining to the same gauging, namely the maximal SO(6) theory of [39], and were all already known [36–38]. Among the Minkowski vacua there are the Cremmer–Scherk–Schwarz gaugings [63, 64] with various mass parameters and a supersymmetric vacuum for the SO*(6) theory discovered in [65], but we also find three new vacua with a non-abelian gauge group (like those in [53] for the analogous analysis of maximal supergravity in 4 dimensions). Finally, we also find 2 de Sitter vacua, resulting from gauging of the semisimple groups SO(3,3) [25] and SO*(6), the latter being new. All vacua are reported in table 4.7, together with the number of supersymmetry they preserve, the original gauging, the residual gauge group, and the reference where they were first discovered. In Appendix B we provide for each vacuum one instance of fermion shift values reproducing the critical point mentioned in the table.

vacuum	susy	G_{gauge}	G_{res}	ref.	branching
A1	8	SO(6)	SO(6)	[39, 38]	4,9,10,12
A2	0	SO(6)	SO(5)	[36–38]	4, 10, 12_{cef}
A3	0	SO(6)	SU(3)	[36–38]	9_a, 12_{ef}
A4	2	SO(6)	SU(2) × U(1)	[38]	12_{cef}
A5	0	SO(6)	SU(2) × U(1) × U(1)	[38]	12_{be}
M1	0,2,4,6	$U(1)\ltimes\mathbb{R}^{16}$	U(1)	[64]	5,6,7,10 11,12,13
M2	2	SO*(6)=SU(3,1)	SU(3) × U(1)	[65]	8,9,11 12_{abcef}, 13_b
M3	4	$SO^*(4)\ltimes\mathbb{R}^8$	U(2)	[66]	12_{abcef} 11, 13_{bc}
M4	0	$[SO(3,1)\times SO(2,1)]\ltimes\mathbb{R}^8$	U(2)	[66]	10_b
M5	4	$SO^*(4)\ltimes\mathbb{R}^8$	SO(3)	[66]	12_{be}
D1	0	SO(3,3)	$SO(3)^2$	[25]	9_b,10_{ab}
D2	0	SO*(6)=SU(3,1)	SU(2)	[66]	9_b

Table 4.7 Summary of vacua found in this work.

Given the nature of the gaugings generating such vacua, we can also see how some of these could be obtained from string theory reductions. All AdS vacua appear in the SO(6) theory, which is a consistent truncation of type IIB supergravity compactified on S^5 [67]. A subset of CSS gaugings and their vacua M1 are known to be the result of a twisted torus reduction [63], while the most general gauging and vacuum in this class may admit an uplift through a generalised Scherk–Schwarz Ansatz analogous to the ones described for four-dimensional CSS gaugings in [57].

It is interesting to notice that for the first time in a maximal 5 dimensional theory we find a gauging that produces at the same time vacua with different types of cosmological constants. This is the $SO^*(6) = SU(3,1)$ gauging, which contains at the same time a Minkowski and a de Sitter vacuum. Our claim that they reside in the same model follows both from the analysis of the embedding tensors that generate them, and the direct identification of a truncated scalar potential for the SU(3,1) theory where both vacua are easily found. From the embedding tensors we find which generators of $\mathfrak{e}_{6(6)}$ are involved in their corresponding model, and analysing the commutants we find in both cases that the representation **27** decomposes into the representations $\mathbf{15}+\mathbf{6}+\mathbf{6}$ of the gauge group. This corresponds to the correct branching under SU(3,1) and since the adjoint is unique in the branching, we argue that the gaugings are the same. Moreover, if we directly decompose the **351** representation of $\mathfrak{e}_{6(6)}$ from the branching above for the **27** we see that there is a unique singlet with respect to SU(3,1) and therefore there is a unique possible form of embedding tensor leading to this gauging up to duality transformations.

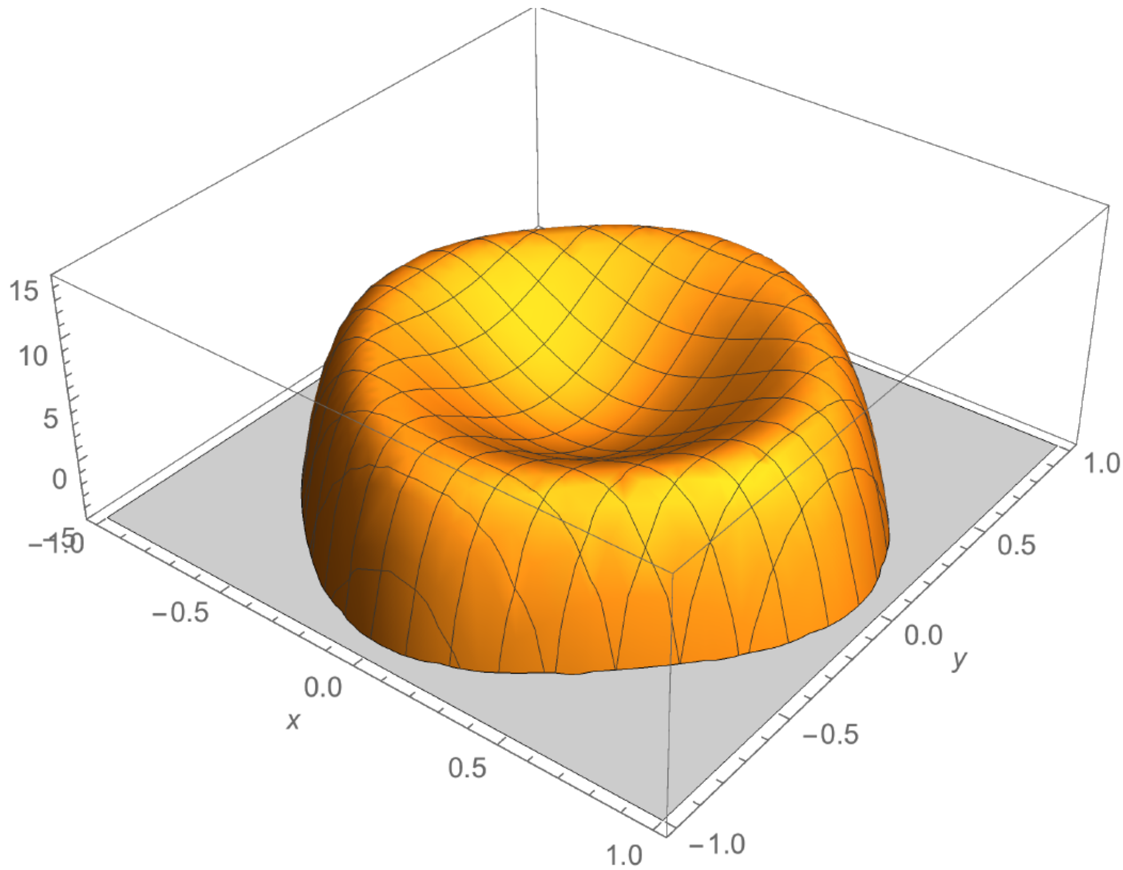


Fig. 4.2 scalar potential for the two common scalars invariant under the residual symmetries of the vacua (M2) and (D2). We see a Minkowski vacuum at the centre of the picture, surrounded by a family of de Sitter vacua with a massless modulus.

Actually, for this specific model, we can provide a truncated scalar potential, where we make explicit the dependence on the two scalar fields that are singlets of both symmetry groups. Furthermore, both vacua arise as different solutions of the 9_b case and the commutator of the residual U(2) group with the non-compact generators of $\mathfrak{e}_6$ leaves only two generators g_1 and g_2, for which we can provide a truncated scalar potential where both vacua can be found. We construct the coset representative

$$L(x,y) = \exp\left(g_1 x + g_2 y\right), \tag{4.25}$$

which induces the scalar potential

$$V = -\frac{27}{16}\left(12 - 16\cosh(2x)\cosh(2y) + 4\cosh^2(2x)\cosh^2(2y)\right), \tag{4.26}$$

where x and y are canonically normalised scalar fields. The scalar potential has two vacua, a Minkowski one at $x = y = 0$ and a line of unstable de Sitter vacua at $\cosh(2x)\cosh(2y) = 2$. At any point in the family of de Sitter vacua, we see that the masses of the two fluctuations are, in fact, zero and $m^2/\Lambda = -24$. These coincide with one of the moduli and one of the unstable directions of the full scalar spectrum about the de Sitter vacuum (see table 4.14).

A similar discussion could apply to the vacua (M3) and (M5). They both have the same gauge group, though in this case they do not belong to the same model. In fact, there are four U(2) invariant scalar fields in both models, but the scalar potentials show only a single vacua in each of the potentials constructed from (M3) and (M5) by introducing the appropriate coset representatives. For example, using canonically normalised fields, the potential of (M3) is

$$\begin{aligned}
V =& \frac{x_1^{-\frac{4}{\sqrt{3}}} x_3^{-3\sqrt{2}}}{8192 x_2^2 x_4^2}\left(x_3^{\sqrt{2}} - 1\right)^2 \Big[-8 m_1 m_2 x_1^{\sqrt{3}} x_2 \left(x_2^2 - 1\right)\left(3x_3^{2\sqrt{2}} + 2x_3^{\sqrt{2}} + 3\right)^2 \left(x_4^4 - 1\right) \\
&+ m_1^2 x_1^{2\sqrt{3}} \left(\left(3x_3^{2\sqrt{2}} + 2x_3^{\sqrt{2}} + 3\right)\left(x_4^2 + 1\right)\left(1 + x_2^2\right) + 4\left(x_3^{\sqrt{2}} - 1\right)^2 x_4 x_2\right) \\
&\left(\left(3x_3^{2\sqrt{2}} + 2x_3^{\sqrt{2}} + 3\right)\left(x_4^2 + 1\right)\left(1 + x_2^2\right) - 4\left(x_3^{2\sqrt{2}} + 6x_3^{\sqrt{2}} + 1\right) x_4 x_2\right) \\
&+ m_2^2 \left(\left(3x_3^{2\sqrt{2}} + 2x_3^{\sqrt{2}} + 3\right)\left(x_4^2 + 1\right)\left(1 + x_2^2\right) - 4\left(x_3^{\sqrt{2}} - 1\right)^2 x_4 x_2\right) \\
&\left(\left(3x_3^{2\sqrt{2}} + 2x_3^{\sqrt{2}} + 3\right)\left(x_4^2 + 1\right)\left(1 + x_2^2\right) + 4\left(x_3^{2\sqrt{2}} + 6x_3^{\sqrt{2}} + 1\right) x_4 x_2\right)\Big].
\end{aligned} \tag{4.27}$$

This shows a single critical point at $x_i = 1$, where the scalars $x_{1,2,4}$ are moduli, while the scalar x_3 is massive with mass m_2. In fact, x_1 is a modulus that simply rescales the mass parameters. While the gauge group is the same, the two vacua indeed pertain to two different gaugings. This is possible because the decomposition of the **351** of $E_{6(6)}$ under SO*(4) shows 6 singlets and therefore one could find inequivalent embeddings of the same gauge group.

Mass spectra

In this final section, we present the mass spectra of all of the vacua listed in the previous table. The masses for backgrounds with a non-vanishing cosmological constant are normalised in terms of the (A)dS radius squared $L^2 = |6/V|$, so that supersymmetric gravitinos have a normalised squared mass of $9/4$.

For the AdS vacua, which are not new, most of these spectra were already known from previous work, though the spectrum of the non-supersymmetric ones (A2), (A3) and (A5) was lacking some states that we provide.

$L^2m^2_{3/2}$	$\left[\frac{9}{4}\right]_8$
$L^2m^2_{vec}$	$[0]_{15}$
$L^2m^2_{tens}$	$[1]_{12}$
$L^2m^2_{1/2}$	$\left[\frac{1}{4}\right]_{40}$, $\left[\frac{9}{4}\right]_8$
$L^2m^2_{scal}$	$[-4]_{20}$, $[-3]_{20}$, $[0]_2$

Table 4.8 Masses for the AdS vacuum A1

$L^2m^2_{3/2}$	$\left[\frac{8}{3}\right]_8$
$L^2m^2_{vec}$	$[0]_{10}$, $\left[\frac{8}{3}\right]_5$
$L^2m^2_{tens}$	$\left[\frac{2}{3}\right]_{10}$, $[6]_2$
$L^2m^2_{1/2}$	$[0]_{32}$, $\left[\frac{8}{3}\right]_8$, $\left[\frac{675}{128}\right]_8$
$L^2m^2_{scal}$	$\left[-\frac{16}{3}\right]_{14}$, $[-2]_{20}$, $[0]_7$, $[8]_1$

Table 4.9 Masses for the AdS vacuum A2

$L^2m^2_{3/2}$	$\left[\frac{49}{18}\right]_6$, $\left[\frac{9}{2}\right]_2$
$L^2m^2_{vec}$	$[0]_8$, $\left[\frac{32}{9}\right]_6$, $[8]_1$
$L^2m^2_{tens}$	$\left[\frac{8}{9}\right]_6$, $\left[\frac{32}{9}\right]_6$
$L^2m^2_{1/2}$	$[0]_8$, $\left[\frac{1}{2}\right]_{16}$, $\left[\frac{25}{18}\right]_{18}$, $\left[\frac{121}{18}\right]_6$
$L^2m^2_{scal}$	$\left[-\frac{40}{9}\right]_{12}$, $\left[-\frac{16}{9}\right]_{12}$, $[0]_{17}$, $[8]_1$

Table 4.10 Masses for the AdS vacuum A3

The spectrum of the vacuum (A4) is particularly interesting in the context of the AdS/CFT correspondence, as it fixes the anomalous dimensions of the operators of the corresponding N=1 deformation of super-Yang–Mills in 4 dimensions [68].

$L^2 m^2_{3/2}$	$\left[\frac{49}{16}\right]_4,\ [4]_2,\ \left[\frac{9}{4}\right]_{\times 2}$
$L^2 m^2_{vec}$	$[0]_4,\ \left[\frac{9}{16}\right]_{\times 4},\ \left[\frac{5}{4}\right]_{\times 2},\ \left[\frac{65}{16}\right]_{\times 4},\ [6]_{\times 1}$
$L^2 m^2_{tens}$	$\left[\frac{9}{4}\right]_{\times 2},\ \left[\frac{9}{16}\right]_{\times 4},\ \left[\frac{25}{16}\right]_{\times 4},\ \left[\frac{25}{4}\right]_{\times 2},$
$L^2 m^2_{1/2}$	$\left[\frac{1}{16}\right]_{\times 4},\ \left[\frac{1}{4}\right]_{\times 6},\ \left[\frac{9}{16}\right]_{\times 4},\ [1]_{\times 2},\ \left[\frac{25}{16}\right]_{\times 4},\ \left[\frac{9}{4}\right]_{\times 2},$ $\left[\frac{49}{16}\right]_{\times 8}, [4]_{\times 2},\ [0]_{\times 12},\ \left[\frac{29}{4} \pm \sqrt{7}\right]_{\times 2}$
$L^2 m^2_{scal}$	$[0]_{\times 13},\ [-4]_3,\ \left[-\frac{15}{4}\right]_{\times 12},\ \left[-\frac{55}{16}\right]_{\times 4},\ [-3]_{\times 2},\ \left[-\frac{39}{16}\right]_{\times 4},\ [3]_{\times 2},\ [4 \pm 2\sqrt{7}]_{\times 1}$

Table 4.11 Masses for the AdS vacuum A4

$L^2 m^2_{3/2}$	$\left[\frac{81}{25}\right]_{\times 4},\ \left[\frac{18}{5}\right]_{\times 4}$
$L^2 m^2_{vec}$	$[0]_{\times 4},\ \left[\frac{24}{5}\right]_{\times 1},\ \left[\frac{96}{25}\right]_{\times 8},\ \left[\frac{24}{25}\right]_{\times 2}$
$L^2 m^2_{tens}$	$\left[\frac{44}{5}\right]_{\times 2},\ [4]_{\times 2},\ \left[\frac{16}{25}\right]_{\times 8}$
$L^2 m^2_{1/2}$	$[0]_{\times 8},\ \left[\frac{22}{5} \pm 4\sqrt{\frac{2}{5}}\right]_{\times 4},\ \left[\frac{34}{25}\right]_{\times 8},\ \left[\frac{2}{5}\right]_{\times 4},\ \left[\frac{1}{25}\right]_{\times 12},\ \left[\frac{161}{25} \pm \frac{4}{5}\sqrt{34}\right]_{\times 4}$
$L^2 m^2_{scal}$	$\left[\frac{52}{5}\right]_{\times 2},\ \left[\frac{84}{25}\right]_{\times 2},\ \left[\frac{48}{5}\right]_{\times 1},\ \left[-\frac{136}{25}\right]_{\times 6},\ [-4]_{\times 4},\ \left[-\frac{64}{25}\right]_{\times 8},\ \left[-\frac{12}{5}\right]_{\times 6},\ [0]_{\times 13}$

Table 4.12 Masses for the AdS vacuum A5.

The supersymmetric AdS vacua, (A1) and (A4), are perturbatively stable, respecting the Breitenlohner–Freedman bound [69, 70]

$$m^2 \geq -\frac{(d-1)^2}{4}, \tag{4.28}$$

for scalar degrees of freedom in AdS_d. On the other hand, (A2), (A3), and (A5) are all pertubatively unstable, thus corroborating the hypotesis about the instability of non-supersymmetric AdS spacetimes formulated in the contest of the Swampland programme [71]. The full spectra of the de Sitter vacua (D1) and (D2) are new and show that such vacua are unstable with very large instabilities, of the order of the cosmological constant, or larger.

$L^2 m^2_{3/2}$	$[0]_{\times 8}$
$L^2 m^2_{vec}$	$[0]_{\times 6}$, $[8]_{\times 9}$
$L^2 m^2_{tens}$	$[2]_{\times 12}$
$L^2 m^2_{1/2}$	$[0]_{\times 16}$, $[8]_{\times 32}$
$L^2 m^2_{scal}$	$[-8]_{\times 1}$, $[-6]_{\times 2}$, $[0]_{\times 11}$, $[10]_{\times 18}$, $[16]_{\times 10}$

Table 4.13 Masses for the dS vacuum D1.

$L^2 m^2_{3/2}$	$\left[\frac{9}{2}\right]_{\times 2}$, $\left[\frac{81}{2}\right]_{\times 6}$
$L^2 m^2_{vec}$	$[0]_{\times 3}$, $[24]_{\times 1}$, $[96]_{\times 11}$
$L^2 m^2_{tens}$	$[32]_{\times 6}$, $[56]_{\times 6}$
$L^2 m^2_{1/2}$	$[0]_{\times 8}$, $\left[\frac{25}{2}\right]_{\times 6}$, $\left[\frac{121}{2}\right]_{\times 10}$, $\left[\frac{169}{2}\right]_{\times 6}$, $\left[\frac{225}{2}\right]_{\times 8}$, $\left[\frac{289}{2}\right]_{\times 10}$
$L^2 m^2_{scal}$	$[-24]_{\times 1}$, $[0]_{\times 14}$, $[4(29 \pm \sqrt{433})]_{\times 5}$, $[40]_{\times 3}$, $[112]_{\times 12}$, $[120]_{\times 2}$

Table 4.14 Masses for the dS vacuum D2.

For what concerns the Minkowski vacua, since there is no intrinsic scale associated with the vacuum, we parametrised all masses in terms of the ones of the gravitini. We easily reproduced the expected spectrum for the CSS vacua, whereas the results for all the other vacua are new.

By also looking at the fermion shifts collected in Appendix B, is interesting to notice that all the vacua we found show spectra that do not depend on additional parameters except for a few masses (or the cosmological constant, if different from zero). This means that for all the gaugings considered the vacua appear in a unique theory with that gauge group and there are no continuous families of models with the same gauge group containing such vacua. This differs from what was discovered in the 4-dimensional case [22, 72], where it was found that one can have infinite families of gaugings with the same gauge group and vacua whose existence and value of the cosmological constant may depend on the parameter specifying the family of gaugings.

$m^2_{3/2}$	$\left[m_1^2\right]_{\times 2},\ \left[m_2^2\right]_{\times 2},\ \left[m_3^2\right]_{\times 2},\ \left[m_4^2\right]_{\times 2}$
m^2_{vec}	$[0]_{\times 1},\ [(m_1 \pm m_3)^2]_{\times 2},\ [(m_1 \pm m_4)^2]_{\times 2},\ [(m_2 \pm m_3)^2]_{\times 2},\ [(m_2 \pm m_4)^2]_{\times 2}$
m^2_{tens}	$[0]_{\times 2},\ [(m_1 \pm m_2)^2]_{\times 2},\ [(m_3 \pm m_4)^2]_{\times 2}$
$m^2_{1/2}$	$[0]_{\times 8},\ [m_i^2]_{\times 2},\ [(m_3 \pm m_1 \pm m_2)^2]_{\times 2},\ [(m_4 \pm m_1 \pm m_2)^2]_{\times 2},$ $[(m_1 \pm m_3 \pm m_4)^2]_{\times 2},\ [(m_2 \pm m_3 \pm m_4)^2]_{\times 2}$
m^2_{scal}	$[0]_{\times 18},\ \left[(m_1 \pm m_2)^2\right]_{\times 2},\ \left[(m_3 \pm m_4)^2\right]_{\times 2},\ \left[(m_1 \pm m_2 \pm m_3 \pm m_4)^2\right]_{\times 2}$

Table 4.15 Masses for the CSS vacuum M1.

$m^2_{3/2}$	$[0]_2,\ [m_1^2]_2,\ [m_2^2]_2,\ [m_3^2]_2$
m^2_{vec}	$[0]_3,\ [(m_1 \pm m_2)^2]_2,\ [(m_1 \pm m_3)^2]_2,\ [(m_2 \pm m_3)^2]_2,$
m^2_{tens}	$[m_1^2]_4,\ [m_2^2]_4,\ [m_3^2]_4,$
$m^2_{1/2}$	$[0]_{10},\ [m_1^2]_2,\ [m_2^2]_2,\ [m_3^2]_2,\ [(m_1 \pm m_2)^2]_4,\ [(m_1 \pm m_3)^2]_4,$ $[(m_2 \pm m_3)^2]_4,\ [(m_1 \pm m_2 \pm m_3)^2]_2$
m^2_{scal}	$[0]_{14},\ [m_1^2]_4,\ [m_2^2]_4,\ [m_3^2]_4,\ [(m_1 \pm m_2 \pm m_3)^2]_4,$

Table 4.16 Masses for the Minkowski vacuum M2.

The other interesting fact that emerges from the spectra is that also in 5 dimensions, like in 4, Minkowski vacua have moduli. In fact, once we remove the scalars that are eaten by the massive vectors in the usual Higgs mechanism, we see that the vacuum (M2) has two additional massless fields, the vacuum (M3) has 6 additional moduli, the vacuum (M4) 7 and the vacuum (M5) again 6. Like in the 4-dimensional case [53], it may be worth investigating if these gaugings can be connnected to each other by infinite distance limits along their moduli spaces. Quite possibly, the most general such limits may also generate novel gaugings with new Minkowski vacua and residual symmetries other than U(2).

$m^2_{3/2}$	$[0]_{\times 4},\ [m_1^2]_{\times 2},\ [m_2^2]_{\times 2}$
m^2_{vec}	$[0]_{\times 4},\ [m_1^2]_{\times 4},\ [m_2^2]_{\times 4},\ [(m_1 \pm m_2)^2]_{\times 2}$
m^2_{tens}	$[0]_{\times 3},\ [m_1^2]_{\times 4},\ [m_2^2]_{\times 4}$
$m^2_{1/2}$	$[0]_{\times 12},\ [m_1^2]_{\times 10},\ [m_2^2]_{\times 10},\ [(m_1 \pm m_2)^2]_{\times 8}$
m^2_{scal}	$[0]_{\times 18},\ [m_1^2]_{\times 4},\ [m_2^2]_{\times 4},\ [(m_1 \pm m_2)^2]_{8}$

Table 4.17 Masses for the Minkowski vacuum M3.

$m^2_{3/2}$	$[m^2]_{\times 4},\ [3m^2]_{\times 4}$
m^2_{vec}	$[0]_{\times 4},\ [4m^2]_{\times 10},\ [8m^2]_{\times 3}$
m^2_{tens}	$[0]_{\times 2},\ [4m^2]_{\times 8}$
$m^2_{1/2}$	$[0]_{\times 8},\ [m^2]_{\times 8},\ [3m^2]_{\times 12},\ [7m^2]_{\times 8},\ [9m^2]_{\times 12}$
m^2_{scal}	$[0]_{\times 20},\ [4m^2]_{\times 10},\ [8m^2]_{\times 6},\ [12m^2]_{\times 6}$

Table 4.18 Masses for the Minkowski vacuum M4.

$m^2_{3/2}$	$[0]_{\times 4},\ [m_1^2]_{\times 2},\ [m_2^2]_{\times 2}$
m^2_{vec}	$[0]_{\times 3},\ [m_1^2]_{\times 4},\ [m_2^2]_{\times 4},\ [m_1^2 + m_2^2 \pm m_3^2]_{\times 2}$
m^2_{tens}	$[0]_{\times 4},\ [m_1^2]_{\times 4},\ [m_2^2]_{\times 4}$
$m^2_{1/2}$	$[0]_{\times 12},\ [m_1^2]_{\times 10},\ [m_2^2]_{\times 10},\ [m_1^2 + m_2^2 \pm m_3^2]_{\times 8}$
m^2_{scal}	$[0]_{\times 18},\ [m_1^2]_{\times 4},\ [m_2^2]_{\times 4},\ [m_1^2 + m_2^2 \pm m_3^2]_{\times 8}$

Table 4.19 Masses for the Minkowski vacuum M5.

4.2 Vacua of 7D maximal supergravity

In the case of maximal 7-dimensional supergravities, the number of scalar degrees of freedom is small with respect to the companion theories in 4 or 5 dimensions. In fact, the scalar potential becomes a function of 14 scalars, which depends on 55 parameters. Still, the large space of parameters makes the analysis of vacua very difficult. Employing numerical methods described in the following chapters, we have discovered some new vacua, in addition, by means of the same analytical methods as for the 5 dimensional case, we have been able to restrict the analysis to vacua with at least U(1) residual gauge symmetries and confirm our numerical results. Thus, an analytical scan of these vacua has been possible; in the following, we show a detailed analysis of the U(1) invariant sector and summarise the results in tables that could be easily consulted for future reference.

4.2.1 Some Useful Notation for D=7

In the previous chapter, we made clear some facts about the 7-dimensional maximal supergravity theories, we gave the Lagrangian and the supersymmetry transforations, and reviewed briefly the scalar content of the theory. Just as it has already been shown for the 5 dimensional theories, the scalar potential can be expressed as the square of the fermion shifts:

$$V = \frac{1}{8}|A_2|^2 - 15|A_1|^2 = -\frac{1}{128}\left(15B^2 + 2C^{ab}C_{ab} - 2B^{ab}{}_{cd}B^{cd}{}_{ab} - 2C^{[ab]}{}_{(cd)}C_{[ab]}{}^{(cd)}\right). \quad (4.29)$$

Where, B, $B^{ab}{}_{cd}$, C^{ab} and $C^{ab}{}_{cd}$ are the **1**, **14**, **5** and **35** representations of SO(5)$\equiv$ Usp(4), respectively, which make up the T-tensor of the theory. The equations of motion needed to discover the minima of the scalar potential are obtained by varying the latter under a scalar transformation of the form $\delta_\Sigma \mathbf{V}_M{}^{ab} = \Sigma^{ab}{}_{cd}\mathbf{V}_M{}^{cd}$:

$$\begin{aligned}\delta_\Sigma V = &-\frac{1}{16}B^{[ab]}{}_{[cd]}B^{[cd]}{}_{[ef]}\Sigma^{[ef]}{}_{[ab]} + \frac{1}{32}BB^{[ab]}{}_{[cd]}\Sigma^{[cd]}{}_{[ab]} - \frac{1}{64}C^{[ab]}C_{[cd]}\Sigma^{[cd]}{}_{[ab]} \\ &+ \frac{1}{32}C^{[ab]}{}_{(ef)}C_{[cd]}{}^{(ef)}\Sigma^{[cd]}{}_{[ab]} - \frac{1}{8}C^{[ce]}{}_{(af)}C^{[df]}{}_{(be)}\Sigma^{[ab]}{}_{[cd]}.\end{aligned} \quad (4.30)$$

Once the minima of the scalar potential are found, we proceed to identify the masses of some of the fields by computing the eigenvalues of the squared mass matrices.

For what concerns the gravitini ψ^c_μ, the mass matrix is proportional to the fermion shift A_1:

$$M^{3/2}_{ab} = \frac{3}{2}A_{1ab}. \quad (4.31)$$

The vector mass matrices are contained in the scalar kinetic term $\frac{1}{2}P_{\mu ab}{}^{cd}P^{\mu}_{cd}{}^{ab}$, where $P_{\mu ab}{}^{cd}$ is defined by the gauge covariant space-time derivative of the scalar fields

$$\mathbf{V}_{ab}{}^{M}(\partial_\mu \mathbf{V}_M{}^{cd} - gA_\mu^{PQ}X_{PQ,M}{}^{N}\mathbf{V}^{cd}) \equiv P_{\mu ab}{}^{cd} + 2Q_{\mu[a}{}^{[c}\delta_{b]}^{d]}. \tag{4.32}$$

The $P_{\mu ab}{}^{cd}$ lies in the **14** representation of $\mathfrak{usp}(4)$, while $Q_{\mu a}{}^{c}$ in the **10**, (these must be imposed before computing the masses). Analogously, the mass term for the 2-forms arises from the kinetic term of the vectors, namely $\Omega_{ac}\Omega_{bd}\mathcal{H}_{\mu\nu}{}^{ab}\mathcal{H}^{cd\mu\nu}$, where $\mathcal{H}_{\mu\nu}{}^{ab}$ is given in 3.58. Similarly, the mass term for the 3-forms, $S^N_{\mu\nu\rho}$, arises from the kinetic term of the 2-forms, $\Omega^{ac}\Omega^{bd}\mathcal{H}_{\mu\nu\rho ab}\mathcal{H}^{\mu\nu\rho}_{cd}$. The covariant field strength for the 2-forms is given by:

$$\mathcal{H}_{\mu\nu\rho M} = 3D_{[\mu}B_{\nu\rho]M} + 6\varepsilon_{MNPQR}A^{NP}_{[\mu}\left(\partial_\nu A^{QR}_\rho + \frac{2}{3}gX_{ST,U}{}^{Q}A^{RU}_\nu A^{ST}_{\rho]}\right) + gY_{MN}S^N_{\mu\nu\rho}.$$

Note that the kinetic term for the 3-forms in 3.82 is linear in the derivative, so what one really obtains from this procedure is the mass matrix and not the square mass matrix. Furthermore, this kinetic term is not canonically normalised; indeed, it is in the schematic form $Y_{MN}S^M DS^N$, so the true masses are obtained once one multiplies the mass matrix that arises from the kinetic term of the 2-forms by Y^{-1}_{MN}. For the scalar masses, we note that it is possible to parameterise the scalar fields in terms of the USp(4)-invariant, symmetric unimodular matrix $\mathcal{M}_{MN}$ defined by

$$\mathcal{M}_{MN} \equiv \mathbf{V}_M{}^{ab}\mathbf{V}_N{}^{cd}\Omega_{ac}\Omega_{bd}. \tag{4.33}$$

The scalar potential, written in terms of $\mathcal{M}_{MN}$, is

$$\begin{aligned} V =& \frac{1}{64}\left(2\mathcal{M}^{MN}Y_{NP}\mathcal{M}^{PQ}Y_{QM} - \left(\mathcal{M}^{MN}Y_{MN}\right)^2\right) \\ &+ Z^{MN,P}Z^{QR,S}\left(\mathcal{M}_{MQ}\mathcal{M}_{NR}\mathcal{M}_{PS} - \mathcal{M}_{MQ}\mathcal{M}_{NP}\mathcal{M}_{RS}\right). \end{aligned} \tag{4.34}$$

Therefore, by means of $\delta_\Sigma \mathcal{V}_M{}^{ab} = \Sigma^{ab}{}_{cd}\mathcal{V}_M{}^{cd}$, it is possible to calculate the second variation of the potential and, therefore, the scalar mass matrix, always recalling that $\Sigma^{ab}{}_{cd}$ belongs to the **14** representation of USp(4). Eq. 3.64 has been extensively used to calculate the masses. To check for the residual supersymmetries of the vacua, one has to verify how many non-vanishing ε^a spinors exist that satisfy:

$$A_{2a,bcd}\varepsilon^a = 0. \tag{4.35}$$

We have already explained the procedure for finding the extrema of the scalar potential in the 5 dimensional case, including the XL-Algorithm or extended linearisation (Appendix D) and the use of Grobner Basis, so in the following we will directly deal with the vacua with residual U(1) gauge symmetry.

4.2.2 Vacua with residual U(1) symmetry

Requesting a residual U (1) invariance of the vacuum restricts the allowed coefficients of the embedding tensor. To perform this analysis, we looked at all the in-equivalent embeddings of U(1) in SO(5)∼ Usp(4) and singled out all the different charge assignments. The branchings of the **1**, **14**, **5** and **35** of Usp(4) under this decomposition have been obtained, therefore classifying all in-equivalent cases. When the commutant of the residual symmetry group in USp(4) was non-trivial we used the commuting symmetries to further reduce the number of in-equivalent variables by removing those that could be generated by the action of the commutant. We then proceeded to solve the set of quadratic constraints 3.55, 3.73, and the equations of motion 4.30, and then collected the solutions that could be related by duality transformations. Then we continued by analysing the residual supersymmetry and some of their masses. We first list the possible U(1) branchings, listing also the number of singlets in the T-tensor irreducible USp(4) representations, which are going to be the variables to be fixed by the quadratic constraints and equations of motion.

Branchings

We find 3 different branchings of the fundamental representation of USp(4)∼ SO(5) under U(1), which has been studied separately. The labels on the various factors are self-explanatory: we use letters from the beginning of the alphabet to keep track of the various factors in the decompositions and we use S to specify the symmetric embedding of the group.

Case 1: $\mathbf{4} \to (\pm\mathbf{1},\mathbf{0}) + (\mathbf{0},\pm\mathbf{1})$.

The branching path is

$$USp(4) \to SU(2)_A \times SU(2)_B \to U(1)_A \times U(1)_B \tag{4.36}$$

we classify the branchings by how we choose the embedding of U(1) inside U(1)$_A$× U(1)$_B$

#	**4** decomposition	charges choice	**5** singlets	**14** singlets	**35** singlets
1	$\mathbf{4} \to (\pm 1, 0) + (0, \pm 1)$	(q_A, q_B)	1	2	3
1a	$\mathbf{4} \to (\pm 1) + 2 \cdot (0)$	q_A / q_B	1	4	7
1b	$\mathbf{4} \to 2 \cdot (\pm 1)$	$q_A + q_B$	3	6	11
1c	$\mathbf{4} \to (\pm 1) + (\pm 3)$	$q_A + 3q_B$	1	2	5

Table 4.20 Branchings for case 1.

Case 2: $\mathbf{4} \to (\pm\mathbf{1}, \pm\mathbf{1})$.

The branching path is

$$USp(4) \to SU(2)_A \times SU(2)_B \to U(1)_A \times U(1)_B \tag{4.37}$$

we classify the branchings by how we choose the embedding of U(1) inside U(1)$_A\times$ U(1)$_B$

#	**4** decomposition	charges choice	**5** singlets	**14** singlets	**35** singlets
2	$\mathbf{4} \to (\pm 1, \pm 1)$	(q_A, q_B)	1	2	3
2a	$\mathbf{4} \to 2 \cdot (\pm 1)$	q_A / q_B	3	6	11
2b	$\mathbf{4} \to 2 \cdot (0) + (\pm 2)$	$q_A + q_B$	1	4	7
2c	$\mathbf{4} \to (\pm 1) + (\pm 3)$	$q_A + 2q_B$	1	2	5

Table 4.21 Branchings for case 2.

All these cases were already contained in the first case.

Case 3: $\mathbf{4} \to (\pm\mathbf{3}) + (\pm\mathbf{1})$.

The branching path is

$$USp(4) \to SU(2)_S \to U(1) \tag{4.38}$$

There is 1 singlet in $C_{[ab]}$, 2 in $B^{[ab]}{}_{[cd]}$ and 5 in $C^{[ab]}{}_{(cd)}$, respectively the **5**, **14** and **35** representations of USp(4). Also, this case was contained in the first case. Therefore, we have only three branchings to analyse.

Vacua

The search for vacua has been carried out by solving the sets of quadratic equations for the singlets in the tables above. Once we found solutions, we checked for each candidate vacuum the rank of the embedding tensor, which gives us the dimensions of the gauge group, the signature of the resulting Cartan–Killing matrix, which provides us with the compact and non-compact directions of the gauge group, and some of the masses. Overall, we found 2 different Anti-de Sitter vacua and 5 Minkowski vacua. The vacua with negative cosmological constant are all pertaining to the same gauging, namely the maximal SO(5) theory, and were already known [73]. All vacua are reported in table 4.22, together with the number of supersymmetry they preserve, the original gauging, the residual gauge group, and the reference where they were first discovered. In Appendix C we provide for each vacuum one instance of fermion shift values reproducing the critical point mentioned in the table.

vacuum	susy	G_{gauge}	G_{res}	ref.	branching
A1	4	SO(5)	SO(5)	[73]	1a, 1b, 1c
A2	0	SO(5)	SO(4)	[74]	1a, 1b, 1c
M1	0	$U(1)\ltimes\mathbb{R}^6$	U(1)	[33]	1a, 1b, 1c
M2	0	$U(1)\ltimes\mathbb{R}^4$	U(1)	here	1a
M3	0	$U(1)\ltimes\mathbb{R}^6$	U(1)	here	1b

Table 4.22 Summary of vacua with at least U(1) residual gauge symmetry in 7 dimensions.

All AdS vacua appear in the SO(5) theory, which is a consistent truncation of M-theory compactified on S^4.

Mass spectra

In this section we present some of the masses for all the vacua listed in table 4.22. The masses for the AdS backgrounds are normalised in terms of the AdS radius squared $L^2 = |15/V|$, so that supersymmetric gravitinos have a normalised squared mass of 25/16.

$L^2 m^2_{3/2}$	$\left[\frac{25}{16}\right]_{\times 4}$
$L^2 m^2_{vec}$	$[0]_{\times 10}$
$L^2 m^2_{2-forms}$	$[0]_{\times 5}$
$L^2 m^2_{3-forms}$	$[1]_{\times 5}$
$L^2 m^2_{1/2}$	$\left[\frac{9}{16}\right]_{\times 16}$
$L^2 m^2_{scal}$	$[8]_{\times 14}$

Table 4.23 Masses for the AdS vacuum A1

$L^2 m^2_{3/2}$	$\left[\frac{27}{16}\right]_{\times 4}$
$L^2 m^2_{vec}$	$\left[\frac{3}{4}\right]_{\times 4}$, $[0]_{\times 6}$
$L^2 m^2_{2-forms}$	$[0]_{\times 5}$
$L^2 m^2_{3-forms}$	$\left[\frac{3}{4}\right]_{\times 4}$, $3_{\times 1}$
$L^2 m^2_{1/2}$	$[0]_{\times 4}$, $\left[\frac{3}{16}\right]_{\times 12}$
$L^2 m^2_{scal}$	$[-12]_{\times 1}$, $[12]_{\times 9}$, $[0]_{\times 4}$

Table 4.24 Masses for the AdS vacuum A2

(A1) results stable according to the Breithenloner–Freedman bound [69, 70] while (A2) is perturbatively unstable, as already shown in [74]. For what concerns the Minkowski vacua, since there is no intrinsic scale associated with the vacuum, we parametrised all masses in terms of the ones of the gravitini. More information about these vacua and their spectra can be found in [75].

$m^2_{3/2}$	$[m_1^2]_{\times 2}$, $[m_2^2]_{\times 2}$
m^2_{vec}	$[(m_1+m_2)^2]_{\times 2}$, $[4m_1^2]_{\times 2}$, $[4m_2^2]_{\times 2}$, $[0]_{\times 4}$
$m^2_{2-forms}$	$[0]_{\times 3}$, $[(m_1-m_2)^2]_{\times 2}$
$m^2_{3-forms}$	$[(m_1+m_2)^2]_{\times 2}$, $[0]_{\times 3}$
m^2_{scal}	$[0]_{\times 8}$, $[\pm 16(m_1+m_2)^2]$, $[\pm 16(m_1-m_2)^2]$, $[\pm 4(m_1-m_2)^2]$

Table 4.25 Masses for the Minkowski vacuum M1

$m^2_{3/2}$	$[m^2]_{\times 2}$, $[0]_{\times 2}$
m^2_{vec}	$[(2m)^2]_{\times 2}$, $[(m)^2]_{\times 2}$, $[0]_{\times 6}$
$m^2_{2-forms}$	$[0]_{\times 3}$, $[m^2]_{\times 2}$
$m^2_{3-forms}$	$[m^2]_{\times 2}$, $[0]_{\times 3}$
m^2_{scal}	$[-(4m)^2]_{\times 4}$, [-(2m)^2]$_{\times 2}$, $[0]_{\times 8}$

Table 4.26 Masses for the Minkowski vacuum M2

$m^2_{3/2}$	$\left[\frac{1}{4}(m_1^2 - m_2^2)\right]_{\times 4}$
m^2_{vec}	$[(m_1)^2]_{\times 2}$, $[(m_1 \pm m_2)^2]_{\times 2}$, $[0]_{\times 4}$
$m^2_{2-forms}$	$[0]_{\times 3}$, $[m_2^2]_{\times 2}$
$m^2_{3-forms}$	$[m_1^2]_{\times 2}$, $[0]_{\times 3}$
m^2_{scal}	$[0]_{\times 8}$, $[-(4m_1)^2]_{\times 2}$, $[-(4m_2)^2]_{\times 2}$, $[-(2m_2)^2]_{\times 2}$

Table 4.27 Masses for the Minkowski vacuum M3

Chapter 5

Exceptionality in Supergravity

5.1 Introduction

Various techniques have been adopted during the course of this thesis to tackle the problem of finding and analysing the vacua of supergravity theories. Some of them, which either have been reported previously or will be described in the next chapter, are designed to solve directly the system of quadratic equations. In this chapter, on the other hand, we will focus on certain constraints that may be imposed on the embedding tensor of gauged supergravities in order to reduce the number of parameters in the equations and ease the computational effort to solve the systems. The constraints that we are going to impose are dictated by the principle of upliftability. This concept has gained much attention in recent years, due to the extensive adoption of effective field theory analysis [76–79] and of the Swampland criterion [80–82] by the scientific community. Nowadays, effective field theories which, by the time at which their energy scale reaches the Planck scale, cannot be embedded in String Theory (or more generally in a theory of Quantum Gravity) are considered to be part of the swampland and therefore are ruled out from the set of "good effective field theories".
Driven by this principle, we are going to impose that the possible gaugings of a supergravity theory arise from configurations in String Theory or M-Theory. As we shall see, this will imply further constraints on the embedding tensor.
In order to understand the origin of these constraints, it is necessary to introduce a powerful tool, namely Exceptional Field Theory. These can be seen as natural generalisations of supergravity theories, which relate a specific ungauged theory with M/IIA/IIB theory, by casting 11-dimensional supergravity in an $\mathrm{E}_{n(n)}$ covariant form.

5.2 Exceptional Field Theory at Work

This paragraph is devoted to the explanation of Exceptional Field Theories (EFT) and how they work, in order to achieve this result, we are going to use extensively the example of $E_{6(6)}$-EFT, which relates 5-dimensional ungauged supergravity to M-theory and type IIB string theory, following [83]. EFT is an approach that takes the steps from Double Field Theory (DFT), the latter was created to make the $O(d,d)$ T-duality group of string theory manifest [84, 85]. DFT consists in doubling the number of space-time coordinates, imposing a 'section constraint' or 'strong constraint' and organising the fields into $O(d,d)$ tensors [84–93]. DFT has been used for bosonic string theory, heterotic string [94], and their supersymmetric extensions [95–97]. The fields of M-theory (or better 11-dimensional supergravity) do not organise into tensors of any of $E_{n(n)}$ groups, unlikely the fields of DFT which naturally combine in tensors under O(10, 10). This problem has been overtaken by extending the 'internal' derivatives to transform in the vector representation of $E_{n(n)}$, by imposing the analogue of the DFT's 'strong constraint', by gauge fixing some of the Lorentz symmetry in the 11-D theory, and decomposing the fields as in a Kaluza-Klein complexification (if needed, one also has to dualise some of the fields). In this way, fields can finally be organised as $E_{n(n)}$ tensors. So we achieve a full $E_{n(n)}$ covariance at the cost of sacrificing some of the Lorentz gauge symmetry [77]. For instance, the $E_{6(6)}$ EFT, while maintaining the same fields and the same multiplet structure of the 5-dimensional supergravity, enlarges the coordinate space to a 5 + 27 dimensional space, with coordinates (x^μ, Y^M), where Y^M has dual derivatives ∂_M, which lie in the fundamental representation $\overline{27}$ of $E_{6(6)}$. The number of 'effective' coordinates is restricted by the analogue of the section constraint, though, which can be stated by

$$Y^{MN}{}_{PQ}\partial_M \otimes \partial_N = 0. \tag{5.1}$$

Where, $Y^{MN}{}_{PQ}$ is an $E_{n(n)}$ invariant tensor. In the case of $E_{6(6)}$ EFT, this translates into $Y^{MN}{}_{PQ} = d^{MNR}d_{RPQ}$, and the section constraint becomes

$$d^{PQR}\partial_Q\partial_R A = 0, \qquad d^{PQR}\partial_Q A \partial_R B = 0. \tag{5.2}$$

Where A and B are fields or gauge parameters. This section constraints, unlikely from DFT where it descends from a strong version of the level-matching condition, is a consequence of the closure of the Jacobi identity of the generalised Lie derivative. Indeed, in EFT, the local gauge transformations (diffeomorphisms and q-forms gauge transformations) are all encoded in the 'generalised diffeomorphisms', an infinite set of transformations parametrised through the parameter Λ^P, whose action is given by the generalised Lie derivative:

$$\begin{aligned}\mathbb{L}_\Lambda V^P \equiv&\Lambda^Q\partial_Q V^P - V^Q\partial_Q\Lambda^P + Y^{PQ}{}_{RS}\partial_Q\Lambda^R V^S + (\lambda-\omega)\partial_Q\Lambda^Q V^P\\ =&\Lambda^Q\partial_Q V^P + \alpha\mathbb{P}^P{}_Q{}^R{}_S\partial_R\Lambda^S V^Q + \lambda\partial_Q\Lambda^Q V^P.\end{aligned} \tag{5.3}$$

Where, λ is a weight in order to take into consideration also fields that transform as densities, and $\mathbb{P}^P{}_Q{}^R{}_S \equiv (t_\alpha)_Q{}^P(t^\alpha)_S{}^R$ is the projector onto the adjoint representation and is related to $Y^{PR}{}_{QS}$ by

$$Y^{PQ}{}_{RS} = \delta^P_R\delta^Q_S + \omega\delta^Q_R\delta^P_S - \alpha\mathbb{P}^P{}_R{}^Q{}_S. \tag{5.4}$$

In the case of $E_{6(6)}$, the fundamental representation is **27**, and it contains 2 cubic invariant tensors (fully symmetric), d^{PQR} and d_{PQR}, normalised as $d_{PQR}d^{SQR} = \delta^S_P$, so that $Y^{PQ}{}_{RS} = 10d_{RST}d^{PQU}$, in addition $\omega \equiv \frac{1}{(D-2)} = \frac{1}{3}$, with D the number of spacetime dimensions of the ungauged supergravity, and $\alpha = -6$.
So that the generalised Lie derivative for the $E_{6(6)}$ case is

$$\delta V^P = \mathbb{L}_\Lambda V^P \equiv \Lambda^Q\partial_Q V^P - 6\mathbb{P}^P{}_Q{}^R{}_S\partial_R\Lambda^S V^Q + \lambda\partial_Q\Lambda^Q V^P. \tag{5.5}$$

Some useful properties of the cubic invariants are

$$\begin{aligned}d_{P(QR}d_{ST)U}d^{PUV} &= \frac{2}{15}\delta^V_{(Q}d_{RST)},\\ d_{STR}d^{S(MN}d^{PQ)T} &= \frac{2}{15}\delta^{(M}_R d^{NPQ)}.\end{aligned} \tag{5.6}$$

d-symbols are obviously invariant tensors of weight $\lambda = 0$,

$$\mathbb{L}_\Lambda d_{PQR} = 0. \tag{5.7}$$

Another very useful property is that if V_P is a covariant vector of weight $\lambda(V) = 2\omega = \frac{2}{3}$, then $W^P \equiv d^{PQR}\partial_Q V_R$ is a controvariant vector of weight $\lambda(W) = \omega = \frac{1}{3}$. To better understand the EFT, specifically in the case of $E_{6(6)}$, we note that there are some gauge parameters with trivial gauge transformations $\delta_\Lambda V^P = 0$:

$$\Lambda^P = d^{PQR}\partial_Q\chi_R. \tag{5.8}$$

In order to show this, one has to use the section constraint and eq. 5.6. This relation can be generalised to other $E_{n(n)}$ EFT, just by inserting the right $Y^{MN}{}_{PQ}$ tensor. Thanks to all these properties, it is now possible to study the gauge algebra:

$$[\mathbb{L}_\Lambda, \mathbb{L}_\Sigma]\, V^P = \mathbb{L}_{[\Lambda,\Sigma]_E} V^P. \tag{5.9}$$

Where we have introduced the 'E-bracket':

$$[\Lambda,\Sigma]^P_E = \Lambda^Q \partial_Q \Sigma^P - \Sigma^Q \partial_Q \Lambda^P - \frac{1}{2} Y^{PQ}{}_{RS}(\Lambda^R \partial_Q \Sigma^S - \Sigma^R \partial_Q \Lambda^S) \tag{5.10}$$

Requiring 5.9 implies, as already stated, the section constraint to hold, as well as other constraints:

$$Y^{PQ}{}_{RS} \partial_P \otimes \partial_Q = 0, \tag{5.11a}$$

$$(Y^{PQ}{}_{RS} \delta^T_U - Y^{PQ}{}_{VU} Y^{VT}{}_{RS}) \partial_{(Q} \otimes \partial_{T)} = 0, \tag{5.11b}$$

$$(Y^{MP}{}_{TN} Y^{TQ}{}_{[SR]} + 2Y^{MP}{}_{[R|T} Y^{TQ}{}_{S|N} - Y^{MP}{}_{[RS]} \delta^Q_N - 2Y^{MP}{}_{[S|N} \delta^Q_{R]}) \partial_{(P} \otimes \partial_{Q)} = 0 \tag{5.11c}$$

$$(Y^{MP}{}_{TN} Y^{TQ}{}_{(SR)} + 2Y^{MP}{}_{(R|T} Y^{TQ}{}_{S)N} - Y^{MP}{}_{(RS)} \delta^Q_N - 2Y^{MP}{}_{(S|N} \delta^Q_{R)}) \partial_{[P} \otimes \partial_{Q]} = 0. \tag{5.11d}$$

In all the cases, which encompass all the EFT built so far, in which $Y^{PQ}{}_{RS}$ can be expressed as in 5.4 and the section constraint 5.11a is solved, the previous constraints simplify. Terms with multiplicative coefficients 1 and 2 in eq. 5.11c and 5.11d combine into symmetrisations in three indices, which is also what is needed for eq. 5.11b to hold:

$$Y^{(PQ}{}_{RS} Y^{T)R}{}_{UV} - Y^{(PQ}{}_{UV} \delta^{T)}_S = 0 \tag{5.12}$$

Then the indices can be cycled so that the indices of the derivatives correspond to the indices on the $Y^{PQ}{}_{RS}$ and thus all the other constraints become equivalent to the section constraint, once this is solved. Therefore, the section constraints guarantee that the generalised Lie derivatives satisfy the Lie Algebra 5.9. On the other hand, the E-bracket does not respect the Jacobi Identity, it has a non-vanishing 'Jacobiator':

$$J(\Gamma_1,\Gamma_2,\Gamma_3) \equiv [[\Gamma_1,\Gamma_2]_E,\Gamma_3]_E + [[\Gamma_2,\Gamma_3]_E,\Gamma_1]_E + [[\Gamma_3,\Gamma_1]_E,\Gamma_2]_E. \tag{5.13}$$

However, it can be shown that the jacobiator is in the form of a trivial gauge parameter 5.8. In fact, let us define the Dorfman product among vectors of weight ω

$$(\Gamma_1 \circ \Gamma_2)^P \equiv (\mathbb{L}_{\Gamma_1} \Gamma_2)^P = \Gamma_1^Q \partial_Q \Gamma_2^P - \Gamma_2^Q \partial_Q \Gamma_1^P + Y^{PQ}{}_{RS} \partial_Q \Gamma_1^R \Gamma_2^S \tag{5.14}$$

in the specific case of $E_{6(6)}$, this is given by

$$(\Gamma_1 \circ \Gamma_2)^P \equiv (\mathbb{L}_{\Gamma_1}\Gamma_2)^P = \Gamma_1^Q \partial_Q \Gamma_2^P - \Gamma_2^Q \partial_Q \Gamma_1^P + 10 d^{PQT}{}_{RST}\partial_Q \Gamma_1^R \Gamma_2^S. \tag{5.15}$$

The difference between the Dorfman product and 5.10 is given by a term symmetric in the two arguments, which we call Γ:

$$(\Gamma_1 \circ \Gamma_2)^P = [\Gamma_1, \Gamma_2]_E^P + Y^{PQ}{}_{RS}\partial_Q(\Gamma_1^{(R}\Gamma_2^{S)}). \tag{5.16}$$

We can see that the last factor of the previous relation has the form of a trivial gauge parameter 5.8, so that the Dorfman product and the E-bracket generate the same generalised Lie derivative. Now, we need to prove that the Dorfman product obeys the Jacobi identity:

$$\Gamma_1 \circ (\Gamma_2 \circ \Gamma_3) - \Gamma_2 \circ (\Gamma_1 \circ \Gamma_3) - (\Gamma_1 \circ \Gamma_2) \circ \Gamma_3 = 0. \tag{5.17}$$

Indeed:

$$\begin{aligned} \Gamma_1 \circ (\Gamma_2 \circ \Gamma_3) - \Gamma_2 \circ (\Gamma_1 \circ \Gamma_3) &= \Gamma_1 \circ (\mathbb{L}_{\Gamma_2}\Gamma_3) - \Gamma_2 \circ (\mathbb{L}_{\Gamma_1}\Gamma_3) \\ &= \mathbb{L}_{\Gamma_1}\mathbb{L}_{\Gamma_2}\Gamma_3 - \mathbb{L}_{\Gamma_2}\mathbb{L}_{\Gamma_1}\Gamma_3 \\ &= \mathbb{L}_{[\Gamma_1,\Gamma_2]_E}\Gamma_3 \\ &= \mathbb{L}_{(\Gamma_1 \circ \Gamma_2)}\Gamma_3 \\ &= (\Gamma_1 \circ \Gamma_2) \circ \Gamma_3. \end{aligned} \tag{5.18}$$

then we can compute $[[\Gamma_1, \Gamma_2]_E, \Gamma_3]_E$:

$$[[\Gamma_1, \Gamma_2]_E, \Gamma_3]_E^P = ([\Gamma_1, \Gamma_2]_E \circ \Gamma_3)^P - Y^{PQ}{}_{RS}\partial_Q([\Gamma_1, \Gamma_2]_E^{(R}\Gamma_3^{S)}) \tag{5.19}$$

$$= ((\Gamma_1 \circ \Gamma_2) \circ \Gamma_3)^P - Y^{PQ}{}_{RS}\partial_Q([\Gamma_1, \Gamma_2]_E^{(R}\Gamma_3^{S)}), \tag{5.20}$$

Then using the just proven fact that the Dorfman product satisfies the Jacobi identity, one gets for the Jacobiator of the E-bracket:

$$J^P(\Gamma_1, \Gamma_2, \Gamma_3) = \frac{1}{3} Y^{PQ}{}_{RS}\partial_Q \left([\Gamma_1, \Gamma_2]_E^{(R}\Gamma_3^{S)} + [\Gamma_3, \Gamma_1]_E^{(R}\Gamma_2^{S)} + [\Gamma_2, \Gamma_3]_E^{(R}\Gamma^{S)} \right). \tag{5.21}$$

Which in the case of $E_{6(6)}$ becomes:

$$J^P(\Gamma_1, \Gamma_2, \Gamma_3) = \frac{5}{3} d^{PQT}\partial_Q \left(d_{TRS}([\Gamma_1, \Gamma_2]_E^R \Gamma_3^S + [\Gamma_3, \Gamma_1]_E^R \Gamma_2^S + [\Gamma_2, \Gamma_3]_E^R \Gamma^S) \right). \tag{5.22}$$

Proving that the E-bracket Jacobator is indeed a trivial gauge parameter.

5.2.1 Tensor Hierarchy

The gauge parameters used in the EFT depend on both the internal coordinates Y^P (note that the physical internal coordinates Y^p are a subset of these, chosen by the section constraint) and the external coordinates x^μ. Therefore, there is the need to covariantize the external derivatives and their curvatures, introduce gauge connections and, as we will see, introduce higher-form intertwinings, just as it happens for gauged supergravities. The field content of each $E_{n(n)}$ EFT is fixed by the underlying ungauged supergravity in $11-n$ dimensions, but they all contain vector fields, in particular, for the case of $E_{6(6)}$ we can introduce the gauge field A_μ^P, in order to define the covariant derivative:

$$\mathcal{D}_\mu \equiv \partial_\mu - \mathbb{L}_{A_\mu} \tag{5.23}$$

Consequently, the covariant derivative of a vector of weight λ becomes

$$\mathcal{D}_\mu V^P = \partial_\mu V^P - A_\mu^Q \partial_Q V^P + 6\mathbb{P}^P{}_Q{}^R{}_S \partial_R A_\mu^S V^Q - \lambda \partial_Q A_\mu^Q V^P. \tag{5.24}$$
$$= D_\mu V^P - \lambda \partial_Q A_\mu^Q V^P. \tag{5.25}$$

As usual, gauge transformations for gauge fields A_μ^P can be obtained by demanding the gauge covariance of covariant derivatives:

$$\begin{aligned}\delta A_\mu^P &= \partial_\mu \Lambda^P - A_\mu^Q \partial_Q \Lambda^P + \Lambda^Q \partial_Q A_\mu^P - 10 d^{PQR} d_{STR} \Lambda^T \partial_Q A_\mu^S \\ &= D_\mu \Lambda^P - \frac{1}{3}(\partial_Q A_\mu^Q)\Lambda^P \\ &\equiv \mathcal{D}_\mu \Lambda^P.\end{aligned} \tag{5.26}$$

This also shows that the gauge parameters Λ have weight $\lambda = \omega = \frac{1}{3}$. We introduce a first version of the field strength tensor for gauge fields, we will see that we need to modify it later to introduce higher q-form intertwinings.

$$\begin{aligned}F_{\mu\nu}^P &= 2\partial_{[\mu} A_{\nu]}^P - [A_\mu, A_\nu]_E^P \\ &= 2\partial_{[\mu} A_{\nu]}^P - 2A_{[\mu}^Q \partial_Q A_{\nu]}^P + 10 d^{PQR} d_{STR} A_{[\mu}^S \partial_Q A_{\nu]}^T.\end{aligned} \tag{5.27}$$

We showed before that the E-bracket does not satisfy the Jacobi identity, so this field strength does not transform covariantly. Computing its variation under an arbitrary δA_μ^P, that is a controvariant vector with weight $\lambda = \frac{1}{3}$ one gets

$$\delta F^P_{\mu\nu} = 2\mathcal{D}_{[\mu}\delta A^P_{\nu]} + 10 d^{PQR} d_{STR}\partial_Q (A^S_{[\mu}\delta A^T_{\nu]}). \tag{5.28}$$

It is possible to see, from this expression, that the last term is not covariant and has the form of a trivial gauge parameter 5.8. Therefore, we modify the expression for the gauge field strength in

$$\mathcal{F}^P_{\mu\nu} \equiv F^P_{\mu\nu} + 10 d^{PQR}\partial_R B_{\mu\nu Q} \tag{5.29}$$

in such a way that

$$\delta\mathcal{F}^P_{\mu\nu} \equiv F^P_{\mu\nu} + 10 d^{PQR}\partial_R \Delta B_{\mu\nu Q}, \tag{5.30}$$

with

$$\Delta B_{\mu\nu Q} \equiv \delta B_{\mu\nu Q} + d_{QRS} A^R_{[\mu}\delta A^S_{\nu]}. \tag{5.31}$$

Where $\delta B_{\mu\nu Q}$ comprises the gauge-symmetry transformations of the 2-forms, parametrised by $\Xi_{\mu Q}$. Summarising the gauge transformations are given by

$$\delta A^P_\mu = D_\mu \Lambda^P - \frac{1}{3}(\partial_Q A^Q_\mu)\Lambda^P - 10 d^{PQR}\partial_R \Xi_{\mu Q}, \tag{5.32}$$

$$\Delta B_{\mu\nu P} = 2 D_\mu \Xi_{\nu]P} - \frac{4}{3}(\partial_Q A^Q_{[\mu})\Xi_{\nu]P} + d_{PQR}\Lambda^Q \mathcal{F}^R_{\mu\nu} + \mathcal{O}_{\mu\nu P}, \tag{5.33}$$

with $\mathcal{O}$ unspecified but satisfying:

$$d^{PQR}\partial_R \mathcal{O}_{\mu\nu Q} = 0. \tag{5.34}$$

Under this gauge transformation, the field strength $\mathcal{F}^P_{\mu\nu}$ transform as a controvariant vector with weight $\lambda = \omega = \frac{1}{3}$. From an explicit, tedious computation, one can show that the covariant field strength satisfies the Bianchi identity:

$$2\mathcal{D}_{[\mu}\mathcal{F}^P_{\nu\rho]} = 10 d^{PQR}\partial_R \mathcal{H}_{\mu\nu\rho Q}, \tag{5.35}$$

where, the 3-form field strenght $\mathcal{H}_{\mu\nu\rho Q}$, defined from this equation, up to terms that vanish once contracted with d^{PQR}, is given by

$$\begin{aligned}\mathcal{H}_{\mu\nu\rho Q} = {} & 3\mathcal{D}_{[\mu}B_{\nu\rho]Q} - 3 d_{QRS} A^R_\mu \partial_\nu A^S_{\rho]} + 2 d_{QRS} A^R_\mu A^T_\nu \partial_T A^S_{\rho]} \\ & - 10 d_{QRS} d^{STU} d_{UVP} A^R_{[\mu} A^V_\nu \partial_T A^P_{\rho]} + \ldots\end{aligned} \tag{5.36}$$

5.2.2 The $E_{6(6)}$ Exceptional Field Theory

The field content of the $E_{6(6)}$ EFT is dictated by its underlying ungauged supergravity content, which, for what concern the bosonic field sector has already been extensively described before and contains the graviton e^a_μ, the scalar fields parametrised by a matrix $\mathcal{M}_{PQ}$, the gauge bosons A^P_μ and the two forms $B_{\mu\nu P}$. The Lagrangian is written in such a way as to be invariant under the internal generalised diffeomorphisms as well as the (covariantized) 5-dimensional external diffeomorphisms. The latter relate the various terms of the Lagrangian and fix the relative coefficients, a job usually done by supersymmetry, in this case, supersymmetry has the only job of fixing the field content. The final Lagrangian is given by

$$\begin{aligned} S_{EFT} = \int d^5x d^{27}Y e\Big(\hat{R} + \frac{1}{24} g^{\mu\nu}\mathcal{D}_\mu \mathcal{M}^{PQ}\mathcal{D}_\nu \mathcal{M}_{PQ} \\ - \frac{1}{4}\mathcal{M}_{PQ}\mathcal{F}^{\mu\nu P}\mathcal{F}^{Q}_{\mu\nu} + e^{-1}\mathcal{L}_{top} - V(\mathcal{M}_{PQ}, g_{\mu\nu})\Big). \end{aligned} \tag{5.37}$$

Where, the scalar potential is given by

$$\begin{aligned} V = &- \frac{1}{24}\mathcal{M}^{PQ}\partial_P\mathcal{M}^{RS}\partial_Q\mathcal{M}_{RS} + \frac{1}{2}\mathcal{M}^{PQ}\partial_P\mathcal{M}^{RS}\partial_S\mathcal{M}_{QR} - \frac{1}{2}g^{-1}\partial_P g\partial_Q\mathcal{M}^{PQ} \\ &- \frac{1}{4}\mathcal{M}^{PQ}g^{-1}(\partial_P g)g^{-1}\partial_Q g - \frac{1}{4}\mathcal{M}^{PQ}\partial_P g^{\mu\nu}\partial_Q g_{\mu\nu}. \end{aligned} \tag{5.38}$$

And the topological Lagrangian can be written as an integral of an exact 6-form over a 6-dimensional manifold

$$\begin{aligned} S_{top} &= \int d^5x d^{27}Y \mathcal{L}_{top} \\ &= -\sqrt{\frac{5}{32}}\int d^{27}Y \int_{\mathcal{M}_6} (d_{PQR}\mathcal{F}^P\wedge\mathcal{F}^Q\wedge\mathcal{F}^R - 40 d^{PQR}\mathcal{H}_P\wedge\partial_Q\mathcal{H}_R). \end{aligned} \tag{5.39}$$

We will not show that each of these terms separately is invariant under generalised internal diffeomorphisms, but in order to get a sense for how it works, we will illustrate some examples. For what concerns the kinetic term of the graviton, we note that there is an hat on the Ricci scalar, this is due in order to have appropriate generalised diffeomorphisms transformations. The fünf-bein is a scalar density under generalised diffeomorphisms and has weight $\lambda = \omega = \frac{1}{3}$, its gauge covariant derivative is given by

$$\mathcal{D}_\mu e^a_\nu \equiv \partial_\mu e^a_\nu - A^P_\mu \partial_P e^A_\nu - \frac{1}{3}\partial_P A^P_\mu e^a_\nu. \tag{5.40}$$

The latter enters in the spin connection $\omega_\mu{}^{ab}$, which is a scalar under generalised diffeomorphisms (namely $\lambda = 0$), and consequently the Riemann tensor $R_{\mu\nu}{}^{ab}$ is also a scalar under generalised diffeomorphisms. One problem arises, the covariant derivatives $\mathcal{D}_\mu$ do not commute among themselves, therefore the Riemann tensor does not transform covariantly under local Lorentz transformations $\delta\omega_\mu{}^{ab} = -\mathcal{D}_\mu\lambda^{ab}$. This is the reason why, in the gravity Lagrangian, the Ricci tensor is modified, indeed, we can write an improved Riemann tensor as

$$\hat{R}_{\mu\nu}{}^{ab} \equiv R_{\mu\nu}{}^{ab} + \mathcal{F}^P_{\mu\nu} e^{a\rho}\partial_P e_\rho{}^b, \tag{5.41}$$

which again is a scalar under generalised diffeomorphisms and transforms covariantly under local Lorentz transformations, so the Hilbert–Einstein term

$$S_{EH} = \int d^5x d^{27}Y e e_a{}^\mu e_b{}^\nu \hat{R}_{\mu\nu}{}^{ab}, \tag{5.42}$$

is a scalar with weight one, thus transforming as

$$\delta\mathcal{L} = \Lambda^P\partial_P\mathcal{L} + \mathcal{L}\partial_P\Lambda^P = \partial_P(\mathcal{L}\Lambda^P), \tag{5.43}$$

namely, as a total derivative. The weight of the Lagrangian is composed of the sum of the weights of each term, the determinant of the funfbein contributes as $\lambda = \frac{5}{3}$, while each inverse funf-bein contributes with $\lambda = -\frac{1}{3}$ giving $\lambda(\mathcal{L}_{EH}) = 1$.
For the scalar kinetic term, we have that the matrix M^{PQ} is a symmetric 2-tensor with weight $\lambda = 0$. Then the scalar kinetic term

$$\mathcal{L}_{scal} = \frac{1}{24} e g^{\mu\nu}\mathcal{D}_\mu\mathcal{M}_{PQ}\mathcal{D}_\nu\mathcal{M}^{PQ} \tag{5.44}$$

has total weight 1, indeed, the inverse metric has weight $\lambda = -\frac{2}{3}$, the fünf-bein determinant has always weight $\lambda = \frac{5}{3}$ giving the expected result for the Lagrangian. Analogously, recalling that $\mathcal{F}^P_{\mu\nu}$ has weight $\lambda = \frac{1}{3}$, then the kinetic Lagrangian for gauge bosons

$$\mathcal{L}_{vec} = -\frac{1}{4} e\mathcal{M}_{PQ}\mathcal{F}^P_{\mu\nu}\mathcal{F}^Q_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \tag{5.45}$$

is also generalised diffeomorphic invariant. The proof of the invariance under generalised diffeomorphism for the topological Lagrangian and the potential are more involved and so we will not report these here, in particular the coefficients for each term in the potential are fixed by this invariance.
The relative coefficients of the various terms in the Lagrangian are fixed by the external diffeomorphism invariance, this is due to the Y-dependence of the gauge parameter $\xi^\mu(x,Y)$.

These transformations are given by

$$\begin{aligned}
\delta e_\mu^a &= \xi^\nu \mathcal{D}_\nu e_\mu^a + \mathcal{D}_\mu \xi^\nu e_\nu^a, \\
\delta \mathcal{M}_{PQ} &= \xi^\mu \mathcal{D}_\mu \mathcal{M}_{PQ} \\
\delta A_\mu{}^P &= \xi^\nu \mathcal{F}_{\nu\mu}{}^P + \mathcal{M}^{PQ} g_{\mu\nu} \partial_Q \xi^\nu, \\
\Delta B_{\mu\nu P} &= \frac{e}{2\sqrt{10}} \xi^\rho \varepsilon_{\mu\nu\rho\sigma\tau} \mathcal{F}^{\sigma\tau Q} \mathcal{M}_{PQ}.
\end{aligned} \tag{5.46}$$

They may seem odd, but indeed are equivalent to the conventional covariantized (with respect to the internal diffeomorphism) external diffeomorphisms. This Exceptional Field Theory has 2 inequivalent solutions to the section constraints (besides the trivial one which sets all the internal coordinates to 0 obtaining 5-D ungauged supergravity), corresponding to 11-D and type II-B string theory. In order to show this equivalence, there is the need to decompose the fields of these theories in a Kaluza-Klein-like way(Appendix A), but unlikely from K–K compactifications we retain all the dependance on all the 11/10 coordinates. Let us sketch how this works for 11-dimensional supergravity.

5.2.3 Decomposition of 11-D supergravity

The bosonic sector of 11-dimensional supergravity is formed by the elf-bein $E_{\hat\mu}{}^{\hat a}$ and a 3-form $C_{\hat\mu\hat\nu\hat\rho}$, with all the hatted indices going from 0 to 10. The action is given by

$$S_{11} = \int d^{11}xE \left(R - \frac{1}{12} F^{\hat\mu\hat\nu\hat\rho\hat\sigma} F_{\hat\mu\hat\nu\hat\rho\hat\sigma} + \frac{1}{12\cdot 216} e^{-1} \varepsilon^{\hat\mu_1\dots\hat\mu_{11}} F_{\hat\mu_1\dots\hat\mu_4} F_{\hat\mu_1\dots\hat\mu_8} C_{\hat\mu_9\hat\mu_10\hat\mu_11} \right), \tag{5.47}$$

where the abelian field strength is $F_{\hat\mu\hat\nu\hat\rho\hat\sigma} = 4\partial_{[\hat\mu} C_{\hat\nu\hat\rho\hat\sigma]}$. In addition to being invariant under local Lorentz transformations and 11-dimensional diffeomorphisms, the theory is also invariant under gauge transformations of the 3-form $\delta C_{\hat\mu\hat\nu\hat\rho} = 3\partial_{[\hat\mu}\Lambda_{\hat\nu\hat\rho]}$. As we already anticipated, we need a K–K decomposition in order to show the equivalence between 11-D supergravity and $E_{6(6)}$ EFT, so we will split the hatted indices as

$$\hat\mu = (\mu, m), \qquad \hat a = (a, \alpha), \tag{5.48}$$

with $\mu = 1,...,5$ and $m = 1,...,6$ etc. where $D = d + n = 6 + 5$. Then one partially fixes the local Lorentz invariance SO$(1,10)$ to SO$(1,4)\times$ SO(6), by picking a specific form of the elf-bein:

$$E_{\hat\mu}{}^{\hat a} = \begin{pmatrix} \phi^\gamma e_\mu{}^a & A_\mu{}^m \phi_m{}^\alpha \\ 0 & \phi_m{}^\alpha \end{pmatrix} \tag{5.49}$$

With $\phi = det(\phi_m{}^\alpha)$ and the inverse given by

$$E_{\hat{a}}{}^{\hat{\mu}} = \begin{pmatrix} \phi^{-\gamma} e_a{}^\mu & -\phi^{-\gamma} e_a{}^\nu A_\nu{}^m \\ 0 & \phi_\alpha{}^m \end{pmatrix} \tag{5.50}$$

The parameter γ is fixed by demanding that the 5-dimensional theory is in the Einstein frame, obtaining $\gamma = -\frac{1}{n-2} = -\frac{1}{3}$. Inserting this form of the elf-bein into the Hilbert–Einstein term, we end up with the desired form.

$$\begin{aligned} S_{EH} &= \int d^D x E E_{\hat{a}}{}^{\hat{\mu}} E_{\hat{b}}{}^{\hat{\nu}} R_{\hat{\mu}\hat{\nu}}{}^{\hat{a}\hat{b}} \\ &= \int d^5 x d^6 y e \Big[\hat{R} - \frac{1}{4} \phi^{-2\gamma} \phi_{mn} F^{\mu\nu m} F_{\mu\nu}{}^n - \frac{1}{2} \phi^{mn} g^{\mu\nu} D_\mu \phi_m{}^\alpha D_\nu \phi_{n\alpha} \\ & - \gamma^2 (n-2) \phi^{-2} g^{\mu\nu} D_\mu \phi D_\nu \phi - \frac{1}{2} g^{\mu\nu} (\phi^{\alpha m} D_\mu \phi_m{}^\gamma)(\phi_\gamma{}^n D_\nu \phi_{n\alpha}) - V(\phi, e) \Big]. \end{aligned} \tag{5.51}$$

With n=5, $\phi_{mn} = \phi_{\alpha m} \phi_n^\alpha$ and

$$D_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a - A_\mu{}^m \partial_m e_\nu{}^a + \gamma \partial_n A_\mu{}^n e_\nu{}^a, \tag{5.52}$$

$$D_\mu \phi_m{}^\alpha = \partial_\mu \phi_m{}^\alpha - A_\mu{}^n \partial_n \phi_m{}^\alpha - \partial_m A_\mu{}^n \phi_n{}^\alpha, \tag{5.53}$$

$$F_{\mu\nu}{}^m = \partial_\mu A_\nu{}^m - \partial_\nu A_\mu{}^m - A_\mu{}^n \partial_n A_\nu{}^m + A_\nu{}^n \partial_n A_\mu{}^m, \tag{5.54}$$

that are covariant under "internal" diffeomorphism. Indeed, the 11-dimensional Lagrangian is invariant under 11-D diffeomorphisms, which act on the elf-bein as

$$\delta E_{\hat{\mu}}{}^{\hat{a}} = \xi^{\hat{\nu}} \partial_{\hat{\nu}} E_{\hat{\mu}}{}^{\hat{a}} + \partial_{\hat{\mu}} \xi^{\hat{\nu}} E_{\hat{\nu}} \hat{a}. \tag{5.55}$$

Splitting the gauge parameter into $\xi^{\hat{\mu}} = (\xi^\mu, \Lambda^m)$ one obtains for the internal diffeomorphisms acting on the fields describing the elf-bein:

$$\begin{aligned} \delta_\Lambda e_\mu{}^a &= \Lambda^n \partial_n e_\mu{}^a - \gamma \partial_n \Lambda^n e_\mu{}^a, \\ \delta_\Lambda \phi_m{}^\alpha &= \Lambda^n \partial_n \phi_m{}^\alpha + \partial_m \Lambda^n \phi_n{}^\alpha, \\ \delta_\Lambda \phi &= \Lambda^n \partial_n \phi + \partial_n \Lambda^n \phi, \\ \delta_\Lambda A_\mu{}^m &= \partial_\mu \Lambda^m - A_\mu{}^n \partial_n \Lambda^m + \Lambda^n \partial_n A_\mu{}^m. \end{aligned} \tag{5.56}$$

In addition, we define the improved Riemann tensor, just as we did before for the EFT

$$\hat{R}_{\mu\nu}{}^{ab} = R_{\mu\nu}{}^{ab} + F_{\mu\nu}{}^{m} e^{a\rho} \partial_m e_\rho{}^{b}. \tag{5.57}$$

This again ensures that $\hat{R_{\mu\nu}}{}^{ab}$ preserves local SO$(1,4)$ Lorentz invariance. The scalar potential in 5.51 is given by

$$V_{EH}(\phi, e) = -\phi^{2\gamma}\left(R(\phi) + \frac{1}{4}\phi^{mn}(D_m g^{\mu\nu} D_n g_{\mu\nu} + g^{-1} D_m g g^{-}1 D_n g)\right), \tag{5.58}$$

with

$$\begin{aligned} R(\phi) =& \frac{1}{2}\phi^{mn}\phi^{kl}\phi^{pq}\partial_k\phi_{mq}\partial_p\phi_{nl} - \frac{1}{4}\phi^{mn}\phi^{kl}\phi^{pq}\partial_p\phi_{mk}\partial_q\phi_{nl} - \frac{2}{3}\partial_m\phi^{mn}\phi^{-1}\partial_n\phi \\ & - \frac{21}{9}\phi^{mn}(\phi^{-1}\partial_m\phi)(\phi^{-1}\partial_n\phi) + \partial_m\phi^{mn}e^{-1}\partial_n e + 2\phi^{mn}(e^{-1}\partial_m e)(\phi^{-1}\partial_n\phi). \end{aligned} \tag{5.59}$$

Analogously, one proceeds for the other terms of the 11-dimensional Lagrangian. In general, the procedure is first to flatten all hatted indices and then to ’un-flatten’ with the fünf-bein $E_\mu{}^a$, for instance, for the 3-forms and its field strength, one gets

$$A_{\mu mn} \equiv E_\mu{}^a E_a{}^{\hat{\nu}} C_{\hat{\nu}mn}. \tag{5.60}$$

Expanding each component, one arrives at

$$\begin{aligned} A_{mnk} &= C_{mnk}, \\ A_{\mu mn} &= C_{\mu mn} - A_\mu{}^k C_{kmn}, \\ A_{\mu\nu m} &= C_{\mu\nu m} - 2A_{[\mu}{}^n C_{\nu]mn} + A_\mu{}^n A_\nu{}^k C_{mnk}, \\ A_{\mu\nu\rho} &= C_{\mu\nu\rho} - 3A_{[\mu}{}^m C_{\nu\rho]m} + 3A_{[\mu}{}^m A_\nu{}^n C_{\rho]mn} - A_\mu{}^m A_\nu{}^n A_\rho{}^k C_{mnk}. \end{aligned} \tag{5.61}$$

Similarly, for the field strengths

$$\begin{aligned} F_{mnkl} &= 4\partial_{[m}A_{nkl]}, \\ F_{\mu nkl} &= D_\mu A_{nkl} - 3\partial_{[n}A_{\mu kl]}, \\ F_{\mu\nu mn} &= 2D_{[\mu}A_{\nu]mn} + F_{\mu\nu}{}^k A_{kmn} + 2\partial_{[m}A_{\mu\nu n]}, \\ F_{\mu\nu\rho m} &= 3D_{[\mu}A_{\nu\rho]m} + 3F_{[\mu\nu}{}^n A_{\rho]mn} - \partial_m A_{\mu\nu\rho}, \\ F_{\mu\nu\rho\sigma} &= 4D_{[\mu}A_{\nu\rho\sigma]} + 6F_{[\mu\nu}{}^m A_{\rho\sigma]m}. \end{aligned} \tag{5.62}$$

Then the next steps are inserting these in the kinetic and topological 11-D Lagrangian for the 3-forms and carrying out all the computation to have the explicit terms of the Lagrangian necessary for the comparison, just as we did before for the Einstein–Hilbert term. Having sketched how to K–K decompose 11-dimensional supergravity, we can move to solve the section constraint of the EFT, in order to obtain this subcase.

5.2.4 GL(6) invariant reduction of EFT

In order to obtain the equivalence of EFT with 11-D supergravity in the formulation described above, one has to solve the section constraint in a specific manner. In particular, for 11-dimensional supergravity, we need to break $E_{6(6)}$ to GL(6). GL(6) is embedded in $E_{6(6)}$ as

$$GL(6) = SL(6) \times GL(1) \subset SL(6) \times SL(2) \subset E_{6(6)}, \tag{5.63}$$

such that the fundamental $E_{6(6)}$ representation breaks as

$$\overline{\mathbf{27}} \to 6_{+1} + 15'_0 + 6_{-1}, \tag{5.64}$$

while the adjoint decomposes as

$$\mathbf{78} \to 1_{-2} + 20_{-1} + (1+35)_0 + 20_{+1} + 1_{+2}, \tag{5.65}$$

and the subscripts refer to the GL(1) charges. We can solve the section constraint by asking the internal coordinate to lie only in the 6_{+1} representation of GL(6). Indeed, we have:

$$\{Y^P\} \to \{y^p, y_{pq}, y^{\overline{p}}\}, \tag{5.66}$$

where the indices p,q = 1,...,6. Then, using the summation convention $\Gamma_1^P \Gamma_{2P} = \Gamma_1^p \Gamma_{2p} + \Gamma_{1pq}\Gamma_2^{pq} + \Gamma_1^{\overline{p}}\Gamma_{2\overline{p}}$, one learns the non-vanishing components of the d-symbol (the $Y^{PQ}{}_{RS}$ components in the general case)

$$\begin{aligned} d^{PQR} : d^{p\overline{q}}{}_{rs} &= \frac{1}{\sqrt{5}} \delta^p_{[r} \delta^q_{s]}, \qquad & d_{pqrstu} &= \frac{1}{4\sqrt{5}} \varepsilon_{pqrstu}, \\ d_{PQR} : d_{p\overline{q}}{}^{rs} &= \frac{1}{\sqrt{5}} \delta^r_{[p} \delta^s_{q]}, \qquad & d^{pqrstu} &= \frac{1}{4\sqrt{5}} \varepsilon^{pqrstu}, \end{aligned} \tag{5.67}$$

as usual, the other non-vanishing components are related by the symmetries of the d-symbol $d^{PQR} = d^{(PQR)}$. GL(1) charges imply that all the other components vanish, then the section constraint can be solved by choosing the coordinates in such a way that

$$\{\partial_{\overline{p}}\Gamma_1 = 0,\, \partial^{pq}\Gamma_1 = 0\} \iff \Gamma_1(x^\mu, Y^P) \to \Gamma_1(x^\mu, y^p). \tag{5.68}$$

The vector fields in the EFT, transforming in the $\overline{\mathbf{27}}$, now decompose as in 5.64, while the 2-forms enter the theory, only under the projection with the d-symbol $d^{PQR}\partial_Q B_{\mu\nu R}$, therefore it immediately follows that only $B_{\mu\nu\overline{p}}$ and $B_{\mu\nu}{}^{pq}$ enter the Lagrangian, in addition, they enter with the specific combinations

$$\partial_p B_{\mu\nu\overline{q}} - \partial_q B_{\mu\nu\overline{p}} \qquad \text{and} \qquad \partial_p B_{\mu\nu}{}^{pq}, \tag{5.69}$$

so that there is an additional local shift symmetry in the theory, given by

$$\delta B_{\mu\nu\overline{p}} = \partial_p \Gamma_{\mu\nu}, \qquad\qquad \delta B_{\mu\nu}{}^{pq} = \partial_r \Gamma_{\mu\nu}{}^{[pqr]}, \tag{5.70}$$

for arbitrary $\Gamma_{\mu\nu}$, $\Gamma_{\mu\nu}{}^{[pqr]}$. Once decomposed, the q-forms content is

$$\{A_\mu{}^p, A_{\mu pq}, A_\mu{}^{\overline{p}}\}, \qquad\qquad \{B_{\mu\nu\overline{p}}, B_{\mu\nu}{}^{pq}\}, \tag{5.71}$$

modulo the additional local gauge shifts for the 2-forms. The q-form field content can be linked to the field content of 11-D supergravity in the K–K decomposition we obtained above. We can identify $A_\mu{}^p$ with the K-K vector lying in the 11-D metric, while $\{A_{\mu pq}, B_{\mu\nu\overline{p}}\}$ with the corresponding components sitting in the 11-dimensional 3-form 5.61. For what concerns, instead, $\{B_{\mu\nu}{}^{pq}, A_\mu{}^{\overline{p}}\}$, one can relate them to the components of the 11-D 6-form, namely, describing the degrees of freedom dual to $\{A_{\mu pq}, B_{\mu\nu\overline{p}}\}$. Let us give an explicit example of these identifications. The six vector fields $A_\mu{}^p$ transform under a generalised gauge transformation 5.26, as

$$\delta_\Lambda A_\mu{}^p = \partial_\mu \Lambda^p - A_\mu{}^q \partial_q \Lambda^p + \Lambda^q \partial_q A_\mu{}^p, \tag{5.72}$$

and they remain invariant under higher tensor gauge transformations 5.32, due to the solution of the section constraints that kills the derivative ∂_{pq}, $\partial_{\overline{p}}$. The algebra for this gauge transformation is

$$[\delta_{\Lambda_1}, \delta_{\Lambda_2}] = \delta_{\Lambda_{12}}, \quad \text{with} \quad \Lambda_{12}{}^p = \Lambda_2^q \partial_q \Lambda_1^p - \Lambda_1^q \partial_q \Lambda_2^p, \tag{5.73}$$

which is the algebra for the six-dimensional diffeomorphisms, embedded in the E-bracket. This guarantees that the theory is invariant under 6-dimensional internal diffeomorphisms with parameter Λ^p. The covariant derivative for the internal diffeomorphisms is defined, just as before, as

$$D_\mu = \partial_\mu - \mathcal{L}_{A_\mu}. \tag{5.74}$$

The covariant field strength, evaluated from the $E_{6(6)}$ field strength $\mathcal{F}_{\mu\nu}{}^P$ coincides with the field strength in 5.54, corroborating our identification with the K–K vector field that sits in the 11-D metric,

$$\mathcal{F}_{\mu\nu}{}^p = 2\partial_{[\mu}A_{\nu]}{}^p - A_\mu{}^p\partial_p A_{\nu]}{}^p - A_\mu{}^q\partial_q A_\nu{}^p + A_\nu{}^q\partial_q A_\mu{}^p = F_{\mu\nu}{}^p \tag{5.75}$$

For the other components of $\mathcal{F}_{\mu\nu}{}^P$, one gets

$$\begin{aligned} \mathcal{F}_{\mu\nu pq} &= 2D_{[\mu}A_{\nu]pq} + \partial_p \overline{B}_{\mu\nu\overline{q}} - \partial_q \overline{B}_{\mu\nu\overline{p}}, \\ \mathcal{F}_{\mu\nu}{}^{\overline{p}} &= 2D_{[\mu}A_{\nu]}{}^{\overline{p}} - 2(\partial_q A_{[\mu}{}^q)A_{\nu]}{}^{\overline{p}} - \frac{1}{2}\varepsilon^{pqrstu}A_{[\mu|rs}\partial_{q|}A_{\nu]tu} + 2\partial_q \overline{B}_{\mu\nu}{}^{qp}, \end{aligned} \tag{5.76}$$

with,

$$\begin{aligned} \overline{B}_{\mu\nu\overline{p}} &= \sqrt{5}B_{\mu\nu\overline{p}} + A_{[\mu}{}^q A_{\nu]qp}, \\ \overline{B}_{\mu\nu}{}^{pq} &= \sqrt{5}B_{\mu\nu}{}^{pq} + \frac{1}{2}(A_{[\mu}{}^p A_{\nu]}{}^{\overline{q}} - A_{[\mu}{}^q A_{\nu]}{}^{\overline{p}}). \end{aligned} \tag{5.77}$$

The scalar fields contained in $\mathcal{M}^{PQ}$ decompose according to the section constraint, namely, by choosing a parametrisation $\mathcal{M} = \mathcal{V}\mathcal{V}^T$, with

$$\mathcal{V}^T \equiv exp[\Phi t_{(0)}]\mathcal{V}_6 exp\left[c_{pqr}t_{(+1)}^{pqr}\right]exp\left[\varphi t_{(+2)}\right]. \tag{5.78}$$

Where, $t_{(0)}$ is the $E_{6(6)}$ generator associated with GL(1), $\mathcal{V}_6$ is a general SL(6) matrix and $t_{(+n)}$ represents the $E_6(6)$ generators of positive grading in 5.65. Defining the matrix $m_{pq} \equiv (\mathcal{V}_6\mathcal{V}_6^T)_{pq}$, built with the SL(6) vielbein, we can identify the combination

$$\phi_{pq} = e^{-\Phi/3}m_{pq}, \tag{5.79}$$

that corresponds to the internal part $\phi_{pq} = \phi_p^\alpha \phi_{q\alpha}$ of the 11-D metric. This combination can be proved to transform as a tensor of vanishing weight under 6-dimensional internal diffeomorphisms. Therefore, the set of scalar fields contained in the theory is given by

$$\{\phi_{pq}, c_{pqr}, \varphi\}. \tag{5.80}$$

In order to compare this field content to the 11-D supergravity one in the K–K decomposition we performed, we need to dualise the field φ into a 3-form and eliminate the fields $A_\mu{}^{\overline{p}}$ and $\overline{B}_{\mu\nu}{}^{pq}$. In doing so, one also introduces a kinetic term for the 2-form $\overline{B}_{\mu\nu\overline{p}}$ as expected for the comparison. The dualization procedure is quite long, and can be found in [83], and in the end one finds that the 2 theories are effectively equivalent. There is another solution to the section constraint that reduces $E_{6(6)}$ to SL(6)×SL(2), which, on the other hand, shows the equivalence of the EFT with the full 10-D type IIB string theory. So what we showed is that by means of an EFT, subject to a section constraint, we can always relate d-dimensional maximal supergravity to 11-D supergravity and type II (massless) string theory. This result is depicted in fig. 5.1 for the $E_{6(6)}$ case.

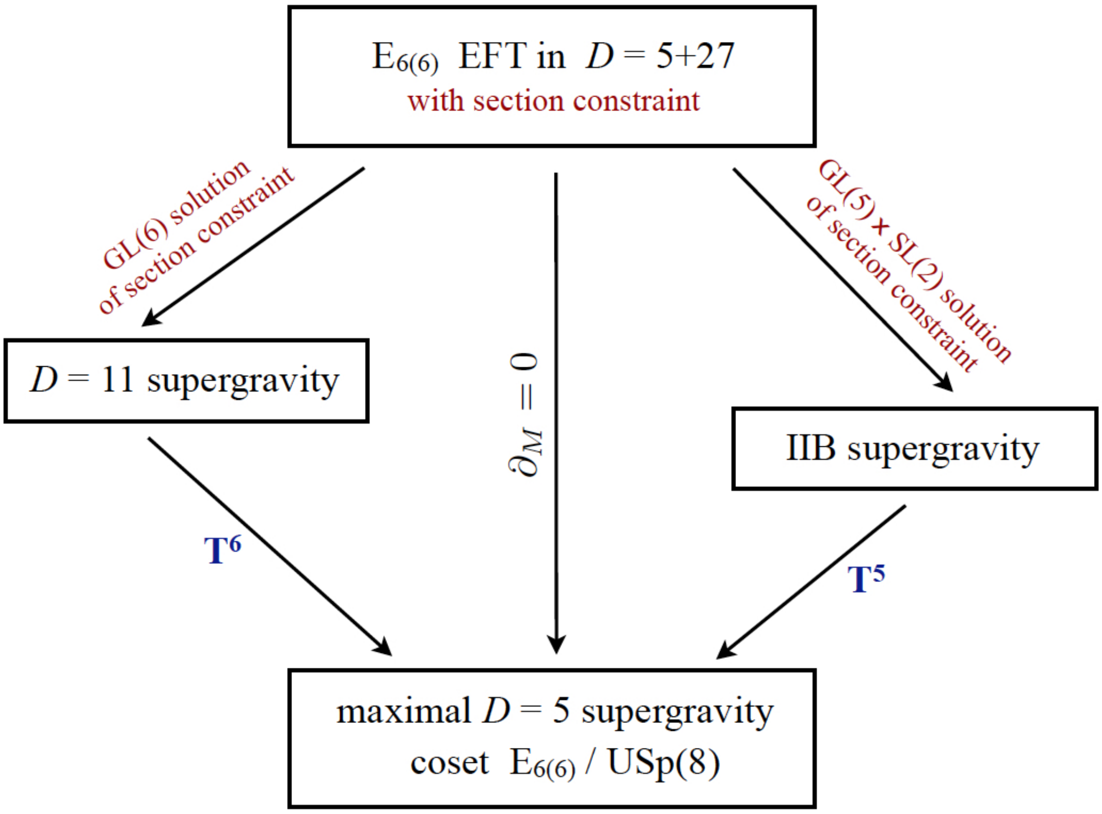


Fig. 5.1 Relation between the $E_{6(6)}$ EFT, 11-D/IIB supergravity and 5-D maximal supergravity. [1]

We are now in a good position to discuss the massive deformations, indeed, we have just shown that it is possible to obtain 11-D and type IIB supergravity, but we could not obtain massive IIA supergravity as a solution of the section constraint. We will now see how to obtain it by adding a deformation to the generalised Lie derivative.

[1] Image taken from [83]

5.2.5 Exceptional IIA supergravity

We will start with an excursus on massless type IIA supergravity [98, 99], which, on the other hand, could be obtained as a solution of the section constraint (it is a subcase of the 11-D supergravity solution). First of all, let us note that the generalised gauge variation for the vectors is usually written as

$$\delta_\Lambda A_\mu{}^P = \mathcal{D}_\mu \Lambda^P = \partial_\mu \Lambda^P - \mathbb{L}_{A_\mu} \Lambda^P, \tag{5.81}$$

can also be written

$$\delta_\Lambda A_\mu{}^P = \partial_\mu \Lambda^P + \mathbb{L}_\Lambda A_\mu{}^P. \tag{5.82}$$

Indeed, this follows from the fact that $\{\Lambda^P, A_\mu{}^Q\}_E$ is a trivial gauge parameter (by explicit computation), and the difference between the two transformations for the vectors is eaten up by the higher q-forms gauge transformations. To be more concrete, in this case we will use the example of 7 + 3 EFT. The q-form fields of the theory are $A_{\hat\mu}$, $A_{\hat\mu\hat\nu}$ and $A_{\hat\mu\hat\nu\hat\rho}$ with gauge transformations given by

$$\delta A_{\hat\mu} = \partial_{\hat\mu}\lambda, \qquad \delta A_{\hat\mu\hat\nu} = 2\partial_{[\hat\mu}\Xi_{\hat\nu]}, \qquad \delta A_{\hat\mu\hat\nu\hat\rho} = 3\partial_{[\hat\mu}\theta_{\hat\nu\hat\rho]} - 3A_{[\hat\mu\hat\nu}\partial_{\hat\rho]}\lambda. \tag{5.83}$$

Just as we did before, for 11-D supergravity, we decompose the fields following a K–K like decomposition in 7+3 dimensions, with coordinates x^μ and y^α, with $\alpha = 1,2,3$ and $\mu = 0,...,6$, but retaining the dependence on all the 10 dimensional coordinates, for instance, the vectors that arise from the q-form fields are

$$A_\mu^{KK} = A_\mu - B_\mu{}^\gamma A_\gamma, \qquad A_{\mu\alpha}^{KK} = A_{\mu\alpha} - B_\mu{}^\gamma A_{\gamma\alpha}, \qquad A_{\mu\alpha\beta}^{KK} = A_{\mu\alpha\beta} - B_\mu{}^\gamma A_{\gamma\alpha\beta}. \tag{5.84}$$

Where A_γ, $A_{\gamma\alpha}$ and $A_{\gamma\alpha\beta}$ are the respective scalars and $B_\mu{}^\gamma$ is the K–K vector field contained in the 10-dimensional vielbein, analogously to 5.61. By performing the redefinition

$$C_\mu = A_\mu^{KK}, \qquad C_{\mu\beta} = A_{\mu\beta}^{KK}, \qquad C_{\mu\beta\gamma} = A_{\mu\beta\gamma}^{KK} + A_\mu^{KK} A_{\beta\gamma}, \tag{5.85}$$

one obtains the following gauge and internal diffeomorphisms (ξ^α) transformations:

$$\begin{aligned}
\delta B_\mu{}^\alpha =& (\partial_\mu - B_\mu{}^\gamma \partial_\gamma)\xi^\alpha + \xi^\gamma \partial_\gamma B_\mu{}^\alpha, \\
\delta C_\mu =& \xi^\gamma \partial_\gamma C_\mu + (\partial_\mu - B_\mu{}^\gamma \partial_\gamma)\lambda, \\
\delta C_{\mu\alpha} =& \xi^\gamma \partial_\gamma C_{\mu\alpha} + C_{\mu\gamma}\partial_\alpha \xi^\gamma + (\partial_\mu - B_\mu{}^\gamma \partial_\gamma)\Xi_\alpha + B_\mu{}^\gamma \partial_\alpha \Xi_\gamma, \\
\delta C_{\mu\alpha\beta} =& \xi^\gamma \partial_\gamma C_{\mu\alpha\beta} + 2C_{\mu\gamma[\beta}\partial_{\alpha]}\xi^\gamma + (\partial_\mu - B_\mu{}^\gamma \partial_\gamma)\theta_{\alpha\beta} + 2B_\mu{}^\gamma \partial_{[\alpha}\theta_{|\gamma|\beta]} \\
& + 2C_\mu \partial_{[\alpha}\Xi_{\beta]} - 2C_{\mu[\alpha}\partial_{\beta]}\lambda.
\end{aligned} \tag{5.86}$$

We can compare this with the $E_{4(4)} \equiv SL(5)$ EFT, where generalised vectors Γ^P live in the $\mathbf{10}'$ representation, respecting $\Gamma^{pq} = -\Gamma^{qp}$, with $p,q = 1,...,5$ living in the fundamental representation of SL(5). The $Y^{PQ}{}_{RS}$ tensor is then given by

$$Y^{mn\,pq}{}_{rstu} = \varepsilon^{mnpqv}\varepsilon_{rstuv}, \tag{5.87}$$

in such a way that the section constraint can be written as

$$\varepsilon^{pqrst}\partial_{pq} \otimes \partial_{rs} = 0. \tag{5.88}$$

Two solutions for this constraint arise, up to SL(5) transformations

$$\begin{aligned}
&\text{M-theory:} \qquad \partial_{\alpha 4} \neq 0,\ \partial_{45} \neq 0 \quad \partial_{\alpha 5} = \partial_{\alpha\beta} = 0, \\
&\text{type IIB:} \qquad \partial_{\alpha\beta} \neq 0, \quad \partial_{\alpha 4} = \partial_{\alpha 5} = \partial_{45} = 0.
\end{aligned} \tag{5.89}$$

Massless type IIA is achieved by imposing in the first case $\partial_{45} = 0$, thus going from 11 to 10 physical coordinates. The EFT contains $\mathbf{10}'$ vectors $A_\mu{}^P \equiv A_\mu{}^{pq}$, by means of the solution of the section constraint for type IIA, namely $\partial_{\alpha 4} \neq 0$, we can compare the vector field content of the EFT with the one of 10-D IIA supergravity:

$$\begin{aligned}
A_\mu{}^{pq} = \big(A_\mu{}^{\alpha 5},\ A_\mu{}^{\alpha 4},\ A_\mu{}^{\alpha\beta},\ A_\mu{}^{45}\big) &= \big(\tfrac{1}{2}\varepsilon^{\alpha\beta\gamma}C_{\mu\beta\gamma},\ B_\mu{}^\alpha,\ \varepsilon^{\alpha\beta\gamma}C_{\mu\gamma},\ C_\mu\big), \\
\Lambda^{pq} = \big(\Lambda^{\alpha 5},\ \Lambda^{\alpha 4},\ \Lambda^{\alpha\beta},\ \Lambda^{45}\big) \qquad &= \big(\tfrac{1}{2}\varepsilon^{\alpha\beta\gamma}\theta_{\beta\gamma},\ \xi^\alpha,\ \varepsilon^{\alpha\beta\gamma}\Xi_\gamma,\ \lambda\big),
\end{aligned} \tag{5.90}$$

Thus, we can see that the EFT generalised Lie derivative comprises internal diffeomorphisms as well as q-form gauge transformations, as we also showed before for the $E_{6(6)}$ case. Taking now into consideration the massive case for type IIA [100–102], so adding the Romans mass parameter m_R (the field strength 'dual' to the 9-form), the gauge transformations 5.86 become

$$\begin{aligned}\delta B_\mu{}^\alpha =&(\partial_\mu - B_\mu{}^\gamma\partial_\gamma)\xi^\alpha + \xi^\gamma\partial_\gamma B_\mu{}^\alpha,\\ \delta C_\mu =&\xi^\gamma\partial_\gamma C_\mu + (\partial_\mu - B_\mu{}^\gamma\partial_\gamma)\lambda - m_R B_\mu{}^\gamma\Xi_\gamma,\\ \delta C_{\mu\alpha} =&\xi^\gamma\partial_\gamma C_{\mu\alpha} + C_{\mu\gamma}\partial_\alpha\xi^\gamma + (\partial_\mu - B_\mu{}^\gamma\partial_\gamma)\Xi_\alpha + B_\mu{}^\gamma\partial_\alpha\Xi_\gamma,\\ \delta C_{\mu\alpha\beta} =&\xi^\gamma\partial_\gamma C_{\mu\alpha\beta} + 2C_{\mu\gamma[\beta}\partial_{\alpha]}\xi^\gamma + (\partial_\mu - B_\mu{}^\gamma\partial_\gamma)\theta_{\alpha\beta} + 2B_\mu{}^\gamma\partial_{[\alpha}\theta_{|\gamma|\beta]}\\ &+2C_\mu\partial_{[\alpha}\Xi_{\beta]} - 2C_{\mu[\alpha}\partial_{\beta]}\lambda - 2m_R C_{\mu[\alpha}\Xi_{\beta]}.\end{aligned} \tag{5.91}$$

The first observation that should come up to the reader's mind while looking at these new gauge transformations is that the new factors do not contain any partial derivatives. This is a problem, because we cannot obtain these transformations from the generalised Lie derivative $\delta\Gamma^P = \mathbb{L}\Gamma^P$, given that the latter contain only terms with partial derivatives. On the other hand, it is well known that massive type IIA supergravity is a geometrically well-defined theory, so there should be an EFT describing it. The problem that we just discovered is also pointing to the solution, indeed a modification to the generalised Lie derivative with a non-derivative term will solve it [103], and defining a new $\tilde{\mathbb{L}}$ such that $\lim_{m_R\to 0}\tilde{\mathbb{L}} = \mathbb{L}$. The new generalised Lie derivative is given by

$$\tilde{\mathbb{L}}_\Lambda\Gamma^P = \mathbb{L}_\Lambda\Gamma^P - F_{QR}{}^P\Lambda^Q\Gamma^R, \tag{5.92}$$

with a constant object $F_{QR}{}^P$, which, as we will see, must satisfy the same constraints of the embedding tensor, this formula is completely general for every $\mathrm{E}_{n(n)}$ EFT. First of all, we note that, taking into account $\mathrm{E}_{n(n)}$ global transformations, consistency requirements imply that $F_{PQ}{}^R = \Theta_P{}^\alpha(t_\alpha)_Q{}^R$, with Θ a constant object selecting the linear combinations of the generators of $\mathrm{E}_{n(n)}$, in addition the linear constraint arises from the consistency conditions when taking into considerations the tensor hierarchy.
A number of constraints, instead, arise by demanding the closure of the algebra of the new generalised Lie derivatives:

$$[\tilde{\mathbb{L}}_{\Gamma_1},\tilde{\mathbb{L}}_{\Gamma_2}] = \tilde{\mathbb{L}}_{[\Gamma_1,\Gamma_2]_X}, \tag{5.93}$$

where we introduced the X-bracket (it is called X because in [103] the deformation took the name $X_{PQ}{}^R$)

$$[\Gamma_1,\Gamma_2]_X^R = [\Gamma_1,\Gamma_2]_E^R - F_{[PQ]}{}^R\Gamma_1^P\Gamma_2^Q. \tag{5.94}$$

Being Γ_1 and Γ_2 arbitrary parameters, the closure of the algebra gives rise to a system of equations, one for each set of terms with different number of derivatives, for instance, terms containing 2 derivatives must vanish by themselves and cannot be used to reduce equations with a different number of derivatives. Indeed, the terms containing 2 derivatives do not contain $F_{PQ}{}^{R}$, so they reduce to the section constraint as before. Computing it explicitly, we arrive at the following expression

$$\begin{aligned}[][\tilde{\mathbb{L}}_{\Gamma_1},\tilde{\mathbb{L}}_{\Gamma_2}]\Gamma_3^P-\tilde{\mathbb{L}}_{[\Gamma_1,\Gamma_2]_X}\Gamma_3^P= {} & A^P_{QRS}\Gamma_1^Q\Gamma_2^R\Gamma_3^S+F_{[QR]}{}^{S}\Gamma_1^Q\Gamma_2^R\partial_S\Gamma_3^P\\ & +B^{PQ}_{RST}(\Gamma_1^R\partial_Q\Gamma_2^S\Gamma_3^T-\partial_Q\Gamma_1^S\Gamma_2^R\Gamma_3^T),\end{aligned} \tag{5.95}$$

with

$$\begin{aligned} A^P_{QRS} ={} & 2F_{[Q|T]}{}^{P}F_{R]S}{}^{T}-F_{TS}{}^{P}F_{[QR]}{}^{T},\\ B^{PQ}_{RST} ={} & F_{(RS)}{}^{P}\delta^Q_T-F_{RT}{}^{Q}\delta^P_S+Y^{PQ}{}_{SU}F_{RT}{}^{U}\\ & -Y^{UQ}{}_{ST}F_{RU}{}^{P}+Y^{PQ}{}_{UT}F_{[RS]}{}^{U}-\frac{1}{2}Y^{UQ}{}_{SR}F_{UT}{}^{P}.\end{aligned} \tag{5.96}$$

Therefore, the new constraints read

$$A^P_{QRS}=0,\qquad F_{[PQ]}{}^{R}\partial_R=0,\qquad B^{PQ}_{RST}\partial_Q=0. \tag{5.97}$$

One immediately notices that the A-constraint is the antisymmetric part of the quadratic constraint:

$$F_{PQ}{}^{R}F_{SR}{}^{T}-F_{SQ}{}^{R}F_{PR}{}^{T}-F_{PS}{}^{R}F_{RQ}{}^{T}=0. \tag{5.98}$$

However, there is another condition we need to impose, indeed, just as it happened for the E-bracket, also the X-bracket does not respect Jacobi identity.

$$[[\Gamma_1,\Gamma_2]_X,\Gamma_3]_X+cycl.=\frac{1}{3}\{[\Gamma_1,\Gamma_2]_X,\Gamma_3\}_X+cycl \tag{5.99}$$

with

$$\{\Gamma_1,\Gamma_2\}^P_X\equiv\frac{1}{2}(\tilde{\mathbb{L}}_{\Gamma_1}\Gamma_2+\tilde{\mathbb{L}}_{\Gamma_2}\Gamma_1)=\{\Gamma_1,\Gamma_2\}^P_E-F_{(QR)}{}^{P}\Gamma_1^Q\Gamma_2^R. \tag{5.100}$$

Therefore, we need to impose that the Jacobiator is a trivial gauge parameter, $\tilde{\mathbb{L}}_{\{\Gamma_1,\Gamma_2\}}=0$.

$$\tilde{\mathbb{L}}_{\{\Gamma_1,\Gamma_2\}_X}\Gamma_3^P=C^{PQ}_{RST}(\Gamma_1^T\partial_Q\Gamma_2^S\Gamma_3^R+\partial_Q\Gamma_1^S\Gamma_2^T\Gamma_3^R)-F_{(ST)}{}^{Q}\Gamma_1^S\Gamma_2^T\partial_Q\Gamma_3^P+F_{(ST)}{}^{Q}F_{QR}{}^{P}\Gamma_1^S\Gamma_2^T\Gamma_3^R, \tag{5.101}$$

where we used 5.100 and

$$C^{PQ}_{RST}=F_{(ST)}{}^{P}\delta^Q_R-Y^{PQ}{}_{UR}F_{(ST)}{}^{U}-\frac{1}{2}Y^{UQ}{}_{ST}F_{UR}{}^{P}. \tag{5.102}$$

Therefore, in order for the Jacobiator to be a trivial gauge parameter, we need the following to hold

$$F_{(PQ)}{}^{R}F_{RS}{}^{T}=0,\qquad F_{(PQ)}{}^{R}\partial_R=0,\qquad C^{PQ}_{RST}\partial_Q=0. \tag{5.103}$$

The first equation is the symmetrization of the quadratic constraint, so together with the A-constraint they produce exactly the quadratic constraint. The second equations of 5.103 and 5.97 combine to form the X-constraint $F_{PQ}{}^{R}\partial_R = 0$. It is also possible to prove that in this case the B and C-constraints are competely equivalent to the X-constraint. In conclusion, the new generalised lie derivative has two sets of constraints that can be summarised as

$$\begin{aligned} Y^{PQ}{}_{RS}\partial_P \otimes \partial_Q = 0 &\qquad \text{Section Constraint,} \\ F_{PQ}{}^{R}\partial_R = 0 &\qquad \text{X-constraint,} \end{aligned} \tag{5.104}$$

Furthermore, $F_{PQ}{}^{R}$ must satisfy the quadratic constraint of the embedding tensor. The X-constraint reduces further the variable dependance, only to those coordinates that remain invariant under the $\mathrm{E}_{n(n)}$ transformations generated by $F_{PQ}{}^{R}$. Let us now apply this to an example that is of interest to us.

SL(5) X-exceptional

We have already given before in 5.87, 5.88 and 5.89 the $Y^{PQ}{}_{RS}$ tensor, the related section constraints and its solutions, respectively, we will report here the solutions for the comfort of the reader:

$$\begin{aligned} \text{M-theory:} &\qquad \partial_{\alpha 4} \neq 0,\ \ \partial_{45} \neq 0, \quad\ \ \partial_{\alpha 5} = \partial_{\alpha\beta} = 0, \\ \text{type IIB:} &\qquad \partial_{\alpha\beta} \neq 0, \quad\ \partial_{\alpha 4} = \partial_{\alpha 5} = \partial_{45} = 0. \end{aligned} \tag{5.105}$$

The Roman mass, which was the reason behind our modification of the generalised Lie derivative, is obtained by choosing the following X-deformation

$$F_{pq\,rs}{}^{tu} = 2F_{pq[r}{}^{[t}\delta^{u]}_{s]} \tag{5.106}$$

with the only non-vanishing entries provided by

$$F_{\alpha\beta\gamma}{}^{5} = -2m_R\varepsilon_{\alpha\beta\gamma}, \tag{5.107}$$

then, the X-constraint reads

$$m_R\partial_{\alpha 5} = m_R\partial_{45} = 0. \tag{5.108}$$

Inserting this solution into the section constraint 5.88 reduces to

$$\varepsilon^{\alpha\beta\gamma}\partial_{\alpha 4} \otimes \partial_{\beta\gamma} = 0. \tag{5.109}$$

The two most obvious solutions to the section constraints are now given by

$$1)\partial_{\alpha 4} \neq 0, \partial_{\alpha\beta} = 0 \quad \text{Type IIA,} \qquad\qquad 4)\partial_{\alpha\beta} \neq 0, \partial_{\alpha 4} = 0 \quad \text{Type IIB.} \tag{5.110}$$

However, there are other solutions with $\alpha \neq \beta \neq \gamma$:

$$\begin{aligned} &2)\partial_{\alpha 4},\ \partial_{\beta 4},\ \partial_{\alpha\beta} \neq 0, \quad \partial_{\gamma 4} = \partial_{\beta\gamma} = \partial_{\gamma\alpha} = 0 \quad \text{Type IIB,} \\ &3)\partial_{\alpha 4},\ \partial_{\alpha\beta},\ \partial_{\gamma\alpha} \neq 0, \quad \partial_{\beta 4} = \partial_{\gamma 4} = \partial_{\beta\gamma} = 0 \quad \text{Typer IIA.} \end{aligned} \tag{5.111}$$

In each of the different solutions, the X deformation has a different higher energy interpretation: in solution 1) it corresponds to the Romans mass, in solution 2) it is identified with the flux of the RR type IIB 1-form $F_{(1)}$ along the coordinate $y^{\alpha\beta}$, in solution 3) it corresponds to the background flux of the RR type IIA 2-form $F_{(2)}$ along the two coordinates $\{y^{\alpha\beta}, y^{\gamma\alpha}\}$ and in solution 4) it is identified with the type IIB background flux for the RR 3-form $F_{(3)}$. It is known that these solutions are related by T-dualities:

$$\text{1) massive IIA} \xrightarrow{T_\gamma} \text{2) IIB with } F_{(1)} \xrightarrow{T_\beta} \text{3) IIA with } F_{(2)} \xrightarrow{T_\alpha} \text{4) IIB with } F_{(3)}, \tag{5.112}$$

with T_α changing $y^{\alpha 4} \leftrightarrow y^{\beta\gamma}$, such that $\alpha \neq \beta \neq \gamma$, as T-duality exchanges coordinates in the extended space (this also happens in Double Field Theory). Obviously, modifying the generalised Lie derivative and consequently the diffeomorphisms algebra has some consequences on the Lagrangian. One needs to modify the field strengths for vectors of the EFT by adding terms with 2-form potentials in order to make contact with the related gauged supergravity, in addition, there is also the necessity to modify the scalar potential and introduce terms proportional to ${X_{PQ}}^R$ (both linear and quadratic) in order for it to be again invariant under the new gauge transformations. An example of this can be found at [103]. Now we are in a good position to introduce the Uplifting constraints.

5.2.6 The Uplifting Constraints

The section constraint of every EFT

$$Y^{PQ}{}_{RS}\partial_P \otimes \partial_Q = 0, \tag{5.113}$$

can be solved in terms of a constant rectangular matrix of maximal rank that selects the physical coordinates from the sets of d-coordinates, with d equal to the dimension of the vector representation for $E_{n(n)}$, namely

$$\partial_P \equiv \mathcal{E}_P{}^p \partial_p, \tag{5.114}$$

with small letters labelling the physical coordinates after the section constraint has been solved. Just as we did before, for the Romans mass, we could include more general fluxes inside the deformations of the generalised Lie derivative, therefore by defining the generalised flux $F_{PQ}{}^R$, which also lies in the same representation of the embedding tensor (in each EFT), we can now write the deformed generalised Lie derivative as

$$\tilde{\mathbb{L}}_\Lambda \Gamma^P = \Lambda^q \partial_q \Gamma^P - \Gamma^q \partial_q \Lambda^P + Y^{Pq}{}_{RS} \partial_q \Lambda^R \Gamma^S - \Lambda^Q \Gamma^R F_{QR}{}^P. \tag{5.115}$$

Where, contraction with $\mathcal{E}_Q{}^q$ is intended whenever small letters appear. Unlike before, now we will not assume that $F_{PQ}{}^R$ is constant and that $F_{PQ}{}^Q = 0$ (which will exclude the Trombone symmetry, a scaling symmetry of the equations of motion of Supergravity and General Relativity [104]), in this case, the constraints that arises from the closure of the algebra and of the Jacobi identity are no more all equivalent to the X-constraint. Indeed, the new constraints are

$$F_{PQ}{}^R \mathcal{E}_R{}^r = 0, \tag{5.116}$$

$$C[F]^{PQ}_{RST} \mathcal{E}_Q{}^q \equiv \left(F_{(ST)}{}^P \delta^Q_R - Y^{PQ}{}_{UR} F_{(ST)}{}^U - \frac{1}{2} Y^{UQ}{}_{ST} F_{UR}{}^P \right) \mathcal{E}_Q{}^q = 0, \tag{5.117}$$

$$\begin{aligned} &F_{PQ}{}^T F_{RT}{}^S - F_{RQ}{}^T F_{PT}{}^S + F_{PR}{}^T F_{TQ}{}^S \\ &\quad - \mathcal{E}_P{}^p \partial_p F_{RQ}{}^S - 2\mathcal{E}_{[R}{}^r \partial_r F_{|P|Q]}{}^S - Y^{ST}{}_{UQ} \mathcal{E}_T{}^t \partial_t F_{PR}{}^U = 0. \end{aligned} \tag{5.118}$$

The first constraint guarantees that $F_{PQ}{}^R$ does not affect the algebra of internal diffeomorphisms generated by Λ^p, the last one instead is the extension of the quadratic constraint to a non-constant generalised flux (when forcing the derivatives to vanish we go back to the usual quadratic constraint). Consequently, the generalised flux induces deformations of the internal gauge symmetry due to mass parameters, background q-form fluxes, twists of the field content by coordinate-dependent $E_n(n) \times \mathbb{R}^+_0$ transformations (R^+_0 being the Trombone scaling) analogous to the Scherk and Schwarz procedure and gaugings presented in Appendix A.

Some components of the generalised flux $F_{PQ}{}^R$, can be reabsorbed into a twist of the covariant tensors (contracted with the generalised flux) by some matrix $C(y)_P{}^Q$ with the property

$$C_P{}^Q \mathcal{E}_Q{}^q = \mathcal{E}_P{}^q. \tag{5.119}$$

Now, let us suppose that we have a gauged supergravity, with an embedding tensor $X_{PQ}{}^R$, satisfying the linear and quadratic constraint. To allow the theory to be upliftable, we need to find a frame $\hat{E}_A{}^P$, analogous to the matrix U_α^β in A.21 in Appendix A introduced to obtain the Scherk and Schwarz dimensional reduction:

$$W^P(x,y) = W^A(x)\hat{E}_A{}^P(y). \tag{5.120}$$

Torsion $T_{AB}{}^C$ induced by a frame $E_A{}^P$ in $\mathrm{E}_{n(n)} \times \mathbb{R}^+$ is defined by

$$\mathbb{L}_{E_A} E_B{}^P \equiv -T_{AB}{}^C E_C{}^P, \tag{5.121}$$

and is in general y-dependent. We can explicitly state this in terms of the Weitzenböck connection coefficients $W_{AB}{}^C$:

$$W_{AB}{}^C \equiv E_A{}^p E_B{}^Q \partial_p E_Q{}^C \tag{5.122}$$

$$T_{AB}{}^C \equiv 2W_{[AB]}{}^C + Y^{CD}{}_{EB} W_{DA}{}^E. \tag{5.123}$$

A generalised flux $\hat{F}^0_{PQ}{}^R$ is also needed together with the frame $\hat{E}_A{}^P$ for upliftability, and they have to satisfy the "generalised" Scherk-Schwarz condition:

$$\mathbb{L}_{\hat{E}_A} \hat{E}_B{}^P - \hat{E}_A{}^Q \hat{E}_B{}^R \hat{F}^0_{QR}{}^P = -X_{AB}{}^C \hat{E}_C{}^P. \tag{5.124}$$

Some part of $\hat{E}_A{}^P$ can be absorbed in the generalised flux, which must still satisfy all the consistency constraints, through a twisting matrix $C(y^p)_P{}^Q$ satisfying $C_P{}^Q \mathcal{E}_Q{}^p = \mathcal{E}_P{}^p$. Consequently, we can study the equivalent condition:

$$\tilde{\mathbb{L}}_{E_A} E_B{}^P = -X_{AB}{}^C E_C{}^P, \qquad \text{with} \qquad \hat{E}_A{}^P \equiv E_A{}^Q C_Q{}^P. \tag{5.125}$$

Now, we will assume that, after a solution to the section constraint $\mathcal{E}_P{}^p$ has been found, a frame $\hat{E}_A{}^Q$ that satisfies 5.124 exists. Then, the vectors $K_A{}^p \equiv \hat{E}_A{}^P \mathcal{E}_P{}^p$ have the standard Lie bracket ($\tilde{\mathbb{L}}_{E_A} E_B{}^P = -X_{AB}{}^C E_C{}^P$):

$$[K_A, K_B] = -X_{AB}{}^C K_C. \tag{5.126}$$

Defining a Lie algebra for a gauge group G. The non-vanishing vector components of $\hat{E}_A{}^P$ are the vectors K_a (this is true in the absence of central charge extension of the algebra, the generalisation including them is straightforward and can be found in [105]), then, given an algebra with generator T_a:

$$[T_A, T_B] = -X_{AB}{}^C T_C, \tag{5.127}$$

its vector representation is by definition given by $R_V(T_A) = \Theta_A{}^a t_a$, so that $\hat{E}_A{}^P \mathcal{E}_P{}^p = \Theta_A{}^a K_a{}^p$, being the K_A respecting the same algebra 5.127. Because $\hat{E}_A{}^P$ is in $\mathrm{E}_{n(n)}$, and therefore its columns are linearly independent, it is everywhere non-vanishing, and there are always d linearly independent vectors among the K_a, with d being the dimension of the internal manifold. This evidence shows the presence of a homogeneous space H\G, with K_a generating the transitive action of G on the manifold. The coset representative $L(y)$ of H\G has the following transformation property

$$L(y)g = h(y')L(y'), \qquad \text{with} \qquad g \in G,\ h(y) \in H. \tag{5.128}$$

Infinitesimally, it is possible to write

$$g = 1 + \varepsilon^a t_a, \qquad t_a \in G \tag{5.129}$$

$$h = 1 - \varepsilon^a W_a{}^i t_i, \qquad t_i \in H \tag{5.130}$$

$$y'^m = y^m + \varepsilon^a K_a{}^m(y), \tag{5.131}$$

where $W_a{}^i$ is the H-compensator, and $K_a{}^m$ are the components of our Killing vector fields:

$$K_a(y) \equiv K_a{}^m \partial_m. \tag{5.132}$$

Expanding 5.128 in terms of the infinitesimal parameter, we have

$$K_a{}^p \partial_p L = L t_a + W_a{}^i t_i L \tag{5.133}$$

then, multiplying this last equation by L^{-1} on the right, one achieves

$$K_a{}^p (\partial_p L) L^{-1} = L t_a L^{-1} + W_a{}^i t_i \tag{5.134}$$

The Maurer-Cartan form is defined by

$$\Omega_p \equiv (\partial_p L) L^{-1} = V_p{}^{\underline{m}} t_{\underline{m}} + Q_p{}^i t_i, \tag{5.135}$$

using it and projecting 5.134 onto the generators of the Coset space, labelled by underlined indices $\underline{m}, \underline{p}$, etc.

$$K_a{}^p V_p{}^{\underline{m}} = (L t_a L^{-1})|^{\underline{m}} = L_a^{-1b} \delta_b^{\underline{m}}, \tag{5.136}$$

where, in the last line, L^{-1} is in the adjoint representation and we used the definition of adjoint representation: $Ad_g(t_a) = gt_a g^{-1}$. Multiplying everything on the left by $\Theta_A{}^a$ and using its gauge invariance, we arrive at

$$\Theta_A{}^a K_a{}^p V_p{}^{\underline{m}} = L_A^{-1B} \Theta_B{}^{\underline{m}} \tag{5.137}$$

which can also be written as:

$$\hat{E}_A{}^P \mathcal{E}_P{}^p V_p{}^{\underline{p}} = L_A^{-1B} \Theta_B{}^{\underline{p}}. \tag{5.138}$$

Note that, because $\hat{E}_A{}^P$ is an $\mathrm{E}_{n(n)}$ matrix, it 'passes through' $Y^{PQ}{}_{RS}$, thus the left-hand side satisfies the section constraint, consequently also the right-hand side must satisfy it, again, L is also in $\mathrm{E}_{n(n)}$ leaving $Y^{PQ}{}_{RS}$ invariant. This implies that $\Theta_A{}^{\underline{p}}$ can only differ from $\mathcal{E}_P{}^p$ by an $\mathrm{E}_{n(n)} \times \mathbb{R}^+$ transformation, which can be reabsorbed in $\hat{E}_A{}^P$, $\mathcal{E}_P{}^p = \delta_P^A \delta_{\underline{p}}^p \Theta_A{}^{\underline{p}}$, and that it has to satisfy the section constraint:

$$Y^{PQ}{}_{RS} \Theta_P{}^{\underline{p}} \Theta_Q{}^{\underline{q}} = 0. \tag{5.139}$$

Note that this equation is only valid when the indices of the embedding tensor running on the adjoint representation are projected along the directions of the coset generators $t_{\underline{m}}$, so one first has to pick the group H, project the embedding tensor along it and then check that the latter projection fulfils the section constraint. The latter equation tells us whether given a frame and a group H we can uplift it to higher-dimensional theories. Further discussions about it can be found in [105]. From now on we will treat as equivalent $\mathcal{E}_A{}^{\underline{p}}$ and $\Theta_A{}^{\underline{p}}$.

Let us now note that the matrix $V_p{}^{\underline{p}}$ and its inverse are elements of GL (d) and therefore have a natural embedding in $\mathrm{E}_{n(n)} \times \mathbb{R}^+$:

$$V_P{}^A \in GL(d) \subset E_{n(n)} \times \mathbb{R}^+. \tag{5.140}$$

It is also possible to observe that any two frames $\hat{E}_A{}^P$, equal to the vector components K_a can differ only by terms that can be absorbed by the generalised flux $F_{PQ}{}^R$ through the matrix $C_P{}^Q$, therefore we can solve 5.125 with $E_A{}^P$ instead of 5.124. The generalised flux that solves 5.125 is

$$F_{PQ}{}^R = E_P{}^A E_Q{}^B (X_{AB}{}^C - T_{AB}{}^C) E_C{}^R, \tag{5.141}$$

however, there is also the need to check that this flux satisfies the consistency conditions 5.118, which generate another linear constraint on the embedding tensor:

$$C[X]^{PQ}_{RST}\Theta_Q{}^{\underline{p}}+\frac{1}{4}\left(Y^{PU}{}_{ST}\delta^V_R-Y^{PV}{}_{MR}Y^{MU}{}_{ST}\right)\left(X_{VU}{}^Q+2\Theta_{(V}{}^{\underline{q}}t_{\underline{q}U)}{}^Q\right)\Theta_Q{}^{\underline{p}}=0 \qquad (5.142)$$

with $C[X]^{PQ}_{RST}$ defined in 5.117. Note that this constraint is also valid only after a projection on the generators of the group H. We are going now to ask ourselves a different question, namely, given a solution of the section constraint $\mathcal{E}_A{}^{\underline{p}}$ (it can correspond to IIB, IIA or 11-D supergravity) we demand whether an embedding tensor $\Theta_A{}^a$ exists such that the generators $X_A \equiv \Theta_A{}^a t_a$ can be decomposed into a set of generators for a coset group H\G and a set of generators for H [106, 107]:

$$X_A=\mathcal{E}_A{}^{\underline{p}}t_{\underline{p}}+\Theta_A{}^i t_i. \qquad (5.143)$$

Where we have used the fact that a necessary condition for uplifting is that $\Theta_A{}^{\underline{m}}$ satisfies the section constraint and therefore we can always write $\mathcal{E}_A{}^{\underline{m}}$ instead of Θ. The embedding tensor must satisfy the quadratic constraint and, therefore, must obey

$$[\Theta_A{}^a t_a,\Theta_B{}^b t_b]=-X_{AB}{}^C\Theta_C{}^c t_c. \qquad (5.144)$$

Now, we define a projector $\overline{\Pi}_A{}^B$ on the space orthogonal to $\mathcal{E}_A{}^{\underline{p}}$ by

$$\overline{\Pi}_A{}^B\mathcal{E}_B{}^{\underline{m}}=0 \qquad \overline{\Pi}_A{}^B\overline{\Pi}_B{}^C=\overline{\Pi}_A{}^C, \qquad (5.145)$$

There may be obviously many projectors solving the previous conditions, for instance **0** is one of them, but we are looking for the maximal projector that comprehends all of the other. If a group H exists, then

$$[\Theta_A{}^i t_i,\Theta_B{}^j t_j]=-\Theta_A{}^i\Theta_B{}^j f_{ij}{}^k t_k, \qquad (5.146)$$

is a necessary condition for an uplifting to exist. Therefore, projecting 5.144 we get

$$[\Theta_A{}^i t_i,\Theta_B{}^j t_j]=\overline{\Pi}_{A'}{}^A\overline{\Pi}_{B'}{}^B[\Theta_A{}^a t_a,\Theta_B{}^b t_b]=-\overline{\Pi}_{A'}{}^A\overline{\Pi}_{B'}{}^B X_{AB}{}^C(\mathcal{E}_C{}^{\underline{p}}t_{\underline{p}}+\Theta_C{}^i t_i). \qquad (5.147)$$

Defining $f_{ij}{}^k$ by the relation

$$\Theta_A{}^i\Theta_B{}^j f_{ij}{}^k=\overline{\Pi}_A{}^{A'}\overline{\Pi}_B{}^{B'}X_{A'B'}{}^C\Theta_C{}^k,$$

we see that a necessary condition for H to be a group, and to an uplift to exist is [107][2].

$$\overline{\Pi}_{A'}{}^{A}\overline{\Pi}_{B'}{}^{B}X_{AB}{}^{C}\mathcal{E}_{C}{}^{\underline{p}}t_{\underline{p}} = 0, \tag{5.148}$$

which is a linear relation for the embedding tensor. There is actually another linear constraint that the embedding tensor has to solve in order for an uplifting to exist:

$$C[X]^{PQ}_{RST}\mathcal{E}_{Q}{}^{\underline{p}} + \frac{1}{4}\left(Y^{PU}{}_{ST}\delta^{V}_{R} - Y^{PV}{}_{MR}Y^{MU}{}_{ST}\right)\left(X_{VU}{}^{Q} + 2\mathcal{E}_{(V}{}^{\underline{q}}t_{\underline{q}U)}{}^{Q}\right)\mathcal{E}_{Q}{}^{\underline{p}} = 0. \tag{5.149}$$

Solving the two previous constraints, for a given solution of the section constraint $\mathcal{E}_{A}{}^{\underline{m}}$ guarantees that the embedding tensor contains only parameters that give rise to upliftable gaugings. For example, in the 11-D case for SL(5) EFT, we know that the section constraint $\varepsilon^{PQRST}\partial_{PQ}\otimes\partial_{RS} = 0$ is solved by

$$\mathcal{E}_{PQ} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 & 0 \end{pmatrix}, \tag{5.150}$$

note that we suppressed the index that runs over the physical coordinates in $\mathcal{E}_{PQ}{}^{a}$ because it is a spectator index in 5.148 and 5.149. It may seem that we need $\mathcal{E}_{UV}{}^{\overline{q}}t_{\overline{q}}$ in 5.149, but that term is given by

$$\mathcal{E}_{UV}{}^{\overline{q}}t_{\overline{q}} = \overline{\Pi}_{//UV}{}^{U'V'}X_{U'V'}, \tag{5.151}$$

where $\overline{\Pi}_{//UV}{}^{U'V'}$ is the projector on the space generated by $\varepsilon_{UV}{}^{\underline{q}}$ that can always be built once $\mathcal{E}$ is known. Using the solution 5.150 of the section constraint in 5.148 and 5.149, with a generic, unconstrained embedding tensor, leads to

[2] Similar results, in a more mathematical formulation are present in [108, 109]

$$
\begin{aligned}
&Y_{MN} = \begin{pmatrix} Y_{1,1} & 0 & 0 & 0 & Y_{1,5} \\ 0 & Y_{2,2} & 0 & 0 & Y_{2,5} \\ 0 & 0 & Y_{3,3} & 0 & Y_{3,5} \\ 0 & 0 & 0 & Y_{4,4} & Y_{4,5} \\ Y_{1,5} & Y_{2,5} & Y_{3,5} & Y_{4,5} & Y_{5,5} \end{pmatrix}, \\
&Z^{MN,5}\varepsilon_{MNQRS} = 0, \\
&Z^{MN,P}\varepsilon_{MNabc} = 0 \qquad \text{with} \quad P = 1,...,5 \quad \text{and} \quad a,b,c = 1,...,4.
\end{aligned} \tag{5.152}
$$

Here, we used $\mathrm{SO}(4) \in \mathrm{GL}(4)$ to diagonalize the 4×4 block in Y_{MN}. For type IIA supergravity, defining $B^P{}_{QRS} = Z^{MN,P}\varepsilon_{MNQRS}$, we have

$$
\begin{aligned}
&Y_{MN} = \begin{pmatrix} Y_{1,1} & 0 & 0 & 0 & Y_{1,5} \\ 0 & Y_{2,2} & 0 & 0 & Y_{2,5} \\ 0 & 0 & Y_{3,3} & 0 & Y_{3,5} \\ 0 & 0 & 0 & 0 & Y_{4,5} \\ Y_{1,5} & Y_{2,5} & Y_{3,5} & Y_{4,5} & Y_{5,5} \end{pmatrix}, \\
&B^m{}_{np5} \to \text{geometric flux}, \\
&B^m{}_{n45} \to \text{locally geometric flux}, \\
&B^4{}_{mnp} \to \text{Romans mass}, \\
&B^4{}_{mn5} \to \text{Vector flux},
\end{aligned} \tag{5.153}
$$

with $m,n,p = 1,...,3$, and we used $\mathrm{SO}(3)$ to diagonalize the 3×3 block in Y_{MN}. Analogously, one can solve the uplifting constraint for type IIB supergravity in the $\mathrm{SL}(5)$ EFT, now the splitting of the variables must respect $\mathrm{GL}(3) \times \mathrm{SL}(2) \in \mathrm{SL}(5)$, thus we label them with indices $m,p,q = 1,...,3$ and $\alpha,\beta = 1,2$. The results in this case are given by

$$
\begin{aligned}
Y_{MN} &= \varepsilon_{npq} f_m{}^{pq} \qquad \text{all the other components vanishing}, \\
B^\alpha{}_{mnp} &= F_3{}^\alpha, \\
B^m{}_{npq} &= \varepsilon_{npq} f_r{}^{mr}, \\
B^\alpha{}_{\beta mn} &= \varepsilon_{mnq} (j^q)^\alpha{}_\beta,
\end{aligned} \tag{5.154}
$$

where ${f_m}^{pq}$ are the traceless geometric fluxes, whose trace is ${f_m}^{mq}$ and represents the flux of the Trombone, ${F_3}^{\alpha}$ is the SL(2)-doublet of 3-fluxes (H_3, F_3) for the 3-forms and ${(j^q)^{\alpha}}_{\beta}$ is the SL(2) axio-dilaton current. Furthermore, these components satisfy ${B^{\alpha}}_{\alpha mn} = -{B^p}_{pmn}$ and $\delta^n_m {B^m}_{np\alpha} = 0$.

Chapter 6

Supergravity Wars, A New Hope: Artificial Intelligence

6.1 Introduction

This chapter will be devoted to the development of tools and methods useful to tackle the problem of finding vacua of supergravity theories directly. We have already seen how the problem of looking for Lorentz invariant vacua can be stated, from the mathematical point of view, as searching for the physically in-equivalent solutions of a system of quadratic equations. If we had to consider the full theory in 5 and 7 dimensions, we would need to solve 1756 equations in 351 variables and 121 equations in 55 variables, respectively (the number of equations is obtained by the representations the quadratic constraints belong to plus 1 corresponding to the minimisation condition for the scalar potential, the number of variables represents just the free parameters of the embedding tensor). Both cases are out of the capabilities to find analytical solutions of any computers (with an appropriate suite that allows such analytical computations, like Wolfram Mathematica). This was the main reason that led us to introduce residual gauge symmetries in the vacua, or to demand for upliftability constraints in the previous chapters. Requiring this sort of further linear constraints on the embedding tensor reduced the number of independent variables and equations down to a feasible system, allowing us to perform analytical computations, even with the use of cryptographic techniques (Appendix D). Sometimes, as in 5 and 4 dimensions, even after having imposed the upliftability constraints, the number of equations and variables is big enough to forbid any advance in the computations. This has led us to dig into the realm of Artificial Intelligence to search for a possible numerical solution to the problem. In recent years, there has been a great deal of interest in the Theoretical

Physics community for applications of Machine Learning techniques and related Artificial Intelligence tools to problems in String Theory and Supergravity [40, 42, 43, 46, 47, 110–119]. Indeed, Artificial Intelligence is a subject that goes back to 1958, when the first perceptron algorithm was invented by Frank Rosenblatt [120], many years before the first papers in String Theory appeared [121–123], but only recently AI developments rendered it appealing to the Theoretical Physics community. The reason behind it lies in the extreme complexity of performing computations in String Theory. Before deepening into it, let us introduce a very well-known problem in the IT community, one of the Millennium problems, P vs NP. The computational complexity is usually estimated by the number of "steps" and the amount of memory used to perform an algorithm. Intermediate steps can be stored in memory and speed up the computations, so let us concentrate only on the number of steps involved in carrying out the algorithms. Problems can be divided mainly into two class of complexity (there are many sublcasses, but they are irrilevant for our discussion), Polynomial running time (P) and Non-deterministic Polynomial running time (NP), we will clarify these with some examples. Imagine one wants to compute the multiplications among two $\mathrm{n}\times\mathrm{n}$ matrices, then we will need to perform n^2 scalar products, each of the latter consisting of n addition and products of 2 scalar numbers, therefore, the complexity for this computation is $\mathcal{O}(\mathrm{n}^3)$. Actually, there are better algorithms to perform this with $\mathcal{O}(\mathrm{n}^{2.376})$ steps, but this will not change the membership of this problem to the class of problems solvable in a polynomial running time. On the other hand, imagine now that you are a salesman that has to visit n houses and you want to know the lowest amount of road you need to travel to visit them all. What one can do is computing the length of every possible trajectory and compare them, this would take at least $\mathcal{O}(\mathrm{n}!)$ steps. The running time consequently grows faster than any possible polynomials in n; we thus say that this problem belongs to the NP class. If we put some numbers in these formulas, we will note the huge difference among these kinds of computations. Let us suppose we have a computer that performs 1000 operations per second, then the multiplication of two 10×10 matrices takes $\frac{1}{10}$ of second, multiplying two 100×100 matrices takes 10 seconds, two 10000×10000 matrices takes 28 hours and two $10^5\times10^5$ matrices takes 115 days, and if we want to speed up our computations we can always use a faster computer or parallelise on more computers. On the other hand, if we want to solve the travel salesman problem (TSP) for 10 houses with our computer, it takes 1 hour, if instead we have a 100 houses it takes 3×10^{137} times the age of the Universe, and already for 27 houses would take 2 human lives on the fastest supercomputer in the world (nowadays running at 2 quintillion operations per second). We immediately see that for NP problems there is nothing we can do, neither using faster computers nor parallelising that can help us in tackling the computations, at a certain point in scaling with the number of houses, we will always be overwhelmed by the number of resources needed to perform the algorithm. At the moment of writing of this thesis it is still debated whether each problem in the NP class can be solved in polynomial time, therefore proving that NP=P, this is actually one of the Millennium Problems and if one would be able to show that TSP can be solved in polynomial time, then any NP problem is in P, because each NP problem can be mapped to TSP. It is interesting, at this point, to note that there is yet another category, which is the class of undecidable problems. The halting problem is one of the most famous puzzles in this category, we can state it as: "Design an algorithm that takes any computer programme as input and decide whether the programme will run forever or will halt at some point". There is no algorithm that can be designed to perform such task, so it is impossible to decide between the two options, and the puzzle therefore lies in the category of undecidable problems. Finding a solution to a system of coupled non-linear Diophantine equations is also an undecidable problem, which has also more relevance for String Theorists. Indeed, finding a physically consistent string vacuum is made up of different tasks which are either NP or undecidable. For instance, finding configurations of fields, branes, planes and fluxes and imposing the tadpole and anomaly cancellation conditions lead to a system of coupled, non-linear Diophantine equations, which is undecidable, or finding the critical points of the scalar potential and controlling if they are minima is in NP. This has led physicists to look for different methods to solve those tasks; obviously Optimisation and Machine Learning algorithms do not change the membership of a problem to one of the previous class, as any other algorithm they are subject to the same complexity rules, but they can be useful when looking at the problems from a different perspective. Indeed, these algorithms are designed to solve different problems which are no more undecidable or in NP, for instance, they can be used to find approximate solutions instead of exact ones (P), some techniques for switching from approximate to exact solutions will be explained later in the chapter, or they may look for patterns which are difficult to be identified by humans. In particular, we can transform our problem of finding solutions to a system of quadratic coupled equations into a minimisation (optimisation) condition. We first need to define the function to be minimised, which is called the fitness or loss function in the Artificial Intelligence jargon. All our equations are homogeneous, so of the form

$$f_j(x[i]) = 0 \qquad \text{with i = 1,...,n} \tag{6.1}$$

with $x[i]$ labelling the parameters of the embedding tensor, n is the number of them, which may depend on the uplifting conditions and/or the residual gauge symmetries and j labelling the equation in the system. Therefore, we define our fitness function by

$$f_{fit} = \sum_j f_j^2. \tag{6.2}$$

Thanks to this definition, we can look for minima of this function and there will be no problem in checking whether a candidate minimum is a global or a local one, because the global minima verify $f_{fit} = 0$, and correspond to solutions of the system of equations. In the following, we are going to expose some of the optimisation algorithms, borrowed from the Artificial Intelligence community, that have been used to look for minima of the fitness function, such as Gradient Descent and Stochastic Gradient Descent, Genetic Algorithms and Covariance Matrix Adaptation, then we will describe how, from the numerical solutions it is possible to pass, in some cases to the exact solutions.

6.2 Optimization Algorithms

Many optimization algorithms have been tested and /or used in order to find solutions to the quadratic constraints and the equations of motion. In this section we will present only the most relevant.

6.2.1 Stocastic Gradient Descent

Stocastic Gradient Descent (SGD) [124–126] is a variation of the Gradient Descent algorithm (GD) invented by Cauchy in 1847 [127]. Given a differentiable function $F(\mathbf{X})$ and an initial point $\mathbf{X}_0 = \{x_1, ..., x_n\}$ in his domain, the algorithm (GD) consists in updating the position of the point $\mathbf{X}$ in the following way

$$\mathbf{X}_{k+1} = \mathbf{X}_k - \gamma \nabla F(\mathbf{X}_k), \tag{6.3}$$

with γ a small positive real parameter $\gamma \in \mathbb{R}^+$, named the learning rate. We can see that at each step, the algorithm tells us to compute the gradient and to update the position by moving in the opposite direction, exactly in the steepest descent direction. The learning rate can be chosen to change in time in such a way that at each successive step it decreases in order not to overshoot when we are near a minimum.

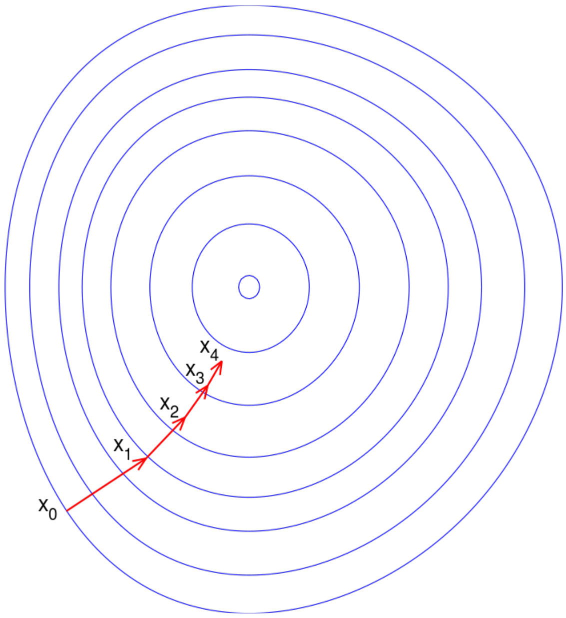


Fig. 6.1 Gradient Descent visual representation

SGD, on the other hand, is designed for functions of the form

$$F(\mathbf{X}) = \sum_i f_i(\mathbf{X}), \tag{6.4}$$

which is our case, or for instance, in supervised learning, each f_i could be the i-th entry of a dataset. The algorithm's prescription is then to update $\mathbf{X}$ at each step by moving in the opposite direction of the gradient of just one of the functions f_i, the latter is changed at each step. If the algorithm did not converge in one sweep of the whole set of f_i, we repeat it changing the order of them, in formula:

$$\mathbf{X}_{k+1} = \mathbf{X}_k - \gamma \nabla f_i(\mathbf{X}_k), \qquad \text{Repeat for each i until a minimum is found.} \tag{6.5}$$

The condition for stopping the algorithm is that no major improvement is done in successive steps for an entire sweep. This method will introduce larger fluctuations in the position of the point with respect to GD. A compromise between GD and SGD is to compute the gradient for a batch of functions f_i and this is called mini-batch gradient descent, reducing the fluctuations in the position. Different techniques to adapt the learning rate have been developed in the years, such as RMSProp or AdaGrad [128], another interesting feature is presented in the "momentum" adaptation [129]. The latter dictates to keep in memeory the variation $\Delta\mathbf{X}$ at each step:

$$\Delta\mathbf{X} = \alpha\Delta\mathbf{X} - \gamma\nabla f_i(\mathbf{X}), \tag{6.6}$$
$$\mathbf{X} = \mathbf{X} + \Delta\mathbf{X}. \tag{6.7}$$

with α a decay factor between 0 and 1, which takes into account how much information coming from the previous step we keep for the next one. It is called "momentum" adaptation because this remembers the physical momentum of a particle that at each step feels the action of a driving force ∇f_i. This is also a method to avoid large fluctuation in the position.
It has been found, in our tests, that these algorithms are not very fast to converge for our problem.

6.2.2 Genetic Algorithms

As many other algorithm techniques in the IT world, Genetic Algorithms (GA) [130] are based on ideas borrowed from nature, indeed, the basic idea behind it is the principle stated by Charles Darwin of the survival of the fittest individual [131]. Therefore, we have an initial population, evaluate the fittest individuals in it by means of the fitness function, make them reproduce, and conclude with a mutation step. This procedure is repeated generation after generation, and in order to avoid reaching a point with a huge population, we remove some individual from time to time based on the fitness function or on their age (measured in number of generations). The reasons why all this procedure works are not certain and are still debated. GAs are most often justified by means of the "Fundamental Theorem of Genetic Algorithms", known also by the name of Holland's Schema Theorem. We will illustrate it after having discussed in detail how GA works.
Let us introduce some vocabulary before, a **phenotype** e (also called individual) is a set of properties (in our case a set of real numbers) that approximate a solution to the problem (the parameters of the embedding tensor in the vacuum), e= $\{a_1,...,a_n\}$. The set of all **phenotypes** is called **population**, $P = \{e_i\}$, and we have a **population** for each **generation** G_t, with t labelling the evolutionary step. Each a_i present in e is called an **allele**, where i is named **locus** and the set of alleles in a specific **phenotype** is called **chromosome** or **genotype** or **genome**. The domain of the **alleles** is named **gene pool** and is usually represented by γ (not to be confused with the learning rate). Therefore, in each evolutionary step, the population evolves by means of **reproduction** and **mutation**. The initial population is selected randomly, by instantiating random alleles from the gene pool, then for each phenotype, we evaluate its fitness function, and based on it, we select the individuals which will reproduce. One of the most common breeding selection criteria is **roulette wheel selection**, which is owing its

name to the fact that the choice for reproduction is similar to a bet on the ball ending in one of the sectors of a roulette wheel. Indeed, declaring that the total number of phenotypes is N and that the fitness function of each individual is f_i and positive (we can always make it positive by adding a positive constant to it), we can define

$$S_j = \sum_{i=1}^{j} f_i. \tag{6.8}$$

The distance between two successive S_j's, $d_j = S_j - S_{j-1}$ is just f_j, the fitness of the phenotype e_j. A random number $r \in [0, S_N]$ is generated, which corresponds to the ball falling in one of the drawers of the roulette wheel, since the fitness functions are all positive, there is a unique label j such that $S_{j-1} \leq r \leq S_j$, consequently the phenotype e_j is picked. Another similar selection algorithm is **stochastic universal sampling**, which, to select k parents, consists of drawing a random number r just once (for the roulette wheel selection, it was chosen k times), then we define k evenly spaced intervals $s_j \equiv (r + j/S_N) \bmod S_N$, with $j = 0, 1, ..., k-1$ giving rise to k phenotypes e_i such that $S_{i-1} \leq s_j \leq S_i$. With this procedure, we increase both the probabilities of the fittest to be chosen at least once, but also of other, less fit individuals to get reproduced. These two selection processes have some drawbacks, mainly they are affected too much by the fitness function. Indeed, imagine there is a phenotype that performs much better than the other, and this happens at the initial stages usually, then it will be selected many more times than other individuals, forcing the algorithm to converge fast and with high chances to end up in a local minimum/maximum, on the other hand at later evolutionary stages, all the phenotypes will have more or less the same fitness and therefore, the ones that are nearer to the minimum/maximum do not have any advantages over the others with these algorithms, basically at the end there is no more evolutionary pressure. One way to avoid this problem is to perform a **tournament** or a **rank** selection. In tournament selection, k phenotypes are randomly selected from the population to compete against each other, the individuals that win most often are chosen for the reproduction stage. This method has the additional advantage of being applicable both to positive and negative fitness functions. Being a random selection process, it does not present the same problems as the previous algorithms based on fitness functions. Another useful advantage of tournament selection is that it can also be used in cases where the fitness function is not defined, and the best phenotype is chosen from the performance of each individual against others (like in the game of chess, where there is no explicit fitness function). The parameter k is in charge to fix the evolutionary pressure, as it grows, the probabilities of less fit individuals getting to reproduce diminish. In addition, it is also possible to decide if each winner competes just once or several times, in the latter case, the evolutionary pressure is high. We can also choose

whether the winners are picked deterministically or that they win with a probability p, in the former case the evolutionary pressure is again high.
Rank selection, instead, assigns a rank r_j to the phenotypes present in the population, based on their fitness f_i, which does not have to be positive definite, in such a way that the fittest has rank 1 and the least fit rank N. Then we define a normalised function:

$$f_i' = -\frac{r_i}{N} + \frac{N+1}{N}. \tag{6.9}$$

The new fitness function is always positive and can be used in a roulette wheel selection, with the fittest element picked N times more often than the least one. Obviously, there are many choices for the newly defined fitness function, we could generalise it to:

$$f_i' = -a\frac{r_i}{N} + b \qquad \text{with} \quad a, b \geq 0, \tag{6.10}$$

in such a way that the ratio between the fittest and the least fit is

$$\frac{f_1'}{f_N'} = \frac{1}{N}\frac{Nb-a}{b-a}. \tag{6.11}$$

By doing so, we can decide the amount of evolutionary pressure that we would like to assign to the system. Indeed, the parameter a allows to modify it, when $a \to 0$ there is no evolutionary pressure at all, $f_1' = f_N'$, when $a \to \infty$, we end up with the previous case in which the fittest is selected N times more frequently than the least one. Another choice for the new fitness function is given by **exponential rank selection**, where the new function is given by

$$f_i' = \frac{1 - e^{r-N}}{C}, \tag{6.12}$$

with C a normalisation constant. In the end, **elitist selection**, can be implemented on top of all the previously listed methods. It consists of carrying to the next generation G_{t+1} the k fittest phenotypes present in the population. This guarantees that the best individuals survive generation after generation, but at the cost of reducing the variance, which can cause troubles when the size of the population is small, causing the algorithm to end prematurely in a local minimum/maximum. The choice of the selection mechanism depends strongly on the problem of interest, for the Travel Salesman Problem has been shown that tournament selection behaves better than any other [132].
Once the selection of the parents has been carried out, we need to decide if any of them survive to the next generation and how we generate the off-springs. For survival criteria we can chose either between **age-based survival** or **fitness-based survival**. In the former,

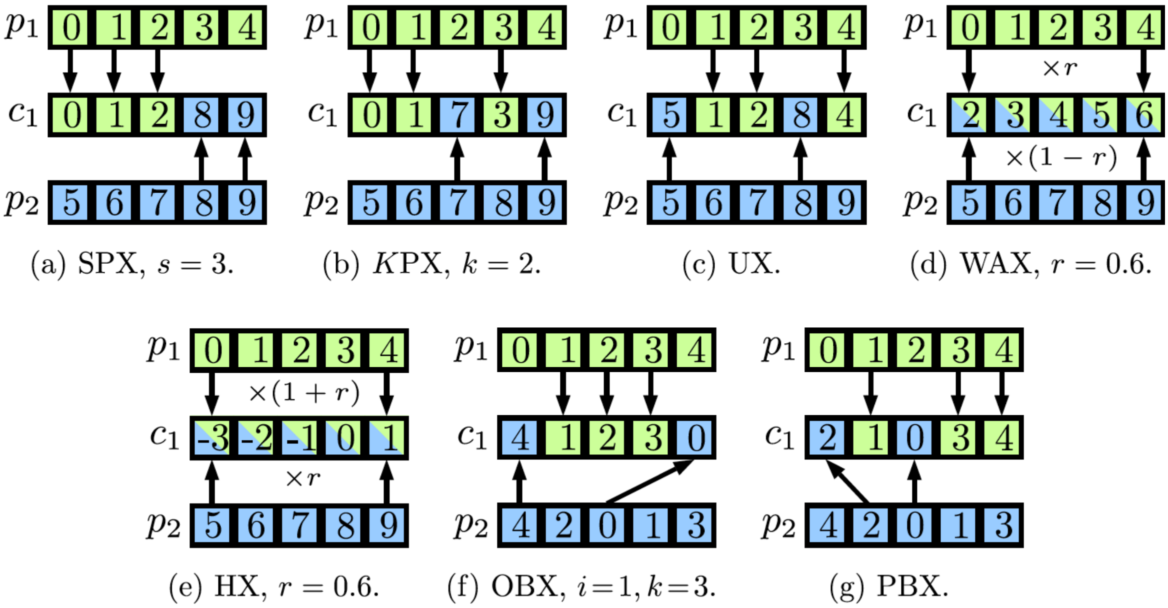


Fig. 6.2 Reproduction mechanisms in Genetic Algorithms

individuals can live for a fixed number of generations, and if we want to keep the size of the population fixed we need to take this into consideration when generating the new offsprings. In the latter case, the k fittest individual of a generation replace the least fit phenotypes from the next one, which can be chosen by tournament, roulette wheel selection, etc...

Reproduction

There are various methods for reproduction, based on two main classes: **cloning** or **elitist selection** and **crossover** where the genotype of the parents can be transmitted to their children in different ways. Let us analyse the case of 2 parents and 2 children, each with a phenotype consisting of N alleles a_i , $i = 1,...,N$:

- **Single Point Crossover (SPX):** a random locus $s \in [1,N)$ is picked, the same for both parents; the first child c_1 takes alleles from 1 to s from the first parent p_1 and alleles from $s+1$ to N from the second parent p_2. The other child c_2 behaves in the opposite way. In fig. 6.2(a) is depicted the case $N = 5$, $s = 3$.

- **K-point Crossover (KPX):** now the genotypes are split at K locations labelled by various loci s_i, $i = 1,...,K$, the alleles are picked alternatingly from p_1 and p_2 for c_1 and in the opposite way for c_2. SPX corresponds to the special case $K = 1$. See fig. 6.2(b) for $K = 3$ and $s_i = (2,3,4)$.

- **Uniform Crossover (UX):** c_1 inherits each allele from p_1 or p_2 with a random choice among the two, the second child c_2 inherits each allele from the parent who did not donate it to c_1. In fig. 6.2(c) you can find the case $(p_2, p_1, p_1, p_2, p_1)$ for c_1.

- **Whole Arithmetic Recombination (WAX):** the alleles of the child c_1 are formed from a linear combination of the alleles of the parents. First, we choose a number $r \in [0,1]$ and then we compute $a_i^{c_1} = ra_i^{p_1} + (1-r)a_i^{p_2}$, analogously for the child c_2 we have $a_i^{c_2} = ra_i^{p_2} + (1-r)a_i^{p_1}$. Obviously, if the alleles are integers, we need to round the result at the nearest integer. $r = 0$ or $r = 1$ coincide with cloning. In fig. 6.2(d) there is the case with $r = 0.6$.

- **Heuristic Crossover (HX):** as it is the case for WAX we pick a number $r \in [0,1]$, the alleles for the children are then formed by $a_i^{c_1} = (1+r)a_i^{p_1} - ra_i^{p_2}$ and $a_i^{c_2} = (1+r)a_i^{p_2} - ra_i^{p_1}$. Just as before, if the alleles are integers we need to round the result, in addition, in this case the linear combination can produce alleles not in the gene pool, if this happens, either we clip them or we repeat the process with a different number r. Fig. 6.2(e) shows the case with $r = 0.6$.

Other types of crossover have been invented to deal specifically with problems such as the travel salesman problem, where the alleles are integers, that can appear only once in the phenotype. Then, the only allowed operation we can do on alleles is permute them, therefore, these types of crossover are usually called **permutation encodings**, examples of these are

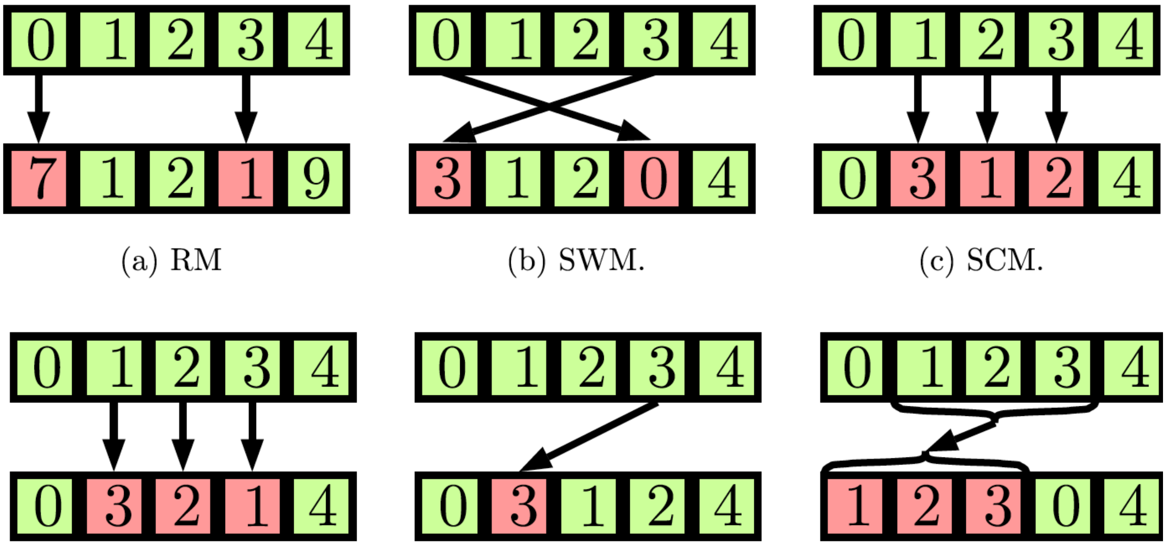
- **Order-based Crossover (OBC):** we pick a number k, which will fix the length of the set of alleles that will be carried from the first parent p_1 to the child. We also pick a locus i for p_1, such that alleles from a_i to a_{i+k} are transmitted to the child, at the same loci i to $i+k$ of the first parent's phenotype, then fill the remaining positions with alleles from p_2 in such a way to avoid repetition of alleles. Fig. 6.2(f) illustrates the case with $i = 1$ and $k = 3$.

- **Position-based Crossover (PBX):** Randomly pick a number of loci from p_1 and carry alleles at the same loci to c_1, then fill the missing ones with alleles from p_2 keeping the order of appearance in p_2. This case is presented in fig. 6.2(g).

Once the reproduction stage has been carried out, we conclude with the mutation stage.

Mutation

Just as was the case for reproduction, even for mutation, there are many possible mechanisms, and we list here only the most relevant ones:



(a) RM (b) SWM. (c) SCM.

(d) INVM. (e) INSM. (f) DM.

Fig. 6.3 Mutation mechanisms in Genetic Algorithms

- **Random Mutation (RM):** a subset of alleles chosen in a random manner are substituted by others picked randomly from the gene pool, this is depicted in fig. 6.3(a).
- **Swap Mutation (SWM):** exchanges two alleles among them, used especially in permutation encodings, because it preserves the distinctiveness of the alleles. Fig. 6.3(b) shows the case with alleles at loci $i = 1$ and $i = 4$.
- **Scramble Mutation (SCM):** pick an interval in the phenotype and randomly permute the elements in it. Again, this can be used for permutation encodings. Fig. 6.3(c) illustrates the case for the interval $[2,4]$.
- **Inversion Mutation (INVM):** select an interval as in SCM and invert the position of the alleles inside it. It is a special case of SCM and it is depicted in fig. 6.3(d) for the interval $[2,4]$.
- **Insertion Mutation (INSM):** pick an allele and move it to another locus of the phenotype, then move all other alleles to the left or to the right. In fig. 6.3(e), the case with allele at locus $i = 4$ moved to locus $i = 2$ is presented.
- **Displacement Mutation (DM):** this is similar to INSM, but now we pick and move a whole interval, as depicted in fig. 6.3(f).

Having reviewed the basics, we can now illustrate the Fundamental Theorem of Genetic Algorithms [133].

Holland's Schema Theorem

First of all, schemas are a set of individuals with the same properties, namely, the same alleles at the same loci, the number of "fixed" alleles in the schemas is called order. For instance, suppose that we have a set of phenotypes named H (schema), all with the same structure $\{1,0,*,*,0,1,*\}$, the order then is 4 and represented by $o(H)$. We need also to introduce the defining length $\delta(H)$ of schemas as the distance between the first and last fixed locus. In our example $\delta(H) = 5$. The statement of the theorem is that short (with small defining length), low-order schemas with above-average fitness become exponentially dominant in the following generations:

$$E(n(H,G_{t+1})) \geq \frac{n(H,G_t)f(H)}{f_t}[1-p]. \tag{6.13}$$

E is the expectation values, $n(H,G_t)$ is the number of individuals with schema H in the population at generation G_t, $f(H)$ is the average fitness for the schema H, while f_t is the average fitness of the population at generation G_t. p is the total probability that crossover or mutation breaks the pattern of the schema:

$$p = \frac{\delta(H)}{l-1}p_c + o(H)p_m, \tag{6.14}$$

with p_c the probability of crossover and p_m the probability of mutation, l is the length of the phenotype. Thus, schemas with shorter defining length or small order are less likely to be disrupted. The inequality in the statement of the theorem is there in order to take care of the non-vanishing possibility that a phenotype with schema H is produced by scratch from mutation of an element not in the schema H in the previous generation. The theorem works in the case of infinite population's size but it fails to describe the reasons why genetic algorithms work (when they indeed work) for finite sizes. Indeed, the theorem does not make any distinction between cases in which GA performs well or poorly [134].
We present an example where GA is capable of finding the global minimum of a fitness function with many local minima. The fitness function is given by

$$f_{fit}(x,y) = 30x^2 + 12y^2 + 90\sin(3x) + 80\sin(3y) + x^2y^2, \tag{6.15}$$

whose level curves are shown in fig. 6.4.

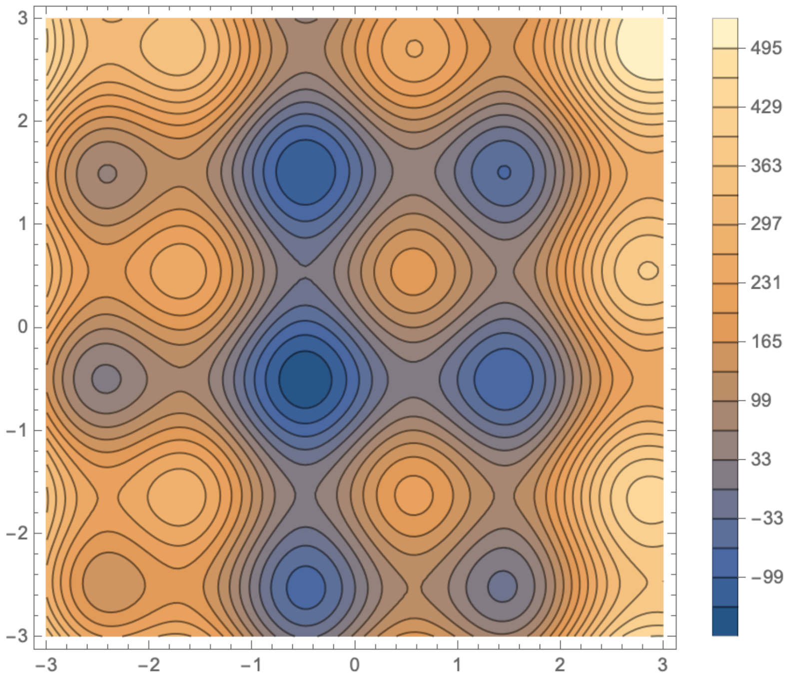


Fig. 6.4 Level curves for the fitness function in eq. 6.15

We immediately see from fig. 6.4 that the fitness function can also become negative, therefore, we use a 2-tournament selection process twice to choose the parents that get to breed and produce 2 children with WAX crossover, where r is picked each time randomly, and a 6-tournament selection mechanism for the survival selection, only among the children. Then mutation is carried out by means of a Random Mutation mechanism, adding a random number $r < 0.2$ to either the first or the second allele (x or y). We used a population of 1000 individuals. The results are shown in fig. 6.5, we can see that already after 4 generations the algorithm converged (within an error ε) to the global minimum. Applying the GA to our problem, for maximal supergravity in 7 dimensions with 45 variables (alleles) in each phenotype, due to the presence of a multitude of sharp local minima, it did not converge to global minima. This led us to search for new algorithms, such as CMA-ES.

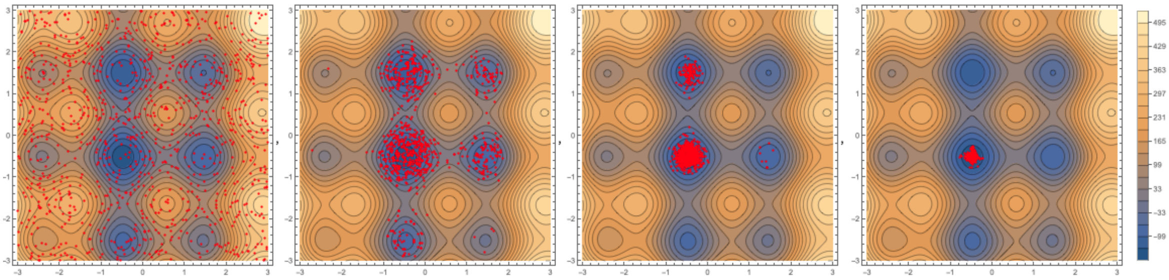


Fig. 6.5 Example of convergence after 4 generations for a Genetic Algorithm

6.2.3 Covariance Matrix Adaptation - Evolutionary Strategy (CMA-ES)

CMA-ES, in short, CMA, is an evolution strategy (ES) algorithm that, as is the case for GA, only needs the fitness function as accessible information [135–143]. Therefore, differently from SGD, we do not require the function to be differentiable, it can also be not continuous. CMA, as the name suggests, is based on the "adaption" of a normal distribution to the fitness function under consideration (with its level curves). Let us analyse the algorithm in depth and consider a multivariate normal distribution, $\mathcal{N}(\mathbf{m},\mathbf{C})$, which is determined by the mean $\mathbf{m} \in \mathbb{R}^n$ (in our case n is the number of free parameters in the embedding tensor) and by the symmetric, positive definite covariance matrix $\mathbf{C} \in \mathbb{R}^{n\times n}$. Covariance matrices are associated with the ellipsoid $\{\mathbf{x} \in \mathbb{R}^n | \mathbf{x}^T\mathbf{C}^{-1}\mathbf{x} = 1\}$, where the latter's principal axis are the eigenvectors of $\mathbf{C}$, and squared axis length are the eigenvalues of the covariance matrix. We can always diagonalize the covariance matrix by means of an orthogonal matrix $\mathbf{B}$ whose columns are the eigenvectors of $\mathbf{C}$ with unit length, $\mathbf{C} = \mathbf{B}(\mathbf{D})^2\mathbf{B}^T$. Then, it is also possible to write the normal distribution as

$$\mathcal{N}(\mathbf{m},\mathbf{C}) \sim \mathbf{m} + \mathcal{N}(\mathbf{0},\mathbf{C}) \sim \mathbf{m} + \mathbf{C}^{-1/2}\mathcal{N}(\mathbf{0},\mathbf{I}) \sim \mathbf{m} + \mathbf{B}\mathbf{D}\mathbf{B}^T\mathcal{N}(\mathbf{0},\mathbf{I}), \qquad (6.16)$$

with $\mathbf{I}$ the $n\times n$ identity matrix. At each step of the process, we generate a new population of points (what we call off-springs in GA) by drawing them from a multivariate normal distribution:

$$\mathbf{x}_j^{g+1} \sim \mathcal{N}\big(\mathbf{m}^g, (\sigma^g)^2\mathbf{C}^g\big) \qquad \text{with} \quad j = 1, ..., \lambda \qquad (6.17)$$

The superscripts g, $g+1$, etc. label the generation, λ is the population size and $\sigma^g \in \mathbb{R}^+$ is the "overall" standard deviation (step size) at generation g. Now we need to explain how the mean, the covariance matrix and standard deviation are computed for the next generation $g+1$.

The mean

The new mean $\mathbf{m}^{g+1}$ is simply selected with a weighted average of the μ best points of the population:

$$\mathbf{m}^{g+1} = \sum_{j=1}^{\mu} w_j \mathbf{x}_{j:\lambda}^{g+1}, \qquad \text{with} \quad \sum_{j=1}^{\mu} w_j = 1 \quad \text{and} \quad w_j > 0. \tag{6.18}$$

$w_j \in \mathbb{R}^+$ with $j = 1, ..., \mu$ are positive ordered weights, that is, $w_1 \geq w_2 \geq ... \geq w_\mu > 0$. If $w_j = 1/\mu$ for each j, we obtain the mean value for the best μ points. $\mathbf{x}_{j:\lambda}^{g+1}$ represents the j-th best individual of the population, meaning, $f(\mathbf{x}_{1:\lambda}^{g+1}) \leq f(\mathbf{x}_{2:\lambda}^{g+1}) \leq ... \leq f(\mathbf{x}_{\lambda:\lambda}^{g+1})$. An essential quantity is the **variance effective selection mass**

$$\mu_{eff} = \Big(\sum_{j=1}^{\mu} w_j^2\Big)^{-1}, \tag{6.19}$$

it is possible to show, from the definition of w_j that $1 \leq \mu_{eff} \leq \mu$ and that $\mu_{eff} = \mu$ only in the case in which all the weights are the same and equal to $1/\mu$. Usually, $\mu \approx \lambda/2$ and $w_i \propto \mu - i + 1$.

The covariance matrix

Let us first define the empirical covariance matrix $\mathbf{C}_{emp}^{g+1}$, which is nothing more than an estimate of the covariance matrix $\mathbf{C}^g$:

$$\mathbf{C}_{emp}^{g+1} = \frac{1}{\lambda - 1} \sum_{i=1}^{\lambda} \left(\mathbf{x}_i^{g+1} - \frac{1}{\lambda} \sum_{j=1}^{\lambda} \mathbf{x}_j^{g+1}\right) \left(\mathbf{x}_i^{g+1} - \frac{1}{\lambda} \sum_{j=1}^{\lambda} \mathbf{x}_j^{g+1}\right)^T. \tag{6.20}$$

We are now going to modify this estimator, in order to obtain a maximum likelihood estimator of $\mathbf{C}^g$, by defining

$$\mathbf{C}_{\lambda}^{g+1} = \frac{1}{\lambda} \sum_{j=1}^{\lambda} \big(\mathbf{x}_j^{g+1} - \mathbf{m}^g\big)\big(\mathbf{x}_j^{g+1} - \mathbf{m}^g\big)^T. \tag{6.21}$$

The difference between $\mathbf{C}_{emp}^{g+1}$ and $\mathbf{C}_{\lambda}^{g+1}$ is what is used as a mean value. The former utilises the mean obtained from the entire population, thus estimating the variance of the sampled points, the latter instead uses the mean obtained by 6.18 therefore estimating the sampled steps, $\mathbf{x}_j^{g+1} - \mathbf{m}^g$. We are going to modify again this estimator, and define

$$C_\mu^{g+1} = \sum_{j=1}^{\mu} w_j \left(\mathbf{x}_{j:\lambda}^{g+1} - \mathbf{m}^g\right)\left(\mathbf{x}_{j:\lambda}^{g+1} - \mathbf{m}^g\right)^T. \tag{6.22}$$

$\mathbf{C}_\mu^{g+1}$ is an estimator for the variance of selected steps (the best/successful μ steps). We have some conditions on μ_{eff} in order for $\mathbf{C}_\mu^{g+1}$ to be a **reliable** estimator. Indeed, μ_{eff} has to be large enough to prevent the condition numbers (that given a matrix A and a linear system Ax=b, with x unknown, measure how sensitive the solution of the system to a change in b is, high condition numbers imply that small changes in b generate huge modifications in the solution) of $\mathbf{C}_\mu^{g+1}$ to be smaller than 10 for the fitness function of the sphere: $f_{sphere}(\mathbf{x}) = \sum_{i=1}^n x_i^2$; empirically, it is seen that $\mu_{eff} \approx 10n$ is a good choice. To avoid this problem for a small population we are going to modify the update of the covariance matrix again.

In order to obtain an algorithm that converges faster, we need a small population, on the other hand, to obtain a more global search the population has to increase. For small population, also $\mu_{eff} \approx \lambda/4$ (which is the choice to take to have reasonable w_j) has to be small, then $\mathbf{C}_\mu^{g+1}$ is not a reliable estimator, in order to circumvent this, we define a new covariance matrix that takes into consideration the information we have from previous generations. We define $\mathbf{C}^0 = \mathbf{I}$ and a learning rate $0 < c_{cov} \leq 1$, then

$$\begin{aligned}\mathbf{C}^{g+1} =& (1-c_{cov})\mathbf{C}^g + c_{cov}\left(\frac{1}{\sigma^g}\right)^2 \mathbf{C}_\mu^{g+1} \\ =& (1-c_{cov})\mathbf{C}^g + c_{cov}\sum_{j=1}^{\mu} w_j \left(\frac{\mathbf{x}_{j:\lambda}^{g+1} - \mathbf{m}^g}{\sigma^g}\right)\left(\frac{\mathbf{x}_{j:\lambda}^{g+1} - \mathbf{m}^g}{\sigma^g}\right)^T.\end{aligned} \tag{6.23}$$

The step-sizes σ^g have been integrated to ensure that $\mathbf{C}_\mu^g$ from different generations are comparable. If $c_{cov} = 1$ the covariance matrix collapses to $\mathbf{C}_\mu^{g+1}$ and no information from previous generations is retained, on the other hand, if $c_{cov} = 0$, $\mathbf{C}^{g+1} = \mathbf{C}^0$ and there is no learning. This kind of update, represented in 6.23 update, is called rank-μ update, because the sum goes from 1 to μ. Eq. 6.23 is iterative and can be expanded as

$$C^{g+1} = (1-c_{cov})^{g+1}\mathbf{C}^0 + c_{cov}\sum_{j=0}^{g}(1-c_{cov})^{g-j}\left(\frac{1}{\sigma^j}\right)^2 \mathbf{C}_\mu^{j+1}. \tag{6.24}$$

Picking high values for c_{cov} leads to degenerate covariance matrices, while small values imply slow learning, a good choice is $c_{cov} \approx \mu_{eff}/n^2$. Small population sizes λ lead to a large number of generations and therefore to a faster adaptation for the covariance matrix. A final step is necessary in the update of the covariance matrix: **cumulation**. Indeed, we did not use the information on the "signs" of the steps the strategy took generation after generation. In order to do so, we introduce the **evolution path**. An evolution path is any sequence of successive steps taken by the strategy, taking the sum of these steps is referred as cumulation; for instance, for three steps we have

$$\frac{\mathbf{m}^{g+1}-\mathbf{m}^{g}}{\sigma^{g}}+\frac{\mathbf{m}^{g}-\mathbf{m}^{g-1}}{\sigma^{g-1}}+\frac{\mathbf{m}^{g-1}-\mathbf{m}^{g-2}}{\sigma^{g-2}}. \tag{6.25}$$

Defining the 0-th-order evolution path $\mathbf{p}_c^0 = \mathbf{0}$, we use exponential smoothing and define iteratively

$$\mathbf{p}_c^{g+1} = (1-c_c)\mathbf{p}_c^g + \sqrt{c_c(2-c_c)\mu_{eff}}\frac{\mathbf{m}^{g+1}-\mathbf{m}^{g}}{\sigma^{g}}, \tag{6.26}$$

with $0 \geq c_c \leq 1$ a new learning rate for the evolution path, the normalisation factor $\sqrt{c_c(2-c_c)\mu_{eff}}$ is dictated by the demand that $\mathbf{p}_c^{g+1}$ is extracted from a normal distribution $\mathcal{N}(\mathbf{0},\mathbf{C})$. When $c_c = 0$ there is no learning and $\mathbf{p}_c^g = \mathbf{0}$. Putting everything toghether, we obtain the update of the covariance matrix:

$$\begin{aligned}\mathbf{C}^{g+1} =&(1-c_{cov})\mathbf{C}^g + \frac{c_{cov}}{\mu_{cov}}\mathbf{p}_c^{g+1}\mathbf{p}_c^{g+1\,T}\\ &+c_{cov}\left(1-\frac{1}{\mu_{cov}}\right)\sum_{j=1}^{\mu} w_j\left(\frac{\mathbf{x}_{j:\lambda}^{g+1}-\mathbf{m}^g}{\sigma^g}\right)\left(\frac{\mathbf{x}_{j:\lambda}^{g+1}-\mathbf{m}^g}{\sigma^g}\right)^T,\end{aligned} \tag{6.27}$$

with $\mu_{cov} \geq 1$ and it should be $\mu_{cov} = \mu_{eff}$. Eq. 6.27 reduces to eq. 6.23 in the case $\mu_{cov} \to \infty$, so information from the last generation is taken into consideration by the rank-μ update and information from previous generations, instead, is exploited by the evolution path update, which is relevant above all for small population's sizes.

The step size

An evolution path is used also to update the step size σ with a method called **cumulative step size adaptation**:

- Whenever the evolution path is long, the steps are going in the same direction (approximatively), so they are correlated. Consequently, we can cover the same distance with longer but fewer steps, and the step size must be increased.
- When the evolution path is short, the steps cancel among each other, and the step size should be decreased.

- The optimal situation is that the steps are totally uncorrelated and orthogonal with respect to the previous and following ones.

We need to define what long- and short-evolution paths mean. In this respect, we compare the latter with the expected length under random selection, which means that the steps are uncorrelated with each other. If our strategy finds that the evolution paths are longer than the uncorrelated ones, σ has to be increased, and vice versa.
The evolution path $\mathbf{p}_c^{g+1}$ depends on its direction, therefore we define the conjugate path

$$\mathbf{p}_\sigma^{g+1} = (1-c_\sigma)\mathbf{p}_\sigma^g + \sqrt{c_\sigma(2-c_\sigma)\mu_{eff}}(\mathbf{C}^g)^{-\frac{1}{2}}\frac{\mathbf{m}^{g+1}-\mathbf{m}^g}{\sigma^g}, \tag{6.28}$$

with $0 < c_\sigma < 1$ a learning rate and $(\mathbf{C}^g)^{-\frac{1}{2}} \equiv \mathbf{B}^g(\mathbf{D}^g)^{-1}\mathbf{B}^{gT}$. Whenever $(\mathbf{C}^g)^{-\frac{1}{2}} \neq \mathbf{I}$ it aligns the step $\mathbf{m}^{g+1}-\mathbf{m}^g$ to the coordinate system produced by $\mathbf{B}^g$. In particular $\mathbf{B}^{gT}$ rotates the system in such a way that the columns of $\mathbf{B}^g$ become the axis. $(\mathbf{D}^g)^{-1}$ rescales the length of the axis so that they measure distances in the same way. $\mathbf{B}^g$ rotates everything back, allowing one to compare the directions of the various steps. By adding the matrix $(\mathbf{C}^g)^{-\frac{1}{2}}$ in eq.6.28 we ensure the independence of $\mathbf{p}_\sigma^{g+1}$ from the direction of the steps. Then we compare the length of $\mathbf{p}_\sigma^{g+1}$ with the expected length of the evolution path obtained from random selection $E\left[||\mathcal{N}(\mathbf{0},\mathbf{I})||\right]$ and define the step size

$$\sigma^{g+1} = \sigma^g \exp\left(\frac{c_\sigma}{d_\sigma}\left(\frac{||\mathbf{p}_\sigma^{g+1}||}{E\left[||\mathcal{N}(\mathbf{0},\mathbf{I})||\right]}-1\right)\right), \tag{6.29}$$

where $d_\sigma \approx 1$ is a damping parameter and $E\left[||\mathcal{N}(\mathbf{0},\mathbf{I})||\right] = \sqrt{2}\Gamma\left(\frac{n+1}{2}\right)/\Gamma\left(\frac{n}{2}\right) \approx \sqrt{n} + \mathcal{O}(1/n)$ is the expectation value of the Euclidean norm for a multivariate normal distribution with identity matrix as covariance matrix. From eq. 6.29 we can see that, whenever $||\mathbf{p}_\sigma^{g+1}|| > E\left[||\mathcal{N}(\mathbf{0},\mathbf{I})||\right]$, σ^g increases and viceversa when $||\mathbf{p}_\sigma^{g+1}|| < E\left[||\mathcal{N}(\mathbf{0},\mathbf{I})||\right]$.
It has been proved, in a survey about Black-Box optimizations [144], that CMA-ES outranked other 31 optimization algorithms, and that its performance is outstanding for rugged, ill-conditioned functions with large search dimensional spaces.

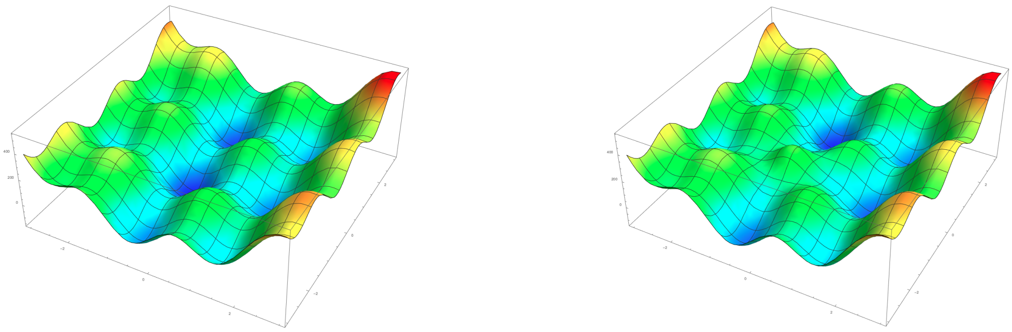
6.3 Results and Analysis

Various numerical analyses have been performed using the different algorithms described above, as well as other not mentioned in this thesis. Upliftability constraints have been used as well, in order to look only for vacua with a direct link to type IIA/IIB or 11-D supergravity. CMA-ES has been able to find some Minkowski and Anti de Sitter vacua in 7-D, whose presence has been confirmed by analytical computations with U(1) residual gauge symmetry and the results have been presented in Chapter 4. Parallelization resulted to be fundamental in making the process faster and allowing a more global search scan. Indeed, with CMA-ES we need to choose a starting mean $\mathbf{m}^0$ and step size σ^0, toghether with some hyper-parameters such as the time allowed to carry the computations ('timeout' parameter), the minimum value accepted for the fitness function in order to declare that a minimum was achieved ('ftarget' parameter), the precision of this result ('tolfun'), the population size λ and whether or not to activate elitist research. Parallelizing gave us the opportunity to choose more initial means $\mathbf{m}^0$ at a time, thus scanning a broader area of the parameter space. Some useful "tricks" have been used in order to adapt the algorithms to our specific problem and to render the analysis of the numerical results faster.

First of all, we are dealing with systems of homogenous quadratic equations, therefore their solutions pass always from the origin, this helps us in restricting the area of research, when setting the initial mean for the multivariate normal distribution. On the other hand, we must pay attention, because the origin of the reference system is a trivial solution for any homogenous system of equations, therefore starting near to it can lead us always there. This could be avoided by modifying the fitness function. Given a system of equation $d_i = 0$, where i labels the equations of the system, the base fitness function is defined by $f_{fit} = \sum_i^l (d_i)^2$ with l being the length of the system, then we can modify it in the following sense $F_{fit} = f_{fit} / \sum_i^n x_i^2$, where x_i represent the embedding tensor parameters and i runs over them. However, we found that the following definition is more efficient:

$$F_{fit} = \begin{cases} f_{fit} & \text{for} \quad \sum_i^n x_i^2 > \text{threshold}, \\ 10000 & \text{for} \quad \sum_i^n x_i^2 \leq \text{threshold}. \end{cases} \tag{6.30}$$

Basically, we create a step function, to avoid the algorithm to converge always in $\mathbf{0}$. The threshold must be chosen in such a way as to leave enough parameter space near the origin to complete a full scan without hitting the barrier too often. A technique to avoid the algorithm to return to minima already found before has also been implemented. It consisted in adding a multivariate normal distribution function centered on the minima on top of the fitness function, as illustrated in fig. 6.6 and fig. 6.7b through the level curves. Basically what one does is to add a series of umbrellas on top of the fitness function to stop the algorithm from going in those directions already analised.

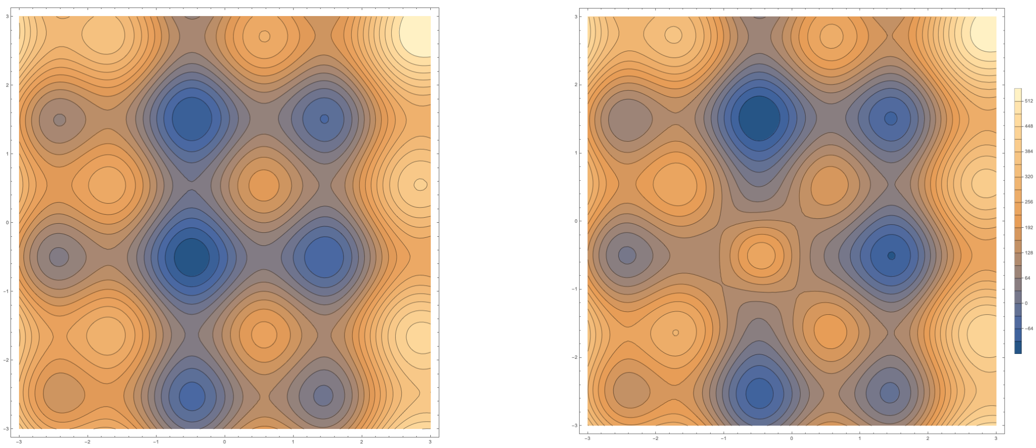


(a) Example of fitness function F_{fit} without modification

(b) Example of fitness function with a Gaussian modification

Fig. 6.6 Modification of the fitness function after a minimum has been found by the algorithm.

In our case though, the solutions to the systems are manifold in the parameter space, so an infinte amount of umbrellas would be needed to prevent the algorithm to find again the vacuum structure under consideration. Just imagine to have a straight line in 2-dimensions and start covering it with 2-dimensional multivariate normal distributions.



(a) level curves for a fitness function F_{fit} without modification

(b) level curves for a fitness function with a Gaussian modification

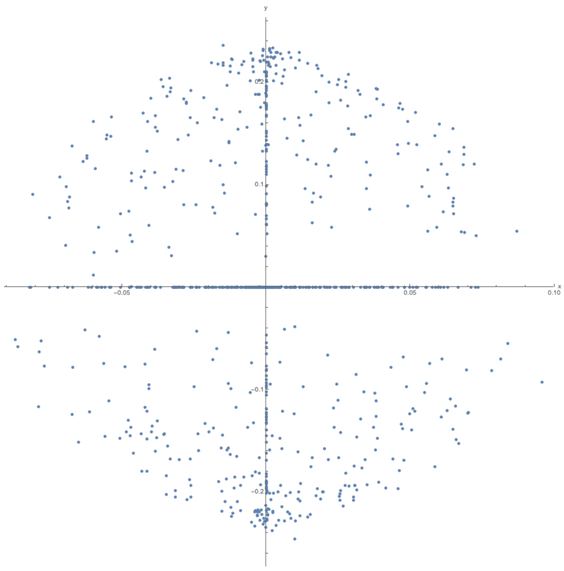
Fig. 6.7 Modification of the fitness function after a minimum has been found by the algorithm, illustrated trough level curves.

In 5 and 7 spacetime dimensions, we were interested, above all, in (Anti)-de-Sitter vacua, this information can be used to simplify further the numerical search. Indeed, taking into consideration the potentials in 3.44, 4.1 and 4.29 written in terms of the fermion shifts 3.33 and 3.79 for 5 and 7 space-time dimensions respectively, we can see that potentials are always written as the difference between 2 squared terms. For AdS we want to impose $V = -k$, with k constant, we can always normalize k to be 1, so we need to add another equation (quadratic) to our system, analogously for dS we need to solve $V = 1$. We can always make a change of variables to solve completely this new constraint, for instance, considering the case of AdS vacua in 7D, calling z_i the variables contained in $A_2^{d,abc}$ and x_i the ones in A_1^{ab}:

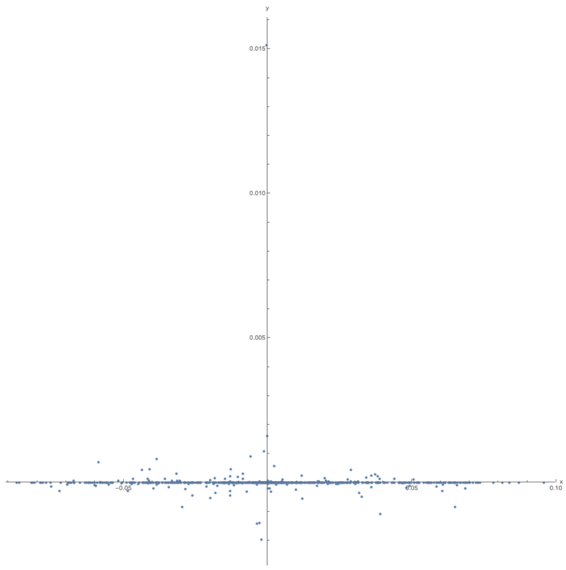
$$\begin{cases} z_i \rightarrow & \frac{\sinh[\psi] w_i}{\sqrt{\sum_j w_j^2}}, \\ x_i \rightarrow & \frac{\cosh[\psi] u_i}{\sqrt{\sum_j u_j^2}}. \end{cases} \tag{6.31}$$

This change of variables introduce one more variable, ψ, but it solves the constraint $V = -1$, thus removing a bunch of solutions from the system. By doing so we will be certain that the algorithm looks only for vacua with negative cosmological constant and the other vacua disappeared from its landscape. Analogously, we can solve the constraint $V = 1$ for dS vacua by exchanging *sinh* with *cosh* in eq. 6.31. For Minkowski vacua we just add one more homogeneous quadratic equation ($V = 0$) to the system. The initial points have been chosen in the following way, for what concerns the variables w_i and u_i in 6.31, we draw them from normal distribution with mean $\mathbf{0}$ and a standard deviation $\sigma = 4$, instead, the starting points for *psi* have been displaced evenly in an interval from 0.1 to 0.5 (note that the initial variables scale exponentially with ψ so there is no need to reach high value of the latter). For the more complicated cases, with huge systems and a lot of unkowns, we first evaluated the fitness function on these starting points and already selected the best ones, setting an acceptance threshold (this has been done after a careful analysis of the fitness function at hand and noting that points who started with huge fitness function never reached an acceptable vacuum). After GA or CMA has been implemented with all these modification, we proceeded to the analysis of the results.
First of all, we removed the solution points which are near to each other, up to a certain threshold which we set to be equal to 3σ, thus 3 times the step size, removing all the candidate solutions corresponding to vacua already present in the set of solutions. Then, we studied the residual amount of supersymmetry, the gravitini masses, the signature of the Cartan matrix (providing the information about the number of compact and non compact generators of the gauge group of the theory) and the rank of ${X_{PQ}}^R$ giving us the dimension of the gauge group. With all these information, it is possible to group the solution points and extract some very useful relations among them. Indeed, by plotting a variable, x_i or z_i, against all the other has been possible to find some analytical relations, as it is shown in fig. 6.8.

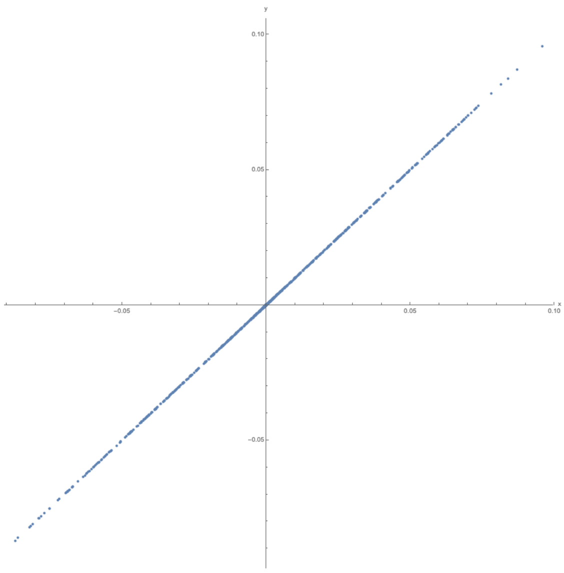
Exploiting this relations, even by mean of linear fitting, it is possible to reduce the number of variables, and in cases where the starting number of unknown is small enough we can find an analytical solution. Indeed, the results found trough CMA, which showed at least an U(1)



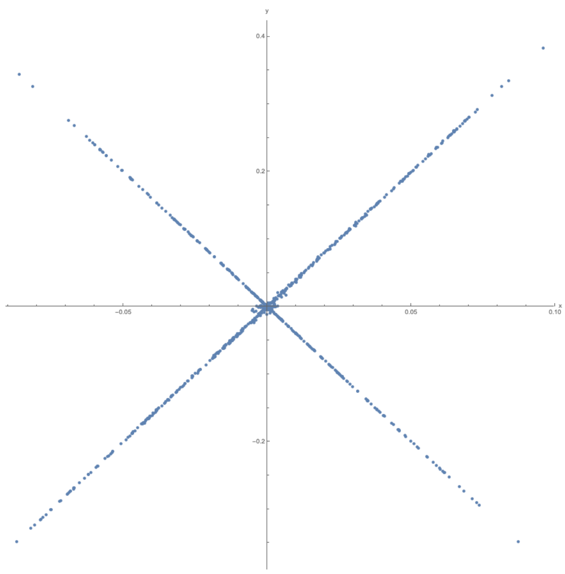
(a) Uncorrelated Variables



(b) No dependance among the variables



(c) variables linearly related



(d) Galois couple

Fig. 6.8 Correlation among the variables gaugings, have been confirmed by the analytical methods described previously and the result are presented in Chapter 4 and Appendix C.

6.4 Summary

In this chapter we presented a number of numerical optimization algoritms, extensively used in the Artificial Intelligence community, especially applied to Machine Learning techniques such as backpropagation. Stochastic Gradient Descent computes the gradient for each equation in the system and update the candidate solution point in the opposite directions, Genetic Algorithms evolve a population based on the principle of the survival of the fittest to find global minima for the fit function and CMA-ES adapt the covariance matrix to the level curves of the fitness function. For our problem CMA-ES resulted to be the most promising optimization algorithm and has been developed further. We provided also some techniques to ease the numerical computations and to adapt these to our problme,in addition we described some methods to analize the results and obtain analytical results from them.

Chapter 7

Conclusion

Since the discovery of supergravity [145], and its extended versions, it has always been interesting to investigate and analyse the structure of the scalar potential and its minima. There are several reasons for this study, spanning from phenomenology to cosmology, and more came out after the formulation of the AdS/CFT conjecture. Due to the great interest around this subject, many techniques have been developed along the years to compute and examine the vacua of those theories. Particular attention has been given from the scientific community to minima of maximally supersimmetric supergravities, the rationale behind it being the fact that the field content and the Lagrangian are completely specified by supersymmetry in these cases.

This work moves in this frame and further develops the community's endeavour aimed at better understanding the structures of supergravities and string theories and relating the latter to cosmological and phenomenological observations. In particular, maximally supersymmetric supergravities have been scrutinised in 5 and 7 spacetime dimensions, led by the idea of relating our results to conformal field theories in 4 and 6 dimensions, respectively. In order to accomplish this highly complicated task, involving NP problems, several methods have been adopted throughout the work, in particular analytic techniques, first adopted in [49], allowed for the scan of all the gaugings and of the whole space of theories in a simple way. We have deepened this analysis, employing algorithms used in the realm of post-quantum cryptography (Appendix D), therefore succeeding in pushing the analytical methods used for scrutinising the vacua of supergravity theories further than what was previously possible. We went even further and adopted a different perspective: instead of solving the system of quadratic equations, we tried a direct minimisation of certain fitness functions related to the system of quadratic constraints. Implementing and developing numerical optimisation algorithms used nowadays, especially in the Artificial Intelligence universe, we have been able to analyse deeply the parameter space for maximal gauged supergravity theories. In certain special cases, we also reconstructed or obtained the analytical solutions from the numerical results.

Many aspects and ideas remain to be further developed; in particular, it would be interesting to study the possible existence of “terminal theories”, namely theories that represent terminal points for the RG flow such that any relevant deformation cannot be added to the CFT. They are argued not to exist, thus, exploring this hypothesis from the gravity side in the case of supersymmetric theories results of uttermost relevance. In addition, analysing the behaviour of the cosmological constant for different space-time dimensions and different amounts of supersymmetry is of interest to understand the behaviour of the cosmological constant and to provide a better grasp on the cosmological constant problem [2]. Besides, very much attention has been given in the last few years to de Sitter solutions of string theories and supergravity theories thanks to the de Sitter conjecture [31], consequently, understending the behaviour of dS spacetimes in various dimensions in supergravity is a priority. In the end, knowing the structure of supergravities and string theory’s compactifications and truncations is an inherently interesting aspect of the theory that needs further clarifications. Many techniques and mathematical tools have been developed along the decades to tackle this problem and hopefully many more will come in the future. There is plenty of work around supergravity vacua at the horizon, and the way is not always paved.

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Appendix A

Dimensional Reduction

Kaluza–Klein reduction

First of all, we observe that all fields in (D+1)-dimensional space, with 1 dimension curled on itself, satisfy the cylinder condition

$$\hat{\Phi}(x^{\mu}, y + 2\pi R_y) = \hat{\Phi}(x^{\mu}, y),$$

where, R_y is the radius of the compactifying circle. Therefore we can Fourier-expand them in terms of eigenfunctions on the circle, namely

$$\hat{\Phi}(x^{\mu}, y) = \sum_n \Phi_n(x^{\mu}) e^{iny/R_y}.$$

Once we insert this ansatz into the (D+1)-dimensional Klein–Gordon equation, we get

$$\hat{\Box}\hat{\Phi}(x^{\mu}, u) = 0 \qquad \Rightarrow \qquad [\Box + \partial_y \partial^y]\hat{\Phi} \equiv \left[\Box - \left(\frac{n}{R_y}\right)\right] \Phi_n(x^{\mu}) = 0.$$

Therefore, an infinite tower of massive states appears in D-dimensions, called Kaluza–Klein modes. In the limit of very small compactification radius, the massive modes decouple from the theory, and only the massless mode contributes to the effective field theory. Keeping this in mind, it is possible to see what fields and simmetries arise when we include the vielbein and p-form fields in the higher-dimensional theory. Let us first start with the reduction of pure gravity from (D+N)-dimensions down to D-dimensions. The coordinates split as $x^M \to (x^{\mu}, y^m)$, $\mu = 0, ..., D-1$, $m = 1, ..., N$, analogously for flat indices $A \to (\alpha, a)$. On the N-torus the vielbein takes the form:

$$E_M{}^A = \begin{pmatrix} e^{\gamma\phi} e_\mu{}^\alpha & e^{\phi/N} V_m{}^a B_\mu^m \\ 0 & e^{\phi/N} V_m{}^a \end{pmatrix} \tag{A.1}$$

All components depend only on x^μ. The matrix $V_m{}^a$ is normalised by det V = 1 and $\gamma = \frac{1}{2-D}$ is chosen such that the lower dimensional action is in Eisntein frame. The field content of the D-dimensional theory is therefore: a vielbein, N vector fields, and N^2 scalar fields. The ansatz A.1 breaks the SO(1, D+N-1) Lorentz symmetry to $\mathrm{SO}(1, D-1) \times \mathrm{SO}(N)$. The second factor can be used to remove part of the scalar fields in $V_m{}^a$, leaving $\frac{1}{2}N(N+1)$ physical scalars in the reduced theory. Let us now see how the (D+N)-dimensional diffeomorphisms act on these fields.

$$\delta E_M{}^A = \xi^N \partial_N E_M{}^A + \partial_M \xi^N E_N{}^A. \tag{A.2}$$

Diffeomorphism with indices along the D-directions $\xi^\mu(x)$ induce D-dimensional diffeomorphisms. While, diffeomorphisms with indices along the internal directions $\xi^m(x)$ induce abelian gauge transformations for the Kaluza–Klein vector fields:

$$\delta B_\mu^m = \partial_\mu \xi^m.$$

There are some other diffeomorphisms that have to be considered. Such as linear diffeomorphisms in the N compactified coordinates, $\xi^m = -\Lambda^m{}_n y^n$, with $\Lambda^m{}_n$ traceless. These do not introduce any y-dependance on the fields of the D-dimensional theory, so are perfectly reasonable. They induce a global SL(N) symmetry acting as

$$\delta_\Lambda V_m{}^a = \Lambda^n{}_m V_n{}^a, \qquad \delta_\Lambda B_\mu^m = -\Lambda^m{}_n B_\mu^n.$$

However, diffeomorphisms corresponding to constant rescaling of the N-Torus $\xi^m = \lambda y^m$ are peculiar. Because they induce an action on the D-dimensional vielbein, they do not constitute an off-shell symmetry in D dimensions. However, combined with a proper rescaling of the (D+N)-dimensional vielbein (the trombone symmetry), they result in an off-shell symmetry:

$$\delta_\lambda \phi = \lambda N(D-2)\phi, \qquad \delta_\lambda B_\mu^m = -\lambda(D-2+N) B_\mu^m,$$

which leaves $e_\mu{}^\alpha$ invariant. Thus, we can see that the scalars live on a coset manifold GL(N)/SO(N).

Reductions of supergravity theories do not end here. Indeed, extended supergravities have larger symmetry groups where GL(N) is embedded as a subgroup. These larger groups are due to the presence of p-forms present in the higher-dimensional theory.

$$A_{M_1\ldots M_p} \to (A_{\mu_1\ldots\mu_p}, A_{m_1\mu_2\ldots\mu_p}, A_{m_1 m_2\mu_3\ldots\mu_p}, \ldots, A_{m_1\ldots m_p})$$

Giving rise to D-dimensional p-forms, (p-1)-forms, (p-2)-forms, etc. Their transformations under SL(N) follow from the index structure of their internal indices. For what concerns the GL(1) scaling transformations, it is possible to show [146] that:

$$A_{m_1\ldots m_k\mu_{k+1}\ldots\mu_p} = \lambda\,((D-2)k + (k-p)N)\,A_{m_1\ldots m_k\mu_{k+1}\ldots\mu_p}.$$

Whenever $N \geq p$ the reduction introduces $\binom{N}{p}$ scalar fields $A_{m_1\ldots m_p}$ in the D-dimensional theory. So the question arises naturally, what happens to the (D+N)-dimensional gauge transformations $\delta A_{M_1\ldots M_p} = p\partial_{[M_1}\Xi_{M_2\ldots M_p]}$? When we consider the transformations that are linear in the compactified coordinates $\Xi_{m_2\ldots m_p} = \xi_{m_1\ldots m_p} y^{m_1}$, new global shift symmetries arise

$$\delta_\xi A_{m_1\ldots m_p} = \xi_{m_1\ldots m_p}.$$

These are not the only possible scalar fields that arise in a dimensional reduction of a p-form. Indeed, it is well known that (D-2)-forms can be dualised into scalar fields. These scalar fields also have a global shift symmetry $\delta_\chi \phi_a = \chi_a$. All these symmetries form a non-semisimple group of the type $GL(N) \ltimes \mathcal{N}$ with nilpotent $\mathcal{N}$ corresponding to the shifts. In addition, the D-dimensional theory has dim$\mathcal{N}$ additional hidden symmetries, with no obvious higher-dimensional origin. These symmetries, together with $GL(N) \ltimes \mathcal{N}$, form the semi-simple group G. The fact that the number of additional hidden symmetries is precisely enough in order to form a semi-simple global symmetry group in D dimensions of course heavily hinges on the field content of the higher-dimensional supergravity theory. Here, the underlying supersymmetric structure that is preserved throughout the reduction plays its role. It is also possible to show that the components of the flux parameters also transform under the GL(1) scaling symmetry as

$$\delta_\lambda \mathcal{F}_{m_1\ldots m_p} = \lambda\,((D-2)p + N)\,\mathcal{F}_{m_1\ldots m_p},$$

for the p-form field strengths with all indices along the compactified directions. Another possible deformation of a compactification is the torsion of the internal torus, namely, a vielbein ansatz made by $E^a = \tilde{E}_m{}^a(x)\eta^m(y)$, with the one-form $\eta^m(y)$ satisfying $d\eta^k = T^k_{mn}\eta^m \wedge \eta^n$, where T^k_{mn} is known as geometric flux. Also, the geometric flux transforms under GL(1) as:

$$\delta_\lambda T^k_{mn} = \lambda(D+N-2)T^k_{mn}.$$

Scherk–Schwarz reduction

Scherk–Schwarz is basically different from Kaluza-Klein reduction in the dependence of the fields on the compactification coordinates. This dependence is not arbitrary but takes a particular form that depends on the symmetry of the higher-dimensional theory. Consider, for example, expanding a (D+1)-dimensional complex scalar field S with the Fourier basis on the circle

$$S(x,y) = e^{imy}\sum_{-\infty}^{\infty} e^{2\pi iny/L},$$

with L the length of the circle. The field, as it is, is not single valued in (D+1)-dimensions, since $S(x,y+L) = e^{imL}S(x,y)$. However, this is acceptable whenever S is part of a theory possessing the phase transformation $S \to e^{i\alpha}S$ as a global symmetry. Free theory has this property. Thus, the multivalued field represents a non-trivial fibre bundle, which is effectively continuous around the circle. In the limit as $L \to 0$, in the effective theory we can ignore all the fields S_n and keep only S_0 which is a D-dimensional field of mass m (just insert this in the (D+1)-dimensional Klein-Gordon equation). So, we see immediately that we generate mass terms with this method of dimensional reduction. If this method is applied to some, but not all the fields in a supermultiplet, supersymmetry gets broken. Now consider a supergravity theory in D + E dimensions. If one can exploit symmetries of the theory along the lines described above so as to generate masses for the gravitinos but not for the graviton, supersymmetry in D dimensions will be broken. The breaking is guaranteed to be spontaneous, because the local supersymmetry transformations in D + E dimensions can be transcribed in D dimensions as part of the dimensional reduction. As an example, we can compactify pure $N = 1$ supergravity from 4 to 3 dimensions. The Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4\kappa^2}ER - \frac{1}{2}\varepsilon^{\lambda\rho\mu\nu}\Psi_\lambda\Gamma_5\Gamma_\mu D_\nu\Psi_\rho.$$

The symmetries of this theory are: local supersymmetry, local Lorentz invariance, and general coordinate transformations. Infinitesimally, they are:

$$\delta E_M{}^A = -i\kappa\bar{\varepsilon}\Gamma^A\Psi_M + \lambda^A{}_B E_M{}^B + \partial_M\xi^N E_N{}^A + \xi^N\partial_N E_M{}^A, \tag{A.3}$$

$$\delta\Psi_M = \frac{1}{\kappa}D_M\varepsilon + \frac{1}{2}\lambda^{AB}\Sigma_{AB}\Psi_M + \partial_M\xi^N\Psi_N + \xi^N\partial_N\Psi_M \tag{A.4}$$

In addition, there is also the global chiral symmetry: $\Psi_M \to e^{ia\Gamma_5}\Psi_M$. Now, $x^\mu = x^0, x^1, x^2$ and $y = x^3$. Choosing the following y dependencies

$$\Psi_M(x,y) = e^{im\Gamma_5 y}\Psi_M(x), \tag{A.5}$$

$$E_M{}^A(x,y) = E_M{}^A(x), \tag{A.6}$$

the theory in 3 dimensions will contain mass terms for the gravitions while leaving the vielbein massless. It is appropriate to decompose the 4-component spinors of four dimensions into 2-component spinors of three dimensions. Thus,

$$\begin{aligned}
\Gamma^\mu &= \gamma^\mu \otimes \tau_3, \qquad \mu = 0,1,2,\\
\Gamma^3 &= \mathbb{1} \otimes i\tau_2,\\
\Gamma^5 &= \mathbb{1} \otimes \tau_1,
\end{aligned}$$

where γ^μ are 2x2 Dirac matrices for $D = 3$ (for example, $\gamma^0 = \sigma^3$, $\gamma^1 = i\sigma^1$, and $\gamma^2 = i\sigma^2$). A Majorana spinor Ψ in four dimensions takes the form $\binom{\psi^1}{i\psi^2}$, where ψ^1 and ψ^2 are 2-components Majorana spinors in D=3. Using local Lorentz transformations, it is possible to put the vielbein in the following gauge:

$$E_M{}^A = \begin{pmatrix} e_\mu{}^\alpha & 2\kappa A_\mu \phi \\ 0 & \phi \end{pmatrix}.$$

This implies $\delta E_3{}^A = 0$ for $A = 0,1,2$, which using A.3 requires $\lambda^A{}_3 = i\kappa\phi^{-1}\bar{\varepsilon}\Gamma^A\Psi_3$, with $A = 0,1,2$. Supersymmetry transformation for the dreibein then becomes:

$$\delta e_\mu{}^\alpha = -i\kappa\bar{\varepsilon}\Gamma^\alpha\Psi_\mu + \lambda^\alpha{}_3 2\kappa A_\mu\phi = -i\kappa\bar{\varepsilon}\Gamma^\alpha\Psi'_\mu,$$

where we introduced $\Psi'_\mu \equiv \Psi_\mu - 2\kappa A_\mu\psi$ and $\Psi \equiv \Psi_3$. Now, to keep $e_\mu{}^\alpha$ independent of y, it is necessary to take $\varepsilon(x,y) = e^{im\Gamma_5 y}\varepsilon(x)$.

Substituting the y-dependence of A.5 and A.6 into a general coordinate transformation with parameter $\xi^3 \equiv \xi$ yields

$$\begin{aligned}
\delta A_\mu &= \frac{1}{2\kappa}\partial_\mu\xi,\\
\delta e_\mu{}^\alpha &= \delta\phi = 0,\\
\delta\Psi'_\mu &= im\Gamma_5\xi\Psi'_\mu,\\
\delta\psi &= im\Gamma_5\xi\Psi.
\end{aligned}$$

We can see that A_μ is the gauge potential of an $U(1)$ symmetry in D=3, and the charge is proportional to the mass parameter. Indeed, putting everything together in the 4-dimensional Lagrangian and dropping primes for Ψ_μ one gets the N=2, D=3 Lagrangian:

$$
\begin{aligned}
\mathcal{L} = & -\frac{1}{4\kappa^2}\phi eR - \frac{1}{4}\phi^3 eF_{\rho\lambda}F^{\rho\lambda} - \frac{1}{2}\phi\varepsilon^{\mu\nu\rho}\overline{\psi}^i_\mu (D_\nu\psi_\rho)^i + me\overline{\psi}^i_\mu\sigma^{\mu\nu}\psi^i_\nu \\
& + 2e\varepsilon^{ij}\overline{\psi}^i\sigma^{\nu\rho}(D_\nu\psi_\rho)^j - \frac{1}{4}i\kappa\phi^2\varepsilon^{ij}\varepsilon^{\mu\nu\rho}\overline{\psi}^i_\mu\gamma^\sigma\psi^i_\rho F_{\nu\sigma} - e\frac{1}{2}i\kappa\phi F_{\mu\nu}\overline{\psi}^i\gamma^\mu\psi^{\nu i} \\
& + e\frac{1}{2}i\phi\overline{\psi}^i_\mu\gamma^\mu\psi^{i\rho}\partial_\rho ln\phi + \mathcal{L}_4,
\end{aligned}
$$

where $\mathcal{L}_4$ represents terms quartic in fermions and

$$
(D_\nu\psi_\rho)^i = \left(\partial_\nu + \frac{1}{2}\omega^0_{\nu rs}\sigma^{rs}\right)\psi^i_\rho + 2\kappa m A_\nu \varepsilon^{ij}\psi^j_\rho.
$$

Inserting the same substitution in the supersymmetry transformation laws gives:

$$
\delta e_\mu{}^\alpha = -i\kappa\overline{\varepsilon}^i\gamma^\alpha\psi^i_\mu, \tag{A.7}
$$

$$
\delta\psi^i_\mu = \frac{1}{k}(D_\mu\varepsilon)^i - \frac{1}{4}i\varepsilon^{ij}\phi\gamma^\nu\varepsilon^j F_{\mu\nu} + O(\psi^2\varepsilon), \tag{A.8}
$$

$$
\delta A_\mu = \phi^{-1}\varepsilon^{ij}\overline{\varepsilon}^i\psi^j_\mu - i\phi^{-2}\overline{\varepsilon}^i\gamma_\mu\psi^i \tag{A.9}
$$

$$
\delta\psi^i = -\frac{m}{\kappa}\varepsilon^{ij}\varepsilon^j + \frac{1}{4}\phi^2 F_{\mu\nu}\sigma^{\mu\nu}\varepsilon^i + \frac{1}{2\kappa}\varepsilon^{ij}\gamma\cdot\partial\phi\varepsilon^j + O(\psi^2\varepsilon), \tag{A.10}
$$

$$
\delta\phi = \kappa\varepsilon^{ij}\overline{\varepsilon}^i\psi^j. \tag{A.11}
$$

The variation of the gravitinos contains a constant in the coefficient of ε which means that ψ^i are two Goldston spinors. Therefore, it is possible to use supersymmetry invariance to set $\psi^i = 0$. Doing this, and letting $\psi^i_\mu \to \phi^{-1/2}\psi^i_\mu$, $e_\mu{}^\alpha \to \phi^{-1}e_\mu{}^\alpha$, and $\phi = e^{\kappa\sigma}$, one finds (putting back the quartic terms)

$$
\begin{aligned}
\mathcal{L} = & -\frac{1}{4\kappa^2}eR + \frac{1}{2}e\partial^\mu\sigma\partial_\mu\sigma - \frac{1}{2}\varepsilon^{\mu\nu\rho}\overline{\psi}^i_\mu(D_\nu\psi_\rho)^i \\
& - \frac{1}{4}ee^{4\kappa\sigma}F_{\mu\nu}F^{\mu\nu} + eme^{-2\kappa\sigma}\overline{\psi}^i_\mu\sigma^{\mu\nu}\psi^i_\nu - \frac{1}{4}i\kappa\varepsilon^{ij}\varepsilon^{\mu\nu\rho}e^{2\kappa\sigma}\overline{\psi}^i_\mu\gamma^\sigma\psi^j_\rho F_{\nu\rho}.
\end{aligned}
$$

It should be noted that a shift of the σ field and scaling of A_μ correspond to a rescaling of the mass m, thus only the combination $m(e^{-2\kappa\sigma_0})$ where σ_0 is the vacuum expectation value of σ, has physical significance.

Scherk and Schwarz Reduction with Difeomorphisms

The Scherk and Schwarz criterion for dimensional reduction can also be applied to general coordinate transformations. We will show its application to Einstein's theory. The algebra for the general coordinate transformations, parametrised in the case of Einstein theory by $\xi^{\hat{\mu}}(x,y)$, is given by

$$[\delta_{\xi_1}, \delta_{\xi_2}] = \delta_{\xi_3}, \tag{A.12}$$

with

$$\xi_3^{\hat{\mu}}(x,y) = \xi_2^{\hat{\sigma}}(x,y)\partial_{\hat{\sigma}}\xi_1^{\hat{\mu}}(x,y) - \xi_1^{\hat{\sigma}}(x,y)\partial_{\hat{\sigma}}\xi_2^{\hat{\mu}}(x,y). \tag{A.13}$$

The reduction works by imposing

$$\begin{aligned} \xi^{\mu}(x,y) &= \xi^{\mu}(x), \\ \xi^{\alpha}(x,y) &= [U^{-1}(y)]^{\alpha}_{\beta}\xi^{\beta}(x), \end{aligned} \tag{A.14}$$

in the limit in which $U \to \mathbb{1}$ one goes back to the usual Kaluza-Klein reduction. The algebra of external diffeomorphisms is then the usual algebra for general coordinate transformations. While, the commutator of an external diffeomorphisms with an internal one produces a new internal transformation with parameter

$$\xi_3^{\alpha} = -\xi_1^{\sigma}(x)\partial_{\sigma}\xi_2^{\alpha}(x), \tag{A.15}$$

from this relation one observes that the parameters for internal transformations are space-time scalars. The algebra of two internal transformations, on the other hand, gives an internal transformation with parameter

$$\xi_3^{\gamma}(x) = f_{\alpha\beta}{}^{\gamma}\xi_1^{\alpha}(x)\xi_2^{\beta}(x), \tag{A.16}$$

with

$$f_{\alpha\beta}{}^{\gamma} = (U^{-1})_{\alpha}{}^{\alpha'}(U^{-1})_{\beta}{}^{\beta'}(\partial_{\beta'}U_{\alpha'}{}^{\gamma} - \partial_{\alpha'}U_{\beta'}{}^{\gamma}). \tag{A.17}$$

Of course, we want $f_{\alpha\beta}{}^{\gamma}$ to be constant, so that A.16 identifies a new internal parameter that depends only on x. Choosing the y-coordinates to be a system of coordinates on the manifold of a Lie group G, we can describe the generators of the group in terms of differential operators

$$L_{\alpha}(y) = (U^{-1})_{\alpha}{}^{\beta}\partial_{\beta}. \tag{A.18}$$

Then, the second theorem by Lie guarantees that if

$$[L_\alpha, L_\beta] = f_{\alpha\beta}{}^{\gamma} L_\gamma, \tag{A.19}$$

the structure constants $f_{\alpha\beta}{}^{\gamma}$ are constant and given by A.17. Therefore, what happened here is that from the internal diffeomorphisms we have been able to extract a Lie-algebra with the same dimensions of the internal space. Obviously, we need to apply the same factorisation technique we applied to the gauge parameters to the other fields in the theory. The general rule is that each lower internal Greek index gets a factor of $U(y)$, while upper internal Greek indices are accompanied by a $U^{-1}(y)$ factor. Thus, the field content of Einstein's theory, contained in the $D+n$ vielbein in a triangular parametrisation

$$E_{\hat{\mu}}{}^{\hat{\alpha}} = \begin{pmatrix} \delta^{\gamma} E_{\mu}{}^{s} & 2\kappa A^{\alpha}_{\mu} \Phi^{a}_{\alpha} \\ 0 & \Phi^{a}_{\alpha} \end{pmatrix}, \tag{A.20}$$

where $\delta = Det(\Phi^a_\alpha(x))$, can be written as

$$\begin{aligned} E^s_\mu(x,y) &= E^s_\mu(x), \\ A^\alpha_\mu(x,y) &= [U^{-1}(y)]^\alpha_\beta A^\beta_\mu(x), \\ \Phi^a_\alpha(x,y) &= U^\beta_\alpha(y)\Phi^a_\beta(x). \end{aligned} \tag{A.21}$$

By doing so, the degrees of freedom do not change, and whenever 2 internal indices are contracted among them, without derivatives, their y-dependance cancels. Recalling the vielbein's behaviour under general coordinate transformations

$$\delta E_{\hat{\mu}}{}^{\hat{\alpha}} = \xi^{\hat{\sigma}} \partial_{\hat{\sigma}} E^{\hat{\alpha}}_{\hat{\mu}} + E^{\hat{\alpha}}_{\hat{\sigma}} \partial_{\hat{\mu}} \xi^{\hat{\sigma}}, \tag{A.22}$$

one obtains the gauge transformations for the various fields:

$$\delta\Phi^a_\alpha(x) = f_{\alpha\beta}{}^{\gamma} \xi^\beta(x) \Phi^a_\gamma(x), \tag{A.23}$$

$$\delta\Phi^\alpha_a(x) = f_{\beta\gamma}{}^{\alpha} \xi^\beta(x) \Phi^\gamma_a(x), \tag{A.24}$$

$$\delta A^\alpha_\mu(x) = \frac{1}{2\kappa} \partial_\mu \xi^\alpha(x) + f_{\beta\gamma}{}^{\alpha} \xi^\beta(x) A^\gamma_\mu(x), \tag{A.25}$$

$$\delta E_\mu{}^s = 0. \tag{A.26}$$

It is immediately possible to see that $A_\mu{}^\alpha$ plays the role of a gauge potential for the Lie group G that has $f_{\alpha\beta}{}^{\gamma}$ as structure constants. The Hilbert-Einstein action is invariant under internal diffeomorphisms only if

$$\partial_\beta[(U^{-1})^\beta_\alpha U]=0, \qquad \text{with} \qquad U(y)=Det(U_\alpha{}^\beta(y)). \tag{A.27}$$

This equation, trnaslate into $f_{\alpha\beta}{}^\beta=0$, which is satisfied by all semi-simple Lie Algebras (as well as other cases), but it must not be imposed if we want to include the Trombone symmetry in our gaugings. Indeed, the Trombone is only a symmetry of the equations of motion in any space-time dimension different from 2, and so does not obey this condition on the trace of the structure constants.
It is possible to compute the D+n spin connections, obtaining

$$\begin{aligned}
\hat{\omega}_{p,qr} =&\delta^{-\gamma}[\omega_{p,qr}+\gamma(\eta_{pq}E^\sigma_r-\eta_{pr}E^\sigma_q)\partial_\sigma ln(\delta)],\\
\hat{\omega}_{p,qa} =&\kappa\delta^{-2\gamma}F^\alpha_{pq}\Phi_{\alpha a},\\
\hat{\omega}_{p,ab} =&\frac{1}{2}\delta^{-\gamma}\Phi^\alpha_a E_p{}^\mu\mathcal{D}_\mu\Phi_{\alpha b}-\frac{1}{2}\delta^{-\gamma}\Phi^\alpha_b E_p{}^\mu\mathcal{D}_\mu\Phi_{\alpha a},\\
\hat{\omega}_{a,pq} =&-\kappa\delta^{-2\gamma}F^\alpha_{pq}\Phi_{\alpha a},\\
\hat{\omega}_{a,pb} =&\frac{1}{2}\delta^{-\gamma}\Phi^\alpha_b\Phi^\beta_a E_p{}^\sigma\mathcal{D}_\sigma h_{\alpha\beta},\\
\hat{\omega}_{a,bc} =&\frac{1}{2}f_{\alpha\beta}{}^\gamma(\Phi^\alpha_b\Phi^\beta_c\Phi_{\gamma a}+\Phi^\alpha_b\Phi^\beta_a\Phi_{\gamma c}-\Phi^\alpha_c\Phi^\beta_a\Phi_{\gamma b}).
\end{aligned} \tag{A.28}$$

with, $h_{\alpha\beta}=\Phi^a_\alpha\delta_{ab}\Phi^b_\beta$, and

$$\begin{aligned}
F^\alpha_{\mu\nu} =&\partial_\mu A^\alpha_\nu-\partial_\nu A^\alpha_\mu-2\kappa f_{\beta\gamma}{}^\alpha A^\beta_\mu A^\gamma_\nu,\\
\mathcal{D}_\mu\Phi^a_\alpha =&\partial_\mu\Phi^a_\alpha-2\kappa f_{\alpha\beta}{}^\gamma A^\beta_\mu\Phi^a_\gamma,\\
\mathcal{D}_\mu\Phi^\alpha_a =&\partial_\mu\Phi^\alpha_a-2\kappa f_{\beta\gamma}{}^\alpha A^\beta_\mu\Phi^\gamma_a,
\end{aligned} \tag{A.29}$$

such that $F^\alpha_{\mu\nu}$ is the gauge covariant field strength and $\mathcal{D}_\mu$ is the gauge covariant derivative. We can see that the y-dependence vanished from the spin connections in A.28, so that the D+n Ricci scalar $\hat{R}$ is y-indpendent. Inserting everything in the Hilbert-Einstein action and integrating on the internal manifold, one gets

$$\begin{aligned}
S=\int d^DxE\Big\{&-\frac{1}{4\kappa^2}R-\frac{1}{4}\delta^{2/(D-2)}F^{\mu\nu\alpha}F^\beta_{\mu\nu}h_{\alpha\beta}-\frac{1}{16\kappa^2}g^{\mu\nu}\mathcal{D}_\mu h_{\alpha\beta}\mathcal{D}_\nu h^{\alpha\beta}\\
&+\frac{1}{4\kappa^2(D-2)}g^{\mu\nu}\partial_\mu ln(\delta)\partial_\nu ln(\delta)-V(h_{\alpha\beta},f_{\alpha\beta}{}^\gamma)\Big\}.
\end{aligned} \tag{A.30}$$

The scalar potential, which depends on the choice of the gauge group, is given by

$$\frac{1}{16\kappa^2}\delta^{-2/(D-2)} f_{\alpha\beta}{}^{\gamma}[2f_{\gamma\lambda}{}^{\alpha}h^{\beta\lambda}+f_{\lambda\rho}{}^{\sigma}h_{\gamma\sigma}h^{\alpha\lambda}h^{\beta\rho}]. \tag{A.31}$$

Imposing that the potential is bounded from below, without the addition of a cosmological constant (in 11 and 10 dimensional supergravities there is no such a term), one restricts the choice of the gauge group from semi-simple Lie group to"flat-group", more information about this and about the reduction procedure in presence of matter fields can be found at [63, 64].

Appendix B

Fermion Shifts at the Vacuum in the 5 dimensional Theory

In this appendix we provide an instance of the value of the fermion shifts generating the vacua of table 4.7. For all examples, we have chosen a basis where either

$$\Omega = \mathbb{1}_4 \otimes i\,\sigma_2 \tag{B.1}$$

or

$$\Omega = i\,\sigma_2 \otimes \mathbb{1}_4. \tag{B.2}$$

A1. In the basis with Ω as in (B.1), the maximal AdS supersymmetric vacuum is easily obtained by setting

$$A_{16} = A_{38} = -A_{25} = -A_{47} = g, \quad A_{i,jkl} = 0. \tag{B.3}$$

A2. In the basis with Ω as in (B.1), the non-supersymmetric SO(5) AdS vacuum follows from choosing

$$A_{16} = A_{38} = -A_{25} = -A_{47} = g, \tag{B.4}$$

and

$$A_{1162} = A_{2251} = A_{3384} = A_{4473} = A_{5562} = A_{6651} = A_{7784} = A_{8873} = \frac{g}{4}, \tag{B.5}$$

$$A_{1238} = A_{1274} = A_{2183} = A_{2147} = A_{3164} = A_{3245} = A_{4136} = A_{4325}$$
$$= A_{5368} = A_{5647} = A_{6385} = A_{6457} = A_{7861} = A_{7258} = A_{8167} = A_{8275} = \frac{3}{16}g, \tag{B.6}$$

$$A_{1364} = A_{2345} = A_{3182} = A_{4127} = A_{5278} = A_{6718} = A_{7456} = A_{8365} = \frac{g}{16}, \tag{B.7}$$

$$A_{1678} = A_{2758} = A_{3568} = A_{4576} = A_{5243} = A_{6134} = A_{7214} = A_{8123} = \frac{5}{16}g. \tag{B.8}$$

A3. In the basis with Ω as in (B.2), the SU(3) invariant AdS vacuum follows from

$$A_{15} = A_{26} = A_{37} = \frac{7}{9}im_1, \quad A_{48} = -im_1, \tag{B.9}$$

and

$$A_{1256} = A_{1357} = A_{2165} = A_{2367} = A_{3517} = A_{3276}$$
$$= A_{5162} = A_{5317} = A_{6125} = A_{6273} = A_{7135} = A_{7236} = \frac{i}{9}g, \tag{B.10}$$

$$A_{1548} = A_{2648} = A_{3748} = A_{5148} = A_{6248} = A_{7348} = \frac{2}{9}ig, \tag{B.11}$$

$$A_{1234} = A_{2314} = A_{3124} = A_{5678} = A_{6587} = A_{7568} = \frac{1}{2\sqrt{3}}g, \tag{B.12}$$

$$A_{4123} = A_{8567} = \frac{\sqrt{3}}{2}g. \tag{B.13}$$

A4. In the basis with Ω as in (B.1), the $N = 2$ AdS vacuum with U(2) residual symmetry follows from

$$A_{14} = -A_{23} = \frac{7}{12}g, \quad A_{56} = -i\frac{g}{2}, \quad A_{78} = i\frac{2}{3}g, \tag{B.14}$$

and

$$A_{1124} = A_{2213} = A_{3324} = A_{4413} = -\frac{5}{24}\,g, \tag{B.15}$$

$$A_{1275} = A_{1268} = A_{2157} = A_{2186} = A_{3457} = A_{3486} = A_{4375} = A_{4368} = \frac{i}{8}\,g, \tag{B.16}$$

$$A_{1475} = A_{1486} = A_{2357} = A_{2368} = A_{3275} = A_{3286} = A_{4157} = A_{4168} = \frac{g}{8}, \tag{B.17}$$

$$A_{1456} = A_{2365} = A_{3265} = A_{4156} = \frac{g}{24}, \tag{B.18}$$

$$A_{1478} = A_{2387} = A_{3287} = A_{4178} = \frac{g}{6}, \tag{B.19}$$

$$A_{7568} = A_{8567} = -i\,\frac{g}{6}, \tag{B.20}$$

$$A_{7152} = A_{7345} = A_{8126} = A_{8436} = i\,\frac{g}{4}, \tag{B.21}$$

$$A_{7154} = A_{7235} = A_{8416} = A_{8236} = \frac{g}{4}, \tag{B.22}$$

$$A_{7128} = A_{7348} = A_{8127} = A_{8347} = i\,\frac{g}{12}. \tag{B.23}$$

A5. In the basis with Ω as in (B.1), the $N = 0$ AdS vacuum with SU(2) $\times$ U(1)^2 residual symmetry follows from

$$A_{23} = -A_{14} = \frac{1}{3}\sqrt{\frac{2}{5}}\,g, \quad A_{56} = -i\frac{g}{5}, \quad A_{77} = -A_{88} = i\,\frac{g}{5}, \tag{B.24}$$

and

$$A_{1124} = A_{2213} = A_{3324} = A_{4413} = \frac{g}{3\sqrt{10}}, \tag{B.25}$$

$$A_{1456} = A_{2365} = A_{3287} = A_{4178} = \frac{1}{12}\sqrt{1-\frac{2}{5}\sqrt{6}}\,g, \tag{B.26}$$

$$A_{1487} = A_{2378} = A_{3256} = A_{4165} = \frac{1}{12}\sqrt{1+\frac{2}{5}\sqrt{6}}\,g, \tag{B.27}$$

$$A_{5164} = A_{5236} = A_{6145} = A_{6253} = A_{7148} = A_{7283} = A_{8174} = A_{8237} = \frac{g}{4\sqrt{15}}, \tag{B.28}$$

$$A_{5621} = A_{6521} = A_{7734} = A_{8843} = i\,\frac{1}{60}\left(2+\sqrt{6}\right)g, \tag{B.29}$$

$$A_{5634} = A_{6534} = A_{7721} = A_{8812} = i\,\frac{1}{60}\left(-2+\sqrt{6}\right)g, \tag{B.30}$$

$$A_{5678} = A_{6578} = A_{7765} = A_{8856} = i\,\frac{g}{15}. \tag{B.31}$$

M1. The general CSS Minkowski vacuum in the basis with Ω as in (B.1), follows from

$$A_{11} = A_{22} = \frac{m_1}{3},\; A_{33} = A_{44} = \frac{m_2}{3},\; A_{55} = A_{66} = \frac{m_3}{3},\; A_{77} = A_{88} = \frac{m_4}{3}, \tag{B.32}$$

and

$$A_{1134} = A_{2234} = -\frac{m_1}{3}, \tag{B.33}$$

$$A_{1156} = A_{1178} = A_{2256} = A_{2278} = \frac{m_1}{6}, \tag{B.34}$$

$$A_{3312} = A_{4412} = -\frac{m_2}{3}, \tag{B.35}$$

$$A_{3356} = A_{3378} = A_{4456} = A_{4478} = \frac{m_2}{6}, \tag{B.36}$$

$$A_{5578} = A_{6678} = -\frac{m_3}{3}, \tag{B.37}$$

$$A_{5512} = A_{5534} = A_{6612} = A_{6634} = \frac{m_3}{6}, \tag{B.38}$$

$$A_{7756} = A_{8856} = -\frac{m_4}{3}, \tag{B.39}$$

$$A_{7712} = A_{7734} = A_{8812} = A_{8834} = \frac{m_4}{6}. \tag{B.40}$$

Obviously the vacua that appear in the context of our analysis have some of the masses either set to zero or proportional to each other, in order to respect the correct U(2) residual symmetry, but they are always subcases of the one presented here.

M2. The Minkowski vacuum from the SU(3,1) gauging appears in the basis with Ω as in (B.1) by choosing

$$A_{34} = i\frac{m_1}{3},\; A_{56} = i\frac{m_2}{3},\; A_{77} = A_{88} = \frac{m_3}{3} \tag{B.41}$$

and

$$A_{3124} = A_{4123} = -i\frac{m_1}{3}, \tag{B.42}$$

$$A_{3456} = A_{3478} = A_{4356} = A_{4378} = i\frac{m_1}{6}, \tag{B.43}$$

$$A_{5126} = A_{6125} = -i\frac{m_2}{3}, \tag{B.44}$$

$$A_{5346} = A_{5678} = A_{6345} = A_{6578} = i\frac{m_2}{6}, \tag{B.45}$$

$$A_{7712} = A_{8812} = -\frac{m_3}{3}, \tag{B.46}$$

$$A_{7734} = A_{7756} = A_{8834} = A_{8856} = \frac{m_3}{6}. \tag{B.47}$$

M3. The first new Minkowski vacuum we found appears in the basis with Ω as in (B.1) by choosing

$$A_{56} = -i\frac{m_2}{3},\ A_{77} = A_{88} = \frac{m_1}{3} \tag{B.48}$$

and

$$A_{5346} = A_{6345} = i\frac{m_2}{3}, \tag{B.49}$$

$$A_{5126} = A_{5678} = A_{6125} = A_{6578} = -i\frac{m_2}{6}, \tag{B.50}$$

$$A_{7734} = A_{8834} = -\frac{m_1}{3}, \tag{B.51}$$

$$A_{7712} = A_{7756} = A_{8812} = A_{8856} = \frac{m_1}{6}. \tag{B.52}$$

M4. The new non-supersymmetric Minkowski vacuum appears in the basis with Ω as in (B.1) by choosing

$$A_{23} = -A_{14} = \frac{m_1}{3},\ A_{58} = -A_{67} = \frac{m_1}{\sqrt{3}} \tag{B.53}$$

and

$$A_{1142} = A_{1478} = A_{2231} = A_{2387} = A_{3342} = A_{3287} = A_{4431} = A_{4178} = \frac{m_1}{6}, \tag{B.54}$$

$$A_{1485} = A_{1467} = A_{2358} = A_{2376} = A_{3258} = A_{3276} = A_{4185} = A_{4167} = \frac{m_1}{3\sqrt{2}}, \tag{B.55}$$

$$A_{5182} = A_{5384} = A_{6127} = A_{6347} = A_{7126} = A_{7346} = A_{8215} = A_{8435} = \frac{m_1}{4\sqrt{3}}, \tag{B.56}$$

$$A_{5146} = A_{5236} = A_{6145} = A_{6235} = A_{7148} = A_{7238} = A_{8147} = A_{8237} = \frac{m_1}{4}, \tag{B.57}$$

$$A_{5568} = A_{6657} = A_{7768} = A_{8857} = \frac{m_1}{2\sqrt{3}} \tag{B.58}$$

$$A_{5542} = A_{6631} = A_{7742} = A_{8831} = \frac{m_1}{2}. \tag{B.59}$$

M5. The new $N = 4$ Minkowski vacuum appears in the basis with Ω as in (B.1) by choosing

$$A_{56} = -i\frac{m_2}{3}, \; A_{77} = A_{88} = \frac{m_1}{3} \tag{B.60}$$

and

$$A_{5126} = A_{6125} = i\frac{m_2}{3}, \tag{B.61}$$

$$A_{5346} = A_{5678} = A_{6345} = A_{6578} = -i\frac{m_2}{6}, \tag{B.62}$$

$$A_{7712} = A_{8812} = -\frac{m_1}{12} - \frac{m_3^2}{8m_2}, \tag{B.63}$$

$$A_{7734} = A_{8834} = -\frac{m_1}{12} + \frac{m_3^2}{8m_2}, \tag{B.64}$$

$$A_{7714} = A_{7732} = A_{8814} = A_{8832} = -\frac{1}{8}\sqrt{4m_1^2 - \frac{m_3^4}{m_2^2}}, \tag{B.65}$$

$$A_{7756} = A_{8856} = \frac{m_1}{6}. \tag{B.66}$$

D1. The de Sitter vacuum associated to the SO(3,3) gauging appears in the basis with Ω as in (B.2) by choosing

$$A_{ij} = 0 \tag{B.67}$$

and

$$A_{1278} = A_{1386} = A_{1476} = A_{2187} = A_{2358} = A_{2457} = A_{3168} = A_{3285} = m, \tag{B.68}$$

$$A_{3465} = A_{4167} = A_{4275} = A_{4356} = A_{5283} = A_{5274} = A_{5346} = A_{6138} = m, \tag{B.69}$$

$$A_{6147} = A_{6354} = A_{7182} = A_{7164} = A_{7245} = A_{8127} = A_{8163} = A_{8235} = m. \tag{B.70}$$

D2. The new de Sitter vacuum associated with the SU(3,1) gauging appears in the basis with Ω as in (B.2) by choosing

$$A_{15} = A_{26} = A_{37} = 3ig, \quad A_{48} = -ig, \tag{B.71}$$

and

$$\begin{aligned}
&A_{1265} = A_{1375} = A_{2156} = A_{2376} = A_{3157} = A_{3267} \\
&= A_{5126} = A_{5137} = A_{6152} = A_{6237} = A_{7153} = A_{7263} = ig, && \text{(B.72)} \\
&A_{1458} = A_{2468} = A_{3478} = A_{5184} = A_{6284} = A_{7384} = 2ig, && \text{(B.73)} \\
&A_{1287} = A_{1368} = A_{2178} = A_{2385} = A_{3186} = A_{3258} = A_{4167} = A_{4275} = A_{4356} \\
&= A_{5247} = A_{5364} = A_{6174} = A_{6345} = A_{7146} = A_{7254} = A_{8127} = A_{8163} = A_{8235} = \frac{\sqrt{3}}{2} g, \\
&A_{1467} = A_{2475} = A_{3456} = A_{5238} = A_{6183} = A_{7128} = \frac{3\sqrt{3}}{2} g. && \text{(B.74)}
\end{aligned}$$

Appendix C

T-tensor at the Vacuum in the 7 dimensional Theory

In this appendix we provide an instance of the value of the irreducible USp(4) representations composing the T-tensor generating the vacua of table 4.22. For all the examples, we have chosen a basis where

$$\Omega = \mathbb{1} \otimes i\sigma_2. \tag{C.1}$$

A1. The maximal supersymmetric Anti de Sitter vacuum was obtained by setting

$$Bs = \kappa, \qquad B^{ab}{}_{cd} = C^{ab}{}_{cd} = C^{ab} = \mathbf{0}. \tag{C.2}$$

A2. The SO(4) non-supersymmetric AdS vacuum follows instead by the choice

$$Bs = \kappa, \qquad C^{ab}{}_{cd} = C^{ab} = \mathbf{0}, \tag{C.3}$$

$$\begin{aligned} B^{12}{}_{12} = B^{12}{}_{43} = B^{34}{}_{21} = B^{34}{}_{34} = \frac{\kappa}{6}, \\ B^{13}{}_{31} = B^{14}{}_{41} = B^{23}{}_{32} = B^{24}{}_{42} = \frac{\kappa}{12}. \end{aligned} \tag{C.4}$$

There are other non-vanishing entries in the **14** representation, which are related by symmetries in the indices and therefore have not been reported here.

M1.

$$Bs = \kappa_1, \qquad C^{12} = C^{43} = \kappa_2, \tag{C.5}$$

$$B^{12}{}_{21} = B^{12}{}_{34} = B^{34}{}_{12} = B^{34}{}_{43} = \frac{\kappa_1}{4}, \tag{C.6}$$

$$B^{13}{}_{13} = B^{24}{}_{24} = \frac{3\kappa_1}{4}, \tag{C.7}$$

$$B^{14}{}_{41} = B^{23}{}_{32} = \frac{\kappa_1}{2}, \tag{C.8}$$

$$C^{13}{}_{13} = C^{24}{}_{24} = C^{32}{}_{23} = C^{41}{}_{14} = \frac{\kappa_2}{2}. \tag{C.9}$$

Again, the other entries are related by symmetries of the representation.

M2.

$$Bs = \kappa, \qquad C^{12} = C^{43} = \frac{5}{4}\kappa, \tag{C.10}$$

$$B^{12}{}_{21} = B^{12}{}_{34} = B^{34}{}_{12} = B^{34}{}_{43} = \frac{1}{4}\kappa, \tag{C.11}$$

$$B^{13}{}_{13} = B^{14}{}_{14} = B^{23}{}_{23} = B^{24}{}_{24} = \frac{1}{8}\kappa, \tag{C.12}$$

$$B^{13}{}_{41} = B^{14}{}_{13} = B^{23}{}_{24} = B^{24}{}_{32} = \frac{5i}{8}\kappa, \tag{C.13}$$

$$C^{14}{}_{13} = C^{23}{}_{24} = C^{31}{}_{14} = C^{42}{}_{23} = \frac{5i}{8}\kappa. \tag{C.14}$$

Here, one needs always to keep in mind that there are other, non-reported entries of these tensors which are related to the ones above by symmetries of representations.

M3.

$$Bs = \kappa_1, \qquad C^{14} = C^{23} = 4i\kappa_2, \tag{C.15}$$

$$B^{12}{}_{21} = B^{12}{}_{34} = B^{34}{}_{12} = B^{34}{}_{43} = \frac{1}{4}\kappa_1, \tag{C.16}$$

$$B^{13}{}_{13} = B^{24}{}_{24} = \frac{3}{4}\kappa_1, \tag{C.17}$$

$$B^{14}{}_{41} = B^{23}{}_{32} = \frac{\kappa_1}{2}, \tag{C.18}$$

$$C^{14}{}_{12} = C^{21}{}_{14} = C^{21}{}_{23} = C^{23}{}_{43} = C^{32}{}_{12} = C^{34}{}_{14} = C^{34}{}_{23} = C^{41}{}_{34} = i\kappa_2, \tag{C.19}$$
$$C^{13}{}_{11} = C^{13}{}_{33} = C^{42}{}_{22} = C^{42}{}_{44} = 2i\kappa_2. \tag{C.20}$$

Again, symmetries must be imposed on these tensors.

Appendix D

Relinearization and X-Linearisation Algorithms

The Relinearization and X-Linearisation algorithms have been discovered with the purpose to provide an attack to the Hidden Field Equations (HFE), a public-key cryptosystem introduced in 1996. HFE is used nowadays for digital signature schemes. It is based on the difficulty to find solutions to systems of multivariate quadratic (MQ) equations over a finite field (NP-Hard). Due to this difficulty, the MQ problem is thought to be an example of Post-Quantum Crittography. Now, we start with a quadratic system of equations:

$$\sum_{1\leq i\leq j\leq n} a_{ijk}x_i x_j = b_k, \qquad \text{with} \quad k = 1,\ldots,m \tag{D.1}$$

with a_{ijk} coefficients and k running over the equations of the system. The first step is to linearise the system, namely, we define a set of new variables $y_{ij} = x_i x_j$, then we can solve the new system of m linear equations in $n(n+1)/2$ variables. This linear system is underdefined, therefore there is the need to introduce new equations. The second step, in fact, consists in creating additional equations that impose the commutativity of the x_i inside the y_{ij}. Let us define $(i,j,k,l,m,n) \sim (i',j',k',l',m',n')$, which means that the two tuples are a permutation of each other, then

$$(x_i x_j)(x_k x_l)\ldots(x_m x_n) = (x_{i'} x_{j'})(x_{k'} x_{l'})\ldots(x_{m'} x_{n'}). \tag{D.2}$$

These can be written in terms of the y_{ij} variables and added to the set of equations. The set of all the permutations can be solved by recurrent relinearization. The problem with this algorithm is that when the degree of linearisation is high, many of these equations become dependent. Indeed, it can be proved that “special” equations, linearly span all the other equations at fixed relinearisation degree, where special equations are of the form

$$y_{i_1 i_2} y_{i_3 i_4} \dots y_{i_{r-1} i_r} = y_{j_1 j_2} y_{j_3 j_4} \dots y_{j_{r-1} j_r} \tag{D.3}$$

such that the y's in this equation are the same except for two of them, whose indices have been permuted.
To introduce XL-algorithm (eXtended Linearizations), on the other hand, we need some further notation. First, given a system of multivariate quadratic equations l_j, equations of the form $\prod_{i=1}^{k} x_{j_1} l_j$ are said to be of type $x^j l$ and we denote the set of terms of degree k, $\prod_{i=1}^{k} x_{j_i}$, by x^k. We will consider the set of all polynomials $\prod_i x_{j_i} l_j$ with total degree $\leq D$. They span $\mathcal{I}_D$, that is, the space generated by $x^k l$ with $0 \leq k \leq D-2$. $\mathcal{I}_D \subset \mathcal{I}$, where $\mathcal{I}$ is the ideal spanned by l_j. XL algorithm contains as a special case Relinearization Algoritm. The algorithm proceeds as follows:

- **Multiply:** Create the equations $\prod_{i=1}^{k} x_{j_i} l_j \in \mathcal{I}_D$, with $k \leq$ D-2. (we are fixing the degree of the equations at D).
- **Linearize:** Linearise the system as described before
- **Solve:** When the linearisation technique produces an equation with only one variable, solve it with Berlekamp's algorithm
- **Repeat:** Insert the root in the system, simplify, and repeat until every root is found.

It has been noted that, when the starting equations are homogeneous, we can use only monomials of either even or odd degrees.