A perturbation result for a Neumann problem in a periodic domain

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Abstract We consider a Neumann problem for the Laplace equation in a periodic domain. We prove that the solution depends real analytically on the shape of the domain, on the periodicity parameters, on the Neumann datum, and on its boundary integral.

1 Introduction

The aim of this paper is to prove the analytic dependence of the solution of a periodic Neumann problem for the Laplace equation, upon joint perturbation of the domain, the periodicity parameters, the Neumann datum, and its integral on the boundary. The domain is obtained as the union of congruent copies of a periodicity cell of edges of length $q_{11}, \ldots, q_{nn}$ with a hole whose shape is the image of a reference domain through a diffeomorphism ϕ. As Neumann datum we take the projection of a function g, defined on the boundary of the reference domain and suitably rescaled, on the space of functions with zero integral on the boundary. As it happens for non-periodic Neumann problems, in order to identify one solution, we impose that the integral of the solution on the boundary is equal to a given real constant k. By means of a periodic version of potential theory, we prove that the solution of

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the problem depends real analytically on the ‘periodicity-domain-Neuman datum-integral’ quadruple $((q_{11},\dots,q_{nn}),\phi,g,k)$.

Many authors have investigated the behavior of the solutions to boundary value problems upon domain perturbations. We mention, e.g., Henry [7] and Sokolowski and Zolésio [18] for elliptic domain perturbation problems. Lanza de Cristoforis [10, 11] has exploited potential theory in order to prove that the solutions of boundary value problems for the Laplace and Poisson equations depend real analytically upon domain perturbation. Moreover, analyticity results for domain perturbation problems for eigenvalues have been obtained for example for the Laplace equation by Lanza de Cristoforis and Lamberti [8], for the biharmonic operator by Buoso and Provenzano [2], and for the Maxwell’s equations by Lamberti and Zaccaron [9].

In order to introduce our problem, we fix once for all a natural number

$$n \in \mathbb{N} \setminus \{0,1\}$$

that represents the dimension of the space. If $(q_{11},\dots,q_{nn}) \in]0,+\infty[^n$ we define a periodicity cell Q and a matrix $q \in \mathbb{D}_n^+(\mathbb{R})$ as

$$Q \equiv \prod_{j=1}^{n}]0,q_{jj}[, \qquad q \equiv \begin{pmatrix} q_{11} & 0 & \cdots & 0 \\ 0 & q_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q_{nn} \end{pmatrix},$$

where $\mathbb{D}_n(\mathbb{R})$ is the space of $n \times n$ diagonal matrices with real entries and $\mathbb{D}_n^+(\mathbb{R})$ is the set of elements of $\mathbb{D}_n(\mathbb{R})$ with diagonal entries in $]0,+\infty[$. Here we note that we can identify $\mathbb{D}_n^+(\mathbb{R})$ and $]0,+\infty[^n$. We denote by $|Q|_n$ the n-dimensional measure of the cell Q, by ν_Q the outward unit normal to ∂Q, where it exists, and by q^{-1} the inverse matrix of q. We find convenient to set

$$\widetilde{Q} \equiv]0,1[^n\,, \qquad \tilde{q} \equiv I_n\,,$$

where I_n denotes the identity $n \times n$ matrix. Then we introduce the reference domain: we take

$$\begin{aligned} &\alpha \in]0,1[\text{ and a bounded open connected subset } \Omega \text{ of } \mathbb{R}^n \\ &\text{of class } C^{1,\alpha} \text{ such that } \mathbb{R}^n \setminus \overline{\Omega} \text{ is connected}\,, \end{aligned} \tag{1}$$

where the symbol ‘$\overline{\cdot}$’ denotes the closure of a set. For the definition of sets and functions of the Schauder class $C^{1,\alpha}$ we refer, e.g., to Gilbarg and Trudinger [6]. In order to model our variable domain we consider a class of diffeomorphisms $\mathcal{A}_{\partial\Omega}^{\widetilde{Q}}$ from $\partial\Omega$ into their images contained in $\widetilde{Q}$ (see (3) below). By the Jordan-Leray separation theorem, if $\phi \in \mathcal{A}_{\partial\Omega}^{\widetilde{Q}}$, the set $\mathbb{R}^n \setminus \phi(\partial\Omega)$ has exactly two open connected components (see, e.g., Deimling [5, Thm. 5.2, p. 26]). We denote by $\mathbb{I}[\phi]$ the bounded open connected component of $\mathbb{R}^n \setminus \phi(\partial\Omega)$. Since $\phi(\partial\Omega) \subseteq \widetilde{Q}$, a topological argument shows that $\widetilde{Q} \setminus \overline{\mathbb{I}[\phi]}$ is also connected (cf., e.g., [3, Theorem

A.10]). We are now in the position to introduce the following two periodic domains (see Figure 1):

$$\mathbb{S}_q[q\mathbb{I}[\phi]] \equiv \bigcup_{z\in\mathbb{Z}^n} (qz + q\mathbb{I}[\phi]), \qquad \mathbb{S}_q[q\mathbb{I}[\phi]]^- \equiv \mathbb{R}^n \setminus \overline{\mathbb{S}_q[q\mathbb{I}[\phi]]}.$$

The set $\mathbb{S}_q[q\mathbb{I}[\phi]]^-$ will be the one where we shall set our Neumann problem. Clearly, a perturbation of q produces a modification of the whole periodicity structure of $\mathbb{S}_q[q\mathbb{I}[\phi]]^-$, while a perturbation of ϕ induces a change in the shape of the holes $\mathbb{S}_q[q\mathbb{I}[\phi]]$.

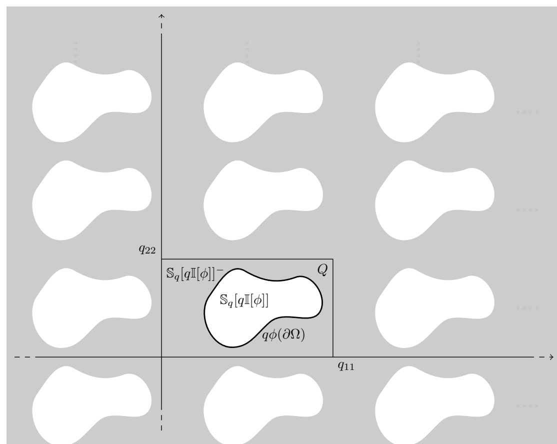

Fig. 1 The sets $\mathbb{S}_q[q\mathbb{I}[\phi]]^-$ (in gray), $\mathbb{S}_q[q\mathbb{I}[\phi]]$ (in white), and $q\phi(\partial\Omega)$ (in black) in case $n = 2$.

If $q \in \mathbb{D}_n^+(\mathbb{R})$, $\phi \in C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}^{\widetilde{Q}}$, $g \in C^{0,\alpha}(\partial\Omega)$ and $k \in \mathbb{R}$, we consider the following periodic Neumann problem for the Laplace equation:

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{S}_q[q\mathbb{I}[\phi]]^-, \\ u(x+qz) = u(x) & \forall x \in \overline{\mathbb{S}_q[q\mathbb{I}[\phi]]^-}, \forall z \in \mathbb{Z}^n, \\ \frac{\partial}{\partial \nu_{q\mathbb{I}[\phi]}} u(x) = g\big(\phi^{(-1)}(q^{-1}x)\big) & \\ \qquad -\frac{1}{\int_{\partial q\mathbb{I}[\phi]} d\sigma} \int_{\partial q\mathbb{I}[\phi]} g\big(\phi^{(-1)}(q^{-1}y)\big)\, d\sigma_y & \forall x \in \partial q\mathbb{I}[\phi], \\ \int_{\partial q\mathbb{I}[\phi]} u\, d\sigma = k. & \end{cases} \tag{2}$$

We note that the function

$$g\big(\phi^{(-1)}(q^{-1}\cdot)\big) - \frac{1}{\int_{\partial q\mathbb{I}[\phi]} d\sigma} \int_{\partial q\mathbb{I}[\phi]} g\big(\phi^{(-1)}(q^{-1}y)\big)\, d\sigma_y$$

clearly belongs to the space

$$C^{0,\alpha}(\partial q\mathbb{I}[\phi])_0 \equiv \left\{\mu \in C^{0,\alpha}(\partial q\mathbb{I}[\phi]) \colon \int_{\partial q\mathbb{I}[\phi]} \mu\, d\sigma = 0\right\}.$$

As a consequence, the solution of problem (2) in the space $C^{1,\alpha}_q(\overline{\mathbb{S}_q[q\mathbb{I}[\phi]]^-})$ of q-periodic functions in $\overline{\mathbb{S}_q[q\mathbb{I}[\phi]]^-}$ of class $C^{1,\alpha}$ exists and is unique and we denote it by $u[q,\phi,g,k]$ (see [3, Thm. 12.23]). Our aim is to prove that $u[q,\phi,g,k]$ depends, in a sense that we will clarify, analytically on (q,ϕ,g,k) (see Theorem 1). Our work originates from Lanza de Cristoforis [10, 11] on the real analytic dependence of the solution of the Dirichlet problem for the Laplace and Poisson equations upon domain perturbations. Moreover, this paper can be seen as the Neumann counterpart of [15], where the authors have proved analyticity properties for the solution of a periodic Dirichlet problem. An analysis similar to the one of the present paper was also carried out for periodic problems related to physical quantities arising in fluid mechanics and in material science (see [4, 14, 16]).

2 Preliminary results

In order to consider shape perturbations, we introduce a class of diffeomorphisms. Let Ω be as in (1). Let $\mathcal{A}_{\partial\Omega}$ be the set of functions of class $C^1(\partial\Omega,\mathbb{R}^n)$ which are injective and whose differential is injective at all points of $\partial\Omega$. The set $\mathcal{A}_{\partial\Omega}$ is well-known to be open in $C^1(\partial\Omega,\mathbb{R}^n)$ (see, e.g., Lanza de Cristoforis and Rossi [13, Lem. 2.5, p. 143]). Then we set

$$\mathcal{A}^{\widetilde{Q}}_{\partial\Omega} \equiv \Big\{\phi \in \mathcal{A}_{\partial\Omega} : \phi(\partial\Omega) \subseteq \widetilde{Q}\Big\}. \tag{3}$$

In order to analyze our boundary value problem, we are going to exploit periodic layer potentials. To define these operators, it is enough to replace the fundamental solution of the Laplace operator by a q-periodic tempered distribution $S_{q,n}$ such that $\Delta S_{q,n} = \sum_{z\in\mathbb{Z}^n}\delta_{qz} - \frac{1}{|Q|_n}$, where δ_{qz} is the Dirac measure with mass in qz (see e.g., [3, Chapter 12]). We can take

$$S_{q,n}(x) = -\sum_{z\in\mathbb{Z}^n\setminus\{0\}} \frac{1}{|Q|_n 4\pi^2 |q^{-1}z|^2} e^{2\pi i (q^{-1}z)\cdot x}$$

in the sense of distributions in $\mathbb{R}^n$ (see e.g., Ammari and Kang [1, p. 53], [3, §12.1]). Moreover, $S_{q,n}$ is even, real analytic in $\mathbb{R}^n \setminus q\mathbb{Z}^n$, and locally integrable in $\mathbb{R}^n$ (see e.g., [3, Thm. 12.4]). We now introduce the periodic single layer potential. Let Ω_Q be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$ for some $\alpha \in \,]0,1[$ such that $\overline{\Omega_Q} \subseteq Q$. We define the following two periodic domains:

$$\mathbb{S}_q[\Omega_Q] \equiv \bigcup_{z\in\mathbb{Z}^n} (qz + \Omega_Q), \qquad \mathbb{S}_q[\Omega_Q]^- \equiv \mathbb{R}^n \setminus \overline{\mathbb{S}_q[\Omega_Q]}$$

and we set

$$v_q[\partial\Omega_Q,\mu](x) \equiv \int_{\partial\Omega_Q} S_{q,n}(x-y)\mu(y)\,d\sigma_y \qquad \forall x \in \mathbb{R}^n$$

and

$$W_q^*[\partial\Omega_Q,\mu](x) \equiv \int_{\partial\Omega_Q} \nu_{\Omega_Q}(x)\cdot DS_{q,n}(x-y)\mu(y)\,d\sigma_y \qquad \forall x\in\partial\Omega_Q$$

for all $\mu \in L^2(\partial\Omega_Q)$. The symbol ν_{Ω_Q} denotes the outward unit normal field to $\partial\Omega_Q$, $d\sigma$ denotes the area element on $\partial\Omega_Q$ and $DS_{q,n}$ denotes the gradient of $S_{q,n}$. The function $v_q[\partial\Omega_Q,\mu]$ is called the q-periodic single layer potential. Now let $\mu \in C^{0,\alpha}(\partial\Omega_Q)$. As is well known, $v_q^+[\partial\Omega_Q,\mu] \equiv v_q[\partial\Omega_Q,\mu]_{|\overline{\mathbb{S}_q[\Omega_Q]}}$ belongs to $C_q^{1,\alpha}(\overline{\mathbb{S}_q[\Omega_Q]})$ and $v_q^-[\partial\Omega_Q,\mu] \equiv v_q[\partial\Omega_Q,\mu]_{|\overline{\mathbb{S}_q[\Omega_Q]^-}}$ belongs to $C_q^{1,\alpha}(\overline{\mathbb{S}_q[\Omega_Q]^-})$ (see [3, Thm. 12.8]). Moreover, the following jump formula for the normal derivative of the q-periodic single layer potential $v_q[\partial\Omega_Q,\mu]$ holds:

$$\frac{\partial}{\partial\nu_{\Omega_Q}} v_q^\pm[\partial\Omega_Q,\mu] = \mp\frac{1}{2}\mu + W_q^*[\partial\Omega_Q,\mu] \qquad \text{on } \partial\Omega_Q.$$

For a proof of the above formula we refer to [3, Thm. 12.11].

Since our approach will be based on integral operators, we need to understand how integrals behave when we perturb the domain of integration. Moreover, we need also to understand the regularity of the normal vector upon domain perturbations. For such reasons, we collect those results in the lemma below (for a proof, see Lanza de Cristoforis and Rossi [13, p. 166]).

Lemma 1 *Let α, Ω be as in* (1). *Then the following statements hold.*

(i) For each $\psi \in C^{1,\alpha}(\partial\Omega,\mathbb{R}^n)\cap\mathcal{A}_{\partial\Omega}$, there exists a unique $\tilde\sigma[\psi]\in C^{0,\alpha}(\partial\Omega)$ such that $\tilde\sigma[\psi]>0$ and

$$\int_{\psi(\partial\Omega)} \omega(s)\,d\sigma_s = \int_{\partial\Omega} \omega\circ\psi(y)\tilde\sigma[\psi](y)\,d\sigma_y, \qquad \forall\omega\in L^1(\psi(\partial\Omega)).$$

Moreover, the map $\tilde\sigma[\cdot]$ from $C^{1,\alpha}(\partial\Omega,\mathbb{R}^n)\cap\mathcal{A}_{\partial\Omega}$ to $C^{0,\alpha}(\partial\Omega)$ is real analytic.

(ii) The map from $C^{1,\alpha}(\partial\Omega,\mathbb{R}^n)\cap\mathcal{A}_{\partial\Omega}$ to $C^{0,\alpha}(\partial\Omega,\mathbb{R}^n)$ which takes ψ to $\nu_{\mathbb{I}[\psi]}\circ\psi$ is real analytic.

3 Analyticity of the solution

Our first goal is to transform problem (2) into an integral equation. In order to analyze the solvability of the obtained integral equation, we need the following lemma.

Lemma 2 *Let α, Ω be as in* (1). *Let $q\in\mathbb{D}_n^+(\mathbb{R})$. Let $\phi\in C^{1,\alpha}(\partial\Omega,\mathbb{R}^n)\cap\mathcal{A}_{\partial\Omega}^{\tilde Q}$. Let N be the map from $C^{0,\alpha}(\partial q\mathbb{I}[\phi])$ to itself, defined by*

$$N[\mu] \equiv \frac{1}{2}\mu + W_q^*[\partial q\mathbb{I}[\phi],\mu] \qquad \forall\mu\in C^{0,\alpha}(\partial q\mathbb{I}[\phi]).$$

Then N is a linear homeomorphism from $C^{0,\alpha}(\partial q\mathbb{I}[\phi])$ to itself. Moreover, N restricts to a linear homeomorphism from $C^{0,\alpha}(\partial q\mathbb{I}[\phi])_0$ to itself.

Proof By [3, Thm. 12.20], we deduce that N is a linear homeomorphism from $C^{0,\alpha}(\partial q\mathbb{I}[\phi])$ to itself. By [3, Prop. 12.15], we have that $\frac{1}{2}\mu+W_q^*[\partial q\mathbb{I}[\phi],\mu]$ belongs to $C^{0,\alpha}(\partial q\mathbb{I}[\phi])_0$ if and only if μ belongs to $C^{0,\alpha}(\partial q\mathbb{I}[\phi])_0$. As a consequence, we also have that N restricts to a linear homeomorphism from $C^{0,\alpha}(\partial q\mathbb{I}[\phi])_0$ to itself.□

Then, in the following proposition, we show how to convert the Neumann problem into an equivalent integral equation.

Proposition 1 *Let α, Ω be as in* (1). *Let $q \in \mathbb{D}_n^+(\mathbb{R})$. Let $\phi \in C^{1,\alpha}(\partial\Omega,\mathbb{R}^n)\cap\mathcal{A}^{\widetilde{Q}}_{\partial\Omega}$. Let $g \in C^{0,\alpha}(\partial\Omega)$. Let $k \in \mathbb{R}$. Then the boundary value problem*

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{S}_q[q\mathbb{I}[\phi]]^-\,, \\ u(x+qz) = u(x) & \forall x \in \overline{\mathbb{S}_q[q\mathbb{I}[\phi]]^-}\,, \forall z \in \mathbb{Z}^n\,, \\ \frac{\partial}{\partial\nu_{q\mathbb{I}[\phi]}}u(x) = g\big(\phi^{(-1)}(q^{-1}x)\big) & \\ \qquad -\frac{1}{\int_{\partial q\mathbb{I}[\phi]}d\sigma}\int_{\partial q\mathbb{I}[\phi]} g\big(\phi^{(-1)}(q^{-1}y)\big)\,d\sigma_y & \forall x \in \partial q\mathbb{I}[\phi]\,, \\ \int_{\partial q\mathbb{I}[\phi]} u\,d\sigma = k & \end{cases} \tag{4}$$

has a unique solution $u[q,\phi,g,k]$ in $C^{1,\alpha}_q(\overline{\mathbb{S}_q[q\mathbb{I}[\phi]]^-})$. Moreover,

$$\begin{aligned} u[q,\phi,g,k](x) &= v_q^-[\partial q\mathbb{I}[\phi],\mu](x) \\ &\quad + \frac{1}{\int_{\partial q\mathbb{I}[\phi]}d\sigma}\left(k-\int_{\partial q\mathbb{I}[\phi]} v_q^-[\partial q\mathbb{I}[\phi],\mu]\,d\sigma\right) \qquad \forall x \in \overline{\mathbb{S}_q[q\mathbb{I}[\phi]]^-}, \end{aligned} \tag{5}$$

where μ is the unique solution in $C^{0,\alpha}(\partial q\mathbb{I}[\phi])_0$ of the integral equation

$$\begin{aligned} \frac{1}{2}\mu(x) + W_q^*[\partial q\mathbb{I}[\phi],\mu](x) &= g\big(\phi^{(-1)}(q^{-1}x)\big) \\ &\quad - \frac{1}{\int_{\partial q\mathbb{I}[\phi]}d\sigma}\int_{\partial q\mathbb{I}[\phi]} g\big(\phi^{(-1)}(q^{-1}y)\big)\,d\sigma_y \qquad \forall x \in \partial q\mathbb{I}[\phi]\,. \end{aligned} \tag{6}$$

Proof By [3, Thm. 12.23] we know that problem (4) has a unique solution. Moreover, by Lemma 2, equation (6) has a unique solution μ which belongs to $C^{0,\alpha}(\partial q\mathbb{I}[\phi])_0$. Then by the properties of the periodic single layer potential (see, *e.g.*, [3, Thm. 12.8]), we deduce that the right hand side of (5) solves problem (4). □

In Proposition 1, we have seen an integral equation on $\partial q\mathbb{I}[\phi]$, namely equation (6), equivalent to problem (2). However, if we want to study the dependence of the solution of the integral equation on the parameters (q,ϕ,g,k), it may be convenient to transform the equation on the (q,ϕ)-dependent set $\partial q\mathbb{I}[\phi]$ into an equation on a fixed domain. We do so in the lemma below.

Lemma 3 *Let* α*,* Ω *be as in* (1)*. Let* $q \in \mathbb{D}_n^+(\mathbb{R})$*. Let* $\phi \in C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}^{\tilde{Q}}$*. Let* $g \in C^{0,\alpha}(\partial\Omega)$*. Then the function* $\theta \in C^{0,\alpha}(\partial\Omega)$ *solves the equation*

$$\begin{aligned}\frac{1}{2}\theta(t) + \int_{q\phi(\partial\Omega)} \nu_{q\mathbb{I}[\phi]}(q\phi(t)) \cdot DS_{q,n}(q\phi(t) - y)\theta\big(\phi^{(-1)}(q^{-1}y)\big)d\sigma_y \\ = g(t) - \frac{1}{\int_{\partial\Omega} \tilde{\sigma}[q\phi]\, d\sigma} \int_{\partial\Omega} g\tilde{\sigma}[q\phi]\, d\sigma \qquad \forall t \in \partial\Omega\,,\end{aligned} \tag{7}$$

if and only if the function $\mu \in C^{0,\alpha}(\partial q\mathbb{I}[\phi])$*, with* μ *delivered by*

$$\mu(x) = \theta\big(\phi^{(-1)}(q^{-1}x)\big) \qquad \forall x \in \partial q\mathbb{I}[\phi], \tag{8}$$

solves the equation

$$\begin{aligned}&\frac{1}{2}\mu(x) + W_q^*[\partial q\mathbb{I}[\phi], \mu](x) \\ &= g\big(\phi^{(-1)}(q^{-1}x)\big) - \frac{1}{\int_{\partial q\mathbb{I}[\phi]} d\sigma} \int_{\partial q\mathbb{I}[\phi]} g\big(\phi^{(-1)}(q^{-1}y)\big)\, d\sigma_y \quad \forall x \in \partial q\mathbb{I}[\phi]\,.\end{aligned}$$

Moreover, equation (7) *has a unique solution* θ *in* $C^{0,\alpha}(\partial\Omega)$ *and the function* μ *delivered by* (8) *belongs to* $C^{0,\alpha}(\partial q\mathbb{I}[\phi])_0$*.*

Proof It is a direct consequence of the theorem of change of variable in integrals, of Lemma 2, and of the obvious equality

$$\int_{\partial q\mathbb{I}[\phi]} \left(g\big(\phi^{(-1)}(q^{-1}x)\big) - \frac{1}{\int_{\partial q\mathbb{I}[\phi]} d\sigma} \int_{\partial q\mathbb{I}[\phi]} g\big(\phi^{(-1)}(q^{-1}y)\big)\, d\sigma_y \right) d\sigma_x = 0\,,$$

which implies that

$$g\big(\phi^{(-1)}(q^{-1}\cdot)\big) - \frac{1}{\int_{\partial q\mathbb{I}[\phi]} d\sigma} \int_{\partial q\mathbb{I}[\phi]} g\big(\phi^{(-1)}(q^{-1}y)\big)\, d\sigma_y$$

is in $C^{0,\alpha}(\partial q\mathbb{I}[\phi])_0$. □

Our next goal is to study the dependence of the solution of the integral equation (7) upon (q, ϕ, g). We wish to apply the implicit function theorem in Banach spaces. Therefore, having in mind equation (7), we introduce the map Λ from $\mathbb{D}_n^+(\mathbb{R}) \times \left(C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}^{\tilde{Q}}\right) \times \left(C^{0,\alpha}(\partial\Omega)\right)^2$ to $C^{0,\alpha}(\partial\Omega)$ by setting

$$
\begin{aligned}
\Lambda[q,\phi,g,\theta](t) &\equiv \frac{1}{2}\theta(t) \\
&+ \int_{q\phi(\partial\Omega)} \nu_{q\mathbb{I}[\phi]}(q\phi(t)) \cdot DS_{q,n}(q\phi(t)-y)\theta\big(\phi^{(-1)}(q^{-1}y)\big)d\sigma_y \\
&- g(t) + \frac{1}{\int_{\partial\Omega}\tilde{\sigma}[q\phi]\,d\sigma}\int_{\partial\Omega} g\tilde{\sigma}[q\phi]\,d\sigma \quad \forall t\in\partial\Omega,
\end{aligned}
$$

for all $(q,\phi,g,\theta)\in\mathbb{D}_n^+(\mathbb{R})\times\Big(C^{1,\alpha}(\partial\Omega,\mathbb{R}^n)\cap\mathcal{A}_{\partial\Omega}^{\widetilde{Q}}\Big)\times\big(C^{0,\alpha}(\partial\Omega)\big)^2$.

We are now ready to apply the implicit function theorem for real analytic maps in Banach spaces to equation $\Lambda[q,\phi,g,\theta]=0$ and prove that the solution θ depends analytically on (q,ϕ,g).

Proposition 2 *Let α, Ω be as in* (1). *Then the following statements hold.*

(i) Λ is real analytic.

(ii) For each $(q,\phi,g)\in\mathbb{D}_n^+(\mathbb{R})\times\Big(C^{1,\alpha}(\partial\Omega,\mathbb{R}^n)\cap\mathcal{A}_{\partial\Omega}^{\widetilde{Q}}\Big)\times C^{0,\alpha}(\partial\Omega)$, there exists a unique θ in $C^{0,\alpha}(\partial\Omega)$ such that

$$\Lambda[q,\phi,g,\theta]=0 \qquad \text{on } \partial\Omega,$$

and we denote such a function by $\theta[q,\phi,g]$.

(iii) The map $\theta[\cdot,\cdot,\cdot]$ from $\mathbb{D}_n^+(\mathbb{R})\times\Big(C^{1,\alpha}(\partial\Omega,\mathbb{R}^n)\cap\mathcal{A}_{\partial\Omega}^{\widetilde{Q}}\Big)\times C^{0,\alpha}(\partial\Omega)$ to $C^{0,\alpha}(\partial\Omega)$ that takes (q,ϕ,g) to $\theta[q,\phi,g]$ is real analytic.

Proof By [17, Thm. 3.2 (ii)], Lemma 1, and standard calculus in Banach spaces, we deduce the validity of statement (i). Statement (ii) follows by Lemmas 2 and 3. In order to prove (iii), since the analyticity is a local property, it suffices to fix (q_0,ϕ_0,g_0) in $\mathbb{D}_n^+(\mathbb{R})\times\Big(C^{1,\alpha}(\partial\Omega,\mathbb{R}^n)\cap\mathcal{A}_{\partial\Omega}^{\widetilde{Q}}\Big)\times C^{0,\alpha}(\partial\Omega)$ and to show that $\theta[\cdot,\cdot,\cdot]$ is real analytic in a neighborhood of (q_0,ϕ_0,g_0) in the product space $\mathbb{D}_n^+(\mathbb{R})\times\Big(C^{1,\alpha}(\partial\Omega,\mathbb{R}^n)\cap\mathcal{A}_{\partial\Omega}^{\widetilde{Q}}\Big)\times C^{0,\alpha}(\partial\Omega)$. By standard calculus in normed spaces, the partial differential $\partial_\theta\Lambda[q_0,\phi_0,g_0,\theta[q_0,\phi_0,g_0]]$ of Λ at $(q_0,\phi_0,g_0,\theta[q_0,\phi_0,g_0])$ with respect to the variable θ is delivered by

$$
\begin{aligned}
&\partial_\theta\Lambda[q_0,\phi_0,g_0,\theta[q_0,\phi_0,g_0]](\psi)(t) \\
&=\frac{1}{2}\psi(t)+\int_{q_0\phi_0(\partial\Omega)} \nu_{q_0\mathbb{I}[\phi_0]}(q_0\phi_0(t))\cdot DS_{q_0,n}(q_0\phi_0(t)-y)\psi\big(\phi_0^{(-1)}(q_0^{-1}y)\big)d\sigma_y \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall t\in\partial\Omega,
\end{aligned}
$$

for all $\psi\in C^{0,\alpha}(\partial\Omega)$. Lemma 2 together with a change of variable implies that $\partial_\theta\Lambda[q_0,\phi_0,g_0,\theta[q_0,\phi_0,g_0]]$ is a linear homeomorphism from $C^{0,\alpha}(\partial\Omega)$ onto $C^{0,\alpha}(\partial\Omega)$. Finally, by the implicit function theorem for real analytic maps in Banach spaces (see, e.g., Deimling [5, Thm. 15.3]) we deduce that $\theta[\cdot,\cdot,\cdot]$ is real analytic in a neighborhood of (q_0,ϕ_0,g_0) in $\mathbb{D}_n^+(\mathbb{R})\times\Big(C^{1,\alpha}(\partial\Omega,\mathbb{R}^n)\cap\mathcal{A}_{\partial\Omega}^{\widetilde{Q}}\Big)\times C^{0,\alpha}(\partial\Omega)$. □

Remark 1 By Lemma 1, Propositions 1 and 2, we have the following representation formula for the solution $u[q, \phi, g, k]$ of problem (2):

$$u[q,\phi,g,k](x) = \int_{\partial\Omega} S_{q,n}(x - q\phi(s))\theta[q,\phi,g](s)\tilde{\sigma}[q\phi](s)\, d\sigma_s$$
$$+ \frac{\left(k - \int_{\partial\Omega}\int_{\partial\Omega} S_{q,n}(q(\phi(t)-\phi(s)))\theta[q,\phi,g](s)\tilde{\sigma}[q\phi](s)d\sigma_s\tilde{\sigma}[q\phi](t)d\sigma_t\right)}{\int_{\partial\Omega}\tilde{\sigma}[q\phi]d\sigma}$$
$$\forall x \in \overline{\mathbb{S}_q[q\mathbb{I}[\phi]]^-},$$

for all $(q, \phi, g, k) \in \mathbb{D}^+_n(\mathbb{R}) \times \left(C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}^{\tilde{Q}}_{\partial\Omega}\right) \times C^{0,\alpha}(\partial\Omega) \times \mathbb{R}$.

By exploiting the representation formula of Remark 1 and the analyticity result for $(q, \phi, g) \mapsto \theta[q, \phi, g]$ of Proposition 2, we are ready to prove our main result on the analyticity of $u[q, \phi, g, k]$ as a map of the variable (q, ϕ, g, k).

Theorem 1 *Let α, Ω be as in* (1). *Let*

$$(q_0, \phi_0, g_0, k_0) \in \mathbb{D}^+_n(\mathbb{R}) \times \left(C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}^{\tilde{Q}}_{\partial\Omega}\right) \times C^{0,\alpha}(\partial\Omega) \times \mathbb{R}.$$

Let U be a bounded open subset of $\mathbb{R}^n$ such that $\overline{U} \subseteq \mathbb{S}_{q_0}[q_0\mathbb{I}[\phi_0]]^-$. Then there exists an open neighborhood $\mathcal{U}$ of (q_0, ϕ_0, g_0, k_0) in

$$\mathbb{D}^+_n(\mathbb{R}) \times \left(C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}^{\tilde{Q}}_{\partial\Omega}\right) \times C^{0,\alpha}(\partial\Omega) \times \mathbb{R}$$

such that the following statements hold.

(i) $\overline{U} \subseteq \mathbb{S}_q[q\mathbb{I}[\phi]]^-$ for all $(q, \phi, g, k) \in \mathcal{U}$.
(ii)Let $m \in \mathbb{N}$. Then the map from $\mathcal{U}$ to $C^m(\overline{U})$ which takes (q, ϕ, g, k) to the restriction $u[q, \phi, g, k]_{|\overline{U}}$ of $u[q, \phi, g, k]$ to $\overline{U}$ is real analytic.

Proof We first note that, by taking $\mathcal{U}$ small enough, we can deduce the validity of (i). The validity of (ii) follows by the representation formula of Remark 1, by Lemma 1, by Proposition 2, by the regularity results of [12] on the analyticity of integral operators with real analytic kernels, and by standard calculus in Banach spaces. □

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