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Weakly tilting bimodules

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Abstract. Tilting modules arose from representation theory of algebras and are known to furnish equivalences between categories of modules. We single out some weaker properties which still guarantee the existence of equivalences between abelian full subcategories of modules given by representable functors and their derived functors. For example, in this general framework and under suitable assumptions, we are able to prove a Gabriel-Popescu type theorem.

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1 Introduction

Tilting modules are a substantial tool in the representation theory of algebras. Their definition has its origin in the works of Gel′fand and Ponomarev, Brenner and Butler, Happel and Ringel (see [2] for a good reference); since then there have been generalizations in several directions.

One of these is the study of tilting modules for arbitrary rings (Menini and Orsatti, Colpi, D'Este, Tonolo and Trlifaj, Colby and Fuller; see [4] for references) or even tilting objects for Grothendieck categories (Colpi [4] and the first author [12]). In particular the papers [7] and [5], together with [3], explained the links between tilting modules and equivalences and established a general form of the "Tilting Theorem".

Many authors have tried to dualize the theory, but the results are not completely satisfactory, mainly because it is difficult to attain the "correct" definition of a cotilting module [1]. The second author [16] tried to single out some properties which could lead to a "Cotilting Theorem" in some class of modules as well as to a duality theory. During a conference held at Ohio University in 1999, Kent Fuller suggested the idea of dualizing those results. This interplay between tilting and cotilting theory quickly gave rise to this paper. Of course some ideas can be found in [16], but the "covariant" setting has many specific features, in particular a good equivalence theory.

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Section 2 deals with the general setting, mainly establishing notations: we fix a bimodule ${}_SP_R$ and consider the functors $H = \mathrm{Hom}_R(P, -)$ and $T = - \otimes_S P$. In Section 3 we introduce the main idea of using the right derived functors of TH and the left derived functors of HT: under some hypotheses, these functors are well behaved on the classes $\mathrm{Gen}(P_R)$ and $\mathrm{Cogen}(P_S^*)$.

In Section 4 we give the definition of a *weakly tilting bimodule* and show that these bimodules define a counter equivalence (in the sense of [3]) between suitable classes of R-modules, the *static* modules, and S-modules, the *costatic* modules: these classes are abelian full subcategories of Mod-R and Mod-S respectively. On these classes we are able to define torsion theories in such a way that

- the functors H and T define an equivalence between the torsion class in the static modules and the torsion-free class in the costatic modules;
- the first right derived functor $H^{(1)}$ of H and the first left derived functor $T_{(1)}$ of T define an equivalence between the torsion-free class in the static modules and the torsion class in the costatic modules.

Two examples are given.

In Section 5 we look at weakly tilting bimodules under additional assumptions and prove several results, among which a generalization to this setting of the Gabriel-Popescu theorem for a projective generator.

Finally Section 6 deals with the relationship between weakly tilting modules and tilting torsion theories; we are able to frame tilting torsion theories [7] in our more general context, providing a counter equivalence where Colpi and Trlifaj only studied an equivalence.

Notations and conventions. We denote by R and S associative rings with 1. Module morphisms are written on the opposite side to the scalars. We also denote by Mod-R the category of right R-modules.

When $\mathscr{C}$ is a subclass of (objects in) a category, we consider it also as a full subcategory. Every class or subcategory is closed under isomorphic objects. All functors we consider are additive and all diagrams we draw are commutative.

2 Generalities

Let ${}_SP_R$ be a bimodule and consider the functors

$$H = \mathrm{Hom}_R(P, -)\colon \text{Mod-}R \to \text{Mod-}S, \quad T = - \otimes_S P\colon \text{Mod-}S \to \text{Mod-}R.$$

Then T is a left adjoint to H; we denote by $\sigma\colon 1_{\text{Mod-}S} \to HT$ and by $\rho\colon TH \to 1_{\text{Mod-}R}$, respectively, the *unit* and the *counit* of this adjunction. It is clear that $\mathrm{im}\, T \subseteq \mathrm{Gen}(P_R)$, the class of P_R-*generated* modules.

Fix an injective cogenerator W_R of Mod-R and set $P_S^* = H(W)$. Then $\mathrm{im}\, H \subseteq \mathrm{Cogen}(P_S^*)$, the class of P_S^*-*cogenerated* modules. It is immediate to see that the class $\mathrm{Cogen}(P_S^*)$ does not depend on the chosen injective cogenerator.

Proposition 2.1. *The class* $\mathrm{Gen}(P_R)$ *is the smallest subclass of* Mod-R *containing* $\mathrm{im}\, T$ *and closed under quotients; the class* $\mathrm{Cogen}(P_S^*)$ *is the smallest subclass of* Mod-S *containing* $\mathrm{im}\, H$ *and closed under subobjects.* □

In view of this proposition, it is natural to consider subcategories of Mod-R and of Mod-S: since we will be using homological arguments, we choose to work in *closed* subclasses, i.e., closed under submodules, epimorphic images and arbitrary direct sums. The smallest such class containing $\mathrm{Gen}(P_R)$ is $\mathrm{Subgen}(P_R)$, which consists of all submodules of modules in $\mathrm{Gen}(P_R)$. Similarly, the smallest closed subclass of Mod-S containing $\mathrm{Cogen}(P^*)$ can be identified with Mod-$S/\mathrm{Ann}_S(P)$, where $\mathrm{Ann}_S(P)$ is the annihilator of ${}_SP$.

Any closed subclass of Mod-R is a Grothendieck category. If the class is closed also under arbitrary products, then it has enough projective objects; it is easy to see that such classes can be identified with Mod-R/I, for a suitable two-sided ideal I of R.

Remark 2.2. If we fix a closed class $\mathscr{G}$ in Mod-R such that $\mathrm{Gen}(P_R) \subseteq \mathscr{G}$, then $\mathscr{G}$ is a hereditary pretorsion class (see [15, Chapter VI]), so it defines a left exact preradical $t_{\mathscr{G}}$. Then all injective cogenerators of $\mathscr{G}$ are of the form $t_{\mathscr{G}}W$, where W_R is an injective cogenerator of Mod-R (see the following lemma). Hence it is clear that $\mathrm{Hom}_R(P, W)$ is canonically isomorphic to $\mathrm{Hom}_R(P, t_{\mathscr{G}}W)$. In conclusion the module P_S^* is the same whether we work in Mod-R or in the class $\mathscr{G}$.

Lemma 2.3. *Let* $\mathscr{G}$ *be a closed subcategory of* Mod-R *and let* $t_{\mathscr{G}}$ *be the left exact preradical associated to* $\mathscr{G}$. *Then a module* $U \in \mathscr{G}$ *is an injective cogenerator of* $\mathscr{G}$ *if and only if* $U \cong t_{\mathscr{G}}W$, *for some injective cogenerator* W_R *of* Mod-R.

Proof. One direction is clear; for the converse, assume U is an injective cogenerator of $\mathscr{G}$ and take $E(U)$, the injective envelope of U in Mod-R. Let X be the direct sum of one copy of each simple module which is not in $\mathscr{G}$; then $E(U) \oplus E(X)$ is an injective cogenerator of Mod-R and it is immediate to see that $t_{\mathscr{G}}(E(U) \oplus E(X)) \cong U$. □

In what follows we will fix a closed subclass $\mathscr{G}$ of Mod-R containing $\mathrm{Gen}(P_R)$.

We could also develop the theory by fixing an ideal I of S contained in $\mathrm{Ann}_S(P)$, but we will not do so. Indeed, in the particular case when $S = \mathrm{End}(P_R)$ the only choice for I is $I = 0$; this is by far the most interesting case. Notice, however, that we do not impose the condition $S = \mathrm{End}(P_R)$, unless this is explicitly stated.

We want also to consider the derived functors of H and T; in particular we set $H^{(i)} = \mathrm{Ext}^i_{\mathscr{G}}(P, -)$, i.e., the left derived functor computed in $\mathscr{G}$, and $T_{(i)} = \mathrm{Tor}_i^S(-, P)$.

Thus $\mathrm{im}\, H^{(i)} \subseteq$ Mod-S and $\mathrm{im}\, T_{(i)} \subseteq \mathrm{Subgen}(P_R)$: given $N \in$ Mod-S, take an exact sequence $0 \to K \to F \to N \to 0$, where F is projective in Mod-S. Then, from the long sequence

$$\cdots \to 0 \to T_{(n+1)}N \to T_{(n)}K \to 0 \to \cdots \to 0$$
$$\to T_{(1)}N \to TK \to TF \to TN \to 0,$$

we get the claim, by induction. This should be a clear reason why we choose closed subclasses.

Remark 2.4. A well-known formula says that

$$\mathrm{Ext}_S^i(-, P^*) \cong \mathrm{Hom}_R(T_{(i)}(-), W), \quad i \geq 0.$$

This formula follows from [14, Theorem 11.40], by setting $F = \mathrm{Hom}_R(-, W)$ and $G = - \otimes_S P$, since W is injective. Moreover, the fact that W is a cogenerator yields that $\ker T_{(i)} = \ker \mathrm{Ext}_S^i(-, P^*)$, $i \geq 0$.

3 Derived functors

The functor TH has, in general, no exactness property; however, it admits right derived functors, which turn out to be useful.

Let $M \in \mathcal{G}$ and consider an injective resolution

$$0 \longrightarrow (M \xrightarrow{\varepsilon}) \, E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} \cdots$$

to which we can apply the functor TH, getting the complex

$$0 \xrightarrow{TH(d_{-1})} TH(E_0) \xrightarrow{TH(d_0)} TH(E_1) \xrightarrow{TH(d_1)} \cdots$$

and so we can define the n-th right derived functor $(TH)^{(n)} = R_n(TH)$ $(n \geq 0)$ by

$$(TH)^{(n)}(M) = \frac{\ker TH(d_n)}{\mathrm{im}\, TH(d_{n-1})}.$$

Then, for all exact sequences $0 \to L \to M \to N \to 0$, we get the long exact sequence

$$\begin{aligned} 0 \to (TH)^{(0)}L \to (TH)^{(0)}M \to (TH)^{(0)}N \to& \\ \to (TH)^{(1)}L \to (TH)^{(1)}M \to (TH)^{(1)}N \to \cdots& \\ \to (TH)^{(n-1)}N \to (TH)^{(n)}L \to (TH)^{(n)}M \to (TH)^{(n)}N \to \cdots& \end{aligned}$$

and, moreover, there exists a natural transformation of functors

$$\alpha\colon TH \to (TH)^{(0)}.$$

Note that, by definition, α_M is an isomorphism, whenever M is injective.

Lemma 3.1. *If* $\mathrm{Cogen}(P^*) \subseteq \ker T_{(1)}$, *then* α_M *is monic, for every* $M \in \mathcal{G}$.

Proof. Consider an exact sequence $0 \to M \to E \to C \to 0$, where E is injective. If we apply the functor H, we get the exact sequence $0 \to HM \to HE \to HC$. Then the cokernel K of $HM \to HE$ belongs to $\mathrm{Cogen}(P^*) \subseteq \ker T_{(1)}$.

Apply T to the exact sequence $0 \to HM \to HE \to K \to 0$. Then we can write the diagram with exact rows

$$\begin{array}{ccccc} 0 & \longrightarrow & THM & \longrightarrow & THE \\ & & \big\downarrow & & \big\downarrow \cong \\ 0 & \longrightarrow & (TH)^{(0)}M & \longrightarrow & (TH)^{(0)}E \end{array}$$

and the thesis follows. □

Proposition 3.2. *Assume* $\mathrm{Cogen}(P^*) \subseteq \ker T_{(1)}$*; then* α_M *is an isomorphism, for all* $M \in \ker H^{(1)}$.

Proof. Proceed as in the proof of Lemma 3.1, and note that $K = HC$; then we can write the diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & THM & \longrightarrow & THE & \longrightarrow & THC & \longrightarrow & 0 \\ & & {\scriptstyle\alpha_M}\big\downarrow & & {\scriptstyle\alpha_E}\big\downarrow (\cong) & & {\scriptstyle\alpha_C}\big\downarrow (\text{monic}) & & \\ 0 & \longrightarrow & (TH)^{(0)}M & \longrightarrow & (TH)^{(0)}E & \longrightarrow & (TH)^{(0)}C & & \end{array}$$

and, by the "five lemma", the proof is complete. □

Proposition 3.3. *Assume* $\mathrm{Cogen}(P^*) \subseteq \ker T_{(1)}$. *Then:*

(1) *there exists a natural transformation* β: $(TH)^{(0)} \to T_{(1)}H^{(1)}$*;*

(2) β_M *is epic, for all* $M \in \mathscr{G}$*;*

(3) *there exists a natural transformation* $\rho^{(0)}$: $(TH)^{(0)} \to 1_{\mathscr{G}}$*;*

(4) $\rho^{(0)}\alpha = \rho$*, the counit of the adjunction.*

Proof. (1) Take an exact sequence $0 \to M \xrightarrow{f} E \xrightarrow{g} C \to 0$, with E_R injective. Then apply H and split the resulting sequence as follows:

$$0 \longrightarrow HM \xrightarrow{Hf} HE \xrightarrow{p} K \longrightarrow 0,$$

$$0 \longrightarrow K \xrightarrow{q} HC \xrightarrow{\partial} H^{(1)}M \longrightarrow 0,$$

where $K = \mathrm{coker}\, Hf$ and $qp = Hg$. We use ∂ to denote any connecting morphism.

By hypothesis, $T_{(1)}K = 0 = T_{(1)}HC$. Thus, applying T to the first sequence, we get the exact sequence

$$0 \longrightarrow THM \xrightarrow{THf} THE \xrightarrow{Tp} TK \longrightarrow 0$$

and, applying T to the second one, we can draw the following diagram:

$$
(*)\qquad
\begin{array}{ccccccccccc}
0 & \longrightarrow & (TH)^{(0)}M & \xrightarrow{(TH)^{(0)}f} & (TH)^{(0)}E & \xrightarrow{(TH)^{(0)}g} & (TH)^{(0)}C & \xrightarrow{\partial} & (TH)^{(1)}M & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \beta_M} & & \Big\downarrow{\scriptstyle (Tp)\alpha_E^{-1}} & & \Big\uparrow{\scriptstyle \alpha_C} & & & & \\
0 & \longrightarrow & T_{(1)}H^{(1)}M & \xrightarrow{\partial} & TK & \xrightarrow{Tq} & THC & \xrightarrow{T\partial} & TH^{(1)}M & \longrightarrow & 0
\end{array}
$$

where $(Tp)\alpha_E^{-1}$ is epic and α_C is monic. A diagram chasing shows that, for any $x \in (TH)^{(0)}M$, there exists a unique $y \in T_{(1)}H^{(1)}M$ such that

$$\partial y = (Tp)\alpha_E^{-1}((TH)^{(0)}f)x$$

and so we can define $\beta_M x = y$.

(2) The morphism β_M is surjective, as another diagram chasing shows.

(3) A direct proof can be given using the definition of $(TH)^{(0)}$. However, it is simpler to resort to the formal theory of derived functors: there exists a natural transformation ρ: $TH \to 1_{\mathscr{G}}$, so there is another one $\rho^{(0)}$: $(TH)^{(0)} \to (1_{\mathscr{G}})^{(0)} = 1_{\mathscr{G}}$.

(4) By calculation. □

Theorem 3.4. *Assume* $\mathrm{Cogen}(P^*) \subseteq \ker T_{(1)}$. *Then, for all* $M \in \mathscr{G}$, *there exists an exact sequence*

$$0 \longrightarrow THM \xrightarrow{\alpha_M} (TH)^{(0)}M \xrightarrow{\beta_M} T_{(1)}H^{(1)}M \longrightarrow 0.$$

Proof. It is sufficient to prove that $\ker\beta_M = \mathrm{im}\,\alpha_M$. We can put together two of the previous diagrams, getting the following one, which has exact rows:

$$
\begin{array}{ccccc}
0 & \longrightarrow & THM & \xrightarrow{THg} & THE \\
 & & \Big\downarrow{\scriptstyle \alpha_M} & & \Big\downarrow{\scriptstyle \alpha_E} \\
0 & \longrightarrow & (TH)^{(0)}M & \xrightarrow{(TH)^{(0)}f} & (TH)^{(0)}E \\
 & & \Big\downarrow{\scriptstyle \beta_M} & & \Big\downarrow{\scriptstyle (Tp)\alpha_E^{-1}} \\
0 & \longrightarrow & T_{(1)}H^{(1)}M & \xrightarrow{\partial} & TK
\end{array}
$$

and the proof follows by diagram chasing. □

Proposition 3.5. *Assume* $\mathrm{Cogen}(P^*) \subseteq \ker T_{(1)}$. *Then:*

(1) *on the subcategory* $\ker H^{(1)}$, *the functors* $(TH)^{(0)}$ *and* TH *are isomorphic;*

(2) *on the subcategory* $\ker H$, *the functors* $(TH)^{(0)}$ *and* $T_{(1)}H^{(1)}$ *are isomorphic;*

(3) *on the subcategory* $\ker H^{(2)}$, *the functors* $(TH)^{(1)}$ *and* $TH^{(1)}$ *are isomorphic;*

(4) *for* $n \geq 2$, *the functors* $(TH)^{(n)}$ *are zero on* $\bigcap_{i\geq 2} \ker H^{(i)}$.

Proof. Statement (1) has already been proved for objects. The statement for morphism is an easy calculation.

(2) Let $M \in \ker H$; take, as usual, an exact sequence $0 \to M \to E \to C \to 0$ with E injective. Then, applying H, we get the exact sequence $0 \to HE \to HC \to H^{(1)}M \to 0$ (since $H^{(1)}E = 0$). Applying now T and recalling that $HC \in \mathrm{Cogen}(P^*)$, we get the diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_{(1)}H^{(1)}M & \longrightarrow & THE & \longrightarrow & THC \\ & & \downarrow & & \downarrow{\scriptstyle \alpha_E} & & \downarrow{\scriptstyle \alpha_C} \\ 0 & \longrightarrow & (TH)^{(0)}M & \longrightarrow & (TH)^{(0)}E & \longrightarrow & (TH)^{(0)}C \end{array}$$

where α_E is an isomorphism and α_C is monic. Hence there exists a unique isomorphism $T_{(1)}H^{(1)}M \to (TH)^{(0)}M$ making the diagram commute. This uniqueness and easy calculations prove also the statement for morphisms.

(3) Let $M \in \ker H^{(2)}$; then $C \in \ker H^{(1)}$, so α_C is an isomorphism. Then we can redraw diagram $(*)$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & (TH)^{(0)}M & \xrightarrow{(TH)^{(0)}f} & (TH)^{(0)}E & \xrightarrow{(TH)^{(0)}g} & \\ & & \downarrow{\scriptstyle \beta_M} & & \downarrow{\scriptstyle (Tp)\alpha_E^{-1}} & & \\ 0 & \longrightarrow & T_{(1)}H^{(1)}M & \xrightarrow{\partial} & TK & \xrightarrow{Tq} & \end{array}$$

$$\begin{array}{ccccc} (TH)^{(0)}C & \xrightarrow{\partial} & (TH)^{(1)}M & \longrightarrow & 0 \\ {\scriptstyle \alpha_C^{-1}}\downarrow & & \downarrow & & \\ THC & \xrightarrow{T\partial} & TH^{(1)}M & \longrightarrow & 0 \end{array}$$

and the existence and naturality of the isomorphism denoted by the broken arrow are guaranteed.

(4) If $M \in \bigcap_{n\geq 2} \ker H^{(n)}$, then $C \in \bigcap_{n\geq 1} \ker H^{(n)}$. In particular $(TH)^{(1)}C \cong TH^{(1)}C = 0$. We can use dimension shifting, since $(TH)^{(n)}C \cong (TH)^{(n+1)}M$, for all $n \geq 1$. □

By dimension shifting we can prove the following extension of statement (3) of Proposition 3.5.

Proposition 3.6. *Assume* $\mathrm{Cogen}(P^*) \subseteq \ker T_{(1)}$ *and let* $n \geq 1$. *Then, on the subcategory* $\ker H^{(n+1)}$, *the functors* $(TH)^{(n)}$ *and* $TH^{(n)}$ *are isomorphic.*

The same arguments as before can be used, with easy modifications, for the *left* derived functors $(HT)_{(n)}$ of HT ($n \geq 0$), making the hypothesis that $\mathrm{Gen}(P_R) \subseteq \ker H^{(1)}$.

We can summarize the results in the following theorems.

Theorem 3.7. *Assume* $\mathrm{Gen}(P_R) \subseteq \ker H^{(1)}$. *Then:*

(1) *there exists a natural transformation* $\delta\colon (HT)_{(0)} \to HT$;

(2) δ_N *is epic, for all* $N \in \text{Mod-}S$;

(3) δ_N *is an isomorphism, for all* $N \in \ker T_{(1)}$, *in particular when* N *is projective;*

(4) *there exists a natural transformation* $\gamma\colon H^{(1)}T_{(1)} \to (HT)_{(0)}$ *and* γ_N *is monic, for all* $N \in \text{Mod-}S$;

(5) *there exists a natural transformation* $\sigma_{(0)}\colon 1_{\text{Mod-}S} \to (HT)_{(0)}$ *and* $\delta\sigma_{(0)} = \sigma$, *the unit of the adjunction;*

(6) *on the subcategory* $\ker T_{(1)}$, *the functors* $(HT)_{(0)}$ *and* HT *are isomorphic;*

(7) *on the subcategory* $\ker T$, *the functors* $(HT)_{(0)}$ *and* $H^{(1)}T_{(1)}$ *are isomorphic;*

(8) *on the subcategory* $\ker T_{(2)}$, *the functors* $(HT)_{(1)}$ *and* $HT_{(1)}$ *are isomorphic;*

(9) *for* $n \geq 2$, *the functors* $(HT)_{(n)}$ *are zero on the subcategory* $\bigcap_{i\geq 2} \ker T_{(i)}$.

Theorem 3.8. *Assume* $\mathrm{Gen}(P_R) \subseteq \ker H^{(1)}$. *Then, for all* $N \in \text{Mod-}S$, *there exists an exact sequence*

$$0 \longrightarrow H^{(1)}T_{(1)}N \xrightarrow{\gamma_N} (HT)_{(0)}N \xrightarrow{\delta_N} HTN \longrightarrow 0.$$

4 Weakly tilting bimodules

We denote by $\mathrm{pd}_{\mathcal{G}} M_R$ the *projective dimension* of M_R in $\mathcal{G}$, i.e., the smallest integer $i \geq 0$ such that $\mathrm{Ext}^i_{\mathcal{G}}(M, -) = 0$ (if such an integer exists). Analogously, $\mathrm{id}_{\mathcal{G}} M_R$ denotes the *injective dimension* of M_R in $\mathcal{G}$, i.e., the smallest integer $i \geq 0$ such that $\mathrm{Ext}^i_{\mathcal{G}}(-, M) = 0$ (if such an integer exists). The *weak dimension* of a module is similarly defined using the vanishing of the Tor functors. We omit the subscript if $\mathcal{G} = \text{Mod-}R$.

It follows from the formula in Remark 2.4 that the injective dimension of P^*_S is the same as the weak dimension of ${}_SP$, so that $\mathrm{id}\, P^*_S \leq \mathrm{pd}\, {}_SP$.

One of the many characterizations of tilting modules is the following (see [7, Proposition 1.3]): a module P_R is tilting if and only if

(a) P_R is finitely generated and

(b) $\mathrm{Gen}(P_R) = \ker \mathrm{Ext}^1_R(P, -)$.

If P_R is a tilting module, $S = \mathrm{End}(P_R)$, W_R is an injective cogenerator of $\mathscr{G}$ and $P^*_S = \mathrm{Hom}_R(P, W)$, then $\mathrm{Cogen}(P^*_S) = \ker \mathrm{Ext}^1_S(-, P^*) = \ker \mathrm{Tor}^S_1(-, P)$; moreover $\mathrm{pd}\, P_R \le 1$ and $\mathrm{id}\, P^*_S \le 1$ [5, Theorem 1.5].

Remark 4.1. In their paper [7], Colpi and Trlifaj call *tilting* a module P_R such that $\mathrm{Gen}(P_R) = \ker H^{(1)}$ and *classical tilting* a tilting module which is finitely generated. We prefer to call *generalized tilting* what they call a tilting module and reserve the unadorned term for the finitely generated tilting modules.

This leads us to set the following definition.

Definition 4.2. A bimodule ${}_SP_R$ is a *weakly $\mathscr{G}$-tilting bimodule* if, setting $P^*_S = \mathrm{Hom}_R(P, W)$, where W_R is an injective cogenerator of $\mathscr{G}$, we have:

(WT1) $\mathrm{Gen}(P_R) \subseteq \ker \mathrm{Ext}^1_{\mathscr{G}}(P, -)$ and $\mathrm{pd}_{\mathscr{G}}\, P_R \le 1$;

(WT2) $\mathrm{Cogen}(P^*_S) \subseteq \ker \mathrm{Tor}^S_1(-, P)$ and $\mathrm{id}\, P^*_S \le 1$.

We say that P_R is a *weakly $\mathscr{G}$-tilting module* if, setting $S = \mathrm{End}(P_R)$, the bimodule ${}_SP_R$ is weakly tilting.

When $\mathscr{G} = \mathrm{Subgen}(P_R)$, we speak of *weakly self-tilting* (bi)modules and, when $\mathscr{G} = \text{Mod-}R$, we speak of *weakly tilting* (bi)modules.

Every tilting module is a weakly tilting module. If $\mathscr{G}'$ is a closed class in Mod-R, $\mathrm{Gen}(P_R) \subseteq \mathscr{G}' \subseteq \mathscr{G}$ and ${}_SP_R$ is a weakly $\mathscr{G}$-tilting bimodule, then it is also a weakly $\mathscr{G}'$-tilting bimodule.

Remark 4.3. The condition $\mathrm{pd}_{\mathscr{G}}\, P_R \le 1$ implies that $\ker H^{(1)}$ is closed under quotients; therefore the condition $\mathrm{Gen}(P_R) \subseteq \ker H^{(1)}$ can be reduced to *every direct sum of copies of P_R belongs to* $\ker H^{(1)}$, i.e.,

$$\mathrm{Ext}^1_R(P, P^{(\kappa)}) = 0, \quad \text{for any cardinal } \kappa.$$

Analogously, in presence of $\mathrm{id}\, P^*_S \le 1$, the condition $\mathrm{Cogen}(P^*_S) \subseteq \ker T_{(1)}$ can be reduced to *every product of copies of P^*_S belongs to* $\ker T_{(1)}$, i.e.,

$$\mathrm{Ext}^1_S((P^*)^\kappa, P^*) = 0, \quad \text{for any cardinal } \kappa.$$

We now give two examples of weakly $\mathscr{G}$-tilting bimodules, one of which is faithfully balanced. Next we give a way to produce weakly tilting bimodules.

Example 4.4. Let p be a prime number and consider $R = S = \mathbb{Z}$ (ring of integers); let $P = \mathbb{Z}(p^\infty)$ be the Prüfer p-group. Then $\mathbb{Z}(p^\infty)^* = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}(p^\infty), \mathbb{Q}/\mathbb{Z}) = J_p$, the ring of p-adic integers, considered as an abelian group. Since the global dimension of $\mathbb{Z}$ is 1, we have that $\operatorname{pd}\mathbb{Z}(p^\infty) \le 1$ and $\operatorname{id}\mathbb{Z}(p^\infty)^* \le 1$. Moreover $\operatorname{Gen}(\mathbb{Z}(p^\infty))$ consists of divisible groups and so $\operatorname{Gen}(\mathbb{Z}(p^\infty)) \subseteq \ker \operatorname{Ext}^1(\mathbb{Z}(p^\infty), -)$. Finally $\operatorname{Cogen}(\mathbb{Z}(p^\infty)^*)$ consists of torsion-free groups and so $\operatorname{Cogen}(\mathbb{Z}(p^\infty)^*) \subseteq \ker \operatorname{Ext}^1(-, \mathbb{Z}(p^\infty)^*)$ since $\mathbb{Z}(p^\infty)^* = J_p$ is a cotorsion group (see [9, Section 9.54]).

Thus ${}_{\mathbb{Z}}\mathbb{Z}(p^\infty)_{\mathbb{Z}}$ is a weakly tilting bimodule, which is not tilting.

Note that $\operatorname{Gen}(\mathbb{Z}(p^\infty)) \neq \ker \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}(p^\infty), -)$, so this module is not tilting even in generalized sense [7].

Example 4.5. The same $P = \mathbb{Z}(p^\infty)$ works if we consider it as a J_p-J_p-bimodule, for again the global dimension is 1 and the class $\operatorname{Gen}(P)$ consists of injective J_p-modules. The class $\operatorname{Cogen}(P^*)$ consists of torsion-free modules and it is true that $\operatorname{Ext}^1_{J_p}(M, J_p) = 0$, for all torsion-free J_p-modules M, by [8, Chapter XII, 1.17]. Thus $\mathbb{Z}(p^\infty)_{J_p}$ is a weakly tilting module, in particular a weakly self-tilting module. Note that, since this module is not finitely generated, it is not a $*$-module (for a complete account of the connections between $*$-modules and tilting objects, see [4]).

Proposition 4.6. *Let ${}_SP_R$ be a weakly $\mathcal{G}$-tilting bimodule and consider $Q = P^{(\kappa)}$, a direct sum of copies of P as an S-R-bimodule. Then ${}_SQ_R$ is a weakly $\mathcal{G}$-tilting bimodule.*

Proof. The classes $\operatorname{Gen}(P_R)$ and $\operatorname{Gen}(Q_R)$ coincide; moreover, for all R-modules M, $\operatorname{Ext}^1_R(Q, M) = \operatorname{Ext}^1_R(P, M)^\kappa$, as S-modules. The projective dimension of P_R and of Q_R are the same.

We have then $Q^* = \operatorname{Hom}_R(Q, W) \cong \operatorname{Hom}_R(P, W)^\kappa$ as S-modules, so $\operatorname{Cogen}(Q^*_S) = \operatorname{Cogen}(P^*_S)$ and $\operatorname{id} Q^*_S = \operatorname{id} P^*_S$. Finally, for all S-modules N, $\operatorname{Tor}^S_1(N, Q) = \operatorname{Tor}^S_1(N, P)^{(\kappa)}$ as R-modules. □

From now on ${}_SP_R$ will be a weakly $\mathcal{G}$-tilting bimodule.

For any module $M \in \mathcal{G}$, there is the diagram with exact row

(S1)
$$\begin{array}{ccccccccc} & & & & M & & & & \\ & & & \overset{\rho_M}{\nearrow} & \uparrow{\scriptstyle \rho_M^{(0)}} & & & & \\ 0 & \longrightarrow & THM & \xrightarrow{\alpha_M} & (TH)^{(0)}M & \xrightarrow{\beta_M} & T_{(1)}H^{(1)}M & \longrightarrow & 0 \end{array}$$

(see Theorem 3.4) and, for any module $N \in \text{Mod-}S$, there is the diagram with exact row

(S2)
$$\begin{array}{ccccccccc} & & & & N & & & & \\ & & & & \downarrow{\scriptstyle \sigma_{(0)N}} & \overset{\sigma_N}{\searrow} & & & \\ 0 & \longrightarrow & H^{(1)}T_{(1)}N & \xrightarrow{\gamma_N} & (HT)_{(0)}N & \xrightarrow{\delta_N} & HTN & \longrightarrow & 0 \end{array}$$

by Theorem 3.8.

The following definition is due to Nauman [13]. We choose to slightly change the terminology, for reasons which will be apparent after the following proposition.

Definition 4.7. A module M_R is called P-0-*static* if ρ_M is an isomorphism; a module N_S is called P-0-*costatic* if σ_N is an isomorphism.

The next proposition is on the line of [16, Section 2].

Proposition 4.8. *A module M_R is P-0-static if and only if α_M and $\rho_M^{(0)}$ are isomorphisms; a module N_S is P-0-costatic if and only if δ_N and $\sigma_{(0)N}$ are isomorphisms.*

Proof. One direction is obvious. Assume that ρ_M is an isomorphism. Then $M \in \mathrm{Gen}(P_R) \subseteq \ker H^{(1)}$ and so α_M is an isomorphism by Proposition 3.2.

The proof for N is similar, using Theorem 3.7. □

Remark 4.9. If M_R is P-0-static, then $H^{(1)}M = 0$; if N_S is P-0-costatic then $T_{(1)}N = 0$.

We want now to define the concept of staticity with respect to the derived functors; if we look at diagram S1, we see that in general it is not possible to define a natural transformation $T_{(1)}H^{(1)} \to 1_{\mathscr{G}}$ or $1_{\mathscr{G}} \to T_{(1)}H^{(1)}$. However, if $\rho_M^{(0)}$ is an isomorphism, then we can compose β_M with $(\rho_M^{(0)})^{-1}$. It turns out that another condition is important, namely that also the first derived of ρ is an isomorphism. Since $\rho^{(1)}$: $(TH)^{(1)} \to (1_{\mathscr{G}})^{(1)} = 0$, this means that we want to consider modules M_R such that $0 = (TH)^{(1)}M \cong TH^{(1)}M$. Of course, we can make similar considerations for S-modules. Note that these conditions hold for P-0-static and P-0-costatic modules (see Remark 4.9).

Definition 4.10. A module M_R is called P-1-*static* if β_M and $\rho_M^{(0)}$ are isomorphisms and $TH^{(1)}M = 0$; a module N_S is called P-1-*costatic* if γ_N and $\sigma_{(0)N}$ are isomorphisms and $HT_{(1)}N = 0$.

A module M_R is called P-*static* if $\rho_M^{(0)}$ is an isomorphism and $TH^{(1)}M = 0$; a module N_S is called P-*costatic* if $\sigma_{(0)N}$ is an isomorphism and $HT_{(1)}N = 0$.

When ${}_SP_R$ is clear from the context, we omit it and speak of (i-)static and (i-)costatic modules ($i = 0, 1$). We denote by

- $\mathrm{St}_i({}_SP_R)$ and $\mathrm{Cost}_i({}_SP_R)$ the classes of P-i-static and P-i-costatic modules ($i = 0, 1$);
- $\mathrm{St}({}_SP_R)$ and $\mathrm{Cost}({}_SP_R)$ the classes of P-static and P-costatic modules;
- $\tilde{\rho}_M = \beta_M(\rho_M^{(0)})^{-1}$, which is defined for $M \in \mathrm{St}({}_SP_R)$;
- $\tilde{\sigma}_N = (\sigma_{(0)N})^{-1}\gamma_N$, which is defined for $N \in \mathrm{Cost}({}_SP_R)$.

For example, when P_R is a tilting module and $S = \mathrm{End}(P_R)$, then (see Theorem 5.1 and Proposition 5.4)

(a) $\mathrm{St}({}_SP_R) = \text{Mod-}R$ and $\mathrm{Cost}({}_SP_R) = \text{Mod-}S$;

(b) $\mathrm{St}_0({}_SP_R) = \mathrm{Gen}(P_R) = \ker H^{(1)}$ and $\mathrm{Cost}_0({}_SP_R) = \mathrm{Cogen}(P_S^*) = \ker T_{(1)}$;

(c) $\mathrm{St}_1({}_SP_R) = \ker H$ and $\mathrm{Cost}_1({}_SP_R) = \ker T$.

We want to show that the functors H, $H^{(1)}$, T and $T_{(1)}$ work between the categories of static and costatic modules.

Remark 4.11. Note first that, by the adjunction, $H\rho_M\sigma_{HM}$ and $\rho_{TN}T\sigma_N$ are identity morphisms, for all modules M_R and N_S. Therefore,

$$H\rho_M^{(0)}H\alpha_M\delta_{HM}\sigma_{(0)HM} = 1_{HM},$$

$$\rho_{TN}^{(0)}\alpha_{TN}T\delta_N T\sigma_{(0)N} = 1_{TN}.$$

In particular, $H\rho_M^{(0)}$ and $\rho_{TN}^{(0)}$ are split epi, whereas $\sigma_{(0)HM}$ and $T\sigma_{(0)N}$ are split mono.

Proposition 4.12. *For all modules M_R, δ_{HM} is an isomorphism and $H\alpha_M$ is monic. Moreover, if $\ker\rho_M^{(0)} \in \ker H$, then $H\alpha_M$ and $\sigma_{(0)HM}$ are isomorphisms and $T_{(1)}H^{(1)}M \in \ker H$.*

Proof. Write the exact sequence (S2) for HM:

$$0 \longrightarrow H^{(1)}T_{(1)}HM \longrightarrow (TH)_{(0)}HM \xrightarrow{\delta_{HM}} HTHM \longrightarrow 0.$$

Since $T_{(1)}HM = 0$, it follows that δ_{HM} is an isomorphism. The fact that $H\alpha_M$ is monic is obvious.

The hypothesis $\ker\rho_M^{(0)} \in \ker H$ implies that $H\rho_M^{(0)}$ is also monic. Hence $H\alpha_M$ must be epic (see Remark 4.11), hence an isomorphism. Finally also $\sigma_{(0)HM}$ is an isomorphism. To end the proof, apply H to the sequence

$$0 \longrightarrow THM \xrightarrow{\alpha_M} (TH)^{(0)}M \longrightarrow T_{(1)}H^{(1)}M \longrightarrow 0$$

and recall that $H\alpha_M$ is epic. Hence $HT_{(1)}H^{(1)}M = 0$. □

We have the analogous result for S-modules.

Proposition 4.13. *For all modules N_S, α_{TN} is an isomorphism and $T\delta_N$ is epic. Moreover, if $\mathrm{coker}\,\sigma_{(0)N} \in \ker T$, then $T\delta_N$ and $\rho_{TN}^{(0)}$ are isomorphisms and $H^{(1)}T_{(1)}N \in \ker T$.*

Note that the conditions "$\ker\rho_M^{(0)} \in \ker H$" and "$\mathrm{coker}\,\sigma_{(0)N} \in \ker T$" are automatically satisfied if M_R is static and N_S is costatic.

The two propositions above have a counterpart for the derived functors.

Proposition 4.14. *For all modules M_R, $H^{(1)}\beta_M$ is an isomorphism and $\sigma_{(0)H^{(1)}M}$ is monic. If $TH^{(1)}M = 0$, then $\gamma_{H^{(1)}M}$ is an isomorphism and*

$$H^{(1)}\rho_M^{(0)}(H^{(1)}\beta_M)^{-1}\gamma_{H^{(1)}M}^{-1}\sigma_{(0)H^{(1)}M} = 1_{H^{(1)}M}.$$

If $TH^{(1)}M = 0$ and $H^{(1)}\rho_M^{(0)}$ is monic, then $H^{(1)}\rho_M^{(0)}$ and $\sigma_{(0)H^{(1)}M}$ are isomorphisms.

Proof. Apply $H^{(1)}$ to the sequence $0 \to THM \to (TH)^{(0)}M \to T_{(1)}H^{(1)}M \to 0$ to get that $H^{(1)}\beta_M$ is an isomorphism, since $H^{(1)}THM = 0$. Take now an exact sequence $0 \to M \xrightarrow{f} E \xrightarrow{g} C \to 0$, with E_R injective. Apply to it the functor H to obtain the exact sequence

$$HE \longrightarrow HC \longrightarrow H^{(1)}M \longrightarrow 0.$$

Apply now $(HT)_{(0)}$, which is right exact: recalling the isomorphisms of Theorem 3.7, we get the diagram with exact rows

$$\begin{array}{ccccccc}
HE & \xrightarrow{Hg} & HC & \xrightarrow{\partial} & H^{(1)}M & \longrightarrow & 0 \\
\downarrow{\scriptstyle \sigma_{HE}} & & \downarrow{\scriptstyle \sigma_{HC}} & & \downarrow{\scriptstyle \sigma_{(0)H^{(1)}M}} & & \\
HTHE & \xrightarrow{HTHg} & HTHC & \xrightarrow{\partial'} & HT_{(0)}M & \longrightarrow & 0 \\
\downarrow{\scriptstyle H\rho_E} & & \downarrow{\scriptstyle H\rho_C} & & & & \\
HE & \xrightarrow{Hg} & HC & & & &
\end{array}$$

and, by Remark 4.11, the compositions of the vertical arrows on the left are identities.

Take now $x \in H^{(1)}M$ such that $x \in \ker \sigma_{(0)H^{(1)}M}$; then $x = \partial y$, for some $y \in HC$. Hence $\partial'\sigma_{HC}y = 0$, so that $\sigma_{HC}y = (HTHg)z$. Therefore

$$x = \partial(H\rho_C)\sigma_{HC}y = \partial(H\rho_C)(HTHg)z = \partial(Hg)(H\rho_E)z = 0.$$

and $\sigma_{(0)H^{(1)}M}$ is monic.

Assume now that $TH^{(1)}M = 0$. The exact sequence (S2) for $H^{(1)}M$ is

$$0 \longrightarrow H^{(1)}T_{(1)}H^{(1)}M \xrightarrow{\gamma_{H^{(1)}M}} HT_{(0)}H^{(1)}M \longrightarrow HTH^{(1)}M = 0 \longrightarrow 0.$$

Consider now the diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_{(1)}H^{(1)}M & \longrightarrow & TK & \longrightarrow & THC & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \beta_M} & & \uparrow & & \| & & \\
0 & \longrightarrow & (TH)^{(0)}M & \longrightarrow & THE & \longrightarrow & THC & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \rho_M^{(0)}} & & \downarrow{\scriptstyle \rho_E} & & \downarrow{\scriptstyle \rho_C} & & \\
0 & \longrightarrow & M & \xrightarrow{f} & E & \xrightarrow{g} & C & \longrightarrow & 0
\end{array}$$

where $K = \operatorname{coker} Hf$. If we apply to it the functor H and $(HT)_{(0)}$ to the sequence $HC \to H^{(1)}M \to 0$, we get

$$
\begin{array}{ccccc}
HC & \xrightarrow{\partial_1} & H^{(1)}M & \longrightarrow & 0 \\
\Big\downarrow \sigma_{(0)HC} & & \Big\downarrow \sigma_{(0)H^{(1)}M} & & \\
(HT)_{(0)}HC & \xrightarrow{(HT)_{(0)}\partial_1} & (HT)_{(0)}H^{(1)}M & \longrightarrow & 0 \\
\Big\downarrow \delta_{HC} & & \Big\uparrow \gamma_{H^{(1)}M} & & \\
HTHC & \xrightarrow{\partial_2} & H^{(1)}T_{(1)}H^{(1)}M & \longrightarrow & 0 \\
\Big\| & & \Big\uparrow H^{(1)}\beta_M & & \\
HTHC & \xrightarrow{\partial_3} & H^{(1)}(TH)^{(0)}M & \longrightarrow & 0 \\
\Big\downarrow H\rho_C & & \Big\downarrow H^{(1)}\rho^{(0)}_M & & \\
HC & \xrightarrow{\partial_1} & H^{(1)}M & \longrightarrow & 0
\end{array}
$$

and the requested identity follows by computation and the fact that ∂_1 is epic.

Assume now also that $H^{(1)}\rho^{(0)}_M$ is monic. Then it is an isomorphism and the claims follow from the above identity. □

A similar result holds for S-modules.

Proposition 4.15. *For all modules N_S, $T_{(1)}\gamma_N$ is an isomorphism and $\rho^{(0)}_{T_{(1)}N}$ is epic. If $HT_{(1)}N = 0$, then $\beta_{T_{(1)}N}$ is an isomorphism and*

$$\rho^{(0)}_{T_{(1)}N}\beta^{-1}_{T_{(1)}N}(T_{(1)}\gamma_N)^{-1}T_{(1)}\sigma_{(0)N} = 1_{T_{(1)}N}.$$

If $HT_{(1)}N = 0$ and $T_{(1)}\sigma_{(0)N}$ is epic, then $T_{(1)}\sigma_{(0)N}$ and $\rho^{(0)}_{T_{(1)}N}$ are isomorphisms. □

We are ready to show that the functors H and $H^{(1)}$ send static modules to costatic ones; analogously, T and $T_{(1)}$ send costatic modules to static ones.

Theorem 4.16. *Let $M_R \in \mathrm{St}({}_SP_R)$ and $N_S \in \mathrm{Cost}({}_SP_R)$. Then:*

(a) *HM and $H^{(1)}M$ belong to $\mathrm{Cost}({}_SP_R)$;*

(b) *TN and $T_{(1)}N$ belong to $\mathrm{St}({}_SP_R)$.*

Proof. Apply Propositions 4.12, 4.14, 4.13 and 4.15. □

The following result is now an easy consequence of the adjunction between H and T.

Proposition 4.17. *The functors H and T induce an equivalence between* $\mathrm{St}_0({}_SP_R)$ *and* $\mathrm{Cost}_0({}_SP_R)$.

Also the functors $H^{(1)}$ and $T_{(1)}$ are well-behaved, when suitably restricted.

Proposition 4.18. *The functor $H^{(1)}$ sends modules in* $\mathrm{St}({}_SP_R)$ *into modules in* $\mathrm{Cost}_1({}_SP_R)$*; the functor $T_{(1)}$ sends modules in* $\mathrm{Cost}({}_SP_R)$ *into modules in* $\mathrm{St}_1({}_SP_R)$*. Moreover, $H^{(1)}$ is a left adjoint to $T^{(1)}$ when the domains of the functors are restricted to* $\mathrm{St}({}_SP_R)$ *and* $\mathrm{Cost}({}_SP_R)$*. In particular $H^{(1)}$ and $T^{(1)}$ induce an equivalence between* $\mathrm{St}_1({}_SP_R)$ *and* $\mathrm{Cost}_1({}_SP_R)$.

Proof. Let $M \in \mathrm{St}({}_SP_R)$; then we have the exact sequence

$$0 \longrightarrow THM \xrightarrow{\rho_M} M \xrightarrow{\tilde{\rho}_M} T_{(1)}H^{(1)}M \longrightarrow 0$$

(see Definition 4.10) and applying H we get that $H^{(1)}\tilde{\rho}_M$ is an isomorphism, whose inverse is $\tilde{\sigma}_{H^{(1)}M}$, by Proposition 4.14. The proof is similar for $N \in \mathrm{Cost}({}_SP_R)$, using Proposition 4.15.

The fact that $H^{(1)}$ is a left adjoint of $T_{(1)}$ follows by easy calculations. □

We want to examine now some properties of the classes $\mathrm{St}({}_SP_R)$ and $\mathrm{Cost}({}_SP_R)$. We give the proofs for the second one.

Lemma 4.19. *Let X be a submodule of $M \in \mathrm{St}({}_SP_R)$; then $(TH)^{(1)}(M/X) = 0$, $\rho_X^{(0)}$ is monic and $\rho_{M/X}^{(0)}$ is epic. If moreover X is a quotient of a module in* $\mathrm{St}({}_SP_R)$*, then $M/X \in \mathrm{St}({}_SP_R)$.*

Let Y be a submodule of $N \in \mathrm{Cost}({}_SP_R)$; then $(HT)_{(1)}(Y) = 0$, $\sigma_{(0)Y}$ is monic and $\sigma_{(0)N/Y}$ is epic. If moreover N/Y is a submodule of a module in $\mathrm{Cost}({}_SP_R)$*, then $Y \in \mathrm{Cost}({}_SP_R)$.*

Proof. We can consider the diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & N & \longrightarrow & N/Y & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\cong} & & \downarrow & & \\
(*) & \longrightarrow & (HT)_{(0)}Y & \longrightarrow & (HT)_{(0)}N & \longrightarrow & (HT)_{(0)}(N/Y) & \longrightarrow & 0
\end{array}$$

where $(*) = 0 \to (HT)_{(1)}Y \to (HT)_{(1)}N \to (HT)_{(1)}(N/Y)$ and the vertical arrows are the suitable instances of $\sigma_{(0)}$. Recalling that $(HT)_{(1)}N = 0$, we have the thesis. If moreover N/Y can be embedded into some object in $\mathrm{Cost}({}_SP_R)$, then $\sigma_{(0)N/Y}$ is also monic and $(HT)_{(1)}(N/Y) = 0$, so that $\sigma_{(0)Y}$ is an isomorphism. □

Theorem 4.20. *The subcategories* $\mathrm{St}({}_SP_R)$ *and* $\mathrm{Cost}({}_SP_R)$ *are closed under kernels, cokernels, images, direct summands and extensions. In particular they are abelian categories.*

Proof. Let $f\colon N_1 \to N_2$ be a morphism in $\mathrm{Cost}({}_SP_R)$; set $X = \ker f$, $Y = \mathrm{im}\, f$ and $Z = \mathrm{coker}\, f$; then we have the exact sequences

$$0 \to X \to N_1 \to Y \to 0 \qquad \text{and} \qquad 0 \to Y \to N_2 \to Z \to 0.$$

By Lemma 4.19, $X \in \mathrm{Cost}({}_SP_R)$ and $(HT)_{(1)}Y = 0$. Then we have the diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \longrightarrow & N_1 & \longrightarrow & Y & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow & & \\
0 = (HT)_{(1)}Y & \longrightarrow & (HT)_{(0)}X & \longrightarrow & (HT)_{(0)}N_1 & \longrightarrow & (HT)_{(0)}Y & \longrightarrow & 0
\end{array}$$

and so $Y \in \mathrm{Cost}({}_SP_R)$. Analogously, we have the diagram with exact rows

$$\begin{array}{ccccccccccc}
 & & 0 & \longrightarrow & Y & \longrightarrow & N_2 & \longrightarrow & Z & \longrightarrow & 0 \\
 & & & & \downarrow\cong & & \downarrow\cong & & \downarrow & & \\
0 & \longrightarrow & (HT)_{(1)}Z & \longrightarrow & (HT)_{(0)}Y & \longrightarrow & (HT)_{(0)}N_2 & \longrightarrow & (HT)_{(0)}Z & \longrightarrow & 0
\end{array}$$

and so $Z = \mathrm{coker}\, f \in \mathrm{Cost}({}_SP_R)$.

The fact that $\mathrm{Cost}({}_SP_R)$ is closed under direct summands follows by applying twice Lemma 4.19. Closure under extensions is another easy consequence of the "five lemma". A similar proof can be used for $\mathrm{St}({}_SP_R)$. □

We are now able to state the main theorem, which is a generalization of the celebrated Brenner and Butler theorem, known also as the "Tilting theorem". We write $F\colon \mathscr{A} \rightleftarrows \mathscr{B} : G$ to denote that F is a left adjoint to the functor G.

Theorem 4.21. *Let* ${}_SP_R$ *be a weakly* $\mathscr{G}$*-tilting bimodule.*

(1) *The functors* H, $H^{(1)}$, T *and* $T_{(1)}$ *induce adjunctions*

$$T\colon \mathrm{Cost}({}_SP_R) \rightleftarrows \mathrm{St}({}_SP_R) : H \qquad \textit{and}$$

$$H^{(1)}\colon \mathrm{St}({}_SP_R) \rightleftarrows \mathrm{Cost}({}_SP_R) : T_{(1)}.$$

(2) *The functors* H *and* T *induce an equivalence between* $\mathrm{St}_0({}_SP_R)$ *and* $\mathrm{Cost}_0({}_SP_R)$, *while the functors* $H^{(1)}$ *and* $T_{(1)}$ *induce an equivalence between* $\mathrm{St}_1({}_SP_R)$ *and* $\mathrm{Cost}_1({}_SP_R)$.

(3) *For every module* $M \in \mathrm{St}({}_SP_R)$*, there exists an exact sequence*

$$0 \to THM \to M \to T_{(1)}H^{(1)}M \to 0$$

where $THM \in \mathrm{St}_0({}_SP_R)$ *and* $T_{(1)}H^{(1)}M \in \mathrm{St}_1({}_SP_R)$*.*

(4) *For every module* $N \in \mathrm{Cost}({}_SP_R)$*, there exists an exact sequence*

$$0 \to H^{(1)}T_{(1)}N \to N \to HTN \to 0$$

where $HTN \in \mathrm{Cost}_0({}_SP_R)$ *and* $H^{(1)}T_{(1)}N \in \mathrm{Cost}_1({}_SP_R)$*.*

(5) *The categories* $\mathrm{St}({}_SP_R)$ *and* $\mathrm{Cost}({}_SP_R)$ *are abelian categories; the pair* $(\mathrm{St}_0({}_SP_R), \mathrm{St}_1({}_SP_R))$ *is a torsion theory in* $\mathrm{St}({}_SP_R)$ *and the pair* $(\mathrm{Cost}_1({}_SP_R), \mathrm{Cost}_0({}_SP_R))$ *is a torsion theory in* $\mathrm{Cost}({}_SP_R)$*.*

(6) *The following equalities hold:*

$$\mathrm{St}_0({}_SP_R) = \mathrm{Gen}(P_R) \cap \mathrm{St}({}_SP_R) = \ker H^{(1)} \cap \mathrm{St}({}_SP_R)$$

$$\mathrm{St}_1({}_SP_R) = \ker H \cap \mathrm{St}({}_SP_R)$$

$$\mathrm{Cost}_0({}_SP_R) = \mathrm{Cogen}(P_S^*) \cap \mathrm{Cost}({}_SP_R) = \ker T_{(1)} \cap \mathrm{Cost}({}_SP_R)$$

$$\mathrm{Cost}_1({}_SP_R) = \ker T \cap \mathrm{Cost}({}_SP_R)$$

Proof. We have to prove only (5) and (6). The fact that $\mathrm{St}({}_SP_R)$ is an abelian category follows from Theorem 4.20. We now see that, for $M \in \mathrm{St}({}_SP_R)$,

$$M \in \mathrm{St}_0({}_SP_R) \quad \text{if and only if} \quad \mathrm{Hom}_R(M, M') = 0, \text{ for all } M' \in \mathrm{St}_1({}_SP_R).$$

Indeed, if $M \in \mathrm{St}_0({}_SP_R)$, $M' \in \mathrm{St}_1({}_SP_R)$ and $f\colon M \to M'$, then $HM' = 0$, so $0 = \rho_{M'}THf = f\rho_M$ and so $f = 0$, since ρ_M is an isomorphism. The converse follows from the fact that $T_{(1)}H^{(1)}M \in \mathrm{St}_1({}_SP_R)$, so $\tilde{\rho}_M = 0$ and ρ_M is epic.

In the same way we prove that

$$M \in \mathrm{St}_1({}_SP_R) \quad \text{if and only if} \quad \mathrm{Hom}_R(M', M) = 0, \text{ for all } M' \in \mathrm{St}_0({}_SP_R).$$

The equalities in (6) are proved as follows. If $M \in \mathrm{St}_0({}_SP_R)$, then $M \in \mathrm{Gen}(P_R)$, so that

$$\mathrm{St}_0({}_SP_R) \subseteq \mathrm{Gen}(P_R) \cap \mathrm{St}({}_SP_R) \subseteq \ker H^{(1)} \cap \mathrm{St}({}_SP_R).$$

If $M \in \ker H^{(1)} \cap \mathrm{St}({}_SP_R)$, then the exact sequence $0 \to THM \to M \to T_{(1)}H^{(1)}M \to 0$ says that M is 0-static.

The same exact sequence says that $M \in \ker H \cap \mathrm{St}({}_SP_R)$ implies $M \in \mathrm{St}_1({}_SP_R)$; if $M \in \mathrm{St}_1({}_SP_R)$, then $HM \cong HT_{(1)}H^{(1)}M = 0$, since $H^{(1)}M \in \mathrm{Cost}({}_SP_R)$. □

The theorem just proved shows that we can do every computation in a smallest class $\mathscr{G}'$, namely the closure of $\mathrm{Subgen}(P_R)$ under extensions in $\mathscr{G}$.

Example 4.22. Let us compute the classes in the case of Example 4.5. The class $\mathscr{G}$ can be taken to be $\mathrm{Subgen}(\mathbb{Z}(p^\infty))$, which is closed under extensions in Mod-J_p, because it consists of all torsion modules, i.e., of all torsion p-groups. As usual we denote by M objects in $\mathrm{Subgen}(\mathbb{Z}(p^\infty))$ and by N objects in Mod-J_p. We also denote by dX and tX respectively the divisible part and the torsion part of any J_p-module. See [10, Chapter 5] and [9, Section 9.54] for the proofs and for unexplained terms.

(1) $HM \cong H(dM)$ and $TN \cong T(N/tN)$;

(2) $H^{(1)}M \cong H^{(1)}(M/dM)$ and $T_{(1)}N \cong T_{(1)}(tN) \cong tN$;

(3) for a torsion module M, $H^{(1)}M$ is the cotorsion hull of M;

(4) for any J_p-module N, $T_{(1)}N = tN$;

(5) by Matlis' equivalence, $M \in \mathrm{St}_0(\mathbb{Z}(p^\infty))$ if and only if $M \in \mathrm{Gen}(\mathbb{Z}(p^\infty))$;

(6) $T_{(1)}H^{(1)}M \cong M$ if and only if M is reduced and torsion, so that $\mathrm{St}_1(\mathbb{Z}(p^\infty))$ coincides with the reduced p-groups [9, Lemma 9.55.1];

(7) $\mathrm{Cost}_0(\mathbb{Z}(p^\infty))$ coincides with the class of completions of free J_p-modules with respect to the p-adic topology;

(8) $\mathrm{Cost}_1(\mathbb{Z}(p^\infty))$ is the class of bounded torsion p-groups.

Notice that, in this case, $\mathrm{St}(\mathbb{Z}(p^\infty)) = \mathrm{Subgen}(\mathbb{Z}(p^\infty))$.

5 Special cases

We could ask whether natural closure properties of the classes $\mathrm{St}(P)$ and $\mathrm{Cost}(P)$ hold.

Theorem 5.1. *Let ${}_SP_R$ be a weakly $\mathscr{G}$-tilting bimodule. The following conditions are equivalent:*

(a) $\mathrm{Cost}({}_SP_R) = $ Mod-*S;*

(b) $\mathrm{Cost}({}_SP_R)$ *contains all projective modules;*

(c) $\mathrm{Cost}_0({}_SP_R)$ *contains all projective modules.*

Proof. (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) are obvious.

(c) $\Rightarrow$ (a) Take $N \in$ Mod-S and an exact sequence $0 \to K \to F \to N \to 0$, where F is projective; then consider the diagram with exact rows

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & K & \longrightarrow & F & \longrightarrow & N & \longrightarrow & 0 \\
 & & & & \Big\downarrow{\scriptstyle \sigma_{(0)K}} & & \Big\downarrow{\scriptstyle \sigma_{(0)F}} & & \Big\downarrow{\scriptstyle \sigma_{(0)N}} & & \\
0 & \longrightarrow & (HT)_{(1)}N & \longrightarrow & (HT)_{(0)}K & \longrightarrow & (HT)_{(0)}F & \longrightarrow & (HT)_{(0)}N & \longrightarrow & 0
\end{array}$$

where the $\sigma_{(0)F}$ is an isomorphism; then $\sigma_{(0)N}$ is epic. Since N is arbitrary, also $\sigma_{(0)K}$ is epic, so they are both isomorphisms and $(HT)_{(1)}N = 0$. □

Lemma 5.2. *If $S = \mathrm{End}(P_R)$, then P_R is 0-static and S_S is 0-costatic.*

Proof. Since $(HT)_{(0)}S \cong HTS \cong S$ and $HT_{(1)}S = 0$, S_S is 0-costatic. Therefore $P_R = TS$ is 0-static. □

Theorem 5.3. *Let P_R be a weakly $\mathcal{G}$-tilting module and set $S = \mathrm{End}(P_R)$. Then the following conditions are equivalent:*

(a) $\mathrm{Cost}({}_SP_R) = \text{Mod-}S$*;*

(b) $\mathrm{Cost}({}_SP_R)$ *is closed under direct sums;*

(c) P_R *is self-small.*

Proof. (a) ⇒ (b) is obvious.

(b) ⇒ (c) For every cardinal κ, $S^{(\kappa)} \in \mathrm{Cost}({}_SP_R)$, hence $S^{(\kappa)} \in \mathrm{Cost}_0({}_SP_R)$. Therefore

$$\mathrm{Hom}_R(P, P^{(\kappa)}) \cong HT(S^{(\kappa)}) \cong S^{(\kappa)} \cong \mathrm{Hom}_R(P, P)^{(\kappa)},$$

which is precisely the definition for self-smallness.

(c) ⇒ (a) It suffices to show that all free S-modules belong to $\mathrm{Cost}({}_SP_R)$; the same chain of morphisms as before proves this. □

We could ask in what cases the class $\mathrm{St}({}_SP_R)$ is big; for example, when is $\mathrm{St}({}_SP_R) = \mathcal{G}$? In this case Theorem 4.21.6 implies that $\mathrm{Gen}(P_R) = \ker H^{(1)}$, so that P_R is a generalized tilting module in the sense of [7, Definition 1.1].

Theorem 5.4. *Let ${}_SP_R$ be a weakly $\mathcal{G}$-tilting bimodule. The following conditions are equivalent:*

(a) $\mathrm{St}({}_SP_R) = \mathcal{G}$*;*

(b) *every injective R-module belongs to* $\mathrm{St}_0({}_SP_R)$*;*

(c) *every injective R-module belongs to* $\mathrm{St}({}_SP_R)$.

Proof. (a) ⇒ (b) If E_R is injective, then $\rho_E^{(0)}$ is an isomorphism.

(c) ⇒ (a) Embed M_R in an injective module E_R; then we have the diagram with exact rows

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & (TH)^{(0)}M & \longrightarrow & (TH)^{(0)}E & \longrightarrow & (TH)^{(0)}(E/M) & \longrightarrow & (TH)^{(1)}M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \rho_M^{(0)}} & & \cong\downarrow{\scriptstyle \rho_E^{(0)}} & & \downarrow{\scriptstyle \rho_{E/M}^{(0)}} & & & & \\
0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & E/M & \longrightarrow & 0 & &
\end{array}$$

and so $\rho_M^{(0)}$ is monic. Since M is arbitrary, also $\rho_{E/M}^{(0)}$ is monic, so that $\rho_M^{(0)}$ is an isomorphism and $(TH)^{(1)}M = 0$. □

The case in which $\mathrm{St}({}_SP_R) = \mathscr{G}$ is very important in view of what follows.

To simplify the notation, set $\mathscr{L} = (HT)_{(0)}$ and $\lambda = \sigma_{(0)}$. Proposition 5.5 says that we can consider $\mathscr{L}$ as a functor $\mathscr{L} : \text{Mod-}S \to \mathrm{Cost}({}_SP_R)$, provided $\mathrm{St}({}_SP_R) = \mathscr{G}$.

Proposition 5.5. *Assume* $\mathrm{St}({}_SP_R) = \mathscr{G}$. *Then, for every* $N \in \text{Mod-}S$, $\mathscr{L}N = (HT)_{(0)}N \in \mathrm{Cost}({}_SP_R)$.

Proof. Consider the exact sequence (S2) for N:

$$0 \longrightarrow H^{(1)}T_{(1)}N \xrightarrow{\gamma_N} (HT)_{(0)}N \xrightarrow{\delta_N} HTN \longrightarrow 0.$$

We need only to show that $H^{(1)}T_{(1)}N$ and HTN are in $\mathrm{Cost}({}_SP_R)$. Indeed, by hypothesis, $TN, T_{(1)}N \in \mathrm{St}({}_SP_R)$, so Theorem 4.21 proves the claim. □

Lemma 5.6. *Assume* $\mathrm{St}({}_SP_R) = \mathscr{G}$. *Then* $\lambda_{\mathscr{L}N} = \mathscr{L}\lambda_N$.

Proof. Assume first that N is projective; then $(HT)_{(0)}N = HTN$ and we can consider $\lambda_N = \sigma_N$. Again, $HTN \in \ker T_{(1)}$ and so $HTN \in \mathrm{Cost}_0({}_SP_R)$ and we can consider $\lambda_{\mathscr{L}N} = \sigma_{HTN}$.

We claim that $\sigma_{HTN} = HT\sigma_N$. Indeed, $HTN \in \mathrm{Cost}_0({}_SP_R)$ implies that σ_{HTN} is an isomorphism, so that $H\rho_{TN}$ is also an isomorphism. But ρ_{TN} is a morphism in $\mathrm{St}_0({}_SP_R)$ and so, by the equivalence, ρ_{TN} is an isomorphism and $T\sigma_N = \rho_{TN}^{-1}$.

Let now N_S be arbitrary and take an epimorphism $g : F \to N$, with F_S projective. Then the following diagrams

$$\begin{array}{ccccc} F & \xrightarrow{f} & N & \longrightarrow & 0 \\ \downarrow{\scriptstyle \lambda_F} & & \downarrow{\scriptstyle \lambda_N} & & \\ \mathscr{L}F & \xrightarrow{\mathscr{L}f} & \mathscr{L}N & \longrightarrow & 0 \end{array} \qquad \begin{array}{ccccc} \mathscr{L}F & \xrightarrow{\mathscr{L}f} & \mathscr{L}N & \longrightarrow & 0 \\ \downarrow{\scriptstyle \mathscr{L}\lambda_F} & & \downarrow{\scriptstyle \mathscr{L}\lambda_N} & & \\ \mathscr{L}^2F & \xrightarrow{\mathscr{L}^2f} & \mathscr{L}^2N & \longrightarrow & 0 \end{array}$$

$$\begin{array}{ccccc} \mathscr{L}F & \xrightarrow{\mathscr{L}f} & \mathscr{L}N & \longrightarrow & 0 \\ \downarrow{\scriptstyle \lambda_{\mathscr{L}F}} & & \downarrow{\scriptstyle \lambda_{\mathscr{L}N}} & & \\ \mathscr{L}^2F & \xrightarrow{\mathscr{L}^2f} & \mathscr{L}^2N & \longrightarrow & 0 \end{array}$$

have exact rows, since $\mathscr{L}$ is right exact. Hence

$$\lambda_{\mathscr{L}N}\mathscr{L}f = \mathscr{L}^2f\lambda_{\mathscr{L}F} = \mathscr{L}\lambda_N\mathscr{L}f$$

and so $\lambda_{\mathscr{L}N} = \mathscr{L}\lambda_N$. □

The hypothesis that $\mathrm{St}({}_SP_R) = \mathscr{G}$ allows us to say that $\mathrm{Cost}({}_SP_R)$ is a reflective subcategory of Mod-S, i.e., the inclusion functor has a left adjoint (see [15, Chapter X]).

Theorem 5.7. *Assume* $\mathrm{St}({}_SP_R) = \mathscr{G}$. *Then* $\mathscr{L} : \text{Mod-}S \to \mathrm{Cost}({}_SP_R)$ *is a left adjoint to the inclusion functor.*

Proof. Let $N \in \text{Mod-}S$ and $L \in \mathrm{Cost}({}_SP_R)$. The required bijections

$$\mathrm{Hom}_S(\mathscr{L}N, L) \to \mathrm{Hom}_S(N, L) \quad \text{and} \quad \mathrm{Hom}_S(N, L) \to \mathrm{Hom}_S(\mathscr{L}N, L)$$

are defined by sending $f \in \mathrm{Hom}_S(\mathscr{L}N, L)$ to $f\lambda_N$ and $g \in \mathrm{Hom}_S(N, L)$ to $\lambda_L^{-1}\mathscr{L}g$. Proposition 5.6 is the key point in showing that these are indeed bijections, inverse of one another. □

Corollary 5.8. *Assume* $\mathrm{St}({}_SP_R) = \mathscr{G}$. *Then* $\mathscr{L}F$ *is projective in* $\mathrm{Cost}({}_SP_R)$, *for any projective module* F_S. *Hence the category* $\mathrm{Cost}({}_SP_R)$ *has a projective generator.*

Proof. Epimorphisms in $\mathrm{Cost}({}_SP_R)$ are surjective. □

Proposition 5.9. *Assume* $\mathrm{St}({}_SP_R) = \mathscr{G}$. *The subcategory* $\mathrm{Cost}({}_SP_R)$ *is a Giraud subcategory of* Mod-S *if and only if* $HT_{(1)} = 0$.

Proof. Since we already know that $\mathrm{Cost}({}_SP_R)$ is reflective, we just look at when $\mathscr{L}$ preserves kernels and this is clearly equivalent to saying that $(HT)_{(0)}$ is exact. □

In the paper [12], it was introduced the notion of tilting equivalence between Grothendieck categories.

Definition 5.10 ([12, Definition 1.1]). If $\mathscr{C}_1$ and $\mathscr{C}_2$ are Grothendieck categories, a tilting equivalence between $\mathscr{C}_1$ and $\mathscr{C}_2$ consists of:

(1) a torsion theory $(\mathscr{T}_i, \mathscr{F}_i)$ in $\mathscr{C}_i$ $(i = 1, 2)$, such that every object of $\mathscr{C}_1$ is a subobject of an object in $\mathscr{T}_1$ and every object of $\mathscr{C}_2$ is a quotient of an object in $\mathscr{F}_2$;

(2) an equivalence $F : \mathscr{T}_1 \to \mathscr{F}_2$, with inverse $G : \mathscr{F}_1 \to \mathscr{T}_2$.

It is readily shown that F and G can be extended to the whole categories, in such a way that G is a left adjoint to F.

When $\mathrm{St}({}_SP_R) = \mathscr{G}$ and $HT_{(1)} = 0$, we have a tilting equivalence, where $\mathscr{C}_1 = \mathscr{G}$, $\mathscr{C}_2 = \mathrm{Cost}({}_SP_R)$, $\mathscr{T}_1 = \mathrm{St}_0({}_SP_R) = \mathrm{Gen}(P_R)$, $\mathscr{F}_1 = \mathrm{St}_1({}_SP_R)$, $\mathscr{T}_2 = \mathrm{Cost}_1({}_SP_R)$, $\mathscr{F}_2 = \mathrm{Cost}_0({}_SP_R)$, $F = H$ and $G = T$. In fact every injective object in $\mathscr{G}$ belongs to $\mathrm{St}_0({}_SP_R)$ and every projective object in $\mathrm{Cost}({}_SP_R)$ belongs to $\mathrm{Cost}_0({}_SP_R)$. The fact that $\mathrm{Cost}({}_SP_R)$ is a Giraud subcategory of Mod-S suggests to use the Gabriel-Popescu theorem.

Definition 5.11. We say that a bimodule ${}_SP_R$ is a *GP-tilting bimodule* if

(1) ${}_SP_R$ is weakly $\mathscr{G}$-tilting;

(2) $\mathrm{St}({}_SP_R) = \mathscr{G}$;

(3) $HT_{(1)} = 0$.

We want to see that any GP-tilting bimodule arises in a tilting equivalence context.

Let $\mathscr{A}$ be a Grothendieck category with a projective generator. Then, fixing a projective generator U of $\mathscr{A}$, we have a *Gabriel-Popescu representation* of $\mathscr{A}$, in the sense that we can define (see [15, Chapter X]):

(1) the functor $\eta = \mathrm{Hom}_{\mathscr{A}}(U, -) : \mathscr{A} \to \text{Mod-}S$, where S is the endomorphism ring of U;

(2) the left adjoint τ of η, which is exact;

(3) a hereditary torsion theory $(\mathscr{X}, \mathscr{Y})$ on Mod-S, where $\mathscr{X} = \ker \tau$;

(4) the Giraud subcategory Mod-$(S, \mathscr{Y})$ of $\mathscr{Y}$-closed modules and the functor $\mathscr{L} : \text{Mod-}S \to \text{Mod-}(S, \mathscr{Y})$, left adjoint to the inclusion i;

(5) equivalence functors $\eta' : \mathscr{A} \to \text{Mod-}(S, \mathscr{Y})$ and $\tau' : \text{Mod-}(S, \mathscr{Y}) \to \mathscr{A}$ such that $i\eta' = \eta$ and $\tau'\mathscr{L} = \tau$.

Note that, in this case, also i is exact, since η is.

In the following theorem we will mention a projective generator U of a Grothendieck category $\mathscr{A}$ and refer to the previous notation.

Theorem 5.12. *Let $\mathscr{G}$ be a closed subcategory of* Mod-R *and $\mathscr{A}$ a Grothendieck category with a projective generator U. Assume we are given a tilting equivalence between $\mathscr{G}$ and $\mathscr{A}$, given by the functors $F : \mathscr{T}_1 \to \mathscr{F}_2$ and $G : \mathscr{F}_2 \to \mathscr{T}_1$. We set $P_R = G\tau S$. Then P is in a natural way an S-R-bimodule and:*

(1) *ηF is isomorphic to* $\mathrm{Hom}_R(P, -)$ *and $G\tau$ is isomorphic to $- \otimes_S P$;*

(2) *${}_SP_R$ is GP-tilting;*

(3) $\mathrm{Cost}({}_SP_R) = \text{Mod-}(S, \mathscr{Y})$.

Proof. It is not restrictive to assume that η' and τ' are the identity. Since $\mathrm{End}(S_S) = S$, it is clear that ${}_SP_R$ is a bimodule. Thus

$$iFM \cong \mathrm{Hom}_S(S, iFM) \cong \mathrm{Hom}_S(\mathscr{L}S, FM) \cong \mathrm{Hom}_R(P, M)$$

by the adjunctions. It follows also that $G\mathscr{L} \cong - \otimes_S P$.

We list some statements which follow from Theorem 3.7 in [12].

(1) $\mathscr{T}_1 = \ker F^{(1)} = \mathrm{im}\, G$ and $\mathscr{F}_1 = \ker F = \mathrm{im}\, G_{(1)}$;

(2) $\mathscr{T}_2 = \ker G = \mathrm{im}\, F^{(1)}$ and $\mathscr{F}_2 = \ker G_{(1)} = \mathrm{im}\, F$;

(3) $F^{(1)}$ is a left adjoint to $G_{(1)}$;

(4) $GF^{(1)} = 0$, $F^{(1)}G = 0$, $FG_{(1)} = 0$ and $G_{(1)}F = 0$;

(5) for every $M \in \mathscr{C}_1 = \mathscr{G}$, there exists an exact sequence

$$0 \to GFM \to M \to G_{(1)}F^{(1)}M \to 0;$$

(6) $F^{(1)}$ and $G_{(1)}$ induce an equivalence between $\mathscr{F}_1$ and $\mathscr{T}_2$.

Let us prove that ${}_SP_R$ is weakly $\mathscr{G}$-tilting. Now $\mathscr{T}_1 = \ker F^{(1)}$, so that $\mathscr{T}_1 = \ker H^{(1)}$, by the exactness of i. Moreover $\mathscr{T}_1 \subseteq \mathrm{Gen}(P_R)$, since, for all $M \in \mathscr{T}_1$, $M \cong GFM \cong G\mathscr{L}iFM = THM$. On the other hand, $\mathscr{T}_1 = \mathrm{im}\, G$, so $P_R \in \mathscr{T}_1$ yields, finally $\mathscr{T}_1 = \mathrm{Gen}(P_R)$. By usual arguments (see [4, Proposition 2.2]), it follows that $\mathrm{pd}_{\mathscr{G}}\, P_R \leq 1$.

Since $G_{(1)}$ is left exact, it is clear that $G_{(2)} = 0$, so $T_{(2)} = 0$ and, from Remark 2.4, $\mathrm{id}\, P_S^* \leq 1$. To show that $\mathrm{Cogen}(P_S^*) \subseteq \ker T_{(1)}$, it is sufficient to prove that $T_{(1)}H = 0$. But $T_{(1)}H = G_{(1)}\mathscr{L}iF \cong G_{(1)}F = 0$.

In the same way, $HT_{(1)} = iFG_{(1)}\mathscr{L} = 0$. Hence we need only to see that $\mathrm{St}({}_SP_R) = \mathscr{G}$ or, equivalently, that $GFM, G_{(1)}F^{(1)}M \in \mathrm{St}({}_SP_R)$. First of all $TH^{(1)} = G\mathscr{L}iF^{(1)} \cong GF^{(1)} = 0$. Now ρ_{GFM} is clearly an isomorphism and similarly $G_{(1)}F^{(1)}M \cong T_{(1)}H^{(1)}G_{(1)}F^{(1)}M$. Hence $GFM \in \mathrm{St}_0({}_SP_R)$ and $G_{(1)}F^{(1)}M \in \mathrm{St}_1({}_SP_R)$.

It remains to show that $\mathrm{Cost}({}_SP_R) = \text{Mod-}(S, \mathscr{Y})$. For any module $N \in \mathscr{F}_2 = \ker G_{(1)} = \ker T_{(1)}$, we have

$$(HT)_{(0)}N \cong HTN \cong iFG\mathscr{L}N \cong \mathscr{L}N;$$

hence, taking any $N \in \text{Mod-}S$ and an exact sequence $0 \to K \to Q \to N \to 0$, with Q projective, we have that $K, Q \in \mathscr{F}_2$ and the diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{L}K & \longrightarrow & \mathscr{L}Q & \longrightarrow & \mathscr{L}N & \longrightarrow & 0 \\ & & \downarrow{\cong} & & \downarrow{\cong} & & & & \\ 0 & \longrightarrow & (HT)_{(0)}K & \longrightarrow & (HT)_{(0)}Q & \longrightarrow & (HT)_{(0)}N & \longrightarrow & 0 \end{array}$$

and so there exists a unique isomorphism $\mathscr{L}N \to (HT)_{(0)}N$, making the diagram commute. Hence $(HT)_{(0)}$ is isomorphic to $\mathscr{L}$ and the claim follows. □

Another possible line of interest is the study of "big" weakly tilting modules, in the sense that they induce a counter equivalence between significant subcategories.

A subcategory $\mathscr{G}'$ of $\mathscr{G}$ is called *finitely closed* if it is closed under submodules, quotients and finite direct sums; it is called *generating* if every object in $\mathscr{G}$ is an epimorphic image of a direct sum of objects in $\mathscr{G}'$.

Definition 5.13. A weakly $\mathscr{G}$-tilting bimodule ${}_SP_R$ is said *fc-tilting* if there exist finitely closed and generating subcategories $\mathscr{G}'$ and $\mathscr{S}'$ of $\mathscr{G}$ and Mod-S, respectively, such that the functors H, $H^{(1)}$, T and $T_{(1)}$ induce a counter equivalence between $\mathscr{G}'$ and $\mathscr{S}'$, where the torsion theory on $\mathscr{G}'$ and $\mathscr{S}'$ are

$$(\mathrm{St}_0({}_SP_R) \cap \mathscr{G}', \mathrm{St}_1({}_SP_R) \cap \mathscr{G}') \quad \text{and} \quad (\mathrm{Cost}_1({}_SP_R) \cap \mathscr{S}', \mathrm{Cost}_0({}_SP_R) \cap \mathscr{S}')$$

respectively. For the definition of counter equivalence, we refer to [3].

For example, the bimodule ${}_{J_p}\mathbb{Z}(p^\infty)_{J_p}$ is fc-tilting, since we can take as $\mathscr{G}'$ the class of artinian modules and as $\mathscr{S}'$ the class of noetherian modules.

It is worth noting that such a counter equivalence need not be extendable to a counter equivalence between the whole categories; indeed our example shows this, since the torsion class in $\mathscr{G}'$ contains all finitely cogenerated modules and the torsion-free class in $\mathscr{S}'$ contains all free modules, so that an extension to the whole categories should be given by a tilting object in Subgen($\mathbb{Z}(p^\infty)$) (see [12]) and this class contains no tilting object.

On the other hand, any *equivalence* between finitely closed subcategories can be extended.

Proposition 5.14. *Let $\mathscr{G}'$ and $\mathscr{S}'$ be finitely closed and generating subcategories of $\mathscr{G}$ and* Mod-S*, respectively and let $F : \mathscr{G}' \rightleftarrows \mathscr{S}' : G$ be an equivalence. Then there exists a progenerator $P \in \mathscr{G}'$ such that $S = \mathrm{End}(P_R)$, $F \cong \mathrm{Hom}_R(P, -)$ and $G \cong - \otimes_S P$. In particular $\mathscr{G}$ and* Mod-S *are equivalent.*

Proof. Since $\mathscr{S}'$ is finitely closed and generating, it contains the module S_S. Take $P_R = G(S_S)$: then $S = \mathrm{End}(P_R)$ and, as usual $F \cong \mathrm{Hom}_R(P, -)$. Now P_R must be finitely generated and quasi-projective and it must generate its submodules. By Fuller's theorem [11], it is a quasiprogenerator and so it defines an equivalence between $\mathscr{G}$ and Mod-S. □

6 Tilting torsion theories

The following lemma uses a technique borrowed from [6, Proposition 2.8].

Lemma 6.1. *Let ${}_SL$ be a module admitting an exact sequence in* S-Mod *of the form*

$$0 \to L' \to L'' \to L \to 0$$

where L' and L'' are finitely presented and L'' is flat. Then the functor $\mathrm{Tor}_S^1(-, L)$ *commutes with direct products.*

Proof. Let N_λ be a family in Mod-S. Then we can build the following diagram where, by the flatness of L'', the rows are exact:

$$\begin{array}{ccccccc}
0 \longrightarrow & \operatorname{Tor}_S^1(\prod N_\lambda, L) & \longrightarrow & (\prod N_\lambda)\otimes_S L' & \longrightarrow & & \\
 & \downarrow & & \downarrow & & & \\
0 \longrightarrow & \prod \operatorname{Tor}_S^1(N_\lambda, L) & \longrightarrow & \prod(N_\lambda \otimes_S L') & \longrightarrow & & \\
 & & \longrightarrow & (\prod N_\lambda)\otimes_S L'' & \longrightarrow & (\prod N_\lambda)\otimes_S L & \longrightarrow 0 \\
 & & & \downarrow & & \downarrow & \\
 & & \longrightarrow & \prod(N_\lambda \otimes_S L'') & \longrightarrow & \prod(N_\lambda \otimes_S L) & \longrightarrow 0
\end{array}$$

and, by the fact that L' and L'' are finitely presented, the three rightmost vertical arrows are isomorphisms (see [17, 12.9]); hence also the leftmost vertical arrow is an isomorphism. □

Remark 6.2. Colpi and Trlifaj defined the concept of a *tilting torsion theory* [7, Definition 2.1] as a torsion theory in Mod-R in which the torsion class is generated by a generalized tilting module. For such a torsion class $\mathscr{T}$ there always exists a generalized tilting module P_R such that, setting $S = \operatorname{End}(P_R)$,

(1) $\mathscr{T} = \operatorname{Gen}(P_R) = \ker H^{(1)}$;

(2) $\rho_M : THM \to M$ is an isomorphism, for all $M \in \operatorname{Gen}(P_R)$;

(3) there exists an exact sequence $0 \to R \to P' \to P'' \to 0$, where P' and P'' are direct summands of finite powers of P_R.

Indeed, this is the content of Corollary 2.18 in [7]. In this case the functors H and T induce an equivalence between $\operatorname{Gen}(P_R)$ and $\operatorname{im} H$. Let us call such a module an *e-tilting module* ("e" for equivalence).

The following theorem extends the above mentioned result by Colpi and Trlifaj by providing a counter equivalence between more explicit categories; for example, $\operatorname{im} H = \operatorname{Cost}_0({}_SP_R)$.

Theorem 6.3. *Let P_R be an e-tilting module; then P_R is weakly tilting and, setting* $S = \operatorname{End}(P_R)$, $\operatorname{St}({}_SP_R) = $ Mod-R.

Proof. Applying $\operatorname{Hom}_R(-, P)$ to the exact sequence $0 \to R \to P' \to P'' \to 0$ of Remark 6.2(3) gives the exact sequence $0 \to \operatorname{Hom}_R(P'', P) \to \operatorname{Hom}_R(P', P) \to {}_SP \to 0$ in S-Mod; now $\operatorname{Hom}_R(P'', P)$ and $\operatorname{Hom}_R(P', P)$ are finitely generated projective left S-modules, hence finitely presented and flat. By Lemma 6.1, $T_{(1)}$ commutes with direct products. Therefore, from $T_{(1)}P^* = 0$ it follows that every power of P^* belongs to $\ker T_{(1)}$. Moreover the projective dimension of ${}_SP$ is ≤ 1, so, *a fortiori*, $\operatorname{id} P_S^* \le 1$.

To end the proof, we see from Remark 6.2(2) that every injective module M_R is in $\operatorname{St}_0({}_SP_R)$. Therefore, from Theorem 5.4, we get $\operatorname{St}({}_SP_R) = $ Mod-R. □

Corollary 6.4. *If $\mathscr{T}$ is a tilting torsion class in* Mod-R, *then there exists a weakly tilting module P_R such that* $\mathscr{T} = \operatorname{Gen}(P_R)$.

We can now give a wide class of examples of weakly tilting bimodules ${}_SP_R$ such that $\mathrm{St}({}_SP_R) = \text{Mod-}R$. Indeed, if R is any right noetherian and right hereditary ring and $E(R)$ is the injective hull of R_R, the module $P_R = E(R) \oplus E(R)/R$ is an e-tilting module by [7, Corollary 2.18].

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