CONSORZIO RFX
Ricerca Formazione Innovazione

Formulation and numerical benchmark of improved magneto-fluid boundary conditions for 3D nonlinear MHD code SPECYL

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1. SPECYL and its Boundary Conditions

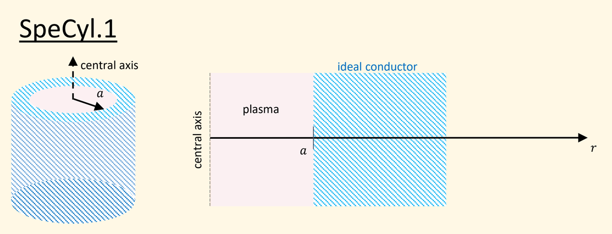
SPECYL:

- Self-consistent
- Non-linear
- Visco-resistive
- Cylindrical plasma

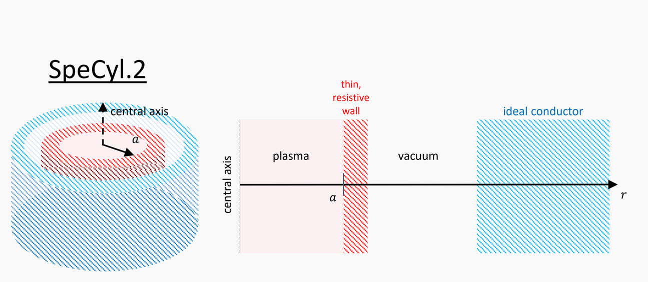
$$\begin{cases} \partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} = \frac{1}{\rho}\boldsymbol{J}\times\boldsymbol{B} + \nu\nabla^2\boldsymbol{v} \\ \partial_t \boldsymbol{B} = \nabla\times(\boldsymbol{v}\times\boldsymbol{B} - \eta\boldsymbol{J}) \\ \boldsymbol{J} = \nabla\times\boldsymbol{B} \\ \nabla\cdot\boldsymbol{B} = 0 \end{cases}$$

Boundary conditions:

SpeCyl.1



SpeCyl.2



Magnetic Part:

On ideal conductor surface: $E_\parallel = B_r = 0$

equilibrium: $E_\theta^{(0,0)} = 0$, $E_z^{(0,0)} = \eta_{pl}\cdot J_z^{(0,0)}$, $B_r^{(0,0)} = 0$ *(loop field)*

modes: $E_\theta^{(m,n)} = 0$, $E_z^{(m,n)} = 0$, $B_r^{(m,n)} = 0$

On resistive thin shell surface: $E_\parallel, \sim B_r \sim time$

$$E_\theta^{(m,n)} = \frac{a}{\tau_w}\left[B_z^{vac} - B_z^{pl}\right]$$
$$E_z^{(m,n)} = \frac{a}{\tau_w}\left[B_\theta^{pl} - B_\theta^{vac}\right]$$
$$\frac{\partial}{\partial t}B_r^{(m,n)} = \frac{1}{\tau_w}\left[\frac{d}{dr}(rB_r^{vac}) - \frac{d}{dr}(rB_r^{pl})\right]$$

Shell transparent to $\vec{B}$ if $\tau_w \sim \tau_A$

Vacuum fields are analytical solutions of Poisson's problem in the cylinder (Bessel's functions)

Fluid Part:

At plasma edge, purely axial V_r (due to $E_z^{(0,0)} \neq 0$)

equilibrium: $V_r^{(0,0)} = -\frac{E_z^{(0,0)}B_\theta^{(0,0)}}{|B|^2}$ ("inward pinch" velocity)

modes: $V_r^{(m,n)} = 0$

Resistive wall permeable to full-spectrum V_r

equilibrium + modes: $(|B|^2\cdot V_r)^{(m,n)} = (E_\theta B_z - E_z B_\theta)^{(m,n)}$ (full-spectrum convolution; full-spectrum convolution)

No chance for finite V_z or V_θ at plasma edge

equilibrium: $V_\theta^{(0,0)} = V_z^{(0,0)} = 0$

modes: $V_\theta^{(m,n)} = V_z^{(m,n)} = 0$

Two alternative sets of BCs for V_θ and V_z

equilibrium + modes:
- No-slip: $V_\theta^{(m,n)} = 0$, $V_z^{(m,n)} = 0$
- No-stress: $\frac{d}{dr}V_\theta^{(m,n)} = 0$, $\frac{d}{dr}V_z^{(m,n)} = 0$

Abstract:

- 3D nonlinear MHD code SPECYL [1] boundary conditions (BCs) have been increasingly made more realistic, from traditional **SpeCyl.1 (ideal conductor facing plasma)** to modelling **plasma-vacuum interface as a resistive thin-shell** [2].
- Linear benchmark against ideal MHD ($\eta_{pl}, \nu_{pl} \to 0$) suggested the need for finite plasma-edge V_r [3].
- We present the **resulting BCs, here dubbed SpeCyl.2**
- **Verification of SpeCyl.2 against another code (Pixie3D** [4]**)**, enforcing analogous physical assumptions in BCs, amends some unphysical idealities of SpeCyl.1 and completes what already done in [5].
- **Linear benchmark against the theory of ideal MHD instabilities** is also presented

Conclusions:

- SpeCyl.2 is a new set of BCs for SPECYL. It combines pre-existing thin-shell like modelling of interface with finite radial (and angular) edge flow.
- Verifications (Br properties and against Pixie3D) found improved self-consistency.
- Benchmark against the theory of linear ideal MHD instabilities (mainly external kinks) show promising behaviour, both concerning modes radial profiles and their exponential growth rates.

2. Verifications of SpeCyl.2

- Components of $\vec{B}_\parallel$ found from Ohm's equation, assuming $E_{\parallel,pl} = E_{\parallel,wall}$

$$B_\theta^{(m,n)} \text{ from } \frac{\eta_{pl}}{\mu_0}\left[\frac{1}{a}\frac{d}{dr}\left(rB_\theta^{(m,n)}\right) - \frac{im}{a}B_r^{(m,n)}\right] - \left(V_r B_\theta - V_\theta B_r\right)^{(m,n)} = \frac{a}{\tau_w}\left[B_\theta^{(m,n)}\right]_{pl}^{vac} \to 0, \quad \text{if } \tau_w \gg t_A$$

$$B_z^{(m,n)} \text{ from } \frac{\eta_{pl}}{\mu_0}\left[\frac{in}{R}B_r^{(m,n)} - \frac{d}{dr}B_z^{(m,n)}\right] - \left(V_z B_r - V_r B_z\right)^{(m,n)} = -\frac{a}{\tau_w}\left[B_z^{(m,n)}\right]_{pl}^{vac} \to 0, \quad \text{if } \tau_w \gg t_A$$

$$B_r^{(m,n)} \text{ from } \frac{\partial}{\partial t}B_r^{(m,n)} = \frac{1}{\tau_w}\left[\frac{d}{dr}(rB_r)\right]_{pl}^{vac} \to 0, \quad \text{if } \tau_w \gg t_A$$

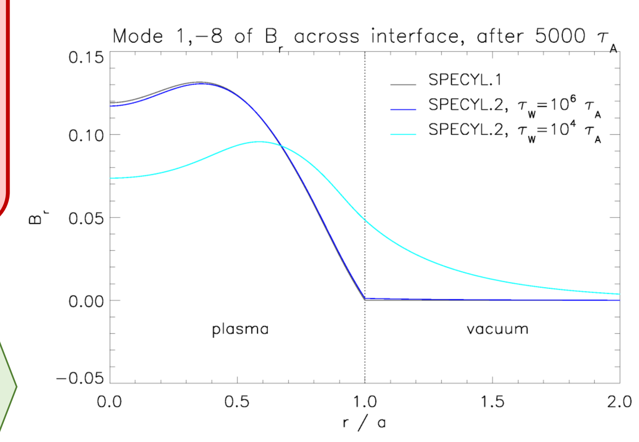
Convolutions now have all terms → self-consistency

Ideal wall limit yields SpeCyl.1 $\vec{B}$ field

Fully consistent magneto-fluid BCs

Magnetic diffusion across interface

- $B_r(r=a)$ slope imbalance levels up in $t \sim \mathcal{O}(\tau_w)$
- If $\tau_w \to \infty$ the shell becomes ideal: SpeCyl.1 limit case
- Vacuum fields are analytical solutions of Poisson's problem



Reversed-Field Pinch (RFP) 2D simulation, 10 modes m/n=1/8, externally resonating (near axis) kink m=1.

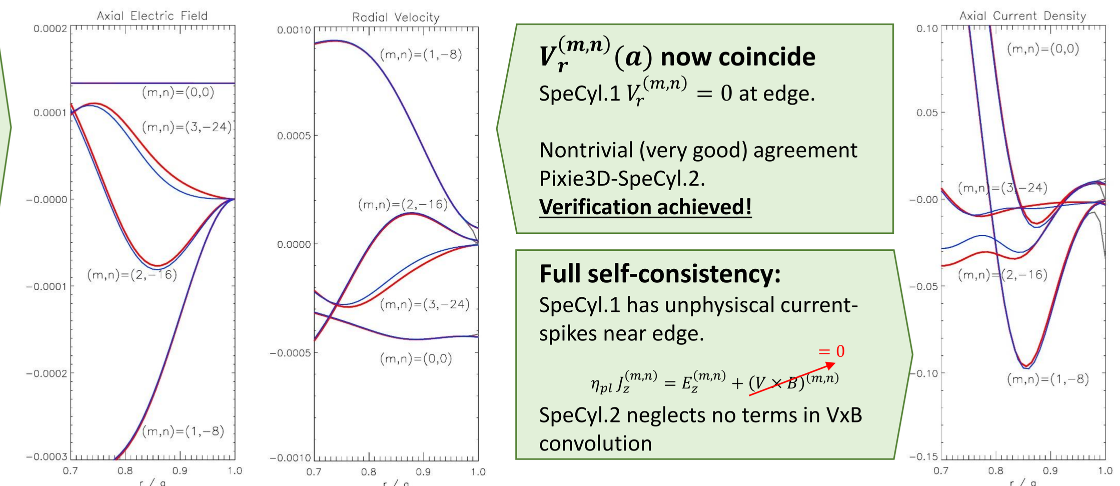
- Successful nonlinear benchmark of SpeCyl.1 and Pixie3D already in [5], now extended to SpeCyl.2 (ideal wall limit)

Verification against Pixie3D

More self-consistent than SpeCyl.1

Same BCs:
Ideal wall in $r = a$
$\to E_\parallel^{(0,0)} = E_\parallel^{(m,n)} = 0$
if not for
$E_z^{(0,0)} = \eta_{pl} J_z^{(0,0)} \neq 0$

— SpeCyl.1
— SpeCyl.2 ($\tau_w \to \infty$)
— Pixie3D (ideal wall)



$V_r^{(m,n)}(a)$ now coincide
SpeCyl.1 $V_r^{(m,n)} = 0$ at edge.
Nontrivial (very good) agreement Pixie3D-SpeCyl.2.
Verification achieved!

Full self-consistency:
SpeCyl.1 has unphysical current-spikes near edge.
$\eta_{pl} J_z^{(m,n)} = E_z^{(m,n)} + (V\times B)_z^{(m,n)}$ ($= 0$)
SpeCyl.2 neglects no terms in VxB convolution

Reversed-Field Pinch (RFP) 2D simulation, 10 modes m/n=1/8, saturated externally resonating (near axis) kink m=1. Profiles taken at final time 5000 · τ_A

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3. Energy Principle and numerical set-up

Energy Principle: what is vacuum [6,7]

Momentum balance equation: $\rho\frac{\partial \boldsymbol{v}}{\partial t} + \rho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \boldsymbol{J}\times\boldsymbol{B} - \nabla p$ (Plasma reaction; Force density $\mathcal{F}$)

$\boldsymbol{v} = \boldsymbol{v}_0 + \partial_t\boldsymbol{\xi}$, $\quad \boldsymbol{\xi} = \begin{pmatrix}\xi_r(r)\\ \xi_\theta(r)\\ \xi_z(r)\end{pmatrix} e^{i(m\theta - \frac{n}{R}z - \omega t)}$

"Newton's law": Leading order: $\rho\ddot{\xi} = \mathcal{F}[\xi]$

"Energy balance": $-\omega^2\int_V \frac{\rho}{2}\boldsymbol{\xi}\cdot\boldsymbol{\xi}^* d^3\boldsymbol{r} = -\frac{1}{2}\int_V \boldsymbol{\mathcal{F}}[\boldsymbol{\xi}]\cdot\boldsymbol{\xi}^* d^3\boldsymbol{r}$ (Kinetic term $K[\xi|\xi]$; System potential variation $\delta W[\xi,\xi^*]$)

Energy balance: $\omega^2 = -\frac{\delta W[\xi,\xi^*]}{K[\xi|\xi]}$

Euler's equation: ξ such that $\delta W[\xi+\delta\xi, \xi^*+\delta\xi^*] - \delta W[\xi,\xi^*] = 0$

Extremant for potential variation ↓ Most favourable displacement

- ξ is chosen after force density $\mathcal{F} = J\times B - \nabla p$ and regardless of ρ ($\nabla p \approx 0$ for large aspect ratio)
- **Modes stability** (=sign of ω^2) depends solely on ξ
- **Growth rate $\gamma = \text{Im}\{\omega\}$** depends also on ρ

Vacuum is the same as $J = 0$

Vacuum requires also $\rho = 0$

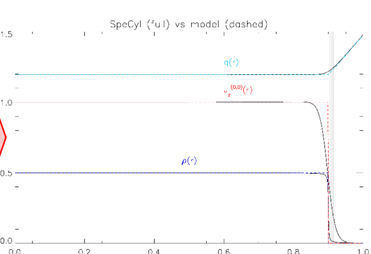
Numeric set-up for linear benchmark

- Profile of $J_z^{(0,0)}$ numerically produced by shaping $\eta_{pl}(r)$
- Step-profile of $\rho(r)$: **Plasma region** $\rho(r<a) \approx 1$; **Pseudo vacuum** $\rho(a<r<r_{wall}) \approx 10^{-3}$ (shaded in figures)

pseudo vacuum is used in many other codes, *e.g.* in JOREK [8]

- Two current models:

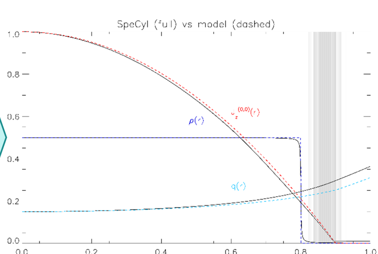
Shafranov $J_z^{(0,0)} = \begin{cases} J_0, & r < r_0 \\ 0, & r_0 < r < r_{wall}\end{cases}$



$a = r_0 = 0{,}9$

Density and current drop at the same radius!

Wesson $J_z^{(0,0)} = \begin{cases} J_0(1-(r/r_0)^2), & r < r_0 \\ 0, & r_0 < r < r_{wall}\end{cases}$



$a = 0{,}8 < r_0 = 0{,}9$

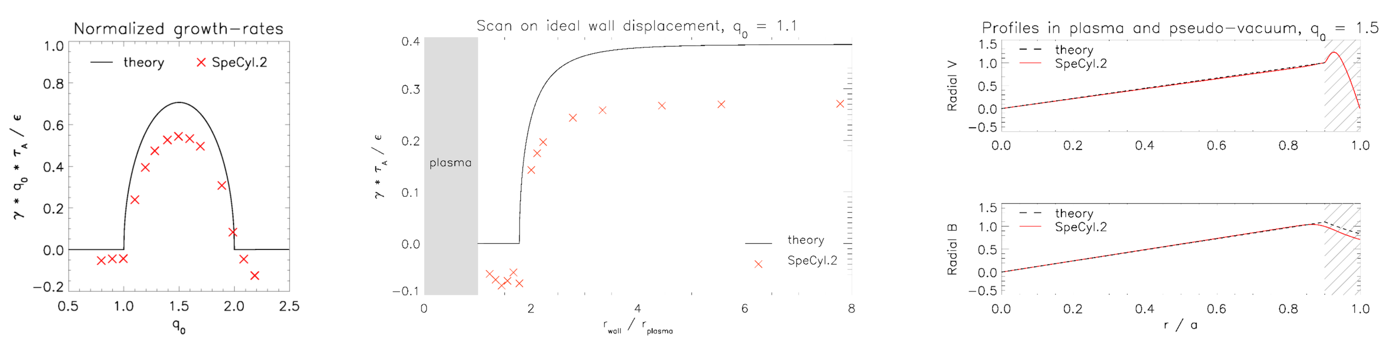
Highly resistive (unideal) region $0{,}8 < r < 0{,}9$ already in pseudo-vacuum.

4. Linear Benchmark against ideal MHD external kink

Shafranov's flat-current model

Growth rate and profiles, mode m=2

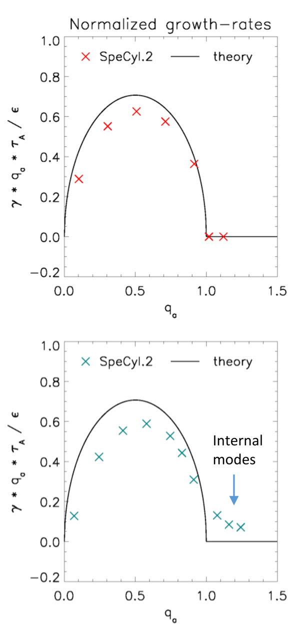
- SpeCyl.2 reliably predicts instability region for kink m=2 ($1 \le q_a \le 2$)
- Little under-estimate of γ. Small $\gamma < 0$ when theory gives stability ($\gamma = 0$)
- Ideal-wall scan qualitatively good. Matching of critical stabilizing distance. Results compatible with [8]
- Profiles in optimal agreement in plasma region and psudo-vacuum (only $B_r^{(2,1)}$)



Shafranov (flat-current) model in tokamak geometry, external kink m=2. Inverse aspect ratio small $\epsilon = 0.05$, $\eta_{pl}(0) = 10^{-7}$, $\nu_{pl} = 10^{-10} \cdot r$, $\tau_w = \tau_A/100$, 2D simulations with m/n=2, no-stress BCs for V_θ and V_z. Growth rates normalized on toroidal Alfvén time τ_A/ϵ and (first figure) edge-plasma safety factor q_a.

Shafranov's and Wesson's models

Growth rate and profiles, mode m=1

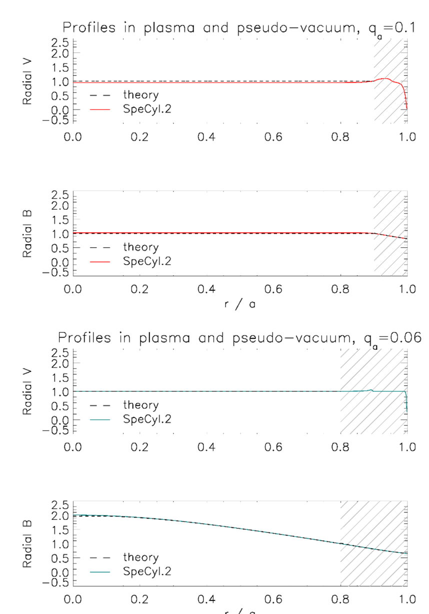


Shafranov (flat-current) model:

- Very good agreement on γ: boundaries, values
- Profiles are well-resolved
- Flat $q(r)$ profile allows no internal modes

Wesson (peaked-current) model:

- Good agreement on γ.
- Profiles are well-resolved.
- Shaped $q(r)$ intercepts also internal resonances



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