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FAMILIES OF SOLUTIONS OF MATRIX RICCATI DIFFERENTIAL EQUATIONS*

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Abstract. The J. C. Willems–Coppel–Shayman geometric characterization of solutions of the algebraic Riccati equation (ARE) is extended to *asymmetric* Riccati *differential* equations with *time-varying coefficients*. The coefficients do not need to satisfy any *definiteness*, *periodicity*, or *system-theoretic condition*. More precisely, given any two solutions $X_1(t)$ and $X_2(t)$ of such equation on a given interval $[t_0, t_1]$, we show how to construct a family of solutions of the same equation of the form $X(t) = (I - \pi(t))X_1(t) + \pi(t)X_2(t)$, where π is a suitable matrix-valued function. Even when specialized to the case of X_1 and X_2 equilibrium solutions of a symmetric equation with constant coefficients, our results considerably extend the classical ones, as no further assumption is made on the pair X_1, X_2 and on the coefficient matrices.

Key words. asymmetric Riccati differential equation, families of solutions, geometric characterization, invariant subspaces, projection-preserving differential equation

AMS subject classifications. 34A05, 34A26, 34A34, 15A24, 49K15

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1. Families of solutions of the RDE. Consider the asymmetric Riccati differential equation (RDE)

$$\dot{X} = AX + XB + XPX + Q, \tag{1.1}$$

where X is $m \times n$ and A, B, P, Q are continuous, matrix-valued functions with real entries on $[t_0, t_1]$ of dimension $m \times m, n \times n, n \times m$, and $m \times n$, respectively. As is well known, the symmetric version of (1.1), i.e., when $n = m$, $B = A^T$, $P = P^T$, $Q = Q^T$, plays a central role in many fields of applied mathematics, including optimal control and estimation, and has therefore been intensively studied. General Riccati equations such as (1.1) arise in the theory of differential games [3], in state-space solutions to H^∞ problems [10], in polynomial factorization [5], in problems of feedback control [1], and in the singular perturbation of boundary value problems [4]; see the introductions of [17, 3, 12] for further information. A further example is provided by equation (1.7) below, which is asymmetric even when (1.1) is symmetric.

All through this paper, X_1 and X_2 denote two fixed but arbitrary solutions of (1.1) on the time interval $[t_0, t_1]$. Moreover, let $\Delta_{12} := X_2 - X_1$. There exists a one-to-one correspondence between solutions of (1.1) and solutions of the homogeneous Riccati equation

$$\dot{\Delta} = A_{X_1}\Delta + \Delta B_{X_1} + \Delta P \Delta, \tag{1.2}$$

where $A_{X_1} := A + X_1 P$ and $B_{X_1} := B + PX_1$, given by $X \leftrightarrow \Delta = X - X_1$. Thus, all results below concerning solutions of (1.1) may also be viewed as results concerning

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solutions of (1.2), where the roles of X_1 and X_2 are played by the zero solution and Δ_{12}, respectively.

Jan Willems' classification of solutions of the ARE [23] was used in [14] to classify all output-induced minimal stochastic realizations of a given process. In [2, Theorem 8.3], this classification was extended to the nonstationary case. Its implications for the RDE, however, were not pursued there. Jan Willems' original derivation of the geometric parametrization of solutions of the ARE relied on first establishing a similarity relation involving two "extreme" closed-loop matrices [23, Lemma 8]. The latter result was generalized to the symmetric, nonsingular (i.e., Δ_{12} invertible), time-varying situation in [18, Theorem 5.5]. It can indeed be extended to our very general setting, and its consequences are far reaching.

LEMMA 1.1. *Let X be any solution of* (1.1) *on* $[t_0, t_1]$ *and let* $\Delta_i := X - X_i$, $i = 1, 2$. *Let* $\phi(\cdot,\cdot)$ *and* $\psi(\cdot,\cdot)$ *be the transition matrices corresponding to* $A_X := A + XP$ *and* $-B_X := -(B + PX)$, *respectively. Let* $\phi_i(\cdot,\cdot)$ *and* $\psi_i(\cdot,\cdot)$, $i = 1, 2$, *be the transition matrices corresponding to* $A_{X_i} := A + X_iP$ *and* $-B_{X_i} := -(B + PX_i)$, *respectively. Then, for* $i = 1, 2$ *and for all* s *and* t *in* $[t_0, t_1]$, *we have*

$$\Delta_i(t)\psi_i(t,s) = \phi(t,s)\Delta_i(s), \tag{1.3}$$

$$\Delta_i(t)\psi(t,s) = \phi_i(t,s)\Delta_i(s). \tag{1.4}$$

Proof. Notice that Δ_i satisfies

$$\begin{aligned} \dot{\Delta}_i &= A_{X_i}\Delta_i + \Delta_i B_{X_i} + \Delta_i P \Delta_i \\ &= A_X\Delta_i + \Delta_i B_{X_i} = A_{X_i}\Delta_i + \Delta_i B_X. \end{aligned} \tag{1.5}$$

From (1.5) it follows that

$$\begin{aligned} \frac{\partial(\Delta_i(t)\psi_i(t,s))}{\partial t} &= \dot{\Delta}_i(t)\psi_i(t,s) + \Delta_i(t)\frac{\partial\psi_i(t,s)}{\partial t} \\ = \dot{\Delta}_i(t)\psi_i(t,s) - \Delta_i(t)B_{X_i}\psi_i(t,s) &= A_X(t)\Delta_i(t)\psi_i(t,s). \end{aligned}$$

Hence, both sides of (1.3) satisfy

$$\frac{\partial W(t,s)}{\partial t} = A_X(t)W(t,s).$$

Since they coincide for $s = t$, they coincide everywhere. Exchanging the roles of X and X_i, we get (1.4) from (1.3). □

COROLLARY 1.2. $\Delta_i(t), i = 1, 2,$ *has constant rank on* $[t_0, t_1]$.

Proof. By (1.3), $\Delta_i(t) = \phi(t, t_0)\Delta_i(t_0)\psi_i(t_0, t)$. □

COROLLARY 1.3. *Let X be any solution of* (1.1) *on* $[t_0, t_1]$, *and let* $i = 1, 2$. *Suppose that* $\ker \Delta_{12}(t_0) \subseteq \ker \Delta_i(t_0)$. *Then* $\ker \Delta_{12}(t) \subseteq \ker \Delta_i(t)$ *for all* $t \in [t_0, t_1]$.

Proof. Let $x \in R^n$ be such that $\Delta_{12}(t)x = 0$. By (1.3), we get $\Delta_{12}(t_0)\psi_i(t_0, t)x = 0$. Thus, $\psi_i(t_0, t)x$ is in the kernel of $\Delta_{12}(t_0)$. By hypothesis, $\Delta_i(t_0)\psi_i(t_0, t)x = 0$. The latter implies $\phi(t, t_0)\Delta_i(t_0)\psi_i(t_0, t)x = 0$. Using equation (1.3) again, we get $\Delta_i(t)x = 0$. □

Obviously, the above result holds true if t_0 is replaced by any other time s in $[t_0, t_1]$. Let us agree that all through the paper $\pi(t)$ denotes an $m \times m$ matrix function on $[t_0, t_1]$.

THEOREM 1.4. *The matrix function* $X(t) = (I - \pi(t))X_1(t) + \pi(t)X_2(t)$ *is a solution of* (1.1) *on* $[t_0, t_1]$ *if and only if* $\pi(t)$ *is a* C^1 *function satisfying*

$$\dot{\pi}\Delta_{12} = [A_{X_1}\pi - \pi A_{X_1} - \pi\Delta_{12}P(I - \pi)]\Delta_{12}. \tag{1.6}$$

Conversely, let $X(t)$ *be a solution of* (1.1) *on* $[t_0, t_1]$ *with* $\ker \Delta_{12}(t_0) \subseteq \ker \Delta_1(t_0)$ *where* $\Delta_1 = X - X_1$. *Then there exists a* C^1 *function* $\pi(t)$ *satisfying* (1.6) *such that* $X(t) = (I - \pi(t))X_1(t) + \pi(t)X_2(t)$. *Moreover, if* $\mathrm{Rank}\Delta_{12}(t_0) = m$, (1.6) *may be replaced by the* auxiliary Riccati differential equation (ARDE)

$$\dot{\pi} = A_{X_1}\pi - \pi A_{X_1} - \pi\Delta_{12}P(I - \pi), \tag{1.7}$$

and there is a one-to-one correspondence between solutions of (1.1) *and solutions of* (1.7).

Proof. Let $\mathcal{R}(X) := AX + XB + XPX + Q$. If $X(t) = (I - \pi(t))X_1(t) + \pi(t)X_2(t)$, we get

$$\begin{aligned}
\mathcal{R}(X) &= A[(I-\pi)X_1 + \pi X_2] + [(I-\pi)X_1 + \pi X_2]B \\
&\quad + [(I-\pi)X_1 + \pi X_2]P[(I-\pi)X_1 + \pi X_2] + Q \\
&= (I-\pi)\mathcal{R}(X_1) + \pi\mathcal{R}(X_2) - (I-\pi)AX_1 - (I-\pi)X_1PX_1 - \pi AX_2 \\
&\quad - \pi X_2PX_2 + A(I-\pi)X_1 + A\pi X_2 + [(I-\pi)X_1 + \pi X_2]P[(I-\pi)X_1 + \pi X_2] \\
&= (I-\pi)\mathcal{R}(X_1) + \pi\mathcal{R}(X_2) - \pi A\Delta_{12} + A\pi\Delta_{12} \\
&\quad - (I-\pi)X_1P\pi X_1 - \pi X_2P(I-\pi)X_2 + \pi X_2P(I-\pi)X_1 + (I-\pi)X_1P\pi X_2 \\
&= (I-\pi)\mathcal{R}(X_1) + \pi\mathcal{R}(X_2) + [-\pi A + A\pi + (I-\pi)X_1P\pi - \pi X_2P(I-\pi)]\Delta_{12} \\
&= (I-\pi)\mathcal{R}(X_1) + \pi\mathcal{R}(X_2) \\
&\quad + [-\pi A + A\pi + (I-\pi)X_1P\pi - \pi\Delta_{12}P(I-\pi) - \pi X_1P(I-\pi)]\Delta_{12} \\
&= (I-\pi)\mathcal{R}(X_1) + \pi\mathcal{R}(X_2) + [A_{X_1}\pi - \pi A_{X_1} - \pi\Delta_{12}P(I-\pi)]\Delta_{12}.
\end{aligned}$$

If π is of class C^1, it then follows that X is a solution of (1.1) if and only if (1.6) holds. Conversely, suppose that X is a solution of (1.1) on $[t_0, t_1]$ such that $\ker \Delta_{12}(t_0) \subseteq \ker \Delta_1(t_0)$. By Corollary 1.3, the inclusion $\ker \Delta_{12}(t) \subseteq \ker \Delta_i(t)$ holds for all $t \in [t_0, t_1]$. Then there exist $m \times m$-valued matrix functions $Z(t)$ such that

$$\Delta_1(t) = Z(t)\Delta_{12}(t) \tag{1.8}$$

for all $t \in [t_0, t_1]$. Notice that (1.8) already implies that $X(t) = (I - Z(t))X_1(t) + Z(t)X_2(t)$. Thus, the proof of the converse will be complete if we can show that among the functions Z satisfying (1.8) there is at least one $\tilde{Z}$ of class C^1. In that case, we can take $\pi = \tilde{Z}$. To this end, notice that, in view of Lemma 1.1, any function Z satisfying (1.8) also satisfies

$$\Delta_1(t)\psi_2(t, t_0) = Z(t)\phi_1(t, t_0)\Delta_{12}(t_0). \tag{1.9}$$

This leads us to introduce the function $\tilde{Z}$ defined by

$$\tilde{Z}(t) = \Delta_1(t)\psi_2(t, t_0)\Delta_{12}^{\#}(t_0)\phi_1(t_0, t),$$

where $\Delta_{12}^{\#}(t_0)$ denotes the Moore–Penrose pseudoinverse of $\Delta_{12}(t_0)$. The function $\tilde{Z}$ is clearly continuously differentiable. We show next that indeed $\tilde{Z}$ satisfies $\Delta_1(t) = \tilde{Z}(t)\Delta_{12}(t)$. Observe that the latter is equivalent to

$$\Delta_1(t)[I - \psi_2(t, t_0)\Delta_{12}^{\#}(t_0)\phi_1(t_0, t)\Delta_{12}(t)] = 0. \tag{1.10}$$

Now let Z be any function satisfying (1.8). Using (1.9) in (1.10), we see that the latter is equivalent to

$$Z(t)\phi_1(t, t_0)\Delta_{12}(t_0)\psi_2(t_0, t)[I - \psi_2(t, t_0)\Delta_{12}^{\#}(t_0)\phi_1(t_0, t)\Delta_{12}(t)] = 0.$$

Obtaining $\Delta_{12}(t)$ from Lemma 1.1 and using properties of transition matrices, it can be verified that the latter equation is in turn equivalent to

$$Z(t)\phi_1(t,t_0)\Delta_{12}(t_0)[I-\Delta_{12}^{\#}(t_0)\Delta_{12}(t_0)]\psi_2(t_0,t)=0. \tag{1.11}$$

Because of $\Delta(I-\Delta^{\#}\Delta)=0$, the preceding identity (1.11) is valid. Finally, suppose that $\mathrm{Rank}\Delta_{12}(t_0)=m$. By Corollary 1.2 the same is true for $\Delta_{12}(t)$, $t\in[t_0,t_1]$. The one-to-one map between the solution sets of (1.1) and (1.7) is then given by $\pi(t):=[X(t)-X_1(t)]\Delta_{12}^{-R}(t)$, where Δ_{12}^{-R} denotes any right inverse of Δ_{12}. ∎

Remark 1.5. Obviously, in Theorem 1.4 (and in the following), we could have considered combinations of the form $X(t)=X_1(t)(I-\sigma(t))+X_2(t)\sigma(t)$. The assumption for the converse part would then read $\ker\Delta_{12}(t_0)^T\subseteq\ker\Delta_1(t_0)^T$. Equation (1.6) would be replaced by the equation

$$\Delta_{12}\dot\sigma=\Delta_{12}[\sigma B_{X_1}-B_{X_1}\sigma-(I-\sigma)P\Delta_{12}\sigma].$$

Remark 1.6. Notice that if π_1 and π_2 are two C^1 functions generating the same solution X of (1.1) on $[t_0,t_1]$, i.e., $X(t)=X_1(t)+\pi_1(t)\Delta_{12}(t)=X_1(t)+\pi_2(t)\Delta_{12}(t)$, then necessarily $[\pi_1(t)-\pi_2(t)]\Delta_{12}(t)=0$ for all t in $[t_0,t_1]$. If $\Delta_{12}(t_0)$ admits a right inverse, then $\pi_1=\pi_2$.

At first sight, the correspondence between solutions of (1.1) and solutions of (1.6) or (1.7) established by Theorem 1.4 appears rather disappointing. Indeed, in the best case, we still have to deal with an asymmetric Riccati equation, the ARDE, with the only apparent advantage that π, A_{X_1}, and $\Delta_{12}P$ are all square $m\times m$-dimensional. Notice that solutions X_1 and X_2 of (1.1) correspond to the equilibrium solutions zero and identity, respectively, of (1.6) and (1.7). Nevertheless, the power of this connection will shortly be apparent. Indeed, (1.6) and (1.7) lend themselves naturally to a geometric characterization of a subclass of their solutions; see Theorems 2.3 and 2.5 below.

We conclude this section with a result relating different ϕ transition matrices. This result, which will not be needed in what follows, appears to be of interest for nonstationary stochastic realization [2]. Indeed, it extends a result for feedback matrices corresponding to different solutions of the symmetric ARE that was applied to stationary stochastic realization in [11, Lemma 4.1].

PROPOSITION 1.7. *Let X be any solution of* (1.1) *on* $[t_0,t_1]$. *If $X(t)-X_1(t)=\Delta_1(t)=\pi(t)\Delta_{12}(t)$ on $[t_0,t_1]$ for some function π, we have, in the notation of Lemma* 1.1,

$$\{\phi(t,s)\pi(s)-\pi(t)\phi_2(t,s)\}\Delta_{12}(s)=0, \tag{1.12}$$

$$\{\phi(t,s)(I-\pi(s))-(I-\pi(t))\phi_1(t,s)\}\Delta_{12}(s)=0. \tag{1.13}$$

If π is projection valued, it follows that

$$(I-\pi(t))\phi(t,s)\pi(s)\Delta_{12}(s)=0, \tag{1.14}$$

$$\pi(t)\phi(t,s)(I-\pi(s))\Delta_{12}(s)=0. \tag{1.15}$$

If, moreover, π is C^1, the latter gives

$$(I-\pi)(\dot\pi-A_X\pi)\Delta_{12}=0, \tag{1.16}$$

$$\pi(\dot\pi+A_X(I-\pi))\Delta_{12}=0. \tag{1.17}$$

Proof. Employing (1.3) twice, once for X and once for X_2, we get

$$\phi(t,s)\pi(s)\Delta_{12}(s) = \phi(t,s)\Delta_1(s) = \Delta_1(t)\psi_1(t,s)$$
$$= \pi(t)\Delta_{12}(t)\psi_1(t,s) = \pi(t)\phi_2(t,s)\Delta_{12}(s),$$

which is (1.12). Similarly, (1.13) is established. If π is a C^1, projection-valued function, differentiating (1.14) and the equation $\pi(t) = \pi(t)^2$ with respect to t, we get $[(I-\pi(t))A_X - \dot{\pi}(t)]\phi(t,s)\pi(s)\Delta_{12}(s) = 0$ and $\dot{\pi}(t)\pi(t) = (I-\pi(t))\dot{\pi}(t)$, respectively. Evaluating the first at $s = t$ and then using the second, we get (1.16). Similarly, we get (1.17) from (1.15). □

2. Geometric results. The first step in establishing a geometric characterization of certain families of solutions of (1.1) consists of rewriting (1.6) and (1.7). Simply rearranging terms, we get that these equations are equivalent to

$$[\dot{\pi} - (I-\pi)A_{X_1}\pi + \pi A_{X_2}(I-\pi)]\Delta_{12} = 0, \tag{2.1}$$
$$\dot{\pi} - (I-\pi)A_{X_1}\pi + \pi A_{X_2}(I-\pi) = 0, \tag{2.2}$$

where $A_{X_2} := A + X_2P = A_{X_1} + \Delta_{12}P$.

LEMMA 2.1. *If π is a projection for all times, i.e., $\pi(t) = \pi(t)^2$ for t in $[t_0, t_1]$, then it satisfies* (2.1) *if and only if it satisfies the system of equations*

$$(I-\pi)[\dot{\pi} - A_{X_1}\pi]\Delta_{12} = 0, \tag{2.3}$$
$$\pi[(I-\dot{\pi}) - A_{X_2}(I-\pi)]\Delta_{12} = 0. \tag{2.4}$$

Proof. Multiplying (2.1) on the left first by $(I-\pi)$ and then by π, we get (2.3) and (2.4), respectively. Conversely, obtaining $\pi\dot{\pi}\Delta_{12}$ from (2.4) and plugging it into (2.3), we get (2.1). □

Remark 2.2. Equations (2.3), (2.4) can be obtained from (1.16), (1.17), observing that

$$(I-\pi)A_{X_1} = (I-\pi)A_X,$$
$$\pi A_{X_2} = \pi A_X.$$

Equations (2.1), (2.2), (2.3), (2.4) enjoy a certain symmetry. Indeed, they are invariant under the permutation $\pi \leftrightarrow (I-\pi)$, $X_1 \leftrightarrow X_2$. Lemma 2.1 above singles out a subclass of solutions of (2.1) and, by Theorem 1.4, of (1.1). This subclass may also be described as the solutions on $[t_0, t_1]$ of the following *implicit system*:

$$[0, I-\pi, \pi]\dot{\pi}\Delta_{12} = [\pi - \pi^2, (I-\pi)A_{X_1}\pi\Delta_{12}, -\pi A_{X_2}(I-\pi)\Delta_{12}]. \tag{2.5}$$

The following result provides a geometric characterization of the *projection-valued* solutions of (2.1). The question of *existence* of such solutions will be addressed in Theorem 2.7 below.

THEOREM 2.3. *Let $X(t) = (I-\pi(t))X_1(t) + \pi(t)X_2(t)$ be a solution of* (1.1) *on $[t_0, t_1]$. Let $M(t) := \pi(t)\Delta_{12}(t)R^n$ and $N(t) := (I-\pi(t))\Delta_{12}(t)R^n$. Then, for s and t in $[t_0, t_1]$, we have*

$$M(t) = \phi_1(t,s)M(s), \tag{2.6}$$
$$N(t) = \phi_2(t,s)N(s). \tag{2.7}$$

Moreover, we also have

$$M(t) = \phi(t,s)M(s), \tag{2.8}$$

$$N(t) = \phi(t,s)N(s). \tag{2.9}$$

Conversely, Let $\{M(t)\}$ *and* $\{N(t)\}$, $t \in [t_0, t_1]$, *be two families of subspaces of* R^m *providing a direct sum decomposition of* $\Delta_{12}(t)R^n$. *Let* π *be a* C^1 *function such that* $\pi(t)x = x\ \forall x \in M(t)$ *and* $\pi(t)y = 0\ \forall y \in N(t)$. *If* (2.6), (2.7) *hold for all* s *and* t *in* $[t_0, t_1]$, *then* $X(t) = (I - \pi(t))X_1(t) + \pi(t)X_2(t)$ *is a solution of the* RDE (1.1) *on* $[t_0, t_1]$.

Proof. By Lemma 1.1, $\Delta_1(t) = \phi_1(t,s)\Delta_1(s)\psi(s,t)$. Replacing $\Delta_1(t)$ with $\pi(t)\Delta_{12}(t)$, we get $\pi(t)\Delta_{12}(t)R^n = \phi_1(t,s)\pi(s)\Delta_{12}(s)R^n$, namely (2.6) holds. Formula (2.7) is proven similarly. Lemma 1.1 also gives $\Delta_1(t) = \phi(t,s)\Delta_1(s)\psi_1(s,t)$. The same argument as above then gives (2.8). Similarly, (2.9) is established. To prove the converse, notice that (2.6), (2.7) imply

$$[I - \pi(t)]\phi_1(t,s)\pi(s)\Delta_{12}(s) = 0, \tag{2.10}$$

$$\pi(t)\phi_2(t,s)[I - \pi(s)]\Delta_{12}(s) = 0. \tag{2.11}$$

Evaluating the derivatives of (2.10) and (2.11) with respect to t on the diagonal $t = s$, we get (2.3) and (2.4). The latter imply that (1.6) holds, and consequently X is a solution of (1.1). □

Remark 2.4. Notice that the first half of the theorem holds for *any* solution X of (1.1) of the form $X(t) = (I - \pi(t))X_1(t) + \pi(t)X_2(t)$, namely even when π is not projection valued. In that case, however, the spaces $M(t)$ and $N(t)$ do not need to form a direct sum. Observing once more that $X - X_1 = \pi\Delta_{12}$ and $X_2 - X = (I - \pi)\Delta_{12}$, we also see that the spaces $M(t)$ and $N(t)$ are *uniquely determined* by the solution X and do not depend on the particular projection π used in the definition.

In the important case where Δ_{12} has full row rank, Theorem 2.3 reads as follows.

THEOREM 2.5. *Assume that* $\Delta_{12}(t_0)$ *has full row rank. Let* $X(t) = (I - \pi(t))X_1(t) + \pi(t)X_2(t)$ *be a solution of* (1.1) *on* $[t_0, t_1]$. *Let* $M(t)$ *and* $N(t)$ *denote the range of* $\pi(t)$ *and the range of* $(I - \pi(t))$, *respectively. Then, for* s *and* t *in* $[t_0, t_1]$, *relations* (2.6), (2.7), (2.8) *and* (2.9) *hold true. Conversely, let* $\pi(\cdot)$ *be a* C^1, *projection-valued function on* $[t_0, t_1]$, *and let* $M(t)$ *and* $N(t)$ *denote the range of* $\pi(t)$ *and the range of* $(I - \pi(t))$, *respectively. If the propagation relations* (2.6) *and* (2.7) *hold for all* s *and* t *in* $[t_0, t_1]$, *then* $X(t) = (I - \pi(t))X_1(t) + \pi(t)X_2(t)$ *is a solution of the* RDE (1.1) *on the same time interval.*

Theorems 2.3 and 2.5 provide the desired geometric characterization of a subclass of solutions of the ARDE (2.2) and, consequently, of the RDE (1.1). Notice that, in the case $m = n$, Remark 2.4 gives that the first half of Theorem 2.5 applies to *any* solution of (1.1) on $[t_0, t_1]$. Indeed, in this case, $\Delta_{12}(t)$ is invertible at all times, and $\ker \Delta_{12}(t) \subseteq \ker \Delta_1(t)$ is trivially satisfied. Hence, any solution X of (1.1) can be expressed as $X(t) = (I - \pi(t))X_1(t) + \pi(t)X_2(t)$. For the purpose of immediate comparison, we state below Jan Willems' classical result; cf. also [6, 19, 20, 21, 13] (the latter should also be compared with Theorems 3.3 and 4.2 below).

THEOREM 2.6. *In equation* (1.1), *let* $n = m$, $B = A^T$, $P = P^T$, $Q = Q^T$. *Suppose moreover that* P *is negative semidefinite and that the pair* (A, P) *is reachable. Let* X_- *and* X_+ *denote two symmetric equilibrium solutions of* (1.1) *such that the corresponding closed-loop matrices* $A_- := A + X_-P$ *and* $A_+ := A + X_+P$ *have all their eigenvalues in the closed right and left half-planes, respectively. Suppose that*

$\Delta := X_+ - X_-$ *is positive definite. Then* X *is another symmetric equilibrium solution of* (1.1) *if and only if it can be expressed as*

$$X = (I - \pi)X_- + \pi X_+$$

where π *projects onto an* A_-*-invariant subspace and* $I - \pi$ *projects onto an* A_+*-invariant subspace.*

We now turn to the question of *existence of projection-valued solutions* of (2.2) ((1.7)) (equivalently, of solutions of the implicit system (2.5) if $\Delta_{12}(t_0)$ has full row rank). The following remarkable result basically says that (2.2) is a *projection-preserving differential equation.*

THEOREM 2.7. *Let* π *be a solution of* (2.2) *on* $[t_0, t_1]$. *Suppose that* $\pi(t_0)$ *is a projection. Then* $\pi(t)$ *is a projection for all* t *in* $[t_0, t_1]$.

Proof. Let us rewrite (2.2) as

$$\dot{\pi} = A_{X_1}\pi - \pi A_{X_2} + \pi\Delta_{12}P\pi.$$

Then

$$\begin{aligned}\frac{d\pi^2}{dt} = \dot{\pi}\pi + \pi\dot{\pi} &= [A_{X_1}\pi - \pi A_{X_2} + \pi\Delta_{12}P\pi]\pi + \pi[A_{X_1}\pi - \pi A_{X_2} + \pi\Delta_{12}P\pi] \\ &= A_{X_1}\pi^2 - \pi A_{X_2}\pi + \pi\Delta_{12}P\pi^2 + \pi A_{X_1}\pi - \pi^2 A_{X_2} + \pi^2\Delta_{12}P\pi.\end{aligned}$$

Hence,

$$\frac{d(\pi^2 - \pi)}{dt} = A_{X_1}(\pi^2 - \pi) - (\pi^2 - \pi)A_{X_2} - (\pi^2 - \pi)\Delta_{12}P(\pi^2 - \pi) + \pi^2\Delta_{12}P\pi^2 - \pi\Delta_{12}P\pi.$$

Adding and subtracting the quantity $\pi^2\Delta_{12}P\pi$ in the right-hand side and rearranging terms, we finally get

$$\frac{d(\pi^2 - \pi)}{dt} = (A_{X_1} + \pi^2\Delta_{12}P)(\pi^2 - \pi) - (\pi^2 - \pi)(A_{X_2} - \Delta_{12}P\pi) - (\pi^2 - \pi)\Delta_{12}P(\pi^2 - \pi).$$

Let $F_1 := A_{X_1} + \pi^2\Delta_{12}P$ and $F_2 := A_{X_2} - \Delta_{12}P\pi$. It follows that, if $\pi(t)$ is a solution of (2.2), then, on the same time interval, $\pi^2 - \pi$ is a solution of the homogeneous Riccati equation

$$\dot{X} = F_1 X - X F_2 - X\Delta_{12}PX, \tag{2.12}$$

and F_1 and F_2 are there bounded. Since $\pi^2(t_0) - \pi(t_0) = 0$, by uniqueness of the solution of equation (2.12) starting at zero, it follows that $\pi^2(t) - \pi(t) = 0$ on all of $[t_0, t_1]$. □

The above proof actually establishes an amplification of Theorem 2.7. We record it below because it is of interest on its own.

PROPOSITION 2.8. *Let* A_1 *and* A_2 *be* $m \times m$ *continuous matrix functions on* $[t_0, t_1]$. *Let* Y *be an* $m \times m$ *matrix function solving the homogeneous Riccati equation*

$$\dot{Y} = A_1 Y - Y A_2 + Y(A_2 - A_1)Y \tag{2.13}$$

on $[t_0, t_1]$. *If there exists a time* $\bar{t} \in [t_0, t_1]$ *such that* $Y(\bar{t}) = Y(\bar{t})^2$, *then* $Y(t) = Y(t)^2$ *on all of* $[t_0, t_1]$.

Remark 2.9. Notice that $Y_1 \equiv 0$ and $Y_2 \equiv I$ are two equilibrium solutions of (2.13). Also notice that the corresponding closed-loop matrices are $A_1 + 0(A_2 - A_1) = A_1$

and $A_1 + I(A_2 - A_1) = A_2$. Now let Y be as in the proposition above—namely, a projection-valued solution of (2.13)—and let $M(t)$ and $N(t)$ be the range spaces of $Y(t)$ and $I - Y(t)$, respectively. Then, by Theorem 2.5, the propagation properties (2.6) and (2.7) hold true, where ϕ_1 and ϕ_2 are the transition matrices corresponding to A_1 and A_2, respectively. Finally, if A_1 and A_2 are constant and $Y = Y^2$ is an equilibrium solution of (2.13), Y projects onto a subspace invariant for A_1 along a subspace invariant for A_2.

Remark 2.10. The geometric results of this section provide a procedure to produce new solutions of (1.1). For instance, in the full-rank case, let π_0 be any projection, and let M_0 and N_0 be the ranges of π_0 and $I-\pi_0$, respectively. Define $M(t) := \phi_1(t,t_0)M_0$ and $N(t) := \phi_2(t,t_0)N_0$. Let $\bar{t}$ be the largest time such that for $t_0 \le t < \bar{t}$, $M(t)$ and $N(t)$ give a direct sum decomposition of R^m (by continuity, $\bar{t} > 0$). Let $\pi(t)$ be the projection such that $M(t)$ and $N(t)$ are the ranges of $\pi(t)$ and $I - \pi(t)$, respectively. Then π solves (2.2) and $X = (I-\pi)X_1 + \pi X_2$ solves (1.1) on $[t_0, \bar{t})$. Using an explicit expression for π in terms of bases for its range and the range of $I - \pi$, it is easily seen that $\pi(t)$ becomes unbounded as t tends to $\bar{t}$. If $\Delta_{12}(t_0)$ has full row rank, it follows that the corresponding solution $X(\cdot)$ has a finite escape time (see, e.g., [16, 7, 8]) at $t = \bar{t}$.

We conclude the section with an example that illustrates Remark 2.10 as well as Proposition 2.8 and Remark 2.9.

Example 2.11. Consider equation (2.13) with $m = 2$ and

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Choose as reference solutions $Y_1 = 0$ and $Y_2 = I$, and let π_0 be given by

$$\pi_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Clearly π_0 is a projection, in fact an orthogonal projection. We have that $M_0 = \binom{R}{0}$ and $N_0 = \binom{0}{R}$. Next notice that the transition matrices $\phi_1(t,s)$ and $\phi_2(t,s)$ are given here by

$$\phi_1(t,s) = e^{A_1(t-s)} = \begin{pmatrix} e^{t-s} & 0 \\ 0 & 1 \end{pmatrix}, \quad \phi_2(t,s) = e^{A_2(t-s)} = \begin{pmatrix} 1 & t-s \\ 0 & 1 \end{pmatrix}.$$

Hence, $M(t) = \phi_1(t,t_0)M_0$ is the span of the vector $\binom{1}{0}$ and $N(t) = \phi_2(t,t_0)N_0$ is the span of the vector $\binom{t-t_0}{1}$. Notice that $M(t)$ and $N(t)$ provide a direct sum decomposition of R^2 for all $t \ge t_0$. The projection $\pi(t)$ with range $M(t)$ and kernel $N(t)$ is given by

$$\pi(t) = \begin{pmatrix} 1 & t_0 - t \\ 0 & 0 \end{pmatrix}.$$

The corresponding solution of (2.13) is $Y(t) = 0 + \pi(t)(I - 0) = \pi(t)$, namely $\pi(t)$ itself. This is no surprise. Since $Y(t_0) = 0 + \pi_0(I - 0) = \pi_0$ is a projection, Theorem 2.7 implies that $Y(t)$ has to be a projection for all t. Notice that $\pi(t)$ is unbounded as t tends to infinity. This is possible because $\pi(t)$ for $t > 0$ is not an *orthogonal* projection, although π_0 is orthogonal.

3. Geometric results: The case where X_1 and X_2 are equilibrium solutions. All through this section we assume that X_1 and X_2 are equilibrium solutions of (1.1). The coefficients A, B, P, and Q may still be time varying.

PROPOSITION 3.1. *Let X be an equilibrium solution of* (1.1) *and let $\Delta_i = X - X_i$, $i = 1, 2$. Then for all t in $[t_0, t_1]$,*

$$A_X(t)\Delta_i = -\Delta_i B_{X_i}(t), \tag{3.1}$$

$$A_{X_i}\Delta_i = -\Delta_i B_X(t). \tag{3.2}$$

It follows that if $(\xi(t), \lambda(t))$ is an eigenvector-eigenvalue pair for $B_{X_i}(t)$ so that $B_{X_i}(t)\xi(t) = \lambda(t)\xi(t)$, then either $\Delta_i\xi(t) = 0$ or $(\Delta_i\xi(t), -\lambda(t))$ is an eigenvector-eigenvalue pair for $A_X(t)$. Similarly, it follows for $A_{X_i}(t)$ and $B_X(t)$. If Δ_i admits a right inverse, we get the relations

$$A_X(t) = -\Delta_i B_{X_i}(t)\Delta_i^{-R},$$
$$A_{X_i}(t) = -\Delta_i B_X(t)\Delta_i^{-R}.$$

In particular, if $m = n$ and Δ_{12} is invertible, we have

$$A_{X_2}(t) = -\Delta_{12} B_{X_1}(t)\Delta_{12}^{-1}. \tag{3.3}$$

Proof. Relations (3.1) and (3.2) are a consequence of (1.5). □

Once more, we compare (3.3) with the corresponding classical result. In the notation of Theorem 2.6, let $X_1 = X_-$ and $X_2 = X_+$. Then (3.3) reads $A_+ = -\Delta A_-^T \Delta^{-1}$ which is precisely [23, Lemma 8]. Let us now assume that the coefficients of (1.1) are constant. Theorems 1.4 and 2.3 yield the following result.

THEOREM 3.2. *Let $X = (I - \pi)X_1 + \pi X_2$ be an equilibrium solution of the RDE* (1.1). *Let $M := \pi\Delta_{12}R^n$ and $N := (I - \pi)\Delta_{12}R^n$. Then M is an invariant subspace for A_{X_1} and N is an invariant subspace for A_{X_2}. Moreover, M and N are both invariant for A_X. Conversely, let M and N be two subspaces of R^m providing a direct sum decomposition of $\Delta_{12}R^n$. Let π be an $m \times m$ matrix such that $\pi x = x$ for any x in M and $\pi y = 0$ for any y in N. If M is an invariant subspace for A_{X_1} and N is an invariant subspace for A_{X_2}, then $X = (I - \pi)X_1 + \pi X_2$ is an equilibrium solution of the RDE* (1.1).

Once more, we state independently the result in the case when Δ_{12} has full row rank.

THEOREM 3.3. *Suppose Δ_{12} has full row rank and let X be an equilibrium solution of* (1.1). *Assume that* $\ker \Delta_{12} \subseteq \ker \Delta_1$. *Then there exists an $m \times m$ matrix π such that $X = (I - \pi)X_1 + \pi X_2$. Moreover, the range M of π is invariant for A_{X_1} and for A_X, and the range N of $I - \pi$ is invariant for A_{X_2} and for A_X. Conversely, if π is any oblique projection onto a subspace invariant for A_{X_1} along a subspace invariant for A_{X_2}, then $X = (I - \pi)X_1 + \pi X_2$ satisfies* (1.1).

In order to compare this result with Theorem 2.6, notice that if $m = n$ and Δ_{12} has full rank, the condition $\ker \Delta_{12} \subseteq \ker \Delta_1$ is always satisfied. The additional assumptions of Theorem 2.6 permit us to conclude that if $X = (I - \pi)X_1 + \pi X_2$ is an equilibrium solution of (1.1), π is always a projection.

4. The symmetric Riccati equation. We finally consider the symmetric case where $n = m$, $B = A^T$, $P = P^T$, $Q = Q^T$ but return to the nonequilibrium situation. Equation (1.1) is now

$$\dot{X} = AX + XA^T + XPX + Q. \tag{4.1}$$

We also assume that the two reference solutions X_1 and X_2 take values in the symmetric matrices. Hence, $\Delta_{12}(t)$ is also symmetric at all times. It is then natural to restrict our attention to *symmetric* solutions of (4.1).

LEMMA 4.1. $\phi_2(t,s)\Delta_{12}(s) = \Delta_{12}(t)\phi_1(s,t)^T$.

Proof. By Lemma 1.1, $\phi_2(t,s)\Delta_{12}(s) = \Delta_{12}(t)\psi_1(t,s)$. The conclusion now follows observing that $B_{X_1} = A_{X_1}^T$ implies that $\psi_1(t,s) = \phi_1(s,t)^T$. ∎

For the sake of simplicity, we only give the main result in the case where Δ_{12} is nonsingular.

THEOREM 4.2. *Let X_1 and X_2 be any two symmetric solutions of* (4.1) *on* $[t_0,t_1]$ *such that $\Delta_{12}(t_0)$ is invertible. Let $X(t) = (I-\pi(t))X_1(t)+\pi(t)X_2(t)$ be a symmetric solution of* (4.1) *on* $[t_0,t_1]$. *Let $M(t)$ and $N(t)$ denote the range of $\pi(t)$ and the range of $(I-\pi(t))$, respectively. Then for s and t in* $[t_0,t_1]$ *the following relations hold true:*

$$\pi(t)\Delta_{12}(t) = \Delta_{12}(t)\pi(t)^T, \tag{4.2}$$

$$(I-\pi(t))\phi_1(t,s)\pi(s) = 0. \tag{4.3}$$

Conversely, let π be a C^1, projection-valued function satisfying for all s and t in $[t_0,t_1]$ (4.2), (4.3). *Then $X(t) = (I-\pi(t))X_1(t)+\pi(t)X_2(t)$ is also a symmetric solution of the RDE* (4.1) *on* $[t_0,t_1]$.

Proof. Let $X(t) = (I-\pi(t))X_1(t)+\pi(t)X_2(t) = X_1(t)+\pi(t)\Delta_{12}(t)$ be a symmetric solution of (4.1) on $[t_0,t_1]$. The symmetry of $X(t)$ implies that (4.2) must hold. Let $M(t)$ and $N(t)$ denote the range of $\pi(t)$ and of $I-\pi(t)$, respectively. By Theorem 2.5, we have $M(t) = \phi_1(t,s)M(s)$ from which (4.3) follows. Conversely, suppose that (4.2) and (4.3) are verified. From (4.3) we get $\phi_1(t,s)M(s) \subseteq M(t)$. Exchanging the roles of s and t, we see that equality, i.e., equation (2.6), must hold. Now, multiplying equation (4.3) (with s and t exchanged) by $\Delta_{12}(s)^{-1}$ on the left and by $\Delta_{12}(t)$ on the right we get

$$\Delta_{12}(s)^{-1}(I-\pi(s))\phi_1(s,t)\pi(t)\Delta_{12}(t) = 0. \tag{4.4}$$

Transposing (4.4) and using (4.2) twice, we get

$$\pi(t)\Delta_{12}(t)\phi_1(s,t)^T\Delta_{12}(s)^{-1}(I-\pi(s)) = 0.$$

The latter equation, together with Lemma 4.1, now gives equation (2.7). The conclusion now follows from Theorem 2.5. ∎

5. Closing comments. As is well known, the Riccati differential equation may be viewed as the description in local coordinates of the restriction to a subset of the Lagrangian Grassmannian manifold $\mathcal{L}$ of a vector field on $\mathcal{L}$; see [15, 19]. Our results may then be readily interpreted in that setting. In fact, some may be also directly derived in that setting; see [8], where the case of $l \geq 2$ reference solutions $X_1, X_2, \ldots, X_l$, is also considered (see also [17, Theorem 4]). Similar results may also be derived in the discrete-time setting [9]. Alternative representation formulas for solutions of (1.1) have been proposed in [22] and references therein.

The classification of the solutions of the ARE via invariant subspaces of the Hamiltonian matrix has the disadvantage, when compared with the J. C. Willems classification, that the invariant subspaces need to be J-neutral and complementary to the subspace Span $\binom{0}{I}$; see [13, pp. 67–68]. In [19] it was observed that the disadvantages of Jan Willems method are that, contrary to the Hamiltonian matrix method, it does not lead naturally to a concept of solution at infinity (phenomenon of

the finite escape time; see, e.g., [16, 7]) and it does not have an obvious generalization to the nonsymmetric Riccati equation. Whereas the first disadvantage persists, we observe that this paper has completely removed the second.

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