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GLOBAL SOLUTIONS TO THE GRADIENT FLOW EQUATION OF A NONCONVEX FUNCTIONAL*

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Abstract. We study the L^2-gradient flow of the nonconvex functional $F_\phi(u) := \frac{1}{2}\int_{(0,1)} \phi(u_x)\, dx$, where $\phi(\xi) := \min(\xi^2, 1)$. We show the existence of a global in time possibly discontinuous solution u starting from a mixed-type initial datum u_0, i.e., when u_0 is a piecewise smooth function having derivative taking values both in the region where $\phi'' > 0$ and where $\phi'' = 0$. We show that, in general, the region where the derivative of u takes values where $\phi'' = 0$ progressively disappears while the region where ϕ'' is positive grows. We show this behavior with some numerical experiments.

Key words. nonconvex functionals, forward-backward parabolic equations, finite element method

AMS subject classifications. 35K55, 35B05

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1. Introduction.

Let $\phi : \mathbb{R} \to [0, +\infty)$ be the nonconvex continuous function defined as

$$\phi(\xi) := \begin{cases} \xi^2 & \text{if } |\xi| \le 1, \\ 1 & \text{otherwise.} \end{cases} \tag{1.1}$$

In this paper we study the L^2-gradient flow of the nonconvex functional

$$F_\phi(u) := \frac{1}{2}\int_{(0,1)} \phi(u_x)\, dx, \qquad u \in BV(0,1), \tag{1.2}$$

where u_x stands for the absolutely continuous part of the distributional derivative of u. Note that $\phi^{**} \equiv 0$, where ϕ^{**} is the convex envelope of ϕ; hence the L^2-lower semicontinuous envelope of F_ϕ is identically zero. Note also that if the initial datum u_0 is smooth and such that $u_{0x}([0,1]) \subset (-1,1)$, it is reasonable to look for a solution of the gradient flow of F_ϕ which coincides with the usual solution of the heat equation starting from u_0. In particular, such a solution cannot coincide with the standing solution $u(x,t) \equiv u_0(x)$ obtained as the gradient flow of the lower semicontinuous envelope of F_ϕ.

The solution $u(x,t)$ of the formal gradient flow of F_ϕ should satisfy the following evolution equation:

$$\begin{cases} u_t = u_{xx}, & \text{where } |u_x| < 1, \\ u_t = 0, & \text{where } |u_x| > 1, \\ u(0) = u_0, \end{cases} \tag{1.3}$$

but the behavior of the interface $\{|u_x| = 1\}$ is not apparent.

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While existence and regularity theories for solutions of gradient flow equations originated by convex energies is well established (see, for instance, [12], [31], [4], [2]), very little is known for nonconvex evolution problems. The main difficulty is due to the fact that nonconvexity of the energy density leads in general to ill-posed (i.e., backward-parabolic) problems and, as a consequence, to instabilities in the evolution. The lack of forward parabolicity of the equation shows that even the local in time existence of a solution (in some reasonable class of functions) is not straightforward, as well as uniqueness and regularity. We refer the reader to [30] and to the papers [26], [27], [32], [29], [23], [24], [5], [6] for some results in this direction and for possible regularization techniques. We point out that variational models involving (1.2) have been used in [11] in the context of image segmentation; see also [14]. See also the papers [28], [20], where other backward-forward parabolic equations, such as the Perona–Malik equation corresponding to the choice $\phi_{PM}(\xi) := \log(1+\xi^2)$, have been used to reconstruct a digital image; see [34], [33], [13], [17], [18], [7], [8], [9].

Among nonconvex energy densities, the function ϕ in (1.1) is maybe the simplest one (despite the fact that it is not of class $\mathcal{C}^1$, there are no points in $\mathbb{R} \setminus \{\pm 1\}$ where ϕ'' is negative), and this motivates our choice of studying the gradient flow of the associated functional F_ϕ.

The aim of the present paper is to prove the existence of a reasonable notion of (discontinuous) *global* solution u to the gradient flow of F_ϕ starting from u_0; we stress that u_0 will be allowed to be of mixed type, i.e., to have points where u_{0x} belongs to the locally convex region $(-1,1)$ of ϕ *and* points where u_{0x} belongs to the region $\mathbb{R} \setminus [-1,1]$. We show that, in general, the interface $\{|u_x| = 1\}$ has a velocity, and that the region where u_x takes values in $(-1,1)$ has the tendency to grow at the expenses of the remaining region, with a well determined speed. Thus we are in the presence of a free boundary problem and, in general,

(a) our solution does not coincide with the standing solution $u(x,t) \equiv u_0(x)$;

(b) our solution does not coincide with the solution of (1.3) obtained by keeping the interface $\{|u_x| = 1\}$ fixed and by imposing the condition

$$\lim_{y \to x,\, y \in \{|u_x(\cdot,t)|<1\}} u_x(y,t) = 0 \qquad \text{for } x \in \{|u_x(\cdot,t)| = 1\}, \tag{1.4}$$

i.e., zero Neumann boundary conditions from the side of $\{|u_x| < 1\}$;

(c) these behaviors appear in numerical experiments; see section 7.

Observe that the lack of forward parabolicity precludes, as far as we know, a direct way to construct global solutions based on the comparison principle, such as viscosity solutions [15] or minimal barriers [10]. Moreover, global solutions obtained by using the usual minimization methods (such as the implicit Euler scheme; see [16]) coincide with the solution $u(x,t) \equiv u_0(x)$; this is due to the fact that, in the minimization procedure, the functional F_ϕ can be equivalently replaced with its lower semicontinuous envelope.

In the present paper we restrict the analysis to periodic boundary conditions, even if the same technique can be adapted to different situations such as Neumann or Dirichlet boundary conditions. We base our approach on the study of the system of ODEs obtained as the gradient flow of the restriction $F_{\phi_{|V_N}}$ of F_ϕ to V_N, the space of continuous piecewise affine functions on a uniformly distributed grid of $[0,1]$ of size $1/N$. The function $F_{\phi_{|V_N}}$ turns out to be Lipschitz continuous; nevertheless, it is possible to give a precise notion to the equation $\dot{u} = -\nabla(F_{\phi_{|V_N}})(u)$. After solving the resulting system of ODEs, we pass to the limit as the discretization step goes to zero ($N \to +\infty$), and we identify the limit problem. This sort of regularization is particularly handleable (as a consequence of the special features of ϕ in (1.1)) since the interior of the region $\{|u_x| > 1\}$ has zero velocity, so that we can focus the attention only at the free boundary $\{|u_x| = 1\}$. This is a remarkable simplification, for instance in comparison with the Perona–Malik equation where the quick formation of microstructures in the region where $|u_x| > 1$ seems to be present.

The plan of the paper is the following. In section 2 we state the main result (Theorem 2.4). We look for a solution in the class of ϕ-admissible functions in the sense of Definition 2.1. Several comments clarify both the definition and the theorem (see, in particular, Remark 2.3 concerning condition (4) of Definition 2.1). In section 3 we motivate from a variational point of view the evolution law. In section 4 we discretize the problem and introduce the discretized operator A_u; see Definition 4.4. The rigorous analysis of the discretized scheme is performed in section 5; in particular, in Theorem 5.4 we prove the basic estimates and comparisons necessary to pass to the limit as $N \to +\infty$. In section 6 we prove Theorem 2.4. In Remark 6.16 we discuss in which sense our solution could provide a solution to the gradient flow of the Mumford–Shah functional in one dimension. In section 7 we implement our scheme and show that the numerical experiments are in agreement with Theorem 2.4. In particular, we show that the free boundary $\{|u_x(\cdot)| = 1\}$ has, in general, nonzero speed.

We conclude this introduction by observing that the analysis of the gradient flow of (1.2) could be considered as a first step toward the understanding of the behavior of the Perona–Malik equation.

2. Statement of the main results.

We now state the main results of the paper (Theorem 2.4). To this purpose we need some preparation. $BV(0,1)$ stands for the space of functions with bounded variation in $(0,1)$. If $u \in BV(0,1)$ and $x \in (0,1)$, $u(x_-)$ (resp., $u(x_+)$) is the left (resp., right) limit of u at x. We always identify the function u with its representative defined pointwise everywhere as the mean value of u; i.e., $u(x) = (u(x_+)+u(x_-))/2$ for any $x \in (0,1)$. We set $u(0) := u(0_+)$ $u(1) := u(1_-)$. We denote by J_u the jump set of u.

We recall that the distributional derivative of $u \in BV(0,1)$ is represented by a measure Du, with finite total variation in $(0,1)$ (which we denote by $\|Du\|$), and that it splits into the sum of an absolutely continuous part (which we denote by u_x or by u') and a singular part. We refer the reader to [3] for the main properties of BV functions. If $u : [0,T) \to \mathbb{R}$, we indicate by $\frac{d}{dt^+}u$ the right derivative of u; i.e., $\frac{d}{dt^+}u(t) := \lim_{h\to 0^+} \frac{u(t+h)-u(t)}{h}$ for any $t \in [0,T)$, provided the limit is finite.

If u depends on $(x,t) \in (0,1) \times (0,T)$, we write $u(t)(\cdot) = u(\cdot,t) = u(t)$.

Given $B \subseteq \mathbb{R}$ we denote by $\overline{B}$ (resp., $\mathrm{int}(B)$, ∂B, $\#(B)$, $|B|$) the closure (resp., the interior part, the topological boundary, the number of elements, the Lebesgue measure) of B. We denote by $d_{\mathcal{H}}(\cdot,\cdot)$ the Hausdorff distance between sets.

Our analysis is restricted to a subset of $BV(0,1)$ given by the *ϕ-admissible* functions, according to the following definition.

DEFINITION 2.1. *Let $u \in BV(0,1)$ with $u(0) = u(1)$. We say that u is ϕ-*admissible*, and we write $u \in \mathcal{A}_\phi(0,1)$, if there exist a natural number $m \geq 0$ and real numbers $0 < a_1 \leq b_1 < \cdots < a_m \leq b_m < 1$ such that, setting*

$$\sigma_B^\phi(u) := \bigcup_{j=1}^{m} [a_j, b_j] \subset (0,1), \qquad \sigma_G^\phi(u) := [0,1] \setminus \sigma_B^\phi(u), \tag{2.1}$$

we have

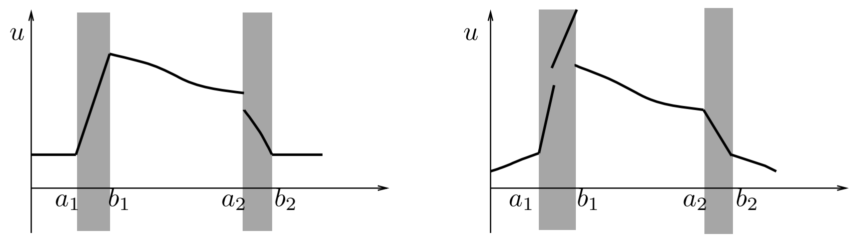

FIG. 2.1. *The gray rectangles correspond to the closed intervals of* $\sigma_B^\phi(u)$. *The function on the left is* ϕ*-admissible. The function on the right is not* ϕ*-admissible, because it is not monotone on the* closed *interval* $[a_1, b_1]$.

(1) $|u(x) - u(y)| \le |x - y|$ *whenever* $[x, y] \subset \sigma_G^\phi(u)$;
(2) *if* $a_j = b_j$ *for some* $j \in \{1, \dots, m\}$, *then* $a_j \in J_u$;
(3) *if* $j \in \{1, \dots, m\}$ *and* $a_j < b_j$, *then* $|u(x) - u(y)| > |x - y|$ *whenever* $x, y \in [a_j, b_j]$, $x \ne y$;
(4) *if* $j \in \{1, \dots, m\}$ *and* $a_j < b_j$, *then* u *is monotone on* $[a_j, b_j]$.

Remark 2.2. Let us clarify Definition 2.1.

(a) Note that $\sigma_G^\phi(u) \ne \emptyset$ for any $u \in \mathcal{A}_\phi(0,1)$. We adopt the convention that there are no points a_i, b_j if $m = 0$; in this case, $\sigma_B^\phi(u) = \emptyset$, $\sigma_G^\phi(u) = (0,1)$ and u is one-Lipschitz in the whole of $(0,1)$. The assumptions $a_1 > 0$ and $b_m < 1$ are not restrictive, since we can always assume (up to a translation) that a one-periodic function $u \in \mathcal{C}^1(\mathbb{R})$ is such that $|u_x(0)| < 1$. Due to our periodicity assumption, the point $\{0\}$ is identified with $\{1\}$ and can be considered as belonging to the interior of $\sigma_G^\phi(u)$.
(b) In each interval I of $\sigma_G^\phi(u)$ we have that u is one-Lipschitz; hence, at almost every $x \in I$ we have that $u_x(x)$ belongs (unless $|u_x(x)| = 1$) to the set where ϕ is twice differentiable and $\phi'' > 0$, i.e., $u_x(x) \in (-1, 1)$.
(c) In each interval I of $\sigma_B^\phi(u)$ we have that

$$Du(A) \ge |A| \quad \forall\, A \subseteq I \qquad \text{or} \qquad Du(A) \le -|A| \quad \forall\, A \subseteq I,$$

with the strict inequalities when $|A| > 0$, where A is any Borel subset of I.
(d) The class $\mathcal{A}_\phi(0,1)$ is L^2-dense in $BV(0,1)$.

The following remark shows some analogy with the entropy condition in hyperbolic conservation laws.

Remark 2.3. Condition (4) in Definition 2.1 is required on the *closed* intervals $[a_j, b_j]$. Hence, since $u(x) = (u(x_+) + u(x_-))/2$ for any $x \in (0,1)$, if u is discontinuous at some a_j and u is nondecreasing on $[a_j, b_j]$ (resp., u is nonincreasing on $[a_j, b_j]$), then $u(a_j) \le u(a_{j_+})$ (resp., $u(a_j) \ge u(a_{j_+})$). Similarly, it happens if u is discontinuous at some b_j; see Figure 2.1. Condition (4) is fulfilled at each time by the solution that we are going to construct in Theorem 2.4 and arises naturally as a consequence of the approximation procedure through spatial discretizations. Ultimately, it can be considered as a consequence of the fact that, once a region in $\sigma_G^\phi(u_N(t))$ appears for the discretized solutions $u_N(t)$ considered in Theorem 5.4 below, it must persist (and possibly increase) with time.

Let us denote by $AC^2([0, +\infty); L^2(0,1))$ the space of absolutely continuous functions u from $[0, +\infty)$ to $L^2(0,1)$ such that $u_t \in L^2((0, +\infty) \times (0,1))$; see, for instance, [2]. Let $V_N \subset H^1(0,1)$ be the N-dimensional vector space of one-periodic continuous functions on $\mathbb{R}$ which are affine on every interval of the form $[i/N, (i+1)/N]$ with

$i = 0, \ldots, N-1$. It is clear that $V_N \subset \mathcal{A}_\phi(0,1)$ and that each function in V_N is N-Lipschitz.

Let us denote by A_u the differential of $F_{\phi|V_N}$ at $u \in V_N$; the linear operator A_u is a discrete Laplace operator with zero blocks corresponding to the region $\sigma_B^\phi(u)$ and zero Neumann boundary conditions on the boundaries; see Remark 4.3 and Definition 4.4 below.

THEOREM 2.4. *Let $u_0 \in \mathcal{A}_\phi(0,1)$, and write*

$$\sigma_B^\phi(u_0) = \bigcup_{j=1}^m [a_j^0, b_j^0].$$

Then there exist a sequence of initial data $(u_0^N) \subset V_N$, a sequence (u^N) of functions taking $[0,+\infty)$ in V_N, and a function $u : (0,1) \times [0,+\infty) \to \mathbb{R}$ with the following properties:

(i) *There exist numbers $0 < a_1^{0N} \le b_1^{0N} < \cdots < a_m^{0N} \le b_m^{0N} < 1$ such that*

$$\sigma_B^\phi(u_0^N) = \bigcup_{j=1}^m [a_j^{0N}, b_j^{0N}], \tag{2.2}$$

and

$$\begin{aligned}
&\lim_{N\to+\infty} \|u_0^N - u_0\|_{L^2} = 0,\\
&\lim_{N\to+\infty} \left(\|u_0^N\|_{BV(0,1)} - \|u_0\|_{BV(0,1)}\right) = 0,\\
&\lim_{N\to+\infty} \Big(d_{\mathcal{H}}(\sigma_G^\phi(u_0^N), \sigma_G^\phi(u_0)) + d_{\mathcal{H}}(\sigma_B^\phi(u_0^N), \sigma_B^\phi(u_0))\Big) = 0,\\
&\lim_{N\to+\infty} F_\phi(u_0^N) = F_\phi(u_0).
\end{aligned} \tag{2.3}$$

(ii) *$u^N : [0,+\infty) \to V_N$ is continuous and right-differentiable, and satisfies*

$$\begin{cases} \dfrac{d}{dt^+} u^N(t) = A_{u^N(t)} u^N(t), & t \in [0,+\infty),\\ u^N(0) = u_0^N. \end{cases} \tag{2.4}$$

(iii) *$u^N, u \in L^\infty((0,+\infty); BV(0,1)) \cap AC^2([0,+\infty); L^2(0,1))$, and $u^N \rightharpoonup u$ weakly in $H^1_{\rm loc}((0,+\infty); L^2(0,1))$ and weakly* in $L^\infty((0,+\infty); BV(0,1))$ as $N \to +\infty$.*

(iv) *$u(t) \in \mathcal{A}_\phi(0,1)$ for any $t \in [0,+\infty)$.*

(v) *For any $j \in \{1,\ldots,m\}$ there exist $T_j \in (0,+\infty]$ and functions $a_j, b_j : [0,T_j) \to (0,1)$ such that*

(v1) *$a_j(0) = a_j^0$, a_j is continuous and nondecreasing;*

(v2) *$b_j(0) = b_j^0$, b_j is continuous and nonincreasing;*

(v3) *$a_j \le b_j$ on $[0,T_j)$, and $\lim_{t\to T_{j\_}} a_j(t) = \lim_{t\to T_{j\_}} b_j(t)$;*

(v4) *$\overline{\bigcup_{j=1}^m (a_j(t), b_j(t))} \subseteq \sigma_B^\phi(u(t)) \subseteq \bigcup_{j=1}^m [a_j(t), b_j(t)]$ for any $t \in [0,+\infty)$, where we have set $(a_j(t), b_j(t)) = [a_j(t), b_j(t)] := \emptyset$ if $t \ge T_j$.*

(vi) *$u_{xx} \in L^2(\Gamma_u)$, where $\Gamma_u := \bigcup_{t\in(0,+\infty)} (\sigma_G^\phi(u(t)) \times \{t\})$, and u is a solution of*

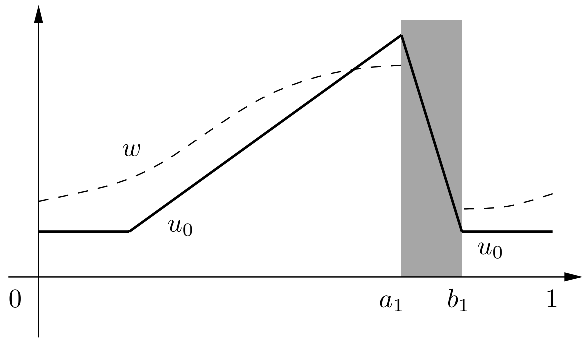

FIG. 2.2. *Remark* 2.6(b). *We construct a function w starting from u_0, such that $w \equiv u_0$ in (a_1, b_1) and that evolves according to the heat equation in $(0, a_1) \cup (b_1, 1)$ with zero Neumann boundary conditions in a_1, b_1 (dashed curve). Recall that we have periodic boundary conditions. Note that $J_{w(t)} = \{a_1, b_1\}$ for $t > 0$, and that $w(t) \notin \mathcal{A}_\phi(0,1)$ for any $t > 0$, since* (4) *of Definition* 2.1 *is violated at a_1, b_1.*

(2.5)
$$\begin{cases} u_t = u_{xx}, & x \in \sigma_G^\phi(u(t)),\ t \in (0,+\infty), \\ u_t = 0, & x \in \mathrm{int}(\sigma_B^\phi(u(t))),\ t \in (0,+\infty), \\ \displaystyle\lim_{y \to x,\, y \in \sigma_G^\phi(u(t))} u_x(y,t) = 0, & x \in \partial\sigma_G^\phi(u(t)) \setminus \{0,1\},\ t \in (0,+\infty), \\ u(x,0) = u_0(x), & x \in (0,1), \\ u(0,t) = u(1,t),\ u_x(0,t) = u_x(1,t), & t \in (0,+\infty). \end{cases}$$

(vii) *For any $t \in (0,+\infty)$ we have*
$\sup_{\sigma_G^\phi(u(t))} |u_x(\cdot,t)| < 1$*;*
$\sup_{[0,1]} u(\cdot,t) \le \sup_{[0,1]} u_0$*;*
$\inf_{[0,1]} u(\cdot,t) \ge \inf_{[0,1]} u_0$*;*
$\|Du(\cdot,t)\| \le \|Du_0\|$.

The proof of Theorem 2.4 is achieved in sections 5 and 6. In particular, (i) is given by Lemma 6.1, (ii) is given by Theorem 5.4, (iii) is the content of Remark 6.5, (iv) is given by Lemma 6.12, (v) is given by Lemma 6.8, Remark 6.6, and Lemma 6.12, and (vi) is the content of Theorem 6.14. Finally, the first inequality in (vii) follows from (vi) and the maximum principle applied to u_x, while the last three inequalities in (vii) are consequences of (c) and (d) of Theorem 5.4.

Remark 2.5.

(a) In general a function u and intervals (a_j, b_j) satisfying (v) and (vi) of Theorem 2.4 are not unique: it is easy to construct a solution w of (2.5) satisfying also the requirement

$$\sigma_B^\phi(w(t)) = \sigma_B^\phi(u_0) \qquad \forall\, t \in (0,+\infty), \tag{2.6}$$

and the function w in general cannot coincide with u. Indeed, $w(t) = u(t)$ for all times t for which $w(t) \in \mathcal{A}_\phi(0,1)$, but the property $w(t) \in \mathcal{A}_\phi(0,1)$ for all $t \in (0,+\infty)$ is in general violated; see Figure 2.2. In fact, condition (4) in Definition 2.1 cannot be satisfied for all times by w (cf. Remark 2.3), unless $\sigma_G^\phi(w(\cdot))$ is allowed to expand, in contrast with (2.6).

(b) If we do not require the functions a_j, b_j to be monotone (nondecreasing and

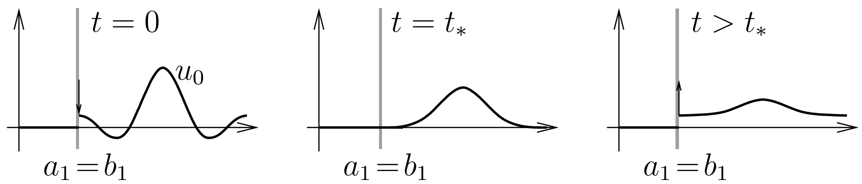

FIG. 2.3. *The "bouncing" solution discussed in Example* 1.

nonincreasing, respectively), several different solutions could be constructed; see Figure 2.4(b).

One can ask whether a function u and intervals (a_j, b_j) satisfying (iv), (v), and (vi) of Theorem 2.4 are unique. This is not the case, as shown by the following example related, in spirit, to the so-called fattening phenomenon in mean curvature flow (see [22] for similar behaviors concerning the evolution of the Mumford–Shah functional in one dimension).

Example 1. Let us construct an initial datum $u_0 \in \mathcal{A}_\phi(0,1)$ as follows:

u_0 has only one jump point $a_1 = b_1 = 1/2$;
$u_0 = 0$ in $(0, 1/2)$;
u_0 is a smooth function in $(1/2, 1)$ with the following property: $|u_{0x}| < 1$ and, if we flow $u_{0|(1/2,1)}$ by the heat equation with zero Neumann boundary conditions in $\{1/2, 1\}$, then there is a first time $t_* > 0$ for which the solution, evaluated at the point $1/2$, touches the horizontal axis with *zero vertical velocity* and then, for t immediately after t_*, becomes *positive* at $1/2$; see Figure 2.3.

Then we can exhibit two functions u_1, u_2, which coincide for $t \in [0, t_*]$ but differ for $t \in (t_*, +\infty)$, and both satisfy (iv), (v), and (vi) of Theorem 2.4. The function u_1 is defined as follows: $u_1 = 0$ in $(0, 1/2) \times [0, +\infty)$; u_1 equals, in $(1/2, 1) \times [0, t_*)$, the solution of the heat equation with zero Neumann boundary conditions in $\{1/2, 1\}$; u_1 equals, in $(0, 1) \times [t_*, +\infty)$, the solution of the heat equation with zero Neumann boundary conditions in $\{0, 1\}$ starting from $u_1(t_{*-})$. Namely, immediately after the time t_* when the two graphs of the solution on the left and on the right of $1/2$ join, the evolution continues with one graph only, and the jump disappears.

The function u_2 is defined as follows: $u_2 = 0$ in $(0, 1/2) \times [0, +\infty)$; u_2 equals, in $(1/2, 1) \times [0, +\infty)$, the solution of the heat equation with zero Neumann boundary conditions in $\{1/2, 1\}$. That is, the function u_2 "bounces" at $1/2$ at time t_*, the evolutions in $(0, 1/2)$ and in $(1/2, 1)$ do not "see" each other, and $1/2$ becomes again a jump point of $u_2(t)$ for t immediately larger than t_*.

Remark 2.6.

(a) As a consequence of (v) of Theorem 2.4, the set-valued map $t \in [0, +\infty) \to \sigma_G^\phi(u(t)) \subseteq (0, 1)$ is nondecreasing up to a finite number of points (at most m), and the number of connected components with nonempty interior of $\sigma_B^\phi(u(\cdot))$ is nonincreasing. Itmay happen that at some time $\bar{t}_j \in (0, T_j)$ the interval $[a_j(\bar{t}_j), b_j(\bar{t}_j)]$ is reduced to a point not belonging to $J_{u(\bar{t}_j)}$ (recall conclusion (iv) of Theorem 2.4 and Definition 2.1(2)) but belonging to $J_{u(t)}$ for some $t \in (\bar{t}_j, T_j)$ (as it happens for the function u_2 in Example 1). At time T_j at least one of the intervals in $\sigma_B^\phi(u(\cdot))$ disappears (provided $T_j < +\infty$).

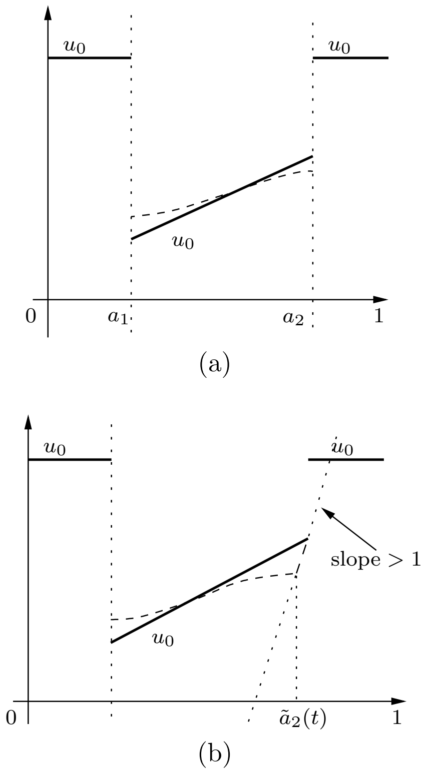

FIG. 2.4. *Remark* 2.5(b). *For the function* u_0 *we have* $a_1^0 = b_1^0$ *and* $a_2^0 = b_2^0$. *In* (a) *is displayed the solution* u *of Theorem* 2.4 *starting from* u_0 *for which* $a_1(t) = b_1(t) \equiv a_1^0$, $a_2(t) = b_2(t) \equiv a_2^0$, u *evolves according to the heat equation in* $[a_1(t), a_2(t)]$ *with zero Neumann boundary conditions (dashed curve), and* $u(t) \equiv u_0$ *in* $[0,1] \setminus [a_1(t), a_2(t)]$. *In* (b) *we construct a function* w *with* $w(t) \in \mathcal{A}_\phi(0,1)$, *such that* w *evolves according to the heat equation in* $[a_1^0, \tilde{a}_2(t)]$ *with zero Neumann boundary conditions, and* $\tilde{a}_2(t)$ *is decreasing in time, in such a way that the corresponding point* $w(\tilde{a}_2(t), t)$ *slides on a line with slope greater than one; hence the function* w *does not satisfy condition* (v1) *of Theorem* 2.4.

(b) A weak formulation of (2.5) is given by

$$\int_{(0,1)\times(0,+\infty)} u\,\psi_t\,dx\,dt - \int_{\mathrm{int}(\Gamma_u)} u_x \psi_x\,dx\,dt = 0 \tag{2.7}$$

for any $\psi \in \mathcal{C}_c^1([0,1] \times [0,+\infty))$.

(c) Solutions verifying conditions (iv), (v), and (vi) of Theorem 2.4 do not satisfy the comparison principle, in the sense that it is easy to find solutions u_1, u_2 such that $u_1(\cdot,0) \le u_2(\cdot,0)$ on $(0,1)$, but $u_1(\bar{x},\bar{t}) > u_2(\bar{x},\bar{t})$ for some $(\bar{x},\bar{t}) \in (0,1) \times (0,+\infty)$; see Figure 2.5.

Remark 2.7.

(a) Under sufficient regularity on u we can predict the speed of the free boundary $\partial\sigma_G^\phi(u(\cdot))$. For instance, assume that a_j is of class $\mathcal{C}^1$ in a neighborhood U of $\bar{t} \in (0, T_j)$ and that $a_j'(\bar{t}) \neq 0$. Assume in addition that $u(\cdot,\cdot)$ is twice differentiable in $\bigcup_{t \in U} \sigma_G^\phi(u(t)) \times \{t\}$ up to the boundary. Then from the equality

$$u(a_j(t), t) = u_0(a_j(t))$$

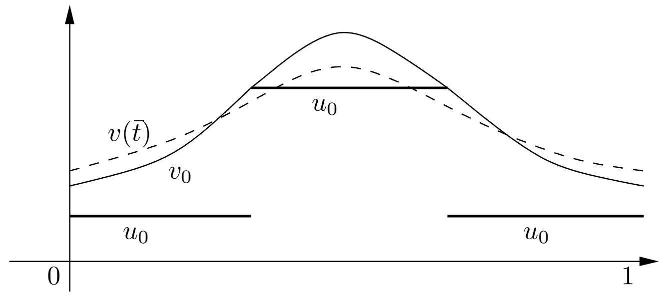

FIG. 2.5. *Remark* 2.6(c). *In general the solution* u *of Theorem* 2.4 *cannot satisfy the comparison principle. Indeed, let* u_0 *and* v_0 *be as in the figure,* $u_0 \le v_0$*, where we assume that the function* v_0 *is one-Lipschitz, so that* $\sigma_G^\phi(v_0) = (0,1)$*. Moreover,* $u(t) \equiv u_0$ *for any* $t \in (0,+\infty)$*. On the other hand, the solution* v *starting from* v_0 *given by Theorem* 2.4 *is the usual solution of the heat equation in* $(0,1)$ *with zero Neumann boundary conditions. Hence, at some time* $\bar{t} > 0$ *and at some* $\overline{x} \in (0,1)$ *it happens that* $v(\overline{x},\bar{t}) < u(\overline{x},\bar{t})$*.*

valid in the neighborhood of $\bar{t}$ it follows, using the third equality in (2.5), that

$$u_t(a_j(t)_-,t) = \frac{d}{dt}u(a_j(t),t) = u_{0x}(a_j(t)_+)a_j'(t). \tag{2.8}$$

Hence, using the first equation in (2.5), we get

$$a_j'(\bar{t}) = \frac{u_{xx}(a_j(\bar{t})_-,\bar{t})}{u_{0x}(a_j(\bar{t})_+)}. \tag{2.9}$$

Similarly, under the corresponding regularity assumptions and provided $b_j'(\bar{t}) \neq 0$, we get

$$b_j'(\bar{t}) = \frac{u_{xx}(b_j(\bar{t})_+,\bar{t})}{u_{0x}(b_j(\bar{t})_-)}. \tag{2.10}$$

(b) We expect that if $u_0 \in \mathcal{C}^{1,1}(\sigma_G^\phi(u_0))$ and $\lim_{y\to x,\, y\in\sigma_G^\phi(u_0)} u_{0x}(y) = 0$ for any $x \in \partial\sigma_G^\phi(u_0)$, then

$$\|u_{xx}\|_{L^\infty(\sigma_G^\phi(u(t)))} \le \|u_{0xx}\|_{L^\infty(\sigma_G^\phi(u_0))}, \qquad t \ge 0. \tag{2.11}$$

Indeed, assuming we can differentiate a_j, b_j in $(0,T_j)$ and $u(\cdot,t)$ in $\sigma_G^\phi(u(t))$ up to the boundary, arguing as in (a) we get

$$\frac{u_{xx}(a_j(t)_-,t)a_j'(t)}{u_{0x}(a_j(t)_+)} \ge 0, \qquad \frac{u_{xx}(b_k(t)_+,t)b_k'(t)}{u_{0x}(b_k(t)_-)} \ge 0 \tag{2.12}$$

for any $t \ge 0$. Differentiating the equalities $u_x(a_j(t)_-,t) = u_x(b_k(t)_+,t) = 0$

with respect to t and using (2.12), we then get

$$
\begin{aligned}
&\frac{u_{xxx}(a_j(t)_-,t)}{u_{0x}(a_j(t)_+)} = -\frac{u_{xx}(a_j(t)_-,t)a_j'(t)}{u_{0x}(a_j(t)_+)} \le 0,\\
&u_{xxx}(a_j(t)_-,t) = 0 \quad \text{if} \quad \frac{u_{xx}(a_j(t)_-,t)}{u_{0x}(a_j(t)_+)} < 0,\\
&\frac{u_{xxx}(b_k(t)_+,t)}{u_{0x}(b_k(t)_-)} = -\frac{u_{xx}(b_k(t)_+,t)b_k'(t)}{u_{0x}(b_k(t)_-)} \le 0,\\
&u_{xxx}(b_k(t)_+,t) = 0 \quad \text{if} \quad \frac{u_{xx}(b_k(t)_+,t)}{u_{0x}(b_k(t)_-)} > 0.
\end{aligned}
\tag{2.13}
$$

Letting $v := u_{xx}$ and differentiating (2.5) twice with respect to x, we obtain

$$
\begin{cases}
v_t = v_{xx}, & x \in \sigma_G^\phi(u(t)),\ t \in (0,+\infty),\\
v_t = 0, & x \in \mathrm{int}(\sigma_B^\phi(u(t))),\ t \in (0,+\infty),\\
v(x,0) = u_{0xx}(x), & x \in (0,1),
\end{cases}
\tag{2.14}
$$

with the boundary conditions on $\partial\sigma_G^\phi(u(t))$ given by (2.13). Note that, from the third equality in (2.5), for any $t \ge 0$ it follows that

$$
\int_{\sigma_G^\phi(u(t))} v(x,t)\,dx = 0 \Longrightarrow \max_{\sigma_G^\phi(u(t))} v(\cdot,t) \ge 0, \qquad \min_{\sigma_G^\phi(u(t))} v(\cdot,t) \le 0.
$$

The boundary conditions (2.13) then imply that $v(\cdot,t)$ assumes its maximum and minimum in the interior of $\sigma_G^\phi(u(t))$; hence (2.11) follows from (2.14) by the maximum principle.

Let us observe that from (2.8) and (2.11) it follows that

$$
\|a_j'\|_{L^\infty(0,T_j)} \le \|u_{0xx}\|_{L^\infty(\sigma_G^\phi(u_0))}, \qquad \|b_j'\|_{L^\infty(0,T_j)} \le \|u_{0xx}\|_{L^\infty(\sigma_G^\phi(u_0))}.
$$

In particular, we also expect that the functions a_j and b_j are Lipschitz continuous on $[0,T_j)$.

Remark 2.8. It is clear that Theorem 2.4 holds also for the function

$$
\overline{\phi}(\xi) := \min(1, \phi_{PM}(\xi)) = \min(1, \log(1+\xi^2)).
$$

In the present paper, solutions u to the gradient flow of $F_{\overline{\phi}}$ are intended as those functions satisfying (iv), (v), and (vi) (with $u_t = u_{xx}$ replaced by $u_t = (\overline{\phi}'(u_x))_x$) of Theorem 2.4. These solutions could be compared with some notion of weak solutions of the gradient flow of $F_{\phi_{PM}}$; see [29]. We can observe that u is not a BV-distributional solution of the Perona–Malik equation in the sense of [29, Definition 1]; see (2.7). However, u turns out to be a Young-varifold solution of the Perona–Malik equation; see [19], [21]. We also observe that if $a = b \in (0,1)$ is a jump point of $u(t)$ and if u is sufficiently smooth in a neighborhood of a (see Remark 2.7), then as a consequence of (2.9), (2.10), we have that $a'(t) = 0$. This is consistent with [29, formula (3)], in connection with the notion of generalized solution. Finally, observe that $a'(t) = 0$ is also a consequence of the $AC_2([0,+\infty); L^2(0,1))$ regularity of u.

3. First variation. In this section we want to identify the L^2-gradient of the functional F_ϕ in (1.2) on a suitable dense subspace X of $L^2(0,1)$; see Definition 3.3. We begin by computing the first variation of F_ϕ along functions $\psi \in \mathrm{Lip}(0,1)$.

PROPOSITION 3.1. *Let $u \in \mathcal{A}_\phi(0,1)$ be such that $\sigma_B^\phi(u) = \bigcup_{j=1}^m [a_j, b_j]$, $a_j < b_j$ for any $j = 1, \dots, m$,*

$$u \in H^2(\sigma_G^\phi(u)) \qquad \text{and} \qquad \sup_{\sigma_G^\phi(u)} |u_x| < 1.$$

Then for any $\psi \in \mathrm{Lip}(0,1)$ with $\psi(0) = \psi(1)$ we have

$$\begin{aligned} \frac{d}{d\lambda} F_\phi(u + \lambda\psi)_{|\lambda=0} &= \int_{\sigma_G^\phi(u)} u_x \psi_x \, dx \\ &= -\int_{\sigma_G^\phi(u)} u_{xx} \psi \, dx \\ &\quad + \sum_{j=1}^m \big(u_x(a_{j-})\psi(a_j) - u_x(b_{j+})\psi(b_j) \big). \end{aligned} \tag{3.1}$$

Proof. Since $\sup_{\sigma_G^\phi(u)} |u_x| < 1$ and $\psi \in \mathrm{Lip}(0,1)$, we have $\sigma_G^\phi(u + \lambda\psi) = \sigma_G^\phi(u)$ for $|\lambda|$ small enough. In addition, $\sigma_B^\phi(u + \lambda\psi) = \sigma_B^\phi(u)$ for $|\lambda|$ small enough. For such λ we have

$$\begin{aligned} F_\phi(u + \lambda\psi) &= \frac{1}{2} \int_{\sigma_B^\phi(u+\lambda\psi)} 1 \, dx + \frac{1}{2} \int_{\sigma_G^\phi(u+\lambda\psi)} (u_x + \lambda\psi_x)^2 \, dx \\ &= \frac{|\sigma_B^\phi(u)|}{2} + \frac{1}{2} \int_{\sigma_G^\phi(u)} (u_x + \lambda\psi_x)^2 \, dx \\ &= \frac{|\sigma_B^\phi(u)|}{2} + \frac{1}{2} \int_{\sigma_G^\phi(u)} (u_x)^2 \, dx + \lambda \int_{\sigma_G^\phi(u)} u_x \psi_x \, dx + O(\lambda^2). \end{aligned}$$

Then (3.1) follows with an integration by parts, using the assumptions $u \in H^2(\sigma_G^\phi(u))$ and $\psi(0) = \psi(1)$. ∎

Remark 3.2. Observe that the variations $u \to u + \lambda\psi$, as in Proposition 3.1, cannot increase the number of singular points of $u \in \mathcal{A}_\phi(0,1)$.

If u is as in Proposition 3.1 it follows that

$$\begin{aligned} &\inf_{\substack{\psi \in \mathrm{Lip}(0,1),\, \psi(0)=\psi(1) \\ \|\psi\|_{L^2} \le 1}} \frac{d}{d\lambda} F_\phi(u + \lambda\psi)_{|\lambda=0} \\ &= \begin{cases} -\|u_{xx}\|_{L^2(\sigma_G^\phi(u))} & \text{if } u_x(a_{j-}) = u_x(b_{j+}) = 0, \quad 1 \le j \le m, \\ -\infty & \text{otherwise.} \end{cases} \end{aligned} \tag{3.2}$$

DEFINITION 3.3. *We denote by X the dense subset of $L^2(0,1)$ consisting of the functions u as in Proposition* 3.1 *and satisfying $u_x(a_{j-}) = u_x(b_{j+}) = 0$ for any $1 \le j \le m$.*

Once we fix $u \in X$, the right-hand side of (3.1), if considered as a function of ψ, is a linear functional defined on the Lipschitz functions ψ in $(0,1)$ with $\psi(0) = \psi(1)$ (which form a dense subset of $L^2(0,1)$), which is continuous with respect to the

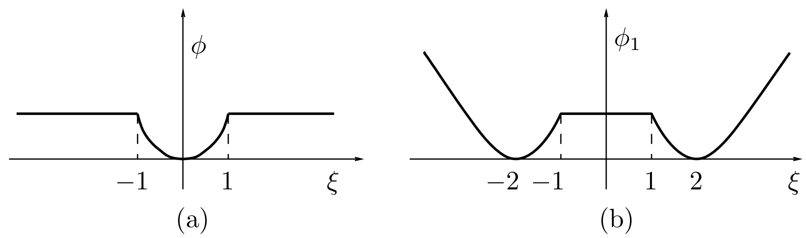

FIG. 3.1. (a) *the function* ϕ *considered in the present paper;* (b) *the function* ϕ_1 *of Remark* 3.6.

$L^2(0,1)$-norm. Therefore it can be extended on the whole of $L^2(0,1)$, thus providing a well-defined unique left-hand side of (3.1) for any $\psi \in L^2(0,1)$, and

$$\inf_{\substack{\psi\in \mathrm{Lip}(0,1),\, \psi(0)=\psi(1)\\ \|\psi\|_{L^2}\leq 1}} \frac{d}{d\lambda} F_\phi(u+\lambda\psi)_{|\lambda=0} = \inf_{\substack{\psi\in L^2(0,1)\\ \|\psi\|_{L^2}\leq 1}} \frac{d}{d\lambda} F_\phi(u+\lambda\psi)_{|\lambda=0}. \tag{3.3}$$

The infimum in (3.3) is attained at $\tilde{\psi} \in L^2(0,1)$, where

$$\tilde{\psi} = \begin{cases} 0 & \text{on } \sigma_B^\phi(u), \\ \|u_{xx}\|^{-1}_{L^2(\sigma_G^\phi(u))} u_{xx} & \text{on } \sigma_G^\phi(u). \end{cases}$$

It follows that the L^2-gradient flow of F_ϕ starting from $u_0 \in X$ is given by the free boundary problem (2.5).

As already observed in the introduction, in general, solutions to problem (2.5) are not unique, since the motion of the free boundary $\partial\sigma_G^\phi(u)$ is not prescribed. However, among all ϕ-admissible solutions we can look for those which most decrease the energy F_ϕ. This is expressed by the following proposition, which follows by a direct computation and recalling that $\phi \equiv 1/2$ on $\sigma_B^\phi(u)$.

PROPOSITION 3.4. *Let u be a solution of* (2.5) *satisfying* (iv) *of Theorem* 2.4. *Then for almost every $t \in (0, +\infty)$ we have*

$$\frac{d}{dt} F_\phi(u(t)) = -\frac{1}{2}\frac{d}{dt}|\sigma_G^\phi(u(t))| - \int_{\sigma_G^\phi(u(t))} |u_{xx}(x,t)|^2\, dx. \tag{3.4}$$

Remark 3.5. Proposition 3.4 implies that in order to most decrease the energy F_ϕ, the region $\sigma_G^\phi(u)$ should expand as fast as possible, compatibly with the ϕ-admissibility of u.

Remark 3.6. Our results can be extended to other integrands. Let us consider, for example, the potential in Figure 3.1(b), i.e.,

$$\phi_1(\xi) := \begin{cases} |\xi-2|^2 & \text{if } \xi \geq 1, \\ |\xi+2|^2 & \text{if } \xi \leq -1, \\ 1 & \text{otherwise,} \end{cases} \tag{3.5}$$

which is related to the ones considered in [26], [1], [36], [35]. Then Definition 2.1 still makes sense, provided that we define $\sigma_B^{\phi_1}(u)$ as the finite union of closed intervals where $|u(x) - u(y)| < |x - y|$, and $\sigma_G^{\phi_1}(u) = [0,1] \setminus \sigma_B^{\phi_1}(u)$ as the finite union of intervals where either $u(x)-u(y) \geq x-y$ or $u(x)-u(y) \leq -(x-y)$. Let us denote by

$\sigma^{\phi_1}_{G,+}(u)$ (resp., $\sigma^{\phi_1}_{G,-}(u)$) the subset of $\sigma^{\phi_1}_G(u)$ where u is increasing (resp., decreasing). The first variation of F_{ϕ_1} can be computed as in Proposition 3.1, and the evolution equation corresponding to (2.5) reads as

$$\begin{cases} u_t = u_{xx}, & x \in \sigma^{\phi_1}_G(u(t)),\ t \in (0,+\infty), \\ u_t = 0, & x \in \mathrm{int}(\sigma^{\phi_1}_B(u(t))),\ t \in (0,+\infty), \\ \displaystyle\lim_{y\to x,\, y\in\sigma^{\phi_1}_{G,\pm}(u(t))} u_x(y,t) = \pm 2, & x \in \partial\sigma^{\phi_1}_{G,\pm}(u(t)) \setminus \{0,1\},\ t \in (0,+\infty), \\ u(x,0) = u_0(x), & x \in (0,1), \\ u(0,t) = u(1,t),\ u_x(0,t) = u_x(1,t), & t \in (0,+\infty). \end{cases} \tag{3.6}$$

Since equality (3.4) still holds, also in this case the region $\sigma^{\phi_1}_G(u(\cdot))$ expands as fast as possible, compatibly with (3.6). We finally observe that the analogue of Theorem 2.4 is not expected to hold in this case; cf. Remark 7.1.

Remark 3.7. Let us consider a continuous function $\phi_2 : \mathbb{R} \to [0,+\infty)$ of the form $\phi_2(\xi) = \xi^2$ for $|\xi| \in [0,1]$, and $\phi_2(\xi) = \alpha\xi + \beta$ for $|\xi| \in [1,+\infty)$, where $\alpha + \beta = 1$ and $\alpha \geq 0$. The computations leading to (3.2) can be repeated for the functional F_{ϕ_2} and give the following result:

$$\begin{aligned} &\inf_{\substack{\psi\in \mathrm{Lip}(0,1),\, \psi(0)=\psi(1) \\ \|\psi\|_{L^2}\leq 1}} \frac{d}{d\lambda} F_{\phi_2}(u+\lambda\psi)_{|\lambda=0} \\ &= \begin{cases} -\|u_{xx}\|_{L^2(\sigma^{\phi_2}_G(u))} & \text{if } |u_x(a_{j-})| = |u_x(b_{j+})| = \alpha/2, \quad 1 \leq j \leq m, \\ -\infty & \text{otherwise,} \end{cases} \end{aligned} \tag{3.7}$$

where the interior Neumann boundary condition, for example in a_j, is equal to $\alpha/2$ (resp., $-\alpha/2$) if u_0 is increasing (resp., decreasing) in $[a_j, b_j]$.

In particular, the resulting PDE arising from (3.7) is different from (2.5) (since the conditions on the free boundary are different) unless $\alpha = 0$, i.e., $\phi_2 = \phi$.

4. Discretization. In this section we define the spatial discretization used to approximate problem (2.5). In particular, in Definition 4.4 we introduce the discretized operator A_v.

Let $N \in \mathbb{N}$ and $i \in \{1,\dots,N\}$. To simplify notation, we set $i+1 = 1$ and $[i,i+1] = [0,1]$ when $i = N$, and $i-1 = N$ and $[i-1,i] = [0,1]$ when $i = 1$.

For any $i = 1,\dots,N$ we define the *hat* function $h^i \in H^1(0,1)$ as

$$h^i(x) := \begin{cases} Nx - (i-1) & \text{if } Nx \in [i-1,i], \\ i+1-Nx & \text{if } Nx \in [i,i+1], \\ 0 & \text{otherwise.} \end{cases}$$

We denote by V_N the N-dimensional vector subspace of $H^1(0,1)$ generated by $h^1,\dots,h^N$. Each function $v \in V_N$ is Lipschitz and is the restriction to $[0,1]$ of an affine continuous periodic function defined on $\mathbb{R}$.

For any $i = 1,\dots,N$ we define the *flat* function $k^i \in L^2(0,1)$ as

$$k^i(x) := \begin{cases} 1 & \text{if } Nx \in (i-1,i], \\ 0 & \text{otherwise.} \end{cases}$$

We denote by W_N the N-dimensional vector subspace of $L^2(0,1)$ generated by $k^1,\dots,k^N$. W_N is the space of all piecewise constant functions on the grid.

The spaces $\bigcup_N V_N$ and $\bigcup_N W_N$ are dense in $BV(0,1)$ with respect to the weak*-topology.

Given $v \in V_N$ (resp., $w \in W_N$) we denote with $v_1,\dots,v_N$ the coordinates of v with respect to the basis $\{h^1,\dots,h^N\}$ (resp., $\{k^1,\dots,k^N\}$), i.e.,

$$v = \sum_{i=1}^N v_i h^i, \qquad v_i = v(i/N),$$

$$w = \sum_{i=1}^N w_i k^i, \qquad w_i = w\left(\frac{i-\frac{1}{2}}{N}\right).$$

We recall that

$$\int_{(0,1)} u\,dx = \frac{1}{N}\sum_{i=1}^N u_i, \qquad u \in V_N \cup W_N.$$

We define the scalar product $\langle\cdot,\cdot\rangle$ on V_N and on W_N as

$$\langle v,\overline{v}\rangle = \frac{1}{N}\sum_{i=1}^N v_i\overline{v}_i, \quad \langle w,\overline{w}\rangle = \frac{1}{N}\sum_{i=1}^N w_i\overline{w}_i, \qquad v,\overline{v}\in V_N,\ w,\overline{w}\in W_N.$$

Recall that

$$\langle w,\overline{w}\rangle = \int_{(0,1)} w\overline{w}\,dx = \frac{1}{N}\sum_{i=1}^N w_i\overline{w}_i, \qquad w,\overline{w}\in W_N.$$

Given $v \in V_N$ we define

$$\begin{aligned}
\|v\|_{L^\infty} &:= \max\{|v_i| \colon i = 1,\dots,N\},\\
\|v\|_{L^2} &:= \langle v,v\rangle^{\frac{1}{2}},\\
\|\nabla v\|_{L^1} &:= \sum_{i=1}^N |v_{i+1}-v_i| = \int_{(0,1)} |v_x|\,dx.
\end{aligned}$$

DEFINITION 4.1. *We define the linear map $D^+ \colon V_N \to W_N$ as the restriction of the weak derivative taking $H^1(0,1)$ in $L^2(0,1)$. In coordinates,*

$$(D^+v)_i = N(v_{i+1}-v_i), \qquad i \in \{1,\dots,N\}.$$

We let $D^- \colon W_N \to V_N$ be the adjoint operator of $-D^+$.

The operator D^- satisfies $\langle D^-w,v\rangle = -\langle w,D^+v\rangle$ for all $v\in V_N$ and $w \in W_N$. In coordinates,

$$(D^-w)_i = N(w_i - w_{i-1}), \qquad i\in\{1,\dots,N\}.$$

DEFINITION 4.2. *Given $v \in V_N$ we define $\Psi_v \in W_N$ in coordinates by*

$$(\Psi_v)_i = \begin{cases} 1 & \text{if } |(D^+v)_i| \le 1,\\ 0 & \text{otherwise,}\end{cases} \qquad i\in\{1,\dots,N\}.$$

If v is ϕ-admissible, the function $\Psi_v \colon (0,1) \to \mathbb{R}$ is the characteristic function of the set $\sigma_G^\phi(v)$.

Note that the restriction of F_ϕ to V_N reads as follows: given $v \in V_N$,

$$
\begin{aligned}
F_\phi(v) &= \frac{1}{2N} \sum_{i=1}^{N} \min\big(((D^+v)_i)^2, 1\big) \\
&= \frac{1}{2}\langle \Psi_v D^+v, D^+v\rangle + \frac{1}{2}\int_{(0,1)} (1-\Psi_v)\,dx,
\end{aligned}
\tag{4.1}
$$

where

$$
\langle \Psi_v D^+v, D^+v\rangle = \sum_{i=1}^{N} (\Psi_v)_i (D^+v)_i (D^+v)_i.
$$

Remark 4.3. The function $F_{\phi|V_N}$ is Lipschitz in V_N and is of class $\mathcal{C}^\infty$ out of the polyhedral hypersurface $H := \bigcup_{i=1}^N H_i$, where $H_i := \{v \in V_N : |(D^+v)_i| = 1\}$. Assume that $v \in V_N \setminus H$. Then, for any $\overline{v} \in V_N$, we have

$$
\lim_{\lambda\to 0} \frac{\Psi_{v+\lambda\overline{v}} - \Psi_v}{\lambda} = 0 \in V_N.
$$

Therefore, using also (4.1), we get

$$
\begin{aligned}
\lim_{\lambda\to 0} \frac{F_\phi(v+\lambda\overline{v}) - F_\phi(v)}{\lambda} &= \frac{1}{2}\langle \Psi_v D^+\overline{v}, D^+v\rangle + \frac{1}{2}\langle \Psi_v D^+v, D^+\overline{v}\rangle \\
&= -\big\langle D^-\big(\Psi_v D^+v\big), \overline{v}\big\rangle.
\end{aligned}
\tag{4.2}
$$

More generally, for $v \in V_N$ there exists the limit

$$
\begin{aligned}
&\lim_{\lambda\to 0^+} \frac{F_\phi(v+\lambda\overline{v}) - F_\phi(v)}{\lambda} \\
&= \sum_{i\colon |(D^+v)_i|<1} (D^+v)_i (D^+\overline{v})_i + \sum_{i\colon |(D^+v)_i|=1} \min\Big(\big((D^+v)_i(D^+\overline{v})_i\big), 0\Big) \\
&= -\big\langle D^-\big(\Psi_v D^+v\big), \overline{v}\big\rangle - \sum_{i\colon |(D^+v)_i|=1} \max\Big(\big((D^+v)_i(D^+\overline{v})_i\big), 0\Big) \\
&\le -\big\langle D^-\big(\Psi_v D^+v\big), \overline{v}\big\rangle.
\end{aligned}
\tag{4.3}
$$

Note that both the limits in (4.2) and (4.3) attain their minimum on $\{v \in V_N : \|v\|_{L^2} = 1\}$ at

$$
\overline{v} = \frac{D^-\big(\Psi_v D^+v\big)}{\big\|D^-\big(\Psi_v D^+v\big)\big\|_{L^2}}.
$$

We are now in a position to define the discretized operator.

DEFINITION 4.4. *Given any $v \in V_N$ we define the linear operator $A_v \colon V_N \to V_N$ as follows: for any $\overline{v} \in V_N$ we let*

$$
A_v\overline{v} := D^-\big(\Psi_v D^+\overline{v}\big).
$$

In coordinates, we have

$$(A_v \overline{v})_i = \frac{(\Psi_v)_i[\overline{v}_{i+1} - \overline{v}_i] - (\Psi_v)_{i-1}[\overline{v}_i - \overline{v}_{i-1}]}{1/N^2}.$$

Remark 4.5. By Remark 4.3, if $v \in V_N \setminus H$, then $A_v = -\nabla(F_{\phi_{|V_N}})(v)$, where ∇ indicates the gradient of the function $F_{\phi_{|V_N}}$ defined in the finite-dimensional space V_N. Note also that the equality holds in the last line of (4.3) if we take $v \in V_N$ and $\overline{v} = A_v v$.

Remark 4.6. If $v, \overline{v} \in V_N$ are such that $\Psi_v = \Psi_{\overline{v}}$, then $A_v = A_{\overline{v}}$.

5. Discretized evolution. Maximum principles. The aim of this section is to prove Theorem 5.4, which is a key step in the proof of Theorem 2.4. We begin with some elementary lemmata.

LEMMA 5.1. *Let $u_1, \dots, u_n$ be real continuous right-differentiable functions in an interval $[0, t_1)$. Define $M(t) := \max_{i=1,\dots,n} u(t)_i$. Then $M(t)$ is continuous and right-differentiable in $[0, t_1)$ and*

$$\frac{d}{dt^+} M(t) = \max_{i=1,\dots,n} \left\{ \frac{d}{dt^+} u(t)_i \colon u(t)_i = M(t) \right\}, \qquad t \in [0, t_1).$$

Proof. It is enough to prove the lemma when $n = 2$. Set $f := u_1$, $g := u_2$, and let $t \in [0, t_1)$. If $f(t) \neq g(t)$, the claim is trivial since $M(t)$ equals one of the two functions in a neighborhood of t. Suppose $f(t) = g(t) = M(t)$. If $\frac{d}{dt^+} f(t) > \frac{d}{dt^+} g(t)$, then for all $h > 0$ sufficiently small $M(t+h) = f(t+h)$; hence $\frac{d}{dt^+} M(t) = \frac{d}{dt^+} f(t)$. If $\frac{d}{dt^+} f(t) = \frac{d}{dt^+} g(t)$, then $M(t+h) - M(t)$ belongs to $[f(t+h) - f(t),\, g(t+h) - g(t)]$ if $f(t+h) \leq g(t+h)$ or to $[g(t+h) - g(t),\, f(t+h) - f(t)]$ if $f(t+h) \geq g(t+h)$. Hence $\frac{d}{dt^+} M(t) = \frac{d}{dt^+} f(t) = \frac{d}{dt^+} g(t)$. ∎

LEMMA 5.2. *Let u be a real continuous right-differentiable function in an interval $[0, t_1)$. If $\frac{d}{dt^+} u \leq 0$ on $[0, t_1)$, then u is nonincreasing.*

Proof. See, for instance, [25, p. 298]. ∎

LEMMA 5.3. *Let u be a real continuous right-differentiable function in an interval $[0, t_1)$, and let $g = |u|$. Then g is right-differentiable on $[0, t_1)$ and*

$$\frac{d}{dt^+} g(t) = \begin{cases} (\operatorname{sign} u(t)) \frac{d}{dt^+} u(t) & \textit{if } u(t) \neq 0, \\ \left| \frac{d}{dt^+} u(t) \right| & \textit{if } u(t) = 0, \end{cases} \qquad t \in [0, t_1).$$

Proof. If $u(t) \neq 0$, the assertion is trivial, since g is right-differentiable at t. Suppose $u(t) = 0$. Given $h > 0$ we have $\frac{g(t+h) - g(t)}{h} = \left|\frac{u(t+h)}{h}\right|$. Being u right-differentiable at t we find that $\frac{d}{dt^+} g(t) = \left|\frac{d}{dt^+} u(t)\right|$. ∎

THEOREM 5.4. *Let $N \in \mathbb{N}$ and $u_0 \in V_N$. Then there exists a unique function u_N such that*

(a) *$u_N : [0, +\infty) \to V_N$ is continuous and right-differentiable, and satisfies*

$$\text{(5.1)} \qquad \begin{cases} \dfrac{d}{dt^+} u_N(t) = A_{u_N(t)} u_N(t), & t \in [0, +\infty), \\ u_N(0) = u_0. \end{cases}$$

In addition, u_N satisfies the following properties:

(b) *The set-valued map* $t \in [0,+\infty) \to \{\Psi_{u_N(t)} = 1\} \subseteq (0,1)$ *is nondecreasing, and the set-valued map* $t \in [0,+\infty) \to \#\partial\{\Psi_{u_N(t)} = 1\}$ *is nonincreasing. Moreover, for any* $t \geq 0$ *there exists* $\varepsilon > 0$ *such that* $\Psi_{u_N(\tau)}$ *is constant for any* $\tau \in [t, t+\varepsilon]$. *In particular,* $\frac{d}{dt^+}\Psi_{u_N(t)} = 0$ *for any* $t \geq 0$.

(c) *The function* $t \in [0,+\infty) \mapsto \sup_{x\in(0,1)} u_N(x,t)$ *is nonincreasing, and the function* $t \in [0,+\infty) \mapsto \inf_{x\in(0,1)} u_N(x,t)$ *is nondecreasing.*

(d) *The function* $t \in [0,+\infty) \mapsto \|\nabla u_N(t)\|_{L^1}$ *is nonincreasing.*

(e) *The function* $t \in [0,+\infty) \mapsto F_\phi(u_N(t))$ *is continuous and right-differentiable, and*

$$\frac{d}{dt^+}F_\phi(u_N(t)) = -\left\|\frac{d}{dt^+}u_N(t)\right\|_{L^2}^2 \leq 0. \tag{5.2}$$

(f) *There exist* $M \in \mathbb{N}$, $M \leq N$, *and positive times* $t_1, \ldots, t_M$ *such that* u_N *is analytic on each interval of* $(0,+\infty)\setminus\{t_1,\ldots,t_M\}$, *and* $\{t_1,\ldots,t_M\}$ *coincides with the jump set of the function* $t \in [0,+\infty) \to \Psi_{u_N(t)}$.

Proof. Let $t_0 := 0$, and consider the function $u : [t_0,+\infty) \to V_N$,

$$u(t) = u_0 \exp((t-t_0)A_{u_0}), \qquad t \geq t_0, \tag{5.3}$$

i.e., the solution of

$$\begin{cases} \frac{d}{dt^+}u(t) = A_{u_0}\,u(t), & t \in (t_0,+\infty), \\ u(t_0) = u_0, \end{cases}$$

where we view the operator A_{u_0} as an $(N \times N)$-matrix.

For any $t \geq t_0$ let

$$\widetilde{M}(t) := \max\left(0, \max_{i=1,\ldots,N}\{(D^+u(t))_i \colon (\Psi_{u_0})_i = 1\}\right),$$
$$\widetilde{m}(t) := \min\left(0, \min_{i=1,\ldots,N}\{(D^+u(t))_i \colon (\Psi_{u_0})_i = 1\}\right).$$

Observe that

$$-1 \leq \widetilde{m}(t_0) \leq \widetilde{M}(t_0) \leq 1. \tag{5.4}$$

In addition, the maps $t \in [t_0,+\infty) \to (D^+u(t))_i$ are continuously differentiable for any $i \in \{1,\ldots,N\}$; hence, by Lemma 5.1, $\widetilde{M}(t)$ and $\widetilde{m}(t)$ are right-differentiable for any $t \geq t_0$.

Claim 1. For any $t \geq t_0$ we have

$$\frac{d}{dt^+}\widetilde{M}(t) \leq 0, \qquad \frac{d}{dt^+}\widetilde{m}(t) \geq 0. \tag{5.5}$$

Since D^+ is a linear operator, for all $t \geq t_0$ we have

$$\frac{d}{dt^+}D^+u(t) = D^+\frac{d}{dt^+}u(t) = D^+A_{u_0}u(t) = D^+D^-\left(\Psi_{u_0}D^+u(t)\right).$$

Therefore, if $i \in \{1, \ldots, N\}$ is such that $(D^+u(t))_i = \widetilde{M}(t)$, we have

$$
\begin{aligned}
\frac{d}{dt^+}(D^+u)_i &= N\left[(D^-(\Psi_{u_0}D^+u))_{i+1} - (D^-(\Psi_{u_0}D^+u))_i\right] \\
&= N^2\left[(\Psi_{u_0})_{i+1}(D^+u)_{i+1} - (\Psi_{u_0})_i(D^+u)_i\right] \\
&\quad - N^2\left[(\Psi_{u_0})_i(D^+u)_i - (\Psi_{u_0})_{i-1}(D^+u)_{i-1}\right] \\
&= N^2\left[(\Psi_{u_0})_{i+1}(D^+u)_{i+1} - \widetilde{M}(t)\right. \\
&\quad \left. + (\Psi_{u_0})_{i-1}(D^+u)_{i-1} - \widetilde{M}(t)\right],
\end{aligned} \tag{5.6}
$$

where both sides are evaluated at $t \geq t_0$. Since $(\Psi_{u_0}D^+u)_j \leq \widetilde{M}(t)$ for all $j \in \{1, \ldots, N\}$, from the previous equation we obtain $\frac{d}{dt^+}(D^+u(t))_i \leq 0$ for all $i \in \{1, \ldots, N\}$ such that $(\Psi_{u_0})_i = 1$ and $(D^+u(t))_i = \widetilde{M}(t)$. As a consequence we get

$$
0 \geq \max_{i=1,\ldots,N}\left\{\frac{d}{dt^+}\left(D^+u(t)\right)_i : (\Psi_{u_0})_i = 1,\ (D^+u(t))_i = \widetilde{M}(t)\right\} = \frac{d}{dt^+}\widetilde{M}(t),
$$

where the last equality follows from Lemma 5.1. In a similar way we can prove that if $i \in \{1, \ldots, N\}$ is such that $(\Psi_{u_0})_i = 1$ and $(D^+u(t))_i = \widetilde{m}(t)$, we have $\frac{d}{dt^+}(D^+u(t))_i \geq 0$; hence $\frac{d}{dt^+}\widetilde{m}(t) \geq 0$. This concludes the proof of Claim 1.

Claim 1 and Lemma 5.2 imply that $t \to \widetilde{M}(t)$ is nonincreasing and that $t \to \widetilde{m}(t)$ is nondecreasing. Recalling (5.4) we conclude that $-1 \leq \widetilde{m}(t) \leq \widetilde{M}(t) \leq 1$ for any $t \geq t_0$. Hence

$$
\Psi_{u(t)} = 1 \quad \text{at those nodes where} \quad \Psi_{u_0} = 1.
$$

It follows that the set-valued map $t \in [t_0, +\infty) \to \{|D^+u(t)| \leq 1\} = \sigma_G^\phi(u(t)) \subseteq (0, 1)$ is nondecreasing.

Let us define

$$
t_1 := \sup\{t \geq t_0 \colon A_{u(s)} = A_{u_0} \quad \forall\, s \in [t_0, t)\}. \tag{5.7}
$$

We want to show that $t_1 > t_0$.

For all $i \in \{1, \ldots, N\}$ such that $|(D^+u_0)_i| \leq 1$ we have $|(D^+u(t))_i| \leq 1$ for all $t \geq t_0$. In addition, $t \to D^+u(t)$ being a continuous function, if $|(D^+u_0)_i| > 1$, then there exists $\varepsilon > 0$ independent of i such that $|(D^+u(t))_i| > 1$ for any $t \in [t_0, t_0 + \varepsilon)$. Hence $\Psi_{u(t)} = \Psi_{u_0}$ for any $t \in [t_0, t_0+\varepsilon]$. From Remark 4.6 it follows that $A_{u(t)} = A_{u_0}$ for any $t \in [t_0, t_0 + \varepsilon)$, which gives $t_1 \geq t_0 + \varepsilon > t_0$.

We have proven that the function $u(t)$ in (5.3) satisfies (5.1) for $t \in [t_0, t_1)$. We have also proven that either $t_1 = +\infty$ or $\Psi_{u(t_1)} \geq \Psi_{u(t_0)}$ and $(\Psi_{u(t_1)})_i > \Psi_{u(t_0)}$ for some $i \in \{1, \ldots, N\}$.

If $t_1 < +\infty$, repeating the previous construction with t_1 in place of t_0 and $u(t_1)$ in place of u_0, we find a time $t_2 > t_1$ and a solution u of (5.1) defined in $[t_1, t_2)$ which satisfies (5.1). Repeating this argument, we can construct an increasing sequence (t_k) of times. Since at step k the number of nodes where $\Psi_{u(t)} = 1$ is nondecreasing, we can only have a finite number $M \leq N$ of steps, and in the last step we find that $t_M = +\infty$. Gluing together the solutions defined in the intervals $[t_k, t_{k+1})$ we find a function u_N defined for all $t \geq 0$ such that (a), (b), and (f) hold.

Let us prove (c), (d), and (e). Write for notational simplicity u in place of u_N. Let $t \in [0, +\infty)$. We say that $i \in \{1, \ldots, N\}$ is a relative maximum (resp., minimum) for $u(t)$ if

$$
u(t)_i \geq \max\{u(t)_{i-1}, u(t)_{i+1}\} \qquad (\text{resp., } u(t)_i \leq \min\{u(t)_{i-1}, u(t)_{i+1}\}).
$$

Claim 2. Let $t \in [0, +\infty)$. If i is a relative maximum (resp., minimum) for $u(t)$, then $\frac{d}{dt^+} u(t)_i \leq 0$ (resp., ≥ 0).

By (5.1),

$$\begin{aligned} \frac{d}{dt^+} u(t)_i &= N\Big[(\Psi_{u(t)})_i (D^+ u(t))_i - (\Psi_{u(t)})_{i-1} (D^+ u(t))_{i-1}\Big] \\ &= N^2\Big[(\Psi_{u(t)})_i (u(t)_{i+1} - u(t)_i) - (\Psi_{u(t)})_{i-1} (u(t)_i - u(t)_{i-1})\Big]. \end{aligned} \tag{5.8}$$

Hence, if i is a relative maximum, we have $\frac{d}{dt^+} u(t)_i \leq 0$ since $u(t)_{i+1} - u(t)_i \leq 0$ and $u(t)_i - u(t)_{i-1} \geq 0$. Similarly, we can reason when i is a relative minimum, and Claim 2 follows.

Assertion (c) then follows from Claim 2.

Consider now the function

$$S_i(t) := \begin{cases} \operatorname{sign}(u(t)_{i+1} - u(t)_i) & \text{if } u(t)_{i+1} \neq u(t)_i, \\ |\frac{d}{dt^+}(u(t)_{i+1} - u(t)_i| & \text{if } u(t)_{i+1} = u(t)_i. \end{cases}$$

By Lemma 5.3 we have

$$\begin{aligned} \frac{d}{dt^+} \|\nabla u(t)\|_{L^1} &= \sum_{i=1}^{N} \frac{d}{dt^+} |u(t)_{i+1} - u(t)_i| = \sum_{i=1}^{N} S_i(t) \left(\frac{d}{dt^+} u(t)_{i+1} - \frac{d}{dt^+} u(t)_i \right) \\ &= \sum_{i=1}^{N} (S_{i-1}(t) - S_i(t)) \frac{d}{dt^+} u(t)_i. \end{aligned}$$

In order to prove that $\frac{d}{dt^+} \|\nabla u(t)\|_{L^1} \leq 0$, it is enough to show that

$$(S_{i-1}(t) - S_i(t)) \frac{d}{dt^+} u(t)_i \leq 0 \qquad \forall\, i \in \{1, \ldots, N\}. \tag{5.9}$$

We divide the proof into four cases. We write for simplicity u in place of $u(t)$ and S in place of $S(t)$.

Case 1: the point i is simultaneously a relative maximum and a relative minimum, i.e., $u_{i-1} = u_i = u_{i+1}$. From (5.8) we deduce that $\frac{d}{dt^+} u_i = 0$, and (5.9) is satisfied.

Case 2: the point i is a relative maximum but not a relative minimum. Then either $u_i > u_{i-1}$ or $u_i > u_{i+1}$. So either $S_{i-1} = 1$ or $S_i = -1$, and in both cases $S_{i-1} - S_i \geq 0$. With $(D^+ u)_i \leq 0$ and $(D^+ u)_{i-1} \geq 0$, from (5.8) we find that $\frac{d}{dt^+} u \leq 0$, and (5.9) follows.

Case 3: the point i is a relative minimum but not a relative maximum. Then either $S_{i-1} = -1$ or $S_i = 1$, while $\frac{d}{dt^+} u \geq 0$.

Case 4: the point i is neither a relative maximum nor a relative minimum. Then either $u_{i-1} < u_i < u_{i+1}$ or $u_{i-1} > u_i > u_{i+1}$. In both cases we have $S_{i-1} = S_i$, and hence (5.9) holds.

Then (d) follows from Claim 1 and Lemma 5.2.

Let us now prove (e). Recalling that $\frac{d}{dt^+} \Psi_u = 0$ and using the expression of $F_\phi(u)$ as

$$F_\phi(u) = \frac{1}{2} \int_{(0,1)} [\Psi_u (D^+ u)^2 + (1 - \Psi_u)]\, dx, \tag{5.10}$$

we have

$$\begin{aligned}\frac{d}{dt^+}F_\phi(u) &= \frac{1}{2}\int_{(0,1)}\Psi_u \frac{d}{dt^+}(D^+u)^2\,dx = \int_{(0,1)}\Psi_u D^+u \frac{d}{dt^+}D^+u\,dx \\ &= \left\langle D^+\frac{d}{dt^+}u, \Psi_u D^+u\right\rangle = -\left\langle \frac{d}{dt^+}u, D^-\left(\Psi_u D^+u\right)\right\rangle \\ &= -\left\langle \frac{d}{dt^+}u, A_u u\right\rangle = -\int_{(0,1)}\left(\frac{d}{dt^+}u\right)^2 dx = -\left\|\frac{d}{dt^+}u\right\|^2_{L^2} \le 0,\end{aligned}$$

which proves (5.2).

For all $t \ge 0$ for which $\Psi_{u(\cdot)}$ is continuous at t, the continuity of $F_\phi(u(\cdot))$ at t is a consequence of (5.10). On the other hand, if $(\Psi_{u(\cdot)})_i$ has a discontinuity at $\bar{t} \ge 0$, we know that there exists $\sigma > 0$ such that $(\Psi_u)_i = 0$ in $(\bar{t}-\sigma, \bar{t})$ and $(\Psi_u)_i = 1$ in $[\bar{t}, \bar{t}+\sigma)$. This implies that $|(D^+u)_i| > 1$ in $(\bar{t}-\sigma, \bar{t})$ and $|(D^+u)_i| \le 1$ in $[\bar{t}, \bar{t}+\sigma)$. Since $(D^+u(\cdot))_i$ is continuous, we deduce that $(D^+u(\bar{t}))_i^2 = 1$. As a result,

$$\lim_{t\to\bar{t}^\pm} \Psi_{u(t)}(D^+u(t))^2 + (1-\Psi_{u(t)}) = 1.$$

This implies the continuity of the map $t \mapsto F_\phi(u(t))$ at $\bar{t}$.

To conclude the proof of the theorem, we need to show that the function u_N is unique. The proof is divided into two steps.

Step 1. Let $\underline{u}_N : [0,+\infty) \to V_N$ be a continuous right-differentiable function satisfying (5.1). Assume, in addition, that for any $t \ge 0$ there exists $\varepsilon > 0$ such that $\Psi_{\underline{u}_N(\tau)}$ is constant for any $\tau \in [t, t+\varepsilon]$. Then $\underline{u}_N = u_N$.

Let $\varepsilon > 0$ be such that $\Psi_{\underline{u}_N}$ is constant on $[0,\varepsilon]$. It follows that $\underline{u}_N = u_N$ in $[0,\varepsilon]$, since the solution of (5.1), in $[0,\varepsilon]$, is uniquely given by (5.3). Without loss of generality, we can assume that

$$\varepsilon < t_1, \tag{5.11}$$

where t_1 is defined in (5.7) and is the first time for which Ψ_{u_N} is discontinuous. Recall that, by definition, $\{\Psi_{u_N(\varepsilon)} = 0\} = \{|D^+u_N(\varepsilon)| > 1\}$.

We claim that

$$\{\Psi_{u_N(\varepsilon)} = 1\} = \{|D^+u_N(\varepsilon)| < 1\}. \tag{5.12}$$

Indeed, denote by $I_j = (j/N, (j+1)/N)$ the generic interval of the grid and by $\sigma_j(t)$ the slope of $u_N(t)$ in I_j. A closer look at the last term in (5.6) reveals that for any $t \in [0, t_1)$, if

$$\widetilde{M}(t) = (D^+u(t))_i = 1, \text{ and either } \sigma_{i-1}(t) \ne 1 \text{ or } \sigma_{i+1}(t) \ne 1, \tag{5.13}$$

then

$$\frac{d}{dt^+}(D^+u(t))_i < 0, \tag{5.14}$$

where we recall that u stands for u_N. Similarly, if

$$\widetilde{m}(t) = (D^+u(t))_i = -1, \text{ and either } \sigma_{i-1}(t) \ne -1 \text{ or } \sigma_{i+1}(t) \ne -1, \tag{5.15}$$

then

$$\frac{d}{dt^+}(D^+u(t))_i > 0. \tag{5.16}$$

Observe that from (5.13), (5.14), (5.15), and (5.16), we already deduce that if $|\sigma_i(t)| = 1$ and if either $|\sigma_{i-1}(t)| \neq 1$ or $|\sigma_{i+1}(t)| \neq 1$, then $|\sigma_i(t+\tau)| < 1$ for any $\tau > 0$ small enough. What remains is the most delicate case; namely, we have to consider those intervals I_i of the grid where $|\sigma_i(t)| = 1$ and also $|\sigma_{i-1}(t)| = |\sigma_{i+1}(t)| = 1$. The following observation again follows from the expression on the right-hand side of (5.6). For any $t \in [0, t_1)$, if

$$\widetilde{M}(t) = (D^+u(t))_i = 1, \text{ and } \sigma_{i-1}(t) = 1 = \sigma_{i+1}(t), \tag{5.17}$$

then

$$\frac{d}{dt^+}(D^+u(t))_i = 0. \tag{5.18}$$

Similarly, if

$$\widetilde{m}(t) = (D^+u(t))_i = -1, \text{ and } \sigma_{i-1}(t) = -1 = \sigma_{i+1}(t), \tag{5.19}$$

then

$$\frac{d}{dt^+}(D^+u(t))_i = 0. \tag{5.20}$$

Hence (5.18) and (5.20) do not allow us to conclude that if $|\sigma_i(t)| = 1$ and if $|\sigma_{i-1}(t)| = 1 = |\sigma_{i+1}(t)|$, then $|\sigma_i(t+\tau)| < 1$ for any $\tau > 0$ small enough. However, such an inequality is valid and can be proved as follows. Let us denote by C the connected component of $\{\Psi_{u(t)} = 1\}$ containing I_i and by I_{i^-} (resp., I_{i^+}) the extremal left (resp., right) interval of the grid belonging to C (note that thanks to the boundary conditions, 0 is not a boundary point of I_{i^-} and 1 is not a boundary point of I_{i^+}). By (5.14) and (5.16) it follows that $|\sigma_{I_{i^-}}(t+\tau)| < 1$ and $|\sigma_{I_{i^+}}(t+\tau)| < 1$ for any $\tau > 0$. Using the previous arguments, we deduce that $|\sigma_{I_{i^-+1}}(t+\tau)| < 1$ and $|\sigma_{I_{i^+-1}}(t+\tau)| < 1$ for any $\tau > 0$ small enough. After a finite number of iterations, we deduce that $|\sigma_i(t+\tau)| < 1$ for any $\tau > 0$ small enough. This concludes the proof of the claim.

We can now repeat the reasoning taking ε as initial time, and we conclude that $\underline{u}_N = u_N$ in $[0, t_1]$. Iterating the argument for any $i = 1, \ldots, M$ we obtain that $\underline{u}_N = u_N$ in $[0, +\infty)$.

Step 2. Let $\underline{u}_N : [0, +\infty) \to V_N$ be a continuous right-differentiable function satisfying (5.1). Then for any $t \geq 0$ there exists $\varepsilon > 0$ such that $\Psi_{\underline{u}_N(\tau)}$ is constant for any $\tau \in [t, t+\varepsilon]$.

Let us consider an interval I_i where the slope $\underline{\sigma}_i(t)$ of $\underline{u}_N(t)$ satisfies $|\underline{\sigma}_i(t)| = 1$. Arguing as in Step 1, independently of the values of $|\underline{\sigma}_{i-1}(t)|$ and $|\underline{\sigma}_{i+1}(t)|$, we deduce that $|\underline{\sigma}_i(t+\tau)| < 1$ for any $\tau > 0$ sufficiently small. This implies that $\Psi_{\underline{u}_N(t)}$ is right continuous and proves Step 2.

Steps 1 and 2 conclude the proof of uniqueness, and hence the proof of the theorem. ◻

Remark 5.5. We have already observed in the introduction that the right-hand side of the ODE's system $\dot{u} = -\nabla(F_{\phi_{|V_N}})$ (see (5.1)) is only a bounded function, since $F_{\phi_{|V_N}}$ is Lipschitz. Nevertheless, due to the special form of F_ϕ the solution in the sense of Theorem 5.4 is unique. This is not the case if we change the notion of solution to (5.1), for instance if we consider solutions to the system (5.1) only for almost all times. This is shown in the following example, which is related to the nonuniqueness example (Example 1 of section 2) and also shows another interesting phenomenon: the solution considered in Theorem 5.4 does not depend continuously on the initial datum.

Example 2. Assume that the initial datum $u_0 = u_{0N} \in V_N$ (with N even, in such a way that $1/2$ is a point of the mesh) is as follows:

$u_0 = 0$ in $(0, 1/2)$;
u_0 is increasing in $(1/2, 1/2 + 1/N)$ with slope exactly 1;
u_0 is piecewise linear, with slopes (in modulus) strictly less than 1 in $(1/2 + 1/N, 1)$.

Note that such an initial datum can be obtained from the discretization of solutions considered in the nonuniqueness example (Example 1 of section 2) at a time slightly smaller than t_* (and converging to t_* as $N \to +\infty$). The (unique) solution u_N of Theorem 5.4 is such that the linear part in the interval $(1/2, 1/2 + 1/N)$ for small positive times decreases its slope to a value less than 1. This solution, in the limit $N \to +\infty$, produces the solution $u_1(\cdot + t_*)$ of Example 1 of section 2.

Given $\varepsilon \in (0, 1)$ let us consider the functions $u_0^{\varepsilon\pm} = u_{0N}^{\varepsilon\pm} \in V_N$ defined as follows: $u_0^{\varepsilon\pm} := u_0$ in $(0, 1/2)$, $u_0^{\varepsilon\pm} := u_0 \pm \frac{\varepsilon}{N}$ in $(1/2 + 1/N, 1)$, and $u_0^{\varepsilon\pm}$ is increasing in $(1/2, 1/2 + 1/N)$ with slope $1 \pm \varepsilon$. Then, if ${u_N^\varepsilon}^\pm$ denotes the solution of Theorem 5.4 having $u_0^{\varepsilon\pm}$ as initial datum, we have

$$\lim_{\varepsilon \to 0^+} {u_N^\varepsilon}^- = u_N,$$

while

$$\lim_{\varepsilon \to 0^+} {u_N^\varepsilon}^+ = \widetilde{u}_N,$$

where $\widetilde{u}_N \in V_N$ satisfies (a) of Theorem 5.4 for any $t > 0$ but *not* for $t = 0$, and

$$\lim_{N \to +\infty} \widetilde{u}_N(\cdot) = u_2(\cdot + t_*),$$

where u_2 is as in Example 1 of section 2. Hence the solution u_N of Theorem 5.4 is not continuous with respect to initial data. We can summarize the above discussion, coupled with the remarks of section 2, with the following conclusion: solutions to (iv), (v), and (vi) of Theorem 2.4 are not unique thanks to Example 1 of section 2 (which, however, we believe to be nongeneric). On the other hand, solutions of Theorem 5.4 are unique; however, they do not depend in a continuous way on the initial data. It is such an instability at the discrete level (i.e., for fixed N) which seems to produce nonuniqueness in the limit $N \to +\infty$.

6. Convergence of the approximating schemes. In this section we prove Theorem 2.4. We begin with the following elementary lemma.

LEMMA 6.1. *Let* $u_0 \in \mathcal{A}_\phi(0, 1)$. *Then there exists a sequence* $(u_0^N) \subset V_N$ *of functions satisfying assertion* (i) *of Theorem* 2.4.

Proof. Define $u_0^N \in V_N$ as $(u_0^N)_i := u_0(i/N)$. Then $\|u_0^N\|_{BV(0,1)} \leq \|u_0\|_{BV(0,1)}$ for any $N \in \mathbb{N}$, (u_0^N) converges to u_0 weakly* in $BV(0, 1)$ and strongly in $L^2(0, 1)$, and $\lim_{N \to +\infty} \|u_0^N\|_{BV(0,1)} = \|u_0\|_{BV(0,1)}$. Note that for any $x \in [0, 1]$ such that $\mathrm{dist}(x, \sigma_B^\phi(u_0)) > 1/N$ (resp., $\mathrm{dist}(x, \sigma_G^\phi(u_0)) > 1/N$), then $x \in \sigma_G^\phi(u_0^N)$ (resp., $x \in$

$\sigma_B^\phi(u_0^N))$. It follows that $\lim_{N\to+\infty} d_{\mathcal{H}}(\sigma_G^\phi(u_0^N), \sigma_G^\phi(u_0)) = 0$. Since any isolated point in $\sigma_B^\phi(u)$ belongs to $\sigma_B^\phi(u_0^N)$ for N large enough, we also have $d_{\mathcal{H}}(\sigma_B^\phi(u_0^N), \sigma_B^\phi(u_0)) \to 0$ as $N \to +\infty$.

Now let $K \subset \sigma_G^\phi(u_0)$ be an interval with $\overline{K} \subset (0,1)$. Then $\|u_0^N\|_{L^2(K)} \le \|u_0\|_{L^2(K_N)}$, where $K_N := \{x \in \mathbb{R} : \mathrm{dist}(x,K) < 1/N\}$ and N is large enough in such a way that $K_N \subset (0,1)$. Hence $\|u_0^N\|_{L^2(K)} \le \|u_0\|_{L^2(K)} + \frac{2}{N}$, (u_0^N) weakly converges to u_0 in $H^1(K)$, and $\|u_0^N{}_x\|_{L^2(K)}$ converges to $\|u_{0x}\|_{L^2(K)}$. Therefore $\lim_{N\to+\infty} F_\phi(u_0^N) = F_\phi(u_0)$, and this concludes the proof. ◻

By construction, $u_0^N \in V_N \subset \mathcal{A}_\phi(0,1)$; moreover, we can assume that if N is large enough, the number of connected components of $\sigma_B^\phi(u_0^N)$ equals m, the number of connected components of $\sigma_B^\phi(u_0)$, and we can uniquely write $\sigma_B^\phi(u_0^N)$ as in (2.2).

Definition 6.2. *Let $u_0 \in \mathcal{A}_\phi(0,1)$, and let (u_0^N) be as in Lemma* 6.1. *We denote by $u^N\colon [0,+\infty) \to V_N$ the solution of*

$$\begin{cases} \dfrac{d}{dt^+} u(t) = A_{u(t)} u(t), & t \in (0,+\infty), \\ u^N(0) = u_0^N \end{cases} \tag{6.1}$$

given by Theorem 5.4 *(with u_0 in* (5.1) *replaced by u_0^N).*

Note that all assertions in Theorem 2.4(ii) are satisfied.

Remark 6.3.

(a) For any $j \in \{1,\dots,m\}$ we define

$$T_j^N := \sup\left\{t \ge 0\colon \sigma_B^\phi(u^N(t)) \cap [a_j^N(0), b_j^N(0)] \ne \emptyset\right\} > 0,$$
$$[a_j^N(t), b_j^N(t)] := \sigma_B^\phi(u^N(t)) \cap [a_j^N(0), b_j^N(0)], \qquad t \in [0, T_j^N).$$

Then $a_j^N(0) = a_j^0$, $b_j^N(0) = b_j^0$, and

$$\sigma_B^\phi(u^N(t)) = \bigcup_{j=1}^m [a_j^N(t), b_j^N(t)], \qquad t \in [0,+\infty),$$

where we have set

$$[a_j^N(t), b_j^N(t)] := \emptyset \qquad \text{if } t \ge T_j^N.$$

(b) The map $t \in [0, T_j^N) \mapsto a_j^N(t)$ is continuous and nondecreasing, and the map $t \in [0, T_j^N) \mapsto b_j^N(t)$ is continuous and nonincreasing.

(c) Since $u^N(\cdot,t) = u_0^N(\cdot)$ on $\sigma_B^\phi(u^N(t))$, for any $j \in \{1,\dots,m\}$ we have that either $u_0^N{}_x(x,t) > 1$ for a.e. $x \in [a_j^N(t), b_j^N(t)]$ or $u_0^N{}_x(x,t) < -1$ for a.e. $x \in [a_j^N(t), b_j^N(t)]$.

Lemma 6.4. *There exists a constant $C > 0$ depending only on u_0 such that*

$$\sup_{t>0} \sup_{N\in\mathbb{N}} F_\phi(u^N(t)) \le C,$$
$$\sup_{N\in\mathbb{N}} \left\| \frac{d}{dt^+} u^N \right\|_{L^2((0,+\infty);L^2(0,1))} \le C,$$
$$\sup_{N\in\mathbb{N}} \|u^N\|_{L^\infty((0,+\infty);BV(0,1))} \le C.$$

Proof. The first two inequalities follow from (2.3) and (5.2). The last one follows from Theorem 5.4 (c) and (d) and (2.3). ◻

Remark 6.5. Thanks to Lemma 6.4 (and extracting if necessary a not relabelled subsequence) the sequence (u^N) converges weakly in $H^1_{\rm loc}((0,+\infty);L^2(0,1))$ and weakly* in $L^\infty((0,+\infty);BV(0,1))$ to a function u as $N\to+\infty$, and this gives assertion (iii) of Theorem 2.4. In particular, for almost every $x\in(0,1)$ the function $t\in[0,+\infty)\to u(x,t)$ is continuous, and $u^N(x,\cdot)\to u(x,\cdot)$ uniformly on $[0,+\infty)$. As a consequence the function $u(\cdot,t)$ is well defined for all $t\in[0,+\infty)$ and $\|Du(\cdot,t)\|\le\|Du_0\|$. It also follows that $u^N(t)\to u(t)$ weakly* in $BV(0,1)$ for almost every $t\ge 0$.

Remark 6.6. Possibly extracting a further subsequence, we can assume that for any $j\in\{1,\dots,m\}$, $T_j^N\to T_j$ as $N\to+\infty$ for some $T_j\in[0,+\infty]$. If $T_j>0$, since the functions $a_j^N(\cdot)$ (resp., $b_j^N(\cdot)$) are nondecreasing (resp., nonincreasing), there exist nondecreasing functions $a_j:[0,T_j)\to[0,1]$ (resp., nonincreasing functions $b_j:[0,T_j)\to[0,1]$) such that $a_j^N\to a_j$ (resp., $b_j^N\to b_j$) weakly* in $BV(0,T_j-\varepsilon)$ as $N\to+\infty$ for all $\varepsilon>0$ small enough. Since $a_j^N(t)<b_j^N(t)$ for all $t\in[0,T_j^N)$, passing to the limit we obtain that $a_j(t)\le b_j(t)$ for all $t\in[0,T_j)$. Recall that $a_j(0)=a_j^0$ and $b_j(0)=b_j^0$ for any $j\in\{1,\dots,m\}$.

In the following, set $\mathcal{J}(0):=\{1,\dots,m\}$.

Definition 6.7. *For any $t\in[0,+\infty)$ we define*

$$\begin{aligned}
\mathcal{J}(t) &:= \{j\in\{1,\dots,m\}: t<T_j\},\\
B(t) &:= \bigcup_{j\in\mathcal{J}(t)} [a_j(t),b_j(t)],\\
G(t) &:= [0,1]\setminus B(t),\\
\widetilde{B}(t) &:= \bigcup_{j\in\mathcal{J}(t):\,a_j(t)<b_j(t)} [a_j(t),b_j(t)] \cup \bigcup_{j\in\mathcal{J}(t):\,a_j(t)=b_j(t)\in J_{u(t)}} \{a_j(t)\},\\
\widetilde{G}(t) &:= [0,1]\setminus\widetilde{B}(t).
\end{aligned}$$

Note that

$$\overline{\mathrm{int}(B(t))}\subseteq\widetilde{B}(t)\subseteq B(t). \tag{6.2}$$

Lemma 6.8. *For any $j\in\{1,\dots,m\}$ we have $T_j>0$, and the functions a_j and b_j are continuous on $[0,T_j)$.*

Proof. Assume by contradiction that there exists $j\in\{1,\dots,m\}$ such that $T_j=0$. Then $[a_j^0,b_j^0]\in\sigma_G^\phi(u(s))$ for any $s>0$. Hence $u(s)$ is one-Lipschitz in $[a_j^0,b_j^0]$ for any $s>0$.

Case 1. Assume that $a_j^0<b_j^0$. Using the triangular property and $u(0)=u_0$, for any $x,x'\in[a_j^0,b_j^0]$, $x\neq x'$, we have

$$\begin{aligned}
|u(x,s)-u(x,0)|+|u(x',s)-u(x',0)| &\ge |u_0(x)-u_0(x')|-|u(x,s)-u(x',s)|\\
&\ge |u_0(x)-u_0(x')|-|x-x'|>0.
\end{aligned}$$

This means that $s\mapsto u(x,s)$ has a discontinuity at $s=0$ for a.e. $x\in[a_j^0,b_j^0]$, and this is in contradiction with $u\in AC^2([0,+\infty);L^2(0,1))$.

Case 2. Assume that $a_j^0=b_j^0$. Let $L:=u_0(a^0_{j+})$ and $l:=u_0(a^0_{j-})$. We can assume $l<L$. Let $\delta:=\min(\frac{(L-l)}{4},a_1^0,(1-b_m^0),\min_{j=1,\dots,m-1}(a_{j+1}^0-b_j^0))>0$, and define $x^{\pm} := a_j^0 \pm \delta$. Note that $u(s)$ is one-Lipschitz in (x^-, x^+) for any $s > 0$. For any $x, x' \in (x^-, x^+)$, $x \neq x'$, we have

$$\begin{aligned}|u(x,s) - u(x,0)| + |u(x',s) - u(x',0)| &\geq |u_0(x) - u_0(x')| - |x - x'| \\ &\geq |u_0(a^0_{j\,-}) - u_0(a^0_{j\,+})| - |u_0(x) - u_0(a^0_{j\,-})| \\ &\quad - |u_0(x') - u_0(a^0_{j\,+})| - |x - x'| \\ &\geq L - l - 4\delta > 0.\end{aligned}$$

As above, this is in contradiction with $u \in AC^2([0, +\infty); L^2(0, 1))$.

Let us now prove that a_j and b_j are continuous. Assume by contradiction that a_j has a discontinuity at $t = \bar{t} \in [0, T_j)$. Since a_j is nondecreasing, $\bar{t}$ is a jump point of a_j. If $\bar{t} = 0$, we can argue in analogy to Case 1. Assume $\bar{t} > 0$, and let $x^- := \lim_{t \to \bar{t}^-} a_j(t) < x^+ := \lim_{t \to \bar{t}^+} a_j(t)$. Since $u^N(\cdot, t)$ coincides with $u_0^N(\cdot)$ in $\sigma_B^\phi(u^N(t))$, it follows that $u(\cdot, t)$ coincides with $u_0(\cdot)$ in each connected component of $\mathrm{int}(B(t))$. In particular, the function $u(t)$ coincides with u_0 in (x^-, x^+) for all $t \in [0, \bar{t})$. We then obtain

$$|u(x,t) - u(x',t)| = |u_0(x) - u_0(x')| > |x - x'| \qquad \forall\, x, x' \in (x^-, x^+).$$

On the other hand, $u(s)$ is one-Lipschitz in (x^-, x^+) for any $s > \bar{t}$. It follows that, for any $x, x' \in (x^-, x^+)$,

$$|u(x,t) - u(x,s)| + |u(x',t) - u(x',s)| \geq |u_0(x) - u_0(x')| - |x - x'| > 0,$$

which contradicts $u \in AC^2([0, +\infty); L^2(0, 1))$. This proves the continuity of a_j. The continuity of b_j follows using a similar argument. ∎

Remark 6.9. Whenever $T_j < +\infty$, arguing as in Lemma 6.8 with $\bar{t} = T_j$, we get $\lim_{t \to T_j^-} a_j(t) = \lim_{t \to T_j^-} b_j(t)$.

Remark 6.10.

(a) Since $u^N(\cdot, t)$ is one-Lipschitz in each connected component of $\sigma_G^\phi(u^N(t))$, it follows that $u(\cdot, t)$ is one-Lipschitz in each connected component of $\widetilde{G}(t)$.

(b) The function $u(\cdot, t)$ coincides with $u_0(\cdot)$ in each connected component of $\mathrm{int}(B(t))$.

Remark 6.11. As a consequence of Lemma 6.8 the sequence (a_j^N) (resp., (b_j^N)) converges to a_j (resp., to b_j) uniformly in $[0, T_j - \varepsilon)$ as $N \to +\infty$ for any $\varepsilon > 0$ small enough. In particular, for any connected component I of $B(t)$ there exists a connected component I_N of $\sigma_B^\phi(u^N(t))$ such that

$$\lim_{N \to +\infty} d_{\mathcal{H}}(I_N, I) = 0.$$

LEMMA 6.12. *The function $u(t)$ is ϕ-admissible for any $t \geq 0$ and*

$$\overline{\mathrm{int}(B(t))} \subseteq \sigma_B^\phi(u(t)) \subseteq B(t) \qquad \forall\, t \in [0, +\infty). \tag{6.3}$$

Proof. Recalling Remark 6.5, let us fix $t \geq 0$ such that $u^N(t) \to u(t)$ weakly* in $BV(0, 1)$. From Remark 6.10(a) it follows that $u(t)$ is one-Lipschitz in each connected component of $\widetilde{G}(t)$; hence

$$\widetilde{G}(t) \subseteq \sigma_G^\phi(u(t)).$$

Moreover, from Remarks 6.3(c) and 6.11 it follows that the assertion in Remark 2.2(c) holds with u replaced by $u(t)$ for any connected component I of $\widetilde{B}(t)$ and any Borel set $A \subseteq I$. Indeed, if A is compactly contained in I, then $Du^N(A) = Du_0^N(A)$ for $N \in \mathbb{N}$ large enough, and by construction (see Lemma 6.1) in the first case $|A| < \lim_{N\to+\infty} Du_0^N(A) = Du(A)$ or in the second case $-|A| > \lim_{N\to+\infty} Du_0^N(A) = Du(A)$. If A is a boundary point of I, then (using Remarks 6.11 and 6.10(a)) in the first case $0 \le \lim_{N\to+\infty} Du^N(A) = Du(A)$ or in the second case $0 \le \lim_{N\to+\infty} Du^N(A) = Du(A)$. To obtain the desired inequalities when A is a generic Borel set in I, it is enough to write $A = (A \cap \mathrm{int}(I)) \cup (A \cap \partial I)$, to approximate $A \cap \mathrm{int}(I)$ with a sequence of subsets of A compactly contained in I, and to use the previous arguments. It follows that

$$\widetilde{B}(t) \subseteq \sigma_B^\phi(u(t)).$$

In particular, for almost every $t \ge 0$, $u(t)$ is ϕ-admissible, $\sigma_B^\phi(u(t)) = \widetilde{B}(t)$, and (6.3) follows from (6.2).

Assume now that $t \ge 0$ is generic, and pick a sequence $(t_n) \subset (0,+\infty)$ converging to t as $n \to +\infty$ such that $u(t_n) \in \mathcal{A}_\phi(0,1)$ and for which (6.3) holds with t_n in place of t. Since $u \in AC^2([0,+\infty); L^2(0,1))$ and $u(t) \in BV(0,1)$, we have $u(t_n) \to u(t)$ weakly* in $BV(0,1)$ as $n \to +\infty$. It is then enough to repeat the previous arguments, and the assertion follows. ☐

Remark 6.13.

(a) We have $\lim_{N\to+\infty} d_{\mathcal{H}}(\Gamma_{u^N}, \Gamma_u) = 0$, where $\Gamma_{u^N} := \bigcup_{t\in(0,+\infty)} (\sigma_G^\phi(u^N(t)) \times \{t\})$. In particular, by Lemma 6.8, for all $t \in [0,+\infty)$ we have

$$\lim_{N\to+\infty} d_{\mathcal{H}}\big(\sigma_G^\phi(u^N(t)), \sigma_G^\phi(u(t))\big) = 0,$$
$$\lim_{N\to+\infty} d_{\mathcal{H}}\big(\mathrm{int}(\sigma_B^\phi(u^N(t))), \mathrm{int}(\sigma_B^\phi(u(t)))\big) = 0. \tag{6.4}$$

(b) Since $u^N \rightharpoonup u$ weakly* in $L^\infty([0,+\infty); BV(0,1))$ and $u^N \equiv u_0^N$ in $[0,1] \times [0,+\infty) \setminus \overline{\Gamma_{u^N}}$ by Remark 6.3(c), we have $u \equiv u_0$ in $[0,1] \times [0,+\infty) \setminus \overline{\Gamma_u}$.

THEOREM 6.14. *The function u satisfies $u_{xx} \in L^2(\Gamma_u)$ and is a solution of*

$$\begin{cases} u_t = u_{xx}, & x \in \sigma_G^\phi(u(t)),\ t \in (0,+\infty), \\ u_t = 0, & x \in \mathrm{int}(\sigma_B^\phi(u(t))),\ t \in (0,+\infty), \\ \displaystyle\lim_{y\to x,\, y\in\sigma_G^\phi(u(t))} u_x(y,t) = 0, & x \in \partial\sigma_G^\phi(u(t)) \setminus \{0,1\},\ t \in (0,+\infty), \\ u(x,0) = u_0(x), & x \in (0,1), \\ u(0,t) = u(1,t),\ u_x(0,t) = u_x(1,t), & t \in (0,+\infty). \end{cases} \tag{6.5}$$

Proof. Let $\psi \in \mathcal{C}_c^1([0,+\infty) \times [0,1])$, and let $\psi^N : [0,+\infty) \to V_N$, $\psi^N \in \mathrm{Lip}_c([0,+\infty) \times [0,1])$, be such that $\psi^N(t) \to \psi(t)$ in $H^1(0,1)$ for any $t \ge 0$. We have

$$
\begin{aligned}
\mathrm{I}_N(t) := \int_{\sigma_G^\phi(u^N(t))} u_x^N(t)\psi_x^N(t)\,dx &= \sum_{i:\,(\Psi_{u^N(t)})_i=1} \frac{D^+u_i^N(t)\,D^+\psi_i^N(t)}{N} \\
&= -\int_{(0,1)} D^-(\Psi_{u^N(t)}D^+u^N(t))\psi^N(t)\,dx \\
&= -\int_{(0,1)} A_{u^N(t)}u^N(t)\psi^N(t)\,dx \\
&= -\int_{(0,1)} \frac{d}{dt^+}u^N(t)\psi^N(t)\,dx =: \mathrm{II}_N(t).
\end{aligned} \tag{6.6}
$$

From (6.4) (which is valid for any $t \geq 0$ thanks to Lemma 6.8) and from the weak $H^1_{\mathrm{loc}}(\mathrm{int}(\Gamma_u))$-convergence of (u^N) to u, using (6.3) it follows that

$$
\lim_{N\to+\infty} \mathrm{I}_N(t) = \int_{\sigma_G^\phi(u(t))} u_x(t)\psi_x(t)\,dx \qquad \text{for a.e. } t \geq 0. \tag{6.7}
$$

On the other hand, $\frac{d}{dt^+}u^N \rightharpoonup \frac{d}{dt^+}u$ in $L^2((0,1)\times(0,+\infty))$ as $N \to +\infty$; hence

$$
\lim_{N\to+\infty} \mathrm{II}_N(t) = \int_{(0,1)} \frac{d}{dt^+}u(t)\psi(t)\,dx \qquad \text{for a.e. } t \geq 0. \tag{6.8}
$$

Recalling also Remark 6.13(b), equalities (6.7), (6.8) coupled with (6.6) imply that u solves the problem

$$
\begin{cases}
u_t = u_{xx} & \text{in } \mathrm{int}(\Gamma_u), \\
u_t = 0 & \text{in } [0,1]\times[0,+\infty) \setminus \overline{\Gamma_u}, \\
u(0) = u_0 & \text{in } [0,1]\times\{0\}.
\end{cases} \tag{6.9}
$$

In particular, we have $u \in \mathcal{C}^\infty(\mathrm{int}(\Gamma_u))$. Moreover, since $u_t \in L^2((0,1)\times(0,+\infty))$, we also get $u_{xx} \in L^2(\Gamma_u)$. It then follows that there exists the limit

$$
\lim_{x\to\bar{x},\, x\in\sigma_G^\phi(u(t))} u_x(x,t) = 0 \qquad \text{for a.e. } t \geq 0,\ \bar{x} \in \partial\sigma_G^\phi(u(t)); \tag{6.10}
$$

i.e., $u|_{\mathrm{int}(\Gamma_u)}$ satisfies zero Neumann boundary conditions on $\partial\Gamma_u$. Problem (6.9), together with the boundary condition (6.10), is equivalent to problem (6.5).

The periodic boundary conditions are a consequence of u being ϕ-admissible. □

Remark 6.15. The same results of Theorem 2.4 hold if we replace in the definition (1.1) of ϕ the function ξ^2 with a function $f \in \mathcal{C}^\infty(\mathbb{R})$ which satisfies $f(0) = 0$, $f(1) = 1$, $f(\xi) = f(-\xi)$, and $f''(\xi) > 0$ for all $\xi \in (-1,1)$. It is clear that the equation $u_t = u_{xx}$ in (2.5) is replaced by $u_t = \frac{1}{2}f''(u_x)u_{xx}$.

Remark 6.16. Let $N \in \mathbb{N}$, and set $\phi^N(\xi) := \min(\xi^2, N)$ for any $\xi \in \mathbb{R}$. Define the functional $F_{\phi^N,N} : L^1(0,1) \to [0,+\infty]$ as

$$
F_{\phi^N,N}(v) := \frac{1}{2N}\sum_{i=1}^N \min\left(((D^+v)_i)^2, N\right), \qquad v \in V_N
$$

(and extended to $+\infty$ elsewhere). In [14] it is proved that the sequence $(F_{\phi^N,N})$ Γ-converges, as $N \to +\infty$, to the Mumford–Shah functional. Let $\overline{u} \in BV(0,1)$, with

$\overline{u}(0) = \overline{u}(1)$, having a finite set $\overline{x}_1, \dots, \overline{x}_n$ of jump points in $(0,1)$, and of class $\mathcal{C}^1(\overline{I})$, for any interval $I \subset (0,1) \setminus \{\overline{x}_1, \dots, \overline{x}_n\}$. Then $\overline{u}$ is ϕ^N-admissible for N large enough; i.e., $\overline{u}$ satisfies Definition 2.1, where (1) is replaced by $|\overline{u}(x) - \overline{u}(y)| \leq \sqrt{N}|x-y|$ whenever $[x,y] \subset \sigma_G^{\phi^N}(\overline{u})$, and where the inequality involving u in (3) is replaced by $|\overline{u}(x) - \overline{u}(y)| > \sqrt{N}|x-y|$. Let us consider the solutions ω^N to the rescaled gradient flow system of ODEs

$$\begin{cases} \omega_t^N = -N\nabla(F_{\phi^N,N|V_N})(\omega^N), \\ \omega^N(0) = \overline{u}^N, \end{cases} \tag{6.11}$$

$\overline{u}^N$ as in Lemma 6.1. Reasoning as in Theorem 2.4 we get that, as $N \to +\infty$, the sequence (ω^N) converges, up to a subsequence, to a function ω which satisfies the heat equation with zero Neumann interior conditions on each interval of $(0,1) \setminus \{\overline{x}_1, \dots, \overline{x}_n\}$ (except in $\{0,1\}$), has periodic conditions in $\{0,1\}$, and keeps the points $\overline{x}_1, \dots, \overline{x}_n$ fixed in time ($\overline{x}_j$ may disappear at time $\overline{t}_j < +\infty$ if $\lim_{x \to \overline{x}_{j\,-}} \omega(x, \overline{t}_j) = \lim_{x \to \overline{x}_{j\,+}} \omega(x, \overline{t}_j)$). Therefore ω can be considered as a reasonable global solution to the gradient flow of the Mumford–Shah functional in one dimension starting from $\overline{u}$ (compare [22], [20]).

7. Numerical simulations. In this section we show a numerical simulation which confirms the behaviors predicted by Theorem 2.4. Let $u_0 \in \mathcal{A}_\phi(0,1)$ be the upper graph in Figure 7.2; see also Figure 7.1. We have

$$\sigma_B^\phi(u_0) = [a_1^0, b_1^0] \cup [a_2^0, b_2^0] \cup [a_3^0, b_3^0],$$

where $a_1^0 = 0.05$, $b_1^0 = 0.2$, $a_2^0 = b_2^0 = 0.6$, $a_3^0 = 0.9$, and $b_3^0 = 0.99$. Note that $J_{u_0} = \{a_2^0, a_3^0\}$.

The sequence of graphs displayed in Figures 7.1 and 7.2 presents the solution u starting from u_0 at subsequent times. The computation solves the discrete evolution presented in section 5 with space discretization $\Delta x = 1/N$ with $N = 500$. The algorithm used is a forward Euler scheme with time step $\Delta t = (\Delta x)^2/10$. Let us list the main features of the computed evolution u, all of which are in accordance with Theorem 2.4.

(1) We have $a_1(t) \equiv a_1^0$ for all $t > 0$, and on the interval $(0, a_1(t))$ the solution u evolves according to the heat equation with zero Neumann boundary condition at $a_1(t)$. In addition,

$$a_1(t) \in J_{u(t)} \qquad \forall\, t > 0.$$

Since $a_1^0 \notin J_{u_0}$, $a_1(t)$ "instantly" becomes a discontinuity point of the solution; see also Figure 7.1.

(2) The function $t \to b_1(t)$ is decreasing for positive times. The interval $[a_1(t), b_1(t)]$ is gradually eroded, from the right, by the interval $[b_1(t), a_2(t)]$, where the solution evolves according to the heat equation, with zero Neumann boundary conditions.

(3) There exists $T_2 > 0$ such that $a_2(t) \equiv a_2^0$ and $a_2^0 \in J_{u(t)}$ for $t \in [0, T_2)$, and then a_2^0 becomes a continuity point of $u(t)$ for $t \geq T_2$. In the region $[b_2^0, a_3(t)]$, for all times $t \in (0, T_2)$, the solution evolves according to the heat equation with zero Neumann boundary conditions at b_2^0 and $a_3(t)$. Note that

$$\sigma_B^\phi(u(t)) = [a_1(t), b_1(t)] \cup [a_3(t), b_3(t)], \qquad t \geq T_2,$$

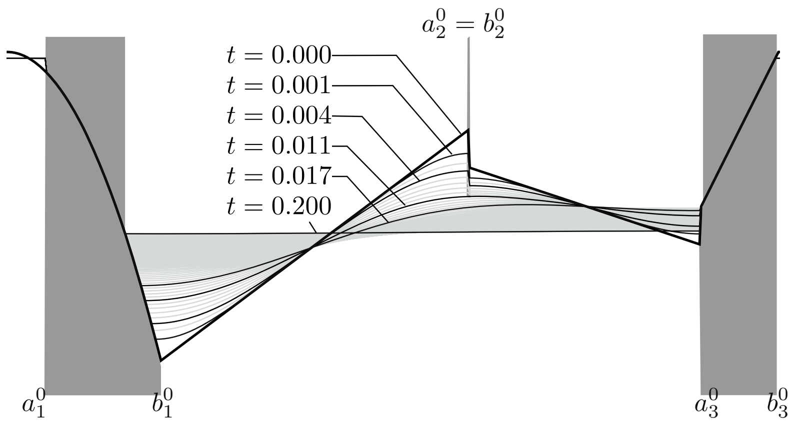

FIG. 7.1. *A simulation of the discretized evolution. The function is plotted in black for some relevant time values. The initial datum u_0 is plotted thick. The gray regions represent the intervals $[a_j(t), b_j(t)]$.*

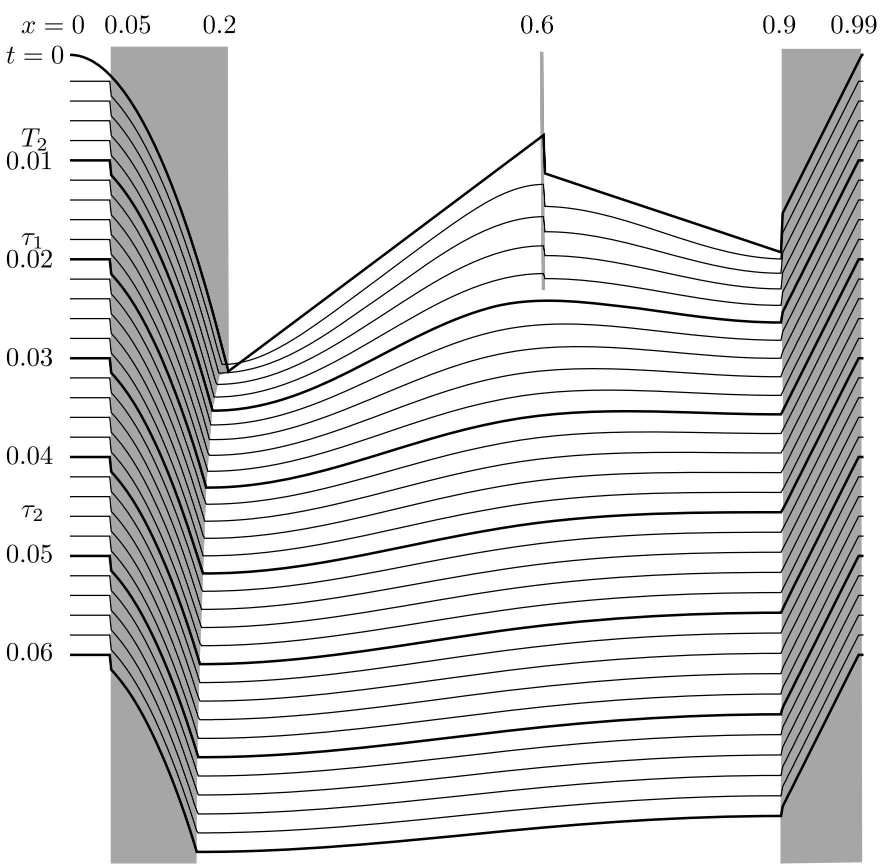

FIG. 7.2. *A vertical translation has been added to the evolution to distinguish the functions.*

and u evolves accordingly to the heat equation in the interval $(b_1(t), a_3(t))$ with zero Neumann boundary conditions.

(4) There exist two positive times $0 < \tau_1 < \tau_2$ such that $a_3(t) \equiv a_3^0 \in J_{u(t)}$ for $t \in [0, \tau_1)$, the point $a_3(t)$ becomes a continuity point of u for $t \in [\tau_1, \tau_2]$, and the function $t \to a_3(t)$ is strictly increasing in that interval, $a_3(t) \equiv a_3(\tau_2) \in J_{u(t)}$ for all $t > \tau_2$. The function $t \to b_3(t)$ is strictly decreasing.

(5) On the interval $(b_3(t), 1)$ the solution u evolves according to the heat equation with zero Neumann boundary conditions for all $t > 0$.

Remark 7.1. We conclude the paper by observing that, for energy densities different from (1.1), in particular for the function ϕ_1 considered in Figure 3.1 (the nonconvex region of which is bounded), the discrete approximation scheme discussed in sections 5 and 6, which keeps *fixed in time* the nodes of the mesh in $(0, 1)$, could converge to functions $\widetilde{u}$ which are not solutions to (3.6). In particular, the functions $\widetilde{u}$ might not satisfy the condition $\widetilde{u}_t = 0$ in $\mathrm{int}(\sigma_B^{\phi_1}(\widetilde{u}(t)))$; see also the comments in [21, p. 590]. This behavior of $\widetilde{u}$, which is related to the interactions of the nonconvex region of ϕ_1 with the numerical scheme with fixed nodes, deserves further investigation.

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