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Liouville Equation and Schottky Problem

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Abstract. An Ansatz for the Poincaré metric on compact Riemann surfaces is proposed. This implies that the Liouville equation reduces to an equation resembling a nonchiral analogous of the higher genus relationships (KP equation) arising within the framework of Schottky's problem solution. This approach connects uniformization (Fuchsian groups) and moduli space theories with KP hierarchy. Besides its mathematical interest, the Ansatz has some applications within the framework of quantum Riemann surfaces arising in 2D gravity.

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1. Schottky Problem and KP Hierarchy

Let us consider a genus h compact Riemann surface Σ. A fundamental object defining the complex structure of Σ is the Riemann period matrix

$$\Omega_{ij} \equiv \oint_{\beta_i} \omega_j, \tag{1.1}$$

where the ω_k's denote the h holomorphic differentials with the standard normalization $\oint_{\alpha_i} \omega_j = \delta_{ij}$. By means of the Riemann bilinear relations, it can be proved that Ω_{ij} is symmetric with positive definite imaginary part (see, for example, [1]). Let us consider the Siegel space

$$\mathscr{A}_h = \mathscr{H}_h / Sp(h, \mathbf{Z}), \tag{1.2}$$

where $\mathscr{H}_h$, called the Siegel upper half-plane, is the space of symmetric $h \times h$ matrices with a positive definite imaginary part. Recognizing the locus in $\mathscr{A}_h$ of the Riemann period matrices is the famous Schottky problem. This problem has been essentially solved by Dubrovin, Mulase and Shiota [2–4]. The solution is based on the proof of the Novikov conjecture stating that

$$u(x, y, t) = 2\partial_x^2 \log \Theta(Ux + Vy + Wt + z_0 | \Omega), \tag{1.3}$$

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satisfies the KP equation if and only if Ω is the period matrix of some Σ. The corresponding equations on Ω (see Equation (2.15)) were derived in [2] where it was proved that they determine an algebraic variety with a component given by the matrices of the β-periods. In [4], Shiota pointed out that if u in Equation (1.3) satisfies the KP equation, then there are vectors U^k such that the function

$$u(t_1, t_2, \ldots) = 2\partial_t^2 \log \Theta\left(\sum_{k=1}^{\infty} U^k t_k | \Omega \right), \quad t_1 = x,\, t_2 = y,\, t_3 = t, \tag{1.4}$$

determines the solutions of the KP hierarchy

$$\left[\frac{\partial}{\partial t_j} - L_j, \frac{\partial}{\partial t_k} - L_k \right] = 0, \tag{1.5}$$

where the order k differential operators L_k have coefficients depending on $\mathbf{t} \equiv (t_1, t_2, \ldots)$ and are determined by the equation $(\partial_{t_k} - L_k)\psi(\mathbf{t}, z) = 0$, ψ being the Baker–Akhiezer function on Σ. Since the space of vectors U^k is h-dimensional, there are two commuting operators of coprime order which are linear combinations of the L_k's. Therefore, one can apply the results in [5] to show that Ω is the Riemann matrix of the surface defined by these operators.

2. The Ansatz

Let Σ be a compact Riemann surface of genus $h > 1$. It is well known that the Liouville equation on Σ

$$\partial_z \partial_{\bar{z}} \varphi(z, \bar{z}) = \tfrac{1}{2} e^{\varphi(z, \bar{z})}, \tag{2.1}$$

is uniquely satisfied by the Poincaré metric (with Gaussian curvature -1). This metric can be written in terms of the inverse map of uniformization

$$J_H^{-1} \colon \Sigma \to H, \tag{2.2}$$

where $H = \{w | \operatorname{Im} w > 0\}$ denotes the upper half-plane. The Poincaré metric on H is

$$ds^2 = \frac{|dw|^2}{(\operatorname{Im} w)^2}, \tag{2.3}$$

so that on $\Sigma \cong H/\Gamma$ (with Γ a hyperbolic Fuchsian group)

$$e^{\varphi(z, \bar{z})} = \frac{|J_H^{-1}(z)'|^2}{(\operatorname{Im} J_H^{-1}(z))^2}, \tag{2.4}$$

which is invariant under $SL(2, \mathbf{R})$ fractional transformations of J_H^{-1}. Unfortunately, no one has succeeded in writing down J_H^{-1} in terms of the moduli of Σ. Here we consider the following Ansatz for the Poincaré metric★ on compact Riemann surfaces of genus $h \geqslant 4$

$$e^{\varphi} = \sum_{i,j=1}^{h} \omega_i A_{ij} \bar{\omega}_j. \tag{2.5}$$

We do not consider this *Ansatz* for lower genus because the Liouville equation for (2.5) is equivalent to nontrivial relations for the holomorphic differentials. On the other hand it is well known that interesting relations for ω_k arise for $h \geqslant 4$ (see, for example, III.8.6. in [1]). To get the inverse map from (2.5) one has to solve the Schwarzian equation

$$\{J_H^{-1}, z\} = T^F(z), \tag{2.6}$$

where

$$T^F(z) = \varphi_{zz} - \tfrac{1}{2}\varphi_z^2, \tag{2.7}$$

is the classical Liouville stress tensor (or Fuchsian projective connection) that by (2.5) reads

$$T^F(z) = \frac{\sum_{i,j=1}^{h} \omega_i'' A_{ij} \bar{\omega}_j}{\sum_{i,j=1}^{h} \omega_i A_{ij} \bar{\omega}_j} - \frac{3}{2}\left(\frac{\sum_{i,j=1}^{h} \omega_i' A_{ij} \bar{\omega}_j}{\sum_{i,j=1}^{h} \omega_i A_{ij} \bar{\omega}_j}\right)^2. \tag{2.8}$$

Observe that Equation (2.1) implies that

$$\partial_{\bar{z}} T^F(z) = 0. \tag{2.9}$$

A crucial property of Equation (2.6) is that it can be reduced to the linear equation

$$(2\partial_z^2 + T(z))\psi = 0. \tag{2.10}$$

Actually, it turns out that, up to SL(2, **C**) linear fractional transformations,

$$J_H^{-1} = \psi_1/\psi_2, \tag{2.11}$$

with ψ_1 and ψ_2 two linearly independent solutions of (2.10) (see [7] for a discussion on this point).

Let us now consider the Liouville equation for (2.5). We have

$$\sum_{i,j,k,l=1}^{h} \omega_l^2 \partial_z(\omega_i/\omega_l) A_{ij} A_{lk} \bar{\omega}_k^2 \partial_{\bar{z}}(\bar{\omega}_j/\bar{\omega}_k) = \left(\sum_{i,j=1}^{h} \omega_i A_{ij} \bar{\omega}_j\right)^3, \quad h \geqslant 4. \tag{2.12}$$

This equation has a strict similarity with the relations between the periods of holomorphic differentials on Riemann surfaces [2]. Thus, one should expect that A_{ij} depends on the moduli through the Riemann period matrix. To show this similarity, we write down the fundamental relations given in [2]. Let us set

$$U_k = -\omega_k(P), \qquad V_k = -\omega_k'(P), \qquad W_k = -\tfrac{1}{2}\omega_k''(P) - \tfrac{1}{2}c(P)U_k, \tag{2.13}$$

★Notice that a possible choice for the matrix to be positive definite is to set $A_{ij} = \Omega_{ij}^{(2)-1}$, in this case (2.5) coincides with the Bergman metric.

where $c(P)$ is a projective connection [2] and P is an arbitrary point on Σ. In [2], Dubrovin proved that the function (1.3) is a solution of the KP equation

$$u_{yy} = (4u_t - 6uu_x - u_{xxx})_x \tag{2.14}$$

if and only if the following relations between U, V, W, Ω and an additional constant d are satisfied (see [2] for notation)

$$\sum_{i,j,k,l=1}^{h} U_i U_j U_k U_l \hat{\Theta}_{ijkl}[n] + \\ + \sum_{i,j=1}^{h} (\tfrac{3}{4} V_i V_j - U_i W_j)\hat{\Theta}_{ij}[n] + \mathrm{d}\hat{\Theta}[n] = 0, \quad n \in \mathbf{Z}_2^h. \tag{2.15}$$

We emphasize that this result is a fundamental step to solve Schottky's problem.

Our remark is that Equation (2.12) looks like a non-chiral generalization of (2.15). In the notation introduced above, Equation (2.12) reads

$$\sum_{i,j,k,l=1}^{h} (U_l V_i - U_i V_l) A_{ij} A_{lk} (\bar{U}_k \bar{V}_j - \bar{U}_j \bar{V}_k) = \left(\sum_{i,j=1}^{h} U_i A_{ij} \bar{U}_j \right)^3. \tag{2.16}$$

We stress that solving this equation is equivalent to solving crucial questions arising in uniformization theory, Fuchsian groups, and related subjects. In particular, Weil–Petersson's 2-form ω_{WP} can be recovered using the fact that its Kähler potential is given by the Liouville action evaluated on the classical solution [6].

Another aspect that should be investigate is whether Equation (2.16) furnishes conditions on the period matrix in a more manageable form than KP equation (2.14)–(2.15).

A possible approach to study Equation (2.12) is using Krichever–Novikov's differentials $\psi_j^{(n)}$ [8]. These differentials are holomorphic on $\Sigma \backslash \{P_+, P_-\}$ with prescribed behaviour at $P_\pm$. In particular, in terms of local coordinates $z_\pm$ vanishing at $P_\pm \in \Sigma$, we have

$$\psi_j^{(n)}(z_\pm)(\mathrm{d}z_\pm)^n = a_j^{(n)\pm} z_\pm^{\pm j - s(n)} (1 + \mathcal{O}(z_\pm))(\mathrm{d}z_\pm)^n, \quad s(n) = \frac{h}{2} - n(h-1), \tag{2.17}$$

where $j \in \mathbf{Z} + h/2$ and $n \in \mathbf{Z}$. There are few exceptions to (2.17) concerning essentially the $h = 1$ and $n = 0, 1$ cases [8, 9]. By the Riemann–Roch theorem, $\psi_j^{(n)}$ is uniquely fixed by choosing the value of $a_j^{(n)+}$ or $a_j^{(n)-}$. In the following, we set $a_j^{(n)+} = 1$.

These differentials can be written in terms of theta functions★ [9]

$$\psi_j^{(n)}(z) = C_j^{(n)} \Theta(I(z) + \mathfrak{D}^{j;n} | \Omega) \frac{\sigma(z)^{2n-1} E(z, P_+)^{j-s(n)}}{E(z, P_-)^{j+s(n)}}, \tag{2.18}$$

where

$$\mathfrak{D}^{j;n} = (j - s(n)) I(P_+) - (j + s(n)) I(P_-) + (1 - 2n)\Delta,$$

★In the appendix we illustrate a general method to construct differentials in higher genus Riemann surfaces.

and constant $C_j^{(n)}$ is fixed by the condition $a_j^{(n)} = 1$. Let $\mathfrak{C}$ be a homologically trivial contour separating P_+ and P_-. The dual of $\psi_j^{(n)}$ is defined by

$$\frac{1}{2\pi i}\oint_{\mathfrak{C}} \psi_j^{(n)}\psi_{(n)}^k = \delta_j^k, \tag{2.19}$$

which implies

$$\psi_{(n)}^j = \psi_{-j}^{(1-n)}. \tag{2.20}$$

Note that (2.17) provides a basis for the $1 - 2s(n) = (2n-1)(h-1)$ holomorphic n-differentials on $\Sigma\,(h \geqslant 2)$

$$\mathscr{H}^{(n)} = \{\psi_k^{(n)} \mid s(n) \leqslant k \leqslant -s(n)\}, \quad n \geqslant 2. \tag{2.21}$$

Furthermore, from

$$\tilde{\mathscr{H}}^{(m)} = \{\psi_k^{(m)} \mid 1 - s(m) \leqslant k \leqslant s(m) - 1\}, \quad m \leqslant -1, \tag{2.22}$$

one can define the space of generalized Beltrami differentials. They are vanishing everywhere on Σ except in a disk where they coincide with [7]

$$\tilde{\mathscr{B}}^{(m)} = \{\partial_{\bar z}\psi_k^{(m)} \mid 1 - s(m) \leqslant k \leqslant s(m) - 1\}, \quad m \leqslant -1, \tag{2.23}$$

(for $m = -1$, one gets the Beltrami differentials considered in [9]). Observe that the differentials in (2.22) have poles both in P_+ and P_-. In particular, $\tilde{\mathscr{H}}^{(1-n)}$ is the dual space of $\mathscr{H}^{(n)}$.

We now expand the holomorphic 3-differentials in (2.12) in terms of the basis introduced above. We have

$$\begin{aligned} \omega_i^2\,\partial_z(\omega_j/\omega_i) &= \sum_{p=1}^{5h-5} a_{ij}^p \psi_{p+s(3)-1}^{(3)}, \\ a_{ij}^p &= \frac{1}{2\pi i}\oint_{\mathfrak{C}} \psi_{-p-s(3)+1}^{(-2)}\,\omega_i^2\,\partial_z(\omega_j/\omega_i), \end{aligned} \tag{2.24}$$

$$\omega_i\omega_j\omega_k = \sum_{p=1}^{5h-5} b_{ijk}^p \psi_{p+s(3)-1}^{(3)}, \qquad b_{ijk}^p = \frac{1}{2\pi i}\oint_{\mathfrak{C}} \psi_{-p-s(3)+1}^{(-2)}\,\omega_i\omega_j\omega_k. \tag{2.25}$$

Inserting these expansions in (2.12), we get the 'Liouville relations'

$$\sum_{i,j,k,l=1}^{h} a_{ij}^p A_{ik} A_{jl} \bar a_{kl}^q = \sum_{i,j,k,l,m,n=1}^{h} b_{ijm}^p A_{ik} A_{jl} A_{mn} \bar b_{kln}^q. \tag{2.26}$$

Note that a_{kl}^q and b_{kln}^q are functionals of the ω_k's and their derivatives computed at P_+ and coincide with the vectors of β-periods of second-kind differentials.

The above expansions provide relations involving the holomorphic differentials, theta functions, and their derivatives. To see this, it is sufficient to notice that the coefficients a_{ij}^p and b_{ijk}^p are vanishing for $p < 1$ and $p > 5h - 5$. The reason is that in this range the $\psi_{-p-s(3)+1}^{(-2)}$'s are holomorphic in P_- or P_+. This implies that for $p < 1$ and $p > 5h - 5$, the contribution to a_{ij}^p and b_{ijk}^p coming from the poles at P_- or P_+ add to zero. Notice that this 'residue formula' is crucial to get important relations such as Hirota's formulation of the KP hierarchy (see for example [10]).

3. The Accessory Parameters

Here we consider some aspects concerning the Fuchsian accessory parameters. First of all, we introduce the projective connection

$$T^S(z) = \{J_\Omega^{-1}, z\}, \tag{3.1}$$

where $J_\Omega\colon \Omega \to \Sigma$ denote the Schottkian uniformization map. Here, Ω denotes the region of discontinuity in $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ of the Schottky group $\mathscr{S}$ and $\Sigma \cong \Omega/\mathscr{S}$. Let us introduce the following notation for the Krichever–Novikov vector fields and quadratic differentials

$$e_k \equiv \psi_k^{(-1)}, \qquad \Omega^k \equiv \psi_{-k}^{(2)}. \tag{3.2}$$

Let $\mathscr{T}_\Sigma$ be the holomorphic projective connection on Σ obtained from the symmetric bidifferential of the second kind with biresidue 1 and zero α-periods. Let us consider the expansions

$$\begin{aligned} T^F &= \mathscr{T}_\Sigma + \sum_{k=1}^{3h-3} \lambda_k^{(F)} \Omega^{k+1-h_0}, \\ T^S &= \mathscr{T}_\Sigma + \sum_{k=1}^{3h-3} \lambda_k^{(S)} \Omega^{k+1-h_0}, \quad h_0 \equiv \tfrac{3}{2} h. \end{aligned} \tag{3.3}$$

The $\lambda_k^{(F)}$'s ($\lambda_k^{(S)}$'s) are called Fuchsian (Schottkian) accessory parameters.

In order to write $\mathscr{T}_\Sigma$ explicitly, we consider an arbitrary nonsingular point f of the theta divisor, that is $\Theta(f) = 0$ and grad $\Theta(f) \neq 0$. We define

$$H_f(z) = \sum_{k=1}^{h} \Theta_k(f)\omega_k(z), \tag{3.4}$$

$$Q_f(z) = \sum_{j,k=1}^{h} \Theta_{jk}(f)\omega_j(z)\omega_k(z), \tag{3.5}$$

$$T_f(z) = \sum_{i,j,k=1}^{h} \Theta_{ijk}(f)\omega_i(z)\omega_j(z)\omega_k(z). \tag{3.6}$$

The holomorphic projective connection is [11]

$$\mathscr{T}_\Sigma(z) = \left\{\int_{P_0}^{z} H_f, z\right\} + \frac{3}{2}\left(\frac{Q_f(z)}{H_f(z)}\right)^2 - 2\,\frac{T_f(z)}{H_f(z)}. \tag{3.7}$$

At a zero of H_f, we have

$$Q_f(z_0) = \pm H'_f(z_0), \qquad T_f(z_0) = -H''_f(z_0) \pm \tfrac{3}{2} Q'_f(z_0), \tag{3.8}$$

with the sign $\pm$ chosen accordingly as $\Theta(z - z_0 \mp f) \equiv 0$, $\forall z \in \Sigma$.

Besides T^F and T^S, also $\mathscr{T}_\Sigma$ can be expressed as a Schwarzian derivative. To do this, we simply note that, according to the general rule described above, the ratio of two arbitrary linearly independent solutions ϕ_1, ϕ_2 of the equation

$$\left(\frac{\partial}{\partial z^2} + \frac{1}{2}\mathscr{T}_\Sigma(z)\right)\phi(z) = 0, \tag{3.9}$$

is the solution of the Schwarzian equation

$$\{J_\Sigma^{-1}(z), z\} = \mathscr{T}_\Sigma(z), \tag{3.10}$$

where

$$J_\Sigma^{-1}(z) = \phi_2/\phi_1. \tag{3.11}$$

Note that, by (3.3), it follows that

$$\lambda_k^{(F)} = \frac{1}{2\pi i}\oint_{\mathfrak{C}} (\{J_H^{-1}(z), z\} - \{J_\Sigma^{-1}(z), z\})\, \mathrm{e}^{k+1-h_0}, \tag{3.12}$$

and by (2.8), (3.7) we get

$$\lambda_k^{(F)} = \frac{1}{2\pi i}\oint_{\mathfrak{C}} \left(\frac{\sum_{i,j=1}^{h} \omega_i'' A_{ij}\bar{\omega}_j}{\sum_{i,j=1}^{h} \omega_i A_{ij}\bar{\omega}_j} - \frac{3}{2}\left(\frac{\sum_{i,j=1}^{h} \omega_i' A_{ij}\bar{\omega}_j}{\sum_{i,j=1}^{h} \omega_i A_{ij}\bar{\omega}_j}\right)^2 - \left\{\int_{P_0}^{z} H_f, z\right\} - \frac{3}{2}\left(\frac{Q_f(z)}{H_f(z)}\right)^2 + 2\frac{T_f(z)}{H_f(z)} \right) \mathrm{e}^{k+1-h_0}. \tag{3.13}$$

It is interesting to note that by the chain rule for the Schwarzian derivative

$$\{w(t(z)), z\}(\mathrm{d}z)^2 - \{t(z), z\}(\mathrm{d}z)^2 = \{w(t), t\}(\mathrm{d}t)^2, \tag{3.14}$$

it follows that

$$\lambda_k^{(F)} = \frac{1}{2\pi i}\oint_{c} (\{J_H^{-1}(J_\Sigma^{-1}(z)), J_\Sigma^{-1}(z)\}(\partial_z J_\Sigma^{-1}(z))^2)\, \mathrm{e}^{k+1-h_0}. \tag{3.15}$$

In the second reference in [6], where the results for the punctured Riemann sphere are generalized to higher genus Riemann surfaces, a relationship has been established between $c_k^{(h)} = \lambda_k^{(F)} - \lambda_k^{(S)}$, the Liouville action computed on the classical solution and the Weil–Petersson metric. In particular, it turns out that

$$\frac{1}{2}\frac{\partial S_{cl}^{(h)}}{\partial z_i} = c_i^{(h)}, \qquad \frac{\partial c_i^{(h)}}{\partial \bar{z}_j} = -\frac{1}{2}\left\langle \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right\rangle_{\mathrm{WP}}, \tag{3.16}$$

where the brackets denote the Weil–Petersson metric on the Teichmüller space T_h projected onto the Schottky space whose coordinate are $z_1, \ldots, z_{3h-3}$. Since

$$\Theta(z) = T^F(z) - T^S(z) = \sum_{k=1}^{3h-3} c_k^{(h)} \Omega^{k+1-h_0}(z), \tag{3.17}$$

is a holomorphic quadratic differential (i.e. a section of $T^* T_h$), the formulas in Equation (3.16) are equivalent to

$$\partial S_{cl}^{(h)} = 2\Theta, \qquad \bar{\partial}\partial S_{cl}^{(h)} = -2i\omega_{\mathrm{WP}}, \tag{3.18}$$

where $d = \partial + \bar{\partial}$ is the exterior differentiation on the Schottky space and ω_{WP} is the Weil–Petersson 2-form on this space. Because the Schottky projective connection depends holomorphically on the moduli, we have

$$\bar{\partial} T^F = -i\omega_{\mathrm{WP}}, \tag{3.19}$$

that by (2.8) gives

$$\omega_{\mathrm{WP}} = i\bar{\partial}\left(\frac{\sum_{i,j=1}^{h} \omega_i'' A_{ij}\bar{\omega}_j}{\sum_{i,j=1}^{h} \omega_i A_{ij}\bar{\omega}_j} - \frac{3}{2}\left(\frac{\sum_{i,j=1}^{h} \omega_i' A_{ij}\bar{\omega}_j}{\sum_{i,j=1}^{h} \omega_i A_{ij}\bar{\omega}_j}\right)^2\right). \tag{3.20}$$

A result analogous to (3.16)–(3.19) has been derived by Fay [12], in particular

$$\{J_H^{-1}, z\} = \mathscr{T}_\Sigma - 24\pi i \sum_{j,k=1}^{h} \left(\frac{\partial}{\partial \Omega_{jk}} \log c_0\right) \omega_j(z)\omega_k(z), \tag{3.21}$$

where c_0 is the anomaly in the spin-1/2 bosonization formula computed with respect to the Poincaré metric e^φ, i.e.

$$c_0 = \left[\frac{8\pi^2 \det' \Delta}{\det \operatorname{Im} \Omega}\right]^{-1/2}. \tag{3.22}$$

The relation with the Weil–Petersson metric on T_h arises in considering the quasi-conformal mapping

$$\partial_{\bar{z}} f^\rho = \rho \partial_z f^\rho, \quad \rho = t_1 v_1 + t_2 v_2. \tag{3.23}$$

It turns out that

$$-24\pi\partial\bar{\partial} \log c_0 = \langle v_1, v_2 \rangle_{\mathrm{WP}}, \tag{3.24}$$

where $\langle v_1, v_2 \rangle_{\mathrm{WP}} = \int_\Sigma e^\varphi v_1 \bar{v}_2$ and

$$\partial = \partial_{t(p)} = \sum_{j,k=1}^{h} \frac{\partial}{\partial \Omega_{jk}} \delta\Omega_{jk}, \tag{3.25}$$

is the Schiffer variation (see [12] for details).

Another possible way to investigate Equation (2.5) is by noticing that both the first and second variations of the Poincaré area vanish for the deformation of the complex structure induced by the harmonic Beltrami differentials [13, 14]. Applying this condition to (2.5) should give further information on the form of matrix A_{ij}.

As a final remark, we observe that, besides any mathematical interest, the solution for the Poincaré metric is crucial to get explicit expressions for correlators in string theory. In particular, in the 'uniformization approach' to 2D quantum gravity [15, 7], one needs the explicit expression of $S_{cl}^{(h)}$ to compute the 'VEV of quantum Riemann surfaces' $\langle \Sigma \rangle$ (see [16, 17]).

4. Appendix

Here we illustrate a general method to express differentials in terms of theta functions. In particular, we will construct a n-differential $f^{(n)}$ with poles only at $Q_1, \ldots, Q_{p-2n(h-1)}$ and zeroes at $P_{h+1}, \ldots, P_p$. Since the degree of $f^{(n)}$ is $2n(h-1)$, it follows (by Riemann–Roch) that $f^{(n)}$ is uniquely fixed up to a multiplicative constant. As we will see, the remainder h-zeroes are fixed by the singlevaluedness condition.

Let us introduce the theta function with characteristic

$$\Theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z\,|\,\Omega) = \sum_{k \in Z^h} e^{\pi i (k+a)\cdot\Omega\cdot(k+a) + 2\pi i (k+a)\cdot(z+b)}, \qquad \Theta(z\,|\,\Omega) \equiv \Theta\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right](z\,|\,\Omega), \tag{A.1}$$

where $z \in \mathbf{C}^h$, $a, b \in \mathbf{R}^h$. When $a_k, b_k \in \{0, 1/2\}$, $\Theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z\,|\,\Omega)$ is even or odd depending on the parity of $4a\cdot b$. The Θ-function is multivalued under a lattice shift in the z-variable

$$\Theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z + n + \Omega\cdot m\,|\,\Omega) = e^{-\pi i m\cdot\Omega\cdot m - 2\pi i m\cdot z + 2\pi i (a\cdot n - b\cdot m)}\, \Theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z\,|\,\Omega). \tag{A.2}$$

The 'building block' to construct differentials on Σ is the prime form $E(z, w)$. It is a holomorphic $-1/2$-differential both in z and w, vanishing for $z = w$ only

$$E(z, w) = \frac{\Theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](I(z) - I(w)\,|\,\Omega)}{h(z)h(w)}. \tag{A.3}$$

Here $h(z)$ denotes the square root of $\Sigma_{k=1}^{h} \omega_k(z)\partial_{u_k}\Theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](u\,|\,\Omega)|_{u_k=0}$; it is the holomorphic 1/2-differential with nonsingular (i.e. $\partial_{u_k}\Theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](u\,|\,\Omega)|_{u_k=0} \neq 0$) odd spin structure $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$. The function $I(z)$ in (A.3) denotes the Jacobi map

$$I_k(z) = \int_{P_0}^{z} \omega_k, \quad P_0, z \in \Sigma. \tag{A.4}$$

This map is an embedding of Σ into the Jacobian

$$J(\Sigma) = \mathbf{C}^h/L_\Omega,\ \mathbf{Z}^h + \Omega\mathbf{Z}^h. \tag{A.5}$$

By (A.2), it follows that the multivaluedness of $E(z, w)$ is

$$E(z + n\cdot\alpha + m\cdot\beta, z) = \mathrm{e}^{-\pi i m\cdot\Omega\cdot m - 2\pi i m\cdot(I(z) - I(w))}\, E(z, w). \tag{A.6}$$

In terms of E one can construct the following $h/2$-differential with empty divisor

$$\sigma(z) = \exp\left(- \sum_{k=1}^{h} \oint_{\alpha_k} \omega_k(w) \log E(z, w) \right), \tag{A.7}$$

whose multivaluedness is

$$\sigma(z + n\cdot\alpha + m\cdot\beta) = \mathrm{e}^{\pi i(h-1)m\cdot\Omega\cdot m - 2\pi i m\cdot(\Delta - (h-1)I(z))}\, \sigma(z), \tag{A.8}$$

where Δ is (essentially) the *vector of Riemann constants* [11]. Finally, we quote two theorems:

ABEL THEOREM [1]. *A necessary and sufficient condition for $\mathscr{D}$ to be the divisor of a meromorphic function is that*

$$I(\mathscr{D}) = 0 \bmod (L_\Omega) \text{ and } \deg \mathscr{D} = 0. \tag{A.9}$$

RIEMANN VANISHING THEOREM [11]. *The function*

$$\Theta\left(I(z) - \sum_{k=1}^{h} I(P_k) + \Delta \,|\, \Omega \right), \quad z, P_k \in \Sigma, \tag{A.10}$$

either vanishes identically or else it has h zeroes at $z = P_1, \ldots, P_h$.

We are now ready to explicitly construct the differential $f^{(n)}$ defined above. First of all, note that

$$\tilde{f}^{(n)} = \sigma(z)^{2n-1} \frac{\prod_{k=h+1}^{p} E(z, P_k)}{\prod_{j=1}^{p-2n(h-1)} E(z, Q_j)}, \tag{A.11}$$

is a multivalued n-differential with $\mathrm{Div}\,\tilde{f}^{(n)} = \Sigma_{k=h+1}^{p} P_k - \Sigma_{k=1}^{p-2n(h-1)} Q_k$. Therefore, we set

$$f^{(n)}(z) = g(z)\tilde{f}^{(n)}, \tag{A.12}$$

where g is fixed by the requirement that $f^{(n)}$ be single-valued. From the multivaluedness of $E(z, w)$ and $\sigma(z)$, it follows that

$$g(z) = \Theta(I(z) + \mathscr{D} \,|\, \Omega), \tag{A.13}$$

with

$$\mathscr{D} = \sum_{k=h+1}^{p} I(P_k) - \sum_{k=1}^{p-2n(h-1)} I(Q_k) + (1-2n)\Delta. \tag{A.14}$$

By the Riemann vanishing theorem, $g(z)$ has just h-zeroes $P_1, \ldots, P_h$ fixed by $\mathscr{D}$. Thus, the requirement of single-valuedness also fixes the position of the remainder h zeroes. To make manifest the divisor in the RHS of (A.12), we first recall that the image of the canonical line bundle K on the Jacobian of Σ coincides with 2Δ [11]. On the other hand, since

$$[K^n] = \left[\sum_{k=1}^{p} P_k - \sum_{k=1}^{p-2n(h-1)} Q_k \right], \tag{A.15}$$

by Abel theorem we have★

$$\mathrm{Div}\,\Theta(I(z) + \mathscr{D} \mid \Omega) = \mathrm{Div}\,\Theta\left(I(z) - \sum_{k=1}^{h} I(P_k) + \Delta \mid \Omega \right), \tag{A.16}$$

and by Riemann vanishing theorem

$$\mathrm{Div}\,\Theta(I(z) + \mathscr{D} \mid \Omega) = \sum_{k=1}^{h} I(P_k). \tag{A.17}$$

References

1. Farkas, H. M. and Kra, I., *Riemann Surfaces*, 2nd edn, Springer-Verlag, New York, 1992.
2. Dubrovin, B. A., *Russian Math. Surveys* **36**, 11 (1981); *Math. USSR-Izv.* **19**, 285 (1982).
3. Mulase, M., *J. Differential Geom.* **19**, 403 (1984).
4. Shiota, T., *Invent. Math.* **83**, 333 (1986).
5. Krichever, I. M., *Funktsional Anali. Prilozhen* **11**(1), 15 (1977).
6. Zograf, P. G. and Takhtajan, L. A., *Math. USSR Sb.* **60**, 143 (1988); **60**, 297 (1988); Takhtajan, L. A., *Proc. Symp. Pure Math. Amer. Math. Soc.* **49**(1), 581 (1989).
7. Matone, M., *Internat. J. Modern Phys.* **A10**, 289 (1995).
8. Krichever, I. M. and Novikov, S. P., *Funktsional Anali. Prilozhen* **21**(2), 46 (1987); **21**(4), 47 (1987); **23**(1), 24 (1989).
9. Bonora, L., Lugo, A., Matone, M., and Russo, J., *Comm. Math. Phys.* **123**, 329 (1989).
10. Fay, J. D., *Proc. Symp. Pure Math.* **49**, 485 (1989).
11. Fay, J. D., *Theta Functions On Riemann Surfaces*, Lecture Notes in Math. 356, Springer-Verlag, New York, 1973.
12. Fay, J. D., Perturbation of analytic torsion on Riemann surfaces, *Mem. Amer. Math. Soc.* **96**, 464 (1992).
13. Ahlfors, L. V., *Ann. Math.* **74**, 171 (1961); *J. Anal. Math.* **9**, 161 (1961).
14. Wolpert, S. A., *Invent. Math.* **85**, 119 (1986).
15. Takhtajan, L. A. in J. Fröhlich et al., (eds), *New Symmetry Principles in Quantum Field Theory*, Plenum Press, New York, 1992.
16. Takhtajan, L. A., *Modern Phys. Lett. A* **8**, 3529 (1993).
17. Matone, M., *Modern Phys. Lett.* **9**, 2871 (1994).

★The square brackets in (A.15) denote the divisor class associated to the line bundle K^n. Two divisors belong to the same class if they differ by a divisor of a meromorphic function.