16 April 1998

PHYSICS LETTERS B

ELSEVIER

Physics Letters B 425 (1998) 104–106

Beta function, C-theorem and WDVV equations in 4D $N=2$ SYM

Gaetano Bertoldi [1], Marco Matone [2]

Department of Physics "G. Galilei" and INFN, Padova, Italy

Received 15 December 1997
Editor: L. Alvarez-Gaumé

Abstract

We show that the exact *beta*-function of 4D $N=2$ SYM plays the role of the metric whose inverse satisfies the WDVV-like equations $\mathcal{F}_{ikl}\beta^{lm}\mathcal{F}_{mnj}=\mathcal{F}_{jkl}\beta^{lm}\mathcal{F}_{mni}$. The conjecture that the WDVV-like equations are equivalent to the identity involving the u-modulus and the prepotential $\mathcal{F}$, seen as a superconformal anomaly, sheds light on the recently considered c-theorem for the $N=2$ SYM field theories. © 1998 Elsevier Science B.V. All rights reserved.

The Seiberg–Witten results about $N=2$ SUSY Yang–Mills [1] have been recently rederived, in the $SU(2)$ case, in [2]. The approach, based on uniformization theory, uses reflection symmetry of quantum vacua, asymptotics analysis and the identity [3]

$$u=\pi i(\mathcal{F}-a\partial_a\mathcal{F}/2). \tag{1}$$

This identity, first checked up to two-instanton in [4], has been proved to any order in the instanton expansion in [5]. Furthermore, Eq. (1) has been obtained as an anomalous superconformal Ward identity in [6].

As a consequence of the Seiberg–Witten results, it has been possible to derive the explicit expression for the *beta*-function [7–10] which has been recently reconsidered in [11–13].

The above results suggested looking for the analogue of the Zamolodchikov c-theorem [14] in the framework of $4D$ $N=2$ SYM. In particular, very recently, it has been shown in [15] that the results in [12] can be understood from the c-theorem point of view (see [16] for related aspects). Furthermore, it has been shown that for the $SU(n)$ case there is a Lyapunov function which is naturally determined and related to the *classical* discriminant of the Seiberg–Witten curve. It has been also observed that the c-theorem point of view actually fits with the fact that, according to [6], Eq. (1) means that u is proportional to the (super)conformal anomaly.

In this paper, we first shortly consider the role of the *beta*-function in the framework of the WDVV-like equations which have been introduced in [9] for the $SU(3)$ case. While for $SU(3)$ the *beta*-function satisfies a basic equation, derived from the reduced Picard–Fuchs equations, the identification of the inverse of the *beta*-function with the WDVV metric results in the identity

$$\mathcal{F}_{ikl}\beta^{lm}\mathcal{F}_{mnj}=\mathcal{F}_{jkl}\beta^{lm}\mathcal{F}_{mni}. \tag{2}$$

This naturally suggests considering this equation for the groups $SU(n)$ $n>3$. It turns out that in this case

[1] E-mail: bertoldi@padova.infn.it.
[2] E-mail: matone@padova.infn.it.

0370-2693/98/$19.00 © 1998 Elsevier Science B.V. All rights reserved.
PII S0370-2693(98)00200-7

(2) is no longer an identity and actually is a consequence of the WDVV-like equations derived in [17] (see [18] for related aspects).

The appearance of the *beta*-function in (2), and the way we derive it, suggests considering it as related to the superconformal anomaly. This would be in agreement with the general setting considered in [15] and the results in [3,6,19]. A consequence of this identification is the natural conjecture that the WDVV-like Eqs. (2) be equivalent to the higher rank version of the identity (1), namely [19]

$$u = \frac{i}{4\pi b_1}\left(\mathscr{F} - \sum_i \frac{a^i}{2} a_i^D\right), \tag{3}$$

that satisfies the equation [9]

$$\mathscr{L}_\beta u = u, \tag{4}$$

where $\mathscr{L}_\beta$ is a second-order modular invariant operator.

Let us start by recalling the derivation of the WDVV-like equations for the $SU(3)$ case whose curve and its extension to $SU(n)$ has been derived in [20]. Let us denote by $a^i = \langle \phi^i \rangle$ and $a_i^D = \langle \phi_i^D \rangle = \partial \mathscr{F}/\partial a^i$, $i = 1,2$, the vev's of the scalar component of the chiral superfield and its dual. The effective couplings are given by $\tau_{ij} = \partial^2 \mathscr{F}/\partial a^i \partial a^j$. We also set $u^2 \equiv u = \langle \mathrm{tr}\, \phi^2 \rangle$, $u^3 \equiv v = \langle \mathrm{tr}\, \phi^3 \rangle$ and $\partial_k \equiv \partial/\partial a^k$, $\partial_\alpha \equiv \partial/\partial u^\alpha$. The reduced Picard–Fuchs equations (RPFE's) for $SU(3)$ are [21]

$$\mathscr{L}_\beta \begin{pmatrix} a_i^D \\ a^i \end{pmatrix} = 0, \quad \beta = 2,3, \tag{5}$$

where

$$\mathscr{L}_2 = (P/u)\,\partial_u^2 + \mathscr{L}, \quad \mathscr{L}_3 = (P/3)\,\partial_v^2 + \mathscr{L}, \tag{6}$$

and $P = 27(v^2 - \Lambda^6) + 4u^3$, $\mathscr{L} = 12uv\partial_u\partial_v + 3v\partial_v + 1$.

Let us set

$$\mathscr{U} = u_2^2 \partial_{11} - 2u_1u_2\partial_{12} + u_1^2\partial_{22},$$
$$\mathscr{V} = v_2^2 \partial_{11} - 2v_1v_2\partial_{12} + v_1^2\partial_{22},$$
$$\mathscr{C} = (u_1v_2 + v_1u_2)\,\partial_{12} - u_2v_2\partial_{11} - u_1v_1\partial_{22},$$

and $D = u_1v_2 - u_2v_1$, where $\partial_{i_1\ldots i_n} \equiv \partial^n/\partial a^{i_1}\ldots\partial a^{i_n}$, $u_i \equiv \partial_i u$ and $v_i \equiv \partial_i v$. We have

$$\left[12uv\mathscr{C} + \frac{1}{3}P\mathscr{U} + D^2\left(1 - a^i\partial_i\right)\right]\mathscr{F}_l = 0$$
$$= \left[12uv\mathscr{C} + \frac{1}{u}P\mathscr{V} + D^2\left(1 - a^i\partial_i\right)\right]\mathscr{F}_l, \tag{7}$$

where $l = 1,2$ and $\mathscr{F}_{i_1\ldots i_n} \equiv \partial_{i_1\ldots i_n}\mathscr{F}$. Subtracting the LHS from the RHS of Eqs. (7), we obtain

$$A_l \equiv x_{11}\mathscr{F}_{22l} + x_{22}\mathscr{F}_{11l} - 2x_{12}\mathscr{F}_{12l} = 0, \tag{8}$$

where $l = 1,2$ and

$$x_{ij} = 3v_iv_j - uu_iu_j. \tag{9}$$

Next, considering

$$A_1(y_{22}\mathscr{F}_{112} - 2y_{12}\mathscr{F}_{122} + y_{11}\mathscr{F}_{222})$$
$$- A_2(-2y_{12}\mathscr{F}_{112} + y_{11}\mathscr{F}_{122} + y_{22}\mathscr{F}_{111}) = 0,$$

where y_{jk} are arbitrary parameters, we get

$$\mathscr{F}_{ikl}\eta^{lm}\mathscr{F}_{mnj} = \mathscr{F}_{jkl}\eta^{lm}\mathscr{F}_{mni}, \tag{10}$$

for $i,j,k,n = 1,2$, where

$$\eta^{lm} = \begin{pmatrix} 2x_{22}y_{12} - 2x_{12}y_{22} & x_{11}y_{22} - x_{22}y_{11} \\ x_{11}y_{22} - x_{22}y_{11} & 2x_{12}y_{11} - 2x_{11}y_{12} \end{pmatrix}. \tag{11}$$

For any choice of the parameters y_{jk}, there is only one nontrivial equation in (10) which can be rewritten in the form

$$\eta^{11}\Theta_{11} + 2\eta^{12}\Theta_{12} + \eta^{22}\Theta_{22} = 0, \tag{12}$$

where $\Theta_{ij} = (\mathscr{F}_{11i}\mathscr{F}_{22j} + \mathscr{F}_{11j}\mathscr{F}_{22i})/2 - \mathscr{F}_{12i}\mathscr{F}_{12j}$, which satisfies the identity

$$2\mathscr{F}_{12l}\Theta_{12} = \mathscr{F}_{22l}\Theta_{11} + \mathscr{F}_{11l}\Theta_{22}, \quad l = 1,2. \tag{13}$$

The fact that τ_{ij} is dimensionless implies that

$$(\Lambda\partial_\Lambda + \Delta_{u^\gamma})\tau_{ij} = 0, \tag{14}$$

where

$$\Delta_{u^\gamma} = \sum_{\gamma=2}^{n} \gamma u^\gamma \frac{\partial}{\partial u^\gamma}, \tag{15}$$

is the scaling invariant vector field.

Let us consider the *beta*-function

$$\beta_{ij} = \Lambda \frac{\partial \tau_{ij}}{\partial \Lambda}\Big|_{u^\alpha, u^\gamma, \ldots}. \tag{16}$$

By (14) we have for $SU(3)$

$$\beta_{ij} = -\left(2u\frac{\partial a^k}{\partial u} + 3v\frac{\partial a^k}{\partial v}\right)\mathscr{F}_{ijk}. \tag{17}$$

Let us denote by β^{ij} the inverse of the matrix β_{ij}. Setting

$$\eta^{ij} = \beta^{ij}, \tag{18}$$

implies the equation

$$x_{22}\beta^{22} + x_{11}\beta^{11} + 2x_{12}\beta^{12} = 0. \tag{19}$$

This equation, which arises as a consistency condition, is equivalent to $A_1 = 0 = A_2$ and

$$\mathscr{F}_{ikl}\,\beta^{lm}\mathscr{F}_{mnj} = \mathscr{F}_{jkl}\,\beta^{lm}\mathscr{F}_{mni}, \tag{20}$$

for $i,j,k,n = 1,2$, is an identity.

We now consider the β-function for $SU(n)$ with $n \geq 4$. We have

$$\beta_{ij} = -\Delta^k \mathscr{F}_{kij}, \tag{21}$$

where

$$\Delta^k = \Delta_{u^\gamma} a^k = \sum_{\gamma=2}^{n} \gamma u^\gamma \frac{\partial a^k}{\partial u^\gamma}, \quad k = 1,\dots,n-1. \tag{22}$$

Let us set

$$F_i = \mathscr{F}_{ijk}. \tag{23}$$

For the inverse of the beta matrix function β_{ij} we have

$$\beta^{ij} = -\left(\Delta^k F_k\right)^{-1}_{ij}, \tag{24}$$

We now show that Eq. (20) holds also for $SU(n)$ $n > 3$ (and is no longer an identity). In particular, Eq. (20) can be derived by the WDVV equations in [17]. These have the form

$$F_i F_k^{-1} F_j = F_j F_k^{-1} F_i. \tag{25}$$

These equations have the property of being invariant if

$$F_k^{-1} \to \beta^{-1} = -\left(\Delta^k F_k\right)^{-1}. \tag{26}$$

To see this, we simply observe that by (25) one has $F_i^{-1} F_k F_j^{-1} = F_j^{-1} F_k F_i^{-1}$, so that $F_i^{-1}\beta F_j^{-1} = F_j^{-1}\beta F_i^{-1}$, implying

$$F_i\,\beta^{-1} F_j = F_j\,\beta^{-1} F_i, \tag{27}$$

that is

$$\mathscr{F}_{ikl}\,\beta^{lm}\mathscr{F}_{mnj} = \mathscr{F}_{jkl}\,\beta^{lm}\mathscr{F}_{mni}. \tag{28}$$

It is a pleasure to thank G. Bonelli, J. Isidro and M. Tonin for discussions. MM was supported in part by the European Commission TMR programme ERBFMRX–CT96–0045.

References

[1] N. Seiberg, E. Witten, Nucl. Phys. B 426 (1994) 19; B 431 (1994) 484.

[2] G. Bonelli, M. Matone, M. Tonin, Phys. Rev. D 55 (1997) 6466.

[3] M. Matone, Phys. Lett. B 357 (1995) 342; Phys. Rev. D 53 (1996) 7354.

[4] F. Fucito, G. Travaglini, Phys. Rev. D 55 (1997) 1099.

[5] N. Dorey, V.V. Khoze, M.P. Mattis, Phys. Lett. B 390 (1997) 205.

[6] P.S. Howe, P.C. West, Nucl. Phys. B 486 (1996) 425.

[7] J.A. Minahan, D. Nemechansky, Nucl. Phys. B 468 (1996) 72.

[8] G. Bonelli, M. Matone, Phys. Rev. Lett. 76 (1996) 4107.

[9] G. Bonelli, M. Matone, Phys. Rev. Lett. 77 (1996) 4712.

[10] E. D'Hoker, I.M. Krichever, D.H. Phong, Nucl. Phys. B 494 (1997) 89; B 489 (1997) 211; E. D'Hoker, D.H. Phong, Phys. Lett. B 397 (1997) 94.

[11] A. Ritz, hep-th/9710112.

[12] B.P. Dolan, hep-th/9710161.

[13] J.I. Latorre, C.A. Lütken, hep-th/9711150.

[14] A.B. Zamolodchikov, JETP Lett. 43 (1986) 730.

[15] G. Bonelli, M. Matone, hep-th/9712025.

[16] J.L. Cardy, Phys. Lett. B 215 (1988) 749; H. Osborn, Phys. Lett. B 222 (1989) 97; A. Cappelli, D. Friedan, J.I. Latorre, Nucl. Phys. B 352 (1991) 616; D. Anselmi, D.Z. Freedman, M.T. Grisaru, A.A. Johansen, hep-th/9708042; D. Anselmi, J. Erlich, D.Z. Freedman, A.A. Johansen, hep-th/9711035; H. Osborn, J.I. Latorre, hep-th/9703196; F. Bastianelli, Phys. Lett. B 369 (1996) 249.

[17] A. Marshakov, A. Mironov, A. Morozov, Phys. Lett. B 389 (1996) 43.

[18] R. Carroll, Phys. Lett. A 234 (1997) 171; hep-th/9712110; J. Chang, R. Carroll, solv-int/9612010.

[19] J. Sonnenschein, S. Theisen, S. Yankielowicz, Phys. Lett. B 367 (1996) 145. T. Eguchi, S.-K. Yang, Mod. Phys. Lett. A 11 (1996) 131.

[20] A. Klemm, W. Lerche, S. Theisen, S. Yankielowicz, Phys. Lett. B 344 (1995) 169. P. Argyres, A.E. Faraggi, Phys. Rev. Lett. 74 (1995) 3931.

[21] A. Klemm, W. Lerche, S. Theisen, Int. J. Mod. Phys. A 11 (1996) 1929.