Nonperturbative Relations in $N = 2$ Supersymmetric Yang-Mills Theory and the Witten-Dijkgraaf-Verlinde-Verlinde Equation

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We find the nonperturbative relation between $\langle \mathrm{tr}\phi^2\rangle$, $\langle \mathrm{tr}\phi^3\rangle$ the prepotential $\mathcal{F}$ and $\langle \phi_i\rangle$ in $N = 2$ supersymmetric Yang-Mills theory (SYM) with gauge group SU(3). Nonlinear differential equations for $\mathcal{F}$ including the Witten-Dijkgraaf-Verlinde-Verlinde equation are obtained, indicating that $N = 2$ SYM theories are essentially topological field theories which should be seen as the low-energy limit of some topological string theory. Furthermore, we construct relevant modular invariant quantities, derive canonical relations between the periods, and find the β function in terms of the moduli. In doing this we discuss the uniformization problem for the quantum moduli space. [S0031-9007(96)01665-1]

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Seiberg-Witten exact results about $N = 2$ supersymmetric Yang-Mills theory (SYM) [1] concern the low-energy Wilsonian effective action with at most two derivatives and four fermions. These terms are completely described by the so-called prepotential $\mathcal{F}$ whose most important property is holomorphicity [2]. Furthermore, it has been shown in [3] that $\mathcal{F}$ gets perturbative contributions only up to one loop. Higher-order terms in the asymptotic expansion come as instanton contributions implicitly determined in [1].

In [4], where a method to invert functions was proposed, a nonperturbative equation has been derived which relates in a simple way $\mathcal{F}$ and the vev's (vacuum expectation values) of the scalar fields. In [5], proving a conjecture in [6], it has been shown that the above relation underlies the nonperturbative renormalization group equation and the exact expression for the beta function in the SU(2) case has been obtained. The problem of extending these results to the case of higher rank groups is a nontrivial task. An important step in this direction is the result in [6,7] where the nonperturbative relation in [4] has been generalized. However, there remains the problem of finding the nonperturbative relations between $\langle \mathrm{tr}\phi^k\rangle$ for $k > 2$ and $\mathcal{F}$. Also, one should find a set of equations for $\mathcal{F}$ in a similar way to the SU(2) case [4].

In this Letter we will solve these problems for the SU(3) case. In particular, we will find a complete set of nonlinear differential equations completely characterizing $\mathcal{F}$ including the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equation [8]. This indicates that $N = 2$ SYM theories are essentially topological field theories and that they should be seen as the low-energy limit of some topological string theory. Furthermore, we introduce a set of modular invariant quantities which will be useful to find the relation between $\langle \mathrm{tr}\phi^k\rangle$ and $\mathcal{F}$ and to formulate canonical relations between the periods. We also investigate the structure of the beta function and give its explicit form in the moduli coordinates.

The Seiberg-Witten curve for SU(n), $n \geq 3$, has been found in [9]. Let us denote by $a^i = \langle \phi^i\rangle$ and $a_i^D = \langle \phi_i^D\rangle = \partial\mathcal{F}/\partial a^i$, $i = 1, 2$ the vev's of the scalar component of the chiral superfield and its dual. The effective couplings are given by $\tau_{ij} = \partial^2\mathcal{F}/\partial a^i\partial a^j$. We also set $u^2 \equiv u = \langle \mathrm{tr}\phi^2\rangle$, $u^3 \equiv v = \langle \mathrm{tr}\phi^3\rangle$, and $\partial_k \equiv \partial/\partial a^k$, $\partial_\alpha \equiv \partial/\partial u^\alpha$. Our starting point is the reduced Picard-Fuchs equations (RPFE's) for SU(3) introduced in [10]

$$\mathcal{L}_\beta \begin{pmatrix} a_i^D \\ a^i \end{pmatrix} = 0, \quad \beta = 2, 3, \tag{1}$$

where

$$\begin{gathered} \mathcal{L}_2 = \frac{1}{u} P\partial_u^2 + \mathcal{L}, \quad \mathcal{L}_3 = \frac{1}{3} P\partial_v^2 + \mathcal{L}, \\ P = 27(v^2 - \Lambda^6) + 4u^3, \quad \mathcal{L} = 12uv\partial_u\partial_v + 3v\partial_v + 1. \end{gathered} \tag{2}$$

Let us recall that in the SU(n) case, under the action of $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n - 2, \mathbf{Z})$ we have $(\tilde{a}^D, \tilde{a}) = (Aa^D + Ba, Ca^D + Da)$ and [4,6,11] $\tilde{\mathcal{F}}(\tilde{a}) = \mathcal{F}(a) + \frac{1}{2} a^D C^t A a^D + \frac{1}{2} aB^t Da + aB^t Ca^D$.

We now rewrite Eq. (1) as nonlinear differential equations with respect to the a^i coordinates. To this end it is convenient to introduce the following notation:

$$\begin{gathered} \mathcal{U} = u_2^2\partial_{11} - 2u_1u_2\partial_{12} + u_1^2\partial_{22}, \quad \mathcal{V} = v_2^2\partial_{11} - 2v_1v_2\partial_{12} + v_1^2\partial_{22}, \\ \mathcal{C} = (u_1v_2 + v_1u_2)\partial_{12} - u_2v_2\partial_{11} - u_1v_1\partial_{22}, \quad \mathcal{D} = u_1v_2 - u_2v_1, \end{gathered}$$

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where $\partial_{i_1\cdots i_n} \equiv \partial^n/\partial a^{i_1}\cdots\partial a^{i_n}$, $u_i \equiv \partial_i u$, and $v_i \equiv \partial_i v$. We have

$$\left[12uvC + \frac{1}{3}P(u,v,\Lambda)\mathcal{U} + D^2(1 - a^i\partial_i)\right]\mathcal{F}_l = 0 = \left[12uvC + \frac{1}{u}P(u,v,\Lambda)\mathcal{V} + D^2(1 - a^i\partial_i)\right]\mathcal{F}_l\,, \tag{3}$$

where $l = 1,2$ and $\mathcal{F}_{i_1\ldots i_n} \equiv \partial_{i_1\cdots i_n}\mathcal{F}$. Note that D is the Jacobian of $(u,v) \to (a^1,a^2)$ and therefore generally nonvanishing. Subtracting the left-hand side from the right-hand side of Eq. (3), we obtain

$$A_l \equiv x_{11}\mathcal{F}_{22l} + x_{22}\mathcal{F}_{11l} - 2x_{12}\mathcal{F}_{12l} = 0\,, \tag{4}$$

where $l = 1,2$ and

$$x_{ij} = 3v_iv_j - uu_iu_j\,. \tag{5}$$

We stress that (4) has been obtained from (1). Therefore, since $\tilde{a}^i$ and $\tilde{a}_i^D$ are still solutions of (1), it follows that (4) is modular invariant by construction.

In general, it seems that the Picard-Fuchs equations are related to the WDVV equation. The above construction allows us to show that this is actually true for the Picard-Fuchs equations underlying $N = 2$ SYM with gauge group SU(3). Actually, a suitable linear combination of the equations $A_l = 0$, namely,

$$A_1(y_{22}\mathcal{F}_{112} - 2y_{12}\mathcal{F}_{122} + y_{11}\mathcal{F}_{222}) - A_2(-2y_{12}\mathcal{F}_{112} + y_{11}\mathcal{F}_{122} + y_{22}\mathcal{F}_{111}) = 0\,,$$

where y_{jk} are arbitrary parameters, can be written in the WDVV form

$$\mathcal{F}_{ikl}\eta^{lm}\mathcal{F}_{mnj} = \mathcal{F}_{jkl}\eta^{lm}\mathcal{F}_{mni}\,, \tag{6}$$

for $i,j,k,n = 1,2$, where

$$\eta^{lm} = \begin{pmatrix} 2x_{22}y_{12} - 2x_{12}y_{22} & x_{11}y_{22} - x_{22}y_{11} \\ x_{11}y_{22} - x_{22}y_{11} & 2x_{12}y_{11} - 2x_{11}y_{12} \end{pmatrix}. \tag{7}$$

We observe that for each choice of the metric, that is of the parameters y_{jk}, there is only one nontrivial equation in (6) which can be rewritten as

$$\eta^{11}\Theta_{11} + 2\eta^{12}\Theta_{12} + \eta^{22}\Theta_{22} = 0\,, \tag{8}$$

where $\Theta_{ij} = (\mathcal{F}_{11i}\mathcal{F}_{22j} + \mathcal{F}_{11j}\mathcal{F}_{22i})/2 - \mathcal{F}_{12i}\mathcal{F}_{12j}$, which satisfy the identity

$$2\mathcal{F}_{12l}\Theta_{12} = \mathcal{F}_{22l}\Theta_{11} + \mathcal{F}_{11l}\Theta_{22}, \quad l = 1,2\,. \tag{9}$$

Let us introduce some modular invariant quantities which will be used later on. We set

$$I_\beta^\gamma = (\partial_k z)(\partial_\beta\tau)^{-1^{kl}}\partial_l u^\gamma, \tag{10}$$

where β, $\gamma = 2,3$ and z is the modular invariant $z = a^i\partial_i\mathcal{F} - 2\mathcal{F}$. Other useful invariants are

$$v_{(\beta)}^a = I_\beta^\gamma(\partial_\gamma\partial_k u^\alpha)\partial_\beta a^k + a^k\partial_k u^\alpha, \tag{11}$$

where $\alpha,\beta = 2,3$. Let us define the brackets

$$\{X,Y\}_{(\beta)} \equiv \partial_i X(\partial_\beta\tau)^{-1^{ij}}\partial_\beta\partial_j Y - \partial_i Y(\partial_\beta\tau)^{-1^{ij}}\partial_\beta\partial_j X\,. \tag{12}$$

For the vector field components $v_{(\beta)}^\alpha$ we have

$$v_{(\beta)}^\alpha = \{u^\alpha, z\}_{(\beta)}\,. \tag{13}$$

Furthermore, the periods satisfy the canonical relations:

$$\{a^i, a^j\}_{(\beta)} = 0 = \{a_i^D, a_j^D\}_{(\beta)}, \quad \{a^i, a_j^D\}_{(\beta)} = \delta_j^i\,. \tag{14}$$

In order to extract the differential equations for $\mathcal{F}$, we rewrite the operators in (2) in the following general form:

$$\mathcal{L}_\beta = \xi_{(\beta)}\partial_\beta + \eta_{(\beta)} + 1\,, \tag{15}$$

where $\xi_{(\beta)} = \xi_{(\beta)}^\alpha\partial_\alpha = \xi_{(\beta)}^i\partial_i$ and $\eta_{(\beta)} = \eta_{(\beta)}^\alpha\partial_\alpha = \eta_{(\beta)}^i\partial_i$ are vector fields. Considering the action of $\mathcal{L}_\beta$ on f_g with f and g arbitrary functions, we have $\mathcal{L}_\beta fg = g\mathcal{L}_\beta f + f\mathcal{L}_\beta g - fg + \partial_\beta f\xi_{(\beta)}g + \xi_{(\beta)}f\partial_\beta g$, and by Eq. (1)

$$\mathcal{L}_\beta(a^ia_i^D - 2\mathcal{F}) = a^ia_i^D - 2\mathcal{F}\,, \tag{16}$$

that is $\mathcal{L}_\beta z = z$. Note that in (15), as in (2), for each value of β the second-order derivative terms contain always at least one ∂_β [note that Eq. (16) is independent from this peculiarity].

In order to find $\xi_{(\beta)}$ and $\eta_{(\beta)}$, we impose that the operators defined in (15) satisfy (1). From the lower components of (1) we obtain $\eta_{(\beta)}^i = -a^i - \xi_{(\beta)}\partial_\beta a^i$, which substituted in the upper components of (1), yields $\xi_{(\beta)}^i = (\partial_k z)(\partial_\beta\tau)^{-1^{ki}}$. Therefore

$$\mathcal{L}_\beta = I_\beta^\gamma\partial_\gamma\partial_\beta - v_{(\beta)}^\gamma\partial_\gamma + 1\,. \tag{17}$$

Comparing (17) with (2) we obtain a complete set of nonlinear differential equations for $\mathcal{F}$ and its exact relation with the moduli coordinates, namely,

$$v_{(2)}^2 = 0 = v_{(3)}^2\,, \tag{18}$$

$$v_{(2)}^3 = -3v = v_{(3)}^3\,, \tag{19}$$

$$I_2^3 = 12uv = I_3^2, \tag{20}$$

$$uI_2^2 = P = 3I_3^3. \tag{21}$$

The above procedure when applied to the SU(2) case gives [4] $u = -i\pi z/2$, and $I_2^2 = 4(u^2 - \Lambda^4)$, which is the equation for the prepotential obtained in [4,12]. Let us now define the modular invariant 1 form

$$W = (a^i\partial_\beta a_i^D - a_i^D\partial_\beta a^i)du^\beta = dz\,, \tag{22}$$

which, due to the existence of the prepotential, is closed, i.e., $dW = 0$. Substituting in (22) the expression of the periods in terms of Appell's F_4 functions derived in [10], we obtain $W = (3i/\pi)du$. On the other hand, by (16) it follows that the components of $W = W_\beta du^\beta$ satisfy the linear differential equations

$$\xi_{(\beta)} W_\beta = v^\alpha_{(\beta)} W_\alpha, \quad \beta = 2, 3, \tag{23}$$

which are satisfied by $W_2 = 3i/\pi$, $W_3 = 0$. Therefore, we have $z = (3i/\pi)u$; that is,

$$u = \frac{2\pi i}{3}\left(\mathcal{F} - \frac{a^i}{2}\partial_i \mathcal{F}\right), \tag{24}$$

in agreement with [6,7]. Note that by (24), thanks to (13), Eq. (18) is identically satisfied. We now use Eq. (4) to face the problem of finding the explicit relation between v and $\mathcal{F}$. In general, it can be shown that the properties of special geometry imply that $\Theta_{ii} \neq 0$ [13]. By (9) the general solution of Eq. (4) is given by

$$x_{ij} = \rho \Theta_{ij}, \tag{25}$$

where ρ is determined by the compatibility condition $(3v_1 v_2)^2 = (3v_1^2)(3v_2^2)$ applied to (5) and (25); that is,

$$\rho^2 \Delta = u\rho(\Theta_{11} u_2^2 + \Theta_{22} u_1^2 - 2u_1 u_2 \Theta_{12}), \tag{26}$$

where $\Delta = \Theta_{12}^2 - \Theta_{11}\Theta_{22}$. Notice that $\rho \neq 0$; otherwise, $3v_i v_j = u u_i u_j$, which would imply $D = 0$. Since $\rho^2 \Delta = x_{12}^2 - x_{11}x_{22} = 3uD^2$, we have $\Delta \neq 0$, so that (26) can be solved as $\rho = \Delta^{-1} u(\Theta_{11}u_2^2 + \Theta_{22}u_1^2 - 2u_1 u_2 \Theta_{12})$, which implies

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \epsilon \sqrt{\frac{u}{3\Delta}} \begin{pmatrix} \Theta_{12} & -\Theta_{11} \\ \Theta_{22} & -\Theta_{12} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{27}$$

where $\epsilon = \pm 1$ and the relative sign between v_1 and v_2 has been fixed by $x_{12} = \rho\Theta_{12}$. Observe that we can set $\epsilon = 1$ by a suitable transformation on the moduli variables (we use the fact that the RPFE's are invariant under the transformations $u \rightarrow e^{2i\pi k/3}u$ and $v \rightarrow -v$).

In order to find v we first explicitly evaluate the I_β^γ invariants in terms of u_i, v_i, and $\mathcal{F}_{ijk}$. Then, by (24) and (27) we will obtain the relation between v and $\mathcal{F}$ and nonlinear differential equations for $\mathcal{F}$ as well. The essential point is that by (20) and (21) we have

$$v = \frac{I_3^2 I_2^2}{36 I_3^3}. \tag{28}$$

On the other hand (10) can be written as $I_2^\alpha = \frac{3i}{\pi} D(v_2^2\Theta_{11} + v_1^2\Theta_{22} - 2v_1v_2\Theta_{12})^{-1}(v_2 g_1^\alpha - v_1 g_2^\alpha)$, and $I_3^\alpha = \frac{3i}{\pi} D(u_2^2\Theta_{11} + u_1^2\Theta_{22} - 2u_1u_2\Theta_{12})^{-1}(u_1 g_2^\alpha - u_2 g_1^\alpha)$, where $g_i^\alpha = u_2 u_2^\alpha \mathcal{F}_{11i} + u_1 u_1^\alpha \mathcal{F}_{22i} - (u_1 u_2^\alpha + u_2 u_1^\alpha)\mathcal{F}_{12i}$. By (24) and (27) the I_β^α's are explicitly known in terms of a^i and $\mathcal{F}$. It follows that (28) solves the problem of finding the relation between v and $\mathcal{F}$. By (13), we can rewrite Eq. (19) in the form

$$\{v, u\}_{(\beta)} = i\pi v, \quad \beta = 2, 3, \tag{29}$$

which by (28) are two nonlinear differential equations for $\mathcal{F}$, that together with the two-parameter WDVV equations correspond to the four nonlinear differential equations (3).

Let us now consider the modular properties of $\mathcal{F}$ and its homogeneity. The fact that τ_{ij} is dimensionless implies that

$$(\Lambda\partial_\Lambda + \Delta_{u,v})\tau_{ij} = 0, \tag{30}$$

where $\Delta_{u,v} = 2u\partial_u + 3v\partial_v$ is the scaling invariant vector field. Let ξ be an arbitrary modular invariant vector field. We have $\xi\tau \rightarrow \xi\tilde{\tau} = (\tau C^t + D^t)^{-1}\xi\tau(C\tau + D)^{-1}$, implying that (30) is a modular invariant equation. We also have $(\Lambda\partial_\Lambda + \Delta_{u,v})a^i = a^i$, $(\Lambda\partial_\Lambda + \Delta_{u,v})a_i^D = a_i^D$, which are compatible with a pseudohomogeneity of degree 2 for $\mathcal{F}$: $(\Lambda\partial_\Lambda + \Delta_{u,v})\mathcal{F} = 2\mathcal{F} + \Lambda^2 \times \text{const}$. In our case the semiclassical analysis gives const $= 0$.

Let us now discuss in the SU(3) case the uniformization mechanism which generalizes the structure underlying the SU(2) case [4]. The structure of the covering of the quantum moduli space $\mathcal{M}_{\mathrm{SU}(3)}$ is encoded in the properties of the Appell's functions. The Appell system F_4 is a two-dimensional generalization of the hypergeometric system also endowed with algebraic relations involving the functions and their derivatives. It is known [10] that the period matrix τ_{ij} is a rational combination of Appell's functions. By (30), the dependence on u, v, and Λ is of the form $\tau = \tau(u/\Lambda^2, v/\Lambda^3)$. Therefore the τ space is a subvariety S of the genus 2 Siegel upper-half space of complex codimension 1 covering $\mathcal{M}_{\mathrm{SU}(3)}$. S can be characterized by $\mathrm{s}(\tau) = 0$, where the structure of s is related to the equations satisfied by $\mathcal{F}$. Let $M_{\mathrm{SU}(3)} \subset Sp(4, \mathbf{Z})$ be the monodromy group of $N = 2$ SYM with gauge group SU(3) [10]. The above remarks imply that the Picard-Fuchs equations, from which Eq. (1) is derived, are the uniformizing equations for $\mathcal{M}_{\mathrm{SU}(3)}$. Therefore $\mathcal{M}_{\mathrm{SU}(3)} \cong S/M_{\mathrm{SU}(3)}$. The polymorphic matrix function τ is the inverse covering with $M_{\mathrm{SU}(3)}$ monodromy. Let $u/\Lambda^2 = \mathrm{u}(\tau), v/\Lambda^3 = \mathrm{v}(\tau), \tau \in S$, be the covering map. From the above data we now derive the beta function of the theory. Let us consider the following equations:

$$\begin{aligned} 0 &= \Lambda\partial_\Lambda \mathrm{s}(\tau) = \Sigma(\beta)\mathrm{s}(\tau), \\ 0 &= \Lambda\partial_\Lambda u = \Lambda^2[\Sigma(\beta)\mathrm{u}(\tau) + 2\mathrm{u}(\tau)], \\ 0 &= \Lambda\partial_\Lambda v = \Lambda^3[\Sigma(\beta)\mathrm{v}(\tau) + 3\mathrm{v}(\tau)], \end{aligned} \tag{31}$$

where $\beta_{ij} = \Lambda\partial_\Lambda\tau_{ij}$ is the β function and $\Sigma(\beta)$ is the scaling operator $\Sigma(\beta) = \beta_{11}\partial_{\tau_{11}} + \beta_{12}\partial_{\tau_{12}} + \beta_{22}\partial_{\tau_{22}}$. Note that the solution of the system (31) completely determines the β function of the theory. We now derive the exact β function projected on the natural moduli directions in terms of the modular invariants $J_{\alpha\beta\gamma} = \partial_\alpha a^i \partial_\beta \tau_{ij} \partial_\gamma a^j$, which are completely symmetric in their indices. Actually, defining the projected β function $\beta_{\alpha\gamma} = \partial_\alpha a^i \beta_{ij} \partial_\gamma a^j$, and using (30), we have

$$\beta_{\alpha\gamma} = -2uJ_{\alpha 2\gamma} - 3vJ_{\alpha 3\gamma}. \tag{32}$$

The $J_{\alpha\beta\gamma}$'s are related to the I_β^γ's by $I_\beta^\gamma J_{\gamma\beta 2} = \frac{\pi}{3i}, I_\beta^\gamma J_{\gamma\beta 3} = 0$; that is, $\frac{P}{u} J_{223} + 12uvJ_{233} = 0 = \frac{P}{3} J_{333} + 12uvJ_{233}$, $\frac{P}{u} J_{222} + 12uvJ_{223} = \frac{3i}{\pi} = \frac{P}{3} J_{233} + 12uvJ_{223}$. Inserting the solution of this system in (32), we obtain

$$\beta_{22} = \frac{2Au}{3}[P - 54v^2], \tag{33}$$

$$\beta_{23} = \beta_{32} = \frac{3Av}{u}[P - 8u^3], \tag{34}$$

$$\beta_{33} = 2A[P - 54v^2], \tag{35}$$

where $A = \frac{3i}{\pi}[(12uv)^2 - P^2/3u]^{-1}$.

Let us make some concluding remarks. First of all we note that similar structures can be generalized to the case of gauge group SU(n), $n > 3$. Furthermore the condition $s(\tau) = 0$ and the WDVV equation suggest a relation with the condition on the lattices obtained in [14] [see Eq. (5.22) there]. In this framework one should be able to connect the mass formula with the area of degenerate metrics on a suitable Riemann surface. This surface should be related to the two-dimensional space which arises in compactifying $N = 1$ in $D = 6$ to obtain $N = 2$ in $D = 4$. In [12] a similar structure for the SU(2) case has been obtained.

We also observe that the way we use the Picard-Fuchs equations could be useful in investigating some algebraic-geometrical structure and some aspects concerning mirror symmetry (see [15] for related aspects). In this context we note that by considering the branching points of the hyperelliptic Riemann surfaces as punctures on the Riemann sphere, it should be possible to describe the Seiberg-Witten moduli space in terms of moduli space of Riemann spheres with punctures. Observe that already in the SU(2) case the moduli space is the Riemann sphere with three punctures which can be essentially seen as $\overline{\mathcal{M}}_{0,4}$, the moduli space of Riemann spheres with four punctures. In this framework one can use relevant structures such as the Deligne-Knudsen-Mumford compactification $\overline{\mathcal{M}}_{h,p}$, were "punctures never collide," which allows us to consider natural embeddings (this problem is of interest also for softly supersymmetry breaking [16]). We also observe that the WDVV equation can be seen as an associativity condition for divisors on $\overline{\mathcal{M}}_{0,p}$ [17]. These structures together with the restriction phenomenon of the Weil-Petersson metric, whose Kähler potential is the on-shell Liouville action, are at the basis of recursion relations arising in 2D quantum gravity [18].

In conclusion, we obtained nonperturbative relations for $N = 2$ SYM with gauge groups SU(3) which generalize the results in [4] where the relation between u and the prepotential has been found in the SU(2) case. This relation has been recently proved in [19]. The results of our investigation should be similarly verified for a more complete proof of the Seiberg-Witten theory.

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