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On a Class of Strongly Quasi Injective Modules.

ALBERTO TONOLO (*)

0. Introduction.

0.1 Let R be a ring with $1 \neq 0$, ${}_RK$ a unitary left R-module, $A = \mathrm{End}({}_RK)$; denote by $\mathcal{D}(K_A)$ the full subcategory of Mod-A cogenerated by K_A and by $\mathcal{C}({}_RK)$ the full subcategory of R-TM consisting of all modules which are topologically isomorphic to a closed submodule of a topological product ${}_RK^X$, where X is a set and ${}_RK$ is endowed with the discrete topology. The modules belonging to $\mathcal{D}(K_A)$ are called *K-discrete*, those belonging to $\mathcal{C}({}_RK)$ are called *K-compact*.

0.2 Let $M \in \mathcal{D}(K_A)$; M^* will denote the module $\mathrm{Hom}_A(M, K_A)$ with the topology that has as basis of neighbourhoods of zero $W(F) = \{\xi \in \mathrm{Hom}_A(M, K_A): \xi(F) = 0\}$, where F is a finite subset of M; it will be called the character module or the *dual* of M. Let now $N \in \mathcal{C}({}_RK)$; the abstract right A-module $\mathrm{Chom}(N, {}_RK)$ of continuous R-morphisms of N into ${}_RK$, called the character module or the *dual* of N, will be denoted by N^*. Associating to each K-discrete module its dual and to each morphism its transposed, gives a contravariant functor $\Delta_1: \mathcal{D}(K_A) \to \mathcal{C}({}_RK)$. In a similar way we define a contravariant functor $\Delta_2: \mathcal{C}({}_RK) \to \mathcal{D}(K_A)$. Let $\Delta_K = (\Delta_1, \Delta_2)$; we say that Δ_K is a *duality* if for each $M \in \mathcal{D}(K_A)$ and for each $N \in \mathcal{C}({}_RK)$, the natural canonical morphisms $\omega_M: M \to M^{**}$, $\omega_N: N \to N^{**}$ are respectively an isomorphism and a topological isomorphism. Next we call

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Δ_K a *good duality* if Δ_K is a duality and $\mathcal{C}({}_RK)$ has the *extension property* of characters (in short E.P.), i.e. if, for each $M \in \mathcal{C}({}_RK)$ and each topological submodule L of M, any character of L extends to a character of M. A (topological) R-module M is *quasi-injective* (in short q.i.) if every (continuous) morphism of any submodule of M into M extends to a (continuous) endomorphism of M. A (topological) R-module M is *strongly quasi-injective* (in short s.q.i.) if for every (closed) submodule B of M and for every element $x_0 \in M\text{-}B$ any (continuous) morphism $\xi: B \to M$ extends to a (continuous) endomorphism $\xi^\wedge$ of M such that $x_0\, \xi^\wedge \neq 0$. Claudia Menini and Adalberto Orsatti [M.O.1] proved that Δ_K is a good duality if and only if ${}_RK$ is s.q.i.

0.3 The purpose of this paper is to study the s.q.i. modules ${}_RK$ for which Δ_K is a good duality between $\mathcal{C}({}_RK)$ and Mod-A; we have achieved the following results:

THEOREM A (Th. 1.6). *$\mathcal{C}({}_RK)$ is an abelian category if and only if $\mathcal{D}(K_A) =$ Mod-A, i.e. K_A is a cogenerator of* Mod-A.

In order to obtain more precise results we have introduced the notion of strongly abelian category of topological modules and we have proved:

THEOREM B (Th.s 1.8-1.9). *$\mathcal{C}({}_RK)$ is a strongly abelian category if and only if K_A is an injective cogenerator of* Mod-A.

When K_A is an injective cogenerator of Mod-A, we have a complete description:

THEOREM C (Th. 1.11). *Let R_τ be a left l.t. Hausdorff ring, ${}_RK \in \mathcal{C}_\tau$ an injective cogenerator of $\mathcal{C}_\tau$ with essential socle, $A =$* End$({}_RK)$. *The following conditions are equivalent:*

a) $\mathcal{C}({}_RK)$ *is a strongly abelian category,*

b) $\mathcal{C}({}_RK) = R_\tau\text{-}LC_*$,

c) ${}_RK$ *is l.c.d.,*

d) K_A *is an injective cogenerator of* Mod-A,

e) A_A *is l.c.d. and every f.g. submodule of* ${}_RK$ *is l.c.d.,*

f) A_A *is l.c.d. and* K_A *is q.i.,*

g) Δ_K *is a good duality between* Mod-A *and* $R_\tau\text{-}LC_*$.

0.4 In the second part we carefully investigate the case when K_A is a cogenerator of Mod-A. We have a description of the exact sequences in $\mathcal{C}(_RK)$, (Th.s 2.1-2.4); we prove that in this case A_A is l.c.d., $A/J(A)$ is semisimple artinian, $\mathrm{Soc}(_RK) = \mathrm{Soc}(K_A)$ they are both essential (Prop. 2.7) and we obtain a structure theorem for $_RK$ (Th. 2.10). Although the conditions on $_RK$ are very particular, it is not clear if they are sufficient to characterize the s.q.i. modules $_RK$ such that $\mathcal{C}(_RK)$ is an abelian category.

Finally in the third part we have obtained an example of a good duality Δ_K between $\mathcal{C}(_RK)$ and Mod-A where K_A is a cogenerator not-injective of Mod-A that justifies the different treatment in the cases K_A cogenerator and K_A injective cogenerator of Mod-A.

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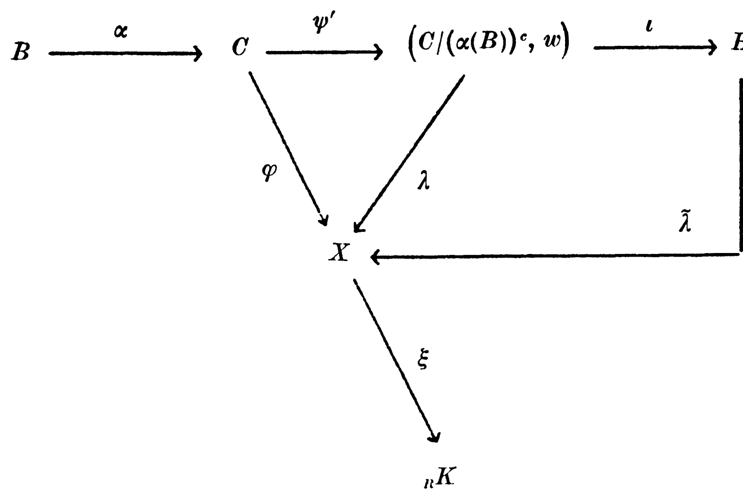
1. Δ_K dualities and abelian categories.

1.1 Let R be a ring, $_RK$ a left R-module; endow R with the K-topology τ and denote by $R^\wedge_{\tau^\wedge}$ the Hausdorff completion of R_τ. From the topological embeddings $R/\mathrm{Ann}_R(K) \leqslant R^\wedge \leqslant K^K$ it follows that the topology $\tau^\wedge$ of $R^\wedge$ coincides with the K-topology of $R^\wedge$. Let R_τ-LT the category of all l.t. Hausdorff left modules over R_τ; if $M \in R_\tau$-LT is a complete module, then in natural way $M \in R^\wedge_{\tau^\wedge}$-$LT$ and each continuous R-morphism between complete modules belonging to R_τ-LT is a $R^\wedge$-morphism. Since $\mathrm{End}(_{R^\wedge}K) = \mathrm{End}(_RK)$ and $\mathcal{C}(_{R^\wedge}K) = \mathcal{C}(_RK)$, we may assume, without loss of generality, R_τ complete and Hausdorff.

1.2 The category $\mathcal{C}(_RK)$ of K-compact R-modules is obviously preadditive and closed under topological products; given any morphism in $\mathcal{C}(_RK)$ there exists the kernel (the usual one) and, if $_RK$ is s.q.i., also the cokernel. Let $\alpha: B \to C$ be a morphism in $\mathcal{C}(_RK)$, we denote by H the Hausdorff completion of $\big(C/(\alpha(B)^c, w)\big)$, where w is the weak topology of characters of $C/(\alpha(B))^c$ endowed with quotient topology. By proposition 2.6 of [M.O.1] it is easy to prove that H is an object of $\mathcal{C}(_RK)$.

1.3 Proposition. *Let $_RK$ be a s.q.i. module; given a morphism $\alpha: B \to C$ in $\mathcal{C}(_RK)$, we have* Coker $\alpha = \big(C/(\alpha(B)^c, w)\big)^\wedge$.

Proof. Let $\varphi\colon C \to X$ be a morphism in $\mathcal{C}({}_RK)$ with $\alpha\varphi = 0$ and ξ be a character of X; let us consider the diagram



where ψ' and ι are respectively the natural projection and embedding. Set $\psi = \psi'\iota$, obviously ψ is continuous and $\alpha\psi = \alpha(\psi'\iota) = 0$. Being $\varphi|_{(\alpha(B))^c} = 0$, there exists an algebraic morphism $\lambda\colon C/(\alpha(B))^c \to X$ with $\varphi = \psi'\lambda$. $\varphi\xi$ is a character of C equal to zero on $(\alpha(B))^c$, then $\lambda\xi$ is a character of $\big(C/(\alpha(B))^c, w\big)$ hence it is continuous; having X the weak topology of characters, λ is continuous for the arbitrary choice of ξ. Being X complete and Hausdorff, λ extends to a continuous morphism $\tilde{\lambda}\colon H \to X$ with $\varphi = \psi\tilde{\lambda}$.

For what we have seen above $\mathcal{C}({}_RK)$ is an abelian category if and only if for any morphism $\alpha\colon B \to C$ in $\mathcal{C}({}_RK)$, Coker(ker α) and Ker(coker α) are isomorphic; having previously identified $B/\operatorname{Ker}\alpha$ and $\alpha(B)$, this happen only when the weak topology w_1 of characters of $B/\operatorname{Ker}\alpha$, endowed of quotient topology, and the topology w_2 of $\alpha(B)$, as topological submodule of C, coincide.

1.4 Definition. If in the above context w_1 and w_2 coincide and are complete, we say that $\mathcal{C}({}_RK)$ is a *strongly abelian category*.

1.5 Proposition. *In the category $\mathcal{D}(K_A)$ monomorphisms are injective; if $\mathcal{D}(K_A)$ is an abelian category its epimorphisms are surjective.*

PROOF. The first statement is obvious; next if $f\colon M \to N$ is an epimorphism then, remembering that $\mathfrak{D}(K_A)$ is closed under submodules, $f(M) \to N$ is a monomorphism and an epimorphism, hence an isomorphism in $\mathfrak{D}(K_A)$, i.e. a usual bijective morphism of modules.

1.6 THEOREM. *Let $\mathcal{C}({}_RK)$ be an abelian category; if Δ_K is a duality between $\mathcal{C}({}_RK)$ and $\mathfrak{D}(K_A)$, then $\mathfrak{D}(K_A) = \text{Mod-}A$, i.e. K_A is a cogenerator.*

PROOF. Let $M \in \text{Mod-}A$, M is an homomorphic image of $A^{(X)}$; since ${}_RK^* = A$ we have $A^{(X)} \in \mathfrak{D}(K_A)$. The kernel L in Mod-A of $A^{(X)} \to M$ belongs to $\mathfrak{D}(K_A)$. The dualities preserves the abelian categories, hence $\mathfrak{D}(K_A)$ is abelian; consider then the exact sequence in $\mathfrak{D}(K_A)$

$$(*) \qquad 0 \to L \xrightarrow{f} A^{(X)} \xrightarrow{\psi} \text{Coker}_{\mathfrak{D}(K)}(f) \to 0 ;$$

By the above proposition f is injective and ψ is surjective. Obviously $f(L) \subseteq \text{Ker}\,\psi$; next we consider $\iota\colon f(L) \to \text{Ker}\,\psi = \text{Ker}(\text{coker}\, f) = \text{Im}\, f$ in $\mathfrak{D}(K_A)$: ι is a monomorphism and an epimorphism in $\mathfrak{D}(K_A)$ and so it is an isomorphism. Then the sequence $(*)$ is exact also in Mod-A and it results $M \cong A^{(X)}/L \cong \text{Coker}_{\mathfrak{D}(K)}(f) \in \mathfrak{D}(K_A)$.

1.7 PROPOSITION. *If $\mathcal{C}({}_RK)$ is a strongly abelian category, then epimorphisms in $\mathcal{C}({}_RK)$ are surjective.*

PROOF. Let $f\colon M \to N$ be an epimorphism in $\mathcal{C}({}_RK)$; we consider the exact sequence in $\mathcal{C}({}_RK)$ $0 \to \text{Ker}\, f \xrightarrow{i} M \xrightarrow{f} N \to 0$; if w is the weak topology of characters on $(M/\text{Ker}\, f, q)$ then $N \cong (M/\text{Ker}\, f, w)$ topologically, for $(M/\text{Ker}\, f, w) \in \mathcal{C}({}_RK)$ is the cokernel of i.

1.8 THEOREM. *Let $\mathcal{C}({}_RK)$ be a strongly abelian category and Δ_K a duality between $\mathfrak{D}(K_A)$ and $\mathcal{C}({}_RK)$; then K_A is an injective cogenerator of* Mod-A.

PROOF. By theorem 1.6, K_A is a cogenerator of Mod-A; we consider the injective hull $E = E(K_A)$ of K_A in Mod-A. The functor $\Delta_1 = \text{Hom}(\cdot, K_A)$ transposes the inclusion $K_A \xrightarrow{i} E$ in an epimorphism $\text{Hom}_A(E, K_A) \xrightarrow{i^*} \text{Hom}_A(K_A, K_A)$ of $\mathcal{C}({}_RK)$. By proposition 1.5, i^* is surjective, hence the identity morphism of K_A extends to a morphism $E \to K_A$; then K_A is a direct summand of E and so $K_A = E = E(K_A)$.

1.9 THEOREM. *Let Δ_K be a good duality between $\mathcal{D}(K_A)$ and $\mathcal{C}({}_RK)$; if K_A is an injective cogenerator of* Mod-A, *then $\mathcal{C}({}_RK)$ is a strongly abelian category.*

PROOF. $\mathcal{C}({}_RK)$ is an abelian category since $\mathcal{D}(K_A) = \text{Mod-}A$ and Δ_K is a duality. By theorem 17.1 of [M.O.2] ${}_RK$ is l.c.d., hence each module belonging to $\mathcal{C}({}_RK)$ is l.c.; given $M \in \mathcal{C}({}_RK)$ and a closed submodule L of M, $(M/L,q)$ is linearly compact, since it is a Hausdorff quotient of a l.c. module; moreover M/L endowed with the weak topology of characters, wich is coarser than q, is still l.c. and hence complete.

1.10 Let $M_\varepsilon \in R_\tau\text{-}LT$; we denote by ε_* the Leptin topology, i.e. the topology on M having as a basis of neighbourhoods of 0 all the open cofinite submodules of M_ε. We denote by $R_\tau\text{-}LC_*$ the full subcategory of $R_\tau\text{-}LT$ consisting of all $M_\varepsilon \in R_\tau\text{-}LT$ such that M_ε is l.c. and $\varepsilon = \varepsilon_*$. If M is l.c. it is known (see [W.]) that among all topologies equivalent to ε there exists a finest one which will be denoted by ε^*. The topology ε^* has as a basis of neighbourhoods of 0 in M the closed submodules H of M_ε such that M/H is l.c.d. We indicate with $\mathcal{T}_\tau$ the class of the τ-torsion left R_τ-modules, i.e.

$$\mathcal{T}_\tau = \{M \in R_\tau\text{-}TM : \forall x \in M, \ \text{Ann}_R(x) \text{ is open in } R_\tau\}.$$

1.11 THEOREM. *Let R_τ be a left l.t. Hausdorff ring, ${}_RK \in \mathcal{T}_\tau$ an injective cogenerator of $\mathcal{T}_\tau$ with essential socle, $A = \text{End}({}_RK)$. The following conditions are equivalent:*

i) $\mathcal{C}({}_RK) = R_\tau\text{-}LC_*$,

ii) ${}_RK$ *is l.c.d.*,

iii) K_A *is an injective cogenerator of* Mod-A,

iv) A_A *is l.c.d. and every f.g. submodule of* ${}_RK$ *is l.c.d.*,

v) A_A *is l.c.d. and* K_A *is q.i.*,

vi Δ_K *is a good duality between* Mod-A *and* $R_\tau\text{-}LC_*$.

PROOF. i) $\Rightarrow$ ii) ${}_RK$ endowed with the discrete topology belongs to $\mathcal{C}({}_RK)$, hence it is l.c.d.

ii) $\Rightarrow$ i) Let us prove that $\mathcal{C}({}_RK) \subseteq R_\tau\text{-}LC_*$. Let $M_\varepsilon \in \mathcal{C}({}_RK)$: since ${}_RK$ is l.c.d., M_ε is l.c. Next $\varepsilon = \varepsilon_*$: in fact ${}_RK$, being l.c.d. with essential socle, is finitely generated and so for each character f of M, being $M/\mathrm{Ker}\, f$ a submodule of ${}_RK$, $\mathrm{Ker}\, f$ is cofinite. $\mathcal{C}({}_RK) \supseteq R_\tau\text{-}LC_*$: let $M_\varepsilon \in R_\tau\text{-}LC_*$, since ${}_RK$ is an injective cogenerator of $R_\tau\text{-}LT$, the K-characters of M_ε separate the points of M; then, by the minimality of ε, $M_\varepsilon \in \mathcal{C}({}_RK)$.

ii) $\Rightarrow$ iii) ${}_RK$ is an injective cogenerator of $\mathcal{C}_\tau$, then ${}_RK$ is s.q.i. and hence a selfcogenerator; by theorem 9.4 of [M.O.1] K_A is injective. Let S_A be a simple module; we consider the exact sequence $0 \to P \xrightarrow{\iota} A \to S \to 0$ with P a right maximal ideal of A. K_A injective implies that $\mathrm{Hom}(\cdot, K_A)$ is an exact functor, so that we have the exact sequence $0 \to \mathrm{Hom}_A(S, K_A) \to {}_RK \xrightarrow{\iota^*} \mathrm{Hom}_A(P, K_A) \to 0$. If $\mathrm{Hom}_A(S, K_A) = 0$, ι^* is a continuous isomorphism from ${}_RK$ into $\mathrm{Hom}_A(P, K_A)$; being the discrete topology equal to the Leptin topology, it is the only Hausdorff linear one on ${}_RK$ and $\mathrm{Hom}_A(P, K_A) \cong {}_RK$ topologically. Since Δ_K is a duality, ι must be an isomorphism: absurd!

iii) $\Rightarrow$ ii) Clear by theorem 9.4 of [M.O.1].

ii) $\Rightarrow$ iv) Let

$$(*) \qquad a \equiv a_i \bmod J_i (i \in I)$$

be a finitely solvable system of congruences with $(J_i)_{i\in I}$ a family of right ideals of A. Let $L = \sum_{i\in I} \mathrm{Ann}_K(J_i) \leqslant {}_RK$; we define a R-morphism $g: L \to {}_RK$ by setting $g\left(\sum_{i\in F} x_i\right) = \sum_{i\in F} x_i a_i$ where F is a finite subset of I and, for each $i \in I$, $x_i \in \mathrm{Ann}_K(J_i)$; this is a good definition because $(*)$ is finitely solvable. ${}_RK$ is s.q.i. hence q.i., and so g extends to an endomorphism $g^\wedge$ of ${}_RK$; $g^\wedge$ is the right moltiplication by an element $a \in A$, thus for each $i \in I$ and for each $x \in \mathrm{Ann}_K(J_i)$ we have $g(x) = xa = xa_i$ and hence $a - a_i \in \mathrm{Ann}_A(\mathrm{Ann}_K(J_i)) = J_i$ since K_A is a cogenerator.

iv) $\Rightarrow$ ii) By theorem 9.4 of [M.O.1] it is sufficient to prove that K_A is injective. Let H be a right ideal of A and $f: H \to K_A$ a morphism; set σ equal to the K-topology of A; since A is l.c.d. every right ideal of A is closed in σ. Being $R \leqslant K^K$, it is l.c. with the K-topology; $\mathrm{Soc}({}_RK)$ is essential in ${}_RK$, ${}_RK$ is s.q.i. therefore by theorems 2.8 and 2.10 of [D.O.1] we find that K_A is s.q.i. and $\mathrm{Soc}(K_A)$ is essential in K_A: then $\mathrm{Im}\, f$ is finitely cogenerated. There exists a finite number of simple A-submodules S_i with $i = 1, \ldots, N$ of K_A such that $\operatorname{Im} f \leqslant \leqslant \bigoplus_1^N E(S_i)$ and hence f extends to a morphism $f^\wedge : A \to \bigoplus_1^N E(S_i)$. Let $x = f^\wedge(1)$: then $x = x_1 + \ldots + x_N$ with $x_i \in E(S_i)$ and $\bigcap_1^N \operatorname{Ann}_A(x_i) = = \operatorname{Ann}_A(x) = \operatorname{Ker} f^\wedge \geqslant \operatorname{Ker} f$; moreover $\operatorname{Ann}_A(x_i)$ is closed in A_σ and completely irriducible for all i, hence it is open in A_σ. Therefore $\operatorname{Ker} f = H \cap \operatorname{Ker} f^\wedge$ is open in H with the relative topology of σ and, K_A being s.q.i., f extends to a morphism $A \to K_A$.

iv) $\Leftrightarrow$ v) See proposition 1.5 of [M.1]

i) $\Leftrightarrow$ vi) is obvious.

1.12 COROLLARY. *Let R_τ be a left l.t. Hausdorff ring, ${}_RK \in \mathcal{C}_\tau$ an injective cogenerator of $\mathcal{C}_\tau$ with essential socle, $A = \operatorname{End}({}_RK)$; then $\mathcal{C}({}_RK)$ is a strongly abelian category if and only if it is closed with respect to Hausdorff quotients.*

PROOF. R_τ-LC_* is closed with respect to Hausdorff quotients.

2. Structure theorems.

In the rest of the paper ${}_RK_A$ will be a faithfully balanced bimodule with ${}_RK$ s.q.i. and K_A cogenerator; under this assumptions Δ_K will be a good duality between Mod-A and $\mathcal{C}({}_RK)$.

2.1 THEOREM. *Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be an exact sequence in* Mod-A; *if we consider the trasposed sequence $N^* \xrightarrow{g^*} M^* \xrightarrow{f^*} L^*$, then*

a) *g^* is a topological embedding,*

b) $\operatorname{Ker} f^* = \mathit{Im}\, g^*$,

c) $(f^*(M^*))^c = L^*$.

PROOF. *a)* Clearly g^* is injective and continuous, in addition it is also open: any neighbourhood of zero in N^* is of the form $W(F) = = \{\varphi \in N^* : \varphi|_F = 0\}$ with $F = \langle x_1, \ldots, x_n \rangle$ a finitely generated submodule of N; let $y_i \in M$ be such that $g(y_i) = x_i$ $(i = 1, \ldots, n)$ and set $G = \langle y_1, \ldots, y_n \rangle$. We claim that $g^*(W(F)) \supseteq W(G) \cap \operatorname{Im} g^*$: if $\xi \in W(G) \cap \operatorname{Im} g^*$, $\xi = g^*(\eta)$ it is $\xi = \eta \circ g$ with $\eta \in N^*$; in this way we have $0 = \xi(y_i) = \eta \circ g(y_i) = \eta(x_i)$, consequently $\eta \in W(F)$ and then $\xi \in g^*(W(F))$.

b) It is obvious since $\Delta_1 = \text{Hom}(\cdot, K_A)$.

c) Let $\xi \in L^*$ and F be a finitely generated submodule of L; we show that $(\xi + W(F)) \cap f^*(M^*) \neq 0$. Set $\eta = \xi|_F$: by theorem 2.5 of [D.O.1], η extends to a character η' of M and obviously $\eta' - \xi \in W(F)$.

2.2 Remark. If F is finitely generated in $\mathfrak{D}(K_A)$, F^* is discrete since $0 = W(F)$ is a neighbourhood of 0 in F^*. If Mod-$A = \mathfrak{D}(K_A)$, then it is true also the converse: let $M = N^*$ be discrete, there exists a finitely generated submodule F of N such that $W(F) = 0$. If $F \neq N$, and $x \in N - F$, we would find, being K_A a cogenerator, a morphism φ with $\varphi(x) \neq 0$ and $\varphi|_F = 0$: absurd!

2.3 Duality Lemma. *Let* $\alpha: N \to M$ *and* $f: L \to M$ *morphisms in* Mod-A; *then* $\text{Im}\,\alpha \leqslant \text{Im}\,f$ *if and only if* $\text{Ker}\,\alpha^* \geqslant \text{Ker}\,f^*$.

Proof. $(\Rightarrow)$ Let $\xi \in \text{Ker}(f^*)$, then $f^*(\xi) = 0$, i.e. $\xi \circ f = 0$ hence $\alpha^*(\xi) = \xi \circ \alpha = 0$ and consequently $\xi \in \text{Ker}\,\alpha^*$.

$(\Leftarrow)$ Now we assume that for each $\xi \in M^*$, $f^*(\xi) = 0$ implies $\alpha^*(\xi) = 0$, i.e. $\text{Im}\,f \leqslant \text{Ker}\,\xi$ implies $\text{Im}\,\alpha \leqslant \text{Ker}\,\xi$; we claim that $\text{Im}\,\alpha \leqslant \text{Im}\,f$: if $x \in \text{Im}\,\alpha$ and $x \notin \text{Im}\,f$, being K_A a cogenerator of Mod-A, there exists $\xi \in M^*$ such that $\xi(f(L)) = 0$ and $\xi(x) \neq 0$, so $f(L) \leqslant \text{Ker}\,\xi$ and $\text{Im}\,\alpha \nleqslant \text{Ker}\,\xi$, absurd!

2.4 Theorem. *Let* $L \xrightarrow{f} M \xrightarrow{g} N$ *be a sequence in* $\mathcal{C}({}_RK)$ *such that*

a) *f is a topological embedding,*

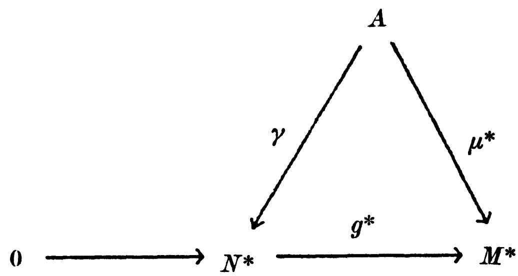
b) $\text{Im}\,f = \text{Ker}\,g$,

c) $(g(M))^c = N$;

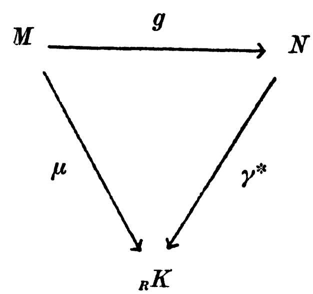
then the sequence $0 \to N^* \xrightarrow{g^*} M^* \xrightarrow{f^*} L^* \to 0$ *is an exact sequence in* Mod-A.

Proof. Let $\nu \in \text{Chom}_R(N, {}_RK)$ be such that $g^*(\nu) = \nu \circ g = 0$; since $\text{Ker}\,\nu$ is closed in N, $N = (\text{Im}\,g)^c \leqslant \text{Ker}\,\nu$ and g^* is injective. If $\lambda \in L^*$, by *a*) and the E.P., there exists a character $\mu: M \to {}_RK$ such that $\mu \circ f = \lambda$; then $\lambda = f^*(\mu)$ and consequently f^* is surjective. Finally we have to prove $\text{Im}\,g^* = \text{Ker}\,f^*$: if $\mu \in \text{Im}\,g^*$, then $\mu = g^*(\nu) = \nu \circ g$ with $\nu \in N^*$; it results $f^*(\nu \circ g) = (\nu \circ g) \circ f = \nu \circ (g \circ f) = 0$, hence $\mu \in \text{Ker}\,f^*$. Let $\mu \in \text{Ker}\,f^*$, $0 = f^*(\mu) = \mu \circ f$ hence $\text{Im}\,f \leqslant \text{Ker}\,\mu$ that implies $\text{Ker}\,g \leqslant$

$\leqslant \operatorname{Ker} \mu$. We consider in Mod-$A$ the diagram with exact row



Being $\operatorname{Ker} g \leqslant \operatorname{Ker} \mu$, by Lemma 2.3 we have $\operatorname{Im} \mu^* \leqslant \operatorname{Im} g^*$ and hence there exists a unique morphism $\gamma: A \to N^*$ such that $\mu^* = g^* \circ \gamma$. We obtain the commutative diagram



with γ^* continuous morphism; then $\mu = \gamma^* \circ g = g^*(\gamma^*)$ and hence $\mu \in \operatorname{Im} g^*$.

2.5 Definition. A module $M \in R$-Mod is called *weakly quasi-injective* (in short w.q.i.) if for any $n \in \mathbb{N}$ and any finitely generated submodule H of M^n, each morphism of H in M extends to M^n.

2.6 We will denote by $(V_\gamma)_{\gamma \in \Gamma}$ a system of representatives of the isomorphism classes of the simple τ-torsion left R_τ-modules, we set $\operatorname{End}(V_\gamma) = D_\gamma$ and $n_\gamma = \dim_{D_\gamma}(V_\gamma)$. Being V_γ a simple module, D_γ is a division ring and V_γ is a vector space over D_γ. We call *isotypic component* of $\operatorname{Soc}({}_RK)$ relative to V_γ the sum of all simple submodule of ${}_RK$ that are isomorphic to V_γ; it will be denoted by $\sum V_\gamma$.

Let A be a ring and $J(A)$ be its Jacobson radical, i.e. the intersection of all maximal left ideals of A.

2.7 PROPOSITION. *Let* $_RK$ *be s.q.i. and* $\mathfrak{D}(K_A) = \text{Mod-}A$; *then*

i) K_A *is a cogenerator of* Mod-A,

ii) A_A *is l.c.d.*,

iii) $A/J(A)$ *is semisimple artinian, hence* Mod-A *has only a finite number of non isomorphic simple modules*,

iv) $\mathrm{Soc}(K_A) = \mathrm{Soc}(_RK)$ *and they are both essential.*

PROOF. i) It is obvious.

ii) K_A is a cogenerator of Mod-A, $_RK_A$ is faithfully balanced and, since $_RK$ is s.q.i., we conclude by Corollary 17.9 of [M.O.2].

iii) Since A is l.c.d., then $A/J(A)$ is semiprimitive and l.c.d., hence, by Theorem of Leptin [O. Th. 5.10], it is artinian semisimple.

iv) K_A is a cogenerator of Mod-A, $_RK$ is s.q.i. hence it is a self-cogenerator; then K_A is w.q.i., and so Theorem 2.6 of [D.O.1] applies.

2.8 PROPOSITION. *Let* $_RK_A$ *be faithfully balanced,* $_RK$ *s.q.i. and* $\mathfrak{D}(K_A) = \text{Mod-}A$; *then*

i) *for each* $\gamma \in \Gamma$ $_RK$ *has a submodule that is isomorphic to* V_γ,

ii) V_γ^* *is a simple module belonging to* Mod-A, $V_\gamma^* \leqslant \mathrm{Soc}(K_A)$ *and all the simple submodules of* K_A *are of this form*,

iii) *The modules* V_γ^*, $\gamma \in \Gamma$ *are a system of representatives of the isomorphism classes of the simple modules belonging to* Mod-A.

Moreover Γ *is finite.*

PROOF. i) Let $0 \neq x \in V_\gamma$; since $_RK$ is s.q.i. there exists $f\colon V_\gamma \to {}_RK$ with $f(x) \neq 0$ and, being V_γ a simple module, f is an embedding.

ii) and iii) $V_\gamma^* = \mathrm{Hom}_R(V_\gamma, {}_RK) \cong \mathrm{Hom}_R(R/\mathcal{M}, {}_RK) \cong \mathrm{Ann}_K(\mathcal{M})$ with $\mathcal{M}$ maximal ideal of R; $\mathrm{Ann}_K(\mathcal{M})$ is a simple submodule of K_A and so V_γ^* is isomorphic to a submodule of $\mathrm{Soc}(K_A)$. Since K_A is a cogenerator of Mod-A, each simple module has this form, for the dual of a simple submodule of K_A is a simple R-module.

2.9 Let $S = \mathrm{Soc}(_RK)$ and $S = \bigoplus_{\lambda \in \Lambda} S_\lambda$ be a fixed decomposition of S as direct sum of simple modules. Consider the sequence $0 \to S \to {}_RK \to {}_RK/S \to 0$; since $_RK$ is q.i., each morphism from S into $_RK$

extends to an endomorphism of ${}_RK$, then we have the exact sequence

$$0 \to \mathrm{Hom}_R({}_RK/S, {}_RK) \to \mathrm{End}({}_RK) \to \mathrm{Hom}_R(S, {}_RK) \to 0 .$$

Next it is $\mathrm{Hom}_R(S, {}_RK) \cong \mathrm{End}_R(S)$ and $\mathrm{Hom}_R({}_RK/S, {}_RK) \cong J(A)$, for $\mathrm{Hom}_R({}_RK/S, {}_RK)$ is isomorphic to the subgroup of $\mathrm{End}({}_RK)$ consisting of all f such that $f|_S = 0$, i.e. to the subgroup of all $a \in A$ such that $\mathrm{Soc}({}_RK)\cdot a = 0$; since $\mathrm{Soc}({}_RK) = \mathrm{Soc}(K_A)$, $\mathrm{Hom}_R({}_RK/S, {}_RK)$ is isomorphic to $\mathrm{Ann}_A(\mathrm{Soc}(K_A)) = \bigcap_\lambda \mathrm{Ann}_A(S_\lambda) = J(A)$. We have so the exact sequence $0 \to J(A) \to A \to \mathrm{End}_R(S) = A/J(A) \to 0$ and the following isomorphisms of right A-module

$$A/J(A) \cong \mathrm{Hom}_R(S, {}_RK) = \\ = \mathrm{Hom}_R\Big(\bigoplus_\lambda S_\lambda, {}_RK\Big) \cong \prod_\lambda \mathrm{Hom}_R(S_\lambda, {}_RK) = \prod_\lambda S_\lambda^* .$$

Since $A/J(A)$ is l.c.d., $\prod_\lambda \mathrm{Hom}_R(S_\lambda, {}_RK)$ is l.c.d. and hence $\bigoplus_\lambda \mathrm{Hom}_R(S_\lambda, {}_RK)$ is l.c.d.; therefore Λ is finite and being $S = \bigoplus_{\gamma\in\Gamma} V_\gamma^{(\nu_\gamma)} = \bigoplus_{\gamma\in\Gamma} (V_\gamma^*)^{(\nu_\gamma)} = \bigoplus_\lambda S_\lambda^*$, Γ is finite and ν_γ is finite for all $\gamma \in \Gamma$.

2.10 THEOREM. *Let ${}_RK$ be s.q.i. and* Mod-$A = \mathfrak{D}(K_A)$; *then*

$${}_RK = \bigoplus_{\gamma\in\Gamma} E_\tau(V_\gamma)^{\nu_\gamma},$$

Γ is finite and ν_γ are positive integer numbers. Moreover $|\Gamma|$ and the ν_γ are uniquely determined.

PROOF. Owing to the above considerations we have $\mathrm{Soc}({}_RK) = \bigoplus_{\gamma\in\Gamma} \sum V_\gamma = \bigoplus_{\gamma\in\Gamma} V_\gamma^{\nu_\gamma}$; since $\mathrm{Soc}({}_RK)$ is essential in ${}_RK$ which is s.q.i.; it turns out that ${}_RK = E_\tau(\mathrm{Soc}({}_RK)) = E_\tau\Big(\bigoplus_{\gamma\in\Gamma} V_\gamma^{\nu_\gamma}\Big) = \bigoplus_{\gamma\in\Gamma} E_\tau(V_\gamma)^{\nu_\gamma}$, for Γ and ν_γ are finite.

3. Example.

3.1 In this part we give an example of a good duality Δ_K between $\mathcal{C}({}_RK)$ and Mod-A, where K_A is a cogenerator not injective of Mod-A.

Let $\mathbb{Z}(p^\infty)$ be the p-primary component of $\mathbb{Q}/\mathbb{Z}$ and J_p its endomorphisms ring. Let us consider the set $J_p \times \mathbb{Z}(p^\infty)$; the positions $(a, b) + (c, d) = (a + c, b + d)$ and $(a, b)(c, d) = (ac, ad + bc)$ define a ring structure on $J_p \times \mathbb{Z}(p^\infty)$; it will be called the *trivial extension* of $\mathbb{Z}(p^\infty)$ by J_p and will be denoted by $J_p \ltimes \mathbb{Z}(p^\infty)$.

Let $A = J_p \ltimes \mathbb{Z}(p^\infty)$, $K = \mathbb{Z}(p^\infty)^{(\mathbb{N})}$ and $R = \mathrm{End}(K_A)$; we will prove that K_A is a non injective cogenerator of Mod-A and that ${}_RK$ is s.q.i., hence Δ_K is a good duality between Mod-A and $\mathcal{C}({}_RK)$.

A is a local l.c.d. ring; $\mathbb{Z}(p^\infty)$, being the injective hull of the unique simple A-module, is the minimal injective cogenerator of Mod-A. Obviously $\mathbb{Z}(p^\infty)^{(\mathbb{N})}$ is a cogenerator of Mod-A and it is not injective: for, denoted by c_i ($i \in \mathbb{N}$) the system of generators of $\mathbb{Z}(p^\infty)$ with $pc_1 = 0$ and $pc_i = c_{i-1}$, the morphism $\mathbb{Z}(p^\infty) \to K$ $c_i \to (c_i, c_{i-1}, \ldots, c_1, 0, \ldots)$ does not extend to a morphism of A in K.

By Corollary 22.8 of [M.O.2], set $R = \mathrm{End}\,(K_A)$, the bimodule ${}_RK_A$ is faithfully balanced and ${}_RK$ is q.i. The ring R is isomorphic to the ring $T_\mathbb{N}$ of the matrices $\mathbb{N} \times \mathbb{N}$ with summable columns with entries in $\mathrm{End}\,(\mathbb{Z}(p^\infty)) = J_p$ endowed with the $\mathbb{Z}(p^\infty)$-topology. It is the ring of all matrices $(\alpha_{ij})_{i,j\in\mathbb{N}}$ with $a_{ij} \in J_p$ such that for each k, $n \in \mathbb{N}$ there exists $l \in \mathbb{N}$ with $\alpha_{jk} \in p^n J_p$ $\forall j \geqslant l$. If R is endowed with the K-topology τ and $T_\mathbb{N}$ with the topology having the left ideals $W(F; I) = \{(\alpha_{ij})_{i,j\in\mathbb{N}}\colon (\alpha_{i\mu})_{i\in\mathbb{N}} \in I^\mathbb{N}\ \forall \mu \in F\}$, with I open left ideal of J_p (i.e. $I = p^n J_p$ for a suitable $n \in \mathbb{N}$) and F finite subset of $\mathbb{N}$, as a basis of neighbourhoods of 0, the isomorphism is also topological (see [D.O.2], Th. 4.4).

3.2 PROPOSITION. *The maximal open left ideal of $T_\mathbb{N}$ are precisely those of the form*

$$I_{\mathcal{F},\Lambda} = \left\{(\alpha_{ij}) \in T_\mathbb{N}\colon 0 \equiv \sum_{r\in\mathcal{F}} \lambda_r \alpha_{ir} \quad (pJ_p)\ \ \forall i \in \mathbb{N}\right\},$$

where $\mathcal{F}$ is a finite subset of $\mathbb{N}$, $\Lambda = \{\lambda_r\colon r \in \mathcal{F}\} \subseteq J_p$, and $\Lambda \not\subseteq pJ_p$.

PROOF. Obviously these are proper open left ideals, for $W(\mathcal{F}, pJ_p) \subseteq I_{\mathcal{F},\Lambda}$. Let I be a maximal open left ideal of $T_\mathbb{N}$, then $I \supseteq pT_\mathbb{N}$: in fact suppose that $pT_\mathbb{N} \not\subseteq I$, then $I + pT_\mathbb{N} = T_\mathbb{N}$ hence

$$A = \begin{bmatrix} 1 + pb_{11} & pb_{12} & \cdots \\ pb_{21} & 1 + pb_{22} & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$

belongs to I; now I is open, therefore it contains $W(F; p^n J_p)$; set $s = \max(F)$, then

$$B = \begin{bmatrix} 1+pb_{11} & pb_{12} & \dots & pb_{1s} & 0 & & \\ pb_{21} & 1+pb_{22} & \dots & pb_{2s} & 0 & & \\ \dots & \dots & \ddots & \dots & 0 & \mathbf{0} & \\ pb_{s1} & pb_{s2} & \dots & 1+pb_{ss} & 0 & & \\ \dots & \dots & \dots & \dots & 1 & & \\ & & & & & 1 & \\ & & & & & & \ddots \end{bmatrix} \in I$$

for $B = A + [B - A]$ where $A \in I$ and $[B - A] \in W(F; p^n J_p) \subseteq I$. Now $1 + pb_{ii}$ is a unit in J_p, hence

$$C = \begin{bmatrix} 1 & pa_{12} & \dots & pa_{1s} & 0 & & \\ pa_{21} & 1 & \dots & pa_{2s} & 0 & & \\ \dots & \dots & \ddots & \dots & 0 & \mathbf{0} & \\ pa_{s1} & pa_{s2} & \dots & 1 & 0 & & \\ \dots & \dots & \dots & \dots & 1 & & \\ & & & & & 1 & \\ & & & & & & \ddots \end{bmatrix} \in I$$

Now multiplying C on the left by

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 & & \\ -pa_{21} & 1 & 0 & \dots & 0 & 0 & & \\ -pa_{31} & 0 & 1 & \dots & 0 & 0 & \mathbf{0} & \\ \dots & \dots & \dots & \ddots & \dots & 0 & & \\ -pa_{s1} & 0 & 0 & \dots & 1 & 0 & & \\ \dots & \dots & \dots & \dots & \dots & 1 & & \\ & & & & & & 1 & \\ & & & & & & & \ddots \end{bmatrix}$$

we find

$$\begin{bmatrix} 1 & pa_{12} & \cdots & \cdots & pa_{1s} & 0 & & \\ 0 & 1-p^2a_{21}a_{12} & \cdots & \cdots & \cdots & 0 & & \\ 0 & p(\ldots) & \ddots & \cdots & \cdots & 0 & \text{\Large 0} & \\ \cdots & \cdots & \cdots & \ddots & \cdots & 0 & & \\ 0 & \cdots & \cdots & \cdots & 1-p^2a_{s1}a_{1s} & 0 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & 1 & & \\ & & & & & & 1 & \\ & & & & & & & \ddots \end{bmatrix} \in I$$

Next $1\text{-}p^2(..)$ is a unit in J_p, hence

$$\begin{bmatrix} 1 & pa'_{12} & \cdots & pa'_{1s} & 0 & & \\ 0 & 1 & \cdots & \cdots & 0 & & \\ \cdots & \cdots & \ddots & \cdots & 0 & \text{\Large 0} & \\ 0 & \cdots & \cdots & 1 & 0 & & \\ \cdots & \cdots & \cdots & \cdots & 1 & & \\ & & & & & 1 & \\ & & & & & \text{\Large 0} & \ddots \end{bmatrix} \in I$$

and multiplying the last matrix by

$$\begin{bmatrix} 1 & -pa'_{12} & 0 & \cdots & \cdots & 0 & & \\ 0 & 1 & \cdots & \cdots & \cdots & 0 & & \\ 0 & -pa'_{32} & \ddots & \cdots & \cdots & 0 & \text{\Large 0} & \\ \cdots & \cdots & \cdots & \ddots & \cdots & 0 & & \\ 0 & \cdots & 0 & \cdots & 1 & 0 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & 1 & & \\ & & & & & & 1 & \\ & & & & & & \text{\Large 0} & \ddots \end{bmatrix}$$

and repeating the above arguments we have

$$\begin{bmatrix} 1 & 0 & pa''_{13} & \cdots & pa''_{1s} & & \\ 0 & 1 & pa''_{23} & & pa''_{2s} & & \mathbf{0} \\ 0 & 0 & 1 & & \cdots & & \\ & & pa''_{43} & & & & \\ \cdots & & & & & & \\ \cdots & & & & 1 & & \\ & & & & & 1 & \\ \cdots & & & & & & \ddots \\ \cdots & & & & & & \\ & & & & & & \mathbf{0} \end{bmatrix} \in I .$$

Carrying over the previous machinery finitely many times we reach the identity matrix belongs to I: absurd! Now let us consider the ring morphism $\varphi\colon T_{\mathbf{N}} \to T_{\mathbf{N}}/pT_{\mathbf{N}}$; there is a bijective correspondence between the ideals of $T_{\mathbf{N}}$ containing $\operatorname{Ker}\varphi = pT_{\mathbf{N}}$ and the ideals of $T_{\mathbf{N}}/pT_{\mathbf{N}}$; moreover this correspondence respects the inclusion. $T_{\mathbf{N}}/pT_{\mathbf{N}}$ is isomorphic to the ring B of matrices with the entries in the field $D = J_p/pJ_p$ with infinitely many rows and columns where the elements of each column are almost all zero. Next B is isomorphic to the ring of endomorphisms of the vector space $V = D^{(\mathbf{N})}$; the maximal ideal of B are $I_v = \{(\alpha_{ij}) \in B\colon (\alpha_{ij})v = 0\}$ with $v \in V$ then all open maximal left ideals of $T_{\mathbf{N}}$, since they contain $pT_{\mathbf{N}}$, they are equal to $\varphi^{-1}(I_v) = I_{\mathcal{F},\Lambda}$ where, set $v = (v_i)_{i\in\mathbf{N}}$, $\mathcal{F} = \{i \in \mathbf{N}\colon v_i \neq 0\}$ and $\Lambda = \{v_i\colon v_i \neq 0\}$.

Now $T_{\mathbf{N}}/I_{\mathcal{F},\Lambda}$ is isomorphic to the $T_{\mathbf{N}}$-module of matrices

$$\begin{bmatrix} 0 & \cdots & 0 & l_{1k} & 0 & 000 \\ \vdots & & & & & \\ 0 & \cdots & 0 & l_{kk} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

with $l_{ik} \in J_p/pJ_p \cong \mathbb{Z}(p)$ almost all zero, where the scalar multiplication is defined rows by columns. It is obvious that if $\mathcal{G}$ is another finite subset of $\mathbb{N}$ and $M = \{\mu_r : r \in \mathcal{G}\}$ is another subset of J_p, $T_{\mathbb{N}}/I_{\mathcal{F},\Lambda} \cong T_{\mathbb{N}}/I_{\mathcal{G},M}$ as $T_{\mathbb{N}}$-modules. Being $T_{\mathbb{N}}/I_{\mathcal{F},\Lambda} \cong \mathbb{Z}(p)^{(\mathbb{N})}$, we conclude that there is only one simple τ-torsion R-module and it is contained in $\mathbb{Z}(p^\infty)^{(\mathbb{N})}$. Then $\mathbb{Z}(p^\infty)^{(\mathbb{N})}$ is a s.q.i. R-module by theorem 6.7 of [M.O.1] and the example is made.

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