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Quantum mechanics from an equivalence principle

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Abstract

We postulate that physical states are equivalent under coordinate transformations. We then implement this equivalence principle first in the case of one-dimensional stationary systems showing that it leads to the quantum analogue of the Hamilton–Jacobi equation which in turn implies the Schrödinger equation. In this context the Planck constant plays the role of covariantizing parameter. The construction is deeply related to the $GL(2,\mathbf{C})$-symmetry of the second-order differential equation associated to the Legendre transformation which selects, in the case of the quantum analogue of the Hamiltonian characteristic function, self-dual states which guarantee its existence for any physical system. The universal nature of the self-dual states implies the Schrödinger equation in any dimension.

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While general relativity is based on a simple fundamental principle, a similar geometrical structure does not seem to underlie quantum mechanics. This suggests that the problems arising in quantizing gravity are deeply connected with the apparently different origin of the two theories. In this letter we postulate that physical systems are equivalent under coordinate transformations. We will see that while the equivalence principle cannot be consistently implemented in the Classical Stationary Hamilton–Jacobi Equation (CSHJE), it leads to its quantum analogue and then to the Schrödinger equation. This quantum stationary Hamilton–Jacobi equation is a third-order differential equation whose solution defines $\mathscr{S}_0$, denoting the quantum analogue of the Hamilton characteristic function, or reduced action, $\mathscr{S}_0^{cl}$. Here we consider the case of stationary one-dimensional systems. The higher dimensional, time dependent systems will be considered in forthcoming papers.

Our formulation is strictly related to the $GL(2,\mathbf{C})$-symmetry underlying the recently observed relationship between second-order differential equations and Legendre transformation. In particular, as we will see, this identifies in the case of the reduced

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action $\mathscr{S}_0$, a basic self-dual state which guarantees the existence of its Legendre transformation for any system. This is the starting point in a chain of deductions culminating with the Schrödinger equation. In particular, the existence of the self-dual state implies that for any one-dimensional stationary state with potential V and energy E, there is always a coordinate choice $\tilde{q}$ for which $\mathscr{W}(q) \equiv V(q) - E$ corresponds to $\tilde{\mathscr{W}}(\tilde{q}) = 0$. In this context the Planck constant, which determines the universal self-dual state, naturally arises as a covariantizing parameter.

Let us denote by q the coordinate and by p the momentum of a stationary physical system. We assume the existence of $\mathscr{T}_0$, the Legendre dual of $\mathscr{S}_0$, that is

$$p = \partial_q \mathscr{S}_0(q), \quad q = \partial_p \mathscr{T}_0(p), \tag{1}$$

$$\mathscr{S}_0 = p\partial_p \mathscr{T}_0 - \mathscr{T}_0. \tag{2}$$

Let us consider the $GL(2,\mathbf{C})$-transformations

$$\tilde{q} = \frac{Aq+B}{Cq+D}, \quad \tilde{p} = \rho^{-1}(Cq+D)^2 p, \tag{3}$$

where $\rho = AD - BC \neq 0$. These transformations are equivalent to say that $\mathscr{S}_0$ is $GL(2,\mathbf{C})$-invariant up to an additive constant

$$\tilde{\mathscr{S}}_0(\tilde{q}) = \mathscr{S}_0(q). \tag{4}$$

Note that

$$\tilde{\mathscr{T}}_0(\tilde{p}) = \mathscr{T}_0(p) + \rho^{-1}\left(ACpq^2 + BDp + 2BCpq\right). \tag{5}$$

The transformations (3) are equivalent to ($\epsilon = \pm\sqrt{1/\rho}$)

$$\tilde{q}\sqrt{\tilde{p}} = \epsilon\left(Aq\sqrt{p} + B\sqrt{p}\right),$$
$$\sqrt{\tilde{p}} = \epsilon\left(Cq\sqrt{p} + D\sqrt{p}\right), \tag{6}$$

and can be seen as a rotation of the elements in the kernel of a second-order operator. The second derivative of (2) with respect to $\mathrm{s} = \mathscr{S}_0(q)$ gives the ''canonical equation''

$$\left(\partial_{\mathrm{s}}^2 + \mathscr{U}(\mathrm{s})\right)q\sqrt{p} = 0 = \left(\partial_{\mathrm{s}}^2 + \mathscr{U}(\mathrm{s})\right)\sqrt{p}, \tag{7}$$

where $\mathscr{U}(\mathrm{s}) = \{q,\mathrm{s}\}/2$, with $\{h(x),x\} = h'''/h' - (3/2)(h''/h')^2$ denoting the Schwarzian derivative.

The above method, used in the framework of the Schrödinger equation in [1], has been introduced in [2] for deriving the inversion formula in $N=2$ super Yang-Mills, and has been further investigated in [3].

Involutivity of the Legendre transformation and the duality

$$\mathscr{S}_0 \leftrightarrow \mathscr{T}_0, \quad q \leftrightarrow p, \tag{8}$$

imply another $GL(2,\mathbf{C})$-symmetry, with the dual versions of (7) being ($\mathrm{t} = \mathscr{T}_0(p)$)

$$\left(\partial_{\mathrm{t}}^2 + \mathscr{V}(\mathrm{t})\right)p\sqrt{q} = 0 = \left(\partial_{\mathrm{t}}^2 + \mathscr{V}(\mathrm{t})\right)\sqrt{q}, \tag{9}$$

where $\mathscr{V}(\mathrm{t}) = \{p,\mathrm{t}\}/2$. Now observe that in the case in which $p = \gamma/q$, the solutions of (7) and (9) coincide and $\mathscr{U}(\mathrm{s}) = -1/4\gamma^2 = \mathscr{V}(\mathrm{t})$. We will call self-dual the states parametrized by γ. These states correspond to

$$\mathscr{S}_0 = \gamma \ln \gamma_q q, \quad \mathscr{T}_0 = \gamma \ln \gamma_p p. \tag{10}$$

Since

$$\mathscr{S}_0 + \mathscr{T}_0 = pq = \gamma, \tag{11}$$

it follows that the dimensional constants γ_p, γ_q and γ, satisfy the relation

$$\gamma_p \gamma_q \gamma = e. \tag{12}$$

We observe that as q and p above are not considered independent, the transformations in (3) are not canonical ones. Nevertheless, note that, as in the search for canonical transformations leading to a system with vanishing Hamiltonian one obtains the Hamilton–Jacobi equation, we may similarly look for the equation one obtains by considering the transformation of q, which induces the transformation of the dependent variable $p = \partial_q \mathscr{S}_0(q)$, reducing to the free system with vanishing energy. Answering this basic question will lead to the formulation of the equivalence principle and then to the quantum analogue of the Hamilton–Jacobi equation.

Let us first generalize (3) to arbitrary coordinate transformations $q \to \tilde{q}(q)$. Note that setting $\tilde{\mathscr{S}}_0(\tilde{q}) = \mathscr{S}_0(q(\tilde{q}))$ is a natural way to associate a new reduced action to the coordinate transformation. As $\tilde{p} = \partial_{\tilde{q}} \tilde{\mathscr{S}}_0(\tilde{q})$, it follows that in passing from $\mathscr{S}_0$ to $\tilde{\mathscr{S}}_0$, p transforms as ∂_q. Similarly, the dual version $q \sim \partial_p$ arises by associating to an arbitrary transformation $p \to \tilde{p}(p)$ the state $\tilde{\mathscr{T}}_0$ defined by $\tilde{\mathscr{T}}_0(\tilde{p}) = \mathscr{T}_0(p(\tilde{p}))$.

Under $q \to \tilde{q}(q)$, the associated Legendre transformation $\tilde{\mathscr{S}}_0(\tilde{q}) = \tilde{p}\partial_{\tilde{p}} \tilde{\mathscr{T}}_0(\tilde{p}) - \tilde{\mathscr{T}}_0(\tilde{p})$ generates Eq.(7) with the ''canonical potential'' $\mathscr{U}(\tilde{\mathrm{s}})$. While for a Möbius transformation both $\tilde{q}\sqrt{\tilde{p}}$ and $\sqrt{\tilde{p}}$ are by (6) still solutions of (7), so that $\tilde{\mathscr{U}}(\tilde{s}) = \mathscr{U}(s)$, this is no longer the case for arbitrary coordinate transformations. This is a consequence of the properties of the Schwarzian derivative, as $\{\tilde{q},s\} = \{q,s\}$ if and only if $\tilde{q} = (Aq+B)/(Cq+D)$ [4]. Observe that for a given $\mathscr{U}$, the ratio of two linearly independent solutions of (7) gives, up to a Möbius transformation, $q = f(s)$. Inverting it we get the solution of the equation of motion $s = \mathscr{S}_0(q)$. Hence, states with the same $\mathscr{U}$ correspond to specifying different initial conditions of (7). However, under arbitrary transformations we have $\tilde{\mathscr{U}}(\tilde{s}) \neq \mathscr{U}(s)$, unless one considers the transformations (3). It follows that different $\mathscr{U}$'s can be connected by coordinate transformations. Similarly, as we noticed, two systems $\tilde{\mathscr{S}}_0$ and $\mathscr{S}_0$ are related by the transformation $q \to \tilde{q}(q)$ defined by $\tilde{\mathscr{S}}_0(\tilde{q}) = \mathscr{S}_0(q(\tilde{q}))$. Therefore, to find the coordinate transformation connecting $\mathscr{S}_0$ with $\tilde{\mathscr{S}}_0$ is equivalent to solving the inversion problem

$$q \to \tilde{q} = \tilde{\mathscr{S}}_0^{\,-1} \circ \mathscr{S}_0(q). \tag{13}$$

This suggests the following ''equivalence principle'':

For each pair $\mathscr{W}^a,\mathscr{W}^b$ there is a coordinate transformation such that $\mathscr{W}^a(q) \to \tilde{\mathscr{W}}^a(\tilde{q}) = \mathscr{W}^b(\tilde{q})$.

Observe that this implies that there always exists a coordinate transformation reducing to $\mathscr{W} = 0$ corresponding to the free system with vanishing energy.

We now show the basic fact that this principle is not consistent with classical mechanics. Let us consider the CSHJE

$$\frac{1}{2m}\left(\frac{\partial \mathscr{S}_0^{cl}(q)}{\partial q}\right)^2 + V(q) - E = 0. \tag{14}$$

Under (3) we have $\partial_{\tilde{q}}\tilde{\mathscr{S}}_0^{cl}(\tilde{q}) = (Cq+D)^2\partial_q\mathscr{S}_0^{cl}(q)$ and $\tilde{\mathscr{U}}(\tilde{s}^{cl}) = \{\tilde{q},\tilde{s}^{cl}\}/2 = \mathscr{U}(s^{cl})$. On the other hand, as $\frac{1}{2m}(\partial_{\tilde{q}}\tilde{\mathscr{S}}_0^{cl}(\tilde{q}))^2 + \tilde{\mathscr{W}}(\tilde{q}) = 0$, consistency, *i.e.* covariance, implies that $\tilde{\mathscr{W}}(\tilde{q}) = (Cq+D)^4\mathscr{W}(q)$. Similarly, in the case of v^{cl}-transformations $q \to \tilde{q} = v^{cl}(q)$, defined by $\tilde{\mathscr{S}}_0^{cl}(\tilde{q}) = \mathscr{S}_0^{cl}(q(\tilde{q}))$, the state corresponding to $\tilde{\mathscr{W}}$ associated to $\tilde{\mathscr{S}}_0^{cl}$, satisfies $\tilde{\mathscr{W}}(\tilde{q})(d\tilde{q})^2 = \mathscr{W}(q)(dq)^2$. In other words $\mathscr{W}(q)$ would belong to $\mathscr{Q}^{cl}$, the space of functions transforming as quadratic differentials under v^{cl}-transformations. It follows that

$$\mathscr{W}(q) = 0 \to \tilde{\mathscr{W}}(\tilde{q}) = \left(\partial_q\tilde{q}\right)^{-2}\mathscr{W}(q) = 0, \tag{15}$$

that is, due to the homogeneity of the transformation properties of the quadratic differentials, the state corresponding to $\mathscr{W} = 0$ is a fixed point in the space $\mathscr{H}$ of all possible $\mathscr{W}$. In other words, in classical mechanics the space $\mathscr{H}$ cannot be reduced to a point upon factorizing by the v^{cl}-transformations.

In the following we will derive a differential equation for $\mathscr{S}_0$ with the following properties

1. Covariance, i.e. consistency, under the v-transformations $q \to \tilde{q} = v(q)$, defined by $\tilde{\mathscr{S}}_0(\tilde{q}) = \mathscr{S}_0(q(\tilde{q}))$.
2. In a suitable limit it reduces to the CSHJE.
3. All the states $\mathscr{W} \in \mathscr{H}$ are equivalent under the v-transformations.

While point **1.** is nothing else but a consistency condition and **2.** is a consequence of the existence of classical mechanics, point **3.** has a highly nontrivial dynamical content as will play the basic role in fixing the differential equation for $\mathscr{S}_0$.

Without loss of generality, we can write the equation we are looking for in the form

$$\frac{1}{2m}\left(\frac{\partial \mathscr{S}_0}{\partial q}\right)^2 + \mathscr{W}(q) + Q(q) = 0. \tag{16}$$

Observe that if $\mathscr{Q}$ denotes the space of functions transforming as quadratic differentials under the v-transformations, then as $\frac{1}{2m}\left(\partial_{\tilde{q}}\tilde{\mathscr{S}}_0(\tilde{q})\right)^2 + \tilde{\mathscr{W}}(\tilde{q}) + \tilde{Q}(\tilde{q}) = 0$ we have by consistency that $(\mathscr{W} + Q) \in \mathscr{Q}$. On the other hand, Eq. (15) and point **3.** imply that $\mathscr{W} \notin \mathscr{Q}$, so that we also have $Q \notin \mathscr{Q}$. We also note that the classical limit $Q \to 0$, for which Eq. (16) reduces to Eq. (14), corresponds to the covariance breaking limit, so that Q has the geometrical nature of a covariantizing term.

Let us now consider the free system with vanishing energy. In this case Eq. (16) becomes $\left(\partial_q\mathscr{S}_0\right)^2 = -2mQ$. Observe that as $\left(\partial_q\mathscr{S}_0\right)^2 \in \mathscr{Q}$, and $Q \notin \mathscr{Q}$, covariance would apparently imply $Q = 0$ so that $\mathscr{S}_0 = cnst$. Therefore, as $\tilde{\mathscr{S}}_0(\tilde{q}) = \mathscr{S}_0(q)$, any choice of coordinates would always give $\tilde{\mathscr{S}}_0 = cnst$, so that, in contradiction with **3.**, $\mathscr{S}_0 = cnst$ would be a fixed point in the space $\mathscr{K}$ of all possible $\mathscr{S}_0$. This aspect is related to the existence of the Legendre transformation. In particular, $\mathscr{S}_0$-$\mathscr{T}_0$ duality holds unless $\mathscr{S}_0 = cnst$ or $\mathscr{S}_0 \propto q$, for which the formalism breaks down. On the other hand, one expects that the basic properties of the equations underlying physical systems should be independent from the specific system one considers. In particular, we have that the formalism breaks down for the system corresponding to $\mathscr{W}=0$. Similarly, whereas $\mathscr{S}_0$-$\mathscr{T}_0$ duality holds for an accelerated particle, this would not be the case in its rest frame. We will see that there is a remarkable mechanism, direct consequence of the equivalence principle, which solves the above problems.

Let us first introduce the basic identity

$$\left(\frac{\partial \mathscr{S}_0}{\partial q}\right)^2 = \frac{\beta^2}{2}\left(\left\{e^{\frac{2i}{\beta}\mathscr{S}_0},q\right\} - \{\mathscr{S}_0,q\}\right), \tag{17}$$

which forces us to use the dimensional constant β. By (16) and (17) we have

$$\mathscr{W}(q) = \frac{\beta^2}{4m}\left(\{\mathscr{S}_0,q\} - \left\{e^{\frac{2i}{\beta}\mathscr{S}_0},q\right\}\right) - Q(q). \tag{18}$$

Since there is no universal constant in the CSHJE with the dimension of an action, we see that β is the only natural parameter we can use in order to reach the covariance breaking phase in which $Q=0$.

We now consider the natural solution

$$Q(q) = \frac{\beta^2}{4m}\{\mathscr{S}_0,q\}, \tag{19}$$

which we will show in Ref. [5] to be unique. It follows from (18) and (19) that

$$\mathscr{W}(q) = -\frac{\beta^2}{4m}\left\{e^{\frac{2i}{\beta}\mathscr{S}_0},q\right\}, \tag{20}$$

which is equivalent to the differential equation

$$\frac{1}{2m}\left(\frac{\partial \mathscr{S}_0}{\partial q}\right)^2 + V(q) - E + \frac{\beta^2}{4m}\{\mathscr{S}_0,q\} = 0, \tag{21}$$

that in the $\beta \to 0$ limit reduces to the CSHJE (14).

Eq. (20) and the identities

$$\partial_x h'^{1/2} h'^{-1/2} = 0 = \partial_x h'^{-1}\partial_x h'^{1/2} h'^{-1/2} h, \tag{22}$$

and $h'^{1/2}\partial_x h'^{-1}\partial_x h'^{1/2} = \partial_x^2 + \{h,x\}/2$, imply

$$e^{\frac{2i}{\beta}\mathscr{S}_0} = \frac{A\psi^D + B\psi}{C\psi^D + D\psi}, \tag{23}$$

$AD - BC \neq 0$, where ψ^D and ψ are linearly independent solutions of the stationary Schrödinger equation

$$\left[-\frac{\beta^2}{2m}\frac{\partial^2}{\partial q^2} + V(q)\right]\psi = E\psi. \tag{24}$$

Thus, for the ''covariantizing parameter'' β we have

$$\beta = \hbar, \tag{25}$$

where $\hbar = h/2\pi$ with h the Planck constant.

The formulation manifests explicit $\mathscr{S}_0$-$\mathscr{T}_0$ duality as both $\mathscr{S}_0 = cnst$ and $\mathscr{S}_0 \propto q$ do not belong to $\mathscr{K}$. Due to the Möbius invariance of the Schwarzian derivative [4], instead of $\mathscr{S}_0 = \sqrt{2mE}\,q$, corresponding to $\mathscr{W}(q) = -\frac{\hbar^2}{4m}\{e^{\frac{2i}{\hbar}\mathscr{S}_0},q\} = -E \neq 0$, we can choose

$$\mathscr{S}_0 = \frac{\hbar}{2i}\ln\frac{Ae^{\frac{2i}{\hbar}\sqrt{2mE}\,q} + B}{Ce^{\frac{2i}{\hbar}\sqrt{2mE}\,q} + D}, \tag{26}$$

where the constants are chosen in such a way that $\mathscr{S}_0 \not\propto q$.

For $\mathscr{W}=0$, the equation $(\partial_q \mathscr{S}_0)^2 = -\hbar^2\{\mathscr{S}_0,q\}/2$, (by (17) equivalent to $\{e^{\frac{2i}{\hbar}\mathscr{S}_0},q\}=0$) has the solutions $\mathscr{S}_0 = \frac{\hbar}{2i}\ln(Aq+B)/(Cq+D)$. We therefore have the important fact that $\mathscr{S}_0$ is never a constant! Comparing with (10) and relaxing the reality condition on $\mathscr{S}_0$, we can choose for $\mathscr{W}=0$ the pair of self-dual states

$$\mathscr{S}_0^{sd} = \pm\frac{\hbar}{2i}\ln\gamma_q q, \tag{27}$$

that for $\hbar \to 0$ reduce to the classical result. Physical solutions for $\mathscr{S}_0$ correspond to values of A,B,C,D in (23) such that $\mathscr{S}_0$ is real, that is we have

$$e^{\frac{2i}{\hbar}\mathscr{S}_0} = e^{i\alpha}\frac{\psi^D + i\bar{\ell}\psi}{\psi^D - i\ell\psi}, \tag{28}$$

where $\alpha \in \mathbf{R}$ and $\mathrm{Re}\,\ell \neq 0$. Thus, while (27) is a complex solution and corresponds to the state with $\mathscr{W}=0$, the physical solution, still corresponding to $\mathscr{W}=0$, is given by

$$e^{\frac{2i}{\hbar}\mathscr{S}_0} = e^{i\alpha}\frac{q + i\bar{\ell}}{q - i\ell}. \tag{29}$$

The above analysis shows that while in the standard approach the solutions corresponding to the state with $\mathscr{W} = cnst$ coincide with the classical ones, here we have a basic difference related to the existence of the Legendre transformation of $\mathscr{S}_0$ for any system. The solutions (26)(27) have been overlooked in the literature. Note that by (23) the general solution of (24) is

$$\psi = \frac{1}{\sqrt{\mathscr{S}_0'}}\left(Ae^{-\frac{i}{\hbar}\mathscr{S}_0} + Be^{\frac{i}{\hbar}\mathscr{S}_0}\right), \tag{30}$$

that for (26)(27) gives, as it should, the solutions $\psi = Ae^{-\frac{i}{\hbar}\sqrt{2mE}\,q} + Be^{\frac{i}{\hbar}\sqrt{2mE}\,q}$ and $\psi = Aq + B$.

Let us compare the above equations with those of the standard notation [6]. While Eq. (21) is written in terms of $\mathscr{S}_0$ only, substituting $\psi = R\exp(i\hat{\mathscr{S}}_0/\hbar)$ in (24) yields

$$\left(\partial_q \hat{\mathscr{S}}_0\right)^2/2m + V(q) - E - \hbar^2\left(\partial_q^2 R\right)/2mR = 0, \tag{31}$$

$$\partial_q\left(R^2\partial_q\hat{\mathscr{S}}_0\right) = 0. \tag{32}$$

We can distinguish the cases $\bar{\psi} \not\propto \psi$ and $\bar{\psi} \propto \psi$. In the first one we can choose $\psi^D = \bar{\psi}$, i.e.

$$\psi(q) = R(q)e^{\frac{i}{\hbar}\hat{\mathscr{S}}_{0(q)}}, \quad \psi^D(q) = R(q)e^{-\frac{i}{\hbar}\hat{\mathscr{S}}_{0(q)}}, \tag{33}$$

so that we can set $\mathscr{S}_0 = \hat{\mathscr{S}}_0$. The continuity Eq. (32) gives $R = 1/\sqrt{\mathscr{S}_0'}$ so that $Q(q) = \hbar^2\{\mathscr{S}_0, q\}/4m = -\hbar^2(\partial_q^2 R)/2mR$ and Eq. (31) corresponds to Eq. (21).

In the $\bar{\psi} \propto \psi$ case one has that $\hat{\mathscr{S}}_0$ is a constant, and we can set $\hat{\mathscr{S}}_0 = 0$. This fact shows that identifying the wave function with $R\exp(i\hat{\mathscr{S}}_0/\hbar)$, typical of Bohmian mechanics, is problematic as it would imply a rather involved classical limit. Since the case $\bar{\psi} \propto \psi$ corresponds to bound states, we would have systems, such as the harmonic oscillator, in which in the $\hbar \to 0$ limit one has to recover a nontrivial classical reduced action from $\hat{\mathscr{S}}_0 = 0$. This fact can be seen as further evidence that the quantum analogue of the classical reduced action is $\mathscr{S}_0$ rather than $\hat{\mathscr{S}}_0$. This also implies that Q is the genuine quantum potential rather than $-\hbar^2(\partial_q^2 R)/2mR$.

Let us further consider the $\bar{\psi} \propto \psi$ case. Since $\hat{\mathscr{S}}_0 = 0$, we have

$$\psi(q) = R(q). \tag{34}$$

Furthermore, Eqs. (31) and (32) give

$$V(q) - E - \hbar^2\left(\partial_q^2 R\right)/2mR = 0, \tag{35}$$

so that in this case the relation between the standard quantum potential

$$\hat{Q} = -\frac{\hbar^2}{2m}\frac{\partial_q^2 R}{R}, \tag{36}$$

and Q is

$$\hat{Q} = Q + \frac{1}{2m}\left(\frac{\partial \mathscr{S}_0}{\partial q}\right)^2. \tag{37}$$

The existence of the self-dual state makes it possible to find a coordinate $\tilde{q}$, solution of the Schwarzian equation

$$\{\tilde{q}, q\} + 4m(V(q) - E)/\hbar^2 = 0, \tag{38}$$

in which any $\mathscr{W} \in \mathscr{H}$ reduces to $\tilde{\mathscr{W}}(\tilde{q}) = 0$. In complete analogy with the fact that the existence of the classical trivializing conjugate variables (Q, P), defined by the canonical transformation

$$q \to Q, \quad p \to P = cnst = -\partial_Q \mathscr{S}_0^{cl}(q, Q)|_{Q=cnst}, \tag{39}$$

implies the CSHJE $H(q, \partial_q \mathscr{S}_0^{cl}) - E = \tilde{H}(Q, P) = 0$, we have that (38) is a consequence of the existence of the trivializing map

$$q \to \tilde{q} = \gamma_q^{-1} e^{\frac{2i}{\hbar}\mathscr{S}_0}, \quad p \to \tilde{p} = \left(\partial_q \tilde{q}\right)^{-1} p = i\hbar/2\tilde{q}, \tag{40}$$

leading to the free system with vanishing energy. Eq. (40) is the solution of the inversion problem (13) when $\tilde{\mathscr{S}}_0$ is the reduced action of the state with $\tilde{\mathscr{W}} = 0$. Therefore, given an arbitrary state $\mathscr{W} \in \mathscr{H}$, the transformation (40) gives $\tilde{\mathscr{W}}(\tilde{q}) = 0$, and by (30) Eq. (24) becomes $(\partial_q \tilde{q})^{3/2}\partial_{\tilde{q}}^2 \tilde{\psi}(\tilde{q}) = 0$, where

$$\tilde{\psi}(\tilde{q})(d\tilde{q})^{-1/2} = \psi(q)(dq)^{-1/2}. \tag{41}$$

We note that the trivializing map can be transformed to a real map by performing a Cayley transformation of $e^{\frac{2i}{\hbar}\mathscr{S}_0}$. Since this map is a Möbius transformation, it is a symmetry of $\mathscr{W} = -\hbar^2\{e^{\frac{2i}{\hbar}\mathscr{S}_0},q\}/4m$.

Remarkably, the quantum correction to the CSHJE (14), can be also seen as modification by a ''conformal factor'' defined by the canonical potential

$$\frac{1}{2m}\left(\frac{\partial \mathscr{S}_0}{\partial q}\right)^2\left[1-\hbar^2\mathscr{U}(\mathscr{S}_0)\right]+V(q)-E=0, \tag{42}$$

where we used the identity $\{q,\mathscr{S}_0\} = -(\partial_q\mathscr{S}_0)^{-2}\{\mathscr{S}_0,q\}$. This shows the basic role of the purely quantum mechanical self-dual states (27) as in this case

$$1-\hbar^2\mathscr{U}\left(\pm\frac{\hbar}{2i}\ln\gamma_q q\right)=0, \tag{43}$$

which are two possible solutions of (42) for $\mathscr{W}=0$, the other possible solutions are given by $S_0 = \frac{\hbar}{2i}\ln(Aq+B)/(Cq+D),\ AD-BC\neq 0$.

We note that an additional term in (19) would imply a differential equation for $\mathscr{S}_0$ which could not satisfy conditions **1.-3.** [5].

We observe that by (41) it follows that, in general, diffeomorphisms do not preserve the transition amplitudes and are not unitary. This is of course expected as these transformations connect any pair of different physical systems.

We have seen that the requirement of preserving the original structures observed in the Legendre transformation can be consistently satisfied. This brings us to the Schrödinger equation which can be characterized by the equation $\mathscr{W}(q) = -\hbar^2\{e^{\frac{2i}{\hbar}\mathscr{S}_0},q\}/4m$, or equivalently by the term $Q(q)=\hbar^2\{\mathscr{S}_0,q\}/4m$ which added to the CSHJE leads to the Schrödinger equation. Recalling the structure of the canonical potential, namely $\mathscr{U}(\mathscr{S}_0) = \{q,\mathscr{S}_0\}/2$, we explicitly see how the basic Möbius symmetry, a characteristic property of the Schwarzian derivative, still survives in quantum theory. Thus, the canonical formalism by itself unavoidably contains an intrinsic Möbius ambiguity which actually turns out to be at the heart of quantum mechanics. In particular, the fact that the relevant equations remain invariant under Möbius transformations of the canonical variables and the related existence of the self-dual states, characterized by $\gamma=\pm\hbar/2i$, reflect in the reconsideration of these classical variables.

We stress that an important aspect in our construction concerns the identity (17) which contains both the classical and quantum parts $\mathscr{W}$ and Q respectively. In particular, note that it includes in the same equation both $e^{\frac{2i}{\hbar}\mathscr{S}_0}$ and $\mathscr{S}_0$. If one considers $\mathscr{S}_0$ as a scalar field operator, then the ''vertex'' $e^{\frac{2i}{\hbar}\mathscr{S}_0}$ resembles the bosonization of a fermionic operator. It is amusing that inspired by duality in SUSY Yang-Mills [1,2], we obtained a quantum mechanical expression reminiscent of supersymmetry.

Though it may seem specifically one-dimensional, our formulation implies quantum mechanics also in higher dimensions [7]. This is just like the Heisenberg uncertainty relations $\Delta p_k \Delta q_k \geq \hbar/2$, which, in spite of being intrinsically one-dimensional, actually encode quantum mechanics in any dimension. In particular, since the formulation trivially extends to the case when

$$V(q)=\sum_{k=1}^{D}V_k(q_k), \tag{44}$$

we have that the state with $\mathscr{W}=0$ still corresponds to the nontrivial universal solution

$$\mathscr{S}_0^{sd}=\pm\frac{\hbar}{2i}\sum_{k=1}^{D}\ln\gamma_q q_k. \tag{45}$$

This guarantees that the Legendre transformation

$$\mathscr{S}_0=\sum_{k=1}^{D}p_k\frac{\partial\mathscr{T}_0}{\partial p_k}-\mathscr{T}_0, \tag{46}$$

is defined for any physical system and, as in the one-dimensional case, its involutivity implies $\mathscr{S}_0$-$\mathscr{T}_0$ duality. Therefore, in higher dimensions one should derive an equation that, for potentials of the form (44), is equivalent to decoupled one-dimensional Schrödinger equations. Furthermore, in the classical limit it should reproduce the CSHJE. In Ref. [7] it will be shown how these conditions yield the Schrödinger equation in any dimension.

Finally, in the time-dependent case the equation for the action $\mathscr{S}$ is determined by considering that in the classical limit it should correspond to the Hamilton–Jacobi equation and that in the time-independent case it reproduces the above results. This implies the quantum Hamilton–Jacobi equation in the general case and then the time-dependent Schrödinger equation [7].

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References

[1] A.E. Faraggi, M. Matone, Phys. Rev. Lett. 78 (1997) 163.
[2] M. Matone, Phys. Lett. B 357 (1995) 342.
[3] R. Carroll, hep-th/9607219; hep-th/9610216; hep-th/9702138; hep-th/9705229; Nucl. Phys. B 502 (1997) 561; Lect. Notes Phys., vol. 502, Springer, 1998, pp. 33-56.
[4] Z. Nehari, Conformal Mapping, Dover, New York, 1975.
[5] A.E. Faraggi, M. Matone, Phys. Lett. B 437 (1998) 369, hep-th/9711028.
[6] L.D. Landau, E.M. Lifschitz, Quantum Mechanics, Pergamon Press, Oxford, 1958.
[7] G. Bertoldi, A.E. Faraggi, M. Matone, paper in preparation.