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ON THE ATTAINABLE SET FOR TEMPLE CLASS SYSTEMS WITH BOUNDARY CONTROLS*

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Abstract. Consider the initial-boundary value problem for a strictly hyperbolic, genuinely nonlinear, Temple class system of conservation laws

$$u_t + f(u)_x = 0, \qquad u(0,x) = \overline{u}(x), \qquad \begin{cases} u(t,a) = \widetilde{u}_a(t), \\ u(t,b) = \widetilde{u}_b(t), \end{cases} \tag{1}$$

on the domain $\Omega = \{(t,x) \in \mathbb{R}^2 : t \geq 0,\ a \leq x \leq b\}$. We study the mixed problem (1) from the point of view of control theory, taking the initial data $\overline{u}$ fixed and regarding the boundary data $\widetilde{u}_a$, $\widetilde{u}_b$ as control functions that vary in prescribed sets $\mathcal{U}_a$, $\mathcal{U}_b$, of $\mathbf{L}^\infty$ boundary controls. In particular, we consider the family of configurations

$$\mathcal{A}(T) \doteq \{u(T,\cdot);\ u \text{ is a sol. to (1)},\ \widetilde{u}_a \in \mathcal{U}_a,\ \widetilde{u}_b \in \mathcal{U}_b\}$$

that can be attained by the system at a given time $T > 0$, and we give a description of the attainable set $\mathcal{A}(T)$ in terms of suitable Oleinik-type conditions. We also establish closure and compactness of the set $\mathcal{A}(T)$ in the $\mathbf{L}^1$ topology.

Key words. hyperbolic systems, conservation laws, Temple class systems, boundary control, attainable set

AMS subject classifications. 35L65, 35B37

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1. Introduction.

Consider the initial-boundary value problem for a strictly hyperbolic, genuinely nonlinear, system of conservation laws in one space dimension

$$u_t + f(u)_x = 0, \tag{1.1}$$

$$u(0,x) = \overline{u}(x), \tag{1.2}$$

$$u(t,a) = \widetilde{u}_a(t), \tag{1.3}$$

$$u(t,b) = \widetilde{u}_b(t) \tag{1.4}$$

on the strip $\Omega = \{(t,x) \in \mathbb{R}^2;\ t \geq 0,\ x \in [a,b]\}$. Here, $u = u(t,x) \in \mathbb{R}^n$ is the vector of the conserved quantities, $\widetilde{u}_a, \widetilde{u}_b$ are measurable, bounded boundary data, and the flux function $f : U \mapsto \mathbb{R}^n$ is a smooth vector field defined on some open set $U \subseteq \mathbb{R}^n$ that belongs to a class of fields introduced by Temple [29, 28] for which rarefaction and Hugoniot curves coincide. We recall that, for problems of this type, classical solutions may develop discontinuities in finite time, regardless of the regularity of the initial and boundary data. Hence, it is natural to consider weak solutions in the sense of distributions. Moreover, since, in general, the Dirichlet conditions (1.3)–(1.4) cannot be fulfilled pointwise a.e. (see [7, 19]), different weaker formulations of the boundary condition have been considered in the literature (see [1, 20, 27] and references therein). Here, following Dubois and LeFloch [19], we will adopt a formulation of (1.3)–(1.4)

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based on the definition of a time-dependent set of *admissible boundary data* that is related to the notion of the Riemann problem.

In the present paper, having in mind applications of Temple systems to problems of oil reservoir simulation, multicomponent chromatography, and traffic flow models, we study the effect of the boundary conditions (1.3)–(1.4) on the solution of (1.1)–(1.2) from the point of view of control theory. Namely, following the same approach adopted in [4, 5, 21] for scalar conservation laws, we fix an initial data $\overline{u} \in \mathbf{L}^\infty([a,b])$ and consider the family of configurations

$$\mathcal{A}(T;\mathcal{U}_a,\mathcal{U}_b) \doteq \{u(T,\cdot);\ u \text{ sol. to } (1.1)\text{–}(1.4),\ \widetilde{u}_a \in \mathcal{U}_a,\ \widetilde{u}_b \in \mathcal{U}_b\} \tag{1.5}$$

that can be attained at a given time $T > 0$ by solutions to (1.1)–(1.4), with boundary data $\widetilde{u}_a$, $\widetilde{u}_b$ that vary in prescribed sets $\mathcal{U}_a$, $\mathcal{U}_b \subset \mathbf{L}^\infty(\mathbb{R}^+)$ of *admissible boundary controls.* In the case of scalar, convex conservation laws, it was proved in [4], by using the theory of generalized characteristics [17], that the profiles $w(x)$ which can be attained at a fixed time $T > 0$ are only those for which the map $x \mapsto \frac{f'(w(x))}{x}$ is nonincreasing. Under the assumption that $f'(u) \geq 0$ for all u, and for solutions of the mixed problem (1.1)–(1.4) on the region Ω, this condition is equivalent to the Oleinik-type inequalities

$$D^+ w(x) \leq \frac{f'(w(x))}{(x-a)f''(w(x))} \quad \text{for a.e. } x \in [a,b] \tag{1.6}$$

($D^+ w$ denoting the upper Dini derivative of w). For general $n \times n$ systems, a complete characterization of the attainable set does not seem possible, due to the complexity of repeated wave-front interactions. However, in the particular case of Temple systems, wave interactions can change only the speed of wave-fronts without modifying their amplitudes, due to the special geometric features of such systems. Therefore, the only restriction to boundary controllability is the decay due to genuine nonlinearity. We then consider here a convex, compact set $\Gamma \subset U$ and provide a description of the attainable set

$$\mathcal{A}(T) \doteq \mathcal{A}(T;\mathcal{U}^\infty,\mathcal{U}^\infty), \qquad \mathcal{U}^\infty \doteq \mathbf{L}^\infty([0,T],\Gamma)$$

in terms of certain Oleinik-type conditions. We also establish the compactness of $\mathcal{A}(T)$ in the $\mathbf{L}^1$ topology, as was shown in [4] for the scalar case. These results are useful in applications to calculus of variations and optimal control problems where the cost functional depends on the profile of the solution to (1.1)–(1.4) at a fixed time T. An example is given by a model of two-component chromatography that describes a liquid flowing through a tube packed with solid particles that absorbs (with different rates of adsorption) two interacting chemical substances dissolved in the liquid. In case one is interested in producing the separation of the two substances, the controller acts by varying the concentration of the solutes entering the tube to maximize the concentration of each substance in the liquid phase on opposite sides of the tube.

The paper is organized as follows. Section 2 contains the basic definitions and the statement of the main results, together with a discussion of the two-component chromatography problem. We also provide in this section a review of the existence and well-posedness theory for the mixed problem (1.1)–(1.4), and a description of a front tracking algorithm that will be used throughout the paper. In section 3 we establish some preliminary estimates and a regularity result concerning the global structure of solutions to the mixed problem (1.1)–(1.4) generated by a front tracking algorithm. The proof of the main results is contained in section 4.

2. Preliminaries and statement of the main results.

2.1. Formulation of the problem. Let $f : U \mapsto \mathbb{R}^n$ be the flux function of the strictly hyperbolic system (1.1) defined on a neighborhood of the origin $U \subseteq \mathbb{R}^n$. Denote by $\lambda_1(u) < \cdots < \lambda_n(u)$ the eigenvalues of the Jacobian matrix $Df(u)$, and let $\{r_1(u), \ldots, r_n(u)\}$ be a basis of right eigenvectors of $Df(u)$. By possibly considering a sufficiently small restriction of the domain U, we may assume that the following *uniform* strict hyperbolicity condition holds.

(SH) For every $u, v \in U$, the characteristic speeds at these points satisfy

$$\lambda_i(u) < \lambda_j(v) \quad \forall 1 \leq i < j \leq n. \tag{2.1}$$

We also assume that there is a fixed set of characteristic lines entering the interior of the strip $[a, b] \times \mathbb{R}^+$ at the boundaries $x = a$, $x = b$, i.e., that for some index $p \in \{1, \ldots, n\}$, there holds

$$\lambda_p(u) < 0 < \lambda_{p+1}(u) \quad \forall u \in U, \tag{2.2}$$

and we let $\lambda^{\min}$, $\lambda^{\max}$ denote the minimum and maximum characteristic speed so that there holds

$$0 < \lambda^{\min} \leq |\lambda_i(u)| \leq \lambda^{\max} \quad \forall u \in U. \tag{2.3}$$

Moreover, we assume that each ith characteristic field r_i is *genuinely nonlinear* in the sense of Lax [22], and that system (1.1) is of Temple class according to the following definition.

DEFINITION 2.1. *A system of conservation laws is of Temple class if there exists a system of coordinates $w = (w_1, \ldots, w_n)$ consisting of Riemann invariants, and such that the level sets $\{u \in U;\ w_i(u) = constant\}$ are hyperplanes (see [28]).*

By possibly performing a translation of coordinates, it is not restrictive to assume that the Riemann invariants are chosen so that $\partial_i \lambda_i(w) > 0$, $i = 1, \ldots, n$, for all $w = w(u)$, $u \in U$. Throughout the paper, we will often write $w_i(t, x) \doteq w_i(u(t, x))$ to denote the ith Riemann coordinate of a solution $u = u(t, x)$ to (1.1). We recall that, for a Temple class system, Hugoniot curves and rarefaction curves coincide and are straight lines [29]. Moreover, as observed in [3], thanks to the existence of Riemann coordinates one can show that the assumption (SH) implies the invertibility of the map $f : U \mapsto f(U)$.

We next introduce a definition of weak solution to (1.1)–(1.4) which includes an entropy admissibility condition of Oleinik type on the decay of positive waves, to achieve uniqueness. The boundary conditions (1.3)–(1.4) are formulated in terms of the weak trace of the flux $f(u)$ at the the boundaries $x = a$, $x = b$ and are related to the notion of the Riemann problem in the same spirit of [19]. To this purpose, letting $u(t, x) = W(\xi = x/t; u_L, u_R)$, $u_L, u_R \in U$, denote the self-similar solution of the Riemann problem for (1.1) with initial data

$$u(0, x) = \begin{cases} u_L & \text{if } x < 0, \\ u_R & \text{if } x > 0 \end{cases}$$

for any given boundary state $\widetilde{u} \in U$, we define the set of *admissible states at the boundaries*

$$\begin{aligned} \mathcal{V}_a(\widetilde{u}) &\doteq \{W(0+; \widetilde{u}, u_R);\ u_R \in U\}, \\ \mathcal{V}_b(\widetilde{u}) &\doteq \{W(0-; u_L, \widetilde{u});\ u_L \in U\}. \end{aligned} \tag{2.4}$$

DEFINITION 2.2. *A function $u : [0,T] \times [a,b] \mapsto U$ is an entropy weak solution of the initial-boundary value problem* (1.1)–(1.4) *on $\Omega_T \doteq [0,T] \times [a,b]$ if it is continuous as a function from $]0,T]$ into $\mathbf{L}^1$, and the following properties hold:*

(i) *u is a distributional solution to the Cauchy problem* (1.1)–(1.2) *on Ω_T in the sense that, for every test function $\phi \in \mathcal{C}^1_c$ with compact support contained in the set $\{(t,x) \in \mathbb{R}^2;\ a < x < b,\ t < T\}$, there holds*

$$\int_0^T \int_a^b (u(t,x) \cdot \phi_t(t,x) + f(u(t,x)) \cdot \phi_x(t,x)) dx\, dt + \int_a^b \overline{u}(x) \cdot \phi(0,x) dx = 0;$$

(ii) *the flux $f(u)$ admits weak* traces at the boundaries $x = a$, $x = b$, i.e., there exist two measurable functions $\Psi_a, \Psi_b : [0,T] \mapsto \mathbb{R}^n$ such that*

$$f(u(\cdot,x)) \xrightarrow[x\to a^+]{*} \Psi_a, \qquad f(u(\cdot,x)) \xrightarrow[x\to b^-]{*} \Psi_b \quad \textit{in } \mathbf{L}^\infty([0,T]), \tag{2.5}$$

and the boundary conditions (1.3)–(1.4) *are satisfied in the following sense:*

$$\Psi_a(t) \in f(\mathcal{V}_a(\widetilde{u}_a(t))), \qquad \Psi_b(t) \in f(\mathcal{V}_b(\widetilde{u}_b(t))) \quad \textit{for a.e. } 0 \le t \le T; \tag{2.6}$$

(iii) *u satisfies the following entropy conditions on the decay of positive waves in time and space. There exists some constant $C > 0$, depending only on the system* (1.1)*, so that*

(a) *for any $0 < t \le T$, and for a.e. $a < x < y < b$, there holds*

$$w_i(t,y) - w_i(t,x) \le C \cdot \left\{ \frac{y-x}{t} + \log\left(\frac{y-b}{x-b}\right) \right\} \quad \textit{if } i \in \{1,\dots,p\}, \tag{2.7}$$

$$w_i(t,y) - w_i(t,x) \le C \cdot \left\{ \frac{y-x}{t} + \log\left(\frac{y-a}{x-a}\right) \right\} \quad \textit{if } i \in \{p+1,\dots,n\}; \tag{2.8}$$

(b) *for a.e. $a < x < b$, and for a.e. $0 < \tau_1 < \tau_2 \le T$, there holds*

$$w_i(\tau_2,x) - w_i(\tau_1,x) \le C \cdot \left\{ \frac{\tau_2-\tau_1}{x-b} + \log\left(\frac{\tau_2}{\tau_1}\right) \right\} \quad \textit{if } i \in \{1,\dots,p\}, \tag{2.9}$$

$$w_i(\tau_2,x) - w_i(\tau_1,x) \le C \cdot \left\{ \frac{\tau_2-\tau_1}{x-a} + \log\left(\frac{\tau_2}{\tau_1}\right) \right\} \quad \textit{if } i \in \{p+1,\dots,n\}. \tag{2.10}$$

Remark 2.3. The set of admissible flux values at the boundaries $x = a$, $x = b$, can be expressed in Riemann coordinates as

$$\begin{aligned} f(\mathcal{V}_a(\widetilde{u})) &= \{f(u);\ w_i(u) = w_i(\widetilde{u}) \qquad \forall i = p+1,\dots,n\}, \\ f(\mathcal{V}_b(\widetilde{u})) &= \{f(u);\ w_i(u) = w_i(\widetilde{u}) \qquad \forall i = 1,\dots,p\}. \end{aligned} \tag{2.11}$$

Hence, by the invertibility of the map $f : U \mapsto f(U)$, the above boundary conditions (2.6) are equivalent to the set of equalities

$$\begin{aligned} w_i(f^{-1}(\Psi_a(t))) &= w_i(\widetilde{u}_a(t)) \quad \text{for a.e. } 0 \le t \le T,\ di = p+1,\dots,n, \\ w_i(f^{-1}(\Psi_b(t))) &= w_i(\widetilde{u}_b(t)) \quad \text{for a.e. } 0 \le t \le T,\ i = 1,\dots,p. \end{aligned} \tag{2.12}$$

This means that the boundary conditions (2.6) guarantee that, at almost every time $t \in [0,T]$, the solution to the Riemann problem for (1.1), having left and right initial states $u^L = \widetilde{u}_a(t)$, $u^R = f^{-1}(\Psi_a(t))$, contains only waves with negative speeds, while the solution to the Riemann problem with initial states $u^L = f^{-1}(\Psi_b(t))$, $u^R = \widetilde{u}_b(t)$, contains only waves with positive speeds. Thus, in particular, such solutions do not contain any front entering the domain $[t, +\infty[\times]a, b[$.

In the present paper we regard the boundary data as admissible controls and, in connection with a fixed convex, compact set $\Gamma \subset U$ having the form

$$\Gamma = \{u \in U;\ w_i(u) \in [\alpha_i, \beta_i],\ i = 1, \dots, n\}, \tag{2.13}$$

we study the basic properties of the *attainable set* for (1.1)–(1.2), i.e., of the set

$$\mathcal{A}(T) \doteq \{u(T,\cdot);\ u \text{ is a sol. to (1.1)–(1.4)},\ \widetilde{u}_a, \widetilde{u}_b \in \mathbf{L}^\infty([0,T],\Gamma)\} \tag{2.14}$$

which consists of all profiles that can be attained at a fixed time $T > 0$ by entropy weak solutions of (1.1)–(1.4) (according to Definition 2.2) with a fixed initial data $\overline{u} \in \mathbf{L}^\infty([a,b],\Gamma)$ and boundary data $\widetilde{u}_a$, $\widetilde{u}_b$ that vary in

$$\mathcal{U}_T^\infty \doteq \mathbf{L}^\infty([0,T],\Gamma). \tag{2.15}$$

We will establish a characterization of (2.14) in terms of certain Oleinik type estimates on the decay of positive waves, and we will prove the compactness of (2.14) in the $\mathbf{L}^1$ topology.

2.2. Statements of the main results. For any $\rho > 0$, consider the set of maps

$$K^\rho \doteq \left\{ \varphi \in \mathbf{L}^\infty([a,b],\Gamma);\ \begin{array}{l} \dfrac{w_i(\varphi(y)) - w_i(\varphi(x))}{y-x} \le \dfrac{\rho}{x-a} \begin{cases} \text{for a.e. } a < x < y < b, \\ \text{if } i \in \{p+1,\dots,n\} \end{cases} \\ \dfrac{w_i(\varphi(y)) - w_i(\varphi(x))}{y-x} \le \dfrac{\rho}{b-y} \begin{cases} \text{for a.e. } a < x < y < b, \\ \text{if } i \in \{1,\dots,p\} \end{cases} \end{array} \right\}. \tag{2.16}$$

The inequalities in (2.16) reflect the fact that positive waves entering through the boundaries $x = a$, $x = b$ decay in time. Therefore, their density (expressed in terms of Riemann coordinates) is inversely proportional to their distance from their entrance point on the boundary.

THEOREM 2.4. *Let* (1.1) *be a system of Temple class with all characteristic fields genuinely nonlinear, and assume that the strict hyperbolicity condition* (SH) *is verified. Then, for every fixed $\overline{\tau} > 0$, there exists $\rho = \rho(\overline{\tau}) > 0$ such that*

$$\mathcal{A}(\tau) \subseteq K^\rho \quad \forall \tau \ge \overline{\tau}. \tag{2.17}$$

Moreover, taking $T \doteq \frac{4(b-a)}{\lambda^{\min}}$, there exists $\rho' < \rho(T)$ such that

$$K^{\rho'} \subseteq \mathcal{A}(\tau) \quad \forall \tau > T. \tag{2.18}$$

Remark 2.5. Observe that, given $\varphi \in K^\rho$, any map $x \mapsto w_i(\varphi(x))$, $i \in \{1,\dots,n\}$, is essentially bounded and has finite total increasing variation on subsets of $[a,b]$ bounded away from the end points a, b. Hence, any map $x \mapsto w_i(\varphi(x))$, $i \in \{1,\dots,n\}$, also has finite total variation on such sets and, in particular, it admits left and right limits in any point $x \in]a,b[$. Moreover, since an element φ of K^ρ is defined up to $\mathbf{L}^1$

equivalence, we may always assume that there is a right continuous representative of $w_i(\varphi)$, $i \in \{1, \dots, n\}$, that satisfies the inequalities appearing in the definition of K^ρ.

Remark 2.6. It can be easily seen that, no matter how small we choose ρ, it is not possible, in general, to reach at a time $T < \frac{2(b-a)}{\lambda^{\max}}$ any prescribed profile $\varphi \in K^\rho$ with an entropy weak solution u to (1.1)–(1.4) that starts with a fixed initial data $\overline{u}$. This is due to the fact that in this case the domains of determinacy of the line segments $t = 0$, $a \leq x \leq b$, and $t = T$, $a \leq x \leq b$ have a nonempty intersection and hence the determination of the possible solution u induced by $u(T, x) = \varphi(x)$, $x \in [a, b]$, and $u(0, x) = \overline{u}(x)$, $x \in [a, b]$ would be, in general, inconsistent. We expect that the same type of noncontrollability result holds for time larger than $\frac{2(b-a)}{\lambda^{\max}}$. This raises the question concerning the identification of a minimum time $T_m < \frac{4(b-a)}{\lambda^{\min}}$ for which the inclusion (2.18) of Theorem 2.4 is verified, although a definite answer to such a problem does not seem possible due to the complexity of the wave-front structure of a solution to the mixed problem (1.1)–(1.4).

THEOREM 2.7. *Under the same assumptions of Theorem* 2.4*, the set* $\mathcal{A}(T)$ *is a compact subset of* $\mathbf{L}^1([a, b], \Gamma)$ *for each* $T > 0$*.*

2.3. An application: Chromatography of two solutes. Chromatography is a process used by chemists and engineers to separate two (or more) chemical species in a fluid phase by selective adsorption on a solid medium. We consider here the case of a mixture of two interacting solutes S_1 and S_2 dissolved in a liquid with concentrations c_1 and c_2 which passes through the interstices of a solid bed of particles packed in a tube. The solid surface of the filtering bed absorbs different amounts of the two solutes, while it is possible for the particles of each substance to pass from the fluid to the solid phase, and vice versa. The different rates of adsorption of the two solutes causes the less strongly adsorbed solute, say S_2, to move ahead of the more strongly adsorbed one S_1, thus inducing a separation of the two chemical substances. If we make the assumption of local equilibrium in the tube between the liquid and solid phase for each substance, one can express the solid concentrations of the two substances n_1, n_2 as functions of both liquid concentrations. In the case of the Langmuir isotherm equilibrium, the solid concentrations take the form (see [24])

$$n_i \doteq \frac{N_i k_i c_i}{1 + k_1 c_1 + k_2 c_2}, \quad i = 1, 2, \tag{2.19}$$

where k_1, k_2 are constitutive constants depending on the temperature, while N_i denotes the limiting value of n_i (representing the maximum concentration of the solute that can be adsorbed by the solid medium). After performing a suitable transformation of the independent variables, one can express the mass conservation equations for the two species as (see [26])

$$\begin{aligned} [c_1]_x + \left[\frac{\gamma c_1}{1 + c_1 + c_2}\right]_t &= 0, \\ [c_2]_x + \left[\frac{c_2}{1 + c_1 + c_2}\right]_t &= 0 \end{aligned} \tag{2.20}$$

for some constant $\gamma \in]0, 1]$. Notice that the mathematical roles played by the spacelike variable x and by the timelike variable t in (2.20) is the opposite of the typical roles played by such variables in most physical hyperbolic systems. Because of the particular nonlinearity relation (2.19) of the Langmuir isotherm, (2.20) enjoy the special geometric properties of Temple systems. Moreover, by direct computations, one can verify that (2.20) satisfies all the other assumptions made in Theorems 2.4–2.7, namely the following hold (see [26]):

- system (2.20) is strictly hyperbolic if we assume that the state variables c_i take values in some fixed interval $[\overline{c}_i, \overline{d}_i]$ (which corresponds to requiring that the concentrations are nonnegative and do not approach infinity);
- both characteristic speeds of (2.20) are positive and genuinely nonlinear if $\gamma \in]0,1[$ (while, in the case $\gamma = 1$, the first characteristic speed is genuinely nonlinear and the second is linearly degenerate).

A general introduction to the mathematical modeling of chromatography and a detailed analysis of the equilibrium system in the special case of Langmuir isotherm is provided by Rhee, Aris, and Amundson in [25, 26].

We are concerned here with the problem of controlling the feed data c_1^f, c_2^f, i.e., the concentrations of the two solutes S_1, S_2 entering the tube in a time interval $[0,T]$, to maximize the separation of the two substances at time T. Namely, we are interested in maximizing the difference between the distributions of S_1 and S_2 on opposite sides of the tube at a fixed time T. Let $x = 0$ and $x = L$ denote the locations, respectively, of the inlet and outlet of the tube, and suppose that $\overline{c}_1$, $\overline{c}_2$ are the initial distributions in the liquid of S_1 and S_2 at time $t = 0$. Then, consider the functional

$$J(x, c^f) \doteq \int_0^x (c_1(T,\xi) - c_2(T,\xi))d\xi + \int_x^L (c_2(T,\xi) - c_1(T,\xi))d\xi, \tag{2.21}$$

where $(c_1, c_2)(t,x)$ denotes the solution to the mixed problem for (2.20) on the strip $[0,\infty[\times[0,L]$, with initial data $\overline{c} = (\overline{c}_1, \overline{c}_2)$ and boundary data $c^f = (c_1^f, c_2^f)$. Notice that, since both characteristic speeds are positive, the set $\mathcal{V}_L(\widetilde{c})$ of admissible states at the right boundary $x = L$ (defined as in (2.4)) turns out to be equal to the whole space $\mathbb{R}^2$ for every $\widetilde{c} \in \mathbb{R}^2$. Hence, the only significant boundary condition is assigned at the left boundary $x = 0$. Then, assuming that C_i denotes the maximum amount of concentration of the solute S_i that can be introduced in the tube in the time interval $[0,T]$, we are led to study the maximization problem

$$\max_{x\in[0,1],\, c^f\in\mathcal{U}_T^\infty\, c^f\in\mathcal{U}_T^\infty} J(x, c^f), \tag{2.22}$$

where the set of admissible boundary controls is given by

$$\mathcal{U}_T^\infty \doteq \left\{ \widetilde{c} \in \mathbf{L}^\infty([0,T]) : \widetilde{c}_i(t) \in [\overline{c}_i, \overline{d}_i],\ \int_0^T \widetilde{c}_i(t) \leq C_i \right\}. \tag{2.23}$$

With the same arguments we will use to establish Theorem 2.7 one can prove the compactness of the attainable set $\mathcal{A}(T)$ in the case of a system like (2.20) where the mathematical roles played by the x-t variables is reversed w.r.t. (1.1) and in connection with a set of admissible boundary controls that satisfy an additional integral constraint as in (2.23). Then, observing that the map

$$(x, c) \mapsto \int_0^x (c_1(\xi) - c_2(\xi))\, d\xi + \int_x^L (c_2(\xi) - c_1(\xi))\, d\xi \tag{2.24}$$

is continuous as a functional from $[0,L] \times \mathbf{L}^1([0,T])$ to $\mathbb{R}$, we deduce that the maximization problem (2.22) admits a solution.

2.4. Existence and uniqueness of solutions. We describe here a front tracking algorithm that generates approximate solutions to (1.1) on the strip $[a,b]\times\mathbb{R}^+$ continuously depending on the initial and boundary data, which represents a natural extension of [3, 12]. Fix an integer $\nu\ge 1$ and consider the discrete set of points in Γ whose coordinates are integer multiples of $2^{-\nu}$:

$$\Gamma^\nu \doteq \{u\in\Gamma;\ w_i(u)\in 2^{-\nu}\mathbb{Z},\ i=1,\dots,n\}. \tag{2.25}$$

Moreover, consider the domain

$$\mathcal{D}^\nu \doteq \{(u,u',u'');\ u\in\mathbf{L}^\infty([a,b],\Gamma^\nu),\ u',u''\in\mathbf{L}^\infty(\mathbb{R}^+,\Gamma^\nu),\ u,u',u''\text{ piecew. const.}\}. \tag{2.26}$$

On $\mathcal{D}^\nu$ we now construct a flow map E^ν whose trajectories are front tracking approximate solutions of (1.1). To this end, we first describe how to solve a Riemann problem with left and right initial states $u^L,u^R\in\Gamma^\nu$. In Riemann coordinates, assume that

$$w(u^L)\doteq w^L=(w_1^L,\dots,w_n^L),\qquad w(u^R)\doteq w^R=(w_1^R,\dots,w_n^R).$$

Consider the intermediate states

$$z^0\doteq u^L,\dots,\qquad z^i\doteq u(w_1^R,\dots,w_i^R,w_{i+1}^L,\dots,w_n^L),\dots,\qquad z^n\doteq u^R. \tag{2.27}$$

The solution to the Riemann problem (u^L,u^R) is constructed by piecing together the solutions to the simple Riemann problems (z^{i-1},z^i), $i=1,\dots,n$. If $w_i^R<w_i^L$, the solution of the Riemann problems (z^{i-1},z^i) will contain a single i-shock, connecting the states z^{i-1}, z^i and traveling with the Rankine–Hugoniot speed $\lambda_i(z^{i-1},z^i)$. Here and in what follows, by $\lambda_i(u,u')$ we denote the ith eigenvalue of the averaged matrix

$$A(u,u')\doteq\int_0^1 Df(\theta u+(1-\theta)u')\,d\theta. \tag{2.28}$$

If $w_i^R>w_i^L$, the exact solution of the Riemann problem (z^{i-1},z^i) would contain a centered rarefaction wave. This is approximated by a rarefaction fan as follows. If $w_i^R=w_i^L+p_i2^{-\nu}$ we insert the states

$$z^{i,\ell}\doteq(w_1^R,\dots,w_i^L+\ell 2^{-\nu},w_{i+1}^L,\dots,w_n^L),\qquad \ell=0,\dots,p_i, \tag{2.29}$$

so that $z^{i,0}=z^{i-1}$, $z^{i,p_i}=z^i$. Our front tracking solution will then contain p_i fronts of the ith family, each connecting a couple of states $z^{i,\ell-1}$, $z^{i,\ell}$ and traveling with speed $\lambda_i(z^{i,\ell-1},z^{i,\ell})$.

For any given triple of (piecewise constant) initial and boundary data $(\overline{u},\widetilde{u}_a,\widetilde{u}_b)\in\mathcal{D}^\nu$, the approximate solution $u(t,\cdot)\doteq E_t^\nu(\overline{u},\widetilde{u}_a,\widetilde{u}_b)$ is now constructed as follows. At time $t=0$, for $a<x<b$ we solve the initial Riemann problems determined by the jumps in $\overline{u}$ according to the above procedure, while at $x=a$ we construct the solution to the Riemann problem with left and right initial states $u^L=\widetilde{u}_a(0+)$, $u^R=\overline{u}(a+)$ and take its restriction to the interior of the domain Ω. In the same way, at $x=b$ we take the restriction to the interior of Ω of the solution to the Riemann problem with initial states $u^L=\overline{u}(b-)$, $u^R=\widetilde{u}_b(0+)$. This yields a piecewise constant function with finitely many fronts, traveling with constant speeds. The solution is then prolonged up to the first time t_1 at which one of the following events takes place:

(a) two or more discontinuities interact in the interior of Ω;
(b) one or more discontinuities hit the boundary of Ω;

(c) the boundary data $\widetilde{u}_a$ has a jump;

(d) the boundary data $\widetilde{u}_b$ has a jump.

If case (a) occurs, we then solve the resulting Riemann problems applying again the above procedure, while in cases (b), (c), and (d) we construct the solution to the Riemann problem with left and right initial states $u^L = \widetilde{u}_a(t_1+)$, $u^R = u(t_1, a+)$, or $u^L = u(t_1, b-)$, $u^R = \widetilde{u}_b(t_1+)$ and take its restriction to the interior of the domain Ω. This determines the solution $u(t,\cdot)$ until the time $t_2 > t_1$ where one of the events (a), (b), or (c) again takes place, etc. Notice that at any time where case (b) occurs but (c) or (d) do not take place, no new wave is generated. Therefore, waves entering the domain Ω at the boundaries $x = a$, $x = b$ are produced only by the jumps of the boundary data $\widetilde{u}_a$, $\widetilde{u}_b$.

As in [3, 12], one checks that the approximate solution u constructed with this algorithm is well defined for all times $t \geq 0$. Indeed, the following properties hold.

- The total variation of $u(t,\cdot)$, measured w.r.t. the Riemann coordinates $w_1(t,\cdot), \dots, w_n(t,\cdot)$ is nonincreasing in time.
- The number of wave-fronts in $u(t,\cdot)$ is nonincreasing at each interaction. Hence, the total number of wave-fronts in $u(t,\cdot)$ remains finite.

It is then possible to define a flow map

$$\mathbf{p} \mapsto E_t^\nu \mathbf{p}, \quad \mathbf{p} \doteq (\overline{u}, \widetilde{u}_a, \widetilde{u}_b) \in \mathcal{D}^\nu, \quad t \geq 0, \tag{2.30}$$

of approximate solutions of (1.1). By construction, each trajectory $t \mapsto E_t^\nu \mathbf{p}$ is a weak solution of (1.1) (because all fronts of $u(t,\cdot) \doteq E_t^\nu \mathbf{p}$ satisfy the Rankine–Hugoniot conditions) but may contain discontinuities that do not satisfy the usual Lax stability conditions (due to the presence of rarefaction fronts). On the other hand, one can verify as in [3, Lemma 4.4] that, due to genuine nonlinearity, the amount of positive waves in $u(t,\cdot)$, measured w.r.t. the Riemann coordinates $w_1(t,\cdot), \dots, w_n(t,\cdot)$, decays in time and space. Hence, for a.e. $a < x < y < b$, one obtains the Oleinik-type estimates

$$\begin{aligned} w_i(t,y) - w_i(t,x) &\leq C \cdot \left\{ \frac{y-x}{t} + \log\left(\frac{y-b}{x-b}\right) \right\} + N_\nu 2^{-\nu} \quad \text{if } i \in \{1,\dots,p\}, \\ w_i(t,y) - w_i(t,x) &\leq C \cdot \left\{ \frac{y-x}{t} + \log\left(\frac{y-a}{x-a}\right) \right\} + N_\nu 2^{-\nu} \quad \text{if } i \in \{p+1,\dots,n\}, \end{aligned} \tag{2.31}$$

where N_ν denotes the maximum number of shocks of each family present in the initial data $\overline{u}$, and in the boundary data $\widetilde{u}_a$, $\widetilde{u}_b$. Similarly, one can check that along the x-sections, for a.e. $0 < \tau_1 < \tau_2$, there holds

$$\begin{aligned} w_i(\tau_2,x) - w_i(\tau_1,x) &\leq C \cdot \left\{ \frac{\tau_2-\tau_1}{x-b} + \log\left(\frac{\tau_2}{\tau_1}\right) \right\} + N_\nu 2^{-\nu} \quad \text{if } i \in \{1,\dots,p\}, \\ w_i(\tau_2,x) - w_i(\tau_1,x) &\leq C \cdot \left\{ \frac{\tau_2-\tau_1}{x-a} + \log\left(\frac{\tau_2}{\tau_1}\right) \right\} + N_\nu 2^{-\nu} \quad \text{if } i \in \{p+1,\dots,n\}. \end{aligned} \tag{2.32}$$

Remark 2.8. Observe that if $u(t,x)$ is a front tracking solution of the Cauchy problem for (1.1) (with initial data $\overline{u}(x) \doteq u(0,x)$) constructed by the algorithm in [12] on the upper half plane $\mathbb{R}^+ \times \mathbb{R}$, then the restriction of $u(t,\cdot)$ to the interval $[a,b]$ coincides with the front tracking solution $E_t^\nu(\overline{u}, \widetilde{u}_a, \widetilde{u}_b)$ of the mixed problem for (1.1), with boundary data $\widetilde{u}_a(t) \doteq u(t,a)$, $\widetilde{u}_b(t) \doteq u(t,b)$.

As $\nu \to \infty$, the domains $\mathcal{D}^\nu$ become dense in

$$\mathcal{D} \doteq \{(\overline{u}, \widetilde{u}_a \widetilde{u}_b);\ \overline{u} \in \mathbf{L}^\infty([a,b],\Gamma),\ \widetilde{u}_a, \widetilde{u}_b \in \mathbf{L}^\infty(\mathbb{R}^+,\Gamma)\}. \tag{2.33}$$

Thus, following the same technique adopted in [3], one can define a flow map E_t on $\mathcal{D}$ as a suitable limit of the flows E_t^ν in (2.30) that depends Lipschitz continuously on the initial and boundary data. Namely, the following holds.

THEOREM 2.9. *Let* (1.1) *be a system of Temple class with all characteristic fields genuinely nonlinear, and assume that the strict hyperbolicity condition* (SH) *holds. Then, there exists a continuous map*

$$(t, \overline{u}, \widetilde{u}_a, \widetilde{u}_b) \mapsto E_t(\overline{u}, \widetilde{u}_a, \widetilde{u}_b), \quad t \geq 0,\ (\overline{u}, \widetilde{u}_a, \widetilde{u}_b) \in \mathcal{D}, \tag{2.34}$$

and some constant $C > 0$ depending only on the system (1.1) *and on the domain* Γ*, so that, for every fixed $0 < \delta < (b-a)/2$ and for all $\mathbf{p}_1 \doteq (\overline{u}, \widetilde{u}_a, \widetilde{u}_b)$, $\mathbf{p}_2 \doteq (\overline{v}, \widetilde{v}_a, \widetilde{v}_b) \in \mathcal{D}$, letting $L_t \doteq L_t(\delta) = C(1 + \log(t/\delta))$, there holds*

$$\begin{aligned} &\|E_t\mathbf{p}_1 - E_t\mathbf{p}_2\|_{\mathbf{L}^1([a+\delta, b-\delta])} \\ &\qquad \leq L_t \cdot \Big\{ \|\overline{u} - \overline{v}\|_{\mathbf{L}^1([a,b])} + \|\widetilde{u}_a - \widetilde{v}_a\|_{\mathbf{L}^1([0,t])} + \|\widetilde{u}_b - \widetilde{v}_b\|_{\mathbf{L}^1([0,t])} \Big\} \end{aligned} \tag{2.35}$$

for all $t \geq \delta$. Moreover, the map $(t,x) \mapsto E_t(\overline{u}, \widetilde{u}_a, \widetilde{u}_b)(x)$ yields an entropy weak solution (in the sense of Definition 2.2*) to the initial-boundary value problem* (1.1)–(1.4) *on* Ω *that admits strong* $\mathbf{L}^1$ *traces at the boundaries $x = a$ and $x = b$, i.e., there exist two measurable maps $\psi_a, \psi_b : \mathbb{R}^+ \mapsto U$ such that*

$$\begin{aligned} &\lim_{x \to a^+} \int_0^\tau |E_t(\overline{u}, \widetilde{u}_a, \widetilde{u}_b)(x) - \psi_a(t)|\, dt = 0, \\ &\lim_{x \to b^-} \int_0^\tau |E_t(\overline{u}, \widetilde{u}_a, \widetilde{u}_b)(x) - \psi_b(t)|\, dt = 0 \end{aligned} \tag{2.36}$$

$\forall \tau \geq 0$.

The proof of Theorem 2.9 can be obtained with arguments entirely similar to those used to establish [3, Theorem 2.1], where a continuous flow of solutions to (1.1) is constructed in the case of a mixed problem on the quarter of plane $\{(t,x) \in \mathbb{R}^2;\ t \geq 0,\ x \geq 0\}$, with a single boundary at $x = 0$.

Concerning uniqueness, with the same arguments in [3] one obtains the following result which is the extension of [3, Theorem 2.2] to the present case of a domain Ω with two boundaries at $x = a$ and at $x = b$.

THEOREM 2.10. *Let* (1.1) *be a system of Temple class satisfying the same assumptions as in Theorem* 2.9*. Let $u = u(t,x)$ be an entropy weak solution to the mixed problem* (1.1)–(1.4) *on the region $\Omega_T \doteq [0,T] \times [a,b]$ (in the sense of Definition* 2.2*). Assume that the following conditions hold.*

(i) *The map $(t,x) \to (u(t,\cdot), u(\cdot,x))$ takes values within the domain*

$$\mathcal{D}_T \doteq \{(\overline{u}, \widetilde{u}_a, \widetilde{u}_b);\ \overline{u} \in \mathbf{L}^\infty([a,b],\Gamma),\ \widetilde{u}_a, \widetilde{u}_b \in \mathbf{L}^\infty([0,T],\Gamma)\}. \tag{2.37}$$

(ii) *There holds*

$$\operatorname*{ess\,sup}_{t \to 0^+} \int_a^b |u(t,x) - \overline{u}(x)|\, dx = 0. \tag{2.38}$$

(iii) *There holds*

$$\operatorname*{ess\,sup}_{x\to a^+}\int_0^T |w_i(u(t,x)) - w_i(\widetilde{u}_a(t))|\,dt = 0 \quad \forall i = p+1,\dots,n, \tag{2.39}$$

$$\operatorname*{ess\,sup}_{x\to b^-}\int_0^T |w_i(u(t,x)) - w_i(\widetilde{u}_b(t))|\,dt = 0 \quad \forall i = 1,\dots,p. \tag{2.40}$$

Then, u coincides with the corresponding trajectory of the flow map E_t provided by Theorem 2.9*, namely one has*

$$u(t,\cdot) = E_t(\overline{u},\widetilde{u}_a,\widetilde{u}_b)(\cdot) \quad \forall 0 \le t \le T. \tag{2.41}$$

The next result shows that the conditions (2.38)–(2.40) are certainly satisfied by entropy weak solutions to the mixed problem (1.1)–(1.4) obtained as the limit of front tracking approximations.

THEOREM 2.11. *Let* (1.1) *be a system of Temple class satisfying the same assumptions as in Theorem* 2.9*. Consider a sequence $u^\nu(t,\cdot) : [a,b] \mapsto \Gamma^\nu$ of wave-front tracking approximate solutions of the mixed problem for* (1.1) *(constructed with the above algorithm) that converges in $\mathbf{L}^1$, as $\nu\to\infty$, to some function $u(t,\cdot) : [a,b]\mapsto\Gamma$, for every $t\in[0,T]$, and assume that the corresponding sequences of boundary data $\widetilde{u}_a^\nu$, $\widetilde{u}_b^\nu$ converge in $\mathbf{L}^1$ to $\widetilde{u}_a \doteq u(\cdot,a), \widetilde{u}_b \doteq u(\cdot,b)$. Then, there exist the right limit at $x = a$ and the left limit at $x = b$ of the map $x \to u(t,x)$ for every $t\in[0,T]$, and the right limit at $t = 0$ of the map $t\to u(t,x)$ for every $x\in[a,b]$. Moreover, there is a countable set $\mathcal{N}\subset\mathbb{R}$ such that $u(t,a) = u(t,a+), u(t,b) = u(t,b-)$ for all $t\in[0,T]\setminus\mathcal{N}$, and $u(0,x) = u(0+,x)$ for all $x\in[a,b]\setminus\mathcal{N}$, and setting $\overline{u}\doteq u(0,\cdot)$, there holds* (2.41)*.*

Remark 2.12. It was shown in [3, Lemma 2.1] that an alternative way to prove the essential limits (2.39)–(2.40) is to employ the distributional entropy inequalities associated with the “boundary entropy pairs” for (1.1), introduced by Chen and Frid in [14, 15].

In order to prove Theorem 2.11, we will show in the next section that, for Temple systems, solutions of the mixed problem (1.1)–(1.4) with possibly unbounded variation enjoy the same regularity property (of being continuous outside a countable number of Lipschitz curves) possessed by solutions with small total variation of a general system, thus extending the regularity results obtained under the smallness assumption of the total variation by DiPerna [18] and Liu [23] (for solutions constructed by the Glimm scheme) and by Bressan and LeFloch [13] (for solutions generated by a front tracking algorithm).

PROPOSITION 2.13. *In the same setting as Theorem* 2.11*, consider a sequence $u^\nu(t,\cdot) : [a,b]\mapsto\Gamma^\nu$ of wave-front tracking approximate solutions of the mixed problem for* (1.1) *(constructed with the above algorithm) that converges in $\mathbf{L}^1$, as $\nu\to\infty$, to some function $u(t,\cdot) : [a,b]\mapsto\Gamma$, for every $t\in[0,T]$. Then, there exist a countable set of interaction points $\Theta\doteq\{(\tau_l,x_l);\ l\in\mathbb{N}\}\subset\Omega_T\doteq[0,T]\times[a,b]$, and a countable family of Lipschitz continuous shock curves $\Upsilon\doteq\{x = y_m(t);\ t\in]r_m,s_m[,\ m\in\mathbb{N}\}$, such that the following hold.*

(i) *For each $m\in\mathbb{N}$, and for any $\tau\in]r_m,s_m[$ with $(\tau,y_m(\tau))\notin\Theta$, there exist the derivative $\dot{y}_m(\tau)$ and the left and right limits*

$$\lim_{(s,y)\to(\tau,y_m(\tau)),\, y<y_m(\tau)} u(s,y) \doteq u^-, \qquad \lim_{(s,y)\to(\tau,y_m(\tau)),\, y>y_m(\tau)} u(s,y) \doteq u^+. \tag{2.42}$$

Moreover, these limits satisfy the Rankine–Hugoniot relations

$$\dot{y}_m(\tau) \cdot (u^+ - u^-) = f(u^+) - f(u^-) \tag{2.43}$$

and for some $i \in \{1, \dots, n\}$ *there hold the Lax entropy inequalities*

$$\lambda_i(u^+) < \dot{y}_m(t) < \lambda_i(u^-). \tag{2.44}$$

(ii) *The map* u *is continuous outside the set* $\Theta \cup \Upsilon$*.*

3. Preliminary results. In this section we first provide some estimates on the distance between two rarefaction fronts of a front tracking solution (constructed by the algorithm described in section 2.4) similar to [12, Lemma 4], [8, Proposition 4.5]. We next show how to approximate the profile $u(t, \cdot)$ of a solution of the mixed problem (1.1)–(1.4), with a function taking values in the discrete set Γ^ν defined at (2.25), which enjoys the same type of estimates on the positive waves as $u(t, \cdot)$. We conclude the section establishing the regularity result stated in Proposition 2.13 on the global structure of solutions to the mixed problem for (1.1), which in turn yields Theorem 2.11.

LEMMA 3.1. *There exists some constant* $C_1 > 0$ *depending only on the system* (1.1) *such that the following holds. Consider a front tracking solution* $u(t, x)$ *with values in* Γ^ν*, constructed by the algorithm of section* 2.4 *on the region* $[\tau, \tau'] \times [a, b]$*. Then, given any two adjacent rarefaction fronts of* u *located at* $x(t) \le y(t)$*,* $t \in [\tau, \tau']$*, and belonging to the same family, there holds*

$$|y(\tau') - x(\tau')| \le |y(\tau) - x(\tau)| + C_1(\tau' - \tau)2^{-\nu}. \tag{3.1}$$

Proof. Consider two adjacent rarefaction fronts of the kth family $x(t) \le y(t)$, $t \in [\tau, \tau']$, and let $\tau_1 < \cdots < \tau_N$ be the interaction times of $x(t)$ in the interval $[\tau, \tau']$. Set $\tau_0 \doteq \tau$, $\tau_{N+1} \doteq \tau'$, and fix $\alpha \in \{0, \dots, N\}$. Let $t \to z(t; s, x)$ be the characteristic curve of the kth family starting at (s, x), i.e., the solution to the ODE

$$\dot{z} = \lambda_k(u(t, z)), \qquad z(s; s, x) = x.$$

Notice that, although the above ODE has a discontinuous right-hand side (because of the discontinuities in the front tracking solution u), its solution $z(\cdot; s, x)$ is unique and depends Lipschitz continuously on the initial data x since it crosses only a finite number of jumps (see [10]). Choose $t_0 < t_1 < \tau_{\alpha+1}$ so that the characteristic curve $z(\cdot; t_0, x(t_0))$ does not cross any wave-front of the other families in the interval $[t_0, t_1]$, and then, by induction, define a sequence of times $\{t_i\}_{i \in \mathbb{Z}} \subset]\tau_\alpha, \tau_{\alpha+1}[$ so that

$$\begin{gathered} \tau_\alpha < t_{-i-1} < t_{-i} \le t_0 \le t_i < t_{i+1} < \tau_{\alpha+1}, \quad i \in \mathbb{N}, \\ \lim_{i \to -\infty} t_i = \tau_\alpha, \qquad \lim_{i \to +\infty} t_i = \tau_{\alpha+1}, \end{gathered} \tag{3.2}$$

with the properties that the characteristic curve of the kth family starting at $(t_i, x(t_i))$, does not cross any wave-front of the other families in the interval $[t_i, t_{i+1}]$, for each $i \in \mathbb{Z}$. Thus, setting

$$u_i^+ \doteq u(t_i, x(t_i)+), \qquad u_i^- \doteq u(t_i, x(t_i)-)$$

and observing that, by construction, one has $|w(u_i^+) - w(u_i^-)| < 2^{-\nu}$, we derive

$$\begin{aligned} |z(t_{i+1}; t_i, x(t_i)) - x(t_{i+1})| &\le (t_{i+1} - t_i) \cdot |\lambda_k(u_i^+) - \lambda_k(u_i^+, u_i^-)| \\ &\le c \cdot (t_{i+1} - t_i) \cdot |w(u_i^+) - w(u_i^-)| \\ &\le c \cdot (t_{i+1} - t_i) \cdot 2^{-\nu} \end{aligned} \tag{3.3}$$

for some constant $c > 0$ depending only on the system. Relying on (3.3), and since $z(\tau'; t_{i+1}, x)$ depends Lipschitz continuously on the initial data x, we deduce that there exists some other constant $c' > 0$, depending only on the system and on the set Γ, so that there holds

$$\begin{aligned} |z(\tau'; t_i, x(t_i)) - z(\tau'; t_{i+1}, x(t_{i+1}))| &\leq c' \cdot |z(t_{i+1}; t_i, x(t_i)) - x(t_{i+1})| \\ &\leq c' \cdot c \cdot (t_{i+1} - t_i) \cdot 2^{-\nu} \end{aligned} \tag{3.4}$$

for any $i \in \mathbb{Z}$. Thus, by (3.2) and thanks to (3.4), we obtain

$$\begin{aligned} |z(\tau'; \tau_\alpha, x(\tau_\alpha)) - z(\tau'; \tau_{\alpha+1}, x(\tau_\alpha))| &\leq \sum_{i \in \mathbb{Z}} |z(\tau'; t_i, x(t_i)) - z(\tau'; t_{i+1}, x(t_{i+1}))| \\ &\leq c' \cdot c \cdot (\tau_{\alpha+1} - \tau_\alpha) \cdot 2^{-\nu}. \end{aligned} \tag{3.5}$$

Repeating this computation for every interval $]\tau_\alpha, \tau_{\alpha+1}[$, $\alpha \in \{0, \dots, N\}$, we get

$$\begin{aligned} |z(\tau'; \tau, x(\tau)) - x(\tau')| &\leq \sum_{\alpha=0}^{N} |z(\tau'; \tau_\alpha, x(\tau_\alpha)) - z(\tau'; \tau_{\alpha+1}, x(\tau_\alpha))| \\ &\leq c' \cdot c \cdot (\tau' - \tau) \cdot 2^{-\nu}. \end{aligned} \tag{3.6}$$

Clearly, one obtains the same type of estimate as (3.6) for the other rarefaction front $y(t)$, i.e., there holds

$$|z(\tau'; \tau, y(\tau)) - y(\tau')| \leq c' \cdot c \cdot (\tau' - \tau) \cdot 2^{-\nu}. \tag{3.7}$$

On the other hand, by (2.3), we have

$$|z(\tau'; \tau, x(\tau)) - z(\tau'; \tau, y(\tau))| \leq |x(\tau) - y(\tau)| + 2\lambda^{\max} \cdot (\tau' - \tau). \tag{3.8}$$

Thus, (3.6)–(3.8) together yield (3.1), concluding the proof. ∎

In the following, in connection with any (right continuous) piecewise constant map $\psi : [a, b] \mapsto 2^{-\nu}\mathbb{Z}$, we will let $\pi(\psi) = \{x_0 = a < x_1 < \cdots < x_{\overline{\ell}} = b\}$ denote the partition of $[a, b]$ induced by ψ, in the sense that $\psi(x)$ is constant on every interval $[x_\ell, x_{\ell+1}[$, $0 \leq \ell < \overline{\ell}$. Then, given $\rho > 0$, for any $\nu \geq 1$, consider the set of piecewise constant maps

$$K_\nu^\rho \doteq \left\{ \varphi : [a, b] \mapsto \Gamma^\nu ; \begin{array}{l} \dfrac{w_i(\varphi(x_k)) - w_i(\varphi(x_h))}{x_k - x_h} \leq \dfrac{5\rho}{x_h - a} \begin{cases} \text{for} & \begin{array}{l} a < x_h < x_k < b, \\ x_h, x_k \in \pi(w_i \circ \varphi), \end{array} \\ \text{if } i \in \{p+1, \dots, n\} \end{cases} \\ \dfrac{w_i(\varphi(x_k)) - w_i(\varphi(x_h))}{x_k - x_h} \leq \dfrac{5\rho}{b - x_k} \begin{cases} \text{for} & \begin{array}{l} a < x_h < x_k < b, \\ x_h, x_k \in \pi(w_i \circ \varphi), \end{array} \\ \text{if } i \in \{1, \dots, p\} \end{cases} \end{array} \right\}. \tag{3.9}$$

The next lemma shows that we can approximate in $\mathbf{L}^1$ any map $\varphi \in K^\rho$ with a piecewise constant function $\varphi_\nu \in K_\nu^\rho$.

LEMMA 3.2. *For any given $\varphi \in K^\rho$, there exists a sequence of right continuous maps $\varphi_\nu \in K_\nu^\rho$, $\nu \geq 1$, such that*

(a) *for every $i \in \{1, \dots, n\}$, and for any $x_h \in \pi(w_i \circ \varphi_\nu)$, there holds*

$$w_i(\varphi_\nu(x_{h+1})) > w_i(\varphi_\nu(x_h)) \Longrightarrow w_i(\varphi(x_{h+1})) = w_i(\varphi(x_h)) + 2^{-\nu}; \tag{3.10}$$

(b) *there holds*

$$\varphi_\nu \to \varphi \qquad in\ \mathbf{L}^1([a,b]). \tag{3.11}$$

Proof. (1) First observe that, by Remark 2.5, any map $x \mapsto w_i(\varphi(x))$, $i \in \{1,\dots,n\}$, has finite total variation on $[a+\varepsilon, b-\varepsilon]$, $\varepsilon > 0$. Hence, we may assume that $w_i(\varphi(\cdot))$ admits left and right limits in any point $x \in]a,b[$ and that $w_i(\varphi(x)) = w_i(\varphi(x^+)) \doteq \lim_{\xi\to x^+} w_i(\varphi(\xi))$ for all $i \in \{1,\dots,n\}$. Let $\{y_{i,m};\ m \in \mathbb{N}\}$ be the countable set of discontinuities of $w_i(\varphi(\cdot))$, $i \in \{1,\dots,n\}$. Then, we can find a partition $\xi^1_{i,m} = y_{i,m} < \xi^2_{i,m} < \cdots < \xi^{\ell_{i,m}}_{i,m} = y_{i,m'}$ of each interval $[y_{i,m}, y_{i,m'}[$ where $x \mapsto w_i(\varphi(x))$ is continuous, so that

(i) for every $1 < \ell < \ell_{i,m}$ there holds

$$w_i(\varphi(\xi^\ell_{i,m})) \in 2^{-\nu}\mathbb{Z}; \tag{3.12}$$

(ii) for every $1 \le \ell < \ell_{i,m}$ one has

$$|w_i(\varphi(x)) - w_i(\varphi(\xi^\ell_{i,m}))| \le 2^{-\nu} \quad \forall x \in [\xi^\ell_{i,m}, \xi^{\ell+1}_{i,m}[. \tag{3.13}$$

Notice that the Oleinik-type conditions stated in the definition of K^ρ imply that, at any discontinuity point $y_{i,m}$ of $w_i(\varphi(\cdot))$, one has

$$\lim_{\xi\to y^-_{i,m}} w_i(\varphi(\xi)) > w_i(\varphi(y_{i,m})). \tag{3.14}$$

(2) Let $\varphi_\nu : [a,b] \mapsto \Gamma^\nu$ be the piecewise constant, right continuous map defined by setting, for every $i \in \{1,\dots,n\}$ and for any interval $[y_{i,m}, y_{i,m'}[$, where $w_i(\varphi(\cdot))$ is continuous,

$$w_i(\varphi_\nu(x)) \doteq \begin{cases} 2^{-\nu}\lfloor 2^\nu w_i(\varphi(\xi^1_{i,m}))\rfloor & \text{if } \begin{cases} x \in [\xi^1_{i,m}, \xi^2_{i,m}[, \text{ and} \\ w_i(\varphi_\nu(\xi^1_{i,m})) \\ \le 2^{-\nu}(\lfloor 2^\nu w_i(\varphi(\xi^1_{i,m}))\rfloor + 2^{-1}), \end{cases} \\ 2^{-\nu}(\lfloor 2^\nu w_i(\varphi(\xi^1_{i,m}))\rfloor + 1) & \text{if } \begin{cases} x \in [\xi^1_{i,m}, \xi^2_{i,m}[, \text{ and} \\ w_i(\varphi_\nu(\xi^1_{i,m})) \\ > 2^{-\nu}(\lfloor 2^\nu w_i(\varphi(\xi^1_{i,m}))\rfloor + 2^{-1}), \end{cases} \\ w_i(\varphi(\xi^\ell_{i,m})) & \text{if } \quad x \in [\xi^\ell_{i,m}, \xi^{\ell+1}_{i,m}[, \quad 1 < \ell < \ell_{i,m}, \end{cases} \tag{3.15}$$

where $\lfloor\cdot\rfloor$ denotes the integer part. Notice that, by construction and because of (3.12)–(3.14), the map $\varphi_\nu : [a,b] \mapsto \Gamma^\nu$ enjoys the following property:

$$\left.\begin{matrix} w_i(\varphi_\nu(x_k)) > w_i(\varphi_\nu(x_h)) \\ x_h < x_k \in \pi(w_i \circ \varphi_\nu) \end{matrix}\right\} \Longrightarrow w_i(\varphi(x_k)) > w_i(\varphi(x_h)) + 2^{-(\nu+1)}. \tag{3.16}$$

Therefore, since $\varphi \in K^\rho$, relying on (3.13), (3.16), we deduce that for every $w_i(\varphi_\nu(\cdot))$, $i \in \{p+1,\dots,n\}$, and for any $x_h < x_k \in \pi(w_i \circ \varphi_\nu)$ such that $w_i(\varphi_\nu(x_k)) > w_i(\varphi_\nu(x_h))$, there holds

$$\begin{aligned} \frac{w_i(\varphi_\nu(x_k)) - w_i(\varphi_\nu(x_h))}{x_k - x_h} &\le \frac{w_i(\varphi(x_k)) - w_i(\varphi(x_h)) + 2^{-(\nu-1)}}{x_k - x_h} \\ &\le \frac{5(w_i(\varphi(x_k)) - w_i(\varphi(x_h)))}{x_k - x_h} \\ &\le \frac{5\rho}{x_h - a}. \end{aligned} \tag{3.17}$$

Clearly, with the same computations we can show that, for every $w_i(\varphi_\nu(\cdot))$, $i \in \{1,\dots,p\}$, and for any $x_h < x_k \in \pi(w_i \circ \varphi_\nu)$, there holds

$$\frac{w_i(\varphi_\nu(x_k)) - w_i(\varphi_\nu(x_h))}{x_k - x_h} \le \frac{5\rho}{b - x_k}. \tag{3.18}$$

The estimates (3.17)–(3.18), together, imply that $\varphi_\nu \in K_\nu^\rho$, while (3.13) yields (3.11). On the other hand, observe that, by construction and because of (3.14), the map φ_ν satisfies condition (3.10), which completes the proof of the lemma. $\square$

We now provide a further estimate on the distance between two rarefaction fronts of a front tracking solution that, at a fixed time τ, attains a profile belonging to the set (3.9).

LEMMA 3.3. *Consider a front tracking solution $u(t,x)$ with values in Γ^ν, $\nu \ge 1$, constructed by the algorithm of section* 2.4 *on the region $[\tau,\tau'] \times [a,b]$. Assume that $u(\tau',\cdot)$ is right-continuous, verifies condition* (a) *of Lemma* 3.2*, and satisfies*

$$u(\tau',\cdot) \in K_\nu^{\rho'}, \qquad \rho' \doteq \frac{\lambda^{\min}}{6C_1}, \tag{3.19}$$

where $\lambda^{\min}$, C_1 are the minimum speed in (2.3) *and the constant of Lemma* 3.1*. Then, given any two adjacent rarefaction fronts of u located at $x(t) \le y(t)$, $t \in [\tau,\tau']$, and belonging to the same family, there holds*

$$x(\tau) < y(\tau). \tag{3.20}$$

Proof. To fix the ideas, assume that $x(t) \le y(t)$ are the locations of two adjacent rarefaction fronts of the $k \in \{p+1,\dots,n\}$th family and hence, by (2.2), have positive speeds. Observe that, by condition (a) of Lemma 3.2, one has

$$w_k(u(\tau',y(\tau'))) - w_k(u(\tau',x(\tau'))) = 2^{-\nu}. \tag{3.21}$$

Moreover, since u is a front tracking solution constructed by the algorithm of section 2.4 on the region $[\tau,\tau'] \times [a,b]$, we can apply Lemma 3.1. Thus, using (2.3), (3.1), and (3.21), and recalling the definition (3.9) of $K_\nu^{\rho'}$, we deduce

$$\begin{aligned} y(\tau') - x(\tau') &\le y(\tau) - x(\tau) + C_1(\tau'-\tau)2^{-\nu} \\ &\le y(\tau) - x(\tau) + C_1 \frac{x(\tau') - x(\tau)}{\lambda^{\min}} \cdot \big(w_k(\varphi_\nu(y(\tau'))) - w_k(\varphi_\nu(x(\tau')))\big) \\ &\le y(\tau) - x(\tau) + C_1 \frac{5\rho'}{\lambda^{\min}} \cdot (y(\tau') - x(\tau')) \end{aligned}$$

which, because of (3.19), implies

$$y(\tau) - x(\tau) \ge \left(1 - C_1 \frac{5\rho'}{\lambda^{\min}}\right) \cdot (y(\tau') - x(\tau')) > 0,$$

proving (3.20). $\square$

We next derive a regularity property enjoyed by general solutions of Temple systems with boundary variation defined as a limit of front tracking approximations, which allows us to establish Proposition 2.13. This is an extension of the regularity results obtained in [18, 23, 13] for the solution with small total variation of general systems. The arguments of the proof are quite similar to the corresponding result in [13], but we will repeat some of them for completeness, referring to [13] (see also [9, Theorem 10.4]) for further details.

LEMMA 3.4. *Let* (1.1) *be a system of Temple class satisfying the same assumptions as in Theorem* 2.9. *Consider a sequence* $u^\nu(t,\cdot) : [c,d] \mapsto \Gamma^\nu$, $t \in [r,s]$, *of front tracking approximate solutions of the mixed problem for* (1.1) *(constructed by the algorithm of section* 2.4*) that converges in* $\mathbf{L}^1$, *as* $\nu \to \infty$, *to some function* $u(t,\cdot) : [c,d] \mapsto \Gamma$, *for every* $t \in [r,s] \subset \mathbb{R}^+$. *Assume that*

$$\text{(3.22)} \qquad Tot.Var.(u^\nu(t,\cdot)) \le M, \quad Tot.Var.(u^\nu(\cdot,x)) \le M \quad \forall t,x,\nu,$$

for some constant $M > 0$, *here and throughout the following,* $Tot.Var.(w)$ *denotes the total variation of the function* w. *Then, there exist a countable set of interaction points* $\Theta \doteq \{(\tau_l, x_l);\ l \in \mathbb{N}\} \subset D \doteq [r,s] \times [c,d]$, *and a countable family of Lipschitz continuous shock curves* $\Upsilon \doteq \{x = y_m(t);\ t \in]r_m, s_m[,\ m \in \mathbb{N}\}$, *such that the following hold.*

(i) *For each* $m \in \mathbb{N}$, *and for any* $\tau \in]r_m, s_m[$ *with* $(\tau, y_m(\tau)) \notin \Theta$, *there exist the left and right limits* (2.42) *of* u *at* $(\tau, y_m(\tau))$ *and the shock speed* $\dot{y}_m(\tau)$. *Moreover, these limits satisfy the Rankine–Hugoniot relations* (2.43) *and the Lax entropy inequality* (2.44), *for some* $i \in \{1,\dots,n\}$.

(ii) *The map* u *is continuous outside the set* $\Theta \cup \Upsilon$.

Proof. 1. To establish (i) we need to recall some technical tools introduced in [13] (see also [9, Theorem 10.4]). For every front tracking solution u^ν, we define the *interaction and cancellation measure* μ_ν^{IC} that is a positive, purely atomic measure on D, concentrated on the set of points P where two or more wave-fronts of u^ν interact. Namely, if the incoming fronts at P have size $\sigma_1,\dots,\sigma_\ell$ (w.r.t. the Riemann coordinates) and belong to the families $i_1,\dots,i_\ell$, respectively, we set

$$\text{(3.23)} \qquad \mu_\nu^{IC}(P) \doteq \sum_{\alpha,\beta} |\sigma_\alpha \sigma_\beta| + \sum_i \left(\sum_{\{i_\alpha;\, i_\alpha = i\}} |\sigma_\alpha| - \left| \sum_{\{i_\alpha;\, i_\alpha = i\}} \sigma_\alpha \right| \right).$$

Since μ_ν^{IC} have a uniformly bounded total mass, by possibly taking a subsequence we can assume the weak convergence

$$\text{(3.24)} \qquad \mu_\nu^{IC} \rightharpoonup \mu^{IC}$$

for some positive, purely atomic measure μ^{IC} on D. Call Θ the countable set of atoms of μ^{IC}, i.e., set

$$\Theta \doteq \left\{ P \in D;\ \mu^{IC}(P) > 0 \right\}.$$

For every approximate solution u^ν taking values in Γ^ν, $\nu \ge 1$, and for any fixed $\varepsilon \ge 2^{-\nu}$, by an *ε-shock front of the ith family* in u^ν we mean a polygonal line in D, with nodes $(\tau_0, x_0), \dots, (\tau_N, x_N)$, having the following properties.

(I) The nodes (τ_h, x_h) are interaction points or lie on the boundary of D, and the sequence of times is increasing $\tau_0 < \tau_1 < \cdots < \tau_N$.

(II) Along each segment joining (τ_{h-1}, x_{h-1}) with (τ_h, x_h), the function u^ν has an i-shock with strength $|\sigma_h| \ge \varepsilon$.

(III) For $h < N$, if two (or more) incoming i-shocks of strength $\ge \varepsilon$ interact at the node (τ_h, x_h), then the shock coming from (τ_{h-1}, x_{h-1}) has the larger speed, i.e., is the one coming from the left.

An ε-shock front, which is maximal with respect to the set theoretical inclusion, will be called a *maximal ε-shock front*. Observe that, because of (III), two maximal ε-shock fronts of the same family either are disjoint or coincide. Moreover, by (3.22), the number of maximal ε-shock fronts that start at the boundary of D is uniformly bounded by $3M/\varepsilon$. On the other hand, the special geometric features of Temple class systems guarantee that no new shock front can arise in the interior of D. Indeed, the coinciding shock and rarefaction assumption together with the existence of Riemann invariants prevents the creation of shocks of other families than those of the incoming fronts at any interaction point. Therefore, for fixed $\varepsilon > 0$, and $i \in \{1, \dots, n\}$, the number of maximal ε-shock fronts of the ith family remains uniformly bounded by $M_\varepsilon \doteq 3M/\varepsilon$ in all u^ν, $\nu \geq 1$. Denote such curves by

$$y^\varepsilon_{\nu,m} : [t^{\varepsilon,-}_{\nu,m}, t^{\varepsilon,+}_{\nu,m}] \mapsto \mathbb{R}, \qquad m = 1, \dots, M_\varepsilon.$$

By possibly extracting a further subsequence, we can assume the convergence

$$y^\varepsilon_{\nu,m}(\cdot) \longrightarrow y^\varepsilon_m(\cdot), \qquad t^{\varepsilon,\pm}_{\nu,m} \longrightarrow t^{\varepsilon,\pm}_m, \qquad m = 1, \dots, M_\varepsilon,$$

for some Lipschitz continuous paths $y^\varepsilon_m : [t^{\varepsilon,-}_m, t^{\varepsilon,+}_m] \mapsto \mathbb{R},\ m = 1, \dots, M_\varepsilon$. Repeating this construction in connection with a sequence $\varepsilon_k \to 0$, and taking the union of all the paths thus obtained, we find, for each characteristic family $i \in \{1, \dots, n\}$, a countable family of Lipschitz continuous curves $y_m : [t^-_m, t^+_m] \mapsto \mathbb{R},\ m \in \mathbb{N}$. Call Υ the union of all such curves.

2. Consider now a point $P = (\tau, y_m(\tau)) \notin \Theta$ along a curve $y_m \in \Upsilon$ of a family $i \in \{1, \dots, n\}$. Notice that, by construction and because of (3.24), no curve in Υ can cross y_m at P. Moreover, by (3.22), the function $u(\tau, \cdot)$ has bounded variation, and hence there exist the limits

$$\lim_{x \to y_m(\tau)-} u(\tau, x) \doteq u^-, \qquad \lim_{x \to y_m(\tau)+} u(\tau, x) \doteq u^+. \tag{3.25}$$

We claim also that the limits (2.42) exist and thus coincide with those in (3.25). To this end observe that, by construction, there exists a sequence of shocks curves $y_{\nu,m}$ of the ith family converging to y_m, along which each approximate solution u^ν has a jump of strength $\geq \varepsilon^*$, for some $\varepsilon^* > 0$. Then, relying on the assumption

$$\mu^{IC}(\{P\}) = 0 \tag{3.26}$$

and letting $B(P, r)$ denote the ball centered at P with radius r, one can establish the limits

$$\lim_{r \to 0+} \limsup_{\nu \to +\infty} \left(\sup_{(t,x) \in B(P,r) x < y_{\nu,m}(t)} |u^\nu(t,x) - u^-| \right) = 0, \tag{3.27}$$

$$\lim_{r \to 0+} \limsup_{\nu \to +\infty} \left(\sup_{(t,x) \in B(P,r) x > y_{\nu,m}(t)} |u^\nu(t,x) - u^-| \right) = 0, \tag{3.28}$$

which clearly yield (2.42). Indeed, if, for example, (3.27) do not hold, by possibly taking a subsequence we would find $\varepsilon > 0$ and points $P_\nu \doteq (t_\nu, \xi_\nu) \to P$ on the left of $y_{\nu,m}$ such that

$$|u^\nu(t_\nu, \xi_\nu) - u^-| \geq \varepsilon \quad \forall \nu.$$

On the other hand, by the first limit in (3.25), and since $u^\nu(\tau, x) \to u(\tau, x)$ for a.e. $x \in [\alpha, \beta]$, we could also find points $Q_\nu \doteq (\tau, \xi'_\nu) \to P$ on the left of $y_{\nu,m}$ such that

$$u^\nu(\tau, \xi'_\nu) \to u^-, \quad \frac{|\xi_\nu - \xi'_\nu|}{|t_\nu - \tau|} > \lambda^{\max} \quad \forall \nu,$$

where $\lambda^{\max}$ denotes the maximum speed at (2.3). But then, for each solution u^ν, the segment $\overline{P_\nu Q_\nu}$ would be crossed by an amount of waves of strength $\geq \varepsilon$. Hence, by strict hyperbolicity and genuine nonlinearity, this would generate a uniformly positive amount of interaction and cancellation within an arbitrary small neighborhood of P (see [9, Theorem 10.4, Step 5]) which by the definition (3.23) and because of (3.24) contradicts the assumption (3.26).

To complete the proof of (i), observe that, by construction, the states $u^-_{\nu,m}(\tau)$, $u^+_{\nu,m}(\tau)$ to the left and right of the jump in u^ν at $y_{\nu,m}(\tau)$ satisfy the Rankine–Hugoniot conditions. Thus, relying on (3.27)–(3.28) and on the convergence $y_{\nu,m} \to y_\nu$, one deduces (2.43). The proof of (ii) can be established with the same type of arguments (see [9, Theorem 10.4, Step 8]). □

As an immediate consequence of Lemma 3.4, we derive Proposition 2.13, stated in section 2.4.

Proof of Proposition 2.13. Consider a sequence $u^\nu(t, \cdot) : [a, b] \mapsto \Gamma^\nu$ of front tracking approximate solutions of the mixed problem for (1.1) on the region $\Omega_T \doteq [0, T] \times [a, b]$ that converges in $\mathbf{L}^1$, as $\nu \to \infty$, to some function $u(t, \cdot) : [a, b] \mapsto \Gamma$ for every $t \in [0, T]$. Observe that by Theorem 2.9 one can find another sequence $\{v^\nu\}_{\nu \geq 1}$ of approximate solutions of (1.1) on the region Ω_T, whose initial and boundary data have a number of shocks $N_\nu \leq \nu$ for each characteristic family, and such that

$$\|u^\nu(t, \cdot) - v^\nu(t, \cdot)\|_{\mathbf{L}^1([a,b])} \leq 1/\nu \quad \forall t \in [1/\nu, T].$$

Then, thanks to the Oleinik estimates (2.31)–(2.32) and because all v^ν take values in the compact set (2.13), there will be, for every fixed $\varepsilon > 0$, some constant $M_\varepsilon > 0$ such that

$$\begin{aligned} \text{Tot.Var.}\{v^\nu(t, \cdot);\ [a + \varepsilon, b - \varepsilon]\} &\leq M_\varepsilon \quad \forall t \in [\varepsilon, T], \\ \text{Tot.Var.}\{v^\nu(\cdot, x);\ [\varepsilon, T]\} &\leq M_\varepsilon \quad \forall x \in [a + \varepsilon, b - \varepsilon] \end{aligned} \tag{3.29}$$

$\forall \nu \in \mathbb{N}$. Thus, writing Ω_T as the countable union

$$\Omega_T = \cup_k D_k, \quad D_k \doteq [1/k, T] \times [a + (1/k), b - (1/k)],$$

and applying Lemma 3.4 to each sequence of maps $v^\nu_k \doteq v^\nu \restriction_{D_k}, \nu \geq 1$, defined as the restriction of v^ν to the domain D_k, we clearly reach the conclusion of Proposition 2.13. □

We are now in position to establish Theorem 2.11, relying on Proposition 2.13 and Theorem 2.10.

Proof of Theorem 2.11. Let $u^\nu(t, \cdot) : [a, b] \mapsto \Gamma^\nu$ be a sequence of front tracking approximate solutions of the mixed problem for (1.1) on the region $\Omega_T \doteq [0, T] \times [a, b]$ that converges in $\mathbf{L}^1$, as $\nu \to \infty$, to some function $u(t, \cdot) : [a, b] \mapsto \Gamma$ for every $t \in [0, T]$. Since, by construction, each u^ν is a weak solution of (1.1) and because $u^\nu(0, \cdot) \to u(0, \cdot) = \overline{u}$, the limit function u also is a weak solution of the Cauchy problem (1.1)–(1.2) on the region Ω_T. Moreover, applying Proposition 2.13, we deduce that u admits at $t = 0$ and at $x = a$, $x = b$ the left and right limits stated in Theorem 2.11. On the other hand, by the same arguments used in the proof of Proposition 2.13, we may assume that the initial and boundary data of each approximate solution u^ν have at most $N_\nu \leq \nu$ shocks for every characteristic family. Then, letting $\nu \to \infty$ in (2.31)–(2.32), by the lower semicontinuity of the total variation we find that u satisfies the entropy conditions (2.7)–(2.10) on the decay of positive waves. It follows that u is an entropy weak solution of the mixed problem (1.1)–(1.4) according to Definition 2.2. Hence, observing that by construction the map $(t, x) \to (u(t, \cdot), u(\cdot, x))$ takes values within the domain $\mathcal{D}_T$ defined in (2.37), and applying Theorem 2.10, we deduce that (2.41) is verified. ∎

4. Proof of Theorems 2.4–2.7.

Proof of Theorem 2.4. We shall first prove that, for every fixed $\overline{\tau} > 0$, there exists some constant $\rho = \rho(\overline{\tau}) > 0$ so that (2.17) holds. Given $\widetilde{u}_a \in \mathcal{U}_\tau^\infty$, $\widetilde{u}_b \in \mathcal{U}_\tau^\infty$, $\tau \geq \overline{\tau}$, let $u = u(t, x)$ be an entropy weak solution of (1.1)–(1.4) on the region $[0, \tau] \times [a, b]$ according to Definition 2.2. Then, the Oleinik-type estimates (2.8) on the decay of positive waves imply that, for $i \in \{p+1, \ldots, n\}$, $\tau \geq \overline{\tau}$, and for a.e. $a < x < y < b$, there holds

$$\begin{aligned} \frac{w_i(\tau, y) - w_i(\tau, x)}{y - x} &\leq C \cdot \left\{ \frac{y - x}{\tau} + \log\left(\frac{y - a}{x - a}\right)\right\} \\ &\leq (b - a)C \cdot \left\{\frac{1}{\overline{\tau}} + \frac{1}{x - a}\right\} \\ &\leq \frac{C(b - a)((b - a) + \overline{\tau})}{\overline{\tau}} \cdot \frac{1}{x - a}. \end{aligned} \tag{4.1}$$

Clearly, with the same computations, relying on the Oleinik-type estimates (2.7), we deduce that, for $i \in \{1, \ldots, p\}$, $\tau \geq \overline{\tau}$, and for a.e. $a < x < y < b$, there holds

$$\frac{w_i(\tau, y) - w_i(\tau, x)}{y - x} \leq \frac{C(b - a)((b - a) + \overline{\tau})}{\overline{\tau}} \cdot \frac{1}{b - y}. \tag{4.2}$$

Hence, taking

$$\rho \geq \frac{C(b - a)((b - a) + \overline{\tau})}{\overline{\tau}} \tag{4.3}$$

from (4.1)–(4.2), we derive $u(\tau, \cdot) \in K^\rho$, which proves (2.17).

Concerning the second statement of the theorem, we will show that, letting $\lambda^{\min}$ and ρ' be the minimum speed in (2.3) and the constant (3.19) of Lemma 3.1 and taking

$$T \doteq \frac{4(b - a)}{\lambda^{\min}}, \tag{4.4}$$

the relation (2.18) is verified, i.e., given $\varphi \in K^{\rho'}$ and $\tau > T$, there exist $\widetilde{u}_a \in \mathcal{U}_\tau^\infty$, $\widetilde{u}_b \in \mathcal{U}_\tau^\infty$, and a solution $u(t, x)$ of (1.1)–(1.4) on $[0, \tau] \times [a, b]$ (according to Definition 2.2), such that $u(\tau, \cdot) \equiv \varphi$. Notice that, by Remark 2.5, we may assume that $w_i(\varphi(x))$ admits left and right limits in any point $x \in]a, b[$ and that $w_i(\varphi(x)) = w_i(\varphi(x^+)) \doteq \lim_{\xi \to x^+} w_i(\varphi(\xi))$ for all $i \in \{1, \ldots, n\}$. The proof is divided into two steps.

Step 1. *Backward construction of front tracking approximations.* Letting $\rho' > 0$ be the constant in (3.19), consider a sequence $\{\varphi_\nu\}_{\nu \geq 1}$ of (right continuous) piecewise constant maps in $K_\nu^{\rho'}$, satisfying the conditions (a) and (b) of Lemma 3.2, and take a piecewise constant approximation $\overline{u}^\nu : [a,b] \mapsto \Gamma^\nu$ of the initial data $\overline{u}$, so that $\overline{u}^\nu \to \overline{u}$ in $\mathbf{L}^1$. Given $\tau > T$ (T being the time defined in (4.4)), for each $\nu \geq 1$, we will construct here a front tracking solution $u^\nu(t,x)$ of (1.1) on the region $[0,\tau]\times[a,b]$, with initial data $u^\nu(0,\cdot) = \overline{u}^\nu$, so that

$$u^\nu(\tau,\cdot) = \varphi_\nu. \tag{4.5}$$

This goal is accomplished by proving the following two lemmas.

LEMMA 4.1. *Let $T, \rho' > 0$ be the constants in* (4.4) *and* (3.19). *Then, for every (right continuous) $\varphi_\nu \in K_\nu^{\rho'}$, $\nu \geq 1$, satisfying condition* (a) *of Lemma* 3.2 *and for any $\tau > T$, there exists a front tracking solution $u^\nu(t,x)$ of* (1.1) *on the region $[(3/4)T,\tau]\times[a,b]$, with boundary data $\widetilde{u}_a^\nu \doteq u^\nu(\cdot,a)$, $\widetilde{u}_b^\nu \doteq u^\nu(\cdot,b) \in \mathbf{L}^\infty([(3/4)T,\tau],\Gamma^\nu)$, so that*

$$u^\nu((3/4)T,x) \equiv \omega, \quad u^\nu(\tau,x) = \varphi_\nu(x) \quad \forall x \in [a,b] \tag{4.6}$$

for some constant state $\omega \in \Gamma^\nu$.

Proof. Given $\tau > T$ and $\varphi_\nu \in K_\nu^{\rho'}$, $\nu \geq 1$, satisfying condition (a) of Lemma 3.2, we will use the algorithm described in section 2.4 to construct backward in time a front tracking solution that takes value φ_ν at time τ. To this end, we first observe that according to the algorithm of section 2.4, we can always construct the backward solution of a Riemann problem with terminal data

$$u(t,x) = \begin{cases} u^L & \text{if } x < \xi, \\ u^R & \text{if } x > \xi \end{cases} \tag{4.7}$$

if the terminal states $u^L, u^R \in \Gamma^\nu$ have Riemann coordinates

$$w(u^L) \doteq w^L = (w_1^L,\ldots,w_n^L), \qquad w(u^R) \doteq w^R = (w_1^R,\ldots,w_n^R)$$

that satisfy

$$w_i^L < w_i^R \Longrightarrow w_i^R = w_i^L + 2^{-\nu} \quad \forall i. \tag{4.8}$$

Indeed, if we consider the intermediate states

$$z^i = \begin{cases} u^L & \text{if } i = 0, \\ u(w_1^L,\ldots,w_{n-i}^L,w_{n-i+1}^R,\ldots,w_n^R) & \text{if } 0 < i < n, \\ u^R & \text{if } i = n, \end{cases} \tag{4.9}$$

we realize that, because of (4.8), the solution of every Riemann problem with initial states (z^{i-1},z^i) (defined as in section 2.4) contains only a single front. Thus, we can construct the solution to the Riemann problem with terminal data (4.7) in a backward neighborhood of (t,ξ) by piecing together the solutions to the simple Riemann problems (z^{i-1},z^i), $i = 1,\ldots,n$.

A front tracking solution u^ν can now be constructed backward in time starting at $t = \tau$ and piecing together the backward solutions of the Riemann problems determined by the jumps in φ_ν. The resulting piecewise constant function $u^\nu(\tau-,\cdot)$ is then prolonged for $t < \tau$ tracing backward the incoming fronts at $t = \tau$, up to the first time $\tau_1 < \tau$ at which two or more discontinuities cross in the interior of Ω. Observe that, since u^ν is a front tracking solution constructed by the algorithm of section 2.4 on the region $[\tau_1,\tau]\times[a,b]$, we can apply Lemma 3.3. Hence, it follows that the left and

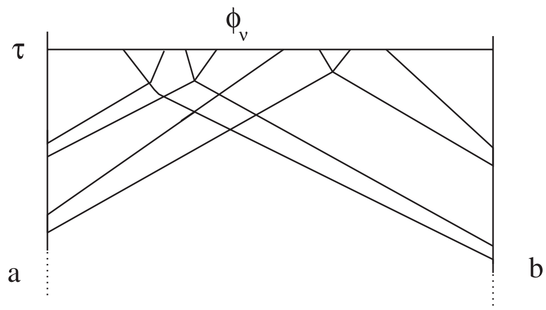

FIG. 1.

right states of the jumps occurring in $u^\nu(\tau_1, \cdot)$ satisfy condition (4.8), because (3.20) guarantees that two (or more) adjacent rarefaction fronts of the same family cannot cross at time τ_1. We then solve backward the resulting Riemann problems applying again the above procedure. This determines the solution $u^\nu(t, \cdot)$ until the time $\tau_2 < \tau_1$ at which another intersection between its fronts takes place in the interior of Ω, and so on (see Figure 1).

With this construction we define a front tracking solution $u^\nu(t, x)$ on the whole region $[(3/4)T, \tau] \times [a, b]$ that verifies the first equality in (4.6) and corresponds to the boundary data $\widetilde{u}_a^\nu \doteq u^\nu(\cdot, a)$, $\widetilde{u}_b^\nu \doteq u^\nu(\cdot, b) \in \mathbf{L}^\infty([(3/4)T, \tau], \Gamma^\nu)$. Clearly, the total number of wave-fronts in $u^\nu(t, \cdot)$ decreases, as $t \downarrow (3/4)T$, whenever a (backward) front crosses the boundary points $x = a$, $x = b$. Since (2.3) implies that the maximum time taken by fronts of u^ν to cross the interval $[a, b]$ is $(b - a)/\lambda^{\min}$, the definition (4.4) of T guarantees that all the (backward) fronts of u^ν will hit the boundaries $x = a$, $x = b$ within some time $\tau' \in](3/4)T, \tau[$, which shows also that the second equality in (4.6) is verified, thus completing the proof. ◻

LEMMA 4.2. *Let $T > 0$ be the constant in* (4.4). *Then, for any piecewise constant function $\overline{u}^\nu \in \mathbf{L}^\infty([a, b], \Gamma^\nu)$ and for every state $\omega \in \Gamma^\nu$, there exists a front tracking solution $u^\nu(t, x)$ of* (1.1) *on the region $[0, (3/4)T] \times [a, b]$, corresponding to some boundary data $\widetilde{u}_a^\nu, \widetilde{u}_b^\nu \in \mathbf{L}^\infty([0, (3/4)T], \Gamma^\nu)$, so that*

$$u^\nu(0, x) = \overline{u}^\nu(x), \quad u^\nu((3/4)T, x) \equiv \omega \quad \forall x \in [a, b]. \tag{4.10}$$

Proof. The approximate solution u^ν is constructed as follows. By Remark 2.8, for $t \in [0, T/4]$, we can define $u^\nu(t, x)$ as the restriction to the region $[0, T/4] \times [a, b]$ of the front tracking solution to the Cauchy problem for (1.1), with initial data

$$\overline{u}(x) = \begin{cases} \overline{u}^\nu(a+) & \text{if } x < a, \\ \overline{u}^\nu(x) & \text{if } a \le x \le b, \\ \overline{u}^\nu(b-) & \text{if } x > b \end{cases}$$

(constructed as in [12] with the same type of algorithm described in section 2.4). Observe that, since u^ν contains only fronts originated at the points of the segment $\{(0, x);\ x \in [a, b]\}$, because of (2.3), (4.4), these wave-fronts cross the whole interval $[a, b]$ and exit from the boundaries $x = a$, $x = b$ before time $T/4$ (see Figure 2). Hence, there will be some state $\omega' \in \Gamma^\nu$ such that

$$u^\nu(T/4, x) \equiv \omega' \quad \forall x \in [a, b]. \tag{4.11}$$

Thus, introducing the intermediate state

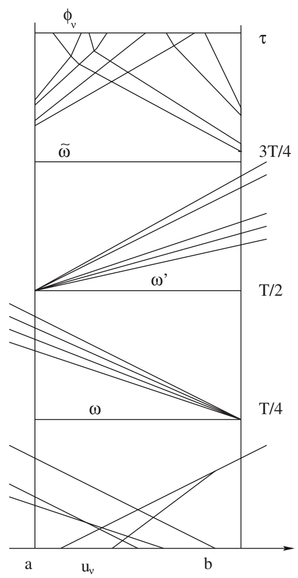

FIG. 2.

$$\widetilde{\omega} \doteq (\omega_1, \ldots, \omega_p, \omega'_{p+1}, \ldots, \omega'_n)$$

between ω' and ω, we will define $u^\nu(t,x)$, for $t \in [T/4, T/2]$, as the restriction to the region $[T/4, T/2] \times [a,b]$ of the approximate solution to the Riemann problem for (1.1), with initial data

$$u^\nu(T/4, x) = \begin{cases} u(\omega') & \text{if } x < b, \\ u(\widetilde{\omega}) & \text{if } x > b, \end{cases} \tag{4.12}$$

while, for $t \in [T/2, (3/4)T]$, we will let $u^\nu(t,x)$ be the restriction to the region $[T/2, (3/4)T] \times [a,b]$ of the approximate solution to the Riemann problem for (1.1), with initial data

$$u^\nu(T/2, x) = \begin{cases} u(\omega) & \text{if } x < a, \\ u(\widetilde{\omega}) & \text{if } x > a. \end{cases} \tag{4.13}$$

By the definition of $\widetilde{\omega}$, and because of (2.3), (4.4), on $[T/4, T/2]$ the solution of the Riemann problems with initial data (4.12) contains only wave-fronts originated at the point $(T/4, b)$ that cross the whole interval $[a,b]$ and exit from the boundary $x = a$ before time $T/2$. Similarly, still by (2.3), (4.4), for $t \in [T/2, (3/4)T]$ the solution of the Riemann problem with initial data (4.13) contains only wave-fronts originated at $(T/2, a)$ that cross the whole interval $[a,b]$ and exit from the boundary $x = b$ before time $(3/4)T$ (see Figure 2). Hence, $u^\nu(t,x)$ is a front tracking solution defined on the whole region $[0, (3/4)T] \times [a,b]$ that corresponds to the boundary data $\widetilde{u}^\nu_a \doteq u^\nu(\cdot, a)$, $\widetilde{u}^\nu_b \doteq u^\nu(\cdot, b) \in \mathbf{L}^\infty([0, (3/4)T], \Gamma^\nu)$, and verifies the conditions (4.10). □

Step 2. *Convergence of the approximate solutions.* By Step 1, for a given $\varphi \in K^{\rho'}$ (with ρ' as in (3.19)), we have found a sequence of initial data $\overline{u}^\nu$ and boundary data $\widetilde{u}^\nu_a, \widetilde{u}^\nu_b \in \mathcal{U}^\infty_\tau$, so that, letting $u^\nu(\tau, \cdot) \doteq E^\nu_\tau(\overline{u}^\nu, \widetilde{u}^\nu_a, \widetilde{u}^\nu_b)$ be the corresponding front tracking solution, there holds

$$\overline{u}^\nu \to \overline{u}, \qquad u^\nu(\tau, \cdot) \to \varphi \quad \text{in } \mathbf{L}^1([a,b]). \tag{4.14}$$

By the same arguments used in the proof of Proposition 2.13, we may assume that the initial and boundary data of each approximate solution u^ν have at most $N_\nu \le \nu$ shocks for every characteristic family. Then, thanks to the Oleinik-type estimates (2.31) and because u^ν are uniformly bounded since they take values in the compact set (2.13), for every fixed $\varepsilon > 0$, there will be some constant $C_\varepsilon > 0$ such that

$$\begin{aligned} &\text{Tot.Var.}\{u^\nu(t,\cdot);\ [a+\varepsilon, b-\varepsilon]\} \le C_\varepsilon \quad \forall t \in [\varepsilon, \tau], \\ &\int_{a+\varepsilon}^{b-\varepsilon} |u^\nu(t,x) - u^\nu(s,x)|\ dx \le C_\varepsilon |t-s| \quad \forall t,s \in [\varepsilon, \tau] \end{aligned} \tag{4.15}$$

$\forall \nu \in \mathbb{N}$. Hence, applying Helly's theorem, we deduce that there exists a subsequence $\{u^{\nu_j}\}_{j\ge 0}$ that converges in $\mathbf{L}^1([a,b],\Gamma)$ to some function $u_\varepsilon(t,\cdot)$, for any $t \in [\varepsilon, \tau]$. Therefore, repeating the same construction in connection with a sequence $\varepsilon_k \to 0+$ and using a diagonal procedure, we obtain a subsequence $\{u^{\nu'}(t,\cdot)\}_{\nu'\ge 0}$ that converges in $\mathbf{L}^1([a,b],\Gamma)$ to some function $u(t,\cdot)$ for any $t \in [0,\tau]$. Then, by Theorem 2.11, there holds (2.41), with $\widetilde{u}_a \doteq u(\cdot,a), \widetilde{u}_b \doteq u(\cdot,b) \in \mathcal{U}^\infty_\tau$, while (4.14) implies $u(\tau,\cdot) = \varphi$, which shows $\varphi \in \mathcal{A}(\tau)$. This completes the proof of Theorem 2.4. ∎

We next establish the compactness of the attainable set (2.14) stated in Theorem 2.7. The proof is quite similar to that of [3, Theorem 2.3]. We repeat it for completeness.

Proof of Theorem 2.7. Fix $T > 0$, and consider a sequence $\{u^\nu\}_{\nu\ge 0}$ of entropy weak solutions to the mixed problem for (1.1) on $\Omega_T \doteq [0,T] \times [a,b]$ (according to Definition 2.2), with fixed initial data $\overline{u} \in \mathbf{L}^\infty([a,b],\Gamma)$. Since all u^ν are uniformly bounded and because of the Oleinik-type estimates (2.7)–(2.8), one can find, for every $\varepsilon > 0$, some constant $C_\varepsilon > 0$ so that (4.15) holds. Thus, with the same arguments used in Step 2 of the previous proof, we can construct a subsequence $\{u^{\nu'}\}_{\nu'\ge 0}$ so that, for any $t \in [0,T]$, $u^{\nu'}(t,\cdot)$ converges in $\mathbf{L}^1$ to some function $u(t,\cdot)$, which is continuous as a map from $]0,T]$ into $\mathbf{L}^1([a,b],\Gamma)$ and satisfies the entropy conditions (2.7)–(2.10) on the decay of positive waves. On the other hand, the weak traces $\Psi_a^{\nu'}, \Psi_b^{\nu'}$ of the fluxes $f(u^{\nu'})$ at the boundaries $x = a$, $x = b$ are uniformly bounded, and hence are weak* relatively compact in $\mathbf{L}^\infty([0,T])$. Thus, by possibly taking a further subsequence, we have

$$\Psi_a^{\nu'} \overset{*}{\rightharpoonup} \Psi_a, \qquad \Psi_b^{\nu'} \overset{*}{\rightharpoonup} \Psi_b \quad \text{in } \mathbf{L}^\infty([0,T]) \tag{4.16}$$

for some maps $\Psi_a, \Psi_b \in \mathbf{L}^\infty([0,T])$. Notice that, by the properties of the Riemann invariants, the set $f(\Gamma)$ is closed and convex, and hence also the weak limits Ψ_a, Ψ_b take values in $f(\Gamma)$. Moreover, since each u^ν is a distributional solution of (1.1)–(1.2) on Ω_T, the limit function u also is a distributional solution of the Cauchy problem (1.1)–(1.2) on the region Ω_T. Then, setting $\widetilde{u}_a \doteq f^{-1} \circ \Psi_a, \widetilde{u}_b \doteq f^{-1} \circ \Psi_b$, it follows that u is an entropy weak solution of the mixed problem (1.1)–(1.4) (with boundary data in $\mathcal{U}^\infty_T$) according to Definition 2.2, which shows that $u(T,\cdot) \in \mathcal{A}(T)$. This completes the proof of Theorem 2.7. ∎

5. Conclusion. The results presented in this paper represent a contribution to the development of a general theory on boundary controllability for systems of nonlinear hyperbolic equations within the context of entropy weak solutions. As is shown in [11] there is no hope of establishing exact controllability results for general systems of conservation laws due to the wave-front structure of the weak solutions, which may present shock waves that can never be canceled by interactions with rarefaction waves of the same characteristic family and that at the same time give rise to new shock fronts by interacting with shock waves of the other characteristic families. Here we have analyzed the exact boundary controllability for the simplest class of nonlinear hyperbolic systems: the class of Temple systems with genuinely nonlinear characteristic fields, whose study is motivated by applications to multicomponent chromatography.

A natural direction in which to pursue this analysis is to consider Temple systems with linearly degenerate characteristic fields (or with a general "nonconvex" flux) which appear in several traffic flow models [6, 16] where one is usually interested in controlling the inflow of cars at the entry of a given road. Another relevant direction worthy of investigation is the controllability of systems of balance laws, i.e., of systems of conservation laws with the presence of source terms. Systems of balance laws belonging to Temple class arise, for example, in modeling chromatography reactors where chemical reactions take place allowing the different solutes (dissolved in the liquid) to transform into each other (see [25, 26]).

All of this type of analysis refers to the case of boundary control problems where total control on the boundary values is available. Of course one may consider more general controllability problems where the control acts only on some of the boundary conditions. For example, we may consider the system of isentropic gas dynamics describing a gas confined in a cylinder within two pistons. In this case it is reasonable to expect that, by controlling only the speed of one piston, it is possible to asymptotically stabilize the system at any constant state. To this purpose it is natural to study first the boundary controllability of the linearized system. A generic condition that guarantees the exact boundary controllability in finite time of a linear hyperbolic system with constant coefficients is obtained in [2].

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