Continuous Control-Lyapunov Functions for Asymptotically Controllable Time-Varying Systems

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Abstract

This paper shows that, for time varying systems, global asymptotic controllability to a given closed subset of the state space is equivalent to the existence of a continuous control-Lyapunov function with respect to the set.

1 Introduction

We will study continuous-time systems with dynamics given by differential equations of the type:

$$\dot{x}(t) = f(t, x(t), u(t)), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ represents the state variable and $u(t) \in U$ is the control variable. (Technical assumptions on f and U are described below.)

We are interested in questions of stabilization relative to a subset $\mathcal{A}$ of the state space $\mathbb{R}^n$. For example, the set $\mathcal{A}$ may be just an equilibrium point, or it may represent a target subset of a different kind. This target set might be a desired periodic orbit, or, in the context of designing observers, the Equations (1) might represent a composite state $x = (x_1, x_2)$, consisting of the state x_1 of the original system together with the state x_2 of an observer; in that case, $\mathcal{A}$ would be the set of states x for which identification has been achieved, that is, the set consisting of those x for which $x_1 = x_2$. (Note that in this last example, the set $\mathcal{A}$ is not bounded.) There are several motivations for considering time-varying dynamics. For instance,

*Supported in part by US Air Force Grant F49620-98-1-0242

in this manner one may encompass problems of tracking, in which the difference between some variables and a signal to be followed evolves according to a differential equation which depends explicitly in t. The purpose of this paper is to generalize the results of the paper [9], which dealt only with $\mathcal{A} = \{0\}$ and time-invariant systems, to the general model (1).

A classical technique for stabilization is to look for Lyapunov-type functions, which play the role of abstract "energy" or "cost" functions that can be made to decrease in directions corresponding to possible controls, as long as the state is away from $\mathcal{A}$. For smooth such functions V, and taking for simplicity the case when V and f are time-invariant, and $\mathcal{A} = \{0\}$, this amounts to the requirement that, for each nonzero state x, there must be some control value $u = u_x$ so that $\nabla V(x).f(x,u) < 0$. One uses the generic term "control-Lyapunov function" (clf) for such a function. The clf approach, assuming a V has been found, allows the search for stabilizing inputs by iteratively solving a static nonlinear programming problem: when at state x, find u such that this inequality holds. Recent expositions of results about (smooth) clf's can be found in the textbooks [5, 6, 7, 10]. An obvious question is whether the existence of a continuously differentiable clf is equivalent to the possibility of driving every state asymptotically to the set $\mathcal{A}$ (zero in this particular case). The answer is negative; for instance, if controls are in $\mathbb{R}^m$, and $f(x,u) = f_0 + \sum_{i=1}^m u_i f_i(x)$ is affine in u, the existence of a (smooth) V would imply that there is some feedback law $u{=}k(x)$ so that the origin is a globally asymptotically stable state for the closed-loop system $\dot{x} = f(x,k(x))$ and k is continuous on $\mathbb{R}^n \setminus \{0\}$. This was proved by Artstein in [2]; see also [10] for an exposition. But continuous feedback may fail to exist, even for very simple controllable systems (see e.g. [10], Section 5.9). Thus smooth clf's do not always exist, even if the system is asymptotically controllable.

However, it was shown in [9] that *continuous* clf's do exist. Of course, one must modify the statement of the clf condition, since the gradient is not well-defined if V is not differentiable. This modification can be done in various ways. Here, we proceed as in [9], asking basically that that for each state $x \neq 0$ (or rather, for each state not in $\mathcal{A}$, in the general case) there exist a trajectory which is defined on a small interval of time and which decreases the value of V. (An additional technical condition, insuring that controls do not "blow up" as one approaches the set $\mathcal{A}$, is also imposed.)

The proofs are in general based on the ideas in [9], but when dealing with time varying systems, and especially with possibly non-compact sets $\mathcal{A}$, many technical complications arise. (Of course, one must allow now for time-dependent V, and the definition of asymptotic controllability must be in some sense uniform on time, in order to obtain a necessary and sufficient result.) Although not at all surprising, the results in this paper are relevant in so far the elucidation of the precise technical assumptions and constructions needed in the generalization are concerned. We also employ several ideas from the paper [8], which dealt with set stability but only for systems with no controls (which allows constructing *smooth* V's).

The existence of continuous clf's is a basic ingredient in the construction of stabilizing feedbacks with respect to the origin for time-invariant systems, as done in [3] and in [11]. We expect to obtain generalizations of these stabilization results, to time varying systems and general attractor sets, in future work, using the result from this paper in the same role that [9] is used in [3].

2 Basic Definitions and Main Results

We take U to be a locally compact metric space, with a distinguished "0" element. We denote $U_r = \{u \,|\, d(u,0) \le r\}$. The map f is supposed to be measurable in t, and locally Lipschitz in (x,u) uniformly for t in a bounded interval. The set of control maps $\mathbf{u} \in \mathbf{U}$ are measurable essentially bounded functions $\mathbf{u} : [t_0, \infty) \to U$, with $t_0 \in \mathbb{R}$. We denote by $\|\mathbf{u}\|$ the essential supremum of the map $\mathbf{u}$. Notice that with these assumptions we guarantee the local existence and uniqueness of solutions of (1). For a given $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, and a control map $\mathbf{u} : [t_0, \infty) \to U$, we denote by $x(t; t_0, x_0, \mathbf{u})$ the maximal solution of (1) with initial condition $x(t_0, t_0, x_0, \mathbf{u}) = x_0$. In general this solution will be defined on an interval of the form $[t_0, \bar{t})$.

A function $\gamma : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is called a $\mathcal{K}$-function if it is continuous, strictly increasing, and $\gamma(0) = 0$; a $\mathcal{K}$-map γ is called a $\mathcal{K}_\infty$-function if $\lim_{s\to\infty} \gamma(s) = \infty$. A function $\beta : \mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is said to be a $\mathcal{KL}$-function if for each fixed $t \ge 0$ the map $\beta(\cdot, t)$ is a $\mathcal{K}$-function, and if for each fixed s the map $\beta(s, \cdot)$ is decreasing to zero as $t \to \infty$. Given a non-empty closed set $\mathcal{A} \subset \mathbb{R}^n$, we denote by $|x|_{\mathcal{A}}$ the distance from x to $\mathcal{A}$.

Definition 2.1 Given a non-empty closed set $\mathcal{A} \subset \mathbb{R}^n$, we say that $\mathcal{A}$ is *weakly invariant* if there exists a positive constant μ such that for all $x_0 \in \mathcal{A}$ and all $t_0 \in \mathbb{R}$, there exists a control map $\mathbf{u}_0$ with $\|\mathbf{u}_0\| \le \mu$ such that $x(t; t_0, x_0, \mathbf{u}_0)$ is defined and lies in $\mathcal{A}$ for all $t \ge t_0$.

Definition 2.2 Let $\mathcal{A} \subset \mathbb{R}^n$ be a closed, weakly invariant, and nonempty set. We say that (1) is *globally asymptotically controllable* (GAC) *to* $\mathcal{A}$ if there exist a $\mathcal{KL}$-function $\beta(\cdot,\cdot)$, and a continuous, positive and increasing function $\gamma(\cdot)$ such that: for each $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ there exists a control function $\mathbf{u} : [t_0, \infty) \to U$, with $\|\mathbf{u}\| \le \gamma(|x_0|_{\mathcal{A}})$, such that the corresponding trajectory $x(t) = x(t, t_0, x_0, \mathbf{u})$ exists for all $t \ge t_0$ and satisfies:

$$|x(t)|_{\mathcal{A}} \;\le\; \beta(|x_0|_{\mathcal{A}}, t - t_0) \qquad \text{for all } \; t \ge t_0. \tag{2}$$

In the previous definition, the $\mathcal{KL}$-function $\beta(\cdot,\cdot)$ captures both the stability and the attraction properties of the invariant set $\mathcal{A}$. An equivalent way to define GAC is given by the following proposition, whose proof is postponed to section 4.

Proposition 2.3 Let $\mathcal{A} \subset \mathbb{R}^n$ be a closed, weakly invariant, and nonempty set. The system (1) is GAC to $\mathcal{A}$ if and only if there exists a continuous, positive and increasing function γ_1 such that:

1. there exists a $\mathcal{K}_\infty$-map γ_2 such that for all $\varepsilon > 0$, if $|x_0|_{\mathcal{A}} \leq \gamma_2(\varepsilon)$ and $t_0 \in \mathbb{R}$, then there exists $\mathbf{u} : [t_0, +\infty) \to U$, with $\|\mathbf{u}\| \leq \gamma_1(|x_0|_{\mathcal{A}})$ such that $|x(t, t_0, x_0, \mathbf{u})|_{\mathcal{A}} \leq \varepsilon$ for all $t \geq t_0$;

2. for any r, ε there exists a nonnegative $T \in \mathbb{R}$ such that given $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, with $|x_0|_{\mathcal{A}} \leq r$, then there exists $\mathbf{u} : [t_0, \infty) \to U$ as in 1 such that $|x(t, t_0, x_0, \mathbf{u})|_{\mathcal{A}} \leq \varepsilon$ for all $t \geq T + t_0$.

We now recall some standard notions regarding relaxed controls. The set of relaxed controls $\mathbf{W}$ is the set of measurable functions $\mathbf{w} : [t_0, \infty) \to P(U)$ where $P(U)$ is the set of probability measures on U equipped with the weak topology. We let $\mathbf{W}_r$ denote the set of those relaxed controls $\mathbf{w}$ such that $\mathbf{w}(t)$ has support on U_r for almost all t. An ordinary control $\mathbf{u} \in \mathbf{U}$ can be seen as a relaxed one if $\mathbf{u}(t)$ is identified with the Dirac measure concentrated in $\mathbf{u}(t)$. One defines a topology on $\mathbf{W}$ characterized by weak convergence: $\mathbf{w}_k \to \mathbf{w}$ if and only if

$$\int_T \int_U g(t, u) d\mathbf{w}_k(u) dt \to \int_T \int_U g(t, u) d\mathbf{w}(u) dt$$

for all functions $g : T \times U \to \mathbb{R}$ which are continuous in u, measurable in t, and such that $\max\{|g(t, u)|, u \in U\}$ is integrable on T, where T is a bounded interval. With this topology $\mathbf{U}_r$ is dense in $\mathbf{W}_r$, and $\mathbf{W}_r$ is sequentially compact. We let

$$\|\mathbf{w}\| = \inf\{r \mid \mathbf{w}(t) \in P(U_r) \text{ for a.e. } t\}$$

(notice that for ordinary controls this is the essential supremum). Given $w \in P(U_r)$, one extends f to relaxed controls by defining

$$f(t, x, w) := \int_{U_r} f(t, x, s) dw(s) .$$

As for ordinary controls, given $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, and a relaxed control $\mathbf{w} : [t_0, \infty) \to P(U)$, we denote by $x(t; t_0, x_0, \mathbf{w})$ the solution of (1) with initial condition $x(t_0; t_0, x_0, \mathbf{w}) = x_0$. For more details on relaxed controls see [1].

Definition 2.4 A continuous function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a *control-Lyapunov function* for the model (1) with respect to the closed, weakly invariant, and non-empty set $\mathcal{A}$ if the following two properties hold:

1. There exist two $\mathcal{K}_\infty$-maps α_1, α_2 such that, for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$:

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(t, x) \leq \alpha_2(|x|_{\mathcal{A}}).$$

2. There exist a continuous, positive and increasing map ϕ, and a $\mathcal{K}$-function α_3, such that for each $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, there exist $t_1 > t_0$, and $\mathbf{w} : [t_0, t_1) \to P(U)$ with $\|\mathbf{w}\| \leq \phi(|x_0|_{\mathcal{A}})$ and:

$$\liminf_{t \to t_0^+} \frac{V(t, x(t)) - V(t_0, x_0)}{t - t_0} \leq -\alpha_3(|x_0|_{\mathcal{A}}), \tag{3}$$

where $x(t) = x(t, t_0, x_0, \mathbf{w})$.

An alternative manner to define control-Lyapunov functions is via an integral inequality, as follows.

Definition 2.5 A continuous function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a *control-Lyapunov integral function* for the model (1) with respect to the closed, weakly invariant, and non-empty set $\mathcal{A}$ if property 1 of definition 2.4 holds, and there exist a continuous, positive and increasing map ϕ, and a $\mathcal{K}$-function α_3, such that for each $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ there exists a relaxed control $\mathbf{w} : [t_0, +\infty) \to P(U)$ with $\|\mathbf{w}\| \leq \phi(|x_0|_{\mathcal{A}})$ such that $x(t) = x(t, t_0, x_0, \mathbf{w})$ is defined for all $t \geq t_0$ and it satisfies, for all $t \geq t_0$:

$$V(t, x(t)) - V(t_0, x_0) \leq -\int_{t_0}^{t} \alpha_3(|x(s)|_{\mathcal{A}}) ds. \tag{4}$$

Definition 2.6 Given a closed and non-empty set $\mathcal{A}$, we say that the function $f(t, x, u)$ satisfies the *boundedness assumption* with respect to $\mathcal{A}$ if it holds that, for each $r_1, r_2 > 0$ there exists a positive constant $M_{r_1,r_2} \in \mathbb{R}$ such that:

$$\sup_{\substack{|x|_{\mathcal{A}} \leq r_1 \\ d(u,0) \leq r_2}} |f(t, x, u)| \leq M_{r_1,r_2} \quad \text{a.e.} \quad t \in \mathbb{R}. \tag{5}$$

Notice that, for example, if f is independent of t and $\mathcal{A}$ is compact, then (5) holds.

Theorem 1 *Let Σ be a given model of type (1), and $\mathcal{A} \subset \mathbb{R}^n$ be a closed, weakly invariant, and non-empty set. Assume that the model satisfies the boundedness assumption with respect to $\mathcal{A}$ (definition 2.6). Then the following are equivalent:*

1. *Σ is* GAC *to $\mathcal{A}$.*
2. *There exists a control-Lyapunov function V for (1) with respect to $\mathcal{A}$ (definition 2.4).*
3. *There exists a control-Lyapunov integral function V for (1) with respect to $\mathcal{A}$ (definition 2.5).*

Remark 2.7 Assume that we give a new definition of control-Lyapunov function which is equal to definition 2.4 except from the fact that we use in equation (3) the "limsup" instead of "liminf". Then it follows from Theorem 1 that this new definition is indeed equivalent to the one given in 2.4. To see this it is sufficies to notice that if V is a control-Lyapunov integral function then obviously V is control-Lyapunov function which satisfies inequality (3) even if the "limsup" is used.

The next section is devoted to the proof that 1 and 3 are equivalent. It is obvious that 3 implies 2, while the proof that 2 implies 3 is given in section 4.

3 Control-Lyapunov function characterization

3.1 Sufficiency part

In this section, we assume given a system Σ of type (1), a closed, weakly invariant, and non-empty set $\mathcal{A} \subset \mathbb{R}^n$, and a control-Lyapunov integral function V (together with the maps α_1, α_2, ϕ, and α_3). Our aim is to prove that Σ is GAC to $\mathcal{A}$.

Fix any $\varepsilon_0 > 0$, and construct a decreasing sequence $\{\varepsilon_p\}_{p\in\mathbb{Z}}$ such that $\lim_{p\to+\infty} \varepsilon_p = 0$, $\lim_{p\to-\infty} \varepsilon_p = +\infty$, and for all $p \in \mathbb{Z}$, $2\alpha_1^{-1}(\alpha_2(\varepsilon_{p+1})) < \varepsilon_p$. For each fixed $p \in \mathbb{Z}$, let $t_p > 0$ be defined by:

$$t_p = \frac{\alpha_2(\varepsilon_{p-1}) - \alpha_1(\varepsilon_{p+1})}{\alpha_3(\varepsilon_{p+1})}.$$

Lemma 3.1 Fix any $p \in \mathbb{Z}$. Assume given any (t_0, x_0), with $x_0 \notin \mathcal{A}$. If $|x_0|_{\mathcal{A}} < \varepsilon_{p-1}$, then there exists a relaxed control $\mathbf{w} : [t_0, +\infty) \to P(U)$ with $\|\mathbf{w}\| \leq \phi(|x_0|_{\mathcal{A}})$ such that:

1. $|x(t)|_{\mathcal{A}} \leq \alpha_1^{-1}(\alpha_2(|x_0|_{\mathcal{A}}))$ for all $t \in [t_0, +\infty)$;
2. there exists $\bar{t} \in [t_0, t_0 + t_p]$, such that $|x(\bar{t})|_{\mathcal{A}} < \varepsilon_{p+1}$.

Proof. Our assumption implies in particular that for any (t_0, x_0) there exists a relaxed control $\mathbf{w} : [t_0, +\infty) \to P(U)$ with $\|\mathbf{w}\| \leq \phi(|x_0|_{\mathcal{A}})$ such that:

$$|x(t)|_{\mathcal{A}} \leq \alpha_1^{-1}\left(\alpha_2(|x_0|_{\mathcal{A}}) - \int_{t_0}^{t} \alpha_3(|x(\tau)|_{\mathcal{A}})d\tau\right). \tag{6}$$

For this particular control $\mathbf{w}$ the first requirement of the lemma clearly holds, since α_3 is a positive function. Assume, by the way of contradiction, that for all $t \in [t_0, t_0 + t_p]$, $|x(t)|_{\mathcal{A}} \geq \varepsilon_{p+1}$. Then, since α_3 is an increasing function, we have $\alpha_3(|x(t)|_{\mathcal{A}}) \geq \alpha_3(\varepsilon_{p+1})$. This fact, together with equation (6), implies:

$$|x(t_0 + t_p)|_{\mathcal{A}} \leq \alpha_1^{-1}(\alpha_2(|x_0|_{\mathcal{A}}) - \alpha_3(\varepsilon_{p+1})t_p) < \alpha_1^{-1}(\alpha_2(\varepsilon_{p-1}) - \alpha_3(\varepsilon_{p+1})t_p) = \varepsilon_{p+1},$$

which gives a contradiction. ∎

For each $p \in \mathbb{Z}$, let

$$T(p) = \sum_{k \geq p} t_k.$$

Notice that $T(p) \in \mathbb{R}^{\geq 0} \cup \{+\infty\}$.

Lemma 3.2 Given any (t_0, x_0), with $x_0 \notin \mathcal{A}$. If $|x_0|_{\mathcal{A}} < \varepsilon_{p-1}$, then there exists an ordinary control $\mathbf{u} : [t_0, +\infty) \to U$ with $\|\mathbf{u}\| \leq \phi(|x_0|_{\mathcal{A}}) + \mu$ such that:

for all $k \geq p$, and for all $t \in [t_0 + \sum_{j=p}^{k-1} t_j, t_0 + \sum_{j=p}^{k} t_j)$:

$$|x(t)|_{\mathcal{A}} \leq 2\alpha_1^{-1}(\alpha_2(\varepsilon_{k-1})). \tag{7}$$

Moreover if $T(p) < +\infty$, then for all $t \geq t_0 + T(p)$, we have $x(t) \in \mathcal{A}$.

Proof. Assume that we have already constructed a control $\mathbf{u}(t)$ for $t \in [t_0, t_0 + \sum_{j=p}^{k-1} t_j]$, with $\|\mathbf{u}\| \leq \phi(|x_0|_{\mathcal{A}}) + \mu$ and such that for all $p \leq h \leq k-1$ we have:

(a) $|x(t)|_{\mathcal{A}} \leq 2\alpha_1^{-1}(\alpha_2(\varepsilon_{h-1}))$, for all $t \in [t_0 + \sum_{j=p}^{h-1} t_j, t_0 + \sum_{j=p}^{h} t_j]$,

(b) $\left|x(t_0+\sum_{j=p}^{h} t_j)\right|_{\mathcal{A}} < \varepsilon_h$.

We want to extend this control to the interval $[t_0+\sum_{j=p}^{k-1} t_j, t_0+\sum_{j=p}^{k} t_j]$. Let $\tilde{t}=t_0+\sum_{j=p}^{k-1} t_j$, then by property (b) we have that $\left|x(\tilde{t})\right|_{\mathcal{A}} < \varepsilon_{k-1}$. By applying lemma 3.1 with initial condition $(\tilde{t}, x(\tilde{t}))$, we get a relaxed control $\tilde{\mathbf{w}} : [\tilde{t}, +\infty) \to P(U)$ such that: $\|\tilde{\mathbf{w}}\| \leq \phi(\left|x(\tilde{t})\right|_{\mathcal{A}}) \leq \phi(|x_0|_{\mathcal{A}})$, and, for all $t \geq \tilde{t}$:

$$|x(t)|_{\mathcal{A}} \leq 2\alpha_1^{-1}\left(\alpha_2(\left|x(\tilde{t})\right|_{\mathcal{A}})\right) \leq 2\alpha_1^{-1}\left(\alpha_2(\varepsilon_{k-1})\right).$$

Moreover there exists $\bar{t} \in [\tilde{t}, \tilde{t}+t_k]$ such that $|x(\bar{t})|_{\mathcal{A}} < \varepsilon_{k+1}$. Now, we apply again lemma 3.1 with initial condition $(\bar{t}, x(\bar{t}))$. Since $|x(\bar{t})|_{\mathcal{A}} < \varepsilon_{k+1}$, there exists another relaxed control $\hat{\mathbf{w}} : [\bar{t}, +\infty) \to P(U)$ such that $\|\hat{\mathbf{w}}\| \leq \phi(|x(\bar{t})|_{\mathcal{A}}) \leq \phi(|x_0|_{\mathcal{A}})$, and, for all $t \geq \bar{t}$:

$$|x(t)|_{\mathcal{A}} \leq 2\alpha_1^{-1}\left(\alpha_2(|x(\bar{t})|_{\mathcal{A}})\right) \leq 2\alpha_1^{-1}\left(\alpha_2(\varepsilon_{k+1})\right) < \varepsilon_k,$$

where the last inequality holds by construction. So, in particular, concatenating the two relaxed controls $\tilde{\mathbf{w}}$ and $\hat{\mathbf{w}}$, we get a relaxed control $\mathbf{w} : [\tilde{t}, +\infty) \to P(U)$ such that:

$$\left|x(\tilde{t})\right|_{\mathcal{A}} \leq 2\alpha_1^{-1}\left(\alpha_2(\varepsilon_{k-1})\right), \quad \text{and} \quad \left|x(\tilde{t}+t_k)\right|_{\mathcal{A}} < \varepsilon_k.$$

Now, since $\forall\ r > 0$ $\mathbf{U}_r$ is dense in $\mathbf{W}_r$, it is clear that we can extend the control $\mathbf{u}$ to the interval $[t_0+\sum_{j=p}^{k-1} t_j, t_0+\sum_{j=p}^{k} t_j]$ preserving the required properties.

If $T(p) = +\infty$ we have finished our construction. Assume that $T(p) < +\infty$. Then, since $\|\mathbf{u}\| \leq \phi(|x_0|_{\mathcal{A}}) + \mu$ and $|x(s)|_{\mathcal{A}} \leq 2\alpha_1^{-1}\left(\alpha_2(\varepsilon_{p-1})\right)$, for all $s \in [t_0, t_0+T(p))$, from the boundedness assumption we have that there exists $L > 0$ such that:

$$|f(s, x(s), u(s))| \leq L, \quad \forall\, s \in [t_0, t_0+T(p));$$

which implies, for all $t \in [t_0, t_0+T(p))$:

$$|x(t)| \leq |x_0| + LT(p).$$

So we can extend our trajectory to the endpoint $\hat{t} = t_0 + T(p)$. Moreover, we have:

$$\lim_{t \to t_0+T(p)} |x(t)|_{\mathcal{A}} = 0,$$

which implies that $x(t_0+T(p)) \in \mathcal{A}$. Thus, for $t > t_0+T(p)$ we may extend the control $\mathbf{u}$ by using the control $\mathbf{u}_0$ given by the weakly invariant assumption which is in norm $\leq \mu$. ∎

Lemma 3.3 There exists a $\mathcal{KL}$-function $\beta(\cdot,\cdot)$ such that, for each $p \in \mathbb{Z}$, if $\varepsilon_p \leq s < \varepsilon_{p-1}$, then:

1. $\beta(s, 0) \geq 2\alpha_1^{-1}\left(\alpha_2(\varepsilon_{p-1})\right)$;

2. if $t \in [\sum_{j=p}^{k-1} t_j, \sum_{j=p}^{k} t_j)$, then $\beta(s,t) \geq 2\alpha_1^{-1}\left(\alpha_2(\varepsilon_{k-1})\right)$.

Proof. Let $l_p = 2\alpha_1^{-1}(\alpha_2(\varepsilon_p))$, and, for $k \geq i$, let $s_{i,k} = \sum_{j=i}^{k} t_j$. First, for $i \in \mathbb{Z}$, we let:

$$\tilde{\beta}(\varepsilon_i, t) = \begin{cases} l_{i-1} & t \in [0, t_i) \\ l_{k-1} & t \in [s_{i,k}, s_{i,k+1}) \\ 0 & t \geq T(i) \end{cases}$$

Then clearly $\lim_{t\to+\infty} \tilde{\beta}(\varepsilon_i, t) = 0$, and $\tilde{\beta}(\varepsilon_i, t)$ is decreasing as a function of t. Moreover it holds that, if $\varepsilon_i < \varepsilon_j$ then

$$\tilde{\beta}(\varepsilon_i, t) \leq \tilde{\beta}(\varepsilon_j, t). \tag{8}$$

To prove (8) we argue as follows. If $t \geq T(i)$ then the inequality is obvious. If $t \in [0, t_i)$ then $t < t_j + \ldots + t_i$ thus

$$\tilde{\beta}(\varepsilon_j, t) \geq \tilde{\beta}(\varepsilon_j, s_{j,i}) = l_{i-1} = \tilde{\beta}(\varepsilon_i, t).$$

Otherwise, there exists $k \geq i$ such that $t \in [s_{i,k}, s_{i,k+1})$. In particular $t < s_{j,k+1}$, thus, again we have:

$$\tilde{\beta}(\varepsilon_j, t) \geq \tilde{\beta}(\varepsilon_j, s_{j,k+1}) = l_{k-1} = \tilde{\beta}(\varepsilon_i, t).$$

So (8) is proved. Now we let:

$$\tilde{\beta}(s, t) = \begin{cases} 0 & \text{if } s = 0 \quad ; \\ \tilde{\beta}(\varepsilon_i, t)\frac{s-\varepsilon_{i-1}}{\varepsilon_i-\varepsilon_{i-1}} + \tilde{\beta}(\varepsilon_{i-1}, t)\frac{s-\varepsilon_i}{\varepsilon_{i-1}-\varepsilon_i} & \varepsilon_i \leq s < \varepsilon_{i-1}. \end{cases}$$

Clearly the function $\tilde{\beta}(s, t)$ satisfies both required properties 1 and 2. Moreover, it satisfies all the requirements of being a $\mathcal{KL}$-function, except possibly for the fact that it is only non-decreasing in the s variable and it can be zero. So, to have the desired $\mathcal{KL}$-function, we define:

$$\beta(s, t) = \tilde{\beta}(s, t) + \hat{\beta}(s, t),$$

where $\hat{\beta}(s, t)$ is any $\mathcal{KL}$-function. ∎

Proof of Sufficiency Let Σ be a model of type (1), and V be a control-Lyapunov integral function for Σ with respect to the closed, weakly invariant, and non-empty set $\mathcal{A} \subset \mathbb{R}^n$. Let $\beta(s, t)$ be the $\mathcal{KL}$-function given by lemma 3.3, and let $\gamma(p) = \phi(p) + \mu$. Then, combining together the results given by lemmas 3.2 and 3.3, we conclude that Σ is GAC to $\mathcal{A}$.

3.2 Necessity part

In this section, we assume given a model Σ of type (1) and a closed, weakly invariant, and non-empty set $\mathcal{A} \subset \mathbb{R}^n$. Moreover we assume Σ to be GAC to $\mathcal{A}$. Our aim is to construct a control-Lyapunov integral function for Σ with respect to the set $\mathcal{A}$. The idea of the construction is similar to the one given in [9]. The next proposition establishes a technical property of $\mathcal{KL}$-functions.

Proposition 3.4 Let $\beta(\cdot,\cdot)$ be $\mathcal{KL}$-function, and choose a strictly increasing sequence of positive real numbers, $\{\varepsilon_i\}_{i\geq 0}$, such that $\lim_{i\to+\infty}\varepsilon_i = +\infty$. Then there exists a continuous, strictly decreasing function $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, and a strictly increasing sequence $\{T_i\}_{i\geq 0}$, with $T_0 = 0$, such that:

i) $\lim_{t\to+\infty} g(t) = 0$;

ii) $\lim_{i\to+\infty} T_i = +\infty$;

iii) if $p < \varepsilon_i$ then $\beta(p,t) \leq g(t)$ for all $t \geq T_i$.

Proof. Let $T_0 = 0$ and, for $i \geq 1$, define, inductively on i:

$$\alpha_i = \beta(\varepsilon_{i-1}, T_{i-1} + 1/i), \quad \text{and} \quad T_i \text{ be such that } \beta(\varepsilon_i, T_i) \leq \alpha_i/2 = g_i.$$

Note that such a T_i exists since $\beta(s,\cdot)$ is decreasing to zero as $t \to \infty$. Since $\beta(\varepsilon_i, T_{i-1}+1/i) > \alpha_i$, it holds that:

$$T_i > T_{i-1} + 1/i \quad \Rightarrow \quad T_i > \sum_{j=1}^{i} 1/j, \quad \text{for } i \geq 1.$$

Thus the sequence $\{T_i\}_{i\geq 0}$ is strictly increasing and satisfies ii). Moreover, since:

$$\alpha_{i+1} = \beta(\varepsilon_i, T_i + \frac{1}{i+1}) < \beta(\varepsilon_i, T_i) \leq \frac{\alpha_i}{2},$$

we have:

$$\alpha_{i+1} \leq \frac{\alpha_1}{2^i} \; \forall\, i \geq 1, \;\; \Rightarrow \;\; \lim_{i\to+\infty} \alpha_i = 0. \tag{9}$$

Let g_{-1} and g_0 be two constants such that $g_{-1} > g_0 > 2g_1$, and, for $i \geq 0$, let P_i be the point (T_i, g_{i-1}). Let $l_{P_iP_{i+1}}(\cdot)$ be the linear function such that $l_{P_iP_{i+1}}(T_i) = g_{i-1}$, and $l_{P_iP_{i+1}}(T_{i+1}) = g_i$. It is easy to see that, by choosing g_{-1} sufficiently large, we may assume:

$$\beta(x,t) < l_{P_0P_1}(t) \; \forall\, t \in [T_0, T_1], \forall\, x < \varepsilon_0. \tag{10}$$

Now let:

$$g(t) = l_{P_iP_{i+1}}(t) \quad \text{for } t \in [T_i, T_{i+1}].$$

Then clearly the map $g(\cdot)$ is continuous and strictly decreasing. Moreover, by equation (9), we have $\lim_{t\to+\infty} g(t) = 0$ (thus i) holds). Now we establish iii). Let $p \in \mathbb{R}$, and assume $p < \varepsilon_i$. If $i = 0$ and $t \in [T_0, T_1]$, then property iii) holds by construction (equation (10)). So we may assume $t \geq T_1$. Let $t \geq T_i$, then $t \in [T_j, T_{j+1}]$ with $j \geq i \geq 1$. Then we have:

$$g(t) \geq g_j = \alpha_j/2 \geq \beta(\varepsilon_j, T_j).$$

On the other hand, if $x < \varepsilon_i$ and $t \in [T_j, T_{j+1}]$, we also have:

$$\beta(x,t) < \beta(\varepsilon_i, t) \leq \beta(\varepsilon_j, t) \leq \beta(\varepsilon_j, T_j) \leq g_j \leq g(t);$$

so iii) holds. ∎

Let $\gamma(\cdot)$ and $\beta(\cdot,\cdot)$ be respectively the continuous, positive and increasing, and $\mathcal{KL}$ maps given by definition 2.2. Choose any $\varepsilon_0 > 0$, and let $\{\varepsilon_i\}_{i\geq 0}$ be any strictly increasing sequence such that:

$$\lim_{i\to+\infty} \varepsilon_i = +\infty,\ \varepsilon_{i+1} > \min\{\beta(\varepsilon_i,0),\ \gamma(\varepsilon_i)\}, \quad \text{and} \quad \varepsilon_{i+2} > 2\varepsilon_{i+1} - \varepsilon_i.$$

Now let $\{T_i\}_{i\geq 0}$, and $g(\cdot)$ be the sequence and the function given by proposition 3.4. Since the map g is strictly decreasing, and $g : \mathbb{R}_{\geq 0} \to (0, g(0)]$, we may let:

$$h(p) = \begin{cases} g^{-1}(p) & \text{if } p \in (0, g(0)], \\ 0 & \text{if } p > g(0). \end{cases}$$

Notice that $h : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ is decreasing, continuous, and $\lim_{p\to 0^+} h(p) = +\infty$. Now we define:

$$N(p) = \begin{cases} 0 & \text{if } p = 0, \\ p\exp\{-h(p)\} & \text{if } p > 0. \end{cases} \tag{11}$$

Notice that:

i) N is a continuous and strictly increasing map;

ii) $\lim_{p\to+\infty} N(p) = +\infty$;

iii) if $p \leq \alpha$ then $N(p) \leq \alpha$.

For each $i \geq 0$, let $M_i = M(\varepsilon_{i+2}, \varepsilon_{i+2})$, where $M(\varepsilon_{i+2}, \varepsilon_{i+2})$ are the constants given by the boundedness assumption, i.e.:

$$\sup_{\substack{|x|_{\mathcal{A}} \leq \varepsilon_{i+2} \\ d(u,0) \leq \varepsilon_{i+2}}} |f(t,x,u)| \leq M_i \qquad \text{for a.e.} \quad t \in \mathbb{R}.$$

Clearly $M_i \leq M_{i+1}$.

Now let $M : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $\delta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be any two continuous and increasing maps such that:

$$M(p) \geq \begin{cases} M_0 & \text{if } p < \frac{\varepsilon_0}{2}, \\ M_i & \text{if } \frac{\varepsilon_{i-1}}{2} \leq p < \frac{\varepsilon_i}{2}, \end{cases} \tag{12}$$

$$\delta(p) \geq \gamma(p) \quad \text{for all } p \geq 0. \tag{13}$$

Moreover, for $i \geq 1$, we require that, if $\varepsilon_{i+1} < p \leq \varepsilon_{i+2}$, then:

$$\delta(p) > \delta(\gamma(\varepsilon_i)) + M(\beta(\varepsilon_i,0))\left[N(\beta(\varepsilon_i,0))T_i + \beta(\varepsilon_i,0)\right] \tag{14}$$

and, for $i \geq 0$,:

$$M(\varepsilon_{i+1}) > \left(\frac{2M_i}{N(\varepsilon_{i+1})}\frac{1}{\varepsilon_{i+2}-\varepsilon_{i+1}}\right)\left\{M(\beta(\varepsilon_i,0))\left[N(\beta(\varepsilon_i,0))T_i + \beta(\varepsilon_i,0)\right] + \delta(\gamma(\varepsilon_i))\right\}. \tag{15}$$

Notice that, since $\min\{\beta(\varepsilon_i,0),\ \gamma(\varepsilon_i)\} < \varepsilon_{i+1}$, one shows easily that two continuous and increasing functions satisfying inequalities (12), (13), (14), and (15) exist.

Finally, we let:

$$F(p) \;=\; M(p)N(p) \;=\; M(p)p\exp\{-h(p)\}. \tag{16}$$

Fix $(t_0, x_0) \in \mathbb{R}\times\mathbb{R}^n$. For each relaxed control $\mathbf{w}$, denote by $x(t) = x(t; t_0, x_0, \mathbf{w})$ and let:

$$Q(t_0, x_0, \mathbf{w}) \;=\; \int_{t_0}^{+\infty} F(|x(t)|_{\mathcal{A}})dt \;+ \max\{\delta(\|\mathbf{w}\|) - \delta(\gamma(\varepsilon_0)),\ 0\}, \tag{17}$$

if $x(t)$ exists for all $t \geq t_0$, let $Q(t_0, x_0, \mathbf{w}) \;=\; +\infty$ otherwise. Then we let

$$V(t_0, x_0) \;=\; \inf_{\mathbf{w}\in\mathbf{W}} Q(t_0, x_0, \mathbf{w}). \tag{18}$$

We want to prove that the function V defined above is a control-Lyapunov integral function for the model (1) with respect to the set $\mathcal{A}$.

Lemma 3.5 Let $(t_0, x_0) \in \mathbb{R}\times\mathbb{R}^n$, and $i \geq 0$ be the first index such that $|x_0|_{\mathcal{A}} < \varepsilon_i$. Then there exists an ordinary control $\mathbf{u}$, with $\|\mathbf{u}\| \leq \gamma(|x_0|_{\mathcal{A}})$, such that:

$$Q(t_0, x_0, \mathbf{u}) \;\leq\; F(\beta(|x_0|_{\mathcal{A}}, 0))T_i + M(\beta(|x_0|_{\mathcal{A}}, 0))\beta(|x_0|_{\mathcal{A}}, 0) + \max\{\delta(\gamma(|x_0|_{\mathcal{A}})) - \delta(\gamma(\varepsilon_0)), 0\}, \tag{19}$$

and $|x(t)|_{\mathcal{A}} < \varepsilon_{i+2}$ for all $t \geq t_0$.

Proof. Given $(t_0, x_0) \in \mathbb{R}\times\mathbb{R}^n$ with $|x_0|_{\mathcal{A}} < \varepsilon_i$, let $\mathbf{u}$ be the control function given by the GAC property. Then $\|\mathbf{u}\| \leq \gamma(|x_0|_{\mathcal{A}})$, moreover the corresponding trajectory $x(t)$ is defined for all $t \geq t_0$ and satisfies

$$|x(t)|_{\mathcal{A}} \;\leq\; \beta(|x_0|_{\mathcal{A}}, t - t_0) \;<\; \beta(\varepsilon_i, 0) \;< \varepsilon_{i+2}.$$

Moreover, for $t \geq T_i + t_0$, we have:

$$|x(t)|_{\mathcal{A}} \;\leq\; \beta(|x_0|_{\mathcal{A}}, t - t_0) \;<\; \beta(\varepsilon_i, t - t_0) \;< g(t - t_0).$$

Thus, for all $t \geq T_i + t_0$, $h(|x(t)|_{\mathcal{A}}) \geq h(g(t - t_0)) = t - t_0$. So we have:

$$\begin{aligned} Q(t_0, x_0, \mathbf{u}) \;&=\; \int_{t_0}^{t_0+T_i} F(|x(t)|_{\mathcal{A}})dt \;+ \int_{t_0+T_i}^{+\infty} F(|x(t)|_{\mathcal{A}})dt + \max\{\delta(\|\mathbf{u}\|) - \delta(\gamma(\varepsilon_0)), 0\} \\ &\leq F(\beta(|x_0|_{\mathcal{A}}, 0))T_i \;+ M(\beta(|x_0|_{\mathcal{A}}, 0))\beta(|x_0|_{\mathcal{A}}, 0)) + \max\{\delta(\gamma(|x_0|_{\mathcal{A}})) - \delta(\gamma(\varepsilon_0)), 0\}. \end{aligned}$$

Thus the lemma is proved. ∎

Remark 3.6 Notice that, from the previous lemma, we have in particular:

(a) if $|x_0|_{\mathcal{A}} < \varepsilon_0$, then $T_0 = 0$ and the max is also zero, so: $V(t_0, x_0) \;\leq M(\beta(|x_0|_{\mathcal{A}}, 0))\beta(|x_0|_{\mathcal{A}}, 0)$,

(b) if $\varepsilon_{i-1} \leq |x_0|_{\mathcal{A}} < \varepsilon_i$, with $i \geq 1$, then

$$V(t_0, x_0) \leq \delta(\gamma(\varepsilon_i)) - \delta(\gamma(\varepsilon_0)) + \; M(\beta(\varepsilon_i, 0))\left[N(\beta(\varepsilon_i, 0))T_i + \beta(\varepsilon_i, 0)\right].$$

Lemma 3.7 There exists a $\mathcal{K}_\infty$-map $\alpha_2(\cdot)$ such that, for all $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, it holds that:

$$V(t_0, x_0) \ \leq \ \alpha_2(|x_0|_{\mathcal{A}}). \tag{20}$$

Proof. For $\varepsilon_{i-1} \leq p < \varepsilon_i$ ($\varepsilon_{-1} = 0$), let

$$\tilde{\alpha}_2(p) = F(\beta(p,0))T_i \ + \ M(\beta(p,0))\beta(p,0)) + \max\{\delta(\gamma(p)) - \delta(\gamma(\varepsilon_0)), 0\}.$$

Then $\tilde{\alpha}_2(\cdot)$ is increasing and $\tilde{\alpha}_2(0) = 0$. Now let $\alpha_2(\cdot)$ be any $\mathcal{K}_\infty$-function such that $\tilde{\alpha}_2(p) \leq \alpha_2(p)$. For each $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, let $\mathbf{u}$ be the control map given by lemma 3.5. Then, from equation (19), it follows that:

$$V(t_0, x_0) \ \leq \ Q(t_0, x_0, \mathbf{u}) \leq \tilde{\alpha}_2(|x_0|_{\mathcal{A}}) \leq \alpha_2(|x_0|_{\mathcal{A}}).$$

∎

Lemma 3.8 For each $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, let $i \geq 0$ be the first index such that $|x_0|_{\mathcal{A}} < \varepsilon_i$, and let $\mathbf{u}_0$ be the control map given by lemma 3.5. If $\mathbf{w}$ is any control such that:

$$Q(t_0, x_0, \mathbf{w}) \ \leq \ Q(t_0, x_0, \mathbf{u}_0),$$

then $\|\mathbf{w}\| \leq \varepsilon_{i+2}$, and the corresponding trajectory $x(t)$ is such that:

$$|x(t)|_{\mathcal{A}} \ < \ \varepsilon_{i+2}. \tag{21}$$

Proof. First notice that if $Q(t_0, x_0, \mathbf{w}) < +\infty$, then, in particular $x(t)$ is defined for all $t \geq t_0$. Assume that $\|\mathbf{w}\| > \varepsilon_{i+2}$. Then it holds that:

$$\begin{aligned} &Q(t_0, x_0, \mathbf{w}) \geq \delta(\|\mathbf{w}\|) - \delta(\gamma(\varepsilon_0)) \geq \delta(\varepsilon_{i+2}) - \delta(\gamma(\varepsilon_0)) \ > \\ &F(\beta(\varepsilon_i, 0))T_i + M(\beta(\varepsilon_i, 0))\beta(\varepsilon_i, 0)) + \delta(\gamma(\varepsilon_i)) - \delta(\gamma(\varepsilon_0)), \end{aligned}$$

where, to get the last inequality, we have used (14). This last inequality contradicts the assumption $Q(t_0, x_0, \mathbf{w}) \leq Q(t_0, x_0, \mathbf{u}_0)$. Now, assume that $\|\mathbf{w}\| \leq \varepsilon_{i+2}$, but that the inequality (21) is not satisfied. Then there exist $T_1 < T_2$ such that:

$$|x(T_1)|_{\mathcal{A}} = \varepsilon_{i+1}, \qquad |x(t)|_{\mathcal{A}} \leq \varepsilon_{i+2} \ \forall \, t \in [t_0, T_2],$$

and

$$|x(T_2)|_{\mathcal{A}} = \varepsilon_{i+2}, \qquad |x(t)|_{\mathcal{A}} \geq \varepsilon_{i+1} \ \forall \, t \in [T_1, T_2].$$

It must hold that

$$T_2 - T_1 \geq \frac{\varepsilon_{i+2} - \varepsilon_i}{2M_i}, \tag{22}$$

otherwise

$$|x(T_2)|_{\mathcal{A}} \leq |x(T_1)|_{\mathcal{A}} + |x(T_1) - x(T_2)| < \varepsilon_{i+1} + \frac{\varepsilon_{i+2} - \varepsilon_i}{2} < \varepsilon_{i+2}.$$

So, we have:

$$Q(t_0, x_0, \mathbf{w}) \;\geq\; \int_{T_1}^{T_2} F(|x(t)|_{\mathcal{A}})dt \;\geq M(\varepsilon_{i+1})N(\varepsilon_{i+1})\frac{\varepsilon_{i+2}-\varepsilon_{i+1}}{2M_i} \;\geq$$
$$F(\beta(\varepsilon_i, 0))T_i + M(\beta(\varepsilon_i, 0))\beta(\varepsilon_i, 0) + \delta(\gamma(\varepsilon_i)),$$

where the last inequality holds by equation (15). Thus, the assumption that

$$Q(t_0, x_0, \mathbf{w}) \;\leq\; Q(t_0, x_0, \mathbf{u}_0)$$

is again contradicted. ∎

Remark 3.9 Fix any $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ and let $i \geq 0$ be the first index such that $|x_0|_{\mathcal{A}} < \varepsilon_i$. From the result proved in the previous lemma, we get:

$$V(t_0, x_0) \;=\; \inf_{\mathbf{w}\in\mathbf{W}} Q(t_0, x_0, \mathbf{w}) \;=\; \inf_{\substack{\mathbf{w}\in\mathbf{W}_{\varepsilon_{i+2}},\\ |x(t)|_{\mathcal{A}} < \varepsilon_{i+2}}} Q(t_0, x_0, \mathbf{w}). \tag{23}$$

Lemma 3.10 There exists a $\mathcal{K}_\infty$-map $\alpha_1(\cdot)$ such that, for all $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, it holds that:

$$\alpha_1(|x_0|_{\mathcal{A}}) \;\leq\; V(t_0, x_0). \tag{24}$$

Proof. Fix any $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, and let $i \geq 0$ be the first index such that $|x_0|_{\mathcal{A}} < \varepsilon_i$. Let $\mathbf{w}$ be any control function such that $\|\mathbf{w}\| \leq \varepsilon_{i+2}$ and the corresponding trajectory is defined for all $t \geq t_0$ and satisfies $|x(t)|_{\mathcal{A}} \leq \varepsilon_{i+2}$. Let:

$$\delta_{x_0} \;=\; \frac{|x_0|_{\mathcal{A}}}{2M_i}.$$

Fact. If $t \in [t_0, t_0 + \delta_{x_0})$, then $|x(t)|_{\mathcal{A}} > |x_0|_{\mathcal{A}}/2$.

Proof of Fact. If the conclusion does not hold, then, by continuity, there exists a $\bar{t} \in [t_0, t_0 + \delta_{x_0})$, such that $|x(\bar{t})|_{\mathcal{A}} = |x_0|_{\mathcal{A}}/2$. However, we have:

$$|x(\bar{t}) - x_0| \;\leq M_i(\bar{t} - t_0) < M_i\delta_{x_0} = |x_0|_{\mathcal{A}}/2,$$

which implies:

$$|x_0|_{\mathcal{A}} \leq |x(\bar{t})|_{\mathcal{A}} + |x(\bar{t}) - x_0| < \frac{|x_0|_{\mathcal{A}}}{2} + \frac{|x_0|_{\mathcal{A}}}{2} = |x_0|_{\mathcal{A}}.$$

So the fact is established.

Now we have:

$$Q(t_0, x_0, \mathbf{w}) \;\geq\; \int_{t_0}^{t_0+\delta_{x_0}} M(|x(t)|_{\mathcal{A}})N(|x(t)|_{\mathcal{A}})dt \geq M(|x_0|_{\mathcal{A}}/2)N(|x_0|_{\mathcal{A}}/2)\frac{|x_0|_{\mathcal{A}}}{2M_i}.$$

Next, by combining equation (12) with the previous inequality, one gets:

$$Q(t_0, x_0, \mathbf{w}) \;\geq\; \frac{|x_0|_{\mathcal{A}}}{2}N(|x_0|_{\mathcal{A}}/2). \tag{25}$$

So, by using equation (23), we conclude:

$$\frac{|x_0|_{\mathcal{A}}}{2} N(|x_0|_{\mathcal{A}}/2) \leq \inf_{\substack{\mathbf{w} \in \mathbf{W}_{\varepsilon_{i+2}}, \\ |x(t)|_{\mathcal{A}} < \varepsilon_{i+2}}} Q(t_0, x_0, \mathbf{w}) = V(t_0, x_0).$$

Thus, by letting $\alpha_1(p) = p/2N(p/2)$, the conclusion follows. ∎

Thus the function $V(\cdot, \cdot)$ satisfies property 1 of Definition 2.4. It remains to show that V is continuous and that equation (4) holds. First, we prove continuity at the points $(t_0, x_0) \in \mathbb{R} \times \mathcal{A}$.

Lemma 3.11 The function $V(\cdot, \cdot)$ is continuous at $(t_0, x_0) \in \mathbb{R} \times \mathcal{A}$. Moreover it holds that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|x|_{\mathcal{A}} \leq \delta$ then

$$|V(t, x)| \leq \varepsilon \qquad \forall\, t \in \mathbb{R}. \tag{26}$$

Proof. It is clear that it is suffices to prove (26), since $V(t_0, x_0) = 0$ for all $(t_0, x_0) \in \mathbb{R} \times \mathcal{A}$. Given $\varepsilon > 0$, choose $0 < \delta < \varepsilon_0$ such that $M(\beta(p, 0))\beta(p, 0) \leq \varepsilon$ for all $p \leq \delta$. Now if $|x|_{\mathcal{A}} \leq \delta$, then by Remark 3.6, property (a), we have:

$$V(t, x) \leq M(\beta(|x|_{\mathcal{A}}, 0))\beta(|x|_{\mathcal{A}}, 0) \leq \varepsilon;$$

as desired. ∎

Lemma 3.12 Let $t_n \to t_0$, $x_n \to x_0$, and $\mathbf{w}_n \to \mathbf{w}_0$, be such that $\|\mathbf{w}_n\| \leq r_1$, and all the solutions $x_n(t) = x(t; t_n, x_n, \mathbf{w}_n)$ exist for all $t \geq t_n$ and satisfy $|x_n(t)|_{\mathcal{A}} \leq r_2$, where r_1, r_2 are two positive real constants. Then the solution $x(t) = x(t; t_0, x_0, \mathbf{w}_0)$ exists for all $t \geq t_0$, it is such that:

$$x_n(t) \to x(t) \;\text{ as }\; n \to \infty, \quad \text{and} \quad |x(t)|_{\mathcal{A}} \leq r_2,$$

again for all $t \geq t_0$.

Proof. Since $x_n \to x_0$, we have $|x_0|_{\mathcal{A}} \leq r_2$. Given $\mathbf{w}_0$, by local existence of solutions, there exists a maximal interval $[t_0, \bar{t})$, in which $x(t)$ is defined. Moreover, we may assume that $x_n \in \operatorname{clos} B(x_0, 1)$, and $t_n \in [t_0 - 1, t_0 + 1]$ for all $n \geq 1$. For each $t > t_0$, we let:

$$A(t) \;=\; \{\, x_n(\tau) \,|\, \tau \in [t_n, t] \,\} \subset \mathbb{R}^n.$$

Then $A(t)$ is bounded, and in fact:

$$|x_n(\tau)| \leq |x_n| + \int_{t_n}^{t} |f(s, x_n(s), u(s))| ds \leq |x_n| + M_{r_1 r_2}(\tau - t) \leq |x_0| + 1 + M_{r_1 r_2}\delta(t),$$

where $M_{r_1 r_2} = \sup |f(t, x, u)|$ for $|x|_{\mathcal{A}} \leq r_2$, and $\|u\| \leq r_1$, and $\delta(t) = (t - t_0 + 1)$. First we prove the following fact.

Fact. If $x(t)$ exists for all $t \in [t_0, t']$, then $x_n(t) \to x(t)$ as $n \to \infty$ for all $t \in [t_0, t']$.

Proof of Fact. Let:

$$K \;=\; \operatorname{clos} A(t') \cup \{\, x(t) \,|\, t \in [t_0, t'] \,\},$$

then K is a compact set. Let L_{K,r_1}, and $\tilde{M}$ be the Lipschitz constant and a bound for f with $x \in K$, $\|u\| \le r_1$, and $t \in [t_0 - 1, t']$. Then one gets:

$$|x_n(t) - x(t)| \;\le |x_n - x_0| + \tilde{M}|t_n - t_0| + L_{k,r_1} \int_{t_0 \vee t_n}^{t'} \|\mathbf{w}_n(s) - \mathbf{w}_0(s)\| + |x_n(s) - x(s)| ds,$$

where $t_0 \vee t_n$ indicates $\max\{t_0, t_n\}$. Using the Bellman-Gronwall inequality, one gets:

$$|x_n(t) - x(t)| \;\le \Big(|x_n - x_0| + \tilde{M}|t_n - t_0| + L_{k,r_1} \int_{t_0 \vee t_n}^{t'} \|\mathbf{w}_n(s) - \mathbf{w}_0(s)\| \, ds\Big) e^{L_{k,r_1}(t' - t_0 \vee t_n)}.$$

From this last inequality the fact easily follows.

Now we prove that $x(t)$ exists for all $t \ge t_0$. Assume that $x(t)$ exists only for $t \in [t_0, \bar{t})$. Since, for $t \in [t_0, \bar{t})$, we have $x_n(t) \to x(t)$ (from the previous fact), it holds that:

$$x(t) \in \operatorname{clos} A(\bar{t}).$$

Since $\operatorname{clos} A(\bar{t})$ is a compact set, this contradicts the fact that $x(t)$ does not exist for $t = \bar{t}$. ∎

Lemma 3.13 Let $(t_n, x_n) \to (t_0, x_0)$, and $\mathbf{w}_n \to \mathbf{w}_0$. Let $i \ge 0$ be the first index such that $|x_0|_{\mathcal{A}} < \varepsilon_i$. Assume that all $Q(t_n, x_n, \mathbf{w}_n)$ are finite, that $\|\mathbf{w}_n\| \le \varepsilon_{i+2}$, and that $|x_n(t)|_{\mathcal{A}} \le \varepsilon_{i+2}$. Then:

$$Q(t_0, x_0, \mathbf{w}_0) \le \liminf_{n \to +\infty} Q(t_n, x_n, \mathbf{w}_n).$$

Proof. From lemma 3.12, it holds that $x_0(t)$ is defined for all $t \ge t_0$. Moreover it is also true that:

$$F(|x_n(t)|_{\mathcal{A}}) \;\to\; F(|x_0(t)|_{\mathcal{A}}).$$

Given $\varepsilon > 0$ there exists $\delta > 0$ such that:

$$\int_{t_0}^{t_0+\delta} F(|x_0(t)|_{\mathcal{A}}) dt < \varepsilon.$$

By Fatou's Lemma, we get (notice that for n large $t_n \le t_0 + \delta$):

$$\int_{t_0+\delta}^{\infty} F(|x_0(t)|_{\mathcal{A}}) dt \le \liminf_{n \to \infty} \int_{t_0+\delta}^{\infty} F(|x_n(t)|_{\mathcal{A}}) dt.$$

On the other hand, one easily sees that:

$$\max\{\delta(\|\mathbf{w}_0\|) - \delta(\gamma(\varepsilon_0)),\, 0\} \;\le \liminf_{n \to +\infty} \max\{\delta(\|\mathbf{w}_n\|) - \delta(\gamma(\varepsilon_0)),\, 0\}.$$

Summing up, we conclude:

$$Q(t_0, x_0, \mathbf{w}_0) \le \varepsilon + \int_{t_0+\delta}^{\infty} F(|x_0(t)|_{\mathcal{A}}) dt + \max\{\delta(\|\mathbf{w}_0\|) - \delta(\gamma(\varepsilon_0)),\, 0\},$$

which is upper bounded by

$$\varepsilon + \liminf_{n\to\infty} \int_{t_0+\delta}^{\infty} F(|x_n(t)|_{\mathcal{A}})dt + \liminf_{n\to+\infty} \max\{\delta(\|\mathbf{w}_n\|) - \delta(\gamma(\varepsilon_0)),\, 0\}$$

and is itself bounded by

$$\varepsilon + \liminf_{n\to+\infty} Q(t_n, x_n, \mathbf{w}_n).$$

From which the conclusion follows since ε was arbitrary. ∎

The arguments used in the proofs of next lemmas are similar to the one used in [9] to prove the corresponding results.

Lemma 3.14 For each $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ there exists $\mathbf{w}_0$ such that:

$$V(t_0, x_0) \;=\; Q(t_0, x_0, \mathbf{w}_0).$$

Proof. Assume that $i \geq 0$ is the first index such that $|x_0|_{\mathcal{A}} < \varepsilon_i$. Let $\mathbf{w}_n$ be a minimizing sequence, then, by lemma 3.8, it must holds that $\|\mathbf{w}_n\| \leq \varepsilon_{i+2}$ and $|x_n(t)|_{\mathcal{A}} < \varepsilon_{i+2}$. Thus, by sequential compactness of $\mathbf{W}_{\varepsilon_{i+2}}$, we may assume that $\mathbf{w}_n \to \mathbf{w}_0$ (possibly extracting a subsequence). From lemma 3.13, we have:

$$Q(t_0, x_0, \mathbf{w}_0) \leq \liminf_{n\to\infty} Q(t_0, x_0, \mathbf{w}_n) = \; V(t_0, x_0) \leq Q(t_0, x_0, \mathbf{w}_0).$$

∎

Lemma 3.15 The function $V(\cdot,\cdot)$ is lower semicontinuous.

Proof. Let $(t_n, x_n) \to (t_0, x_0)$. Then if $i \geq 0$ is the first index such that $|x_0|_{\mathcal{A}} < \varepsilon_i$, we may also assume that $|x_n|_{\mathcal{A}} < \varepsilon_i$. Let $\mathbf{w}_n$ be such that $V(t_n, x_n) = Q(t_n, x_n, \mathbf{w}_n)$, then, by lemma 3.8, $\|\mathbf{w}_n\| \leq \varepsilon_{i+2}$ and $|x_n(t)|_{\mathcal{A}} < \varepsilon_{i+2}$. Since $\mathbf{W}_{\varepsilon_{i+2}}$ is sequentially compact, we may assume that $\mathbf{w}_n \to \mathbf{w}_0$. By applying lemma 3.12, we know that the trajectory $x_0(t) = x(t; t_0, x_0, \mathbf{w}_0)$ exists for all $t \geq t_0$ and it is the limiting trajectory. Moreover by lemma 3.13, we have:

$$V(t_0, x_0) \leq Q(t_0, x_0, \mathbf{w}_0) \leq \liminf_{n\to\infty} Q(t_n, x_n, \mathbf{w}_n) = \liminf_{n\to\infty} V(t_n, x_n).$$

∎

Lemma 3.16 The function $V(\cdot,\cdot)$ is continuous.

Proof. We need only to prove upper semicontinuity. Fix $\varepsilon > 0$, and let $0 < \delta < \varepsilon_0$ be such that if $|x|_{\mathcal{A}} \leq \delta$ then $V(t, x) \leq \varepsilon/3$ for all $t \in \mathbb{R}$ (use lemma 3.11). Fix any $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$. Let $i \geq 0$ be the first index such that $|x_0|_{\mathcal{A}} < \varepsilon_i$, and $\mathbf{w}_0$ be such that $V(t_0, x_0) = Q(t_0, x_0, \mathbf{w}_0)$. Then $\|\mathbf{w}_0\| \leq \varepsilon_{i+2}$ (lemma 3.8). Denote by $x(t) = x(t; t_0, x_0, \mathbf{w}_0)$, then there exists T such that $|x(T)|_{\mathcal{A}} < \delta$. By continuity, there exist a neighborhood $J \times H$ of (t_0, x_0) such that, for all $(t', z) \in J \times H$, $|z|_{\mathcal{A}} < \varepsilon_i$, $z(t) = x(t; t', z, \mathbf{w}_0)$ exists for all $t \in [t', T]$, $|z(T)|_{\mathcal{A}} < \delta$, and

$$\int_{t'}^{T} F(|z(t)|_{\mathcal{A}}) < \int_{t_0}^{T} F(|x(t)|_{\mathcal{A}}) + \varepsilon/3.$$

Now we have:

$$V(t', z) \leq \varepsilon/3 + \int_{t'}^{T} F(|z(t)|_{\mathcal{A}}) + \max\{\delta(\|\mathbf{w}_0\|) - \delta(\gamma(\varepsilon_0)), 0\}$$

$$< \int_{t_0}^{T} F(|x(t)|_{\mathcal{A}}) + \frac{2\varepsilon}{3} + \max\{\delta(\|\mathbf{w}_0\|) - \delta(\gamma(\varepsilon_0)), 0\} \leq V(t_0, x_0) + \varepsilon.$$

Thus V is upper semicontinuous. ∎

Lemma 3.17 The function $V(\cdot, \cdot)$ is a control-Lyapunov integral function.

Proof. We have already shown that V is continuous and satisfies property 1. of Definition 2.4. We need only to prove that equation (4) holds. Fix any $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$. Let $i \geq 0$ be the first index such that $|x_0|_{\mathcal{A}} < \varepsilon_i$, and $\mathbf{w}_0$ be such that $V(t_0, x_0) = Q(t_0, x_0, \mathbf{w}_0)$. Let γ_1 be any continuous, positive and increasing map such that $\gamma_1(p) \geq \varepsilon_{i+2}$ for $\varepsilon_{i-1} \leq p < \varepsilon_i$, for $i \geq 0$ (set $\varepsilon_{-1} = 0$). Then $\|\mathbf{w}_0\| \leq \varepsilon_{i+2} \leq \gamma_1(|x_0|_{\mathcal{A}})$. Denote by $x(t) = x(t; t_0, x_0, \mathbf{w}_0)$, and by $\mathbf{w}'_0$ the translation of $\mathbf{w}_0$ by $(-t)$. It holds:

$$V(t, x(t)) \leq \int_{t}^{\infty} F(|x(s)|_{\mathcal{A}}) ds + \max\{\delta(\|\mathbf{w}'_0\|) - \delta(\gamma(\varepsilon_0)), 0\},$$

thus (notice that $\|\mathbf{w}'_0\| \leq \|\mathbf{w}_0\|$):

$$V(t, x(t)) - V(t_0, x_0) \leq -\int_{t_0}^{t} F(|x(s)|_{\mathcal{A}}) ds.$$

So also property 2. holds with $\alpha_3 = F$, and $\phi = \gamma_1$. ∎

4 Remaining Proofs, and Comments

To prove proposition 2.3 the following technical lemma is needed. Although not explicitly stated in this form, this is what was being proved in Section 3 of [8].

Lemma 4.1 Let $\Phi(r, t) : (\mathbb{R}_{\geq 0})^2 \to \mathbb{R}_{\geq 0}$ be a map such that

(a) for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $r \leq \delta$ then $\Phi(r, t) < \varepsilon$ for all $t \geq 0$,

(b) for all $\varepsilon > 0$ and for all $R > 0$ there exists T such that $\Phi(r, t) < \varepsilon$ for all $0 \leq r \leq R$ and for all $t \geq T$.

Then there exists a $\mathcal{KL}$-function $\beta(\cdot, \cdot)$ such that

$$\Phi(r, t) \leq \beta(r, t) \ \ \forall\, r, \ t.$$

Proof of proposition 2.3. We need to establish only the sufficiency part, the necessity part being obvious. We assume given a model Σ of type (1) satisfying properties 1 and 2.

Fix $(t_0, x_0) \in \mathbb{R} \times (\mathbb{R}^n \setminus \mathcal{A})$. Let $\varepsilon_0 > 0$ be such that $|x_0|_{\mathcal{A}} = \gamma_2(\varepsilon_0)$ (such an ε_0 exists, γ_2 being a $\mathcal{K}_\infty$-map). Then, by property 1, we may choose a control $\mathbf{u}_{t_0,x_0}$ such that $\|\mathbf{u}_{t_0,x_0}\| \leq \gamma_1(|x_0|_{\mathcal{A}})$ and $|x(t, t_0, x_0, \mathbf{u}_{t_0,x_0})|_{\mathcal{A}} \leq \varepsilon_0$ for all $t \geq t_0$. Let

$$\Psi_{t_0}(r,t) = \sup_{\substack{|x_0|_{\mathcal{A}} \leq r \\ t - t_0 \geq 0}} |x(t, t_0, x_0, \mathbf{u}_{t_0,x_0})|_{\mathcal{A}}, \quad \forall\, r \geq 0,\ \forall\, t \geq t_0,$$

and

$$\Phi(r,t) = \sup_{t_0} \Psi_{t_0}(r, t + t_0), \quad \forall\, r \geq 0,\ \forall\, t \geq 0.$$

Assume that $\Phi(r,t)$ satisfies both requirements (a) and (b) of lemma 4.1, and let $\beta(\cdot,\cdot)$ be the $\mathcal{KL}$-function given by the same lemma. Then we have that for all $(t_0, x_0) \in \mathbb{R} \times (\mathbb{R}^n \setminus \mathcal{A})$ there exists a control $\mathbf{u}_{t_0,x_0}$, with $\|\mathbf{u}_{t_0,x_0}\| \leq \gamma_1(|x_0|_{\mathcal{A}})$ such that

$$|x(t, t_0, x_0, \mathbf{u}_{t_0,x_0})|_{\mathcal{A}} \leq \Psi_{t_0}(|x_0|_{\mathcal{A}}, t) \leq \Phi(|x_0|_{\mathcal{A}}, t - t_0) \leq \beta(|x_0|_{\mathcal{A}}, t - t_0).$$

Thus Σ is GAC to $\mathcal{A}$. So, to conclude, we just need to show that both requirements (a) and (b) are satisfied.

(a) Fix any $\varepsilon > 0$. Let $\delta = \gamma_2(\varepsilon)$. Then for all $|x_0|_{\mathcal{A}} \leq \delta$ and for all $t_0 \in \mathbb{R}$, by property 1, we know that

$$|x(t, t_0, x_0, \mathbf{u}_{t_0,x_0})|_{\mathcal{A}} \leq \varepsilon \quad \forall\, t \geq t_0.$$

which clearly implies

$$\Phi(r,t) \leq \varepsilon, \quad \forall r \leq \delta,\ \forall\, t \geq 0.$$

(b) Fix any $\varepsilon > 0$ and any $R > 0$. By property 2, we know that there exists $T > 0$ such that, for all $|x_0|_{\mathcal{A}} \leq R$, for all $t_0 \in \mathbb{R}$,

$$|x(t, t_0, x_0, \mathbf{u}_{t_0,x_0})|_{\mathcal{A}} \leq \varepsilon \quad \forall\, t \geq T + t_0.$$

thus also

$$\Phi(r,t) \leq \varepsilon, \quad \forall r \leq R,\ \forall\, t \geq T,$$

as desired.

Remark 4.2 Observe that, in order to conclude the equivalence between definition 2.2 of GAC and properties 1, 2 of proposition 2.3, it is essential that the control realizing the stability part (property 1) and the one realizing the attraction part (property 2) can be chosen to be the same. In fact, it is possible to give examples of systems (even time-invariant and with $\mathcal{A} = \{0\}$) where both properties 1 and 2 of proposition 2.3 hold but with *different* control maps, and the system is not GAC. We do not provide all the details here, but merely sketch the construction of such an example, as follows. It is possible to build an autonomous two-dimensional model in which the origin is an attractor but not a stable point. In particular, there exist two C^∞-functions $f_1,\ f_2 : \mathbb{R}^2 \to \mathbb{R}$ such that if we look at the system (without control):

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2), \\ \dot{x}_2 &= f_2(x_1, x_2), \end{aligned}$$

it has the following properties.

(a) The origin is an attractor point, i.e. for all $x_0 \in \mathbb{R}^2$ the corresponding trajectory $x(t)$ is defined for all $t \geq 0$ and it is such that $\lim_{t\to+\infty} x(t) = 0$.

(b) The origin is an unstable point, i.e. there exists $\Delta > 0$ such that, for all $n \geq 1$, there exist $x_n \in \mathbb{R}^2$ with $|x_n| < 1/n$ and $t_n > 0$ such that, if we denote by $x_n(t)$ the trajectory with initial state x_n, it holds

$$|x_n(t_n)| > \Delta. \tag{27}$$

For the precise expression of the two functions $f_1,\ f_2$ and for the proof of properties 1, 2 we refer to [4] (Chapter 5, page 191), where such an example studied in detail. Now we look at the system:

$$\Sigma = \begin{cases} \dot{x}_1 &= f_1(x_1, x_2)u, \\ \dot{x}_2 &= f_2(x_1, x_2)u, \end{cases}$$

with state space $\mathbb{R}^2$ and control space $U = \{0, 1\}$. Since this is an autonomous model, we may set $t_0 = 0$ in the definitions. Since for this model $\mathcal{A} = \{0\}$, we will use $|\cdot|$ instead of $|\cdot|_{\mathcal{A}}$.

By choosing $u \equiv 0$ all the states are equilibria, thus property 1 of proposition 2.3 clearly holds with γ_2 equal to the identity function. Now for any fixed r, ε, if we choose $u \equiv 1$, by property (a) of the uncontrolled dynamics, for all $x \in \mathbb{R}^2$ with $|x| \leq r$ there exists $T_x > 0$ such that $|x(T_x)| < \varepsilon$. By continuity of trajectories with respect to the initial state, there exists a neighborhood U_x of x such that, for all $y \in U_x$, the corresponding trajectory, again with $u \equiv 1$, is such that $y(T_x)| < \varepsilon$. As $\bigcup_{|x|\leq r} U_x$ is an open covering of the closed ball of radius r, by compactness there exists a finite subcovering by sets $U_{x_1}, \ldots, U_{x_p}$. Now letting $T_{r,\varepsilon} = \max\{T_{x_1}, \ldots, T_{x_p}\}$, for this constant $T_{r,\varepsilon}$ property 2 of proposition 2.3 holds. In fact if $|x_0| \leq r$ then $|x_0| \in U_{x_i}$ for some $i = 1, \ldots, p$. Choose

$$u(t) = \begin{cases} 1 & \text{for } \ t \in [0, T_{x_i}), \\ 0 & \text{for } \ t \geq T_{x_i}, \end{cases}$$

then, clearly, the corresponding trajectory is such that $x_0(t) < \varepsilon$ for all $t \geq T_{r,\varepsilon}$. However Σ is *not* GAC. To see this we argue as follows. If the model would be GAC, then for each initial state $x \in \mathbb{R}^2$ there would exist a control map $u(t)$ such that the corresponding trajectory would satisfy

$$|x(t)| \leq \beta(|x_0|, t). \tag{28}$$

for a given $\mathcal{KL}$-function.

Fix $r > 0$ small enough such that $\beta(r, 0) < \Delta$. Then, by property (b) there exists x_n, with $|x_n| \leq r$, such that the corresponding trajectory with $u \equiv 1$ satisfies $|x_n(t_n)| > \Delta$. Let $L = \min\{|x_n(t)|,\ t \in [0, t_n]\}$, then there exists $T > 0$ such that $\beta(r, T) < L$. Since the trajectories of Σ either follow the trajectories of the uncontrolled dynamics or stay in equilibrium, it is clear that for x_n it is impossible to find a control maps such that the corresponding trajectory satisfies equation (28) for $t \geq T$.

We now conclude the proof of theorem 1 by showing that if V is a control-Lyapunov function, then V is also an integral control-Lyapunov function (i.e. we prove that $2 \Rightarrow 3$).

Assume given a control-Lyapunov function V, together with maps $\alpha_1,\ \alpha_2,\ \alpha_3$, and ϕ satisfying properties 1 and 2 of definition 2.4. Let $\hat{\phi}$ be any continuous positive and increasing function such that $\hat{\phi}(0) \geq \mu$ and

$$\hat{\phi}(r) \geq \begin{cases} \phi(r) \\ \alpha_1^{-1}\left(\alpha_2(r)\right) \end{cases}$$

Fix any $(t_0, x_0) \in \mathbb{R} \times (\mathbb{R}^n \setminus \mathcal{A})$. Define

$$M(t_0, x_0) = \left\{ t > t_0 \;\middle|\; \begin{array}{l} \exists\, \mathbf{w} : [t_0, t) \rightarrow P(U) \;\; \text{such that} \\ \|\mathbf{w}\| \leq \hat{\phi}(|x_0|_{\mathcal{A}}) \;\; \text{and} \;\; \forall\, s \in [t_0, t) \\ (\star)\, V(s, x(s)) - V(t_0, x_0) \leq -\int_{t_0}^{s} \frac{\alpha_3(|x(\tau)|_{\mathcal{A}})}{8} d\tau \end{array} \right\}, \tag{29}$$

where $x(s) = x(s, t_0, x_0, \mathbf{w})$. Notice that $M(t_0, x_0) \subseteq \mathbb{R}_{\geq 0} \cup \{+\infty\}$. We will prove:

(a) $M(t_0, x_0) \neq \emptyset$,

(b) $\sup M(t_0, x_0) \in M(t_0, x_0)$,

(c) $\sup M(t_0, x_0) = +\infty$.

Properties (b) and (c) clearly imply that V is also an integral control-Lyapunov function (using the maps α_1, α_2, $1/8\alpha_3$, and $\hat{\phi}$).

(a) By property 2 of definition 2.4 there exists a relaxed control $\mathbf{w} : [t_0, \tilde{t}) \rightarrow P(U)$ with $\|\mathbf{w}\| \leq \phi(|x_0|_{\mathcal{A}}) \leq \hat{\phi}(|x_0|_{\mathcal{A}})$ such that equation (3) holds. In particular, this fact implies the existence of a sequence $t_k \rightarrow t_0^+$ such that

$$V(t_k, x(t_k)) - V(t_0, x_0) \leq -\frac{\alpha_3(|x_0|_{\mathcal{A}})}{2}(t_k - t_0). \tag{30}$$

By continuity there exists $\bar{t} > t_0$ (we may assume $\bar{t} \leq \tilde{t}$) such that for all r, $s \in [t_0, \bar{t})$ we have:

$$\frac{\alpha_3(|x(s)|_{\mathcal{A}})}{2} < \alpha_3(|x_0|_{\mathcal{A}}), \tag{31}$$

$$|V(r, x(r)) - V(s, x(s))| < \frac{\alpha_3(|x_0|_{\mathcal{A}})}{4}. \tag{32}$$

Choose $t_0 < \hat{t} \leq \bar{t}$ such that for some index $\bar{k}$ it holds $t_{\bar{k}} < \bar{t}$ and $\hat{t} - t_0 < t_{\bar{k}} - t_0 < 1$. Then for all $t \in [t_0, \hat{t})$ we have:

$$\begin{aligned} V(t, x(t)) - V(t_0, x_0) &\leq |V(t, x(t)) - V(t_{\bar{k}}, x(t_{\bar{k}}))| + V(t_{\bar{k}}, x(t_{\bar{k}})) - V(t_0, x_0) \\ &< \frac{\alpha_3(|x_0|_{\mathcal{A}})}{4}(t_{\bar{k}} - t_0) - \frac{\alpha_3(|x_0|_{\mathcal{A}})}{2}(t_{\bar{k}} - t_0) < -\frac{\alpha_3(|x_0|_{\mathcal{A}})}{4}(t - t_0). \end{aligned}$$

Thus, by equation (31) we have:

$$V(t, x(t)) - V(t_0, x_0) \leq -\int_{t_0}^{t} \frac{\alpha_3(|x_0|_{\mathcal{A}})}{4} ds \leq -\int_{t_0}^{t} \frac{\alpha_3(|x(s)|_{\mathcal{A}})}{4} ds.$$

Therefore $\hat{t} \in M(t_0, x_0)$.

(b) Let $T = \sup M(t_0, x_0)$, then either $T \in \mathbb{R}$ or $T = +\infty$. In any case there exists a sequence $t_n \in M(t_0, x_0)$ such that $t_n \to T$. Thus for all $n > 0$ there exists $\mathbf{w}_n : [t_0, t_n) \to P(U)$ with $\|\mathbf{w}_n\| \leq \hat{\phi}(|x_0|_{\mathcal{A}})$ and such that $x_n(t) = x(t, t_0, x_0, \mathbf{w}_n)$ satisfies equation $(\star)$ for all $t \in [t_0, t_n)$. By sequential compactness of $\mathbf{W}_{\hat{\phi}(|x_0|_{\mathcal{A}})}$ we may assume that $\mathbf{w}_n \to \mathbf{w}_0$, where $\mathbf{w}_0$ is defined for all $t \in [t_0, T)$ and $\|\mathbf{w}_0\| \leq \hat{\phi}(|x_0|_{\mathcal{A}})$. Let $x_0(t) = x(t, t_0, x_0, \mathbf{w}_0)$, which by local existence is defined on an interval of the type $[t_0, \bar{t})$. We want to prove:

$$\bar{t} = T \quad \text{and} \quad x_n \to x_0. \tag{33}$$

From equation (33) we may conclude $T \in M(t_0, x_0)$ since equation $(\star)$, which holds for all $n > 0$, will still hold after having taken the limit as $n \to +\infty$.

The proof of (33) is only sketched since the arguments are quite similar to the one used to prove lemma 3.12. Let

$$A(t) = \{x_n(\tau) \mid t < t_n, \, \tau \in [t_0, t]\},$$

it is not hard to prove that $A(t)$ is bounded. Then one proves that:

$$\text{if } \; x_0(t) \; \text{ exists } \; \forall\, t \in [t_0, t'] \; \text{ then } \; x_n(t) \to x_0(t) \; \forall\, t \in [t_0, t']. \tag{34}$$

Let $K = \operatorname{clos} A(t') \cup \{x_0(t) \,|\, t \in [t_0, t']\}$; then K is compact. Let L be a Lipschitz constant for f with $x \in K$ and $||u|| \leq \hat{\phi}(|x_0|_{\mathcal{A}})$. Then one has:

$$|x_n(t) - x_0(t)| \leq L \int_{t_0}^{t} [||\mathbf{w}_n(s) - \mathbf{w}_0(s)|| + |x_n(s) - x_0(s)|] \, ds,$$

from which, using the Bellman-Gronwall inequality equation (34) follows. Now to prove $\bar{t} = T$ one argues as follows. Assume $\bar{t} < T$. Then for all $t \in [t_0, \bar{t})$, by (34), $x_n(t) \to x_0(t)$ thus $x_0(t) \in \operatorname{clos} A(\bar{t})$ which is compact, so the trajectory may be extended also to $t = \bar{t}$.

(c) Now we want to prove that $T = \sup M(t_0, x_0) = +\infty$. Assume that $T < +\infty$. Since $T \in M(t_0, x_0)$ by (b), there exists a relaxed control $\mathbf{w} : [t_0, T) \to P(U)$ which satisfies all the requirements of equation (29) (call $x(t)$ its corresponding trajectory). Let M be a bound for f when $|x|_{\mathcal{A}} \leq \alpha_1^{-1}(V(t_0, x_0))$ and $||u|| \leq \hat{\phi}(|x_0|_{\mathcal{A}})$. Then it holds that $|x(t)| \leq |x_0| + M(T - t_0)$. So the trajectory can be extended to the endpoint T. Now if $x(T) \in \mathcal{A}$, then we are done, since we may extend the control map $\mathbf{w}$ with the one given by the GAC assumption and all the requirements of (29) continue to be satisfied. On the other hand, if $x(T) \notin \mathcal{A}$, then by what we have seen in part (a), there exists $T_1 > T$ with $T_1 \in M(T, x(T))$ and a corresponding control $\mathbf{w}_1 : [T, T_1) \to P(U)$ with $\|\mathbf{w}_1\| \leq \phi(|x(T)|_\mathcal{A})$ such that the corresponding trajectory satisfies equation $(\star)$. Since $|x(T)|_\mathcal{A} \leq \alpha_1^{-1}(\alpha_2(|x_0|_\mathcal{A}))$, we have $\|\mathbf{w}_1\| \leq \phi(|x(T)|_\mathcal{A}) \leq \hat{\phi}(|x_0|_\mathcal{A})$, thus by concatenating the two control maps $\mathbf{w}$ and $\mathbf{w}_1$ we get that $T_1 \in M(t_0, x_0)$ contradicting the fact that $T = \sup M(t_0, x_0)$.

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