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Analytic Discs and Extension of CR Functions

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Abstract. Let M be a manifold of $X = \mathbb{C}^n$, A a small analytic disc attached to M, z^o a point of ∂A where A is tangent to M, z^1 another point of ∂A where M extends to a germ of manifold M_1 with boundary M. We prove that *CR* functions on M which extend to M_1 at z^1 also extend at z^o to a new manifold M_2. The directions M_1 and M_2 point to, are related by a sort of connection associated to A which is dual to the connection obtained by attaching 'partial analytic lifts' of A to the co-normal bundle to M in X.

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1. Introduction

Let $X = \mathbb{C}^N$ and let M be a real submanifold of X of codimension l in a neighborhood of a point z^o. We assume that M is generic that is $(TM + iTM)_{z^o} = \mathbb{C}^N$. We can then take coordinates $z = (z', z'')$ in $\mathbb{C}^N$ with $z = x + iy$ such that $z^o = 0$ and M is defined by $y'_j = h_j(x', z'')$, $j = 1, \ldots, l$, with $h_j(0) = 0$ and $\partial h_j(0) = 0$. Let M_1 be a germ of a manifold of codimension $l - 1$ with boundary M possibly at a point different from z^o. This can be described for example by $y'_j = h_j(x', z'', t)$ for $t \in \mathbb{R}^+ \cup \{0\}$ with $(\partial_t h_j)_j \neq 0$. We set $r_j = y'_j - h_j$, $r = (r_j)$, $h = (h_j)$. We assume that M and M_1 have a suitable regularity, take an analytic disc A regular up to the boundary, parametrized by $A = \{A(\tau), \tau \in \Delta\}$ (where Δ is the standard disc of $\mathbb{C}$) with $z^o = A(1) \in M$, $z^1 = A(-1) \in M_1 \setminus M$. We assume that the boundary ∂A of A is contained in $M_1 \cup M$ with $\partial A \subset M$ at z_o and $\partial A \subset M_1 \setminus M$ at z^1. Let $\partial'_z r$ be the $l \times l$ Jacobian matrix of $r = (r_j)$ with respect to $\partial'_z = (\partial_{z'_1}, \ldots, \partial_{z'_l})$. Sometimes we also write $\partial' r$ instead of $\partial'_z r$. Associated to A there is an $l \times l$ real matrix $G(\tau)$, $\tau \in \partial\Delta$, with $G(1) = \mathrm{id}_{l \times l}$, such that $G \,.\, (\partial'_z r \circ A)$ extends holomorphically from $\partial\Delta$ to Δ. Such G, which is a small perturbation of the identity, can be easily found by the implicit function theorem. We will write G_{z^1} instead of $G(-1)$ (due to $z^1 = A(-1)$) all through this paper. At last we observe that according to [B-T], we can find an open domain $V \subset M_1 \cup M$ such that CR functions over $M_1 \cup M$ are approximated on V by polynomials. We also assume that $\partial A \subset V$, suppose that $TA_{z^o} \subset T^{\mathbb{C}} M_{z^o}$, and let $v^1 := i\partial_t h(z^1)$. Then for any ε we can find a germ M_2 of a manifold with boundary M at z^o, which 'points' to a direction $v^2 \in i\mathbb{R}^l$ normal to M at z^o verifying $|v^2 + G_{z^1}v^1| < \varepsilon$, such that CR functions f on $M_1 \cup M$ extend as CR to M_2. Here for $z \in M$, we identify $T_M X_z$, the normal space to M at z, with $i\mathbb{R}^l$ by $[v] \xrightarrow{\sim} i\big(\Re e\langle \partial r_j(z), v\rangle\big)_j$ and if M_1 (or M_2) is a manifold with boundary M, we say that $T_M M_1$ (or $T_M M_2$) are the directions to which it points. We can rephrase the above statement in terms of propagation. If $\partial A \subset M$ and if f extends as CR at z^1 to a manifold M_1 which points to the normal vector v^1, then for any ε, f extends at z^o to a manifold M_2 which points to v^2 verifying $|v^2 - G_{z^1}v^1| < \varepsilon$. In particular we regain the classical theorem by Hanges and Treves [H-T] on propagation of holomorphic extendibility along complex curves in M.

The geometry of our propagation is closely related to [T2]. Assume that M is 'non-minimal' in the sense that there is $S \subset M$ such that $T^{\mathbb{C}}S = T^{\mathbb{C}}M|_S$ ($T^{\mathbb{C}}$ denoting the complex tangent bundle), and put

$$E^* = T^*_M X|_S \cap iT^*_S X, \qquad E = \frac{TX|_S}{TM|_S + iTS}.$$

Then a partial connection is defined in [T2] by $E_{z^1} \to E_{z^o}$, $v \mapsto G(z^1)v$. By this, CR extendibility at z^1 to $v^1 \in E_{z^1}$ implies CR extendibility at z^o to directions arbitrarily close to $G_{z^1}v^1$. Note that to prove this result Tumanov uses the essential fact, which is bypassed in the present paper, that when a covector $\zeta_o \partial r(z^o)$ belongs to $E^*_{z^o}$, then the full vector function $\zeta_o G(\tau)(\partial r \circ A(\tau))$, and not only its component $\zeta_o G(\tau)(\partial' r \circ A(\tau))$, extends holomorphically to Δ. (As a representative of a form, ζ_o is here a row l-vector.) Hence, propagation of extendibility in the only E-directions can be treated by his method, whereas the automatic extension in the complementary directions of $TM|_S + iTS$ must be handled by the techiques of his earlier paper [T1].

Another significant difference is that our theorem is indeed a theorem of automatic extension (rather than propagation) by discs which are attached to $M_1 \cup M$ (rather than M). Also the argument of our proof is independent.

2. Automatic CR Extension and Propagation of CR Extendibility

In $X = \mathbb{C}^N$ we take coordinates $z = (z', z'')$, $z = x + iy$ with $z' \in \mathbb{C}^l$, $z'' \in \mathbb{C}^{N-l}$. Let $z^o = 0$ and define an l codimensional submanifold $M \subset X$ by

$$y'_j = h_j(x', z''), \quad j = 1, \ldots, l, \tag{1}$$

with $h_j(0) = 0$, $\partial h_j(0) = 0$. We also use the notations $r_j = y'_j - h_j$, $r = (r_j)$, $h = (h_j)$. Let M_1 be a manifold with boundary M, possibly in a neighborhood of a point different from z^o, of codimension $l - 1$. This is defined, e.g., by introducing a new parameter $t \in \mathbb{R}^+ \cup \{0\}$, and extending the domain of h from $\mathbb{R}^l \times \mathbb{C}^{N-l}$ to $\mathbb{R}^l \times \mathbb{C}^{N-l} \times (\mathbb{R}^+ \cup \{0\})$ with $\partial_t h \neq 0$. Hence, M_1 will be defined by $y' = h(x', z'', t)$, $t \in \mathbb{R}^+ \cup \{0\}$. We shall consider analytic discs A in X 'attached' to $M_1 \cup M$ that is verifying $\partial A \subset M_1 \cup M$ and containing z^o in their boundaries.

We will denote by A both the discs themselves and their parametrizations $A(\tau)$, $\tau \in \Delta$, where Δ is the standard disc of $\mathbb{C}$. We also write $A(\tau) = (u(\tau) + iv(\tau), w(\tau))$ and assume $z^o = A(1)$. We call $w(\tau)$ the 'z components' of A and define the 't components' $t(\tau)$ by the equation $h(u(\tau), w(\tau), t(\tau)) - v(\tau) = 0$. Hence, the condition '$A(\tau) \in M_1 \setminus M$' is equivalent to '$t(\tau) > 0$'. We shall also let the function $t(\tau)$ depend on a small parameter $\eta \in \mathbb{R}^+ \cup \{0\}$, and denote it by $t^\eta(\tau)$.

For $k \geqslant 1$ integer, and $0 < \alpha < 1$ fractional, we denote by $C^{k,\alpha}$ the class of functions whose derivatives up to order k are α-Hölder continuous. Existence of attached discs with prescribed components $w(\tau)$ and $t^\eta(\tau)$ is assured by the following statement due to Tumanov (cf. [T2]).

LEMMA 1. *Let h belong to $C^{k,\alpha}$, $k \geqslant 1$, $0 < \alpha < 1$, and let $w = w(\tau)$ (resp. $t = t^\eta(\tau)$) be $C^{k,\alpha}$ in τ (resp. in τ, η) and small (in $C^{k,\alpha}$-norm). We also suppose $w(1) = 0$, $t^\eta(1) = 0$ and take $w_o \in \mathbb{C}^{N-l}$ and $s \in \mathbb{R}^l$ small. Then we can find an unique solution $u = u^\eta(\tau)$ in $C^{k,\alpha}(\partial\Delta)$ of the equation*

$$u = -T_1(h(u, w + w_o, t^\eta) + s). \tag{2}$$

Moreover, if we put $v^\eta = T_1(u^\eta) + h(s, w_o)$ and $A^\eta(\tau) = (u^\eta(\tau) + iv^\eta(\tau), w(\tau) + w_o)$, we have that A^η is $C^{k,\alpha'}$ and $\partial_\tau A^\eta$ is $C^{k-1,\alpha'}$ with respect to η, s for any $\alpha' < \alpha$.

The proof can be found in [T4, Propositions 1.1 and 1.2].

For a fixed $z^1 \in M_1 \cup M$, we denote by t^1 the value of t which corresponds to z^1, and define $M_{t^1} = \{r = 0,\ t = t^1\}$. In particular for $z^1 \in M$ we have $t^1 = 0$ and $M_{t^1} = M$. Using the basis $\partial_z r_j$, $j = 1, \ldots, l$, for $T^*_{M_{t^1}} X$, we can identify $T_{M_{t^1}} X$ (the normal bundle to M_{t^1}) to $M_{t^1} \times i\mathbb{R}^l$ by $[v] \mapsto i(\Re\mathrm{e}\langle\partial_z r_j, v\rangle)_j$ where $[v]$ is the equivalence class modulo TM_{t^1}. If $z \in M_{t^1}$ with $t^1 > 0$, we have clearly $(TM_1)_z = (TM_{t^1})_z + \mathbb{R}v^1$ where $v^1 = i(\partial_t h(z)) \in i\mathbb{R}^l (\simeq (T_{M_{t^1}} X)_z)$. When $z \in M$ we have clearly $TM_{1\,z} = TM_{\,z} + \mathbb{R}^+ v^1$; in this case we say that M_1 is 'attached' to M at (z, v^1) or that M_1 is an extension of M which 'points' to the normal direction v^1 at z.

We assume now that we are given a small analytic disc A with $z^o \in \partial A$ which contains another point z^1 in its boundary with $z^1 \in M_1 \setminus M$. Let $z^1 = A(-1)$, let $t = t^1 > 0$ at $\tau = -1$, and denote by $w(\tau)$ and $t(\tau)$ the 'z and t components' of A respectively. Let $\partial'_z r$ be the square $l \times l$ Jacobian matrix of r with respect to the $z' = (z_1, \ldots, z_l)$ variables. It is easy to find a real $l \times l$ matrix $G(\tau)$, $\tau \in \partial\Delta$, with $G(1) = \mathrm{id}_{l\times l}$ and such that $G\,.(\partial'_z r \circ A)$ extends holomorphically from $\partial\Delta$ to Δ. To prove this we only need to solve the integral (Bishop's) equation $G(\tau) = T_1(G(\tau)(\partial'_x h(u(\tau), w(\tau), t(\tau)))) + \mathrm{id}_{l\times l}$ on $\partial\Delta$ where T_1 is the Hilbert transform normalized by the condition $T_1(\cdot)|_{\tau=1} = 0$. By means of G we can define an isomorphism $(T_{M_{t_1}} X)_{z^1} \to (T_M X)_{z^o}$ which is defined, in the bases dual to $\partial_z r_j(z^1)$, $j = 1, \ldots, l$ and $\partial_z r_j(z^o)$, $j = 1, \ldots, l$, by $v \mapsto G_{z^1} v$ (where G_{z^1} stands, as always, for $G(-1)$).

Let $\chi(\tau)$ be a real positive smooth function on $\partial\Delta$ with $\chi(-1)=1$ and whose support $\mathrm{supp}(\chi)$ is contained in a small neighborhood of -1 for which $\partial A\subset M_1\setminus M$ holds. Define $t^\eta(\tau)=t(\tau)-\eta\chi(\tau)$ for small η so that $t^\eta(\tau)\geqslant 0$. Let A^η be the family of discs of Lemma 1 for such a data $w(\tau)$ and $t^\eta(\tau)$ and for $s=0$, and let $\dot{A}$ be the derivative in η at $\eta=0$.

THEOREM 2. *Let M be $C^{k,\alpha}$, A be $C^{k,\alpha}$ in $\bar{\Delta}$, small in $C^{k,\alpha}$ norm, attached to $M_1\cup M$, and let z^o, $z^1\in\partial A$ with $z^o\in M$, $z^1\in M_1\setminus M$. We also assume $\partial A\subset M$ at z^o and $TA_{z^o}\subset T^{\mathbb{C}}M_{z^o}$. Let $v^1=i\big(\partial_t h_j(z^1)\big)_j\in i\mathbb{R}^l$, $v^o=G_{z^1}v^1\in i\mathbb{R}^l$. Then*

$$|\partial_\tau\dot{A}|_1=cv^o|<\varepsilon, \tag{3}$$

where $c>0$ and ε is an error vector which can be made arbitrarily small if we correspondingly shrink supp (χ).

Proof. We have for any $i=1,\dots,l$, and with $z=z(\tau)$

$$\begin{aligned}&\textstyle\sum_j g_{ij}\,\Re\mathrm{e}\langle\partial_z r_j\circ A,\dot{A}\rangle=-\sum_j g_{ij}\partial_t h_j\chi,\\&\textstyle\sum_j g_{ij}\,\langle\partial_z r_j\circ A,\dot{A}\rangle \text{ extends holomorphically from } \partial\,\Delta \text{ to } \Delta.\end{aligned}\tag{4}$$

The first can be checked directly. The second follows from the fact that $\langle\partial_z r_j\circ A,\dot{A}\rangle=\langle\partial'_z r_j\circ A,\dot{A}\rangle$, since the z'' components of each A^η are constant in η. Recall that $i\partial_t h_j(z^1)=v^1$, and that $\mathrm{supp}(\chi)$ is contained in a arbitrarily small neighborhood of -1. Hence, applying Hopf's Lemma to the harmonic function whose boundary value is $(\sum_j g_{ij}\Re\mathrm{e}\langle\partial_z r_j\circ A,\dot{A}\rangle)_i$, and recalling that $(g_{ij})_{ij}(1)=\mathrm{id}_{l\times l}$, we get that $i\big(\langle\partial r_j\circ A,\partial_\tau\dot{A}\rangle|_1\big)_j$ has direction arbitrarily close to $G_{z^1}v^1$ provided that $\mathrm{supp}(\chi)$ is small. (Note that $(\langle\partial r_j\circ A,\partial_\tau\dot{A}\rangle|_1)_j$ is real because $\partial A^\eta\subset M$ at z^o and therefore for $\tau=e^{i\theta}$ we have $\partial_\theta\dot{A}|_1\in TM_{z^o}$.) □

We recall the conclusions of Lemma 1. The Taylor expansion of $\partial_\tau A^\eta$ with respect to η gives

$$\Re\mathrm{e}\langle\partial r_j\circ A,\partial_\tau A^\eta\rangle|_1=\eta\Re\mathrm{e}\langle\partial r_j\circ A,\partial_\tau\dot{A}\rangle|_1+\mathrm{o}(\eta), \tag{5}$$

where we have used the basic hypothesis $TA_{z^o}\subset T^{\mathbb{C}}M_{z^o}$. Here $\partial_\tau\dot{A}$ satisfies the conclusions of Theorem 2. It follows

$$\big(\Re\mathrm{e}\langle\partial_z r_j(z^o),\partial_\tau A^\eta\rangle|_1\big)_j=\Big(+cG_{z^1}\big(\partial_t h_j(z^1)\big)_j+\varepsilon\Big)\eta+\mathrm{o}(\eta), \tag{6}$$

where ε is small if $\mathrm{supp}(\chi)$ is small. Hence, the vector v in the right hand side of (6) verifies $v=+c'\eta(v^o+\varepsilon)$ where ε is small when η and $\mathrm{supp}(\chi)$ are so (and c is possibly a new constant). As we have already seen, we have $\Im\mathrm{m}\big(\langle\partial r_j\circ A,\partial_\tau\dot{A}\rangle|_1\big)_j=0$ or, equivalently, for $\tau=\mathrm{e}^{i\theta}\in\partial\Delta$

$$\big(\Re\mathrm{e}\langle\partial r_j\circ A,\partial_\theta A^\eta\rangle|_1\big)_j=0. \tag{7}$$

We then choose a plane Σ in TM_{z^o} transversal to $i\partial_\tau A^\eta|_1$, e.g.

$$\Sigma = \mathbb{R}_{x_2} \times \cdots \times \mathbb{R}_{x_l} \times \mathbb{C}^{N-l} \subset TM_{z^o} = \mathbb{R}^l \times \mathbb{C}^{N-l},$$

and let $w(\tau)$ and $t^\eta(\tau) = t(\tau) - \eta\chi(\tau)$ be the 'CR' and 'normal' components of A^η respectively. (Note that the CR components $w(\tau)$ are the same for A^η and the initial disc A.) We consider the Bishop equations

$$u = -T_1(h(u, w + w_o, t^\eta) + s) \quad \forall (s, w_o) \in \Sigma,\ \tau \in \partial\Delta.$$

We denote by $u = u^\eta_{sw_o}(\tau)$ the solutions of the above equation. Let $v = T_1 u + h(s, w_o)$. Then $u + iv$ extends holomorphically from $\partial\Delta$ to Δ and form the z' components of a disc $A^\eta_{sw_o}(\tau) = (u(\tau) + iv(\tau), w(\tau) + w_o)$ which verifies $\partial A^\eta_{sw_o} \subset M_1$. Note that $A^\eta_{sw_o}|_{s=0, w_o=0} = A^\eta$. We define

$$M_2 = \bigcup_{sw_o} A^\eta_{sw_o} \tag{8}$$

and denote by $D: \Sigma \times \Delta \to M_2, (s, w_o, \tau) \mapsto A^\eta_{sw_o}(\tau)$ the parametric representation of M_2. We have

$$\operatorname{rank}_{\mathbb{R}} \partial_{sw_o\tau} D|_{(s=0,\ w_o=0,\ \tau=1)} \ = \ 2N - l + 1$$

due to

$$\begin{aligned} &\big(\Re\mathrm{e}\langle \partial r_j \circ A, \partial_\tau A^\eta\rangle|_1\big)_j \neq 0,\\ &\big(\Re\mathrm{e}\langle \partial r_j \circ A, \partial_\theta A^\eta\rangle|_1\big)_j = 0. \end{aligned} \tag{9}$$

Because of (5) the first of the vectors in (9) has nearly the direction of $v^o := G_{z^1} v^1$ that is we can find a vector v^2 parallel to it such that $|v^2 + v^o| < \varepsilon$. Here ε is arbitrarily small provided that we correspondingly shrink η and $\mathrm{supp}(\chi)$. Hence M_2 is a germ of a submanifold at z^o with boundary M and codimension $l - 1$ which points to the normal direction v^2 which verifies $|v^2 + v^o| < \varepsilon$.

We recall now that, according to the celebrated Baouendi–Treves approximation Theorem ([B-T]), there exists a neighborhood $\tilde{M}$ of z^o in M such that any $f \in CR(M)$ is uniformly approximated by polynomials over $\tilde{M}$. Also, for any germ of manifold M_1 with boundary M (at some other point of $\tilde{M}$), we can shrink M_1 to $\tilde{M}_1$ so that any $f \in CR(M_1 \cup M)$ is approximated by polynomials in $V := \tilde{M}_1 \cup \tilde{M}$. We summarize our hypotheses and state our main theorem which is just a rearrangement of what has already been proved. Let M be a generic $C^{k,\alpha}$ manifold in a neighborhood of z^o, M_1 a germ of $C^{k,\alpha}$ manifold at z^1 with boundary M. Let A be a small disc, $C^{k,\alpha}$ in $\bar{\Delta}$, attached to $M_1 \cup M$ with $z^o = A(1)$ and $z^1 = A(-1)$ such that $\partial A \subset M$ at z^o and $\partial A \subset M_1 \setminus M$ at z^1. Let $t = t^1$ at z^1, let M_{t^1} be the submanifold of M_1 defined by $t = t^1$, and let v^1 be the normal direction to M_{t^1} in M_1 at z^1 with the orientation induced by that of M_1 with respect to M.

THEOREM 3. *Let $\partial A \subset M_1 \cup M$ with $\partial A \subset M$ at z^o, $\partial A \subset M_1 \setminus M$ at z^1, and assume that $TA_{z^o} \subset T^{\mathbb{C}}M_{z^o}$. Let V be the open domain of $M_1 \cup M$ in which CR functions are approximated by polynomials, and assume $\partial A \subset V$. Then for any ε there is M_2, manifold with boundary M at z^o, which points to an additional direction v^2 with*

$$|v^2 + v^o| < \varepsilon \text{ for } v^o := G(z^1)v^1, \tag{10}$$

such that any $f \in CR(M_1 \cup M)$ extends as CR to M_2.

Proof. We approximate f over V by a sequence of polynomials P_ν. Since $\partial A \subset V$, then V is a neighborhood of $\partial A^\eta_{sw_o}$ $\forall s,\ w_o$. Hence by maximum principle there exists a subsequence P_μ which converges in $M_2 = \bigcup_{sw_o} A^\eta_{sw_o}$ to an analytic function which is the desired extension of f. □

We can restate the above extension result in terms of propagation. For this we need discs which are indeed attached to M and not to $M_1 \cup M$. We also need to define what a 'wedge' W with 'edge' M is. In a coordinate system in $X = \mathbb{C}^N$ and for an open cone $i\Gamma \subset (T_M X)_{z^o}$, a wedge W with edge M and profile $i\Gamma$, is an open set which contains $\forall \Gamma' \subset\subset \Gamma$ and for a suitable neighborhood V of z^o the set $((M \cap V) + i\Gamma') \cap V$.

THEOREM 4. (i) *Let $\partial A \subset M$, $TA_{z^o} \subset T^{\mathbb{C}}M_{z^o}$, A small. Let v^1 be the normal to M at z^1 which points to M_1, and put $v^o = G_{z^1}v^1$. Then for any ε there is M_2 which points at z^o to a direction v^2 satisfying $|v^2 - v^o| < \varepsilon$, such that any CR function f on M which extends to M_1 at z^1 also extends to M_2 at z^o.*

(ii) *In particular if f extends to a full wedge W_1 with profile $i\Gamma_1$ at z^1, then for any ε, it extends to a wedge W_2 at z^o with profile $i\Gamma_2$ which verifies $(\Gamma_2)_\varepsilon \supset G_{z^1}(\Gamma_1)$. (Here $(\Gamma_2)_\varepsilon$ denotes the ε conical neighborhood of Γ_2.)*

Proof. (i): We make a deformation of M in the v^1-direction at z^1, that we still call M, which is contained in the region where f has CR extension, and such that $\partial A \not\subset M$ at z^1. With this new M, we have that f extends now from M to a manifold M_1 which points to the $-v^1$-direction, and such that $\partial A \subset M_1 \setminus M$ at z^1. (We can also assume that the condition for the approximation neighborhood V is fulfilled.) Then the conclusion follows from Theorem 3.

(ii): We can find a set of l manifolds M_1^j $j = 1, ..., l$ of the type described in (i), which are contained in W_1 at z^1 and point to l directions whose convex hull is an approximation of the cone $i\Gamma_1$. Then f will extend to the corresponding manifolds M_2^j at z^o. Finally we conclude by the Ayrapetian–Henkin edge of the wedge theorem. □

COROLLARY 5. (Hanges–Treves [H-T]). *Let $\gamma \hookrightarrow M$ be a complex curve, let $f \in CR(M)$ and assume that f extends holomorphically at a point $z^1 \in \gamma$. Then it extends at any other point $z^o \in \gamma$.*

Proof. Easy consequence of Theorem 4 (ii) for $i\Gamma_2 = i\mathbb{R}^l$, $i\Gamma_1 = i\mathbb{R}^l$, by a classical compactness argument. □

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