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A category of compositional domain-models for separable Stone spaces☆

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Abstract

In this paper we introduce SFP^M, a category of SFP domains which provides very satisfactory *domain-models*, i.e. "partializations", of separable Stone spaces (2-*Stone spaces*). More specifically, SFP^M is a subcategory of SFP^{ep}, closed under direct limits as well as many constructors, such as lifting, sum, product and Plotkin powerdomain (with the notable exception of the function space constructor). SFP^M is "structurally well behaved", in the sense that the functor MAX, which associates to each object of SFP^M the Stone space of its maximal elements, is compositional with respect to the constructors above, and ω-continuous. A correspondence can be established between these constructors over SFP^M and appropriate constructors on Stone spaces, whereby SFP^M domain-models of Stone spaces defined as solutions of a vast class of recursive equations in SFP^M, can be obtained simply by solving the corresponding equations in SFP^M. Moreover any continuous function between two 2-Stone spaces can be extended to a continuous function between any two SFP^M domain-models of the original spaces. The category SFP^M does not include all the SFP's with a 2-Stone space of maximal elements (CSFP's). We show that the CSFP's can be characterized precisely as suitable retracts of SFP^M objects. Then the results proved for SFP^M easily extends to the wider category having CSFP's as objects.

Using SFP^M we can provide a plethora of "partializations" of the space of finitary hypersets (the hyperuniverse $\mathcal{N}_\omega$ (Ann. New York Acad. Sci. 806 (1996) 140). These includes the classical ones proposed in Abramsky (A Cook's tour of the finitary non-well-founded sets unpublished manuscript, 1988; Inform. Comput. 92(2) (1991) 161) and Mislove et al. (Inform. Comput.

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93(1) (1991) 16), which are also shown to be *non-isomorphic*, thus providing a negative answer to a problem raised in Mislove et al.

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0. Introduction

The problem of finding satisfactory "partializations" of topological spaces, arises in several areas of Mathematics and Computer Science, when dealing with computable approximations of classical notions. A "partialization", or equivalently a *domain-model*, of a topological space $(X, \Omega(X))$, is a domain whose subspace of maximal points endowed with the induced Scott topology is homeomorphic to $(X, \Omega(X))$. The points of the original space appear then as *total*, or *maximal*, elements of its domain-model, and the extra *partial* elements can be seen either as *approximations* of the former, or, equivalently, as the representatives of possibly *intentional properties* of the original space.

Following the pioneering work of Scott, domain-models of real numbers and other metric spaces have been used extensively to study generalized computability on those structures (see e.g., [25,26,10,12]). The interest in domain-models of metric spaces arises also in the study of the relations between metric semantics and order-theoretic semantics of programming languages [26,12,5,23,8].

The problem of defining suitable domain-models of a given topological space, has an inverse. Namely, the problem of characterizing the topological spaces determined by the maximal points of a given class of domains. These spaces, called *maximal spaces* in [18], have been widely studied in the literature. Kamimura and Tang, in [17], characterize the maximal spaces of bounded complete continuous (and algebraic) CPO's. Lawson, in [18], gives an elegant characterization of the maximal spaces of ω-continuous CPO's which are *coherent at the top*, i.e. for which the Scott and Lawson topologies on maximal elements coincide. These are precisely the Polish spaces. Flagg and Kopperman, in [13], prove that the maximal spaces of ω-algebraic CPO's coherent at the top, are exactly the complete separable ultrametric spaces (or equivalently the Polish zero-dimensional spaces). Finally, Martin, in [19], shows that this latter class of spaces is obtained also restricting to the maximal spaces of ω-algebraic Scott domains.

However, even if there has been considerable interest in recent years in domain-models of metric spaces and, conversely, in maximal spaces of domains, little attention has been given so far to investigating how tight can be made the structural correspondence between a space and its "partialization".

In this paper, following an idea originally suggested by Abramsky (see [1,2]), we address this latter issue for the categories of separable Stone spaces (2-*Stone spaces* for short), i.e. compact Hausdorff spaces with a countable basis of clopen sets, and SFP *domains*. This is a very significant situation in the semantics of programming languages. Both categories, in fact, play prominent roles in metric semantics (see [9]) and order theoretic semantics (see [21]), respectively.

The crucial fact which allows to establish a tight correspondence between 2-Stone spaces and their SFP domain-models is that both have a *finitary* nature, i.e. they are limits of sequences of finite structures, namely finite discrete spaces and finite partial orders, respectively. At the level of *finite* structures, we have the following pleasing situation:

1. the subspace of maximal elements of a partial order is a discrete topological space, and every discrete space can be viewed as such a subspace, for suitable partial orders;
2. the functor MAX, which associates to each partial order the space of its maximal elements, is "compositional" with respect to many constructors, e.g., lifting $(.)_\perp$, separated sum $+$, product $\times$ and Plotkin powerdomain $\mathscr{P}_{Pl}$;
3. any function on maximal elements can be extended to a monotone function on the partial orders.

Thus, one can define *compositionally* domain-models of (at least) finite discrete topological spaces.

In this paper we show that what happens at finite level can be generalized to the ω-limit. In particular we introduce a suitable (non-full) subcategory SFP^M of SFP^{ep} closed under direct limits as well as under the above mentioned constructors. The maximal space of every SFP^M object is a 2-Stone space and, conversely, every 2-Stone space can be viewed as the subspace of maximal elements of an object in SFP^M. This category provides very satisfactory domain-models of 2-Stone spaces, since the functor MAX, from SFP^M to 2-Stone, is ω-continuous and "compositional" with respect to several domain constructors such as those listed in the following correspondence table:

SFP^M	$(\cdot)_\perp$	$\times$	$+$	$\mathscr{P}_{Pl}$
2-Stone	Id	$\times$	$\uplus$	$\mathscr{P}_{nco}$

where, for a 2-Stone space X, $\mathscr{P}_{nco}(X)$ denotes the set of non-empty compact subsets of X endowed with the Vietoris topology. More precisely we introduce a class $\mathscr{F}$ of constructors in SFP^M including the above and closed under composition and minimalization. For each $F \in \mathscr{F}$ we show that the "corresponding" constructor $\bar{F}$ over 2-Stone (defined inductively according to the above table) is modelled by F, i.e., $\mathsf{MAX} \circ F = \bar{F} \circ (\mathsf{MAX}, \ldots, \mathsf{MAX})$. Thus we can provide naturally SFP domain-models of Stone spaces, defined as solutions of a vast class of domain equations in 2-Stone, by simply solving the corresponding equations in SFP^M. Furthermore any continuous function between 2-Stone spaces can be extended to a continuous function between any two SFP^M domain-models of the original spaces.

The category SFP^M does not include all CSFP's, i.e., all the SFP's with a 2-Stone space of maximal elements. We show that the CSFP's can be characterized precisely as the retracts of SFP^M objects via M-pairs. The corresponding category $\mathsf{cSFP}^{\mathsf{M}}$ of CSFP's and M-pairs, which has SFP^M as a full subcategory, enjoys properties analogous to those proved for SFP^M. First, it is closed under direct limits and under the constructors in the class $\mathscr{F}$. Moreover, the functor MAX extends to a well-defined ω-continuous functor over cSFP^M, compositional with respect to the constructors in $\mathscr{F}$.

Unfortunately, the mentioned results cannot be extended to include the *function space constructor*: neither SFP^M nor 2-Stone are closed under the function space constructor and the attempt of finding a functor over topological spaces which models the function space constructor over domains appears to be hopeless also in wider categories of topological spaces and SFP's.

Using SFP^M as an ambient category, we can obtain various partializations of finitary hypersets, i.e., of the closure with respect to the "bisimulation metric" of the space of hereditarily finite hypersets, including those proposed in the literature by Abramsky [2,3] and Mislove et al. [20]. The space of finitary hypersets is homeomorphic to the hyperuniverse $\mathscr{N}_\omega$ of [14] and it appears quite frequently in topology under different perspectives, e.g., as the Cantor-1 space, i.e., the union of Cantor's discontinue, obtained by the standard middle third removal construction plus the centers of all the removed intervals. Abramsky in [2] defines his domain directly by picking the initial solution of an appropriate equation in SFP^{ep}. The same equation is used in [3] to define the domain *Synchronization trees with divergence* (over a single action). Mislove, Moss and Oles, on the other hand, introduce their domain as the initial *continuous set algebra* [20]. These two domains arise as solutions of *different* domain equations in SFP^{ep}. The well-known fact that the solutions of such domain equations have homeomorphic maximal spaces comes also as an immediate application of the results in this paper. Actually, our results show that there is indeed a *plethora* of reflexive domain equations whose initial solutions have the hyperuniverse $\mathscr{N}_\omega$ as maximal space. There being so many different domain equations yielding domain-models for the finitary hypersets the natural question arises as to whether such domain-models are isomorphic. A special case of this question was formally raised in [20] concerning the two domains mentioned above. In this paper we show that such domains are not isomorphic and that, more generally, there exists an infinite number of non-isomorphic domain-models for the space of finitary hypersets. However, it is a matter of further investigation to find out if such domain-models have significant independent characterizations as those in [3,20].

Throughout the paper we use standard notation and basic facts of Domain Theory and Topology (see [22,11,4,24]). In Section 1 we give the basic definitions and we recall some useful properties of Stone spaces and SFP domains. In Section 2 we define the category SFP^M, providing two alternative characterizations for its objects. In Section 3 we show that SFP^M is closed under direct limits as well as under a significant family of constructors. In Section 4 we establish a tight structural relation between SFP^M and the category of 2-Stone spaces, by introducing the functor MAX. In Section 5 we discuss the problem of extending continuous functions between 2-Stone spaces to their SFP^M domain-models. In Section 6 we study the retracts of SFP^M objects, providing a characterization for the class of CSFP's. In Section 7 we study domain-models of finitary hypersets, focusing on those of [20] and [2,3]. Final remarks appear in Section 8.

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1. Stone spaces and SFP domains

In this section we recall some notations, definitions and basic facts about Stone spaces and SFP domains (see e.g., [22,11,4,24] for more details). Both kinds of objects are *finitary* in the sense that they can be obtained as limits of sequences of finite objects in the corresponding categories.

1.1. Topological spaces and Stone spaces

A topological space will be denoted by $(X, \Omega(X))$ where X is the underlying set and $\Omega(X)$ the topology, or simply by X when the topology is clear from the context. The category of topological spaces and continuous functions will be denoted by Top.

Let $\langle X_n, f_n\rangle_n$ be an inverse sequence in Top, i.e., a sequence $X_0 \overset{f_0}{\longleftarrow} X_1 \overset{f_1}{\longleftarrow} X_2 \cdots$ of topological spaces and continuous functions. The (*inverse*) *limit* of $\langle X_n, f_n\rangle_n$, denoted by $\varprojlim\langle X_n, f_n\rangle_n$, is the categorical limit of the sequence. It can be characterized as the set $X = \{(x_n)_n \in \Pi_n X_n : \forall n \geqslant 0.\ f_n(x_{n+1}) = x_n\}$, considered as a subspace of the product $\Pi_n X_n$, together with the obvious projections $\pi_n : X \to X_n$.

Definition 1 (2-Stone spaces). A 2-*Stone* space is a compact, Hausdorff space with a countable basis of clopen sets. We denote by $\mathsf{2\text{-}Stone}$ the full subcategory of Top consisting of 2-Stone spaces.

The following proposition recalls some alternative characterizations of 2-Stone spaces which will be useful in the sequel.

Proposition 2. *Let $(X, \Omega(X))$ be a topological space. The following are equivalent*:
1. $(X, \Omega(X))$ *is a* 2-*Stone space*;
2. $(X, \Omega(X)) = \varprojlim\langle (X_n, \Omega(X_n)), f_n\rangle_n$ (X_n *finite*, $\Omega(X_n)$ *discrete topology*);
3. $(X, \Omega(X))$ *is compact and ultrametrizable with a distance function* $d : X \times X \to \{0\} \cup \{2^{-n} : n \in \mathbb{N}\}_n$.

1.2. Partial orders, CPO's and SFP's

A *complete partial order* (or CPO for short) will be denoted by $(D, \sqsubseteq)$ or simply by D. Given an element $d \in D$ we will write $\uparrow d$ for the *upper set* $\{x \in D : d \sqsubseteq x\}$ and $\downarrow d$ for the *lower set* $\{x \in D : x \sqsubseteq d\}$. Given two CPO's D and E, an *embedding-projection pair* (*ep-pair*) $p : D \to E$ is any pair of continuous functions $\langle i : D \to E,\ j : E \to D\rangle$ such that $i \circ j \sqsubseteq id_E$ and $j \circ i = id_D$. We denote by CPO^{ep} the category of CPO's and embedding-projection pairs. Let $\langle D_n, p_n\rangle_n$ be a directed sequence in CPO^{ep}, namely a sequence $D_0 \stackrel{p_0}{\to} D_1 \stackrel{p_1}{\to} D_2 \cdots$ of CPO's and ep-pairs $p_n = \langle i_n, j_n \rangle$. The (*direct*) *limit* of $\langle D_n, p_n \rangle_n$, denoted by $\lim\limits_{\to} \langle D_n, p_n \rangle_n$, is the categorical colimit of the sequence. It can be characterized as the set $D = \{(d_n)_n \in \Pi_n D_n : \forall n \geqslant 0.\ j_n(x_{n+1}) = x_n\}$, endowed with the pointwise order, together with the canonical ep-pairs $\gamma_n : D_n \to D$. Typically, we will denote by i_n and j_n the components of each ep-pair p_n and by $\gamma_n = \langle \alpha_n, \beta_n \rangle$ the canonical ep-pair from each D_n into the direct limit. Moreover, for $n, m \in \mathbb{N}$ we will write $p_{n,n+k}$ for the ep-pair $p_{n+k-1} \circ \cdots \circ p_n : D_n \to D_{n+k}$ with components $i_{n,n+k}$ and $j_{n,n+k}$. For $k = 0$ it is intended that $p_{n,n}$ represent the identity pair.

Definition 3 (SFP Domains). A *sequence of finite posets* (SFP domain or simply SFP) is a partial order which is the direct limit of a directed sequence of finite *CPO*'s in CPO^{ep}. We denote by SFP^{ep} the full subcategory of CPO^{ep} consisting of *SFP* domains.

Let D be an algebraic CPO and let $\mathsf{K}(D)$ be the set of its compact elements. Given $X \subseteq \mathsf{K}(D)$, we write $\mathscr{U}(X)$ for the set of *minimal upper bounds* of X. The set $\mathscr{U}(X)$ is said to be *complete* if for each upper bound y of X there exists $x \in \mathscr{U}(X)$ such that $x \sqsubseteq y$. Moreover $\mathscr{U}^*(X)$ denotes the smallest set containing X and closed under $\mathscr{U}$. The following proposition gives a well known alternative characterization of SFP domains.

Proposition 4. Let $(D, \sqsubseteq)$ be a partial order. Then D is an SFP if and only if (i) D is an ω-algebraic CPO and for every finite $X \subseteq \mathsf{K}(D)$, (ii) the set of minimal upper bounds $\mathscr{U}(X)$ is finite and complete and (iii) $\mathscr{U}^*(X)$ is finite.

If D satisfies only the first two of the three conditions above it is called a 2/3 SFP (or a *coherent ω-algebraic domain*).

Given an ω-algebraic CPO D and an enumeration $\mathsf{K}(D) = \{a_0, a_1, a_2, \ldots\}$ of its compact elements, a subbasis for the *Lawson topology* on D is given by the sets

$$\{\uparrow a, (\uparrow a)^c \ :\ a \in \mathsf{K}(D)\},$$

where X^c denotes the complement of X in D, i.e. $D \backslash X$. The Lawson topology is always metrizable with an ultrametric

$$d(x, y) = \inf\{2^{-n} \colon \forall i \leqslant n.\ a_i \in \downarrow x \text{ iff } a_i \in \downarrow y\}, \qquad \text{for } x, y \in D.$$

The following lemma shows that when restricted to the space of maximal elements of a 2/3 SFP, the Scott and Lawson topologies coincide, or, according to the terminology of [19], every 2/3 SFP is *coherent at the top*. A similar proof is used in [17] (Lemma 3.1) to show that bounded and directed complete ω-continuous CPO's are coherent at the top. Both results can be seen as a consequence of Corollary 3.4 in [18], where it is shown that coherence at the top holds of any ω-continuous CPO for which the Lawson topology is compact. Indeed, the explicit proof provided here essentially relies on the fact that, by the 2/3 SFP theorem (see [22, Theorem 7.8]), if (and only if) D is a 2/3 SFP then the Lawson topology on D is compact. Hereafter the topologies induced by the Scott and Lawson topologies over the maximal space of a domain D will be denoted by $\mathscr{S}_D$ and $\mathscr{L}_{\mathscr{D}}$, respectively.

Lemma 5 (Coherence at the top). *Let D be a 2/3 SFP and let $\mathsf{Max}(D)$ be the subset of maximal elements of D. Then the induced topologies $\mathscr{L}_{\mathscr{D}}$ and $\mathscr{S}_D$ over $\mathsf{Max}(D)$ coincide.*

Proof. The inclusion $\mathscr{S}_D \subseteq \mathscr{L}_{\mathscr{D}}$ is trivial. In order to show the converse inclusion we prove that $\mathsf{Max}((\uparrow a)^c)$ is open in $(\mathsf{Max}(D), \mathscr{S}_D)$. Consider $x \in \mathsf{Max}((\uparrow a)^c)$. Since D is ω-algebraic there exists a chain $(a_n)_n$ of compact elements such that $x = \bigsqcup_n a_n$. We state that $\mathsf{Max}(\uparrow a_n) \subseteq \mathsf{Max}((\uparrow a)^c)$ for some n. In fact, suppose by contradiction that for every n there exists $y_n \in \mathsf{Max}(\uparrow a_n) \cap \mathsf{Max}(\uparrow a)$. Since D is a 2/3 SFP, the Lawson topology is compact. Thus $(y_n)_n$ admits a converging subsequence $(y_{n_k})_k$, whose limit y must be in $\uparrow a$, since $\uparrow a$ is Lawson closed. Now, $(a_{n_k})_k$ is a chain, hence $a_{n_k} \sqsubseteq a_{n_h} \sqsubseteq y_{n_h}$ for all $h \geqslant k$ and thus, since $\uparrow a_{n_k}$ is Lawson closed, $a_{n_k} \sqsubseteq y$ for all k. Thus $\bigsqcup_k a_{n_k} = x \sqsubseteq y$. By maximality of x we have that $x = y$, contradicting $y \in \uparrow a$.

Summing up, for each $x \in \mathsf{Max}((\uparrow a)^c)$ there exists a compact element $b \sqsubseteq x$ such that $x \in \mathsf{Max}(\uparrow b) \subseteq \mathsf{Max}((\uparrow a)^c)$. Thus $\mathsf{Max}((\uparrow a)^c)$ is open in $(\mathsf{Max}(D), \mathscr{S}_D)$. □

The next proposition will be used to prove that, when dealing with a direct limit in SFP^{ep}, certain properties of compact elements can be tested at a finite level. In the sequel we will write $A \subseteq_{fin} B$ to mean that A is a finite subset of B.

Proposition 6. *Let $\langle D_n, p_n\rangle_n$ be a directed sequence in SFP^{ep} and let $D = \varinjlim \langle D_n, p_n\rangle_n$ and let $\gamma_n = \langle \alpha_n, \beta_n\rangle : D_n \to D$ be the canonical ep-pairs from each D_n into the limit. Then*:
1. $u \subseteq_{fin} \mathsf{K}(D) \Leftrightarrow \exists n.\ \exists u_n \subseteq_{fin} \mathsf{K}(D_n).\ u = \alpha_n(u_n)$;
2. $\forall n.\ \forall u_n \subseteq_{fin} \mathsf{K}(D_n).\ \mathscr{U}^*(\alpha_n(u_n)) = \alpha_n(\mathscr{U}_n^*(u_n))$.

2. The category SFP^M

In this section we introduce the category SFP^M, a subcategory of SFP^{ep} which provides domain-models, exactly, for the class of 2-Stone spaces. Objects in SFP^M are defined as special direct limits in SFP^{ep}, but we provide also an “intrinsic” characterization of SFP^M and a characterization in terms of retractions. Besides shedding some light on the structure of SFP^M domains, such characterizations will be helpful in the next section to prove some interesting closure properties of SFP^M.

2.1. *Definition of* SFP^M

A first basic observation which guides us to the definition of the category SFP^M is a direct consequence of Lemma 5.

Proposition 7 (Maximal spaces of SFP’s). *Let $(D, \sqsubseteq)$ be a (2/3) SFP. Then $(\mathsf{Max}(D), \mathscr{S}_D)$ is a Hausdorff space, with a countable basis of clopen sets.*

By the above result, if the maximal space of an SFP is not a 2-Stone space the only possible reason is the lack of compactness. Indeed, not all SFP's have a compact maximal space. For instance $\mathbb{N}_\perp$ is clearly an SFP and the space of maximal elements $(\mathsf{Max}(\mathbb{N}_\perp), \mathscr{S}_{\mathbb{N}_\perp})$ is a discrete infinite space, hence it is not compact.

We will show that a sufficient, although not necessary (see Section 2.5), condition on D which ensures the compactness of $\mathsf{Max}(D)$ is the existence of a directed sequence of finite posets with limit D, where *projections preserve maximal elements*. For a sufficient and *necessary* condition the reader is referred to Section 6. First we need the following definition.

Definition 8 (M-pair). Let D and E be *SFP*'s. An ep-pair $p = \langle i, j\rangle : D \to E$ is called a *maximals preserving pair*, or *M-pair*, if $j(\mathsf{Max}(E)) \subseteq \mathsf{Max}(D)$.

Notice that if $p = \langle i, j\rangle : D \to E$ is an M-pair then $j(\mathsf{Max}(E)) = \mathsf{Max}(D)$. In fact, by surjectivity of j, for all $x \in \mathsf{Max}(D)$ there exists $y \in E$ such that $j(y) = x$. Hence if $y' \in \mathsf{Max}(\uparrow y)$ we have $j(y') = x$. Moreover, it is immediate to verify that M-pairs are closed under composition. Finally we can give the first definition of the category SFP^M.

Definition 9 (Category SFP^M). The category SFP^M has as objects SFP's that are limits of directed sequences of finite *CPO*'s and M-pairs (in SFP^{ep}). Morphisms are M-pairs. Identities and composition are standard.

2.2. SFP^M *provides domain-models for* 2-Stone*, precisely!*

In this subsection we prove that for each SFP^M object D the maximal space $\mathsf{Max}(D)$ is a 2-Stone space, and vice versa, that each 2-Stone space X has a domain-model in SFP^M. The first part amounts essentially to proving that the maximal space of any SFP^M object is a Lawson closed subspace of the whole domain. Thus, exploiting the compactness of the Lawson topology for an SFP, we immediately conclude that also the maximal space is compact. Vice versa, given any 2-Stone space X, an SFP^M domain-model for X is constructed by taking the set of closed balls of X, ordered by reverse inclusion.

A first technical lemma shows that given a directed sequence $\langle D_n, p_n\rangle_n$ of SFP's and M-pairs, if each D_n has a compact maximal space then the maximal elements of the direct limit are sequences of maximal elements of the single D_n's.

Lemma 10. *Let* $\langle D_n, p_n\rangle_n$ *be a directed sequence of* SFP's *and M-pairs, and let* $D = \lim\limits_{\to}\langle D_n, p_n\rangle_n$. *Suppose that for each n, the maximal space* $(\mathsf{Max}(D_n), \mathscr{S}_{D_n})$ *is compact. Then for any* $x = (x_n)_n \in D$,

$$x \in \mathsf{Max}(D) \;\; \textit{iff} \;\; x_n \in \mathsf{Max}(D_n), \quad \textit{for all } n \in \mathbb{N}.$$

Proof. ($\Leftarrow$) Assume $x_n \in \mathsf{Max}(D_n)$, for all $n \in \mathbb{N}$. Given $y \in D$, if $x \sqsubseteq y$, i.e., $x_n \sqsubseteq y_n$ for all n, then by maximality of x_n we have $x_n = y_n$ for all n and thus $x = y$.

($\Rightarrow$) Let $x=(x_n)_n \in \mathsf{Max}(D)$ and, for all n, let $y_n \in \mathsf{Max}(D_n)$ such that $x_n \sqsubseteq y_n$. We build, for all k, a sequence $z^{(k)} \in \Pi_n \mathsf{Max}(D_n)$ whose components $(z^{(k)})_n \in \mathsf{Max}(D_n)$ are defined as follows:

$$(z^{(k)})_n = \begin{cases} j_{n,k}(y_k) & \text{if } n < k, \\ y_k & \text{if } n = k, \\ \text{any } z \in j_{n-1}^{-1}((z^{(k)})_{n-1}) \cap \mathsf{Max}(D_n) & \text{if } n > k. \end{cases}$$

Notice that, by definition of M-pair, $j_{n,k}(y_k)$ is maximal in D_k. Furthermore $j_{n-1}(\mathsf{Max}(D_n)) = \mathsf{Max}(D_{n-1})$ and thus $j_{n-1}^{-1}((z^{(k)})_{n-1}) \cap \mathsf{Max}(D_n)$ is not empty.

By hypothesis each $\mathsf{Max}(D_n)$ is compact and thus, by the Tychonoff Theorem, $\Pi_n \mathsf{Max}(D_n)$, with the product topology, is compact. Therefore $z^{(k)}$ admits a subsequence $z^{(k_m)}$ converging to $z \in \Pi_n \mathsf{Max}(D_n)$.

Let $\gamma_n = \langle \alpha_n, \beta_n \rangle$ be the canonical ep-pair from each D_n into the direct limit D. By definition of $z^{(k)}$ and taking into account that $\alpha_n(x_n) \sqsubseteq \alpha_{n+n'}(x_{n+n'})$, it follows that, for $k \geqslant n$, $\alpha_n(x_n) \sqsubseteq \alpha_k(x_k) \sqsubseteq \alpha_k(y_k) \sqsubseteq z^{(k)}$. In particular, since for each h the single component $(z^{(k_m)})_h$ converges to z_h w.r.t. the Lawson metric and $\Uparrow(\alpha_n(x_n))_h$ is Lawson closed, $(\alpha_n(x_n))_h \sqsubseteq z_h$ and thus $\alpha_n(x_n) \sqsubseteq z$. Therefore $x \sqsubseteq z$ and thus, by maximality of x, $x = z$. Recalling that $z_n \in \mathsf{Max}(D_n)$ for each $n \in \mathbb{N}$, we get the thesis. $\square$

Observe that the above lemma applies, in particular, when $\langle D_n, p_n \rangle_n$ is a directed sequence of *finite* posets, since in this case each $\mathsf{Max}(D_n)$ is obviously compact.

One can easily check that projections are Lawson continuous. In fact, if $p = \langle i, j \rangle : D \to E$ is an ep-pair, then, for any $d \in D$, $\uparrow i(d) = j^{-1}(\uparrow d)$ and thus $(\uparrow i(d))^c = j^{-1}((\Uparrow d)^c)$. This simple remark is useful in proving the following lemma.

Lemma 11. *Let $\langle D_n, p_n \rangle_n$ be a directed sequence of finite posets and M-pairs, and let $D = \varinjlim \langle D_n, p_n \rangle_n$ in SFP^{ep}. Then $\mathsf{Max}(D)$ is Lawson closed in D, hence compact.*

Proof. Let $(x_k)_k$ be a sequence in $\mathsf{Max}(D)$ converging to $x \in D$. Since projections are Lawson continuous, for each n, the sequence $(\beta_n(x_k))_k$ converges to $\beta_n(x)$. Therefore, by finiteness of D_n, there exists k_0 such that $\beta_n(x_k) = \beta_n(x)$ for all $k \geqslant k_0$ and, by Lemma 10, $\beta_n(x_k) \in \mathsf{Max}(D_n)$. Hence $\beta_n(x) \in \mathsf{Max}(D_n)$ for all n, and thus, again by Lemma 10, $x \in \mathsf{Max}(D)$. $\square$

By exploiting the above lemma we can finally prove the main theorem of this section.

Theorem 12. *For any SFP^M object D the space $(\mathsf{Max}(D), \mathscr{S}_D)$ is a 2-Stone space. Vice versa for any 2-Stone space X there exists an SFP^M object D such that $X \simeq (\mathsf{Max}(D), \mathscr{S}_D)$.*

Proof. For the first part, since D is an SFP, by Proposition 7, $(\mathsf{Max}(D), \mathscr{S}_D)$ is a Hausdorff space, with a countable basis of clopen sets. Moreover, by Lemma 11, the Lawson topology on $\mathsf{Max}(D)$ is compact. Recalling that, by Lemma 5, the Scott and the Lawson topologies coincide on the maximal space of a 2/3 SFP, we conclude that $(\mathsf{Max}(D), \mathscr{S}_D)$ is compact and therefore a 2-Stone space.

Vice versa, let X be a 2-Stone space. We know by Proposition 2 that X is metrizable with an ultrametric $d : X \times X \to \{0\} \cup \{2^{-n}\colon n \in \mathbb{N}\}$. Following a classical idea (see, e.g., [26,7,5,12]) one can consider the ideal completion of the partial order of closed balls of a metrization of X, ordered by reverse inclusion, namely

$$D_X = \mathsf{Idl}(\{\mathsf{B}(x, 2^{-n})\colon x \in X \wedge n \in \mathbb{N}\}, \supseteq),$$

where $\mathsf{B}(x,r)$ denotes the closed ball with centre x and radius r, i.e., $\{y \in X : d(x,y) \leqslant r\}$. Then D^X is an ω-algebraic CPO where incomparable elements have no upper bounds, i.e. D^X is a finitary (finitely branching) tree. Hence D^X is in SFP^M, since it can be obtained as the limit of a directed sequence $\langle D_n, p_n \rangle_n$, where D_n is the subtree of D including elements of height less than n and $i_n : D_n \to D_{n+1}$ is the inclusion. Maximal elements of D^X can be identified with maximal chains in $(\{\mathsf{B}(x, 2^{-n}) : x \in X \wedge n \in \mathbb{N}\}, \supseteq)$ and the function $f : (\mathsf{Max}(D^X), \mathscr{S}_{D^X}) \to (X, \Omega(X))$ mapping a chain $(B_n)_n$ to the sole point in $\bigcap_n B_n$ is a homomorphism. □

Observe that the domain-model D^X defined in the proof above contains only elements corresponding to a system of disjoint clopen sets. However it is not a "minimal" domain-model. In fact, a "minimal" domain-model does not exist, in general, since we can always remove in the tree "branches" of level less than n for a fixed n.

2.3. An intrinsic characterization of SFP^M

We give now an "intrinsic" characterization of SFP^M objects in terms of an order-theoretic property, that amounts, basically, to a "compactness" requirement. This will be essential later in proving the closure of SFP^M with respect to direct limits.

Definition 13 (M-condition). We say that an *SFP D* satisfies the *M-condition* if for all $u \subseteq_{fin} \mathsf{K}(D)$ there exists $v \subseteq_{fin} \mathsf{K}(D)$ such that:
1. $u \subseteq v$,
2. $\mathsf{Max}(\mathscr{U}^*(v)) \sqsubseteq_s \mathsf{Max}(D)$,

where $\sqsubseteq_s$ is the Smyth preorder (i.e., $u \sqsubseteq_s v$ iff $\forall y \in v.\ \exists x \in u.\ x \sqsubseteq y$).

In order to show that SFP^M objects are exactly those SFP's which satisfy the M-condition we proceed as follows. First we prove that the M-condition is preserved under direct limits. Then, noticing that every finite CPO satisfies the M-condition, we can conclude that each SFP^M object satisfies the M-condition. For the converse, given an SFP satisfying the M-condition, we explicitly show how it can be obtained as direct limit of a directed sequence of finite CPO's and M-pairs.

Lemma 14. *Let* $D = \lim\limits_{\to} \langle D_n, p_n \rangle_n$, *with* $\langle D_n, p_n \rangle_n$ *directed sequence of* SFP^M *objects and M-pairs. If each* D_n *satisfies the M-condition then also D satisfies the M-condition.*

Proof. Let $u \subseteq_{fin} \mathsf{K}(D)$. By Proposition 6(1), there exist $n \in \mathbb{N}$ and $u_n \subseteq_{fin} \mathsf{K}(D_n)$ such that $u = \alpha_n(u_n)$. Since each D_n satisfies the M-condition, there exists $v_n \subseteq_{fin} \mathsf{K}(D_n)$ such that $u_n \subseteq v_n$ and $\mathsf{Max}(\mathscr{U}^*(v_n)) \sqsubseteq_s \mathsf{Max}(D_n)$.

We show that $v = \alpha_n(v_n)$ is the finite set of compact elements required by the M-condition. In fact, clearly, $u \subseteq v$. Moreover, $\mathsf{Max}(\mathscr{U}^*(v)) \sqsubseteq_s \mathsf{Max}(D)$. In fact, let $x \in \mathsf{Max}(D)$. By Theorem 12, each $\mathsf{Max}(D_n)$ is compact, and thus, by Lemma 10, $\beta_n(x) \in \mathsf{Max}(D_n)$. Hence, by construction, there exists $a_n \in \mathsf{Max}(\mathscr{U}^*(v_n))$ such that $a_n \sqsubseteq \beta_n(x)$. By Proposition 6(2), $\alpha_n(a_n) \in \mathsf{Max}(\mathscr{U}^*(\alpha_n(v_n))) = \mathsf{Max}(\mathscr{U}^*(v))$ and $\alpha_n(a_n) \sqsubseteq \alpha_n(\beta_n(x)) \sqsubseteq x$. $\square$

Theorem 15 (Intrinsic characterization of SFP^M objects). *Let D be an* SFP. *Then D is an SFP^M object iff D satisfies the M-condition.*

Proof. ($\Rightarrow$) Let D be an SFP^M object; hence D is the limit of a directed sequence $\langle D_n, p_n \rangle_n$ of finite CPO's and M-pairs. Since each D_n is finite, it is trivially an SFP^M object and it satisfies the M-condition. Thus, by Lemma 14, also D satisfies the M-condition.

($\Leftarrow$) Let D be an SFP that satisfies the M-condition and let $a_0\,(=\bot), a_1, a_2, \ldots$ be an enumeration of its compact elements. Define inductively a sequence $(D_n)_n$ of finite subspaces of D as follows:

$$D_0 = \{a_0\} \quad \text{and} \quad D_{n+1} = \mathscr{U}^*(v_n), \quad \text{for all } n \in \mathbb{N},$$

where $v_n \subseteq_{fin} \mathsf{K}(D)$ is such that $D_n \cup \{a_{n+1}\} \subseteq v_n$ and $\mathsf{Max}(\mathscr{U}^*(v_n)) \sqsubseteq_s \mathsf{Max}(D)$ (such a v_n exists since D satisfies M-condition). For all n, let $p_n = \langle i_n, j_n \rangle : D_n \to D_{n+1}$, defined by

$$i_n(d_n) = d_n, \quad \text{for all } d_n \in D_n,$$

$$j_n(d_{n+1}) = \bigsqcup \{x \in D_n : x \sqsubseteq d_{n+1}\}, \quad \text{for all } d_{n+1} \in D_{n+1}.$$

One can easily check that p_n is a well defined ep-pair. In particular, from the fact that $\bot \in D_n$, using the definition of $\mathscr{U}$, it follows that for $d_{n+1} \in D_{n+1}$, the set $\{x \in D_n : x \sqsubseteq d_{n+1}\}$ is non-empty and directed.

Given $d_{n+1} \in \mathsf{Max}(D_{n+1})$ we show that there is a unique $d_n \in \mathsf{Max}(D_n)$ such that $d_n \sqsubseteq d_{n+1}$. First we prove the existence of such d_n. Let $x \in \mathsf{Max}(D) \cap (\uparrow_D d_{n+1})$. Since $\mathsf{Max}(D_n) \sqsubseteq_s \mathsf{Max}(D)$, there exists $d_n \in D_n$ such that $d_n \sqsubseteq x$. Recalling that $D_n \subseteq D_{n+1}$, we deduce that $\mathscr{U}(\{d_n, d_{n+1}\})$ is included in D_{n+1}, and it is non-empty, otherwise $(\uparrow d_n) \cap (\uparrow d_{n+1})$ should be empty. Since d_{n+1} is maximal in D_{n+1}, it follows that $d_{n+1} \in \mathscr{U}(\{d_n, d_{n+1}\})$, hence $d_n \sqsubseteq d_{n+1}$. As for uniqueness, if $d'_n \in \mathsf{Max}(D_n)$, $d'_n \sqsubseteq d_{n+1}$, then $\mathscr{U}(\{d_n, d'_n\}) \subseteq D_n$ is non-empty. But, since d_n and d'_n are maximal, they must coincide. Therefore $j_n(d_{n+1}) = \bigsqcup \{x \in D_n : x \sqsubseteq d_{n+1}\}$ is such unique d_n and thus p_n is indeed an M-pair.

Finally for each n we define an ep-pair $\langle \alpha_n, \beta_n \rangle : D_n \to D$:

$$\alpha_n(d_n) = d_n, \quad \text{for all } d_n \in D_n,$$

$$\beta_n(d) = \bigsqcup \{x \in D_n : x \sqsubseteq d\}, \quad \text{for all } d \in D.$$

One can easily check that $\langle D, \langle \alpha_n, \beta_n \rangle_n \rangle$ is a cocone for the directed sequence $\langle D_n, p_n \rangle_n$, and it is initial since $\bigsqcup_n \alpha_n \circ \beta_n(d) = \bigsqcup_n \bigsqcup \{x \in D_n : x \sqsubseteq d\} = d$. Hence $D \simeq \varinjlim \langle D_n, p_n \rangle_n$. Since all D_n are finite CPO's and all p_n are M-pairs we conclude that D is an SFP^M object. $\square$

2.4. A characterization of SFP^M based on retractions

Finally, we provide a characterization of SFP^M objects in terms of retractions. More precisely we characterize such domains as those SFP domains having a finitely branching finitary tree as continuous retract via a special kind of M-pair. Intuitively, since an SFP^M object is the limit of a directed sequence where projections preserve maximality of points, a maximal point added in certain approximation must dominate a single maximal point of the previous approximation. Hence the set of maximal elements of every approximation, endowed with the induced order, forms a finitely branching finitary tree. The retraction projects each point x of the original domain to the greatest element dominated by x in the tree. This result will be used in Section 5 to prove that any continuous function between 2-Stone spaces extends to a continuous function between any SFP^M domain-models of such spaces.

We first prove that for all and only the SFP^M objects it is possible to single out a special subset of the compact elements, called a *skeleton*, which is a finitely branching finitary tree. Then (the completion of) each skeleton is shown to be a retract of the original domain via an M-pair which restricts to a homeomorphism between the maximal spaces. Conversely, any SFP which can be projected over (the completion of) a finitely branching finitary tree via a retraction of this kind is shown to have a skeleton, and thus to be an SFP^M object.

Before introducing the notion of skeleton, we fix the notation. A *tree* is a poset T where compatible elements are totally ordered, i.e., for any $a, a' \in T$, if a and a' are compatible, written $a \Uparrow a'$, then $a \sqsubseteq a'$ or $a' \sqsubseteq a$. A tree T is called *finitary* if for any $a \in T$, $\downarrow a$ is finite, and *finitely branching* if for any $a \in T$, the set $\mathsf{Succ}(a) = \{b \in T : a \sqsubset b \wedge \forall x.\ (a \sqsubseteq x \sqsubseteq b \Rightarrow x = a \vee x = b)\}$ is finite.

Definition 16 (Skeleton). A *skeleton* of an *SFP* domain D is a subset of its compact elements $\mathsf{SK}(D) \subseteq \mathsf{K}(D)$ such that
1. $\mathsf{SK}(D)$, with the induced order, is a finitely branching finitary tree;
2. for any $a \in \mathsf{K}(D)$. $\exists d \in \mathsf{SK}(D)$. $a \sqsubseteq d$.

For each $d \in D$ we define $\mathsf{K}(d) = \mathsf{K}(D) \cap (\downarrow d)$ and $\mathsf{SK}(d) = \mathsf{SK}(D) \cap (\downarrow d)$. Observe that $\mathsf{SK}(d) \subseteq \mathsf{K}(d)$ and $\mathsf{SK}(d)$ is a totally ordered subset of $\mathsf{SK}(D)$. In fact, if $d \in \mathsf{K}(D)$, by definition of skeleton, $d \sqsubseteq d'$ for some $d' \in \mathsf{SK}(D)$, and thus $\mathsf{SK}(d)$ is included in

$\mathsf{SK}(d')$, which is totally ordered since $\mathsf{SK}(D)$ is a tree. If d is not compact, just use the fact that, by algebraicity of D, $d = \bigsqcup \mathsf{K}(d)$. The next proposition shows that a skeleton contains enough information to "reconstruct" the maximal elements of the original space.

Proposition 17. *Let D be an* SFP *and let $\mathsf{SK}(D)$ be a skeleton of D. For any $x \in \mathsf{Max}(D)$, $x = \bigsqcup \mathsf{SK}(x)$. Furthermore the space $(\mathsf{Max}(D), \mathscr{S}_D)$ is compact.*

Proof. Let $x \in \mathsf{Max}(D)$. By ω-algebraicity of D, there exists a chain $(a_n)_n$ in $\mathsf{K}(D)$ such that $d = \bigsqcup_n a_n$ and by definition of skeleton, for any n, there is $b_n \in \mathsf{SK}(D)$ such that $a_n \sqsubseteq b_n$. Since $\mathsf{SK}(D)$ is a finitely branching finitary tree, the sequence $(b_n)_n$ surely includes a chain $(b_{n_k})_k$. Taking its least upper bound, we obtain $x \sqsubseteq \bigsqcup_k b_{n_k}$, and thus, by maximality of x, $x = \bigsqcup_k b_{n_k} = \bigsqcup \mathsf{SK}(x)$.

To prove the compactness of the space $(\mathsf{Max}(D), \mathscr{S}_D)$, observe that, by the previous point, $\{\mathsf{Max}(\Uparrow a) : a \in \mathsf{SK}(D)\}$ is a basis for $(\mathsf{Max}(D), \mathscr{S}_D)$, and use the fact that $\mathsf{SK}(D)$ is a finitely branching finitary tree. □

The next lemma shows that the SFP's which admit a skeleton are exactly the SFP^M objects.

Lemma 18. *Let D be an* SFP. *Then D is an SFP^M object iff there exists a skeleton of D.*

Proof. ($\Rightarrow$) Let D be an SFP^M object and let $a_0 (= \bot), a_1, a_2, \ldots$ be an enumeration of its compact elements. Then, as shown in the proof of Theorem 15, $D = \varinjlim \langle D_n, p_n \rangle_n$, where the D_n's are defined inductively by

$$D_0 = \{a_0\},$$

$$D_{n+1} = \mathscr{U}^*(v_n),$$

where $v_n \subseteq_{fin} \mathsf{K}(D)$ is such that $D_n \cup \{a_{n+1}\} \subseteq v_n$ and $\mathsf{Max}(\mathscr{U}^*(v_n)) \sqsubseteq_s \mathsf{Max}(D)$. We show that $\mathsf{SK}(D) = \bigcup_n \mathsf{Max}(D_n)$ is a skeleton of D.

1. $\mathsf{SK}(D)$ *is a finitely branching finitary tree.*
   First observe that $\mathsf{SK}(D)$ is a tree. Let $a, a' \in \mathsf{SK}(D)$ and let $a \Uparrow a'$. Suppose $a \in \mathsf{Max}(D_n)$ and $a' \in \mathsf{Max}(D_{n'})$. Without loss of generality we can assume $n \leqslant n'$ and thus $a, a' \in D_{n'}$. Since a and a' have a common upper bound, $D_{n'}$ is $\mathscr{U}$-closed and a' is maximal in $D_{n'}$ it is easy to conclude that $a \sqsubseteq a'$.
   Furthermore, $\mathsf{SK}(D)$ is finitary. Given $a \in \mathsf{Max}(D_n)$, just notice that $\downarrow a$ in $\mathsf{SK}(D)$ is a subset of $\downarrow a$ in D_n, which is clearly finite.
   Finally, to see that $\mathsf{SK}(D)$ is finitely branching, take $a \in \mathsf{K}(D)$ and consider $I = \{n \in \mathbb{N} : \exists b \in \mathsf{Max}(D_n).\ a \sqsubset b\}$. If $I = \emptyset$ then $\mathsf{Succ}(a) = \emptyset$. Otherwise, taking $n_0 = \min I$, we have that $\mathsf{Succ}(a) = (\Uparrow a) \cap \mathsf{Max}(D_{n_0})$, which is clearly finite.
2. $\forall a \in \mathsf{K}(D).\ \exists d \in \mathsf{SK}(D).\ a \sqsubseteq d$.
   Let $a \in \mathsf{K}(D)$. Then there exists $n \in \mathbb{N}$ such that $a \in D_n$. Consider any maximal element $x \in \mathsf{Max}(\uparrow a)$. Since, by construction $\mathsf{Max}(D_n) \sqsubseteq_s \mathsf{Max}(D)$, there exists

$d \in \mathsf{Max}(D_n) \subseteq \mathsf{SK}(D)$ such that $d \sqsubseteq x$. Since D_n is $\mathscr{U}$-closed, it must include an upper bound of a and d. By maximality of d in D_n we conclude that such upper bound must be d, i.e., $a \sqsubseteq d$.

($\Leftarrow$) Let $\mathsf{SK}(D)$ be a skeleton of D. Since $\mathsf{SK}(D)$ is a finitely branching finitary tree, $\mathsf{SK}(D) = \bigcup_k M^{[k]}$, where, for any k, $M^{[k]}$ is the set of maximal elements of the truncation of $\mathsf{SK}(D)$ at level k (observe that an element in $M^{[k]}$ can have height $h \leqslant k$ in the tree). Notice that for any compact element $d \in \mathsf{K}(D)$ we can find $k_d \in \mathbb{N}$ such that if we define $z = \uparrow d \cap M^{[k_d]}$ then

$$z \sqsubseteq_s \mathsf{Max}(\uparrow d).$$

In fact, for all $x \in \mathsf{Max}(\uparrow d)$, by Proposition 17, $x = \bigsqcup \mathsf{SK}(x)$, and thus, since d is compact, there exists $d^x \in \mathsf{SK}(D)$ such that $d \sqsubseteq d^x \sqsubseteq x$. Since $\mathsf{Max}(\uparrow d) = \bigcup \{\mathsf{Max}(\uparrow d^x) : x \in \mathsf{Max}(\uparrow d)\}$ and $\mathsf{Max}(\uparrow d)$ is compact (it is a closed subset of $\mathsf{Max}(D)$, which is compact by Proposition 17) we conclude the existence of finite subset $\{d_1, \ldots, d_n\}$ of the elements d^x's such that

$$\mathsf{Max}(\uparrow d) = \bigcup \{\mathsf{Max}(\uparrow d_i) : i \in \{1, \ldots, n\}\}.$$

Therefore we can define k_d as $\max\{k : d_i \in M^{[k]} \wedge i \in \{1, \ldots, n\}\}$.

We are now able to show that D is an SFP^M by proving that it satisfies the M-condition. Given $u \subseteq_{fin} \mathsf{K}(D)$, by the property just proved and the finiteness of $\mathscr{U}^*(u)$, there exists $k \in \mathbb{N}$ such that, if $z = \uparrow\mathscr{U}^*(u) \cap M^{[k]}$ then

$$z \sqsubseteq_s \mathsf{Max}(\uparrow \mathscr{U}^*(u)).$$

Then it easy to see that the set v required by the M-condition can be defined as $v = M^{[k]} \cup \mathscr{U}^*(u)$. In fact it can be checked that $\mathscr{U}^*(v) = v$. Thus $\mathsf{Max}(\mathscr{U}^*(v)) = M^{[k]}$ and clearly $M^{[k]} \sqsubseteq_s \mathsf{Max}(D)$. □

Notice that the tree of balls of a 2-Stone space, as constructed in the proof of Theorem 12, is a domain-model of X which can be taken as the skeleton of itself.

Let D be an SFP^M object and let $\mathsf{SK}(D)$ be any of its skeletons. We write $\overline{\mathsf{SK}}(D)$ to denote the completion of $\mathsf{SK}(D)$, i.e., $\overline{\mathsf{SK}}(D) = \mathsf{Idl}(\mathsf{SK}(D))$. Notice that since $\mathsf{SK}(D)$ is a (countable) tree an ideal x in $\mathsf{SK}(D)$ is a (ω-)chain. Therefore the ideal completion can be thought of as obtained by adding a limit point to each maximal (infinite) branch. Our aim is now to prove that it is possible to project continuously D onto $\overline{\mathsf{SK}}(D)$ via a function which "preserves" maximality of points and which restricts to an homeomorphism between the maximal spaces. We first introduce the corresponding class of M-pairs.

Definition 19 (IM-pair). An M-pair $p = \langle i, j \rangle : D \to E$ is called an *IM-pair* if i and j restricts to homeomorphisms between the maximal spaces of D and E.

Lemma 20. *Let D be an SFP^M object and let $\mathsf{SK}(D)$ be a skeleton of D. Define $j_D : D \to \overline{\mathsf{SK}}(D)$ and $i_D : \overline{\mathsf{SK}}(D) \to D$ as follows: for $d \in D$ and $x \in \overline{\mathsf{SK}}(D)$*

$$j_D(d) = \mathsf{SK}(d) \quad \textit{and} \quad i_D(x) = \bigsqcup_D x.$$

Then the pair $\langle i_D, j_D \rangle : \overline{\mathsf{SK}}(D) \to D$ is an IM-pair.

Proof. The functions i_D and j_D are obviously monotone. Moreover i_D is continuous since, given a chain $(x_n)_n$ in $\overline{\mathsf{SK}}(D)$, it is easily seen that

$$i_D\left(\bigcup_n x_n\right) = \bigsqcup\left(\bigcup_n x_n\right) = \bigsqcup_n \left(\bigsqcup x_n\right).$$

Also j_D is continuous. In fact, given a chain $(d_n)_n$ in D, we have

$$j_D\left(\bigsqcup_n d_n\right) = \mathsf{SK}\left(\bigsqcup_n d_n\right) = \bigcup_n \mathsf{SK}(d_n).$$

To justify the last equality observe that if $a \in \mathsf{SK}(\bigsqcup_n d_n)$, namely $a \in \mathsf{SK}(D)$ and $a \sqsubseteq \bigsqcup_n d_n$, then, by compactness of a, there exists n such that $a \sqsubseteq d_n$; thus $a \in \mathsf{SK}(d_n)$. This proves that $\mathsf{SK}(\bigsqcup_n d_n) \subseteq \bigcup_n \mathsf{SK}(d_n)$. The converse inclusion is trivial.

To show that $\langle i_D, j_D \rangle$ is an M-pair we must prove that (i) $j_D \circ i_D = id_{\overline{\mathsf{SK}}(D)}$, (ii) $i_D \circ j_D \sqsubseteq id_D$ and (iii) $j_D(\mathsf{Max}(D)) \subseteq \mathsf{Max}(\overline{\mathsf{SK}}(D))$. (i) Let $x \in \overline{\mathsf{SK}}(D)$. Clearly $x \subseteq j_D(i_D(x)) = \mathsf{SK}(\bigsqcup x)$. Vice versa, if $a \in \mathsf{SK}(\bigsqcup x)$, then, since a is compact, there exists $a' \in x$ such that $a \sqsubseteq a'$. Since x is downward-closed we obtain $a \in x$, and thus the converse inclusion. (ii) Let $d \in D$. Recalling that $d = \bigsqcup \mathsf{K}(d)$ and $\mathsf{SK}(D) \subseteq \mathsf{K}(D)$ we immediately have $i_D(j_D(d)) = \bigsqcup \mathsf{SK}(d) \sqsubseteq \bigsqcup \mathsf{K}(d) = d$. Point (iii) trivially follows from the fact that the ideals in $\mathsf{SK}(D)$ are totally ordered subsets of $\mathsf{SK}(D)$.

To conclude that $\langle i_D, j_D \rangle$ is an IM-pair, since $j_D \circ i_D = id_{\overline{\mathsf{SK}}(D)}$, the only thing to prove is that $i_{D|\mathsf{Max}(D)} \circ j_{D|\mathsf{Max}(D)} = id_{\mathsf{Max}(D)}$. But this immediately follows from Proposition 17. $\square$

The previous lemma can be equivalently formulated by saying that if D is an SFP^M object then (the completion of) any of its skeletons $\overline{\mathsf{SK}}(D)$ is a continuous retract of D via an IM-pair. Vice versa, if D is an SFP and there exists an IM-pair $p = \langle i, j \rangle : T \to D$, where T is (the completion of) a finitely branching finitary tree, then it is easy to see that $i(\mathsf{K}(T))$ is a skeleton for D. Hence, by Lemma 18, D is an SFP^M object. This gives the announced new characterization of SFP^M objects in term of retractions.

Theorem 21. *Let D be an* SFP. *D is an SFP^M object iff it has* (*the completion of*) *a finitely branching finitary tree as continuous retract*, *via an IM-pair.*

2.5. SFP^M does not include all SFP's with a compact maximal space

As we mentioned earlier, the category SFP^M does not contain all SFP's that model 2-Stone spaces. Consider for instance the functor $+^*$ over SFP^{ep} defined as follows:

$$D +^* E = (\{(d,0)\colon d \in D\} \cup \{(e,1)\colon e \in E\} \cup \{\bot, *\}, \sqsubseteq^*),$$

where for each $x, y \neq *$, $x \sqsubseteq^* y$ if and only if $x \sqsubseteq_{D+E} y$ and $(\bot_D, 0) \sqsubseteq^* *$, $(\bot_E, 1) \sqsubseteq^* *$.

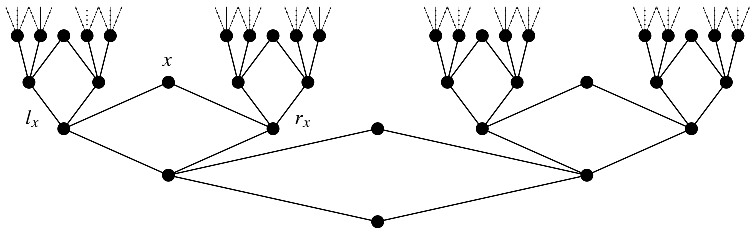

Fig. 1. An SFP $\mathscr{X}$ which is not in SFP^M, but with a 2-Stone space of maximal elements.

Given two *strict* functions $f : D \to D'$, $g : E \to E'$, $f +^* g$ coincides with $f + g$ on all the elements different from $*$ and it maps $*_{D+^*E}$ to $*_{D'+^*E'}$. The action of $+^*$ over M-pairs is defined by $\langle i, j\rangle +^* \langle h, k\rangle = \langle i +^* h, j +^* k\rangle$.

The initial solution $\mathscr{X}$ of the domain equation $X \simeq X +^* X$ (represented in Fig. 1) has a 2-Stone maximal space. In fact, since each compact element of $\mathscr{X}$ has a finite number of successors, it is easy to see that for any sequence $(x_n)_n$ in $\mathsf{Max}(\mathscr{X})$ there exists a chain $(a_n)_n$ in $\mathscr{X}$ such that for any n, $\uparrow a_n$ contains infinitely many elements of the sequence $(x_n)_n$ and the least upper bound $x = \bigsqcup_n a_n$ is a maximal element in $\mathscr{X}$. Thus there exists a subsequence of $(x_n)_n$ converging to x. However, by resorting to Theorem 21 one can prove that $\mathscr{X}$ is not in SFP^M. In fact, assume, by contradiction, that there exists an IM-pair $p = \langle i, j\rangle : T \to D$, where T is (the completion of) a finitely branching finitary tree, and take $f = i \circ j : D \to D$. Then f is the identity over the maximal space and in particular over the set of finite maximal elements of D, i.e., $\mathsf{Max}(\mathsf{K}(D))$. Hence, since $f(D) \simeq T$ is a tree, for any $x \in \mathsf{Max}(\mathsf{K}(D))$ one of the two immediate predecessors l_x and r_x of x (see Fig. 1) must be mapped to a strictly smaller element, i.e. $f(l_x) \sqsubset l_x$ or $f(r_x) \sqsubset r_x$. It is not difficult to see that this fact implies the presence of a chain $(d_n)_n$ in D such that $\bigsqcup_n d_n \in \mathsf{Max}(D)$, but, such that for each n, $f(d_n) = \bot$ and thus by continuity $f(\bigsqcup_n d_n) = \bot$. But this is absurd since f should be the identity on the maximal space.

In Section 6 we will come back to this issue, showing that a precise characterization of the SFP's having a 2-Stone maximal space can be given in term of retracts of SFP^M objects.

3. Closure properties of SFP^M

In this section we show that the category SFP^M is closed under direct limits as well as under a significant family of constructors, obtained from projections, constants, lifting, product, (coalesced) sum, Plotkin powerdomain by composition and minimalization. The function space constructor is instead very problematic. See Section 8 for a brief discussion of this issue.

3.1. Closure under direct limits

The closure of category SFP^M under direct limits is easily proved by resorting to the intrinsic characterization of SFP^M objects given in Theorem 15.

Theorem 22. *The category* SFP^M *is closed under direct limits.*

Proof. Let $\langle D_n, p_n\rangle_n$ be a directed sequence in SFP^M. By Theorem 15, each D_n satisfies the M-condition and thus, by Lemma 14, the direct limit $D = \lim\limits_{\rightarrow}\langle D_n, p_n\rangle_n$ satisfies the M-condition. Therefore D is an SFP^M object. Furthermore, by Lemma 10, the canonical ep-pairs $\gamma_n : D_n \to D$ are M-pairs. Hence, although SFP^M is not a full subcategory of SFP^{ep}, the direct limit of $\langle D_n, p_n\rangle_n$ computed in SFP^{ep} coincides with the direct limit in SFP^M. □

Since the direct limit of a directed sequence $\langle D_n, p_n\rangle_n$ computed SFP^{ep} or in SFP^M is the same, in the following we will not specify in which category we are taking the limit.

3.2. Closure under constructors

Now we prove the closure of category SFP^M under some significant constructors. More precisely we first introduce a class $\mathscr{F}$ of constructors over SFP^{ep}, including constants, identities, lifting, product, (coalesced) sum, Plotkin powerdomain and closed under composition and minimalization. Then we show that each functor in $\mathscr{F}$ restricts (under a mild assumption on the coalesced sum) to a well-defined functor over SFP^M.

Definition 23. For each n, the class $\mathscr{F}^{(n)}$ of n-ary constructors is defined by the following abstract grammar

$$F^{(n)} ::= C_D^{(n)} \mid \Pi_k^{(n)} \mid (F^{(n)})_\perp \mid F^{(n)} \times F^{(n)} \mid F^{(n)} + F^{(n)} \mid$$
$$F^{(n)} \oplus F^{(n)} \mid \mathscr{P}_{pl}(F^{(n)}) \mid \mu F^{(n+1)},$$

where D is any SFP^M object. We denote by $\mathscr{F}$ the set of constructors of any arity, i.e., $\mathscr{F} = \bigcup_n \mathscr{F}^{(n)}$.

Each $F^{(n)} \in \mathscr{F}^{(n)}$ is interpreted as a functor $F^{(n)} : (\mathsf{SFP}^{ep})^n \to \mathsf{SFP}^{ep}$ inductively defined as follows. For any SFP^M object D, $C_D^{(n)}$ denotes the corresponding constant functor. The term $\Pi_k^{(n)}$ denotes the projection on the kth component, $(F_\perp^{(n)})$ the functor $\lambda\vec{x}.\ (F^{(n)}(\vec{x}))_\perp$ and $F_1^{(n)}\,\mathsf{op}\,F_2^{(n)}$ the functor $\lambda\vec{x}.\ F_1^{(n)}(\vec{x})\,\mathsf{op}\,F_2^{(n)}(\vec{x})$ for $\mathsf{op} \in \{\times, +, \oplus\}$. The functor $\mathscr{P}_{Pl}(F^{(n)})$ is defined as $\lambda\vec{x}.\ \mathscr{P}_{Pl}(F^{(n)}(\vec{x}))$ where $\mathscr{P}_{Pl}$ denotes the Plotkin powerdomain. We shall use the characterization of the Plotkin powerdomain $\mathscr{P}_{Pl}(D)$ as the set $\{X \subseteq D:\ X$ non-empty, convex and Lawson closed$\}$, endowed with the Egli–Milner ordering. Let $Con(X)$ denote the least convex set that contains X and let $Cl(\cdot)$ denote the closure operator in Lawson topology. If $f : D \to E$ is a continuous function then $\mathscr{P}_{Pl}(f) : \mathscr{P}_{Pl}(D) \to \mathscr{P}_{Pl}(E)$ is defined as $\mathscr{P}_{Pl}(f)(X) = Con(Cl(f(X)))$. In particular, if f is a projection then $\mathscr{P}_{Pl}(f)(X) = f(X)$. In fact any projection is Lawson continuous and thus $f(X)$ is closed. Moreover $f(X)$ is convex if X is.

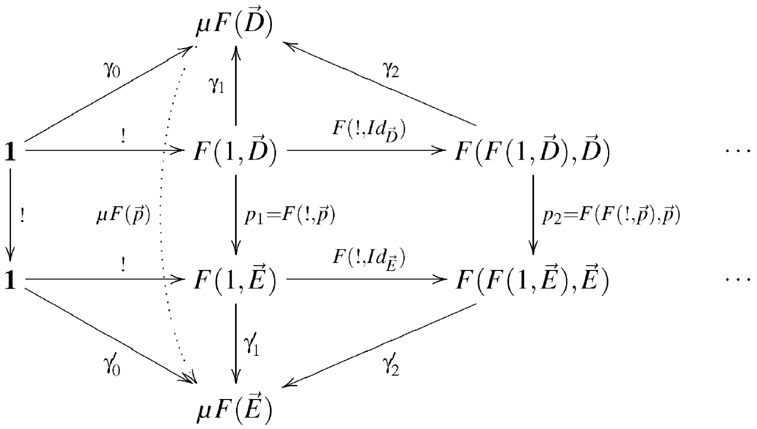

Fig. 2. The minimalization constructor.

Finally, the functor $\mu F^{(n+1)}$ is defined as follows. For any n-tuple of SFP's $\vec{D}$, $\mu F^{(n+1)}(\vec{D})$ is the initial solution of the equation

$$Y = F^{(n+1)}(Y, \vec{D})$$

which is computed as the direct limit

$$\lim_{\rightarrow} \langle D_k, r_k \rangle_k,$$

where $D_0 = \mathbf{1}$ is the one-point SFP (initial object in SFP^{ep}) and $D_{k+1} = F^{(n+1)}(D_k, \vec{D})$, while the ep-pair $r_0 = ! : D_0 \to D_1$ is the unique ep-pair from $D_0 = \mathbf{1}$, and $r_{k+1} = F^{(n+1)}(r_k, id_{\vec{D}})$. Moreover, for any tuple of ep-pairs $\vec{p} : \vec{D} \to \vec{E}$, its image $\mu F^{(n+1)}(\vec{p}) : \mu F^{(n+1)}(\vec{D}) \to \mu F^{(n+1)}(\vec{E})$ is the arrow induced by the universal property of the direct limit construction (see Fig. 2). It is a standard result that this functor is well defined in SFP^{ep} [4].

Remark 24. Observe that $\mathscr{F}$ is closed under composition, in the sense that adding a composition rule $F^{(n)} ::= F^{(k)}(F_1^{(n)}, \ldots, F_k^{(n)})$ would not enlarge the family of functors.

To prove that the constructors in $\mathscr{F}$ are functorial over SFP^M we first show that they preserve the property of being an M-pair. Then, using the characterization of the SFP^M objects as direct limits and the closure of SFP^M under direct limits, we will be able to conclude the desired result. We will see that when considering the coalesced sum $\oplus$ we have to restrict to the subcategory of SFP^M consisting of non-trivial SFP's (i.e., the category without initial object $\mathbf{1}$). Hence, from now on when considering a functor $F \oplus F'$ in $\mathscr{F}$ it will be understood that F and F' do not include $\mathbf{1}$ in their images and, by abuse of notation, we will continue to denote such restricted class of constructors by $\mathscr{F}$.

To deal with the case of the Plotkin powerdomain we need a preliminary technical lemma which provides a characterization of $\mathsf{Max}(\mathscr{P}_{Pl}(D))$ for an SFP^M object D.

Lemma 25. *Let D be an SFP^M object. Then*

$$\mathsf{Max}(\mathscr{P}_{Pl}(D)) = \{X \in \mathscr{P}_{Pl}(D) : X \subseteq \mathsf{Max}(D)\}.$$

Proof. Let $X \in \mathscr{P}_{Pl}(D)$. If $X \subseteq \mathsf{Max}(D)$ then obviously X is maximal. For the converse, let us suppose that in X there is a non-maximal point x. Since X is Lawson compact, it is easy to see that also $\uparrow X$ is Lawson compact. Therefore $\mathsf{Max}(\uparrow X) = (\uparrow X) \cap \mathsf{Max}(D)$ is Lawson closed in D (since $\mathsf{Max}(D)$ is Lawson closed by Lemma 11). Hence $\mathsf{Max}(\uparrow X)$ is in $\mathscr{P}_{Pl}(D)$. Since $X \sqsubseteq_{em} \mathsf{Max}(\uparrow X)$, $X \neq \mathsf{Max}(\uparrow X)$, we have $X \notin \mathsf{Max}(\mathscr{P}_{Pl}(D))$. □

Observe that, since each subset of $\mathsf{Max}(D)$ is clearly convex, the above result implies $\mathsf{Max}(\mathscr{P}_{Pl}(D)) = \{X \subseteq \mathsf{Max}(D) : \emptyset \neq X \text{ Lawson closed}\}$.

We are now ready to prove that the constructors in $\mathscr{F}$ preserve M-pairs.

Lemma 26. *For any $F \in \mathscr{F}^{(n)}$, if $\vec{D}$ and $\vec{E}$ are n-tuples of SFP^M objects and $\vec{p} : \vec{D} \to \vec{E}$ is an n-tuple of M-pairs, then $F(\vec{p}) : F(\vec{D}) \to F(\vec{E})$ is an M-pair.*

Proof. Let $\vec{D}$ and $\vec{E}$ be n-tuples of SFP^M objects and let $\vec{p} : \vec{D} \to \vec{E}$ be an n-tuple of M-pairs. The proof that $F(\vec{p})$ is an M-pair proceeds by induction on the structure of F.

The cases in which F is a constant functor or a projection are trivial. For the cases of $(F)_\perp$, $F \times F'$, $F + F'$, $F \oplus F'$ (with F, F' not including **1** in their images) and $\mathscr{P}_{Pl}(F)$, we argue by using the induction hypothesis and noticing that for all SFP^M objects E and E'

$$\mathsf{Max}(E_\perp) = \mathsf{Max}(E),$$
$$\mathsf{Max}(E \times E') = \mathsf{Max}(E) \times \mathsf{Max}(E'),$$
$$\mathsf{Max}(E + E') = \mathsf{Max}(E) + \mathsf{Max}(E'),$$
$$\mathsf{Max}(E \oplus E') = \mathsf{Max}(E) + \mathsf{Max}(E') \quad (\text{if } |E|, |E'| > 1),$$
$$\mathsf{Max}(\mathscr{P}_{Pl}(E)) = \{X \in \mathscr{P}_{Pl}(E) : X \subseteq \mathsf{Max}(E)\} \quad [\text{by Lemma 25}].$$

Finally, let us consider the case of μF. By induction hypothesis $F : (\mathsf{SFP}^{ep})^{n+1} \to \mathsf{SFP}^{ep}$ preserves M-pairs. Referring to Fig. 2, let $\mu F(\vec{p}) = \langle i', j' \rangle$ and, for any k, $\gamma_k = \langle \alpha_k, \beta_k \rangle$ and $p_k = \langle i_k, j_k \rangle$. For any k, since $\mu F(\vec{p}) \circ \gamma_k = \gamma'_k \circ p_k$, if $x \in \mathsf{Max}(\mu F(\vec{E}))$ then

$$\beta_k(j'(x)) = j_k(\beta'_k(x)).$$

Now, observe that every p_k is an M-pair by induction hypothesis, and γ'_k is an M-pair since SFP^M is closed under direct limits (Theorem 22). Hence $\beta_k(j'(x)) = j_k(\beta'_k(x))$ is a maximal element in the corresponding approximation of $\mu F(\vec{D})$ and thus, by Lemma 10, $j'(x) \in \mathsf{Max}(\mu F(\vec{D}))$. Hence $\mu F(\vec{p})$ is an M-pair. □

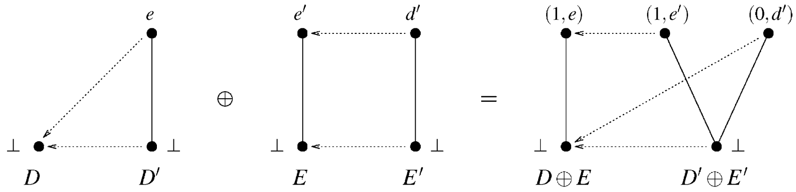

Fig. 3. Coalesced sum is not functorial over SFP^M (dotted arrows represent projections).

We remark that given two M-pairs $p: D \to E$ and $p': D' \to E'$, if either D or D' is the initial object **1** in SFP^{ep} then $p \oplus p'$ can fail to be an M-pair. Consider, for instance, the situation in Fig. 3, where dotted arrows represent the projection components of the corresponding ep-pairs: the coalesced sum of the two M-pairs produce an ep-pair which maps a maximal point in $D' \oplus E'$ to a non maximal point (the bottom) in $D \oplus E$.

To conclude that the constructors in $\mathscr{F}$ are functorial over SFP^M it remains only to show that they map SFP^M objects into SFP^M objects. This will follow easily from the general result below.

Lemma 27. *Let $F : (\mathsf{SFP}^{ep})^n \to \mathsf{SFP}^{ep}$ be a locally continuous functor which preserves M-pairs and finiteness of domains (i.e., $F(\vec{D})$ is finite for any n-tuple $\vec{D}$ of finite SFP's). If $\vec{D}$ is an n-tuple of SFP^M objects, then also $F(\vec{D})$ is an SFP^M object.*

Proof. Let $\vec{D} = D^{(1)}, \ldots, D^{(n)}$ be an n-tuple of SFP^M objects. By definition each $D^{(i)}$ is the limit of a directed sequence of finite CPO's and M-pairs, i.e. $D^{(i)} = \lim_{\to} \langle D_k^{(i)}, p_k^{(i)} \rangle_k$. Therefore

$$
\begin{aligned}
F(D_{(1)}, \ldots, D_{(n)}) &= F(\lim_{\to} \langle D_k^{(1)}, p_k^{(1)} \rangle, \ldots, \lim_{\to} \langle D_k^{(n)}, p_k^{(n)} \rangle) \\
&= \lim_{\to k_1} \ldots \lim_{\to k_n} \langle F(D_{k_1}^{(1)}, \ldots, D_{k_n}^{(n)}), F(p_{k_1}^{(1)}, \ldots, p_{k_n}^{(n)}) \rangle \\
&= \lim_{\to} \langle F(D_k^{(1)}, \ldots, D_k^{(n)}), F(p_k^{(1)}, \ldots, p_k^{(n)}) \rangle.
\end{aligned}
$$

Hence the domain $F(D^{(1)}, \ldots, D^{(n)})$ is obtained as limit of a directed sequence of finite CPO's $F(D_k^{(1)}, \ldots, D_k^{(n)})$ and M-pairs $F(p_k^{(1)}, \ldots, p_k^{(n)})$. Therefore it is an SFP^M object. □

Lemma 28. *Let $F \in \mathscr{F}^{(n)}$ and let $\vec{D}$ be any n-tuple of SFP^M objects. Then $F(\vec{D})$ is an SFP^M object.*

Proof. The proof can be carried out by induction on the structure of F. When F is a constant or a projection thesis is trivial. For the cases $(F)_\perp$, $F \times F'$, $F + F'$, $F \oplus F'$ and $\mathscr{P}_{Pl}(F)$ observe that the basic constructors $(\cdot)_\perp, \times, +, \oplus$ and $\mathscr{P}_{Pl}$ are locally continuous, they preserve M-pairs (by Lemma 26) and finiteness of domains. Hence the induction hypothesis and Lemma 27 allow us to conclude. Finally, for the case of μF just use the closure of SFP^M with respect to direct limits (Theorem 22). □

Now the main result of this section can be obtained as an immediate consequence of Lemmata 26 and 28.

Theorem 29 (Constructors in SFP^M). *The constructors in $\mathscr{F}$ (where the applications of coalesced sum $F \oplus F'$ are restricted to functors F and F' not including* **1** *in their images) are functorial over* SFP^M.

4. Relating SFP^M to 2-Stone

We have already shown that the category SFP^M provides domain-models exactly for 2-Stone spaces. In this section we establish a more structural relation between the categories SFP^M and 2-Stone. First of all we show that it is possible to define an ω-continuous functor $\mathsf{MAX} : \mathsf{SFP}^M \to \textsf{2-Stone}$, which associates to each SFP^M object its subspace of maximal elements with the induced (Scott/Lawson) topology. Then we prove that the functor MAX is "compositional" with respect to the constructors in the class $\mathscr{F}$ introduced in the previous section, in the sense that, for any $F \in \mathscr{F}$,

$$\mathsf{MAX}(F(D_1,\ldots,D_n)) \simeq \bar{F}(\mathsf{MAX}(D_1),\ldots,\mathsf{MAX}(D_n)),$$

where $\bar{F}$ is the functor over 2-Stone "corresponding" to F.

The results in this section illustrate the fact that the connection between SFP^M and 2-Stone is indeed tight, and hence SFP^M constitutes a well-behaved category of domain-models for 2-Stone spaces. For example, an interesting consequence of this correspondence is the fact that a domain-model for the solution of an equation in 2-Stone can be obtained simply by solving the "corresponding" equation in SFP^M, or, equivalently, in SFP^{ep}. This will be exploited in Section 7 to study various partializations of finitary hypersets.

4.1. *The functor* MAX

Definition 30. The (contravariant) functor $\mathsf{MAX} : \mathsf{SFP}^M \to \textsf{2-Stone}$ is defined as follows: for each SFP^M object D, $\mathsf{MAX}(D) = (\mathsf{Max}(D), \mathscr{S}_D)$ and for each M-pair $p = \langle i, j\rangle : D \to E$, $\mathsf{MAX}(p) = j_{|\mathsf{Max}(E)} : \mathsf{MAX}(E) \to \mathsf{MAX}(D)$.

It is straightforward to check that MAX is well-defined. Moreover, as shown below, it is ω-continuous, in the sense that it maps the direct limit of a directed sequence to the inverse limit of the image of the sequence.

Theorem 31 (Continuity of MAX). *Let* $D = \varinjlim \langle D_n, p_n\rangle_n$, *where* $\langle D_n, p_n\rangle_n$ *is a directed sequence in* SFP^M. *Then* $\mathsf{MAX}(D) \simeq \varprojlim \langle \mathsf{MAX}(D_n), \mathsf{MAX}(p_n)\rangle_n$.

Proof. Let us first note that $\varprojlim \langle \mathsf{MAX}(D_n), \mathsf{MAX}(p_n) \rangle_n$ and $\mathsf{MAX}(D)$ contain exactly the same points. In fact, let $p_n = \langle i_n, j_n \rangle$ for all $n \in \mathbb{N}$. Then

$$
\begin{aligned}
& x = (x_n)_n \in \mathsf{MAX}(D) \\
& \quad \Leftrightarrow \forall n.\ (x_n \in \mathsf{MAX}(D_n) \wedge x_n = j_n(x_{n+1})) \quad \text{[by Lemma 10]} \\
& \quad \Leftrightarrow \forall n.\ (x_n \in \mathsf{MAX}(D_n) \wedge x_n = \mathsf{MAX}(p_n)(x_{n+1})) \\
& \quad \Leftrightarrow x \in \varprojlim \langle \mathsf{MAX}(D_n), \mathsf{MAX}(p_n) \rangle_n.
\end{aligned}
$$

We denote by $\pi_i : \varprojlim \langle \mathsf{MAX}(D_n), \mathsf{MAX}(p_n) \rangle_n \to \mathsf{MAX}(D_i)$ the projection over the ith component. A basis for the topology of $\mathsf{MAX}(D_i)$ is given by $\{\mathsf{Max}(\uparrow a_i) : a_i \in \mathsf{K}(D_i)\}$, and thus a subbasis for $\varprojlim \langle \mathsf{MAX}(D_n), \mathsf{MAX}(p_n) \rangle_n$ is given by the sets $\pi_i^{-1}(\mathsf{Max}(\uparrow a_i))$ with $a_i \in \mathsf{K}(D_i)$ and $i \in \mathbb{N}$. Now we have:

$$
\begin{aligned}
\pi_i^{-1}(\mathsf{Max}(\uparrow a_i)) &= \{(y_n)_n \in \varprojlim \langle \mathsf{MAX}(D_n), \mathsf{MAX}(p_n) \rangle_n : a_i \sqsubseteq y_i\} \\
&= \{y \in \mathsf{MAX}(D) : \alpha_i(a_i) \sqsubseteq y\} \\
&= \mathsf{Max}(\uparrow \alpha_i(a_i)).
\end{aligned}
$$

By the characterization of compact elements of the direct limit given in Proposition 6, we immediately conclude that the two topologies coincide. Hence $\mathsf{MAX}(D)$ and $\varprojlim \langle \mathsf{MAX}(D_n), \mathsf{MAX}(p_n) \rangle_n$ are the same space. □

The "correspondence" between constructors in SFP^M and in **2-Stone** is formalized as follows:

Definition 32. We say that a functor $F : (\mathsf{SFP}^M)^n \to \mathsf{SFP}^M$ *models* a functor $G : (\mathsf{2\text{-}Stone})^n \to \mathsf{2\text{-}Stone}$, written $F \propto G$, if there exists a natural isomorphism $\eta : G \circ (\mathsf{MAX}, \ldots, \mathsf{MAX}) \to \mathsf{MAX} \circ F$.

Notice that when $F \propto G$, the functor F can be viewed, so to speak, as a possible "higher order" domain-model for G.

The next definition provides an inductive translation of constructors F in $\mathscr{F}$ to constructors $\bar{F}$ over **2-Stone**. In the rest of this section we will show that each F in $\mathscr{F}$ models the "corresponding" constructor $\bar{F}$ over **2-Stone**. Roughly speaking, the translation leaves the "structural" constructors unchanged and maps $(.)_\perp$, $\times$, $+$ (or $\oplus$) and $\mathscr{P}_{Pl}$ in SFP^M into the "corresponding" constructors Id (identity), $\times$ (product), $\uplus$ (disjoint union) and $\mathscr{P}_{nco}$ (hyperspace of non-empty compact subsets) in **2-Stone**. Recall that the space $\mathscr{P}_{nco}(X)$ is defined as the set $\{K \subseteq X : K \text{ non-empty and compact}\}$ endowed with the Vietoris topology, i.e. the topology having as subbasis the sets $\mathscr{V}_A = \{K \in \mathscr{P}_{nco}(X) : K \subseteq A\}$ and $\mathscr{Z}_A = \{K \in \mathscr{P}_{nco}(X) : K \cap A \neq \emptyset\}$ for $A \in \Omega(X)$. If $\mathscr{B}$ is a basis for X then a subbasis for the Vietoris topology on $\mathscr{P}_{nco}(X)$ is given by the sets $\mathscr{V}_{A_1 \cup \cdots \cup A_n}$ and $\mathscr{Z}_A$, for $A_1, \ldots, A_n$, $A \in \mathscr{B}$.

Definition 33. For any constructor $F \in \mathscr{F}^{(n)}$ the corresponding constructor $\bar{F}$: $(\mathsf{2\text{-}Stone})^n \to \mathsf{2\text{-}Stone}$ is inductively defined as follows:

$$\begin{array}{ll}
\overline{C_D^{(n)}} = C^{(n)}_{\mathsf{MAX}(D)}, & \overline{F_1^{(n)} + F_2^{(n)}} = \overline{F_1^{(n)}} \uplus \overline{F_2^{(n)}}, \\
\overline{\Pi_k^{(n)}} = \Pi_k^{(n)}, & \overline{F_1^{(n)} \oplus F_2^{(n)}} = \overline{F_1^{(n)}} \uplus \overline{F_2^{(n)}}, \\
\overline{(F^{(n)})_\perp} = \overline{F^{(n)}}, & \overline{\mathscr{P}_{Pl}(F^{(n)})} = \mathscr{P}_{nco}(\overline{F^{(n)}}), \\
\overline{F_1^{(n)} \times F_2^{(n)}} = \overline{F_1^{(n)}} \times \overline{F_2^{(n)}}, & \overline{\mu F^{(n+1)}} = \mu \overline{F^{(n+1)}},
\end{array}$$

where the constructors on the right-hand side are interpreted in the natural way as functors over $\mathsf{2\text{-}Stone}$.

To prove that for each $F \in \mathscr{F}$, the constructor F over SFP^M models the constructor $\bar{F}$ over $\mathsf{2\text{-}Stone}$, we first observe that the functor MAX "commutes" for such constructors in the sense that the 2-Stone spaces $\mathsf{MAX}(F(\vec{D}))$ and $\bar{F}(\mathsf{MAX}(\vec{D}))$ are homeomorphic; actually they are the same space if we adopt the usual concrete constructions for $(\cdot)_\perp$, $\times$, $+$, $\oplus$, $\mathscr{P}_{Pl}$ and the direct/inverse limit. Then we will conclude simply observing that the identity is a natural isomorphism between $\bar{F} \circ (\mathsf{MAX}, \ldots, \mathsf{MAX})$ and $\mathsf{MAX} \circ F$.

We start with a preliminary lemma which shows that MAX is "compositional" with respect to the basic constructors $(\cdot)_\perp, \times, +, \oplus$ and $\mathscr{P}_{Pl}$.

Lemma 34. *Let D, D_1 and D_2 be SFP^M objects. Then*
1. $\mathsf{MAX}(D_\perp) = \mathsf{MAX}(D)$;
2. $\mathsf{MAX}(D_1 \times D_2) = \mathsf{MAX}(D_1) \times \mathsf{MAX}(D_2)$;
3. $\mathsf{MAX}(D_1 + D_2) = \mathsf{MAX}(D_1) \uplus \mathsf{MAX}(D_2)$;
4. $\mathsf{MAX}(D_1 \oplus D_2) = \mathsf{MAX}(D_1) \uplus \mathsf{MAX}(D_2)$;
5. $\mathsf{MAX}(\mathscr{P}_{Pl}(D)) = \mathscr{P}_{nco}(\mathsf{MAX}(D))$.

Proof.
1. Clearly $\mathsf{Max}(D_\perp)$ and $\mathsf{Max}(D)$ contain the same elements (if $D_\perp$ is obtained by adding to D an extra element $\perp \notin D$) and the topologies $\mathscr{S}_D$ and $\mathscr{S}_{D_\perp}$, induced by the Scott topology over the maximal space, coincide.
2. We have $\mathsf{Max}(D_1 \times D_2) = \mathsf{Max}(D_1) \times \mathsf{Max}(D_2)$, and also their topologies coincide. In fact $\mathsf{K}(D_1 \times D_2) = \mathsf{K}(D_1) \times \mathsf{K}(D_2)$ and a basis for $\mathsf{MAX}(D_1 \times D_2)$ is

   $$\mathsf{Max}(\uparrow (a_1, a_2)), \quad a_i \in \mathsf{K}(D_i),\ i \in \{1, 2\},$$

   while a subbasis for $\mathsf{MAX}(D_1) \times \mathsf{MAX}(D_2)$ is given by the sets

   $$\pi_i^{-1}(\mathsf{Max}(\uparrow a_i)), \quad a_i \in \mathsf{K}(D_i),\ i \in \{1, 2\}.$$

   Each element $\pi_i^{-1}(\mathsf{Max}(\uparrow a_i))$ in the subbasis of $\mathsf{MAX}(D_1) \times \mathsf{MAX}(D_2)$ is open in $\mathsf{MAX}(D_1 \times D_2)$, since it can be written as $\mathsf{Max}(\uparrow(a_i, \perp))$. Conversely, for any element $\mathsf{Max}(\uparrow(a_1,a_2))$ in the basis of $\mathsf{MAX}(D_1\times D_2)$, we have

$$\mathsf{Max}(\uparrow(a_1,a_2)) = \mathsf{Max}(\uparrow a_1)\times \mathsf{Max}(\uparrow a_2) = \pi_1^{-1}(\mathsf{Max}(\uparrow a_1))\cap \pi_1^{-2}(\mathsf{Max}(\uparrow a_2)),$$

and thus $\mathsf{Max}(\uparrow(a_1,a_2))$ is open in $\mathsf{MAX}(D_1)\times\mathsf{MAX}(D_2)$.

3. Again, we have that $\mathsf{Max}(D_1+D_2)=\mathsf{Max}(D_1)\uplus\mathsf{Max}(D_2)$, and also their topologies coincide. In fact $\mathsf{K}(D_1+D_2)=(\mathsf{K}(D_1)+\mathsf{K}(D_2))\cup\{\bot\}$. Hence a basis for $\mathsf{MAX}(D_1+D_2)$ is

$$\{\mathsf{Max}(\uparrow(i,a))\colon (i,a)\in\mathsf{K}(D_1+D_2),\ i\in\{1,2\}\}\cup\{\mathsf{Max}(\uparrow\bot)\}$$

$$=\{\{i\}\times\mathsf{Max}(\uparrow a)\colon a\in\mathsf{K}(D_i),\ i\in\{1,2\}\}\cup\{\mathsf{Max}(D_1+D_2)\}$$

$$=\{\{i\}\times\mathsf{Max}(\uparrow a)\colon a\in\mathsf{K}(D_i),\ i\in\{1,2\}\}\cup\{\mathsf{Max}(D_1)\uplus\mathsf{Max}(D_2)\},$$

which is also a basis for $\mathsf{MAX}(D_1)\uplus\mathsf{MAX}(D_2)$.

4. The proof is analogous to that for (3).
5. As above we first notice that $\mathsf{Max}(\mathscr{P}_{Pl}(D))=\mathscr{P}_{nco}(\mathsf{Max}(D))$. In fact, by Lemma 25, the maximal elements of $\mathscr{P}_{Pl}(D)$ are non-empty Lawson closed subsets of $\mathsf{Max}(D)$. These are the compact non-empty subsets of $\mathsf{MAX}(D)$, since, by Theorem 12, the Lawson and the Scott topologies coincide on $\mathsf{Max}(D)$ (which is compact).
Let us consider the topologies. The space $\mathsf{MAX}(\mathscr{P}_{Pl}(D))$ is equipped with the induced Scott topology and thus a basis is given by the sets $\mathsf{Max}(\uparrow X)$, with $X\in\mathsf{K}(\mathscr{P}_{Pl}(D))$. Recall that $X\in\mathsf{K}(\mathscr{P}_{Pl}(D))$ iff $X=Con(u)$, where $u\subseteq_{fin}\mathsf{K}(D)$. It is easy to show that for any such X we have:

$$X\sqsubseteq_{em} Y \quad\Leftrightarrow\quad u\sqsubseteq_{em} Y.$$

Thus a basis for $\mathsf{MAX}(\mathscr{P}_{Pl}(D))$ is given by

$$\{\mathsf{Max}(\uparrow u)\}_{u\subseteq_{fin}\mathsf{K}(D)}.$$

On the other hand, a basis for $\mathsf{MAX}(D)$ is $\{\mathsf{Max}(\uparrow a): a\in\mathsf{K}(D)\}$. Since $\mathsf{Max}(\uparrow a_1)\cup\cdots\cup\mathsf{Max}(\uparrow a_n)=\mathsf{Max}(\uparrow\{a_1,\ldots,a_n\})$, a subbasis for the Vietoris topology of $\mathscr{P}_{nco}(\mathsf{MAX}(D))$ is given by the sets

$$\mathscr{V}_{\mathsf{Max}(\uparrow u)}, \mathscr{Z}_{\mathsf{Max}(\uparrow a)} \quad \text{for } u\subseteq_{fin}\mathsf{K}(D),\ a\in\mathsf{K}(D).$$

Let $\mathsf{Max}(\uparrow u)$, where $u\subseteq_{fin}\mathsf{K}(D)$ be an element of the basis of the first topology and let $Y\in\mathsf{Max}(\mathscr{P}_{Pl}(D))$. The following hold:

$$Y\in\mathsf{Max}(\uparrow u) \quad\Leftrightarrow\quad u\sqsubseteq_{em} Y$$

$$\Leftrightarrow (\forall y\in Y.\ \exists a\in u.\ a\sqsubseteq y)\wedge(\forall a\in u.\ \exists y\in Y.\ a\sqsubseteq y)$$

$$\Leftrightarrow (Y\in\mathscr{V}_{\mathsf{Max}(\uparrow u)})\wedge(\forall a\in u.\ Y\in\mathscr{Z}_{\mathsf{Max}(\uparrow a)})$$

$$\Leftrightarrow Y\in\mathscr{V}_{\mathsf{Max}(\uparrow u)}\cap\bigcap_{a\in u}\mathscr{Z}_{\mathsf{Max}(\uparrow a)},$$

hence $\mathsf{Max}(\uparrow u)$ is an open set of the second topology.

As to the converse, an element of the subbasis of the second topology can be either

$$\mathscr{V}_{\mathsf{Max}(\uparrow u)} = \{Y : u \sqsubseteq_s Y\} = \{Y : \exists v \subseteq u.\ v \sqsubseteq_{em} Y\} = \bigcup\{\mathsf{Max}(\uparrow v) : v \subseteq u\}$$

or

$$\mathscr{X}_{\mathsf{Max}(\uparrow a)} = \{Y : Y \cap \mathsf{Max}(\uparrow a) \neq \emptyset\} = \{Y : \{\bot, a\} \sqsubseteq_{em} Y\} = \mathsf{Max}(\uparrow \{\bot, a\}),$$

where $u \subseteq_{fin} \mathsf{K}(D)$ and $a \in \mathsf{K}(D)$. In both cases we conclude that the sets are open in the first topology. Therefore the two topologies coincide. □

We can now extend the compositionality of MAX to the whole family $\mathscr{F}$. In the sequel, given an n-tuple of domains $\vec{D} = D_1, \ldots, D_n$ we will often write $\mathsf{MAX}(\vec{D})$ as a short for $\mathsf{MAX}(D_1), \ldots \mathsf{MAX}(D_n)$.

Lemma 35. *For any constructor $F \in \mathscr{F}^{(n)}$ and n-tuple of SFP^M objects $\vec{D}$*

$$\mathsf{MAX}(F(\vec{D})) = \bar{F}(\mathsf{MAX}(\vec{D})).$$

Proof. Let $F \in \mathscr{F}^{(n)}$ and let $\vec{D}$ be an n-tuple of SFP^M objects. The proof proceeds by induction on the structure of F. As usual, when F is a constant or a projection the thesis is trivial. The cases $(F)_\bot$, $F \times F'$, $F + F'$, $F \oplus F'$, $\mathscr{P}_{Pl}(F)$ are dealt with by exploiting the induction hypothesis and Lemma 34.

Finally, for the case of μF, recall that $\mu F(\vec{D}) = \lim_{\rightarrow} \langle E_k, r_k\rangle_k$, where $E_0 = \mathbf{1}$ and $E_{k+1} = F(E_k, \vec{D})$. Therefore

$$\begin{aligned}\mathsf{MAX}(\mu F(\vec{D})) &= \mathsf{MAX}(\lim_{\rightarrow} \langle E_k, r_k\rangle_k)\\ &= \lim_{\rightarrow} \langle \mathsf{MAX}(E_k), \mathsf{MAX}(r_k)\rangle_k \quad \text{[by Theorem 31]}.\end{aligned}$$

On the other hand, $\overline{\mu F}(\mathsf{MAX}(\vec{D})) = \mu\bar{F}(\mathsf{MAX}(\vec{D}))$ is given by the inverse limit $\lim_{\leftarrow} \langle X_k, f_k\rangle_k$, where $X_0 = \mathbf{1}$ is the final object in 2-Stone and $X_{k+1} = \bar{F}(X_k, \mathsf{MAX}(\vec{D}))$. Now, by exploiting the induction hypothesis, one can prove that, for any k, $\mathsf{MAX}(E_k) = X_k$ and $\mathsf{MAX}(r_k) = f_k$. Hence we conclude that $\mathsf{MAX}(\mu F(\vec{D})) = \overline{\mu F}(\mathsf{MAX}(\vec{D}))$. □

Now, the main result of the section, stating that each constructor $F \in \mathscr{F}$ models the corresponding constructor $\bar{F}$ over 2-Stone, follows as an easy corollary.

Theorem 36. *For any constructor $F \in \mathscr{F}$, $F \propto \bar{F}$.*

Proof. Let $F \in \mathscr{F}$ and let $\vec{D}$ be an n-tuple of SFP^M objects. In view of the previous lemma it is enough to show that, for any choice of the n-tuple of M-pairs $\vec{p} = (p_1, \ldots, p_n)$ in SFP^M, $\mathsf{MAX}(F(\vec{p})) = \bar{F}(\mathsf{MAX}(p_1), \ldots, \mathsf{MAX}(p_n))$. The proof is by induction on the structure of F.
(C_D), (Π_k): Obvious.

($(F)_\perp$): First observe that, if $p: D \to E$ is an M-pair then $\mathsf{MAX}(p_\perp) = \mathsf{MAX}(p)$. In fact, for any $y \in \mathsf{Max}(E_\perp) = \mathsf{Max}(E)$, since $y \neq \perp$, we have $\mathsf{MAX}(p_\perp)(y) = \mathsf{MAX}(p)(y)$. Hence

$$\begin{aligned}\mathsf{MAX}((F)_\perp(\vec{p})) &= \mathsf{MAX}(F(\vec{p}))\\ &= \bar{F}(\mathsf{MAX}(\vec{p})) \quad \text{[by induction hyp.]}\\ &= \overline{(F)_\perp}(\mathsf{MAX}(\vec{p})) \quad \text{[by definition of } \overline{(\cdot)}]\end{aligned}$$

($F \times G$): First observe that if $p_k = \langle i_k, j_k \rangle : D_k \to E_k$ is an M-pair for $k \in \{1,2\}$, then $\mathsf{MAX}(p_1 \times p_2) = \mathsf{MAX}(p_1) \times \mathsf{MAX}(p_2)$. In fact, $\forall (x_1,x_2) \in \mathsf{Max}(E_1 \times E_2) = \mathsf{Max}(E_1) \times \mathsf{Max}(E_2)$, we have:

$$\begin{aligned}(\mathsf{MAX}(p_1 \times p_2))(x_1,x_2) &= (j_1 \times j_2)(x_1,x_2)\\ &= (j_1(x_1), j_2(x_2))\\ &= (\mathsf{MAX}(p_1)(x_1), \mathsf{MAX}(p_2)(x_2))\\ &= (\mathsf{MAX}(p_1) \times \mathsf{MAX}(p_1))(x_1,x_2).\end{aligned}$$

Hence,

$$\begin{aligned}\mathsf{MAX}(F \times G(\vec{p})) &= \mathsf{MAX}(F(\vec{p})) \times \mathsf{MAX}(G(\vec{p}))\\ &= \bar{F}(\mathsf{MAX}(\vec{p})) \times \bar{G}(\mathsf{MAX}(\vec{p})) \quad \text{[by induction hyp.]}\\ &= \overline{F \times G}(\mathsf{MAX}(\vec{p})) \quad \text{[by definition of } \overline{(\cdot)}]\end{aligned}$$

($F + G$), ($F \oplus G$), ($\mathscr{P}_{Pl}(F)$): Same proof as above.
(μF): "Apply" the functor MAX to the diagram in Fig. 2 which defines $\mu F(\vec{p})$. By induction hypothesis and ω-continuity of MAX (Theorem 31) we obtain the diagram for $\mu\bar{F}(\mathsf{MAX}(\vec{p}))$. Since $\mu\bar{F}(\mathsf{MAX}(\vec{p}))$ is defined using the universal property of the limit construction, it is easy to conclude that the equality $\mathsf{MAX}(\mu F(\vec{p})) = \mu\bar{F}(\mathsf{MAX}(\vec{p}))$ holds. □

4.2. Relating solutions of domain equations in SFP^M and 2-Stone

As an application of the previous results, it is now easy to see how domain-models for solutions of domain equations in 2-Stone can be obtained by solving the "corresponding" equations in SFP^M (or equivalently in SFP^{ep}). This fact will be used in Section 7 to study various partializations of finitary hypersets.

Consider any unary functor $F \in \mathscr{F}$. Then the solution of the domain equation $X \simeq \bar{F}(X)$ in 2-Stone is obtained as the inverse limit

$$\lim_{\rightarrow} \langle \bar{F}^n \mathbf{1}, \bar{F}^n f_0 \rangle_n,$$

where $\mathbf{1}$ is the final object in 2-Stone and f_0 is the unique function from $\bar{F}(\mathbf{1})$ to $\mathbf{1}$. In other words, the solution is given by $\mu\bar{F}$. But, by Theorem 36 we know that $\mu\bar{F} \simeq \mathsf{MAX}(\mu F)$, which simply means that μF, the solution of the equation $X \simeq F(X)$ over SFP^M, is a domain-model for $\mu\bar{F}$, the solution of the corresponding equation over 2-Stone.

5. Continuous extensions in SFP^M

It is well-known that domain-models of 2-Stone spaces are useful also for the study of generalized computability over such spaces. To this end it is necessary that continuous functions over the original spaces can be extended to Scott continuous functions over the corresponding domain-models.

A continuous function between SFP^M objects, mapping maximal points into maximal points, clearly restricts to a continuous function between the corresponding maximal spaces. Here we show that also the converse holds, namely that any continuous function between the maximal spaces of two SFP^M objects extends to a continuous function between the whole domains. Several extendability results have appeared in the literature (see, e.g., the classical [16] or [18]) for the case where the target domain is bounded complete. Since an SFP^M object is not, in general, bounded complete we cannot extend those techniques. In our proof we capitalize on the characterization of SFP^M objects as the class of SFP's having (the completion of) a finitely branching finitary tree as continuous retract via an IM-pair (see Theorem 21).

Theorem 37 (Continuous extension). *Let D be an ω-algebraic* CPO, *let E be an SFP^M object, and let $f : \mathsf{MAX}(D) \to \mathsf{MAX}(E)$ be a continuous function. Then there exists a continuous function $g : D \to E$ such that $g_{|\mathsf{Max}(D)} = f$.*

Proof. Since E is an SFP^M object, by Theorem 21 there is an IM-pair

$$\langle i_E, j_E \rangle : T_E \to E,$$

where T_E is (the completion of) a finitely branching finitary tree. The function f induces a continuous function $f' = j_E \circ f$

$$f' : \mathsf{Max}(D) \to \mathsf{Max}(T_E).$$

Since T_E is a tree, the function f' easily extends to a continuous function

$$f'' : D \to T_E,$$

defined by $f''(a) = \sqcap\{f'(x) : x \in \mathsf{Max}(\uparrow a)\}$ for $a \in \mathsf{K}(D)$, and extended by continuity to the non compact points.

Now, by using again the IM-pair $\langle i_E, j_E \rangle$ we can obtain $g = i_E \circ f''$ which is the desired function, namely it is continuous and it coincides with f on the maximal elements. For the last fact, observe that if $d \in \mathsf{Max}(D)$ then

$$\begin{aligned} g(d) &= i_E(f''(d)) \\ &= i_E(f'(d)) \quad \text{[by def. of } f''\text{, since } d \text{ is maximal]} \\ &= i_E(j_E(f(d))) \\ &= f(d) \quad \text{[since } i_E \circ j_E \text{ is the identity on maximal elements].} \qquad \square \end{aligned}$$

Notice, however, that SFP^M is not the largest class of SFP's which satisfies the function extension property. For instance, it is sufficient that the considered SFP's have a retraction, via an IM-pair, onto a generic algebraic bounded complete CPO. More specifically, let D, E be SFP's, such that there exists an IM-pair $p = \langle i,j \rangle : E' \to E$, and assume E' to be bounded-complete. Then any continuous function $f : \mathsf{Max}(D) \to \mathsf{Max}(E)$ admits a continuous extension $f' : D \to E$. In fact, as above, consider the function $g = j \circ f : \mathsf{Max}(D) \to \mathsf{Max}(E')$, which by classical results (see, e.g., [16]) extends to a continuous function

$$g' : D \to E'$$

such that $g'(x) = g(x)$ for any $x \in \mathsf{Max}(D)$. The function g' can be defined on the compact elements $d \in \mathsf{K}(D)$ as $g'(d) = \sqcap\{g(x) : x \in \mathsf{Max}(\uparrow d)\}$, which exists by bounded completeness of E', and then extended by continuity to the non compact points. Then the function $f' : D \to E$ we are looking for can be simply defined as

$$f' = i \circ g'.$$

In fact, for $x \in \mathsf{Max}(D)$ we have $f'(x) = i(g'(x)) = i(j(f(x))) \sqsubseteq f(x)$, where the last inequality follows from the definition of ep-pair. Since $f'(x)$ is maximal we conclude $f'(x) = f(x)$, as desired.

6. Retracts of SFP^M objects

In this section we investigate the possibility of extending the theory developed so far to take into account retracts of 2-Stone spaces and retracts of SFP^M objects. This will lead us to a characterization of the SFP's with a 2-Stone maximal space, called here CSFP's. All the previous results extend to the corresponding category cSFP^M of CSFP's and M-pairs, which has SFP^M as a full subcategory. The category cSFP^M is closed under direct limits and under the constructors in $\mathscr{F}$. Moreover, the functor MAX extends to a well-defined ω-continuous functor over cSFP^M, compositional with respect to the constructors in $\mathscr{F}$.

We notice first that while a continuous retract of a 2-Stone space is still a 2-Stone space, in general the continuous retract of an SFP^M object is not an SFP^M object and it might have a non-compact maximal space. For instance, it is easy to see that $\mathbb{N}_\perp$ is a retract of $\mathbb{N}_{\text{lazy}}$ via an ep-pair. The projection can be the function

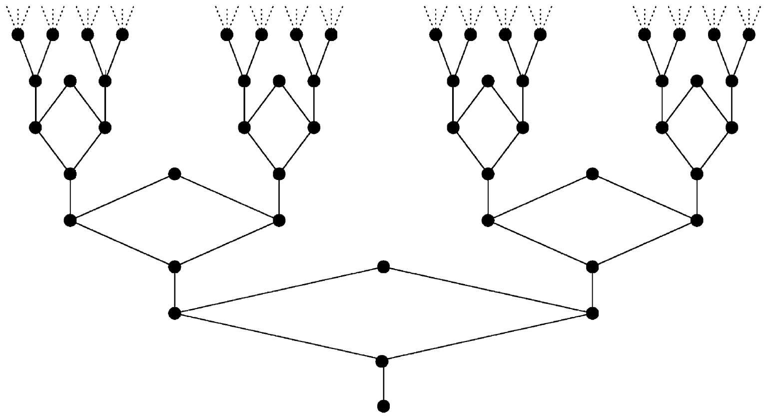

Fig. 4. The initial solution $\mathscr{Z}'$ of the equation $X \simeq (X +^* X)_\perp$.

which maps each lazy number into the corresponding flat number and each intermediate point of $\mathbb{N}_{lazy}$ into $\perp \in \mathbb{N}_\perp$. Thus by continuity $\omega \in \mathbb{N}_{lazy}$ must be mapped to $\perp$.

The observation above suggests that, in this setting, a more natural choice could be to consider retracts via M-pairs: if $\langle i, j \rangle : D \to E$ is an M-pair, D is an SFP and E is an SFP^M object, surely the maximal space $(\mathsf{Max}(D), \mathscr{S}_D)$ is a 2-Stone, since it is a continuous retract of $(\mathsf{Max}(E), \mathscr{S}_E)$, but still D might not be an SFP^M object. For instance, let $+^*$ be the functor defined in Section 2.5. Then it is easy to see that the initial solution $\mathscr{Z}'$ in SFP^{ep} of the equation $X \simeq (X +^* X)_\perp$, depicted in Fig. 4, is an SFP^M object. Moreover, the initial solution of the equation $X \simeq X +^* X$ (see Fig. 1), which is *not* an SFP^M object, is a retract of $\mathscr{Z}'$ via an M-pair. We will show that this is a special case of a more general situation, namely that the CSFP's can be characterized as those SFP's which are retracts of SFP^M objects via M-pairs.

The example just considered suggests that, given a CSFP D, if D is not in SFP^M the reason is that it does not have "enough compact elements", in the sense that it is not possible to express each clopen of its maximal space as the union of a finite disjoint family of clopens of the kind $\mathsf{Max}(\uparrow a)$, for $a \in \mathsf{K}(D)$. For instance, in the domain $\mathscr{Z}$ of Fig. 1, each clopen of the form $\mathsf{Max}(\uparrow l_x) \cup \{x\} \cup \mathsf{Max}(\uparrow r_x)$ cannot be expressed as the union of a finite disjoint family of clopens of the form $\mathsf{Max}(\uparrow a)$. Instead, in the domain $\mathscr{Z}'$, due to the presence of the lifting in the equation, this does not happen.

We will prove that it is always possible to turn a CSFP into an SFP^M object $\mathsf{Sat}(D)$, called the saturation of D, by suitably enriching its set of compact elements. Then we will show that D is a retract of $\mathsf{Sat}(D)$ via an IM-pair, and thus that the CSFP's are exactly the class of retracts of SFP^M objects via M-pairs. In the sequel, given a topological space $(X, \Omega(X))$, we will write $K\Omega_{ne}(X, \Omega(X))$ to denote the set of non-empty compact open subsets of X.

Definition 38 (Saturation). Let D be an SFP domain. The *saturation* of D, denoted by $\mathsf{Sat}(D)$, is defined as the ideal completion of the partial order

$$\mathscr{B}(D) = \{(a,A)\colon a \in \mathsf{K}(D) \wedge A \in K\Omega_{ne}(\mathsf{Max}(D), \mathscr{S}_D) \wedge \mathsf{Max}(\uparrow a) \supseteq A\}$$

ordered by $(a,A) \sqsubseteq (a',A')$ iff $a \sqsubseteq a'$ and $A \supseteq A'$.

For any ideal $I \in \mathsf{Sat}(D) = \mathsf{Idl}(\mathscr{B}(D))$, the projection on the first component, i.e., the set $\varepsilon(I) = \{a \in \mathsf{K}(D) : \exists A \in K\Omega_{ne}(\mathsf{Max}(D)).\ (a,A) \in I\}$ is an ideal in D, and the projection on the second component $\eta(I) = \{A \in K\Omega_{ne}(\mathsf{Max}(D)) : \exists a \in \mathsf{K}(D).\ (a,A) \in I\}$ is a filtered subset of $K\Omega_{ne}(\mathsf{Max}(D))$ with a non-empty (compact) intersection. It is not difficult to see that $\mathsf{Sat}(D)$ is isomorphic to $\{(\bigsqcup \varepsilon(I), \bigcap \eta(I)) : I \in \mathsf{Sat}(D)\}$, ordered in the obvious way. Hence, in the following we will identify $\mathsf{Sat}(D)$ with the latter poset, and thus each ideal $I \in \mathsf{Sat}(D)$ with the pair $(\bigsqcup \varepsilon(I), \bigcap \eta(I))$. In particular, each principal ideal $\downarrow(a,A)$ corresponds to (a,A) itself.

The next proposition shows that the above construction, when applied to a CSFP, produces an SFP^M object, which has the original SFP as continuous retract via an IM-pair.

Lemma 39. *Let D be a* CSFP. *Then* $\mathsf{Sat}(D)$ *is an* SFP^M *object and D is a retract of* $\mathsf{Sat}(D)$ *via an IM-pair.*

Proof. First observe that $\mathsf{Sat}(D)$ is an ω-algebraic CPO. In fact $\mathsf{Max}(D)$ is a 2-Stone space. Hence its basis and thus $K\Omega_{ne}(\mathsf{Max}(D))$ are denumerable, and therefore $\mathsf{K}(\mathsf{Sat}(D)) = \mathscr{B}(D)$ is denumerable. The fact that $\mathsf{Sat}(D)$ is an SFP follows by the observation that, given a finite set of compact elements in $\mathsf{Sat}(D)$, $u = \{(a_i, A_i) : i \in \{1,\dots,n\}\}$, one has

$$\mathscr{U}(u) = \{(a, \mathsf{Max}(\uparrow a) \cap A)\colon a \in \mathscr{U}_D(\{a_1,\dots,a_n\}) \wedge \mathsf{Max}(\uparrow a) \cap A \neq \emptyset\},$$

where $A = \bigcap \{A_1,\dots,A_n\}$ and $\mathscr{U}_D$ gives the set of minimal upper bounds in D. Hence the completeness of $\mathscr{U}(u)$ and the finiteness of $\mathscr{U}^*(u)$ can be proved by exploiting the analogous properties of $\mathscr{U}_D$.

Let us show that D is an SFP^M object. Observe that the maximal elements in $\mathsf{Sat}(D)$ are pairs $(x, \{x\})$ for $x \in \mathsf{Max}(D)$ (corresponding to ideals I such that $\bigsqcup \varepsilon(I) = x \in \mathsf{Max}(D)$ and thus $\bigcap \eta(I) = \{x\}$). Furthermore, for any $(a,A) \in \mathscr{B}(D)$, we have $\mathsf{Max}(\uparrow (a,A)) = \{(x,\{x\}) : x \in A\}$ (i.e., if we identify the maximal spaces of D and $\mathsf{Sat}(D)$, then the set of maximal elements above (a,A) is exactly A). We can now prove that $\mathsf{Sat}(D)$ satisfies the M-condition and thus, by Theorem 15, it is an SFP^M object. Take any $u \subseteq_{fin} \mathscr{B}(D)$. For any $(c,C) \in \mathscr{U}^*(u)$, define

$$r(c,C) = C - \bigcup \{C' : \exists c'.\ (c',C') \in \mathscr{U}^*(u) \wedge (c,C) \sqsubset (c',C')\}.$$

Let $v = \mathscr{U}^*(u) \cup \{(c, r(c,C)) : (c,C) \in \mathscr{U}^*(u) \wedge r(c,C) \neq \emptyset\}$. Then v is $\mathscr{U}$-closed and each element $(c, r(c,C))$ is maximal in v. Hence

$$\mathsf{Max}(v) = \mathsf{Max}(\mathscr{U}^*(u)) \cup \{(c, r(c,C))\colon (c,C) \in \mathscr{U}^*(u) \wedge r(c,C) \neq \emptyset\}$$

and thus $\mathsf{Max}(v) \sqsubseteq_s \{(x,\{x\}) : x \in A\}$. Finally, it is easy to see that $v' = \{(\bot_D, \mathsf{Max}(D) - A)\} \cup v$ is still $\mathscr{U}^*$-closed and that $\mathsf{Max}(v') \sqsubseteq_s \mathsf{Max}(\mathsf{Sat}(D))$. Hence v' can be the set of finite elements required by the M-condition.

To conclude, define an ep-pair $p = \langle i,j \rangle : D \to \mathsf{Sat}(D)$ as follows. For any $a \in D$, $i(a) = (a, \mathsf{Max}(\uparrow a))$ and for any $(a,A) \in \mathsf{Sat}(D)$, $j(a,A) = a$. Then it is easy to see that $\langle i,j \rangle$ is a well-defined M-pair and that $i \circ j$, restricted to the maximal space is the identity. Therefore p is an IM-pair. □

The main result of this section now follows as an easy corollary.

Theorem 40. *The class of* CSFP's *is the class of retracts of* SFP^M *objects via M-pairs.*

Proof. If D is a CSFP then, by the previous lemma, it is the retract via an (I)M-pair of an SFP^M object. Vice versa, let D be an SFP which is the retract of an SFP^M object E via an M-pair $\langle i,j \rangle : D \to E$. Since $\mathsf{Max}(D) = j(\mathsf{Max}(E))$ and $\mathsf{Max}(E)$ is compact in E, then $\mathsf{Max}(D)$ is compact in D. Hence D is a CSFP. □

Let us introduce the category of CSFP's and M-pairs, which has SFP^M as a full subcategory.

Definition 41. We denote by $\mathsf{cSFP}^{\mathsf{M}}$ the category having CSFP's as objects and M-pairs as arrows.

Using the characterization of the CSFP's given in Theorem 40, it is not difficult to verify that $\mathsf{cSFP}^{\mathsf{M}}$ can replace SFP^M as category of compositional models for 2-Stone spaces, i.e., the following facts hold:

- $\mathsf{cSFP}^{\mathsf{M}}$ is *closed under direct limits.*
  In fact the maximal space of a direct limit in $\mathsf{cSFP}^{\mathsf{M}}$ is the inverse limit of the maximal spaces of the domains in the sequence (this result relies essentially on Lemma 10, which uses only the compactness of the maximal spaces of the domains in the sequence).
- $\mathsf{cSFP}^{\mathsf{M}}$ is *closed under the constructors in* $\mathscr{F}$.
  This follows immediately by recalling that the constructors in $\mathscr{F}$ preserves M-pairs (see Lemma 26). Then, for instance, let $F \in \mathscr{F}$ be a unary constructor and let D be a $\mathsf{cSFP}^{\mathsf{M}}$ object. By Theorem 40 there exists an M-pair $p : D \to E$, where E is an SFP^M object. Hence $F(p) : F(D) \to F(E)$ is an M-pair and, since SFP^M is closed under F, $F(E)$ is an SFP^M object. Therefore, by Theorem 40, $F(D)$ is a $\mathsf{cSFP}^{\mathsf{M}}$ object.
- *the functor* $\mathsf{MAX} : \mathsf{cSFP}^{\mathsf{M}} \to \mathsf{2\text{-}Stone}$ *is well-defined and* ω*-continuous.*
  Well-definedness is obvious, while ω-continuity relies on Lemma 10, which, as already observed, only requires the compactness of the maximal spaces of the domains in the directed sequence.
- *for any* $F \in \mathscr{F}$, $F \propto \bar{F}$ *in* $\mathsf{cSFP}^{\mathsf{M}}$;
  The proofs remain the same as for SFP^M.

- *each continuous function* $f:\mathsf{Max}(D)\to\mathsf{Max}(E)$, *where* D *is an* ω*-algebraic* CPO *and* E *is a* $\mathsf{cSFP}^{\mathsf{M}}$ *object, extends to a continuous function* $g:D\to E$.
  In fact, since E is a $\mathsf{cSFP}^{\mathsf{M}}$ object, by Lemma 39, there exists an IM-pair $p=\langle i,j\rangle: E\to E'$, where E' is an SFP^{M} object. By Theorem 37, the function $i\circ f:\mathsf{Max}(D)\to\mathsf{Max}(E')$ admits a continuous extension $g':D\to E'$. The function g can thus be defined as $g=j\circ g'$. In fact, for any $x\in\mathsf{Max}(D)$, we have $g(x)=j(g'(x))=j(i(f(x)))=f(x)$.

We conclude this section by observing that the result on function extendability of Section 5 does not fit nicely with the notion of retract. In fact, note that a retraction between the maximal spaces of two domains does not extend, in general, to a retraction between the original SFP's. It suffices to take $D=\mathbf{1}$ and $E=\mathbf{2}$ and the unique function between the maximal spaces. However, given two 2-Stone spaces X and Y, such that Y is a continuous retract of X via the functions $\langle i,j\rangle:Y\to X$ we can always find in SFP^{M} two domain-models D and E of X and Y, respectively, such that $\langle i,j\rangle$ extends to a retraction between D and E. In fact, observe that, for any 2-Stone space X, the poset $\mathsf{Idl}((K\Omega_{ne}(X),\supseteq))$, which is isomorphic to the set of non-empty compact subsets of X ordered by reverse subset inclusion, is a Scott domain (and thus an SFP^{M} object). Therefore one can take $D=\mathsf{Idl}((K\Omega_{ne}(X),\supseteq))$, $E=\mathsf{Idl}((K\Omega_{ne}(Y),\supseteq))$, and the obvious extensions i^* and j^* of i and j, respectively, to sets, e.g., $i^*:D\to E$ defined by $i^*(A)=\{i(a):a\in A\}$ for any $A\in K\Omega_{ne}(X)$.

7. Domain equations for finitary hypersets

In this section we utilize the machinery developed so far to the study of the metric domain of finitary hypersets, i.e. of the hyperuniverse $\mathcal{N}_\omega$ [15,2,20]. Various domain-models have been proposed in the literature for $\mathcal{N}_\omega$. Mislove et al. in [20], characterized it as the solution of the equation over SFP^{ep}

$$X\simeq\mathbf{1}+\mathscr{P}_{Pl}(X). \tag{1}$$

Another domain-model for $\mathcal{N}_\omega$ can be obtained by considering the "domain equation for bisimulation", introduced by Abramsky in [2,3] as a description of Milner's Synchronization Trees with divergence. In the special case of a language with a single action Abramsky's equation becomes

$$X\simeq\mathbf{2}\oplus\mathscr{P}_{Pl}(X_\perp). \tag{2}$$

We will refer to the initial solution of (1) and (2) above as $\mathscr{M}$ and $\mathscr{A}$, respectively.

The results in Section 4 immediately show that $\mathscr{M}$ and $\mathscr{A}$ are SFP^{M} objects and that the 2-Stone spaces consisting of their maximal elements are homeomorphic. In fact, the functor $F_{\mathscr{M}}$ corresponding to (1) can be expressed as $F_{\mathscr{M}}=C_{\mathbf{1}}+\mathscr{P}_{Pl}$, while the functor $F_{\mathscr{A}}$ corresponding to (2) can be expressed as $F_{\mathscr{A}}=C_{\mathbf{2}}\oplus\mathscr{P}_{Pl}((\Pi_1)_\perp)$ (all the involved functors are unary). Hence both $F_{\mathscr{M}}$ and $F_{\mathscr{A}}$ are in the class $\mathscr{F}$. Furthermore

$$\overline{F_{\mathscr{M}}}=\overline{F_{\mathscr{A}}}=C_{\mathbf{1}}\uplus\mathscr{P}_{nco}$$

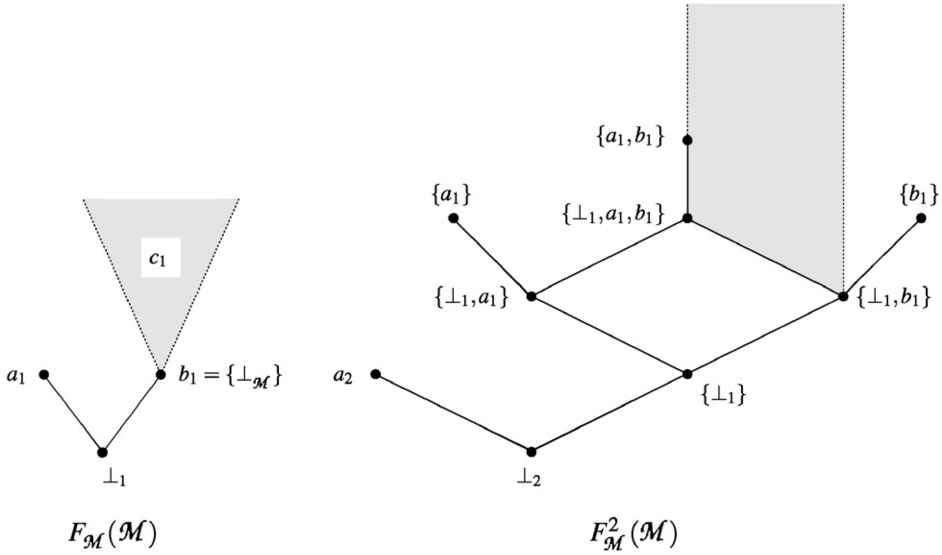

Fig. 5. Structure of the solution $\mathscr{M}$ of $X \simeq 1 + \mathscr{P}_{Pl}(X)$.

and thus a single equation in **2**-Stone, i.e. $X \simeq \mathbf{1} \uplus \mathscr{P}_{nco}(X)$, corresponds both to (1) and (2), and, by Theorem 36, the solution of such an equation is homeomorphic to the maximal spaces of both $\mathscr{M}$ and $\mathscr{A}$.

Using the results in Section 4, we can show in fact that there is a plethora of domain equations whose initial solutions provide a domain-model for the hyperuniverse $\mathscr{N}_\omega$, e.g., $X \simeq \mathbf{2} \oplus (\mathscr{P}_{Pl}(X_\perp))_\perp$ or $X \simeq \mathbf{1} + \mathscr{P}_{Pl}((X_\perp)_\perp)$, etc. More generally, for any SFP^M object D_0 such that $U = \mathsf{MAX}(D_0)$ is a finite discrete space, the initial solutions of the equations $X \simeq (D_0 + \mathscr{P}_{Pl}(X))$, $X \simeq (D_0 \oplus \mathscr{P}_{Pl}(X_\perp))$ (if D_0 has at least two points), etc. are domain-models for the hyperuniverse $\mathscr{N}_\omega(U)$ (see [15] for a definition of $\mathscr{N}_\omega(U)$).

7.1. Non-isomorphism result

In the light of the above considerations, the natural question arises as to whether the domain-models $\mathscr{M}$ and $\mathscr{A}$ of $\mathscr{N}_\omega$, obtained as solutions of Eqs. (1) and (2), are isomorphic. This question was first raised as open problem in [20]. In this section we provide a negative answer to it.

Theorem 42. *The initial solutions of* (1) *and* (2) *are not isomorphic.*

Proof. Let $F_{\mathscr{M}} = C_\mathbf{1} + \mathscr{P}_{Pl}$ be the functor corresponding to equation (1), i.e., $X \simeq \mathbf{1} + \mathscr{P}_{Pl}(X)$. The domain $\mathscr{M}$ is isomorphic to $F^2_{\mathscr{M}}(\mathscr{M})$, which has the shape outlined in Fig. 5. Observe that any point of $F^2_{\mathscr{M}}(\mathscr{M})$ not appearing in the figure is in the upper cone of $\{\perp_1, b_1\}$ since it is a subset of $F_{\mathscr{M}}(\mathscr{M})$ which surely contains an element $c_1 \sqsupseteq b_1$.

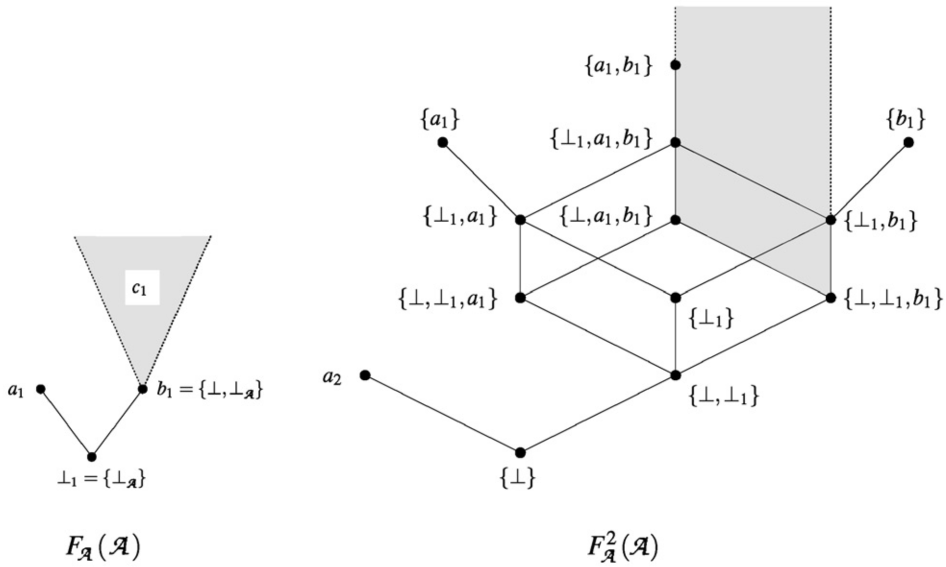

Fig. 6. Structure of the solution $\mathscr{A}$ of $X \simeq 2 \oplus \mathscr{P}_{Pl}(X_\bot)$.

Let $F_{\mathscr{A}} = C_{\mathbf{2}} \oplus \mathscr{P}_{Pl}((\Pi_1)_\bot)$ denote the functor corresponding to equation (2), i.e., $X \simeq 2 \oplus \mathscr{P}_{Pl}(X_\bot)$. The domain $\mathscr{A}$ is isomorphic to $F^2_{\mathscr{A}}(\mathscr{A})$, which has the shape outlined in Fig. 6. As above, any point of $F^2_{\mathscr{A}}(\mathscr{A})$ not appearing in the figure is in the upper cone of $\{\bot, \bot_1, b_1\}$ since it is a subset of $F_{\mathscr{A}}(\mathscr{A})$ which surely contains an element $c_1 \sqsupseteq b_1$.

It is now immediate to conclude that the domains $\mathscr{M}$ and $\mathscr{A}$ are not isomorphic. □

Once we have established that there is a plethora of domain equations providing (possibly non-isomorphic) domain-models of $\mathscr{N}_\omega$ we are left with the problem of determining which of these is the most appropriate. What extra properties do such domain models satisfy?

In the work of Mislove et al. [20] the domain-model of $\mathscr{N}_\omega$, i.e., of the (closure of the space of) hereditarily finite non-well founded sets, is constructed as follows. The key observation is that the class *HF* of hereditarily finite *well-founded* sets is the initial set algebra in Set, namely the initial algebra of the signature comprising the constant e (empty set), the unary function symbol s (singleton) and the binary function symbol $+$ (union). The domain $\mathscr{M}$, proposed as domain-model of $\mathscr{N}_\omega$, is then characterized as the initial *continuous* set algebra and it is proved to be solution of the equation (1).

On the other hand, the domain equation (2) was introduced by Abramsky [3] to show how the notion of bisimulation can be captured in the setting of domain theory. The initial solution $\mathscr{A}$ of such equation (in SFP^{ep}) provides in fact a *fully abstract* denotational semantics for the language SCCS. More precisely, if T_Σ is the class of

SCCS terms and $[\![\cdot]\!] : T_\Sigma \to \mathscr{A}$ is the denotational mapping then for each $t, t' \in T_\Sigma$

$$t \lesssim t' \quad \text{iff} \quad [\![t]\!] \sqsubseteq [\![t']\!],$$

where $\lesssim$ denotes the partial bisimulation relation over SCCS terms. Furthermore, the (recursion free) SCCS terms are shown to provide a notation for the compact elements of $\mathscr{A}$, thus ensuring also a full completeness result.

One could wonder if it is possible to define a different notion of set algebra (by changing the signature and/or the equations) which makes domain $\mathscr{A}$ an initial continuous algebra. This could shed some more light on the interpretation of the points of the domain $\mathscr{A}$ as "partial sets", as it happens for domain $\mathscr{M}$ in [20], where a concrete construction of such domain, based on the idea of murky (partially specified) set, provides an intuitive meaning for the non-total points of the domain. On the other hand one could ask if it is possible to modify the language SCCS and its semantics in order to obtain a fully abstract and complete interpretation of the language in the domain $\mathscr{M}$. More work is necessary to settle these questions.

8. Final remarks

Given any SFP D, the space $\mathsf{MAX}(D)$ is a *Hausdorff space with a countable basis of clopen sets*. One can ask whether Theorem 36 can be extended to SFP^{ep} and **QStone**, the category of zero dimensional Hausdorff spaces and continuous functions. The answer is negative, since there is no functor which models the Plotkin powerdomain constructor when we drop the compactness condition. Let $D_1 = \mathbb{N}_\perp$, $D_2 = \mathbb{N}_\perp + \mathbb{N}_\perp$. Both $\mathsf{Max}(D_1)$ and $\mathsf{Max}(D_2)$ are homeomorphic to $\mathbb{N}$ endowed with the discrete topology. But $\mathsf{Max}(\mathscr{P}_{Pl}(D_1))$ is not homeomorphic to $\mathsf{Max}(\mathscr{P}_{Pl}(D_2))$ since the former has only one limit point, while the latter has more than one. In fact, in $\mathsf{Max}(\mathscr{P}_{Pl}(D_1))$ there is a unique infinite set, namely D_1 itself, while $\mathsf{Max}(\mathscr{P}_{Pl}(D_2))$ contains infinitely many infinite elements.

It would be interesting to extend the results of Section 4 so as to comprise also the *function space* constructor. Unfortunately **2-Stone** is not cartesian closed, in that the space of continuous functions between two 2-Stone spaces endowed with the compact open topology (the unique splitting and conjoining topology), in general, is not compact. One could then try to look at least for the existence of some functor over **QStone** modeling the function space constructor over SFP. But even this is hopeless (also restricting to the *covariant* function space constructor).

First of all maximal functions between SFP's do not necessarily map maximal elements into maximal elements, and thus they do not induce in a natural way functions between the spaces of maximal points. Consider, for instance, the domains $\mathbb{N}_{\text{lazy}}$ and $Bool = \{tt, ff\}_\perp$ and take the continuous function $parity : \mathbb{N}_{\text{lazy}} \to Bool$ (defined in the obvious way). It is a maximal elementin $[\mathbb{N}_{\text{lazy}} \to Bool]$, but it does not map the maximal point $\omega \in \mathbb{N}_{\text{lazy}}$ to a maximal element of $Bool$.

But furthermore, function spaces of SFP's with the same space of maximal elements, can have non-homeomorphic maximal spaces. Consider, for instance,

$$E = \{a, b, \bot\} \cup \{c_i : i \in N\}$$

with the order given by $c_i \sqsubseteq a$, $c_i \sqsubseteq b$ for all $i \in \mathbb{N}$, and $\bot \sqsubseteq x$ for all $x \in E$. Then $\mathsf{Max}(Bool)$ and $\mathsf{Max}(E)$ are the same discrete space, but the maximal spaces $\mathsf{Max}([Bool \to Bool])$ and $\mathsf{Max}([Bool \to E])$ are different. In fact $\mathsf{Max}([Bool \to Bool])$ is a finite discrete space containing only four functions, while $\mathsf{Max}([Bool \to E])$ contains infinitely many functions. Namely, the functions $f_i(tt) = a$, $f_i(ff) = b$, $f_i(\bot) = c_i$, for $i \in \mathbb{N}$, and the constant functions. All these functions are isolated points in a topological sense (since they are compact elements in the SFP) and thus $\mathsf{Max}([Bool \to E])$ is an infinite discrete space and hence it is not compact. This latter example shows also, explicitly, that SFP^M is not closed w.r.t. the function space constructor.

Finally, notice that, differently from what happens for SFP^M objects, we do not have an internal characterization of the CSFP's. It would be interesting to investigate the possibility of characterizing the CSFP's in terms of an order-theoretical property analogous to the M-condition (see Definition 13 and Theorem 15).

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