Supersymmetric classification of cluster states in light nuclei

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Abstract. The Lie superalgebra $u(4|4)$ is proposed and used to classify cluster states in light nuclei by means of a mass formula based on a particular chain of subalgebras. The building blocks, n, p, d and α particles are the superpartners corresponding to the totally supersymmetric $\mathcal{N} = 1$ IRREP of $u(4|4)$. A number of states of other nuclei (from ^{5}He to ^{16}O) are interpreted as cluster configurations formed by 2 or more building blocks and corresponding to $\mathcal{N} = 2, 3, \cdots$ and their energy is reproduced to a reasonable accuracy. The $u(4|4)$ cluster supersymmetry seems therefore to be approximately realized in nature since it accommodates in a single scheme many nuclear states pertaining to different even and odd isotopes. This furnishes a second important example of supersymmetry in nuclear physics.

Since the pioneering work of Wigner on the $su(4)$ symmetry to describe spin and isospin invariance [1], so much has been done in the application of group theoretical concepts to nuclear physics and other fields that it would be impossible to write a comprehensive account here. Afterward, the description of nuclei with algebraic techniques has received a considerable push thanks to the introduction of the Interacting Boson Model [2], where the concept of dynamical symmetries plays a central role. The last development along this line was the extension of Lie algebras to Lie superalgebras to set in a single frame boson and fermion degree of freedoms, and the discovery of supersymmetries in nuclear spectra [3].

We will show, in the following, how the very first application, i.e. Wigner's $u(4)$ and the latest idea, i.e. the boson-fermion symmetries, can be combined to furnish a novel dynamical supersymmetry that, albeit at an approximate level, is realized in the spectra of light nuclei. The most striking feature of what follows is that, through a rather simple mathematical model which can be laid down with ease, it becomes evident that *a subset of energy states in many different isotopes* are linked by the supersymmetry with the hypothesis of a common underlying cluster structure. The degree of approximation to which the present theory is realized in nuclei is the same degree of approximation entailed in Wigner's symmetry: since protons and neutrons are treated on the same level and since their masses differ by almost 1 MeV, one cannot hope for a better accuracy. Nevertheless, here this represents only one side of the coin, because the $su(4)$ boson symmetry, obtained by considering alpha particles and deuteron as elementary bosons, also comes into play. As we will see there are sufficient arguments to establish that this symmetry is also approximately found in the spectra of even nuclei, and therefore the two symmetries are occurring at the same time and one is allowed to speak of a $u(4|4)$ cluster supersymmetry.

An earlier application of supersymmetry to cluster nuclei was proposed in Ref. [4], where the $u(4|12)$ graded Lie algebra was used. They associated the bosonic sector to the relative motion of the α particle with respect to a core (in the spirit of the vibron model [5]) and the fermion sector was used to describe the internal configurations of a p-shell nucleus. A fundamental difference with respect to our work is that we, instead, associate the bosonic sector with the intrinsic spin of the composite bosons and we relegate the relative motion to a substitution that will be discussed below (6). An optimal strategy for the future will be to blend these two ideas together and use a $u(8)$ boson algebra.

We consider the following set of boson (b) and fermion (a) creation and annihilation operators:

$$\begin{array}{ll} b^\dagger_\nu, b_\nu & \nu = 0,1,2,3 \\ a^\dagger_{s_z t_z}, a_{s_z t_z} & s_z, t_z = \pm 1/2 \end{array} \tag{1}$$

which satisfy canonical commutation and anticommutation relations respectively. The bosons are of two types only: a $s-$boson and a $p-$boson (with three components). The fermion may have two components of the spin (s) and two components of the isospin (t), for a total of four possibilities.

The two species of bosons are associated with an α particle ($\ell = 0$) and with a deuteron ($\ell = 1$), while the fermion may be either a proton or a neutron. It may seem artificial to consider ^{2}H and ^{4}He as elementary particles rather than composite particles, but we would like to follow the intuitive idea that cluster configurations in complex nuclei may be formed with simple well-separated building blocks.

The Lie superalgebra $u(4|4)$ can be constructed in a standard way by taking all the bilinear operators that can be formed with the operator in (1). Its maximal boson subalgebra is $u^B(4)$, while the fermion one is $u^F(4)$.

Although the lattice of subalgebras of $u(4|4)$ is quite complex, we wish to focus on the study of the dynamical supersymmetry described by the following chain (2).

$$\begin{array}{ccccccccc} u(4|4) & \supset & u^B(4) & \oplus & u^F(4) & \supset & u^B(3) & \oplus & su^F_{ST}(4) & \supset \\ [\mathcal{N}\} & & [N] & & [1^M] & & n_p & & & \\ & & & & & & & & & \\ & \supset & SO^B(3) & \oplus & su^F_S(2) & \oplus & su^F_T(2) & \supset & SO^{BF}(3) & \\ & & L & & S & & T & & J & \end{array} \tag{2}$$

$$\hat{H} = a_1 C_1[u(4|4)] + a_2 C_2[u(4|4)] + b_1 C_1[u^B(4)] + b_2 C_2[u^B(4)] + c_1 C_1[u^B(3)] + c_2 C_2[u^B(3)] + d C_2[so^{BF}(3)] + e C_2[su^F_T(2)] \tag{3}$$

$$E = a_1 \mathcal{N} + a_2 \mathcal{N}(\mathcal{N}-1) + b_1 N + b_2 N(N+3) + c_1 n_p + c_2 n_p(n_2+2) + dJ(J+1) + eT(T+1) \tag{4}$$

The chain (2) allows for an immediate physical interpretation of the Casimir operators and their eigenvalues. Other interesting chains, like the one passing through $so^B(4)$ should be investigated. The boson part of this chain has been studied in detail in the context of the vibron model (see [5, 6] and references therein). The chain (2) contains the boson $so^B(3)$ algebra, as well as a 'total' or boson-fermion $so^{BF}(3)$ algebra, that insures rotational invariance of the basis states. If needed, a final $u^F_T(1)$ subalgebra might be included, providing an additional Casimir operator with eigenvalue T_z, at the price of introducing also an additional parameter in the mass formula (4). Since our aim is to establish an overall scheme, we will forget, for the moment, the isospin difference between protons and neutrons: in a sense we are extending the Wigner $su(4)$ theory to include also a few bosons that play an important role in the cluster structure of light nuclei.

In the spirit of the IBM and IBFM we construct the analytically solvable hamiltonian (3) by writing a generic linear combination of Casimir of the chain (2), where C_1 and C_2 denote linear and quadratic Casimir operators respectively and $a_1, a_2, b_1, b_2, c_1, c_2, d, e$ are parameters. The necessity of inclusion of the Casimir corresponding to the first (largest) algebra, in contrast to what often happens in the IBM for example, comes from the fact that we are pursuing a description of systems with an arbitrary number of superpartners (i.e. with an arbitrary number of building blocks) and we want to set the scale of their relative energies through a_1 and a_2.

An analytic mass formula, (4), may be directly written by substituting the eigenvalues of the Casimir operators.

The algorithm to calculate the allowed quantum numbers obeys the following set of rules:

$$\begin{array}{l} \mathcal{N} \to (\mathcal{N},0) \oplus (\mathcal{N}-1,1) \oplus \cdots \oplus (\mathcal{N}-4,4) \\ N \to n_p = 0,1,2,\cdots,N \\ M \to \text{see table I} \\ n_p \to L = n_p, n_p-2, n_p-3, \cdots, 1 \; or \; 0 \\ L,S \to J = \mid \vec{L}-\vec{S} \mid, \cdots, \mid \vec{L}+\vec{S} \mid \end{array} \tag{5}$$

Note that, at this stage, J is total angular momentum of a collection of particles each endowed with an intrinsic angular momentum and it does not include the angular momentum of the relative motion of the clusters with each others. To take this into account, albeit in a rough way, we introduce a substitution:

$$\vec{J} \to \vec{J'} = \vec{J} + \sum_{i=1}^{N-1} \vec{\Lambda}_i \tag{6}$$

where $\vec{\Lambda}_i$ are the $N-1$ relative motion angular momenta. Instead of using the substitution above, one would, ideally, enlarge the boson sector by including other boson operators for the relative motion with appropriate multipolarities. This would yield a much larger superalgebra with a larger boson subalgebra without adding much physical information. We prefer to adopt the simple pragmatic prescription (6) to keep the dimensions and number of parameters as small as possible.

By applying the rules given above for $\mathcal{N}=1$ one recovers the list of superpartners that have been chosen as building blocks: in fact (N,M) are either $(1,0)$ or $(0,1)$, i.e. are either bosons or fermions. In the former case $n_p = 0,1$, i.e. our only boson is either the α or the d with $L=0$ and 1 respectively and correspondingly $M=0$ implies $S=T=0$. In the latter case instead $n_p=0$ and $L=0$, but $M=1$ implies $S=T=1/2$ and therefore $J=1/2$, which means that we have recovered our two elementary fermions. For $\mathcal{N}=2$ we have a richer set of possibilities: $(N,M)=(2,0),(1,1),(0,2)$. In the first case we have two bosons that may correspond to the following configurations: $\alpha\alpha$, αd and dd. We already see that the $\mathcal{N}=2$, $dd-$configuration and the $\mathcal{N}=1$, $\alpha-$configuration pertain to the same isotope. This is not a case of double-counting, but rather different supersymmetric states can be put in correspondence with different levels

$u^F_{ST}(4)$	$su_S(2)$	$\oplus$	$su_T(2)$	N. of states
$[0]$	0		0	1
$[1]$	1/2		1/2	4
$[1^2]$	0		1	3
	1		0	3
$[1^3] \equiv [1]$	1/2		1/2	4
$[1^4] \equiv [0]$	0		0	1

Table 1. Branching rules for the fermion spin-isospin quantum numbers. They are, of course, similar to the Wigner ones.

in a nucleus. In this specific case the $\mid \mathcal{N} = 1, N = 1, M = 0, n_p = 0, J' = 0, T = 0\rangle$ state corresponds to the ground state, while the $\mid \mathcal{N} = 2, N = 2, M = 0, n_p = 2, J', T = 0\rangle$ state may correspond to one of the excited states close to the threshold energy for separation into the $d+d$ channel (in accordance with Ikeda's rule). This apparent doubling of states appears frequently for higher quantum numbers, but one can use physical arguments to decide which state is the lowest (not only the position of the energy thresholds, but also cluster transfer intensities may discriminate on the configurations). In table (2) we summarize the results for $\mathcal{N} = 1, 2$ and part of the states with $\mathcal{N} = 3$ (for the sake of brevity).

To fix the eight parameters in the mass formula 4 we use eight experimental energy levels (or rather the mass excess associated to them, taking as a zero the ground state of ^{12}C) in the lightest isotopes with masses $A \leq 8$. This is summarized in fig. . The single-fermion state, f, has been fixed to the average between neutron and proton mass excess and the same for the αf state. The ground statyes of $d, \alpha, {}^6 Li$ and ^{8}Be have been used together with two excited states, the excited 0^+ in the spectrum of the α particle and the first excited 2^+ i nthe spectrum of ^{8}Be.

With the parameters obtained in this way, we looked for predictions on other odd-even and even-even nuclei in figures and respectively.

In the fermionic part the results are less clear than in the bosonic one. This is largely due to the simplication that occurres when neutron and proton states are not distinguished and to the averaging of their masses to fix the parameters of the energy formula. We won't dwell on this apsect, but we concentrate instead on the predictions in the bosonic sector. The famous Hoyle state in ^{12}C as well as the 0^+ at around 15 MeV in ^{16}O come out at about the right energy in our model. This feature is established also in simpler bosonic models taking into account only the α and d.

This is an important prediction because those two states are associated with Bose-Einstein condensed states (See Funaki's, Schuck's and Ohkubo's contributions in these proceedings as well as references [7, 8] and references therein). In other words, without even introducing any alpha-cluster wave function or any phenomenological model or any $\alpha - \alpha$ interaction, we get

$\mathcal{N}$	N	M	n_p	L	S	T	J	T_z	conf.
1	1	0	0	0	0	0	0	0	α
			1	1	0	0	1	0	d
	0	1	0	0	1/2	1/2	1/2	±1/2	p, n
2	2	0	0	0	0	0	0	0	α^2
			1	1	0	0	1	0	αd
			2	2	0	0	2	0	d^2
			2	0	0	0	0	0	d^2
	1	1	0	0	1/2	1/2	1/2	±1/2	$\alpha p, \alpha n$
	1	1	1	1	1/2	1/2	1/2	±1/2	dp, dn
							3/2	±1/2	dp, dn
	0	2	0	0	0	1	0	±1,0	pp, pn_0, nn
					1	0	1	0	pn_1
3	3	0	0	0	0	0	0	0	α^3
			1	1	0	0	1	0	$\alpha^2 d$
			2	2,0	0	0	2,0	0	αd^2
			3	3,1	0	0	3,1	0	d^3

Table 2. Set of quantum numbers that label the first IRREP of $u(4|4)$ along chain (2) and corresponding configurations. The substitution J$\rightarrow$ J' can then be used, according to (6).

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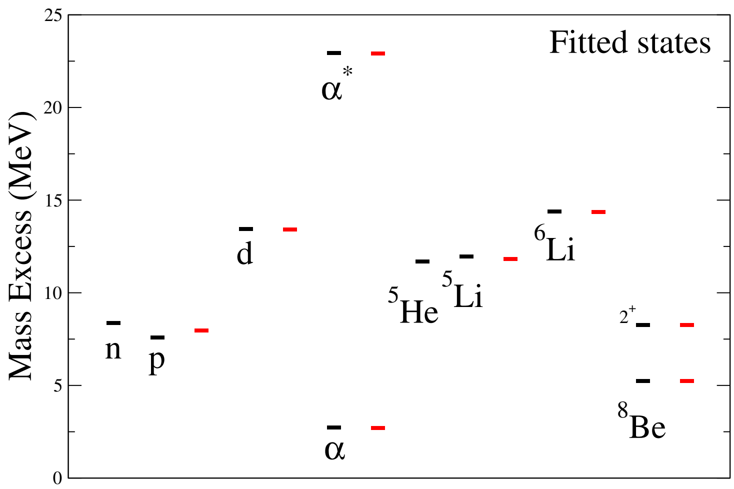

Figure 1. Set of states used to fix the parameters in the mass formula

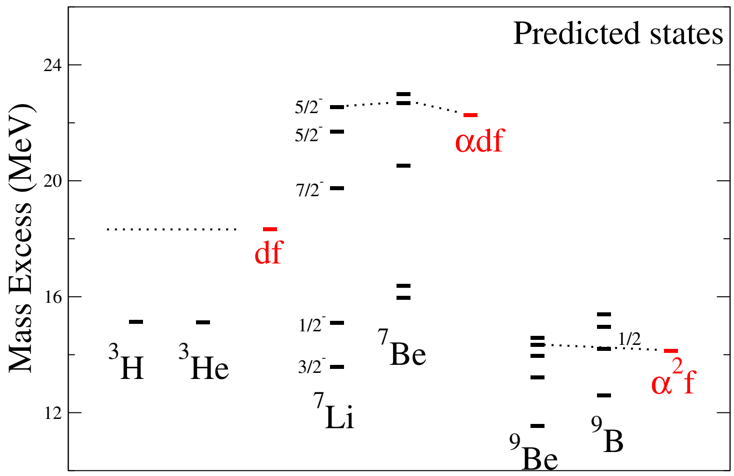

Figure 2. Fermion states predicted by the mass formula

the α^3 and α^4 Bose-Einstein condensed states *only on the basis of a bosonic u(4) dynamical symmetry*. The dimension 4 only reflect the intrinsic spin character ($\ell = 0$ and 1 respectively) of the building blocks, α and d

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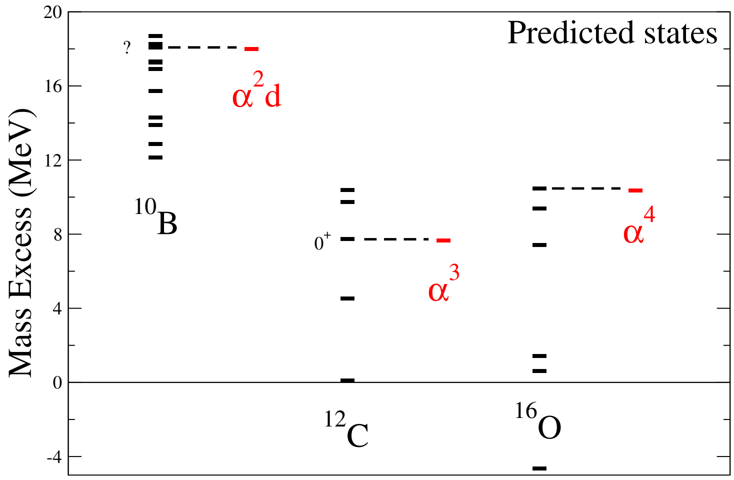

Figure 3. Boson states predicted by the mass formula

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