Pairing Hamiltonian for one pair of identical nucleons bound in a potential well

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The problem of one pair of identical nucleons sitting in $\mathcal{N}$ single particle levels of a potential well and interacting through the pairing force is treated introducing even Grassmann variables. The eigenvectors are analytically expressed solely in terms of these with coefficients fixed by the eigenvalues and the single particle energies. When the latter are those of a harmonic oscillator well an accurate expression is derived for both the collective eigenvalue and for those trapped in between the single particle levels, for any strength of the pairing interaction and any number of levels. Notably the trapped solutions are labeled through an index upon which they depend parabolically.

We have recently obtained, in the framework of the Grassmann algebra, the analytic expressions of the eigenvalues and the eigenvectors of a system of n pairs of like-nucleons interacting through the pairing Hamiltonian and sitting in one single-particle level [1].

We extend the analysis to the case of $\mathcal{N}$ single-particle levels, with energies $e_1, e_2, \ldots, e_{\mathcal{N}}$ and angular momenta $j_1, j_2, \ldots, j_{\mathcal{N}}$ (all the j's being assumed to be different): here the Hamiltonian, for identical particles, reads

$$H = \sum_{\nu=1}^{\mathcal{N}} e_\nu \sum_{m_\nu=-j_\nu}^{j_\nu} \lambda_{j_\nu m_\nu}^* \lambda_{j_\nu m_\nu}$$
$$- G \sum_{\mu,\nu=1}^{\mathcal{N}} \sum_{m_\mu=1/2}^{j_\mu} \sum_{m_\nu=1/2}^{j_\nu} \varphi_{j_\mu m_\mu}^* \varphi_{j_\nu m_\nu}, \qquad (1)$$

where λ_{jm} and λ_{jm}^* are the odd (anticommuting, nilpotent) and [2]

$$\varphi_{jm} \equiv (-1)^{j-m} \lambda_{j-m} \lambda_{jm} \qquad (2)$$

the even (commuting, nilpotent) Grassmann variables. The latter is associated with a pair of fermions with vanishing third component of the total angular momentum ($M=0$). To start with we confine ourselves to consider one pair only.

Notwithstanding the presence of both the λ's and the φ's, the Hamiltonian (1) is diagonalized in the $2\mathcal{N}$ dimensional basis[1]

$$\begin{cases} \Phi_\nu^{(0)} &= \dfrac{1}{\sqrt{\Omega_\nu - 1}} \displaystyle\sum_{m_\nu=1/2}^{j_\nu-1} \varphi_{j_\nu, m_\nu}, \qquad \nu=1,\ldots,\mathcal{N}, \\ \Phi_\nu^{(1)} &= \varphi_{j_\nu, j_\nu} \end{cases} \qquad (3)$$

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[1]If a single particle level with $j_\nu = 1/2$ ($\Omega_\nu = 1$) is present, the dimension of the basis is actually $2\mathcal{N}-1$, because, obviously, $\Phi_\nu^{(0)}$ is identically zero.

which extends the one we introduced in Ref. [1] and describes the two nucleons in the same single-particle level. In Eq. (3), $2\Omega_\nu = 2j_\nu + 1$ is the degeneracy of the level j_ν.

For one pair, only states with seniority $v=0$ and 2 are allowed. In the basis (3) the eigenvalues of the $v=2$ states are trivial, whereas for those of the $v=0$ states one recovers the well-known secular equation [3]

$$\frac{1}{G} + f(E) = 0 \quad \text{with} \quad f(E) = \sum_{\nu=1}^{\mathcal{N}} \frac{\Omega_\nu}{E - 2e_\nu}. \qquad (4)$$

The corresponding eigenvectors are

$$\psi_{v=2}^{(\nu)}(\Phi^*) = \sqrt{1 - \frac{1}{\Omega_\nu}} \left\{ [\Phi_\nu^{(1)}]^* - \frac{1}{\sqrt{\Omega_\nu - 1}} [\Phi_\nu^{(0)}]^* \right\} \qquad (5)$$

and

$$\psi_{v=0}^{(\nu)}(\Phi^*) = \sum_{\mu=1}^{\mathcal{N}} w_\mu^{(\nu)} \sqrt{\Omega_\mu - 1} \left\{ [\Phi_\mu^{(0)}]^* \right.$$
$$\left. + \frac{1}{\sqrt{\Omega_\mu - 1}} [\Phi_\mu^{(1)}]^* \right\}, \qquad (6)$$

with $\nu=1,\ldots,\mathcal{N}$, the coefficients $w_\mu^{(\nu)}$ fulfilling the system of equations

$$(E^{(\nu)} - 2e_\mu) w_\mu^{(\nu)} + G \sum_{\sigma=1}^{\mathcal{N}} \Omega_\sigma w_\sigma^{(\nu)} = 0. \qquad (7)$$

The above system is easily solved and yields the noticeable formula (see also [4])

$$w_\mu^{(\nu)} = \frac{E^{(\nu)} - 2e_{\mathcal{N}}}{E^{(\nu)} - 2e_\mu} w_{\mathcal{N}}^{(\nu)}, \qquad (8)$$

which shows that for a given set of single-particle energies the $v=0$ eigenvectors are fixed by the corresponding eigen-
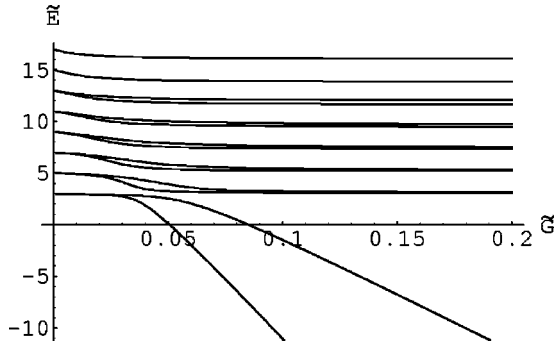
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FIG. 1. The figure shows the solutions $\tilde{E}$ of Eq. (10), for $\mathcal{N}$ $=6$ (upper curves) and 8 (lower curves), as functions of $\tilde{G}$. One can see that with the harmonic oscillator well each trapped solution $\tilde{E}_N$ for $\tilde{G}>0.1$ tends approximately to the single particle energy $2\tilde{e}_N$.

values. When G is large Eq. (4) develops a collective solution with large E: hence the associated w_μ tend to become all equal and correspond to a coherent superposition of the so-called s quasibosons. On the other hand, in the limit $G\to 0$, where $E\simeq 2e_\nu$, only one component of the basis, i.e., the νth one, survives in the wave function of the "trapped" states.

In general the eigenvalues (and hence the eigenvectors) of Eq. (1) stem from an interplay between the single-particle energies and degeneracies. Of course this interplay can be numerically explored. Here we pursue the scope analytically, when the e_ν and the Ω_ν are available. One of the few cases where this occurs is for the harmonic oscillator well, where

$$e_N=\left(N+\frac{3}{2}\right)\hbar\omega \quad \text{and} \quad \Omega_N=\frac{1}{2}(N+1)(N+2),$$
$$N=0,\ldots,\infty. \tag{9}$$

Accordingly the secular equation (4), for $\mathcal{N}$ levels, becomes

$$\sum_{N=0}^{\mathcal{N}-1}\frac{(N+1)(N+2)}{2N+3-\tilde{E}}=\frac{1}{\tilde{G}}, \tag{10}$$

where $\tilde{G}=G/2\hbar\omega$ and the energies are measured in units of $\hbar\omega$ ($2\tilde{e}_N=2N+3$).

In Fig. 1 the numerical solutions of Eq. (10) are displayed for $\mathcal{N}=6$ and 8 versus $\tilde{G}$. Remarkably, the dependence upon $\tilde{G}$ is seen to be lost for $\tilde{G}\gtrsim 0.1$. Furthermore in this regime the trapped solutions are mildly dependent upon $\mathcal{N}$.

Although aware that the solutions of Eq. (10) cannot, in general, be algebraically expressed (for $\mathcal{N}\geqslant 5$), we explore whether the simple ansatz

$$\tilde{E}_{\bar{N}}=a\bar{N}^2+b\bar{N}+c, \qquad \bar{N}=0,\ldots,\mathcal{N}-2 \tag{11}$$

provides a good representation of the trapped solutions (the collective solution $\tilde{E}_c$ will be separately treated).

To fix the coefficients a, b, and c, we recast Eq. (10) in the polynomial form

$$\tilde{E}^{\mathcal{N}}+a_1\tilde{E}^{\mathcal{N}-1}+a_2\tilde{E}^{\mathcal{N}-2}+\cdots a_{\mathcal{N}-1}\tilde{E}+a_{\mathcal{N}}=0, \tag{12}$$

finding for the first three coefficients the expressions

$$a_1=\frac{1}{3}\mathcal{N}(\mathcal{N}+2)[\tilde{G}(\mathcal{N}+1)-3], \tag{13}$$

$$a_2=\frac{1}{6}\mathcal{N}(\mathcal{N}-1)[-\tilde{G}(\mathcal{N}+1)(\mathcal{N}+2)(2\mathcal{N}+3)$$
$$+3\mathcal{N}^2+11\mathcal{N}+11], \tag{14}$$

$$a_3=\frac{1}{90}\mathcal{N}(\mathcal{N}-1)(\mathcal{N}^2-4)[\tilde{G}(\mathcal{N}+1)(15\mathcal{N}^2+40\mathcal{N}+27)$$
$$-15(\mathcal{N}^2+3\mathcal{N}+3)]. \tag{15}$$

Then the first three Viète equations, namely,

$$\sum_{\bar{N}=0}^{\mathcal{N}-2}\tilde{E}_{\bar{N}}=-a_1-\tilde{E}_c, \tag{16}$$

$$\sum_{\bar{N}=0}^{\mathcal{N}-2}\tilde{E}_{\bar{N}}^2=a_1^2-2a_2-\tilde{E}_c^2, \tag{17}$$

$$\sum_{\bar{N}=0}^{\mathcal{N}-2}\tilde{E}_{\bar{N}}^3=-3a_3-a_1(a_1^2-3a_2)-\tilde{E}_c^3, \tag{18}$$

yield a nonlinear system in the unknowns a, b, and c, if $\tilde{E}_c$ is known. This system can be solved by expressing, via Eq. (16), c as a function of a and b

$$c(a,b)=-\frac{1}{\mathcal{N}-1}\left\{\tilde{E}_c+\frac{b}{2}(\mathcal{N}-1)(\mathcal{N}-2)+\frac{a}{6}(\mathcal{N}-1)(\mathcal{N}\right.$$
$$\left.-2)(2\mathcal{N}-3)+\frac{1}{3}\mathcal{N}(\mathcal{N}+2)[\tilde{G}(\mathcal{N}+1)-3]\right\}. \tag{19}$$

In turn Eq. (19), inserted into Eq. (17), yields b as a function of a. One finds

$$b(a)=-\frac{15a(\mathcal{N}^4-6\mathcal{N}^3+13\mathcal{N}^2-12\mathcal{N}+4)+\sqrt{\Delta}}{15(\mathcal{N}-1)^2(\mathcal{N}-2)} \tag{20}$$

with

$$\Delta=-15(\mathcal{N}-1)^2(\mathcal{N}-2)\{a^2(\mathcal{N}-1)^2(\mathcal{N}-1)(\mathcal{N}-2)(\mathcal{N}$$
$$-3)+20[9\tilde{E}_c^2+6\tilde{E}_c(\mathcal{N}+2)(\tilde{G}\mathcal{N}+\tilde{G}-3)-3(\mathcal{N}^3$$
$$-4\mathcal{N}^2-13\mathcal{N}-11)-\tilde{G}^2\mathcal{N}(\mathcal{N}-2)(\mathcal{N}+1)^2(\mathcal{N}+2)^2$$
$$+3\tilde{G}(\mathcal{N}+1)(\mathcal{N}+2)(\mathcal{N}^2-4\mathcal{N}-3)]\}. \tag{21}$$

Finally, from Eq. (18), a cumbersome equation for a follows, not reported here. While Eqs. (16) and (17) are easily solved analytically, the nonlinear equation (18) for a can only be solved numerically and admits, in general, several solutions. The one appropriate for our problem is selected by

TABLE I. Comparison between the exact $\tilde{E}_c^{(e)}$ and the approximate $\tilde{E}_c^{(k)}$ [Eqs. (25) and (28)] collective energies for some values of $\mathcal{N}$ and $\tilde{G} = 0.05$ and 0.1.

	$\tilde{G} = 0.05$					
$\mathcal{N}$	$\tilde{E}_c^{(0)}$	$\tilde{E}_c^{(1)}$	$\tilde{E}_c^{(2)}$	$\tilde{E}_c^{(3)}$	$\tilde{E}_c^{(4)}$	$\tilde{E}_c^{(e)}$
2	2.87617	2.88394	2.88349	2.88348	2.88348	2.88348
3	2.81180	2.87628	2.86196	2.86014	2.86012	2.86012
4	2.70198	2.89185	2.84845	2.82471	2.82056	2.82046
5	2.53580	2.86975	2.80333	2.75484	2.74104	2.74028
6	2.24594	2.63571	2.54951	2.52958	2.52886	2.52886
7	1.61587	1.84837	1.83108	1.83094	1.83094	1.83094
8	0.134714	0.136135	0.136135	0.136135	0.136135	0.136135

	$\tilde{G} = 0.1$					
$\mathcal{N}$	$\tilde{E}_c^{(0)}$	$\tilde{E}_c^{(1)}$	$\tilde{E}_c^{(2)}$	$\tilde{E}_c^{(3)}$	$\tilde{E}_c^{(4)}$	$\tilde{E}_c^{(e)}$
2	2.70757	2.72972	2.72822	2.72822	2.72822	2.72822
3	2.45586	2.61626	2.58336	2.58335	2.58335	2.58335
4	1.96617	2.21918	2.18238	2.18238	2.18238	2.18238
5	0.919942	1.00125	1.00000	1.00000	1.00000	1.00000
6	-1.66966	-1.47625	-1.48025	-1.48025	-1.48025	-1.48025
7	-8.37559	-4.88888	-5.44566	-5.44568	-5.44568	-5.44568
8	-24.4931	-3.26859	-10.7011	-11.0491	-11.0546	-11.0546

requiring that the trapped energies lie in between the single-particle levels of the harmonic oscillators. Moreover, it should be pointed out that, owing to the high degree of nonlinearity of Eq. (18), this solution turns out to be extremely sensitive to the collective energy $\tilde{E}_c$, especially when $\tilde{G}$ is large.

Hence an accurate expression for the collective energy is needed. We look for the latter in the domain of small $\tilde{G}$ (say $\tilde{G} = 0.05$–0.1), as it follows from the empirical determination of G in atomic nuclei and from the experimental nuclear single-particle levels [5].

For this scope we start by realizing that $\tilde{E}_c(0,\mathcal{N}) = 3$ and $\tilde{E}_c(\tilde{G}_0,\mathcal{N}) = 0$, being

$$\tilde{G}_0 = \left[\sum_{N=0}^{\mathcal{N}-1} \frac{(N+1)(N+2)}{(2N+3)} \right]^{-1}. \tag{22}$$

Moreover

$$\left. \frac{\partial \tilde{E}_c(\tilde{G},\mathcal{N})}{\partial \tilde{G}} \right|_{\tilde{G}=0} = -2 \tag{23}$$

and

$$\left. \frac{\partial \tilde{E}_c(\tilde{G},\mathcal{N})}{\partial \tilde{G}} \right|_{\tilde{G}=\tilde{G}_0} = -\left[\sum_{N=0}^{\mathcal{N}-1} \frac{(N+1)(N+2)}{2N+3} \right]^2$$

$$\times \left[\sum_{N=0}^{\mathcal{N}-1} \frac{(N+1)(N+2)}{(2N+3)^2} \right]^{-1}. \tag{24}$$

The above constraints are fulfilled by the cubic function (in $\tilde{G}$)

$$\tilde{E}_c^{(0)}(\tilde{G},\mathcal{N}) = 3 - 2\tilde{G} - \left[9 + \left(\left. \frac{\partial \tilde{E}_c}{\partial \tilde{G}} \right|_{\tilde{G}_0} - 4 \right) \tilde{G}_0 \right] \frac{\tilde{G}^2}{\tilde{G}_0^2}$$

$$+ \left[6 + \left(\left. \frac{\partial \tilde{E}_c}{\partial \tilde{G}} \right|_{\tilde{G}_0} - 2 \right) \tilde{G}_0 \right] \frac{\tilde{G}^3}{\tilde{G}_0^3}, \tag{25}$$

which thus provides an excellent representation of the collective energy (see Table I). If an even better $\tilde{E}_c$ is desired, one can proceed perturbatively setting

$$\tilde{E}_c = \tilde{E}_c^{(0)} + \delta \tag{26}$$

and expanding in the very small parameter $\delta/M(N)$ where

$$M(N) = 2N + 3 - \tilde{E}_c^{(0)}. \tag{27}$$

One thus gets the recursive relation

$$\tilde{E}_c^{(k+1)} = \tilde{E}_c^{(k)} + \left[\frac{1}{\tilde{G}} - \sum_{N=0}^{\mathcal{N}-1} \frac{(N+1)(N+2)}{2N+3-\tilde{E}_c^{(k)}} \right]$$

$$\times \left[\sum_{N=0}^{\mathcal{N}-1} \frac{(N+1)(N+2)}{(2N+3-\tilde{E}_c^{(k)})^2} \right]^{-1}. \tag{28}$$

TABLE II. Comparison between exact (e) ''trapped'' solutions of Eq. (10) and approximate (app) ones, obtained from the ansatz (11) for $\mathcal{N}=5$ levels. The coefficients (a,b,c) of the parabola are (0.086, 1.738, 4.281) when $\tilde{G}=0.05$, (0.014, 2.183, 3.427) when $\tilde{G}=0.1$, (0.027, 2.175, 3.160) when $\tilde{G}=1$, and (0.029, 2.167, 3.151) when $\tilde{G}=5$.

$\bar{N}$	$\tilde{G}=0.05$		$\tilde{G}=0.1$		$\tilde{G}=1$		$\tilde{G}=5$	
	$\tilde{E}_{\bar{N}}^{(e)}$	$\tilde{E}_{\bar{N}}^{(app)}$	$\tilde{E}_{\bar{N}}^{(e)}$	$\tilde{E}_{\bar{N}}^{(app)}$	$\tilde{E}_{\bar{N}}^{(e)}$	$\tilde{E}_{\bar{N}}^{(app)}$	$\tilde{E}_{\bar{N}}^{(e)}$	$\tilde{E}_{\bar{N}}^{(app)}$
0	4.2872	4.2812	3.4245	3.4266	3.1583	3.1601	3.1493	3.1510
1	6.0892	6.1056	5.6302	5.6237	5.3673	5.3621	5.3524	5.3472
2	8.1171	8.1021	7.8422	7.8485	7.6136	7.6190	7.5965	7.6015
3	10.266	10.271	10.103	10.101	9.9314	9.9297	9.9157	9.9140

The energies provided by Eq. (28) rapidly converge to the exact solution, as shown in Table I, but, as mentioned above, a high precision is required, which is obtained after four iterations.

Formula (28) yields $\tilde{E}_c$ also when $\tilde{G}$ is large. Here however the analogous of Eq. (25) follows by expanding Eq. (10) in the parameter $(2N+3)/\tilde{E}$. One thus gets for collective energy, expanded up to terms $1/\tilde{G}$, the expression

$$\tilde{E}_c^{(0)} = -\frac{\tilde{G}}{3}\mathcal{N}(\mathcal{N}+1)(\mathcal{N}+2) + \frac{3}{2}(\mathcal{N}+1)$$
$$-\frac{9(\mathcal{N}-1)(\mathcal{N}+3)}{20\mathcal{N}(\mathcal{N}+1)(\mathcal{N}+2)}\frac{1}{\tilde{G}}. \qquad (29)$$

The above, when inserted in Eq. (28), yields results as accurate as those obtained in the domain of small $\tilde{G}$.

With the collective energy fixed, the coefficients a, b, and c can be found. We quote in Table II, as an example, our predictions for the eigenvalues of the pairing Hamiltonian for one pair in the $\mathcal{N}=5$ case, using as input Eq. (29) when

$\tilde{G}=1$ and $\tilde{G}=5$ and Eq. (25) when $\tilde{G}=0.05$ and $\tilde{G}=0.1$. Our results are seen to agree with the exact ones obtained via the numerical solution of Eq. (10) to better than 0.27%. This occurs as well for all the cases we have explored. Thus the simple ansatz (11) is remarkably accurate. Furthermore the solutions (11), when $\tilde{G}$ is large, scale in $\tilde{G}$. Indeed, in this condition, the right-hand sides of the Viète equations (16)–(18) are easily seen to be $\tilde{G}$ independent when the collective solution is given by Eq. (29).

In conclusion, although the pairing problem can of course, be solved numerically, we believe that our semianalytical solution might be of some help for treating the situation when n pairs, sitting in a harmonic oscillator well, are present. Also of interest is the appearance of the extension of the present analysis to the situation where the pair is made out of a neutron and a proton, particularly when these are in an isospin singlet state [6]. In this case indeed the partners, in order to feel the pairing interaction, are forced to be in different shells, at least $2\hbar\omega$ apart. Whether or not in these conditions our semianalytical solution holds valid as well and a collective mode eventually develops is an issue we are currently exploring.

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