THE PERIODIC PATCH MODEL FOR POPULATION DYNAMICS WITH FRACTIONAL DIFFUSION

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Abstract. Fractional diffusions arise in the study of models from population dynamics. In this paper, we derive a class of integro-differential reaction-diffusion equations from simple principles. We then prove an approximation result for the first eigenvalue of linear integro-differential operators of the fractional diffusion type, and we study from that the dynamics of a population in a fragmented environment with fractional diffusion.

1. Introduction

Consider $\alpha \in (0,1)$ and denote

$$l = (l_1, \cdots, l_N), \quad \mathcal{C}_l = (0,l_1) \times \cdots \times (0,l_N).$$

Let $\mathcal{T}_\alpha$ be the fractional laplacian of order $\alpha \in (0,1)$:

$$\mathcal{T}_\alpha u = c_\alpha P.V.\Big(\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2\alpha}}\, dy\Big), \tag{1}$$

where $P.V.$ denotes the principal value. The symbol of the operator inside the principal value is - see e.g. [5] - proportional to $|\xi|^{2\alpha}$, the constant c_α is chosen so that this symbol is exactly $|\xi|^{2\alpha}$. In the sequel this constant will not play any role and will be omitted in order to make the notations lighter. The precise definition of $\mathcal{T}_\alpha$ is

$$\mathcal{T}_\alpha u = c_\alpha \lim_{\varepsilon\to 0} \int_{|x-y|\geq\varepsilon} \frac{u(x)-u(y)}{|x-y|^{N+2\alpha}}\, dy. \tag{2}$$

2000 *Mathematics Subject Classification.* Primary: 35R11; Secondary 35B40, 35K57, 92D25.
Key words and phrases. Fractional diffusion, reaction-diffusion equation, KPP nonlinearity, persistence.

We will also consider $\mathcal{T}_\alpha$ as an operator acting on the space of smooth functions on Ω which vanish on $\partial\Omega$, where Ω is a a smooth bounded domain of $\mathbb{R}^N$. It will still be given by (1), but it will always be understood that u is extended by 0 outside Ω.

The main contribution of this paper is the analysis of the eigenvalues of the fractional laplacian with a potential:

$$\mathcal{L}u = \mathcal{T}_\alpha u - \mu(x)u, \tag{3}$$

in a bounded domain with Dirichlet conditions outside the domain, or in the whole space with periodicity conditions. As an operator with positive and compact inverse - see again [5], it lends itself to the application of the Kreǐn-Rutman Theorem: the bottom of the spectrum is a real eigenvalue, simple in the geomeric and algebraic sense, with a positive eigenfunction. The main result of this paper is an approximation property of the first periodic eigenvalue by that of the first Dirichlet eigenvalue in a ball, when the radius of the ball becomes infinite. This property is already known for local operators - see [3], [7] for second order elliptic operators, it is less expected for nonlocal oprators such as $\mathcal{T}_\alpha$. It is even somewhat counter-intuitive: one could have guessed that the long range diffusion would have introduced more visible nonlocal effects.

Let $\mathbb{T}^N$ be the torus

$$\mathbb{R}^N/\mathcal{C}_l.$$

Let $\lambda_1(r)$ and λ_{per} denote the principal eigenvalue of $\mathcal{L}$ respectively in B_r under Dirichlet boundary conditions and in $\mathbb{T}^N$, i.e. $\lambda_1(r)$ (resp. λ_{per}) is the smallest λ such that the problem

$$\mathcal{L}u = \lambda u$$

has a solution $u \in C^{1,\alpha}(B_r(0))$ such that $u \equiv 0$ outside $B_r(0)$ (resp. $u \in C^{1,\alpha}(\mathbb{T}^N)$). The approximation result is then

Theorem 1.1. *When r goes to $+\infty$, then $\lambda_1(r)$ converges to λ_{per}.*

One of the motivations of Theorem 1.1 is the long term dynamics of an evolution equation of the KPP type involving $\mathcal{T}_\alpha$. In this context we prove theorems analogous to Theorems 2.1, 2.4 and 2.6 in [3]; the aimed application being to understand a scalar reaction-diffusion model for the interaction of a population with a fragmented environment.

Let us now describe the setting. Consider the semi-linear equation

$$u_t + \mathcal{T}_\alpha u = f(x,u), \qquad t>0,\ x \in \mathbb{R}^N \tag{4}$$

with:

$$x \mapsto f(x,u)\ \mathcal{C}_l\text{-periodic for all } u, \text{ and } s \mapsto \frac{f(x,s)}{s} \text{ is decreasing;}$$

$$\forall x \in \mathbb{R}^N, \quad f(x,0)=0, \quad \text{and} \quad \exists M>0,\ \forall x \in \mathbb{R}^N,\ \forall s \geq M,\ f(x,s) \leq 0.$$

A typical example of such function f is the logistic nonlinearity

$$f(x,u) = \mu(x)u - u^2.$$

In the next section, we will describe how model (4) arises naturally when one wishes to describe the dynamics of a population with integral dispersal. We will then prove the

Theorem 1.2. *Assume that f is as above, set $\mu(x) = f_u(x,0)$ and let $\mathcal{L}$ be given by (3). Equation (4) has a unique bounded positive steady solution u_+ if and only if $\lambda_{per} < 0$. When it exists, the solution is unique and any solution to (4) starting with a bounded nonnegative initial datum $u_0 \not\equiv 0$ tends to u_+ as $t \to +\infty$. For an initial datum bounded away from 0, the convergence is uniform.*

In the opposite case, namely $\lambda_{per} \geq 0$, every solution to (4) with bounded nonnegative initial datum goes to 0 (and there is extinction).

For a detailed biological interpretation, we refer the reader to [3], and we will not reproduce all the details here. Of course, the novelty in this paper resides in the nonlocal dispersal term, that induces different qualitative behaviours. Indeed, while this part of the analysis is qualitatively similar to what happens in [3], the picture changes drastically when we turn to the study of the invasion of the unstable state by the stable one. In the standard diffusion case, it is described in [4]: there are pulsating waves connecting 0 to u_+ and there is a constant average invasion velocity. We already know that such will not be the case here: from [6], the invasion rate will be exponential in time.

The paper is organised as follows. In Section 2 we provide a derivation of (4) as a model for the description of the migration, birth and death of a population. Actually, we point out that most of it is already in [11]; a deeper understanding would involve a probabilistic modelling that we leave out of the scope this paper. The simple presentation that we give here has the advantage of providing a unified point of view that includes local diffusion as well. In Section 3 we give elementary properties of the eigenvalues of $\mathcal{L}$, and the main result about the approximation of the periodic eigenvalue by a sequence of Dirichlet eigenvalues is proved in Section 4. In Section 5 we prove Theorem 1.2.

2. Derivation of the model

The purpose of this secton is to show that our model (4) is obtained as the limit of a dispersal model that yields, in a different limit and different structure assumptions on the coefficients, the classical KPP type equations

$$u_t - D\Delta u = \mu(x)u - u^2 \tag{5}$$

There are many of ways to derive this model. See, for instance, Fife [10] or Murray [12] for a detailed discussion on the various approaches - probabilistic or deterministic - and the modelling assumptions that lead to (5). Our derivation is much more elementary. It follows the lines of the intrduction of [11] and, as said before, has the merit of making quite evident that models (4) and (5) are both limiting cases of a common integro-differential equation. See also [9] for a recent derivation of the classical reaction-diffusion equation from the integral equation.

Consider a population that we may describe - and this is a strong assumption - by its density $u(t,x)$ per time and volume unit. To count the variation of the number of individuals at x and in the time interval $[t, t+\Delta t]$, we first assume - another strong assumption - that it is proportional to Δt. We need to take into account:

- the births and deaths occurring at x during the time interval; we model them by a source term $f(x, u(t,x))\Delta t$.
- The migrations from and to the point x. We assume that, in the interval $[t, t+\Delta t]$ and at the point x, the fraction of the population migrating from

y to x has the form

$$J(x, x-y)\Delta t \tag{6}$$

where $J(x,z)$ is a positive, integrable function. We have chosen J not to depend on time. Putting a time dependence would also be a relevant assumption, and would lead to the same kind of model.

The total increase of the population due to the migrations on the one hand, and to the births and deaths on the other hand, is given by

$$\begin{aligned} u(t+\Delta t, x) - u(t,x) = &\int_{\mathbb{R}^N} u(t,y)J(x,y-x)\Delta t\,dy \\ &- \int_{\mathbb{R}^N} u(t,x)J(y,x-y)\Delta t\,dy + f(x,u(t,x))\Delta t. \end{aligned}$$

The first two integrals represent the individuals moving respectively towards and away from the point x.

Lastly, we make the scaling assumption that

$$J(x,z) = \frac{1}{\varepsilon^N} j(x, \frac{z}{\varepsilon}),$$

with $0 < \varepsilon << 1$. This expresses the fact that the migration range around x - described by the tail of the function $z \mapsto J(x,z)$ - are mostly localised around x with a specified scaling. Notice that this scaling is natural in view of the preservation of the L^1 norm. The additional dependence of J on x expresses that migrations can be favoured - or hampered - by the local environment. Summarising, we obtain

$$u_t(t,x) = \varepsilon^{-N}\left(\int_{\mathbb{R}^N} j(x, \frac{y-x}{\varepsilon})u(t,y)\ dy - \int_{\mathbb{R}^N} j(y, \frac{x-y}{\varepsilon})\ dy\right) + f(x,u(t,x)).$$

We consider two cases. These may not be the most general, but will allow us to recover many classical models of interest as limiting cases.

Case 1. *The distribution j has uniformly bounded third moments.* This means

$$\|y_m y_n y_p j(.,y_1,\dots,y_N)\|_{L^1(\mathbb{R}^N)} \le C$$

for all $m,n,p \in \{1,\dots,N\}$. Write

$$u(t, x+\varepsilon z) = u(t,x) + \varepsilon Du(t,x)\cdot z + \frac{\varepsilon^2}{2}\sum_{i,j=1}^N \partial_{ij}u(t,x)z_i z_j + o(\varepsilon^2)O(|z|^2),$$

and

$$\begin{aligned} &\varepsilon^{-N}\int_{\mathbb{R}^N} (j(x,\frac{y-x}{\varepsilon})u(t,y) - j(y,\frac{x-y}{\varepsilon})u(t,x))\,dy \\ &= \int_{\mathbb{R}^N} (j(x,z)u(t,x+\varepsilon z) - j(x+\varepsilon z, -z)u(t,x))\,dz \\ &= \int_{\mathbb{R}^N} (j(x,z) - j(x+\varepsilon z,-z))\,dz u(t,x) + \varepsilon\int_{\mathbb{R}^N} j(x,z)z\,dz\cdot Du(t,x) \\ &\quad + \frac{\varepsilon^2}{2}\int_{\mathbb{R}^N} j(x,z)z_i z_j\,dz \sum_{i,j=1}^N \partial_{ij}u(t,x) + o(\varepsilon^2) := \mathcal{A}_\varepsilon u + o(\varepsilon^2) \end{aligned}$$

We choose to discard the $o(\varepsilon^2)$ terms, thus obtaining the equation

$$u_t + \mathcal{A}_\varepsilon u = f(x,u). \tag{7}$$

Let us assume that j is smooth enough, at least C^3. We have

$$\mathcal{A}_\varepsilon u = \varepsilon \mathrm{div}_x(u(t,x)V(x)) + \frac{\varepsilon^2}{2}\Big(c(x)u(t,x) + \sum_{i,j} a_{ij}(x)\partial_{ij}u(t,x)\Big) + o(\varepsilon^2)$$

with

$$c(x) = -\int_{\mathbb{R}^N} D_x^2 j(x,z)\,dz, \quad a_{ij}(x) = \frac{1}{2}\int_{\mathbb{R}^N} j(x,z)z_iz_j\,dz, \quad V(x) = \int_{\mathbb{R}^N} j(x,z)z\,dz.$$

Discard the additional $o(\varepsilon^2)$ terms, and define a new time scale $\tau = \varepsilon^2 t$. It is then appropriate to choose the birth and death term as $f(x,u) = \varepsilon^2 g(x,u)$; the final model is thus

$$u_\tau - \sum_{i,j} a_{ij}(x)\partial_{ij}u + \frac{1}{\varepsilon}\mathrm{div}(uV) = h(x,u) := c(x)u + g(x,u). \tag{8}$$

We recognise here a reaction-diffusion model with large drift. When the vector field $V(x)$ is divergence-free (which is not necessarily imposed by the assumptions on J) this kind of models has been studied very much, see [2], [8] for recent advances. When the function j does not depend on x and is radially symmetric with respect to its second argument, we retrieve the classical Fisher-KPP equation

$$u_t = D\Delta u + f(x,u), \qquad D = \frac{1}{2N}\int_{\mathbb{R}^N} |y|^2 j(y)\,dy \tag{9}$$

Case 2. *The distribution j has infinite second moments.* We postulate for the distribution j the following decomposition

$$j(x,z) = j_\alpha(x,z) + j_1(x,z) \tag{10}$$

where j_1 satisfies the assumptions of the above Case 1. The function j_α, with $\alpha \in (0,1)$, has the form

$$j(x,z) = \frac{k(x)\gamma(z)}{|z|^{N+2\alpha}}.$$

where the function $\gamma(z)$ is a smooth nonnegative function, supported outside B_1 and equal to 1 outside B_2. Ideally, we should assume j_α to have the form of Case 1, with an imposed decay at infinity. However, without any further assumption there need not be a limit $\varepsilon \to 0$ for the model under study.

We have, as the preceding case:

$$\varepsilon^{-N}\int_{\mathbb{R}^N} (j_1(x,\frac{y-x}{\varepsilon})u(t,y) - j_1(y,\frac{x-y}{\varepsilon})u(t,x))\,dy = \varepsilon \mathrm{div}_x(u(t,.)V) + O(\varepsilon^2).$$

On the other hand there holds

$$\begin{aligned} &\varepsilon^{-N}\int_{\mathbb{R}^N}\Big(j_\alpha(x,\frac{y-x}{\varepsilon})u(t,y) - j_\alpha(y,\frac{x-y}{\varepsilon})u(t,x)\Big)\,dy \\ = \;& \varepsilon^{2\alpha} P.V.\Big(\int_{\mathbb{R}^N} \frac{k(x)u(t,y)-k(y)u(t,x)}{|x-y|^{N+2\alpha}}\,dy\Big) + o(\varepsilon^{2\alpha}) \\ := \;& \varepsilon^{2\alpha}\mathcal{S}_\alpha u(t,x) + o(\varepsilon^{2\alpha}) \end{aligned}$$

Discarding the $o(\varepsilon^{2\alpha})$, choosing the time scale $\tau = \varepsilon^{2\alpha}t$ and the reaction term $f(x,u) = \varepsilon^{2\alpha}g(x,u)$ we obtain

$$u_\tau + \mathcal{S}_\alpha u + \frac{1}{\varepsilon^{2\alpha-1}}\mathrm{div}\,(uV) = g(x,u). \tag{11}$$

Once again we have a large drift reaction-diffusion model, at least if $\alpha > \frac{1}{2}$. If $\alpha = \frac{1}{2}$ the drift and the diffusion balance exactly, both by the order of magnitude and the order of the derivatives involved. If $\alpha < \frac{1}{2}$, the drift should be discarded for the sake of consistency.

If the function $k(x)$ is constant, and if the dispersal term j_1 is even in its second variable z, we retrieve the reaction-diffusion model (4). From now on, we will concentrate on it, leaving the more general model (11) for a future study.

Remark. The preceding derivation makes precise the role of the tail of the probability density. As soon as it has finite second moments, the limiting equation becomes universal. In the opposite situation, a case by case study is needed. In any event, the characteristic time scale (ε^{-2} in the first case, $\varepsilon^{-2\alpha}$ in the second case) is much shorter in the fractional diffusion case than in the purely local diffusion case. This is made mathematically rigorous in [6].

3. Rayleigh quotients and first eigenvalue

Let us recall that [5], for large enough $\mu > 0$, the operator $\mathcal{T}_\alpha + \mu I$ is invertible from its domain in $C(\mathbb{T}^N)$ to $C(\mathbb{T}^N)$ and, for such a value of μ:

- if $f \in C(\mathbb{T}^N)$ is nonnegative, not everywhere zero, and if $u \in C^{1,\alpha}(\mathbb{T}^N)$ satisfies $\mathcal{T}_\alpha u + \mu u = f$, then $u > 0$ on $\mathbb{T}^N$.
- If Ω is a smooth bounded domain of $\mathbb{R}^N$, $f \in C(\overline{\Omega})$, $f \geq 0$, $f \not\equiv 0$ and if $u \in C^{1,\alpha}(\Omega) \cap C(\overline{\Omega})$ satisfies $\mathcal{T}_\alpha u + \mu u = f$ with $u \equiv 0$ outside Ω, then $u > 0$ in Ω and, for all $x_0 \in \partial\Omega$ and $\delta > 0$:
$$\liminf_{x \to x_0,\ (x-x_0).\nu(x_0) \leq -\delta} \frac{u(x)}{|x - x_0|^\alpha} > 0.$$
where $\nu(x_0)$ is the outer unit normal to Ω at x_0.

Hence, $(\mathcal{T}_\alpha + \mu I)^{-1}$ maps the cone of nonnegative, nonzero functions of $C(\mathbb{T}^N)$ (resp. of $C(\overline{\Omega})$ vanishing on $\partial\Omega$) into its interior. Therefore, the classical Krein-Rutman theorem applies and provides a principal eigenvalue[1] for $\mathcal{L}$, both in the case of periodic - we denote it by λ_{per} - and Dirichlet conditions - we denote it by $\lambda_1(\Omega)$. In this section, we prove some Rayleigh-type formulae.

Proposition 3.1. *$\lambda_1(\Omega)$ is the minimum of*

$$\frac{\displaystyle\frac{1}{2}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{(\phi(x)-\phi(y))^2}{|x-y|^{N+2\alpha}}dy\right)dx - \int_\Omega \mu(x)\phi^2(x)]dx}{\displaystyle\int_\Omega \phi^2(x)dx}, \tag{12}$$

taken over all functions $\phi \in C^1(\Omega) \cap C^0(\overline{\Omega})$, $\phi \not\equiv 0$, vanishing on $\partial\Omega$ and extended by 0 outside Ω, whereas λ_{per} is the minimum of

$$\frac{\displaystyle\frac{1}{2}\int_{\mathcal{C}_l}\left(\int_{\mathbb{R}^N}\frac{(\phi(x)-\phi(y))^2}{|x-y|^{N+2\alpha}}dy\right)dx - \int_{\mathcal{C}_l} \mu(x)\phi^2(x)dx}{\displaystyle\int_{\mathcal{C}_l} \phi^2(x)dx} \tag{13}$$

over all functions $\phi \in C^1(\mathbb{R}^N)$, $\phi \not\equiv 0$, periodic with period $(l_1, \cdots, l_N)$. In both cases the minimum is uniquely (up to a multiplicative constant) attained by the principal eigenfunction (φ_Ω and φ_p respectively).

[1] That is, λ_{per} and $\lambda_1(\Omega)$ are the unique eigenvalues with an associated positive eigenfunction, called principal eigenfunction and denoted in the sequel by φ_p and φ_Ω respectively, which in addition is unique up to a scalar multiple.

Proof. We begin by recalling that, for $\alpha \geq \frac{1}{2}$, the expression $\mathcal{T}_\alpha u$ does not make sense as an integral, but does make sense as a principal value provided that $u \in C^{1,\alpha}$. It will therefore be convenient to work with the truncated kernels

$$k_\varepsilon(x) = \max(\varepsilon, |x|^{N+2\alpha}).$$

The remaining is classical. We notice that

$$\mathcal{T}_\alpha u(x) = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{k_\varepsilon(x-y)}\, dy := \lim_{\varepsilon\to 0} \mathcal{T}_\alpha^\varepsilon u(x)$$

If u is smooth and zero outside Ω, we set

$$\begin{aligned} \mathcal{E}_\Omega(u) = &\ \frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2\alpha}}\, dxdy - \int_\Omega \mu(x)u^2\, dx, \\ \mathcal{E}_{\Omega,\varepsilon}(u) = &\ \frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))^2}{k_\varepsilon(x-y)}\, dxdy - \int_\Omega \mu(x)u^2\, dx \end{aligned} \tag{14}$$

and, if u is $\mathcal{C}_l$-periodic we set

$$\begin{aligned} \mathcal{E}_{per}(u) = &\ \frac{1}{2}\int_{\mathcal{C}_l\times\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2\alpha}}\, dxdy - \int_{\mathcal{C}_l} \mu(x)u^2\, dx, \\ \mathcal{E}_{per,\varepsilon}(u) = &\ \frac{1}{2}\int_{\mathcal{C}_l\times\mathbb{R}^N} \frac{(u(x)-u(y))^2}{k_\varepsilon(x-y)}\, dxdy - \int_{\mathcal{C}_l} \mu(x)u^2\, dx \end{aligned} \tag{15}$$

Now, we have

$$\begin{aligned} &\mathcal{E}_{per,\varepsilon}(u+v) \\ = \ &\mathcal{E}_{per,\varepsilon}(u) + \mathcal{E}_{per,\varepsilon}(v) - 2\int_{\mathcal{C}_l} \mu(x)uv\, dx \\ &+ \int_{\mathcal{C}_l}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{k_\varepsilon(x-y)}\, dxdy \\ = \ &\mathcal{E}_{per,\varepsilon}(u) + \mathcal{E}_{per,\varepsilon}(v) + 2\int_{\mathcal{C}_l} (\mathcal{L}u)v\, dx \end{aligned},$$

whereas, for u, v smooth and vanishing outside Ω,

$$\mathcal{E}_{\Omega,\varepsilon}(u+v) = \mathcal{E}_{\Omega,\varepsilon}(u) + \mathcal{E}_{\Omega,\varepsilon}(v) + 2\int_\Omega (\mathcal{L}u)v\, dx.$$

Passing to the limit $\varepsilon \to 0$ yields that the differentials of $\mathcal{E}_{per}$ and $\mathcal{E}_\Omega$ at u are respectively

$$v \mapsto 2\int_{\mathcal{C}_l} (\mathcal{L}u)v\, dx \quad \text{and} \quad v \mapsto 2\int_\Omega (\mathcal{L}u)v\, dx.$$

Note that minimising the formulae (12) and (13) is equivalent to minimise the energy $\mathcal{E}$ under the constraint $\int_\Omega \phi^2(x)\, dx = 1$ and $\int_{\mathcal{C}_l} \phi^2(x)\, dx = 1$ respectively.

Also, recall that $H_0^\alpha(\Omega)$ - resp. $H^\alpha(\mathbb{T}^N)$ - is relatively compact in $L^2(\Omega)$ - resp. $L^2(\mathbb{T}^N)$. Hence the minimum of the energy in $H_0^\alpha(\Omega)$ - resp. $H^\alpha(\mathbb{T}^N)$ - is attained on the unit sphere of $L^2(\Omega)$. Hence, applying the method of Lagrange multipliers, we see that the mimimum is achieved at an eigenfunction φ. Then, taking $\phi = \varphi$ in the formulae, we see that it is equal to the associated eigenvalue. To see that it is the principal eigenvalue it is enough to prove that $\varphi > 0$. We could directly invoke

Krein-Rutman, but it is worth seeing how it works here: we have

$$\begin{aligned}\mathcal{E}_{per,\varepsilon}(u) = & \quad \mathcal{E}_{per,\varepsilon}(u^+) + \mathcal{E}_{per,\varepsilon}(u^-) + \int_{\mathcal{C}_l \times_R^N} \frac{u^+(x)u^-(y)}{k_\varepsilon(x-y)}\,dxdy \\ \mathcal{E}_{per,\varepsilon}(|u|) = & \quad \mathcal{E}_{per,\varepsilon}(u^+) + \mathcal{E}_{per,\varepsilon}(u^-) - \int_{\mathcal{C}_l \times_R^N} \frac{u^+(x)u^-(y)}{k_\varepsilon(x-y)}\,dxdy\end{aligned}$$

hence $\mathcal{E}_{per}(|u|) < \mathcal{E}_{per}(u)$ if both u^+ and u^- are nonzero. Thus any principal eigenfunction has constant sign, and the same argument works in the Dirichlet case. □

4. The approximation theorem

Consider a family $(\chi_r)_{r>1}$ of cutoff functions in $C^2(\mathbb{R}^N)$, uniformly bounded in $W^{2,\infty}(\mathbb{R}^N)$, such that $0 \le \chi_r \le 1$, $\mathrm{supp}\chi_r \subset \overline{B}_r$ and $\chi_r = 1$ in B_{r-1}. For any $r > 1$ we set $\phi_r := \varphi_p \chi_r$.

Lemma 4.1. *There holds:*

$$\lim_{r\to\infty} \frac{\displaystyle\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{(\phi_r(x)-\phi_r(y))^2}{|x-y|^{N+2\alpha}} dy \right) dx}{\displaystyle\int_{B_r} \phi_r^2(x)dx} = \frac{\displaystyle\int_{\mathcal{C}_l} \left(\int_{\mathbb{R}^N} \frac{(\varphi_p(x)-\varphi_p(y))^2}{|x-y|^{N+2\alpha}} dy \right) dx}{\displaystyle\int_{\mathcal{C}_l} \varphi_p^2(x)dx}.$$

Proof. Step 1. We claim that $\lim\limits_{r\to\infty} \mathcal{E}_r(\phi) = 0$, with

$$\mathcal{E}_r(\phi) := \frac{\displaystyle\int_{\mathbb{R}^N\setminus B_{r+\sqrt{r}}} \left(\int_{\mathbb{R}^N} \frac{(\phi_r(x)-\phi_r(y))^2}{|x-y|^{N+2\alpha}} dy \right) dx}{\displaystyle\int_{B_r} \phi_r^2(x)dx}.$$

Indeed, since $\displaystyle\int_{B_r} \phi_r^2(x)dx \ge |B_{r-1}| \min \varphi_p^2$, there exists a constant $C > 0$ such that

$$\begin{aligned}\mathcal{E}_r(\phi) \le & \quad C r^{-N} \int_{|x|\ge r+\sqrt{r}} \left(\int_{\mathbb{R}^N} \frac{(\phi_r(x)-\phi_r(y))^2}{|x-y|^{N+2\alpha}} dy \right) dx \\ = & \quad C r^{-N} \int_{|x|\ge r+\sqrt{r}} \left(\int_{B_r} \frac{\phi_r^2(y)}{|x-y|^{N+2\alpha}} dy \right) dx \\ \le & \quad C' r^{-N} \int_{B_r} \left(\int_{|x+y|\ge r+\sqrt{r}} |z|^{-N-2\alpha} dz \right) dy,\end{aligned}$$

where C' is another positive constant. Note that if $y \in B_r$ and $z \in \mathbb{R}^N \setminus B_{r+\sqrt{r}}(-y)$ then $|z| \ge \sqrt{r}$. Hence,

$$\frac{\displaystyle\int_{\mathbb{R}^N\setminus B_{r+\sqrt{r}}} \left(\int_{\mathbb{R}^N} \frac{(\phi_r(x)-\phi_r(y))^2}{|x-y|^{N+2\alpha}} dy \right) dx}{\displaystyle\int_{B_r} \phi_r^2(x)dx} \le C' r^{-N} \int_{B_r} \left(\int_{\mathbb{R}^N\setminus B_{\sqrt{r}}} |z|^{-N-2\alpha} dz \right) dy,$$

and the RHS is less than $\le C'' r^{-\alpha}$ for some constant $C'' > 0$.

Step 2. Let us prove that

$$\lim_{r\to\infty}\frac{\displaystyle\int_{B_{r+\sqrt{r}}}\left(\int_{\mathbb{R}^N}\frac{(\phi_r(x)-\phi_r(y))^2-(\varphi_p(x)-\varphi_p(y))^2}{|x-y|^{N+2\alpha}}dy\right)dx}{\displaystyle\int_{B_r}\phi_r^2(x)dx}=0.$$

Let us set for short $\psi_r(x,y):=|(\phi_r(x)-\phi_r(y))^2-(\varphi_p(x)-\varphi_p(y))^2|$. We have that

$$\begin{aligned}\frac{\displaystyle\int_{B_{r+\sqrt{r}}}\left(\int_{\mathbb{R}^N}\frac{\psi_r(x,y)}{|x-y|^{N+2\alpha}}dy\right)dx}{\displaystyle\int_{B_r}\phi_r^2(x)dx} &\leq\; Cr^{-N}\int_{B_{r+\sqrt{r}}}\left(\int_{\mathbb{R}^N}\frac{\psi_r(x,y)}{|x-y|^{N+2\alpha}}dy\right)dx\\ &=\; Cr^{-N}I(r),\end{aligned}$$

where $C>0$ is independent of r. Our aim is to show that

$$\lim_{r\to\infty}r^{-N}I(r)=0, \tag{16}$$

for this we set $I(r)=I_1(r)+I_2(r)$ where $I_1,\ I_2$ are given by

$$I_1(r):=\int_{B_{r+\sqrt{r}}\setminus B_{r-1}}\left(\int_{\mathbb{R}^N}\frac{\psi_r(x,y)}{|x-y|^{N+2\alpha}}dy\right)dx,$$

$$I_2(r):=\int_{B_{r-1}}\left(\int_{\mathbb{R}^N}\frac{\psi_r(x,y)}{|x-y|^{N+2\alpha}}dy\right)dx.$$

Using the equi Lipschitz-continuity of the ϕ_r and φ_p we get

$$\begin{aligned}\int_{\mathbb{R}^N}\frac{\psi_r(x,y)}{|x-y|^{N+2\alpha}}dy= &\quad\int_{\mathbb{R}^N\setminus B_1(x)}\frac{\psi_r(x,y)}{|x-y|^{N+2\alpha}}dy+\int_{B_1(x)}\frac{\psi_r(x,y)}{|x-y|^{N+2\alpha}}dy\\ \leq &\quad k+k'\int_{B_1}|z|^{2-N-2\alpha}dz\leq k'',\end{aligned}$$

for some positive constants $k,\ k',\ k''$. Thus, $I_1(r)=O(r^{N-\frac{1}{2}})$. With a view to estimate $I_2(r)$, we note that $\psi_r=0$ in $B_{r-1}\times B_{r-1}$ and then

$$I_2(r)=\int_{B_{r-1}}\left(\int_{\mathbb{R}^N\setminus B_{r-1}}\frac{\psi_r(x,y)}{|x-y|^{N+2\alpha}}dy\right)dx.$$

Now, consider $\gamma\in(0,1)$ and decompose I_2 into the sum of

$$J_1(r):=\int_{B_{r-1}\setminus B_{(1-\gamma)(r-1)}}\left(\int_{\mathbb{R}^N\setminus B_{r-1}}\frac{\psi_r(x,y)}{|x-y|^{N+2\alpha}}dy\right)dx$$

and

$$J_2(r):=\int_{B_{(1-\gamma)(r-1)}}\left(\int_{\mathbb{R}^N\setminus B_{r-1}}\frac{\psi_r(x,y)}{|x-y|^{N+2\alpha}}dy\right)dx.$$

Since, as we have seen before,

$$\int_{\mathbb{R}^N}\frac{\psi_r(x,y)}{|x-y|^{N+2\alpha}}dy\leq k'',$$

there exists $h>0$ such that $J_1(r)\leq h[1-(1-\gamma)^N](r-1)^N$. On the other hand,

$$J_2(r)\leq h'\int_{B_{(1-\gamma)(r-1)}}\left(\int_{\mathbb{R}^N\setminus B_{r-1}(-x)}|z|^{-N-2\alpha}dz\right)dx,$$

for some constant $h' > 0$. If $x \in B_{(1-\gamma)(r-1)}$ and $z \in \mathbb{R}^N \backslash B_{r-1}(-x)$ then $|z| \geq \gamma(r-1)$. As a consequence,

$$\begin{aligned} J_2(r) &\leq h' \int_{B_{(1-\gamma)(r-1)}} \left(\int_{\mathbb{R}^N \backslash B_{\gamma(r-1)}} |z|^{-N-2\alpha} dz \right) dx \\ &\leq h''(1-\gamma)^N (r-1)^N [\gamma(r-1)]^{-2\alpha} \\ &\leq h'' \gamma^{-2\alpha} (r-1)^{N-2\alpha}, \end{aligned}$$

where $h'' > 0$ is independent of r and γ. For any $\varepsilon > 0$ there exists $\gamma_\varepsilon \in (0,1)$ such that $h[1-(1-\gamma_\varepsilon)^N] \leq \varepsilon$. Thus,

$$\begin{aligned} r^{-N} I_2(r) &\leq r^{-N} \left[h[1-(1-\gamma_\varepsilon)^N](r-1)^N + h'' \gamma_\varepsilon^{-2\alpha} (r-1)^{N-2\alpha} \right] \\ &\leq \varepsilon + h'' \gamma_\varepsilon^{-2\alpha} (r-1)^{-2\alpha}, \end{aligned}$$

Then, owing to the arbitrariness of ε,

$$\lim_{r \to \infty} r^{-N} I_2(r) = 0.$$

Therefore, (16) holds.
Step 3. Let us prove finally that

$$\lim_{r \to \infty} \frac{\displaystyle\int_{B_{r+\sqrt{r}}} \left(\int_{\mathbb{R}^N} \frac{(\varphi_p(x) - \varphi_p(y))^2}{|x-y|^{N+2\alpha}} dy \right) dx}{\displaystyle\int_{B_r} \phi_r^2(x) dx} = \frac{\displaystyle\int_{\mathcal{C}_l} \left(\int_{\mathbb{R}^N} \frac{(\varphi_p(x) - \varphi_p(y))^2}{|x-y|^{N+2\alpha}} dy \right) dx}{\displaystyle\int_{\mathcal{C}_l} \varphi_p^2(x) dx}.$$

Let n_r be the number of cells of the type $\{z\} + \mathcal{C}_l$, $z \in \Pi_{i=1}^N \mathbb{Z} l_i$, contained in B_{r-1}, that is,

$$n_r := |\{z \in \Pi_{i=1}^N \mathbb{Z} l_i \ : \ \{z\} + \mathcal{C}_l \subset B_{r-1}\}|.$$

It is easy to check that $n_r = O(r^N)$ and, since φ_p and $x \mapsto \displaystyle\int_{\mathbb{R}^N} \frac{(\varphi_p(x) - \varphi_p(y))^2}{|x-y|^{N+2\alpha}} dy$ are $(l_1, \cdots, l_N)$ periodic, that

$$\int_{B_r} \phi_r^2(x) dx = n(r) \int_{\mathcal{C}_l} \varphi_p^2(x) dx + O(r^{N-1}), \tag{17}$$

$$\begin{aligned} \int_{B_{r+\sqrt{r}}} \left(\int_{\mathbb{R}^N} \frac{(\varphi_p(x) - \varphi_p(y))^2}{|x-y|^{N+2\alpha}} dy \right) dx = n(r) \int_{\mathcal{C}_l} \left(\int_{\mathbb{R}^N} \frac{(\varphi_p(x) - \varphi_p(y))^2}{|x-y|^{N+2\alpha}} dy \right) dx \\ + O(r^{N-\frac{1}{2}}). \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{\displaystyle\int_{B_{r+\sqrt{r}}} \left(\int_{\mathbb{R}^N} \frac{(\varphi_p(x) - \varphi_p(y))^2}{|x-y|^{N+2\alpha}} dy \right) dx}{\displaystyle\int_{B_r} \phi_r^2(x) dx} \\ &= \frac{\displaystyle\int_{\mathcal{C}_l} \left(\int_{\mathbb{R}^N} \frac{(\varphi_p(x) - \varphi_p(y))^2}{|x-y|^{N+2\alpha}} dy \right) dx + O(r^{-\frac{1}{2}})}{\displaystyle\int_{\mathcal{C}_l} \varphi_p^2(x) dx + O(r^{-1})} \end{aligned}$$

Now, the proof is complete by gathering together steps 1-3. $\square$

Proof of Theorem 1.1. We know that the function $r \mapsto \lambda_1(r)$ is decreasing and that $\lambda_1(r) \geq \lambda_{per}$ for any $r > 0$. Thus, we only need to show that $\lim_{r\to\infty} \lambda_1(r) \leq \lambda_{per}$. The variational formula (12) yields

$$\lambda_1(r) \leq \frac{\displaystyle\frac{1}{2}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{(\phi_r(x)-\phi_r(y))^2}{|x-y|^{N+2\alpha}}dy\right)dx - \int_{B_r}[\mu(x)\phi_r^2(x)]dx}{\displaystyle\int_{B_r}\phi_r^2(x)dx}. \tag{18}$$

Hence, by Lemma 4.1 we obtain

$$\lim_{r\to\infty}\lambda_1(r) \leq \frac{\displaystyle\frac{1}{2}\int_{\mathcal{C}_l}\left(\int_{\mathbb{R}^N}\frac{(\varphi_p(x)-\varphi_p(y))^2}{|x-y|^{N+2\alpha}}dy\right)dx - \int_{\mathcal{C}_l}\mu(x)\varphi_p^2(x)dx}{\displaystyle\int_{\mathcal{C}_l}\varphi_p^2(x)dx} = \lambda_{per},$$

which completes the proof of the theorem. □

5. Application to Theorem 1.2

Recall the following from [1]:

Theorem 5.1. *Set $\mu(x) = f_u(x, 0)$ and $\mathcal{L}$ given by (3). Equation (4), posed either on $\mathbb{T}^N$ or in B_R with Dirichlet conditions, has a unique positive steady solution u_+ if and only if $\lambda_{per} < 0$ (resp. $\lambda_1(R) < 0$). In such a case, any solution to (4) starting from a nonnegative, nontrivial initial datum tends, as $t \to +\infty$, to u_+. The convergence is uniform in $\mathbb{T}^N$ (resp. B_R).*

In [1], it is stated there for a second order elliptic operator but it works just as well with $\mathcal{T}_\alpha$, we only need a strong maximum principle. And, just as in [3], the delicate point is that there is, for a potential positive steady solution to (4), no obvious reason to be periodic. The main argument to bypass the difficulty is the

Lemma 5.2. *Assume that $\lambda_{per} < 0$, let v be a positive steady solution to (4). Then $\inf_{\mathbb{R}^N} v > 0$.*

Proof. From Theorem 1.1 we may fix $R_0 > 0$ large enough so that $\overline{\mathcal{C}}_l \subset B_{R_0}$ and

$$\lambda_1(R_0) \leq \frac{\lambda_{per}}{2} < 0. \tag{19}$$

Let φ_{R_0} be a principal eigenfunction for $\lambda_1(R_0)$ and, for $\varepsilon > 0$, let $u_{R_0,\varepsilon}$ be the solution to

$$u_t + \mathcal{T}_\alpha u = f(x, u) \text{ in } B_{R_0}, \qquad u \equiv 0 \text{ outside } B_{R_0} \tag{20}$$

starting from $\varepsilon\varphi_{R_0}$. Now that we are operating in a bounded domain, we may apply Theorem 5.1 and infer that, because of (19), there is a unique positive steady solution v_{R_0} to (20), which attracts $u_{R_0,\varepsilon}$ as $t \to +\infty$, for any $\varepsilon > 0$. Choosing $\varepsilon > 0$ small enough in such a way that $\varepsilon\varphi_{R_0} \leq v$, we derive, by mean of the parabolic comparison principle,

$$\forall\, x \in B_{R_0}, \quad v(x) \geq \lim_{t\to+\infty} u_{R_0,\varepsilon}(t, x) = v_{R_0}(x).$$

Note that, because of the periodicity, for any $z \in \Pi_{i=1}^N \mathbb{Z}l_i$, the principal eigenvalue of $\mathcal{L}$ in $B_{R_0}(z)$ coincides with $\lambda_1(R_0)$. Moreover, the unique positive steady solution to the problem (20) set in $B_{R_0}(z)$ instead of B_{R_0} is given by $v_{R_0}(\cdot - z)$. As a consequence, repeating the same arguments as before yields $v(x) \geq v_{R_0}(x-z)$ for $x \in B_{R_0}(z)$. We eventually infer that

$$\inf_{\mathbb{R}^N} v = \inf_{z \in \Pi_{i=1}^N \mathbb{Z} l_i} \left(\min_{\{z\}+\overline{\mathcal{C}}_l} v \right) \geq \min_{\overline{\mathcal{C}}_l} v_{R_0} > 0,$$

the desired lower bound. □

Proof of Theorem 1.2. It mimics the proof of Theorem 2.1 of [3].

1. If $\lambda_{per} \geq 0$ we use Theorem 5.1 to deduce that any positive, initially periodic solution of (4) goes to 0 as $t \to +\infty$. If $v(x)$ is a potential steady solution to (4), we put it under a large constant, and apply the above statement to infer that $v \equiv 0$.

2. If $\lambda_{per} < 0$, apply Theorem 5.1 to infer the existence of a steady periodic solution $u(x)$ to (4). Let $v(x)$ be another solution, let $k_0 > 0$ be the smallest k such that $v \leq ku$. If there is a contact point between v and ku, we have $v \equiv ku$ by the strong maximum principle. If not, there is a sequence $(p_n)_n$ of integers, whose size goes to $+\infty$, such that $\lim_{n \to +\infty} \sup_{p_n l + \mathcal{C}_l} (ku - v) = 0$. By the elliptic estimates, a subsequence of $(v(.+p_n l))_n$, denoted by $(v_n)_n$, converges to a steady solution v_∞ of (4) and we have $ku \equiv v_\infty$.

In both cases, if $k \neq 1$, we have constructed a second positive periodic steady solution to (4), namely ku: impossible in view of Theorem 5.1, thus $v \leq u$. The reverse inequality is obtained by exchanging the roles of u and v. Notice that the existence of a constant k such that $u \leq kv$ - which allows one to repeat the previous arguments - follows from Lemma 5.2.

The rest of the theorem is proved by putting a small multiple of the first eigenfunction of $\mathcal{L}$ in a large ball under $u(1,.)$. This implies the local uniform convergence to the positive steady state. If $u(0,.)$ is bounded away from 0, we may put below a small multiple of the periodic eigenfunction, which then yields the uniform convergence. □

Acknowledgments

The first and second authors were supported by the ANR grant 'PREFERED'. The three authors are grateful to the University of Chicago, where they had the occaasion to work together on this paper in a stimulating atmosphere.

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