SIAM J. CONTROL OPTIM.
Vol. 50, No. 2, pp. 991–1011

THE BOUNDED SLOPE CONDITION FOR FUNCTIONALS DEPENDING ON x, u, AND ∇u*

A. FIASCHI† AND G. TREU‡

Abstract. A global regularity result is proved for a class of minimizers of functionals of the form $\mathcal{I}(u) = \int_\Omega f(|\nabla u(x)|) + g(x, u(x))\, dx, u \in \phi + W_0^{1,1}(\Omega)$, where ϕ satisfies the Bounded Slope Condition.

Key words. Bounded Slope Condition, Lipschitz continuity, regularity of minimizers

AMS subject classifications. 49N60, 49K30

DOI. 10.1137/110822980

1. Introduction.

In this paper we address the problem of the regularity of minimizers of scalar integral functionals of the form

$$\int_\Omega L(x, u(x), \nabla u(x))\, dx, \qquad u \in \phi + W_0^{1,1}(\Omega, \mathbb{R}),$$

where Ω is an open bounded subset of $\mathbb{R}^n$ and ϕ is a given boundary datum. Stemming from the fundamental De Giorgi–Moser–Nash theorem a huge literature furnishes many results on the interior regularity. We cannot exhaustively review them here and we refer to [23] for a wide and quite up-to-date reference on the subject. The only fact that we want to underline is that in all these results, the function $L(x, u, \cdot)$ is assumed to have controlled growth, both from below and from above, and to be uniformly convex too.

The situation is quite different when we deal with the global regularity of the minimizers. On one hand we cannot expect to obtain global regularity: in [9] an example can be found of a harmonic function on the unit ball B, coinciding with a Lipschitz function on ∂B, that is not Lipschitz on $\overline{B}$. This shows that even in the case of the Dirichlet functional, depending just on ∇u, having good growth at infinity, and satisfying the uniform convexity assumption, we cannot expect to improve the locally Lipschitz continuity of the minimizer. On the other hand, in some special cases, a different approach that one can say is inspired by works by Hilbert and Haar [17, 14] gives a different perspective of the problem. First we want to recall a very well known result originally due to Stampacchia (we refer to [13] for a presentation of the proof). Consider the functional

$$\int_\Omega L(\nabla u(x))\, dx, \tag{1.1}$$

*Received by the editors January 31, 2011; accepted for publication (in revised form) December 12, 2011; published electronically April 26, 2012.
http://www.siam.org/journals/sicon/50-2/82298.html

†IMATI-CNR, via Ferrata 1 I-27100, Pavia, Italy (alice.fiaschi@imati.cnr.it). This author's work was partially supported by FP7-IDEAS-ERC-StG grant 200497, BioSMA: Mathematics for Shape Memory Technologies in Biomechanics, and by PRIN 2008, "Optimal transport theory, geometric and functional inequalities and applications."

‡Dipartimento di Matematica pura e applicata, Università di Padova, via Trieste 63 I-35121, Padova, Italy (treu@math.unipd.it). This author's work was supported by PRIN 2008, "Elliptic equations and systems: a priori estimates, existence, regularity."

where L is a strictly convex function. If ϕ satisfies the so-called Bounded Slope Condition (BSC) of rank K (see section 4 below), then there exists a Lipschitz function with the same rank of ϕ that is a minimizer in the class of Lipschitz functions coinciding with ϕ on $\partial\Omega$. Under the same assumptions, Cellina has recently proved in [6] that if (1.1) admits a minimum in $\phi + W_0^{1,1}(\Omega)$, then this is Lipschitz of rank K.

We want to sketch in a few words the proof of this result. We recall that the boundary datum ϕ satisfies the BSC if for every point γ on the boundary of Ω there exist two affine functions coinciding in γ and such that $\phi(\gamma')$ is between them for every $\gamma' \in \partial\Omega$. One of the main tools for the proof is the Comparison Principle between minimizers satisfying different boundary conditions, i.e., assume that v and w are minimizers of (1.1), respectively, in $\phi + W_0^{1,1}(\Omega)$ and in $\psi + W_0^{1,1}(\Omega)$, with $\phi(x) \leq \psi(x)$ for every x in $\partial\Omega$, then $v \leq w$ a.e. in Ω. Since any affine function is a minimizer of (1.1) among the functions with the same affine boundary condition, whenever ϕ satisfies the BSC of rank K, the Comparison Principle allows us to box up the minimizer on $\phi + W_0^{1,1}(\Omega)$ between two Lipschitz functions, having again rank K and coinciding on $\partial\Omega$. A Haar–Radò type theorem for Sobolev functions (see [22] for a precise statement) says that the slope of the minimizer is maximum at the boundary, closing the argument.

We underline the fact that in this approach no role is played by either growth properties or uniform convexity of L.

At this point the question of whether we can apply the same method to more general functionals naturally arises. An example (see [11]) of a functional with $p - q$ growth and $(x, \nabla u)$-dependence, whose minimizer has an isolated singularity, leads us not to expect, in the general case, regularity results as strong as those cited above. Moreover, there are other obstructions to the use of the Hilbert–Haar approach. First, the Comparison Principle may fail if we drop the strict convexity assumption on L; see [7] for an example. It has been proved in [7, 4, 8] that it holds, even in this case, for restricted classes of minimizers. This led to proofs of various results on this subject. (See [2, 5, 20, 21, 19] for results concerning the functional (1.1) and [8] for a special case with u-dependence too.) A second remark is that in the case of a function L explicitly depending also on x or u, the affine functions are no longer minimizers of the functional, so that the BSC does not immediately give a “barrier” for the minimizers. A theorem by Miranda [24] ensures that the class of functions satisfying the BSC is quite large: if Ω is uniformly convex and $\partial\Omega$ is $\mathcal{C}^{1,1}$, then any $\mathcal{C}^{1,1}$ function satisfies it.

In this paper we consider the functional

$$\mathcal{I}(u) = \int_\Omega h(\nabla u(x)) + g(x, u(x))\, dx, \qquad u \in \phi + W_0^{1,1}(\Omega), \tag{1.2}$$

where h is a convex function and g is Lipschitz of constant α w.r.t. the second variable. On one hand, in this case, one can expect better regularity results than those of the general case. For example, in the vectorial case, this structure implies better estimates on the Hausdorff dimension of the singular set (see [18] and [12]). On the other hand, for scalar functionals of type (1.2), a Hilbert–Haar approach has already been succesfully used. Stampacchia [25] proved the existence of a minimum in the class of Lipschitz functions under the assumptions that the BSC holds. More recently Bousquet and Clarke under a one-side version of the BSC, but always assuming the uniform convexity of h, proved in [1] the local Lipschitz continuity and some continuity properties up to the boundary for bounded minimizers. In [2, 3] these results are generalized to the case of more general boundary conditions.

In the present paper, we are interested in weakening the uniform convexity assumption. In particular, the hypotheses we make on h allow us to consider functionals obtained via relaxation of nonconvex ones. The first result we prove is a Comparison Principle between any minimizer of the functional $\mathcal{I}$ and the functions

$$\omega_{\pm\alpha}(x) := \frac{n}{\pm\alpha} h^*\Big(\pm\alpha \frac{x - x_0}{n}\Big) + c \tag{1.3}$$

introduced by Cellina in [8]. Then we show that under additional assumptions on h, the validity of the BSC implies that the functions (1.3) provide good barriers for the minimizers of $\mathcal{I}$. Suitable hypotheses on g ((G2) in section 4 below) are needed to apply the Haar–Radò theorem [22], which allows us to conclude that there exist minimizers inheriting the global Lipschitz regularity of the barriers.

We want to explain the difficulties of the proof. To show that the functions (1.3) are suitable to construct barriers we have to proceed in the following way. For any fixed point γ on the boundary of Ω we consider the affine function involved in the BSC from below at the point γ. Assume that it is $a \cdot x + b$. We have to find $x_0 \in \mathbb{R}^n$ and $c \in \mathbb{R}$ such that the set $\Omega_{x_0,c} = \{x \in \mathbb{R}^n : \omega_\alpha(x) - a \cdot x - b < 0\}$ contains Ω and $\partial\Omega_{x_0,c} \cap \partial\Omega$ contains γ. This is essentially a geometric requirement on the two sets $\Omega_{x_0,c}$ and Ω: the normal cone to $\Omega_{x_0,c}$ at γ has to be contained in the normal cone to Ω at the same point and also suitable conditions on the principal curvatures of both sets are needed. For these reasons of an essential technical nature, we restrict our attention to the case of a uniformly convex set Ω and of a radially symmetric function $h(\xi) = f(|\xi|)$. To compute the curvatures of $\Omega_{x_0,c}$ and to guarantee that the estimates on them hold in a suitable neighborhood of γ, further assumptions on f, (F4), or (F5) of section 4 are needed. These hypotheses do not imply the uniform convexity of h, which is instead needed in previous regularity results obtained by barrier techniques for functionals of the type (1.2).

In the last section of the paper we state some results that hold in some special cases that, in our opinion, are interesting on their own. Moreover, we present some examples and remarks with the aim of clarifying the role of the assumptions on the function f. In particular, we underline that constant boundary data trivially satisfy the BSC and that in this case we construct in a easier way the barriers. We can drop assumptions (F4) and (F5) and we obtain the global Lipschitzianity of minimizers (see Theorem 5.1 in section 4) whenever Ω is uniformly convex.

We also observe that a uniform convexity assumption implies condition (F4) and that in this case a simpler construction of the barriers can be provided (see Theorem 5.3). Moreover, in the smooth case (f of class C^2), we can drop hypotheses (F4) and (F5) and the only extra assumption we need to construct the barriers is a growth condition at infinity (see assumption (F6) in Theorem 5.4).

To conclude, we also provide examples of functions satisfying assumptions (F1)–(F3) and (F4) or (F5), obtained via convexification of nonconvex functions.

2. Preliminary results and a Comparison Principle. We consider an open bounded domain $\Omega \subset \mathbb{R}^n$ and an integral functional on $W^{1,1}(\Omega)$ of the form

$$\mathcal{I}(u) := \int_\Omega L(x, u(x), \nabla u(x))\, dx$$

for a function $L: \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ with $L(\cdot, u, \xi)$ measurable for every (u, ξ) and $L(x, \cdot, \cdot)$ continuous for a.e. $x \in \Omega$.

Definition 2.1. *We say that a function* $u \in W^{1,1}(\Omega)$ *is a minimizer of the functional* $\mathcal{I}$ *if* $\mathcal{I}(u) \le \mathcal{I}(v)$ *for every* $v \in u + W_0^{1,1}(\Omega)$.

We recall that given a function $h: \mathbb{R}^n \to \mathbb{R}$, its polar function $h^*: \mathbb{R}^n \to [-\infty, +\infty]$ is defined by $h^*(\xi) := \sup_{x\in\mathbb{R}^n}\{x \cdot \xi - h(x)\}$ for every $\xi \in \mathbb{R}^n$ (see [10]).

We are interested in the particular case where the Lagrangian is in the form $L(x, u, \xi) := h(\xi) + g(x, u)$ for a lower bounded function $h: \mathbb{R}^n \to \mathbb{R}$ and a function $g: \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying the following hypotheses:

(C1) h is convex,

(C2) h is superlinear, that is, $\lim_{|x|\to\infty} \frac{h(x)}{|x|} = +\infty$,

(C3) g is Lipschitz continuous in the second variable, with Lipschitz constant equal to α, i.e., $|g(x, u_1) - g(x, u_2)| \le \alpha|u_1 - u_2|$ for every $x \in \Omega$ and $u_1, u_2 \in \mathbb{R}$.

Remark 2.2. We observe that the hypotheses on h guarantee that the effective domain of its polar function h^* is $\mathbb{R}^n$. Indeed, let us assume by contradiction that $\operatorname{dom} h^* \ne \mathbb{R}^n$. This implies the existence of $\xi \in \mathbb{R}^n$ with

$$\sup_{x\in\mathbb{R}^n}\{x \cdot \xi - h(x)\} = h^*(x) = +\infty.$$

Therefore, we can find a sequence $(x_k)_k \subset \mathbb{R}^n$ such that $h(x_k) + k < x_k \cdot \xi \le |x_k||\xi|$. Hence, $\lim_k |x_k| = +\infty$ and $\lim_k \frac{h(x_k)}{|x_k|} \le |\xi|$, which contradict the superlinearity hypothesis (C2).

We define the functional I on $W^{1,1}(\Omega)$ by

$$I(u) := \int_\Omega \Big[h(\nabla u(x)) + g(x, u(x))\Big]\, dx.$$

A standard application of the Direct Method of the Calculus of Variations ensures the existence of a minimizer of I in $\phi + W_0^{1,1}(\Omega)$ for any $\phi \in W^{1,1}(\Omega)$. It has been shown in [20] that if h is superlinear, then the pointwise minimum and the pointwise maximum of the minimizers of I are in $\phi + W_0^{1,1}(\Omega)$ and are still minimizers of the same functional. We recall here a special case of a Haar–Radò type theorem, which has been proved in its general form in [22, Theorem 5.2].

Theorem 2.3. *Let* h *be convex and superlinear and* g *be measurable and convex in the second variable. Assume moreover that there exists a positive constant* K *such that*

$$\forall x, y \in \mathbb{R}^n,\ \forall u, v \in \mathbb{R} \quad v \ge u + K|y - x| \Rightarrow g_v^+(y, v) \ge g_v^+(x, u),$$

where g_v^+ *denotes the right derivative of* g *with respect to the second variable. If there exist two Lipschitz continuous functions* $l^-, l^+ \in \phi + W_0^{1,1}(\Omega)$ *of rank* L *on* $\bar{\Omega}$ *such that*

$$l^-(x) \le u(x) \le l^+(x) \quad \text{a.e. in } \Omega,$$

where $u \in \phi + W_0^{1,1}(\Omega)$ *is the maximum or the minimum of the minimizers of* I*, then* $|u(x) - u(y)| \le L|x - y|$ *for every Lebesgue point* x *and* y.

We now define the integral functionals $I_{\pm\alpha}$ on $W^{1,1}(\Omega)$ by setting

$$I_{\pm\alpha}(u) := \int_\Omega \Big[h(\nabla u(x)) \pm \alpha u(x)\Big]\, dx,$$

where α is the positive constant appearing in (C3). A result by Cellina (see [8]) states that under our hypotheses on Ω and h, for every $x_0 \in \mathbb{R}^n$ and $c \in \mathbb{R}$ the functions $\omega_{\pm\alpha}(x) : \mathbb{R} \to \mathbb{R}$ defined by

$$\omega_{\pm\alpha}(x) := \frac{n}{\pm\alpha} h^*\Big(\pm\alpha\frac{x - x_0}{n}\Big) + c \tag{2.1}$$

are unique minimizers of $I_{\pm\alpha}$ in the sense that $I_{\pm\alpha}(\omega_{\pm\alpha}) < I_{\pm\alpha}(v)$ for every $v \in \omega_{\pm\alpha} + W_0^{1,1}(\Omega)$. We remark that the hypotheses on h guarantee that $\omega_{\pm\alpha} \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$.

Given $u, v \in W^{1,1}(\Omega)$, we write $u \le v$ on $\partial\Omega$ if $(u-v)^+ \in W_0^{1,1}(\Omega)$. Now we state a comparison result between minimizers of I and minimizers of I_α, $I_{-\alpha}$.

THEOREM 2.4. *Let u be a minimizer of I and ω_α, $\omega_{-\alpha}$ be as in* (2.1) *for some x_0 and c. Under hypotheses* (C1)–(C3)*, if $u \ge \omega_\alpha$ on $\partial\Omega$, then $u \ge \omega_\alpha$ a.e. in Ω, and if $u \le \omega_{-\alpha}$ on $\partial\Omega$, then $u \le \omega_{-\alpha}$ a.e. in Ω.*

Proof. Let us define $E := \{x \in \Omega : u(x) < \omega_\alpha(x)\}$, $v := \min\{u, \omega_\alpha\}$, and $w := \max\{u, \omega_\alpha\}$. We argue by contradiction and assume that E has positive measure. By assumption $u \ge \omega_\alpha$ on $\partial\Omega$, therefore $v \in \omega_\alpha + W_0^{1,1}(\Omega)$ and $w \in u + W_0^{1,1}(\Omega)$. Since ω_α is the unique minimizer of I_α, we have

$$\begin{aligned} I_\alpha(\omega_\alpha) &= \int_{\Omega\setminus E} \Big[h(\nabla\omega_\alpha) + \alpha\omega_\alpha\Big]\,dx + \int_E \Big[h(\nabla\omega_\alpha) + \alpha\omega_\alpha\Big]\,dx \\ &< I_\alpha(v) = \int_{\Omega\setminus E} \Big[h(\nabla\omega_\alpha) + \alpha\omega_\alpha\Big]\,dx + \int_E \Big[h(\nabla u) + \alpha u\Big]\,dx; \end{aligned}$$

therefore, we have

$$\int_E \Big[h(\nabla\omega_\alpha) + \alpha\omega_\alpha\Big]\,dx < \int_E \Big[h(\nabla u) + \alpha u\Big]\,dx. \tag{2.2}$$

Analogously, since u is a minimizer of I, we get

$$\begin{aligned} I(u) &= \int_{\Omega\setminus E} \Big[h(\nabla u) + g(x,u)\Big]\,dx + \int_E \Big[h(\nabla u) + g(x,u)\Big]\,dx \\ &\le I(w) = \int_{\Omega\setminus E} \Big[h(\nabla u) + g(x,u)\Big]\,dx + \int_E \Big[h(\nabla\omega_\alpha) + g(x,\omega_\alpha)\Big]\,dx; \end{aligned}$$

hence it follows that

$$\int_E \Big[h(\nabla u) + g(x,u)\Big]\,dx \le \int_E \Big[h(\nabla\omega_\alpha) + g(x,\omega_\alpha)\Big]\,dx. \tag{2.3}$$

Putting together (2.2) and (2.3), we obtain

$$\begin{aligned} \int_E \Big[&h(\nabla\omega_\alpha) + \alpha\omega_\alpha - h(\nabla u) - \alpha u \\ &+ h(\nabla u) + g(x,u) - h(\nabla\omega_\alpha) - g(x,\omega_\alpha)\Big]\,dx < 0; \end{aligned}$$

i.e.,

$$\int_E \Big[g(x,\omega_\alpha) - g(x,u) - \alpha(\omega_\alpha - u)\Big]\,dx > 0,$$

and this is a contradiction with hypothesis (C3), which implies

$$-\alpha(\omega_\alpha - u) \le g(x,\omega_\alpha) - g(x,u) \le \alpha(\omega_\alpha - u)$$

for every $x \in E$. In the same way we can prove that $u \le \omega_{-\alpha}$ on $\partial\Omega$ implies $u \le \omega_{-\alpha}$ a.e. in Ω. $\square$

3. Basic properties of a class of convex functions. We will focus now on the particular case depending on the norm of the gradient, i.e., $h(\xi) = f(|\xi|)$, where $f : \mathbb{R} \to \mathbb{R}$. Therefore, from now on we will posit the following hypotheses on the function $f : \mathbb{R} \to \mathbb{R}$:

(F1) f is a convex, even function, increasing in $\mathbb{R}^+$, such that $f(0) = 0$;

(F2) the effective domain of f, $\mathrm{dom\,f}$, is equal to $\mathbb{R}$;

(F3) f is superlinear, i.e., $\lim_{|t|\to+\infty} \frac{f(t)}{|t|} = +\infty$.

We recall that assumption (F1) implies that $h^*(\xi) = f^*(|\xi|)$, and we state some basic and well-known facts on the function f that follow from the above assumptions. We will use them in the next section.

LEMMA 3.1. *If* (F1) *and* (F2) *hold, then f^* is superlinear, i.e.,* $\lim_{|\xi|\to+\infty} \frac{f^*(\xi)}{|\xi|} = +\infty$.

Proof. Assumption (F1) implies that f^* is convex and even and $f^*(0) = 0$. Assumption (F2) means that for any $t > 0$ there exists $\xi \geq 0$ such that $t \in \partial f^*(\xi)$. Then, either $f^*(\xi) = +\infty$ for any ξ sufficiently large (and then f^* is superlinear) or $\lim_{\xi\to+\infty} f^{*\prime}(\xi) = +\infty$. The conclusion follows. ∎

Remark 3.2. In Lemma 3.1 we used the simple fact that the superlinearity of an even convex function with effective domain coinciding with $\mathbb{R}$ is equivalent to the fact that its derivative goes to $+\infty$ as the variable goes to $+\infty$. We will use again this property in the rest of the paper.

Remark 3.3. We also recall that, as in Remark 2.2, (F1) and (F3) imply that $\mathrm{dom} f^* = \mathbb{R}$.

LEMMA 3.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function. Assume that there exist $\epsilon > 0$ and $0 < \tau^1 < \tau^2$ such that*

$$f(t) \geq f(s) + \xi_s(t - s) + \frac{\epsilon}{2}(t - s)^2 \tag{3.1}$$

for every $t, s \in (\tau^1, \tau^2)$ and $\xi_s \in \partial f(s)$. Then f^* is $C^{1,1}([\xi^1, \xi^2])$, where $\xi^1 = \sup \partial f(\tau^1)$, $\xi^2 = \inf \partial f(\tau^2)$,

$$\xi^2 - \xi^1 \geq \epsilon(\tau^2 - \tau^1) \quad \textit{and} \quad |f^{*\prime\prime}(\xi)| \leq \frac{1}{\epsilon} \textit{ a.e. in } [\xi^1, \xi^2]. \tag{3.2}$$

Proof. First we observe that 3.1 implies the strict convexity of f in (τ^1, τ^2) and then that f^* is $\mathcal{C}^1(\xi^1, \xi^2)$, where $\xi^i \in \partial f(\tau^i)$, $i = 1, 2$. Indeed let us suppose that f^* is not differentiable in $\xi \in (\xi^1, \xi^2)$, i.e., $\partial f^*(\xi)$ is not a singleton; this means that there exist $s \neq t$ in $\partial f^*(\xi)$. From the monotonicity of ∂f^* we have that $s, t \in (\tau^1, \tau^2)$. Moreover $\xi \in \partial f(s) \cap \partial f(t)$, i.e., f is affine in $[s, t]$ and hence f is not strictly convex. Assume now that $f^* \notin C^1(\mathbb{R})$, so that there exists $\xi \in (\xi^1, \xi^2)$ such that $f^{*\prime}$ is not continuous in ξ; since $f^{*\prime}$ is a monotone function, the left and right limits of $f^{*\prime}$ in ξ exist and they do not coincide; therefore, $\partial f^*(\xi)$, which is the convex envelope of these limits, cannot be a singleton, which is in contradiction with the fact that f^* is differentiable in ξ. By assumption, for any $t, s \in (\tau^1, \tau^2)$, we have

$$f(t) - f(s) - \xi_s(t - s) \geq \frac{\epsilon}{2}(t - s)^2,$$
$$f(s) - f(t) - \xi_t(s - t) \geq \frac{\epsilon}{2}(t - s)^2.$$

By adding term by term, we get

$$(\xi_t - \xi_s)(t - s) \geq \epsilon(t - s)^2.$$

Passing to the limit for $t \to \tau^2$ and $s \to \tau^1$, dividing by $\tau^2 - \tau^1$, we obtain the first inequality in (3.2). Recalling that $\xi \in \partial f(\tau)$ if and only if $\tau \in \partial f^*(\xi) = {f^*}'(\xi)$, we get

$$|{f^*}'(\xi_t) - {f^*}'(\xi_s)| \leq \frac{1}{\epsilon}|\xi_t - \xi_s|,$$

proving the claim. □

LEMMA 3.5. *Let $f: \mathbb{R} \to \mathbb{R}$ be a convex function of class C^2 on $(T, +\infty)$. Let $t > \alpha$ be such that $f''(t) > 0$; then*

$$f^{*\prime\prime}(\xi) = \frac{1}{f''(t)},$$

where $\xi = f'(t)$. Moreover, if f'' is strictly positive in $(T, +\infty)$, $f^{*\prime\prime}$ is of class C^2 on $(f'(T), +\infty)$.

Proof. For every $s, t > T$ we have

$$f(s) - f(t) - f'(t)(s - t) = \frac{f''(t)}{2}(s - t)^2 + o(s - t)^2,$$
$$f(t) - f(s) - f'(s)(t - s) = \frac{f''(s)}{2}(s - t)^2 + o(s - t)^2.$$

By summing up the two equations and dividing by $(s - t)^2$ we get

$$\frac{f'(s) - f'(t)}{s - t} = \frac{f''(s) + f''(t)}{2} + \frac{o(s - t)^2}{(s - t)^2}.$$

As $\xi_t = f'(t)$ if and only if $t = {f^*}'(\xi_t)$, we get

$$\frac{\xi_t - \xi_s}{{f^*}'(\xi_t) - {f^*}'(\xi_s)} = \frac{f''(s) + f''(t)}{2} + \frac{o(s - t)^2}{(s - t)^2}.$$

We now let s tend to t and obtain in the limit $1/f^{*\prime\prime}(\xi_t) = f''(t)$. The last statement is straightforward. □

LEMMA 3.6. *Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function. Assume that there exist $\tau > 0$ such that $\partial f(\tau) = [\xi^1, \xi^2]$, where $\xi^2 > \xi^1$. Then $f^{*\prime\prime}(\xi) = 0$ a.e. in (ξ^1, ξ^2).*

Proof. The assumption $\partial f(\tau) = [\xi^1, \xi^2]$ holds if and only if $\tau \in \partial f^*(\xi^1)$, $i = 1, 2$. It follows that f^* is affine in (ξ^1, ξ^2), proving the claim. □

4. A regularity result. In this section we will consider the functional

$$\mathcal{J}(u) = \int_\Omega f(|\nabla u|) + g(x, u)\, dx$$

defined in $\phi + W_0^{1,1}(\Omega)$. We will assume that the boundary datum is in a special class: the one of the functions satisfying the BSC. We will show that if Ω is uniformly convex, if suitable assumptions hold for f and g, then the results of section 3 and the Comparison Principle of section 2 imply that all the minimizers of the functional are bounded by two Lipschitz barriers that coincide with ϕ on the boundary of Ω. Then the Haar–Radò-type theorem (Theorem 2.3) will guarantee that the maximum and the minimum of the minimizers are Lipschitz continuous.

We recall here the BSC introduced by Hartman and Stampacchia in [15].

Definition 4.1 (BSC). *The function ϕ satisfies the* BSC *of rank $M \geq 0$ if for every $\gamma \in \partial\Omega$ there exist z_γ^-, $z_\gamma^+ \in \mathbb{R}^n$, and $M \in \mathbb{R}$ such that*

$$\forall \gamma' \in \partial\Omega \quad \phi(\gamma) + z_\gamma^- \cdot (\gamma' - \gamma) \leq \phi(\gamma'), \tag{4.1}$$

$$\forall \gamma' \in \partial\Omega \quad \phi(\gamma) + z_\gamma^+ \cdot (\gamma' - \gamma) \geq \phi(\gamma'), \tag{4.2}$$

and $|z_\gamma^\pm| \leq M$ for every $\gamma \in \partial\Omega$.

Remark 4.2. The BSC implies that ϕ is Lipschitz of rank M. Moreover it forces Ω to be convex unless ϕ is affine. Necessary and sufficient conditions to the BSC are studied, respectively, in [16] and [24].

In this section we will use the following set of assumptions on f, g, and Ω. We assume that f satisfies either

(F4) for every $k \in \mathbb{N}$ there exist $\epsilon_k > 0$ and $\tau_k^i > k$, $i = 1, 2$ such that
  (i) $f(t) \geq f(s) + \xi_s(t-s) + \frac{\epsilon_k}{2}(t-s)^2$ for every $t, s \in (\tau_k^1, \tau_k^2)$ and $\xi_s \in \partial f(s)$,
  (ii) $\lim_k \epsilon_k(\tau_k^2 - \tau_k^1) = \lambda > 0$,
  (iii) $\lim_{k\to+\infty} \frac{1}{\epsilon_k(\tau_k^1)^3} = 0$,

or

(F5) for every $k \in \mathbb{N}$ there exist $\tau_k > k$ such that
  (i) $\partial f(\tau_k) = [\xi_k^1, \xi_k^2]$,
  (ii) $\lim_{k\to+\infty} \xi_k^2 - \xi_k^1 = \lambda > 0$.

The function g is assumed to satisfy

(G1) g is Lipschitz continuous in the second variable, with Lipschitz constant equal to α, i.e., $|g(x, u_1) - g(x, u_2)| \leq \alpha|u_1 - u_2|$ for every $x \in \Omega$ and $u_1, u_2 \in \mathbb{R}$;

(G2) $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is measurable and convex in the second variable. Denoting by g_v^+ the right derivative of g with respect to the second variable, we assume that there exists a positive constant K such that

$$\forall x, y \in \mathbb{R}^n,\ \forall u, v \in \mathbb{R} \quad v \geq u + K|y - x| \Rightarrow g_v^+(y, v) \geq g_v^+(x, u).$$

We consider an R-uniformly convex open bounded subset Ω of $\mathbb{R}^n$, where R-uniformly convex means that for every $\gamma \in \partial\Omega$ there exists $b_\gamma \in \mathbb{R}^n$ with $|b_\gamma| = 1$ such that

$$R\, b_\gamma \cdot (\gamma' - \gamma) \geq \frac{1}{2}|\gamma' - \gamma|^2. \tag{4.3}$$

As we previously recalled, in our setting, the existence of a solution of the minimum problem

$$\min_{v \in \phi + W_0^{1,1}(\Omega)} \int_\Omega \left[f(|\nabla v|) + g(x, v)\right] dx$$

follows by the Direct Method of the Calculus of Variations.

The following theorem states the existence of Lipschitz barriers that coincide with the boundary datum on the boundary of Ω.

Theorem 4.3. *Assume that f satisfies hypotheses* (F1)–(F3) *and either* (F4) *or* (F5). *Let g satisfy assumption* (G1). *Let u be a minimizer of the functional*

$$\int_\Omega \left[f(|\nabla v|) + g(x, v)\right] dx, \quad v \in \phi + W_0^{1,1}(\Omega),$$

where Ω is an open bounded R-uniformly convex set and $\phi : \Omega \to \mathbb{R}$ satisfies the BSC *with rank M. Then there exist $\ell^+, \ell^- : \overline{\Omega} \to \mathbb{R}$, both Lipschitz of rank $L = L(R, f, M)$, such that*

$$\ell^-(\gamma) = \phi(\gamma) = \ell^+(\gamma) \quad \textit{for every } \gamma \in \partial\Omega$$

and

$$\ell^-(x) \le u(x) \le \ell^+(x) \quad \text{for almost every } x \in \Omega.$$

Proof. It is sufficient to construct the function ℓ^-, the construction of ℓ^+ going on exactly in the same way.

We fix a point $\gamma \in \partial\Omega$, and we consider also the vector z_γ^- involved in the definition of the BSC at the point γ. The proof of the theorem will be achieved when we will have shown that there exist x_γ, c_γ such that the set

$$\Omega_{x_\gamma,c_\gamma} := \left\{ x \in \mathbb{R}^n : \frac{n}{\alpha} f^*\Big(\frac{\alpha}{n}|x - x_\gamma|\Big) + c_\gamma - z_\gamma^- \cdot (x-\gamma) - \phi(\gamma) < 0 \right\}$$

contains Ω and $\gamma \in \partial\Omega_{x_\gamma,c_\gamma} \cap \partial\Omega$. In fact, by this last property, we have immediately that

$$l_{x_\gamma,c_\gamma}(x) := \frac{n}{\alpha} f^*\Big(\frac{\alpha}{n}|x - x_\gamma|\Big) + c_\gamma \le z_\gamma^- \cdot (x-\gamma) + \phi(\gamma) \le \phi(x)$$

for any $x \in \partial\Omega$. The Comparison Principle (2.4) then implies that

$$l_{x_\gamma,c_\gamma}(x) \le z_\gamma^- \cdot (x-\gamma) + \phi(\gamma) \le u(x) \qquad \text{a.e. on } \Omega.$$

We get the result simply by defining

$$\ell^-(x) = \sup_{\gamma\in\partial\Omega} l_{x_\gamma,c_\gamma}(x).$$

We divide the proof into several steps.

Step 1. In this step we state some properties of the auxiliary domain defined as follows: fix $a \in \mathbb{R}^n$, $b > 0$ and let

$$\Omega_b := \left\{ x \in \mathbb{R}^n : \frac{n}{\alpha} f^*\left(\frac{\alpha}{n}|x|\right) - a\cdot x - b < 0 \right\}.$$

Whenever we assume that (F1) and (F2) hold, Lemma 3.1 implies that Ω_b is bounded for every b and that for $b > 0$, 0 is contained in its interior. Moreover, by the continuity of f^* there exist $x_{1,b}$, $x_{2,b}$ in the set $\partial\Omega_b = \{x \in \mathbb{R}^n : \frac{n}{\alpha} f^*(\frac{\alpha}{n}|x|) - a\cdot x - b = 0\}$ such that $\frac{n}{\alpha} f^*(\frac{\alpha}{n}|x_{1,b}|) \le \frac{n}{\alpha} f^*(\frac{\alpha}{n}|x|) \le \frac{n}{\alpha} f^*(\frac{\alpha}{n}|x_{2,b}|)$ for every $x \in \partial\Omega_b$ and such that $\overline{B}(0, |x_{1,b}|) \subset \Omega_b \subset B(0, |x_{2,b}|)$. It is immediate to see that

$$\lim_{b\to+\infty} |x_{1,b}| = +\infty. \tag{4.4}$$

Indeed, as in Remark 2.2, the superlinearity of f implies that $\operatorname{dom} f^* = \mathbb{R}$; hence $\frac{n}{\alpha} f^*(\alpha|x|/n) - a\cdot x$ is bounded on every bounded set. This implies that if $b \to +\infty$, then $|x| \to +\infty$ for $x \in \partial\Omega_b$; in particular (4.4) holds true.

Step 2. In this step we fix $\gamma \in \partial\Omega$ and η a unit vector in the normal cone to Ω in γ and we aim to select a special domain Ω_{b_η} for $a = z_\gamma^-$.

We first assume that (F4) holds. Let $k \in \mathbb{N}$ be such that $k > M \ge |z_\gamma^-|$ be fixed. Thanks to (4.4), we can choose $\bar{b} > 0$ such that $|x_{1,b}| > k$ for every $b > \bar{b}$. According to assumption (i) of (F4), let $\epsilon_k > 0$ and $\tau_k^i > k$, $i = 1, 2$, be such that $f(t) \ge f(s) + \xi_s(t-s) + \frac{\epsilon_k}{2}(t-s)^2$ for every $t, s \in (\tau_k^1, \tau_k^2)$ and $\xi_s \in \partial f(s)$. We notice that thanks to assumption (F3), we can also assume that $\xi_s > k$ and $\xi_s > |x_{2,\bar{b}}|$ for any $\xi_s \in \partial f(s)$, $s \in [\tau_k^1, \tau_k^2]$. By Lemma 3.4, the function f^* is $C^{1,1}(\xi_k^1, \xi_k^2)$, where

$\xi_k^i \in \partial f(\tau_k^i)$, $i = 1, 2$, and $|f^{*\prime\prime}(\xi)| \le \frac{1}{\epsilon_k}$ for a.e. $\xi \in (\xi_k^1, \xi_k^2)$. In the case where (F5) holds, we immediately obtain that f^* is in $\mathcal{C}^2(\xi_k^1, \xi_k^2)$ and $f^{*\prime\prime} = 0$ in (ξ_k^1, ξ_k^2). In both cases we choose now $\frac{\alpha}{n}|\xi_k| = \frac{\xi_k^1+\xi_k^2}{2}$ and we observe that

$$f^{*\prime}\Big(\frac{\alpha}{n}|\xi_k|\Big) =: \tau_k \ge \tau_k^1 > k > M > |z_\gamma^-|, \tag{4.5}$$

$$\frac{\alpha}{n}|\xi_k| > |x_{2,\bar b}|. \tag{4.6}$$

We prove now that there exist $x_\eta \in \mathbb{R}^n$ and $b_\eta \in \mathbb{R}$ such that $|x_\eta| = |\xi_k|$,

$$\frac{n}{\alpha} f^*\Big(\frac{\alpha}{n}|\xi_k|\Big) - z_\gamma^- \cdot x_\eta - b_\eta = 0, \quad \text{i.e., } x_\eta \in \partial\Omega_{b_\eta}, \tag{4.7}$$

and $f^{*\prime}(\frac{\alpha}{n}|x_\eta|)\frac{x_\eta}{|x_\eta|} - z_\gamma^- = \lambda\eta$ for some $\lambda \ne 0$; i.e., the outward normal to $\partial\Omega_{b_\eta}$ in x_η is parallel to η.

First we observe that the estimate (4.5) guarantees the existence of a one-to-one correspondence between S^{n-1} and $f^{*\prime}(\frac{\alpha}{n}|\xi_k|)S^{n-1} - z_\gamma^-$. Therefore, for every $\eta \in S^{n-1}$, there exists a unique $\nu_\eta \in S^{n-1}$ such that $f^{*\prime}(\frac{\alpha}{n}|\xi_k|)\nu_\eta - z_\gamma^- = \lambda\eta$ for a suitable $\lambda > 0$. Hence, let us define b_η by

$$b_\eta := \frac{n}{\alpha} f^*\Big(\frac{\alpha}{n}|\xi_k|\Big) - z_\gamma^- \cdot \nu_\eta|\xi_k|.$$

We consider now $\Omega_{b_\eta} := \{x \in \mathbb{R}^n \ : \ \frac{n}{\alpha}f^*(\frac{\alpha}{n}|x|) - z_\gamma^- \cdot x - b_\eta < 0\}$; we have that $\nu_\eta|\xi_k| \in \partial\Omega_{b_\eta}$ by definition of b_η and $\partial\Omega_{b_\eta}$ is $\mathcal{C}^1$ in a neighborhood of $\nu_\eta|\xi_k|$. Then the outward normal to $\partial\Omega_{b_\eta}$ in $\nu_\eta|\xi_k|$ is parallel to

$$f^{*\prime}\Big(\frac{\alpha}{n}|\nu_\eta||\xi_k|\Big)\frac{\nu_\eta|\xi_k|}{|\xi_k|} - z_\gamma^- = f^{*\prime}\Big(\frac{\alpha}{n}|\xi_k|\Big)\nu_\eta - z_\gamma^- = \lambda\eta.$$

This proves that $\nu_\eta|\xi_k|$ is the point x_η we were looking for.

By the implicit function theorem we also infer that there exists a neighborhood of the point x_η in which $\partial\Omega_{b_\eta} = \{x \in \mathbb{R}^n : f^*(\frac{\alpha}{n}|x|) - z_\gamma^- \cdot x - b_\eta = 0\}$ is at least $\mathcal{C}^{1,1}$.

We notice that $|x_{1,b_\eta}| > k$, because $b_\eta > \bar b$ by (4.6).

Step 3. We are interested in proving that for k sufficiently large, we can find a ball of radius R contained in Ω_{b_η} that touches $\partial\Omega_{b_\eta}$ in x_η. To reach this aim in this step we compute the principal curvatures of $\partial\Omega_{b_\eta}$ in the neighborhood of x_η.

We can estimate the principal curvatures of $\partial\Omega_{b_\eta}$ in almost every point $x \in \partial\Omega_{b_\eta}$ such that $\xi_k^1 \le \frac{\alpha}{n}|x| \le \xi_k^2$. Each principal curvature is less than or equal to the greater eigenvalue of the Hessian matrix of the implicit function ψ defined by (4.7), and it can be estimated by the norm of the matrix itself. This means to estimate

$$\frac{\partial_{ij}\psi(\hat x)}{(1 + |\nabla\psi(\hat x)|^2)^{3/2}}$$

for $i, j = 1, \dots, n-1$, where we have assumed, without loss of generality, that the n-component x_n of the vector x is implicitly defined with respect to the first $n-1$ components $\hat x := (x_1, x_2, \dots, x_{n-1})$.

From now on, to simplify the notation, we drop the indices η and γ and we denote by F the function $F(x) = \frac{n}{\alpha}f^*(\frac{\alpha}{n}|x|) - z\cdot x - b$. It follows, again by the implicit function theorem, that for $\mathcal{H}^{n-1}$–a.e. $\hat x \in \mathbb{R}^{n-1}$

$$\frac{\partial_{ij}\psi}{(1 + |\nabla\psi|^2)^{3/2}} = \frac{-\partial_{ij}F(\partial_n F)^2 + \partial_{in}F\partial_j F\partial_n F + \partial_{jn}F\partial_i F\partial_n F - \partial_{nn}F\partial_i F\partial_j F}{|\nabla F|^3} \tag{4.8}$$

with F and its derivative computed in the point $x = (\hat{x}, \psi(\hat{x}))$. Hence, we obtain that (4.8) is equal to

$$
\begin{aligned}
&= \Big((f^{*\prime})^2 \Big(\frac{\alpha}{n}|x|\Big) - 2 f^{*\prime}\Big(\frac{\alpha}{n}|x|\Big)\frac{zx}{|x|} + |z|^2 \Big)^{-3/2} \\
&\quad \cdot \Big\{ \Big[-\frac{\alpha}{n} f^{*\prime\prime}\Big(\frac{\alpha}{n}|x|\Big)\frac{x_i x_j}{|x|^2} + f^{*\prime}\Big(\frac{\alpha}{n}|x|\Big)\frac{x_i x_j}{|x|^3} \Big] \cdot \Big[f^{*\prime}\Big(\frac{\alpha}{n}|x|\Big)\frac{x_n}{|x|} - z_n \Big]^2 \\
&\qquad + \Big[\frac{\alpha}{n} f^{*\prime\prime}\Big(\frac{\alpha}{n}|x|\Big)\frac{x_i x_n}{|x|^2} - f^{*\prime}\Big(\frac{\alpha}{n}|x|\Big)\frac{x_i x_n}{|x|^3} \Big] \cdot \Big[f^{*\prime}\Big(\frac{\alpha}{n}|x|\Big)\frac{x_j}{|x|} - z_j \Big] \\
&\qquad \cdot \Big[f^{*\prime}\Big(\frac{\alpha}{n}|x|\Big)\frac{x_n}{|x|} - z_n \Big] \\
&\qquad + \Big[\frac{\alpha}{n} f^{*\prime\prime}\Big(\frac{\alpha}{n}|x|\Big)\frac{x_j x_n}{|x|^2} - f^{*\prime}\Big(\frac{\alpha}{n}|x|\Big)\frac{x_j x_n}{|x|^3} \Big] \cdot \Big[f^{*\prime}\Big(\frac{\alpha}{n}|x|\Big)\frac{x_i}{|x|} - z_i \Big] \\
&\qquad \cdot \Big[f^{*\prime}\Big(\frac{\alpha}{n}|x|\Big)\frac{x_n}{|x|} - z_n \Big] \\
&\qquad + \Big[-\frac{\alpha}{n} f^{*\prime\prime}\Big(\frac{\alpha}{n}|x|\Big)\frac{x_n^2}{|x|^2} - f^{*\prime}\Big(\frac{\alpha}{n}|x|\Big)\frac{|x|^2 - x_n^2}{|x|^3} \Big] \cdot \Big[f^{*\prime}\Big(\frac{\alpha}{n}|x|\Big)\frac{x_i}{|x|} - z_i \Big] \\
&\qquad \cdot \Big[f^{*\prime}\Big(\frac{\alpha}{n}|x|\Big)\frac{x_j}{|x|} - z_j \Big] \Big\}
\end{aligned}
$$

with $(x_1, \ldots, x_{n-1}) = \hat{x}$ and $x_n = \psi(\hat{x})$. By computation, it follows that

$$
\begin{aligned}
(4.9) \quad & \left| \frac{\partial_{ij}\psi(\hat{x})}{(1 + |\nabla\psi(\hat{x})|^2)^{3/2}} \right| \\
&\le \Bigg[\alpha f^{*\prime\prime}\Big(\frac{\alpha}{n}|x|\Big)|z|^2 + \frac{(f^{*\prime})^3\big(\frac{\alpha}{n}|x|\big)n}{|x|} + \frac{(f^{*\prime})^2\big(\frac{\alpha}{n}|x|\big)n|z|}{|x|} + \frac{f^{*\prime}\big(\frac{\alpha}{n}|x|\big)n|z|^2}{|x|} \Bigg] \\
&\quad \cdot \left| f^{*\prime}\Big(\frac{\alpha}{n}|x|\Big)\frac{x}{|x|} - z \right|^{-3}.
\end{aligned}
$$

In the case where (F4) holds we use Lemma 3.4 to estimate

$$
\Big[\alpha f^{*\prime\prime}\Big(\frac{\alpha}{n}|x|\Big)|z|^2 \Big] \left| f^{*\prime}\Big(\frac{\alpha}{n}|x|\Big)\frac{x}{|x|} - z \right|^{-3} \le C \frac{1}{\epsilon_k (\tau_k^1)^3}
$$

for a suitable constant C. In the other case,

$$
\Big[\alpha f^{*\prime\prime}\Big(\frac{\alpha}{n}|x|\Big)|z|^2 \Big] \left| f^{*\prime}\Big(\frac{\alpha}{n}|x|\Big)\frac{x}{|x|} - z \right|^{-3} = 0,
$$

Lemma 3.1 implies also that

$$
\begin{aligned}
&\lim_{|x| \to +\infty} \frac{1}{|x|} \Big[(f^{*\prime})^3\Big(\frac{\alpha}{n}|x|\Big)n + (f^{*\prime})^2\Big(\frac{\alpha}{n}|x|\Big)n|z| + f^{*\prime}\Big(\frac{\alpha}{n}|x|\Big)n|z|^2 \Big] \\
&\qquad \cdot \left| f^{*\prime}\Big(\frac{\alpha}{n}|x|\Big)\frac{x}{|x|} - z \right|^{-3} = 0.
\end{aligned}
$$

Therefore, under either assumption (iii) of (F4) or (i) of (F5), we can choose k such that all the principal curvatures of $\partial\Omega_{b_\eta} \cap \{x \in \mathbb{R}^n : \xi_k^1 \leq \frac{\alpha}{n}|x| \leq \xi_k^2\}$ are bounded by a constant $\frac{1}{R_k}$ less than $\frac{1}{R}$. This implies that for k sufficiently large, the ball $B(x_\eta - R\eta, R)$ contains x_η in its boundary and is "locally" included in $\Omega_{b\eta}$; this means that $U_{x_\eta} \cap B(x_\eta - R\eta, R) \subseteq U_{x_\eta} \cap \Omega_{b_\eta}$ for a suitable neighborhood U_{x_η} of x_η.

Step 4. Now, in the next three steps we want to show that for k sufficiently large, the ball $B(x_\eta - R\eta, R)$ is entirely contained in Ω_{b_η}. In the present step and in the next one, we suppose that $\partial\Omega_{b_\eta}$ is globally $\mathcal{C}^{1,1}$, and we want to compute the norm of the points of $\partial\Omega_{b_\eta}$ in a suitable neighborhood of x_η. First we recall that given a function $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$, such that $\psi(0) = 0$, $\nabla\psi(0) = 0$, and $\psi \in \mathcal{C}^{1,1}(B(0,\delta))$, with the absolute value of the curvature almost everywhere bounded by $1/R_k > 0$, an easy computation shows that

$$|\psi(\zeta)| \leq R_k - \sqrt{R_k^2 - \delta^2} \tag{4.10}$$

for $|\zeta| \leq \delta$. We can assume without loss of generality that the system of coordinates x in $\mathbb{R}^n$ is such that the tangent plane ρ to $B(0,|x_\eta|)$ in x_η is $\{x = (x_1,\dots,x_n) \in \mathbb{R}^n : x_n = x_{\eta_n}\}$, so that $|x_\eta| = |x_{\eta_n}|$. Now we fix the tangent plane π to $\partial\Omega_{b_\eta}$ in x_η. We consider a new system of coordinates (ζ, t), $\zeta \in \mathbb{R}^{n-1}$, $t \in \mathbb{R}$, such that the plane π is the set $\{(\zeta,t) : t = 0\}$, x_η corresponds to $(\zeta,t) = 0$, and $\partial\Omega_{b_\eta}$ is the graph of a function $\psi\colon A \subseteq \mathbb{R}^{n-1} \to \mathbb{R}$ for a suitable open set A. We denote by $T\colon \mathbb{R}^n \to \mathbb{R}^n$ the change of variables that brings back (ζ,t) into x. We have

$$|T(\zeta,\psi(\zeta)) - T(\zeta,0)| \leq R_k - \sqrt{R_k^2 - \delta^2} \quad \text{for every } |\zeta| \leq \delta, \tag{4.11}$$

by (4.10), using the curvature estimate of the previous step. Then, we have

$$\begin{aligned} &\Big|(T(\zeta,0)_1,\dots,T(\zeta,0)_{n-1},x_{\eta_n}) \\ &\quad -\Big(T(\zeta,0)_1,\dots,T(\zeta,0)_{n-1},\operatorname{sign} x_{\eta_n}\sqrt{|x_\eta|^2 - T(\zeta,0)_1^2 + \cdots - T(\zeta,0)_{n-1}^2}\Big)\Big| \\ &\quad \leq |x_\eta| - \sqrt{|x_\eta|^2 - \delta^2} \qquad \text{for } |\zeta| \leq \delta. \end{aligned} \tag{4.12}$$

We also have

$$|T(\zeta,0) - (T(\zeta,0)_1,\dots,T(\zeta,0)_{n-1},x_{\eta_n})| \leq \delta\sin\theta_\eta \quad \text{for } |\zeta| \leq \delta, \tag{4.13}$$

where θ_η is the angle between the planes π and ρ. We observe that θ_η is also the angle between the normal directions to π and ρ, η, and $x_\eta/|x_\eta|$, respectively. We recall that $|x_{1,b_\eta}|$ and $|x_{2,b_\eta}|$ are the points of $\partial\Omega_{b_\eta}$ such that $\frac{n}{\alpha}f^*(\frac{\alpha}{n}|x_{1,b_\eta}|) \leq \frac{n}{\alpha}f^*(\frac{\alpha}{n}|x|) \leq \frac{n}{\alpha}f^*(\frac{\alpha}{n}|x_{2,b_\eta}|)$ for every $x \in \partial\Omega_{b_\eta}$. The set $\overline{\Omega}_{b_\eta}$ contains $\overline{\mathrm{co}}(x_\eta, B(0,|x_{1,b_\eta}|))$ so that the normal cone to $\overline{\mathrm{co}}(x_\eta, B(0,|x_{1,b_\eta}|))$ contains the normal to Ω_{b_η} in x_η, i.e., the angle θ_η between η and $\frac{x_\eta}{|x_\eta|}$ satisfies

$$\begin{aligned} \theta_\eta &= \arccos \eta \cdot \frac{x_\eta}{|x_\eta|} \\ &\leq \arcsin\left(\sup_{|x_{1,b_\eta}|<|x|<|x_{2,b_\eta}|}\ \inf_{y\in B(0,|x_{1,b_\eta}|)} \frac{x-y}{|x-y|}\cdot\frac{x}{|x|}\right). \end{aligned} \tag{4.14}$$

Thus we obtain

$$|T(\zeta,0)-(T(\zeta,0)_1,\dots,T(\zeta,0)_{n-1},x_{\eta_n})| \leq \delta\left(\sup_{|x_{1,b_\eta}|<|x|<|x_{2,b_\eta}|}\inf_{y\in B(0,|x_{1,b_\eta}|)}\frac{x-y}{|x-y|}\cdot\frac{x}{|x|}\right). \tag{4.15}$$

Hence, we can conclude that

$$\begin{aligned}&\Big||T(\zeta,\psi(\zeta))|-|x_\eta|\Big|\\ &\leq\left|T(\zeta,\psi(\zeta))-\Big(T(\zeta,0)_1,\dots,T(\zeta,0)_{n-1},\operatorname{sign}x_{\eta_n}\sqrt{|x_\eta|^2-T(\zeta,0)_1^2+\cdots-T(\zeta,0)_{n-1}^2}\Big)\right|\\ &\leq R_k-\sqrt{R_k^2-\delta^2}+|x_\eta|-\sqrt{|x_\eta|^2-\delta^2}+\delta\left(\sup_{|x_{1,b_\eta}|<|x|<|x_{2,b_\eta}|}\inf_{y\in B(0,|x_{1,b_\eta}|)}\frac{x-y}{|x-y|}\cdot\frac{x}{|x|}\right).\end{aligned} \tag{4.16}$$

Step 5. In this step we show that for fixed δ, by choosing k large enough (in particular $k > R$), we can make the quantity in (4.16) as small as we want. In particular, for $\delta > 4R$ (where R is the constant appearing in (4.3)), we can choose k such that the norm of the points of $\partial\Omega_{b_\eta}$, $T(\zeta,\psi(\zeta))$, with $|\zeta|\leq\delta$, is between $|x_\eta|-\frac{\lambda}{4}$ and $|x_\eta|+\frac{\lambda}{4}$ (where λ is the constant appearing in assumption (ii) of (F4) and (F5)).

We denote by Λ_k the minimum between R_k and $|x_{1,b_\eta}|$. We recall that both are greater than R. We fix $\delta > 4R$. We now want to prove that we can choose k such that

$$\delta\left(\sup_{|x_{1,b_\eta}|<|x|<|x_{2,b_\eta}|}\inf_{y\in B(0,|x_{1,b_\eta}|)}\frac{x-y}{|x-y|}\cdot\frac{x}{|x|}\right)\leq\frac{\lambda}{8} \tag{4.17}$$

and

$$2|\Lambda_k-\sqrt{\Lambda_k^2-\delta^2}|\leq\frac{\lambda}{8}. \tag{4.18}$$

By the estimates in Step 3, we have $\lim_{k\to\infty}R_k=+\infty$. Hence,

$$\lim_k 2|\Lambda_k-\sqrt{\Lambda_k^2-\delta^2}|=0.$$

On the other hand, we remark that

$$\inf_{y\in B(0,|x_{1,b}|)}\frac{x-y}{|x-y|}\cdot\frac{x}{|x|}=\sqrt{1-\frac{|x_{1,b}|^2}{|x|^2}}$$

and $\sqrt{1-\frac{|x_{1,b}|^2}{|x|^2}}\leq\sqrt{1-\frac{|x_{1,b}|^2}{|x_{2,b}|^2}}$. Therefore, if we prove that

$$\lim_{b\to+\infty}\frac{|x_{2,b}|-|x_{1,b}|}{|x_{1,b}|}=0, \tag{4.19}$$

we are done.

For every $t > 0$ we identify $\frac{n}{\alpha} f^*(\frac{\alpha}{n}t) =: \tilde f(t)$. Then $\tilde f$ is convex and satisfies assumptions (F1) and (F2). By Lemma 3.1, recalling Remark 3.2, $\lim_{t\to+\infty} \tilde f'(t) = +\infty$. We can consider the inequality $|x_{2,b}| - |x_{1,b}| \le |y_{2,b}| - |y_{1,b}|$, where

$$|z_\gamma^-||y_{2,b}| + b = \tilde f'(|x_{1,b}|)(|y_{2,b}| - \tilde f^{-1}(b)) + b$$
$$-|z_\gamma^-||y_{1,b}| + b = \tilde f'(|x_{1,b}|)(|y_{1,b}| - \tilde f^{-1}(b)) + b$$

so that $|y_{2,b}| = \frac{\tilde f'(|x_{1,b}|)\tilde f^{-1}(b)}{\tilde f'(|x_{1,b}|) - |z_\gamma^-|}$ and $|y_{1,b}| = \frac{\tilde f'(|x_{1,b}|)\tilde f^{-1}(b)}{\tilde f'(|x_{1,b}|) + |z_\gamma^-|}$. We have $|x_{1,b}| \ge |y_{1,b}|$ and then $\frac{\tilde f^{-1}(b)}{|x_{1,b}|} \le \frac{\tilde f'(|x_{1,b}|) + |z_\gamma^-|}{\tilde f'(|x_{1,b}|)}$ so that $\lim_{b\to+\infty} \frac{\tilde f^{-1}(b)}{|x_{1,b}|}$ is finite; then we obtain

$$\lim_{b\to+\infty} \frac{|x_{2,b}| - |x_{1,b}|}{|x_{1,b}|} \le \lim_{b\to+\infty} \frac{2|z_\gamma^-|\tilde f'(|x_{1,b}|)\tilde f^{-1}(b)}{|x_{1,b}|[(\tilde f'(|x_{1,b}|))^2 - |z_\gamma^-|^2]} = 0.$$

Therefore, we can conclude by putting together estimates (4.16), (4.17), and (4.18) that

$$\Big| |T(\zeta, \psi(\zeta))| - |x_\eta| \Big| = \Big| |T(\zeta, \psi(\zeta))| - \frac{n}{2\alpha}(\xi_k^1 + \xi_k^2) \Big| \le \lambda/4 \quad \text{for } |\zeta| \le \delta. \tag{4.20}$$

Step 6. In this step we conclude the proof of the existence of the ball of radius R contained in Ω_{b_η} and touching $\partial\Omega_{b_\eta}$ in x_η.

We recall that in Steps 4 and 5, we assumed $\partial\Omega_{b_\eta}$ to be globally $C^{1,1}$. We already know that $\partial\Omega_{b_\eta}$ is $C^{1,1}$ near x_η by hypothesis (F4)(i)/(F5)(i). Now, by choosing k sufficiently large, we can make $\partial\Omega_{b_\eta}$ actually $C^{1,1}$, at least in the points $T(\zeta, \psi(\zeta))$, for $|\zeta| \le \delta$. Indeed, hypothesis (F4)(ii) (or (F5)(ii)) implies $\xi_k^2 - \xi_k^1 > \lambda/2$ for k large enough (see (3.2)), and f^* is $\mathcal{C}^{1,1}(\xi_k^1, \xi_k^2)$. Hence, estimate (4.20) and a comparison argument show that $\partial\Omega_{b_\eta}$ is $C^{1,1}$ in its points $T(\zeta, \psi(\zeta))$ for $|\zeta| \le \delta$, since they lie in the set $\{x \in \mathbb{R}^n : \xi_k^1 \le \frac{\alpha}{n}|x| \le \xi_k^2\}$. In particular, in the same points, $\partial\Omega_{b_\eta}$ has curvature less than $1/R_k \le 1/R$.

Now we consider the distances of the points $(T(\zeta, \psi(\zeta))_1, \dots, T(\zeta, \psi(\zeta))_{n-1}, x_{\eta_n})$ from x_η for $|\zeta| = \delta$. We can choose k sufficiently large such that the minimum of these distances, p_δ, can be estimated from below by $\delta/2 > 2R$. Indeed, we have

$$\begin{aligned} p_\delta &= \left[|T(\zeta,0) - T(0,0)| - \frac{|T(\zeta, \psi(\zeta)) - T(\zeta, 0)|}{\cos\theta_\eta} \sin\theta_\eta \right] \cos\theta_\eta \\ &= \left[|\zeta| - \frac{|\psi(\zeta)|}{\cos\theta_\eta} \sin\theta_\eta \right] \cos\theta_\eta \\ &\ge \left[\delta - \frac{R_k - \sqrt{R_k^2 - \delta^2}}{\cos\theta_\eta} \sin\theta_\eta \right] \cos\theta_\eta \\ &= \delta\cos\theta_\eta - \left(R_k - \sqrt{R_k^2 - \delta^2} \right) \sin\theta_\eta, \end{aligned} \tag{4.21}$$

where we recall that θ_η is the angle between the vector η and the direction $x_\eta/|x_\eta|$. Using estimate (4.14), we get

$$p_\delta \ge \delta \frac{|x_{1,b_\eta}|}{|x_{2,b_\eta}|} - \left(R - \sqrt{R^2 - \delta^2} \right) \sqrt{1 - \frac{|x_{1,b_\eta}|^2}{|x_{2,b_\eta}|^2}} > \frac{\delta}{2} > 2R \tag{4.22}$$

for k sufficiently large, thanks to (4.19).

We now choose k such that also $|x_{1,b_\eta}| > 4R$ holds true. We recall that the curvature of $\partial\Omega_{b_\eta}$ in its points $T(\zeta,\psi(\zeta))$ for $|\zeta| \le \delta$ is less than $1/R$. This property together with estimate (4.22) implies that the convex envelope between $B(0,|x_{1,b_\eta}|)$ and the points $T(\zeta,\psi(\zeta))$ for $|\zeta| \le \delta$ contains in its interior the ball $B(x_\eta - R\eta, R)$. Since $\bar\Omega_{b_\eta}$ contains this convex envelope, we conclude that $B(x_\eta - R\eta, R)$ is contained in Ω_{b_η} and touches $\partial\Omega_{b_\eta}$ in x_η.

Step 7. In this step we conclude the proof. It is enough to define $x_\gamma := \gamma - x_\eta$ and $c_\gamma := \phi(\gamma) - \frac{n}{\alpha} f^*(\frac{\alpha}{n}|x_\eta|)$ and to consider the sets $\Omega_{x_\gamma,c_\gamma}$ and the function l_{x_γ,c_γ} defined at the beginning of the proof. In this way, using the fact that Ω is R-uniformly convex, we have

$$\Omega \subseteq B(\gamma - R\eta, R) = \gamma - x_\eta + B(x_\eta - R\eta, R) \subseteq \gamma - x_\eta + \Omega_{b_\eta} = \Omega_{x_\gamma,c_\gamma}.$$

Indeed,

$$\begin{aligned} &\gamma - x_\eta + \Omega_{b_\eta} \\ &\quad = \Big\{x \in \mathbb{R}^n : \frac{n}{\alpha} f^*\Big(\frac{\alpha}{n}|x - \gamma + x_\eta|\Big) - z_\gamma^- \cdot (x - \gamma + x_\eta) - b_\eta < 0\Big\} \\ &\quad = \Big\{x \in \mathbb{R}^n : \frac{n}{\alpha} f^*\Big(\frac{\alpha}{n}|x - x_\gamma|\Big) + \phi(\gamma) - z_\gamma^- \cdot x_\eta - b_\eta - z_\gamma^- \cdot (x - \gamma) - \phi(\gamma) < 0\Big\} \\ &\quad = \Big\{x \in \mathbb{R}^n : \frac{n}{\alpha} f^*\Big(\frac{\alpha}{n}|x - x_\gamma|\Big) + c_\gamma - z_\gamma^- \cdot (x - \gamma) - \phi(\gamma) < 0\Big\}, \end{aligned}$$

since

$$z_\gamma^- \cdot x_\eta + b_\eta = \frac{n}{\alpha} f^*\Big(\frac{\alpha}{n}|x_\eta|\Big).$$

It is immediate to see that $\gamma \in \partial\Omega_{x_\gamma,c_\gamma} \cap \partial\Omega$. ∎

We are now ready to state the main result.

THEOREM 4.4. *Under the same assumption of Theorem* 4.3, *and the additional assumption* (G2)*, the maximum and the minimum of the minimizers of the functional*

$$\int_\Omega \big[f(|\nabla v|) + g(x,v)\big]\,dx \qquad v \in \phi + W_0^{1,1}(\Omega)$$

are Lipschitz continuous of rank $L = L(R,f,M)$.

Proof. The result is immediate by applying Theorem 2.3 with the barriers constructed in Theorem 4.3. ∎

5. Special cases and examples.

In this section we discuss some special cases and examples. In the first theorem the assumption that the boundary datum is constant allows us to drop the hypothesis (F4)/(F5) so that we are able to consider any function which is just convex and satisfies (F1)–(F3). Then, we state a geometrical assumption on the epigraph of f ensuring that whenever we have a Lipschitz minimizer all the others share the same regularity property. The special cases where f is uniformly convex at infinity or where f is C^2 are stated. At the end of the section we discuss assumptions (F4) and (F5) both providing examples of functions satisfying them and discussing their validity for C^2-functions having at least polynomial growth.

The first special case is the one in which we assume that the boundary datum is a constant function. We state the following theorem because we think that it is interesting in itself. Thanks to the fact that here we assume ϕ to be constant we can drop all the "technical" assumptions on f. Above all we underline that assumptions (F4) and (F5) are no longer needed, but not even (F2) has to be assumed.

THEOREM 5.1. *Assume that f satisfies hypotheses* (F1) *and* (F3). *Let g satisfy assumptions* (G1) *and* (G2). *Let u be the maximum or the minimum of the minimizers of the functional*

$$\int_\Omega \big[f(|\nabla v|) + g(x,v)\big]\,dx \quad v \in \phi + W_0^{1,1}(\Omega),$$

where Ω is an open bounded R-uniformly convex set and the boundary datum ϕ is a constant function. Then u is Lipschitz continuous of rank $L = L(R,f,M)$.

Proof. Since ϕ is constant, we can choose in (4.1) and (4.2) $z_\gamma^- = z_\gamma^+ = 0$ for every $\gamma \in \partial\Omega$. In particular, the radial symmetry of f^* implies that Ω_{b_η} is a ball. The conclusion immediately follows. □

The next theorem is in the same flavor of some results in [20, 21], where an extra geometrical assumption on the faces of the epigraph of f allows us to pass from the regularity of one minimizer to the regularity of all minimizers.

THEOREM 5.2. *Assume that f and g are convex functions and that the projections on $\mathbb{R}$ of the faces of the epigraph of f are bounded by a positive constant K. Then, if there exists a minimizer of the functional*

$$\int_\Omega \big[f(|\nabla v|) + g(x,v)\big]\,dx \quad v \in \phi + W_0^{1,1}(\Omega), \tag{5.1}$$

which is Lipschitz continuous of rank L, every minimizer of (5.1) *is Lipschitz continuous of rank $L+K$.*

Proof. We follow closely the argument in [20]. Let u and v be two minimizers and let A be the subset of Ω, where $|\nabla u(x)|$ and $|\nabla v(x)|$ belong to the projection of the same face of epi(f). If $|\Omega \setminus A| > 0$, the convexity of f and g implies

$$\begin{aligned}
&\int_\Omega \left[f\Big(\Big|\frac{1}{2}\nabla u(x) + \frac{1}{2}\nabla v(x)\Big|\Big) + g\Big(x, \frac{1}{2}u(x) + \frac{1}{2}v(x)\Big)\right] \mathrm{d}x \\
&\quad < \int_{\Omega\setminus A} \left[\frac{1}{2}f(|\nabla u(x)|) + \frac{1}{2}f(|\nabla v(x)|) + \frac{1}{2}g(x,u(x)) + \frac{1}{2}g(x,v(x))\right] \mathrm{d}x \\
&\qquad + \int_A \left[\frac{1}{2}f(|\nabla u(x)|) + \frac{1}{2}f(|\nabla v(x)|) + \frac{1}{2}g(x,u(x)) + \frac{1}{2}g(x,v(x))\right] \mathrm{d}x \\
&\quad = \frac{1}{2}\int_\Omega \big[f(|\nabla u(x)|) + g(x,u(x))\big]\,dx + \frac{1}{2}\int_\Omega \big[f(|\nabla v(x)|) + g(x,v(x))\big]\,dx \\
&\quad = \frac{1}{2}\int_\Omega \big[f(|\nabla u(x)|) + g(x,u(x))\big]\,dx,
\end{aligned}$$

which is a contradiction with the fact that u is a minimizer. Therefore, $|\nabla u(x)|$ and $|\nabla v(x)|$ belong to the projection of the same face of the epigraph of f for a.e. $x \in \Omega$. This implies that if there exists a Lipschitz continuous minimizer u of rank L, $u - v$ is a function in $W_0^{1,1}(\Omega)$, with $\nabla(u-v)$ bounded in L^∞ by $L+K$. Hence, v is also Lipschitz continuous of rank $L+K$. □

A very natural question concerns the possibility to construct the barriers in the case where f is uniformly convex. We observe that if f is ϵ-uniformly convex, hypothesis (F4) is automatically satisfied, with $\epsilon_k \equiv \epsilon$ for every k and $\tau_k^i := 2k+1+2(i-1)$, for instance. Therefore the existence of the barriers in this case can be obtained as a corollary of Theorem 4.3; nevertheless a simplified proof of this result can also be provided, as explained below.

THEOREM 5.3. *Assume that f is ϵ-uniformly convex on $[T,+\infty)$, for $T,\epsilon>0$, and satisfies hypotheses* (F1)–(F3). *Let g satisfy assumptions* (G1) *and* (G2). *Let u be any minimizer of the functional*

$$\int_\Omega \big[f(|\nabla v|)+g(x,v)\big]\,dx \quad v\in\phi+W_0^{1,1}(\Omega),$$

where Ω is an open bounded R-uniformly convex set and $\phi:\Omega\to\mathbb{R}$ satisfies the BSC *with rank M. Then u is Lipschitz continuous of rank $L=L(R,f,M)$.*

Proof. As in the proof of Theorem 4.4 it is sufficient to construct the lower and upper barriers $\ell^-(x)$ and $\ell^+(x)$. With the same argument used in Theorem 4.3, fixed $\gamma\in\partial\Omega$, it is enough to show the existence of x_γ, c_γ such that the set

$$\Omega_{x_\gamma,c_\gamma}:=\left\{x\in\mathbb{R}^n:\frac{n}{\alpha}f^*\Big(\frac{\alpha}{n}|x-x_\gamma|\Big)+c_\gamma-z_\gamma^-\cdot(x-\gamma)-\phi(\gamma)<0\right\}$$

contains Ω and $\gamma\in\partial\Omega_{x_\gamma,c_\gamma}\cap\partial\Omega$.

To this end, we observe that since f is ϵ-uniformly convex in $[T,+\infty)$, we have $f^*\in C^{1,1}([\sup\partial f(T),+\infty))$, thanks to Lemma 3.4. Therefore, given $b>0$ sufficiently large, we are able to estimate the principal curvatures of the set $\partial\Omega_b=\{x\in\mathbb{R}^n : f^*(|x|)-z_\gamma^-\cdot x=b\}$ in each point. We recall that $|x_{1,b}|:=\min\{|x| : x\in\partial\Omega_b\}$ and that $\lim_{b\to\infty}|x_{1,b}|=+\infty$. Since $f^{*\prime\prime}\le 1/\epsilon$ in $(\sup\partial f(T),+\infty)$ and f^* is superlinear, we can find M_R such that

$$\begin{aligned}\left\{\frac{1}{|x|}\left[(f^{*\prime})^3\Big(\frac{\alpha}{n}|x|\Big)n+(f^{*\prime})^2\Big(\frac{\alpha}{n}|x|\Big)n|z_\gamma^-|+f^{*\prime}\Big(\frac{\alpha}{n}|x|\Big)n|z_\gamma^-|^2\right]\right.&\\ \left.+\,\alpha f^{*\prime\prime}\Big(\frac{\alpha}{n}|x|\Big)|z_\gamma^-|^2\right\}\cdot\left|f^{*\prime}\Big(\frac{\alpha}{n}|x|\Big)\frac{x}{|x|}-z_\gamma^-\right|^{-3}<\frac{1}{R}&\end{aligned}\tag{5.2}$$

for every x with $|x|>M_R$. In particular (5.2) holds true for every $x\in\partial\Omega_b$ whenever $b>0$ satisfies

$$|x_{1,b}|>M_R.\tag{5.3}$$

Thanks to computations in (4.9), inequality (5.2) gives an estimate of the principal curvatures of the whole $\partial\Omega_b$.

Once we have fixed b with property (5.3), we can easily find $x\in\partial\Omega_b$ such that the outward normal to $\partial\Omega_b$ in x is parallel to a fixed normal η to Ω in γ.

The estimate on the curvatures guarantees that the ball $B(x-R\eta,R)$ is contained in Ω_b and touches $\partial\Omega_b$ in x. Therefore, we can conclude the construction of the barriers as in Step 7 of the proof of the previous theorem. As in Theorem 4.4, we deduce the Lipschitz regularity of the maximum and the minimum of the minimizers of our functional. Since f is strictly convex on $[T,+\infty)$, the assumptions of Theorem 5.2 are automatically satisfied. Therefore, we obtain that every minimizer is Lipschitz continuous. ∎

Another case of some interest by itself is, in our opinion, the one where f is of class C^2.

THEOREM 5.4. *Let f be a function of class C^2 on $(T,+\infty)$, for $T>0$, satisfying hypotheses* (F1)–(F3)*, and moreover*
(F6) $\liminf_{t\to+\infty} f''(t)t^3 = +\infty$.
Let g satisfy assumption (G1) *and* (G2)*. Let u be any minimizer of the functional*

$$\int_\Omega \big[f(|\nabla v|) + g(x,v)\big]\,dx \quad v \in \phi + W_0^{1,1}(\Omega),$$

where Ω is an open bounded R-uniformly convex set and $\phi : \Omega \to \mathbb{R}$ satisfies the BSC *with rank M. Then u is Lipschitz continuous of rank $L = L(R,f,M)$.*

Proof. As in the proof of previous theorems it is sufficient to construct the barriers. To do this, once we have fixed $\gamma \in \partial\Omega$, we need to find x_γ, c_γ such that the set

$$\Omega_{x_\gamma,c_\gamma} := \left\{x \in \mathbb{R}^n : \frac{n}{\alpha} f^*\Big(\frac{\alpha}{n}|x-x_\gamma|\Big) + c_\gamma - z_\gamma^- \cdot (x-\gamma) - \phi(\gamma) < 0\right\}$$

contains Ω and $\gamma \in \partial\Omega_{x_\gamma,c_\gamma} \cap \partial\Omega$. By assumption (F6), there exists $T' \geq T$ such that $f''(t) > 0$ for every $t > T'$. Thanks to Lemma 3.5, f^* is of class C^2 too on $(f'(T'),+\infty)$, and we have

$$f^{*\prime\prime}(\xi) = \frac{1}{f''(t)},$$

whenever $\xi = f'(t)$. Therefore, for every b sufficiently large, $\partial\Omega_b := \{x \in \mathbb{R}^n : f^*(|x|) - z_\gamma^- \cdot x = b\}$ is of class C^2 and we are able to estimate its principal curvatures. Moreover, we have

$$\begin{aligned}
&\alpha f^{*\prime\prime}\Big(\frac{\alpha}{n}|x|\Big)|z_\gamma^-|^2 \cdot \left|f^{*\prime}\Big(\frac{\alpha}{n}|x|\Big)\frac{x}{|x|} - z_\gamma^-\right|^{-3}\\
&\quad = \alpha\frac{1}{f''(t)}|z_\gamma^-|^2 \cdot \left|t\frac{x}{|x|} - z_\gamma^-\right|^{-3}
\end{aligned} \tag{5.4}$$

for $t := f^{*\prime}(\alpha|x|/n)$. Thanks to hypothesis (F6), equality (5.4) gives

$$\lim_{|x|\to+\infty} \alpha f^{*\prime\prime}\Big(\frac{\alpha}{n}|x|\Big)|z_\gamma^-|^2 \cdot \left|f^{*\prime}\Big(\frac{\alpha}{n}|x|\Big)\frac{x}{|x|} - z_\gamma^-\right|^{-3} = 0. \tag{5.5}$$

Hence, due to the superlinearity of f^*, we can find as before $M_R > 0$ such that (5.2) holds true whenever $|x| > M_R$, and we can conclude the construction of the barriers as in the proof of the previous theorem. Also in this case, since f is definitely strictly convex thanks to assumption (F6), the hypotheses of Theorem 5.2 are fulfilled and we get the Lipschitz continuity of every minimizer. ∎

In the general case, where f satisfies assumptions (F4)/(F5), we can deduce the following corollary using Theorem 5.2.

COROLLARY 5.5. *Let us assume that f and g satisfy assumptions* (F1), (F3), (G1), and (G2). *Assume moreover that the projections on $\mathbb{R}^n$ of the faces of the epigraph of f are bounded by a positive constant K. Assume that one of the following hypothesis holds true:*

(a) *ϕ is constant,*
(b) *f satisfies assumptions* (F2) *and* (F4)*,*
(c) *f satisfies assumptions* (F2) *and* (F5)*.*

Then every minimizer of the functional

$$\int_\Omega \big[f(|\nabla v|) + g(x,v)\big]\,dx \quad v \in \phi + W_0^{1,1}(\Omega) \tag{5.6}$$

is Lipschitz continuous of rank $L+K$.

Proof. The result immediately follows from Theorems 4.4 and 5.2. ∎

Remark 5.6. A wide class of elementary functions satisfy either the assumptions of Theorem 5.3 or those of Theorem 5.4. In fact, for example, the functions $f(t)=|t|^p$ and $f(t)=|t|^p\log t$ are uniformly convex in $[T,+\infty)$ for every $p\geq 2$ and satisfy assumption (F6) for every $1<p<2$. Functions with exponential growth such us $f(t)=e^t$ or $f(t)=e^{e^t}$ are uniformly convex.

In the following examples, we want to show that the more general assumptions (F4)/(F5) involved in Theorems 4.3 and 4.4 allow us to consider functions which are not uniformly convex nor C^2; in particular we can also consider functionals obtained via convexification of non convex ones.

Example 5.7. Given $\lambda>0$, let us consider the function $f\colon[0,+\infty)\to\mathbb{R}$ defined by

$$\begin{aligned}&f(0):=0,\\ &f(x):=f(k)+\lambda(k+1)(x-k)\quad\text{whenever } x\in[k,k+1],\ k\in\mathbb{N}.\end{aligned}$$

We extend f to an even function on $\mathbb{R}$. It is easy to see that f has superlinear growth and is convex. Moreover, $\partial f(k)=[k\lambda,(k+1)\lambda]$. Therefore, it is enough to choose $\tau_k:=k$ to obtain (F5).

Example 5.8. Let us observe that a convex superlinear function f which is ϵ-uniformly convex, with $\epsilon>0$, on a countable sequence of open intervals with fixed length in $[0,+\infty)$ and affine on each connected component of the complement of the union of those intervals satisfies (F4).

Example 5.9. Consider the function $f(t)=h_1^{**}(|t|)$, where

$$h_1(s)=\begin{cases} s^p & \text{if } s\in[k-\frac{1}{2k^{p-2}},k+\frac{1}{2k^{p-2}}],\ k\in\mathbb{N},\\ +\infty & \text{otherwise,}\end{cases}$$

for $p>2$. It is easy to check that f is neither C^2 nor uniformly convex but satisfies assumptions (F1)–(F4).

Example 5.10. Consider the function $f(t)=h_2^{**}(|t|)$, where

$$h_2(s)=\begin{cases} e^s & \text{if } s\in[k-\frac{1}{2e^k},k+\frac{1}{2e^k}],\ k\in\mathbb{N},\\ +\infty & \text{otherwise.}\end{cases}$$

Also in this case it is easy to check that f is neither C^2 nor uniformly convex but satisfies assumptions (F1)–(F4).

We want now to stress the fact that some of the properties required in assumption (F4) are very close to those satisfied by a quite large class of function. The next remark is devoted to this aim.

Remark 5.11. Let f be a even convex function of class C^2 such that there exist $a>0$, $b\in\mathbb{R}$ such that $f(t)\geq a|t|^p+b$ for $p\geq 2$. We want to show that f satisfies assumptions (F4)(i) and (iii) so that in this case the only effective assumption is (F4)(ii). The fact that $f(t)\geq a|t|^p+b$ implies that for every $k\in\mathbb{N}$ there exists a nonnegligible set A_k where $f''(t)>\frac{a}{2}p(p-1)|t|^{p-2}$. The continuity of f'' implies that

A_k is an open set, and hence there exists at least an interval $[\tau_k^1, \tau_k^2] \subset A_k$, where $f''(t) \geq \epsilon_k = \frac{a}{2}p(p-1)|\tau_k^1|^{p-2}$. It follows immediately that assumption (F4)(iii) is satisfied too.

The case where $1 < p < 2$ can be handled in a similar but more technical way. For this reason we just sketch it. As in the proof of Theorem 5.4, we can remark that in order to have the principal curvatures of the boundary of the set Ω_b sufficiently small, it is enough to have a slightly weaker condition than (F4)(iii); what we actually need in this case is that

$$\limsup_{t\to+\infty} f''(t)t^3\chi_{(\cup_{k=1}^{\infty}[\tau_k^1,\tau_k^2])}(t) = +\infty,$$

where χ_A denotes the characteristic function of the set $\bigcup_{k=1}^{\infty}[\tau_k^1, \tau_k^2]A$. It is easy to verify that this condition is valid in our case. Therefore, making just the extra assumption (F4)(ii), we can reproduce the argument for the construction of the barriers.

The next example emphasizes the existence of convex functions satisfying neither (F4) nor (F5).

Example 5.12. Let us consider the Cantor–Vitali function $g\colon [0,1] \to \mathbb{R}$, and let us fix $p > 1$. We define $h\colon \mathbb{R} \to \mathbb{R}$ by setting

$$h(x) := \begin{cases} g(x) & \text{if } x \in [0,1], \\ [(n+1)^p - n^p]g(x-n) + h(n) & \text{if } x \in (n, n+1] \end{cases}$$

for every $x \in \mathbb{R}$. Then we set $f(y) := \int_0^y h(x)\,\mathrm{d}x$ for every $y \in [0,+\infty)$ and we extend f to an even function on $\mathbb{R}$. The convexity of f is straightforward from the continuity and the monotonicity of its derivative. It is also immediate to see that f has at least p-growth at infinity; in fact, for any $x \in (n, n+1]$, $f(x) > f(n) = \frac{1}{2}n^p$. Moreover $f'' = h' = 0$ almost everywhere in $\mathbb{R}$. Hence, f is nowhere uniformly convex and does not satisfy (F5) because f' is continuous.

Acknowledgment. The authors wish to thank the referees for interesting comments on the first version of the paper.

REFERENCES

[1] P. Bousquet and F. Clarke, *Local Lipschitz continuity of solutions to a problem in the calculus of variations*, J. Differential Equations, 243 (2007), pp. 489–503.

[2] P. Bousquet, *Boundary continuity of solutions to a basic problem in the calculus of variations*, Adv. Calc. Var., 3 (2010), pp. 1–27.

[3] P. Bousquet, *Continuity of solutions of a problem in the calculus of variations*, to appear.

[4] A. Cellina, C. Mariconda, and G. Treu, *Comparison results without strict convexity*, Discrete Contin. Dyn. Syst. Ser. B, 11 (2009), pp. 57–65.

[5] A. Cellina and M. Vornicescu, *On the existence of solutions to a special variational problem*, Calc. Var. Partial Differential Equations, 35 (2009), pp. 263–270.

[6] A. Cellina, *On the bounded slope condition and the validity of the Euler Lagrange equation*, SIAM J. Control Optim., 40 (2001/02), pp. 1270–1279.

[7] A. Cellina, *Comparison results and estimates on the gradient without strict convexity*, SIAM J. Control Optim., 46 (2007), pp. 738–749.

[8] A. Cellina, *Uniqueness and comparison results for functionals depending on ∇u and on u*, SIAM J. Optim., 18 (2007), pp. 711–716.

[9] F. Clarke, *Continuity of solutions to a basic problem in the calculus of variations*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 4 (2005), pp. 511–530.

[10] I. Ekeland and R. Témam, *Convex Analysis and Variational Problems*, Classics Appl. Math. 28, SIAM, Philadelphia, 1999.

[11] L. Esposito, F. Leonetti, and G. Mingione, *Sharp regularity for functionals with* (p, q) *growth*, J. Differential Equations, 204 (2004), pp. 5–55.
[12] M. Giaquinta and E. Giusti, *Differentiability of minima of nondifferentiable functionals*, Invent. Math., 72 (1983), pp. 285–298.
[13] E. Giusti, *Direct Methods in the Calculus of Variations*, World Scientific, River Edge, NJ, 2003.
[14] A. Haar, *Über das Plateausche Problem*, Math. Ann., 97 (1927), pp. 124–158.
[15] P. Hartman and G. Stampacchia, *On some non-linear elliptic differential-functional equations*, Acta Math., 115 (1966), pp. 271–310.
[16] P. Hartman, *On the bounded slope condition*, Pacific J. Math., 18 (1966), pp. 495–511.
[17] D. Hilbert, *Über das Dirichletsche Prinzip*, Math. Ann., 59 (1904), pp. 161–186.
[18] J. Kristensen and G. Mingione, *The singular set of minima of integral functionals*, Arch. Ration. Mech. Anal., 180 (2006), pp. 331–398.
[19] C. Mariconda and G. Treu, *Lipschitz regularity for minima without strict convexity of the Lagrangian*, J. Differential Equations, 243 (2007), pp. 388–413.
[20] C. Mariconda and G. Treu, *Local Lipschitz regularity of minima for a scalar problem of the calculus of variations*, Commun. Contemp. Math., 10 (2008), pp. 1129–1149.
[21] C. Mariconda and G. Treu, *Hölder regularity for a classical problem of the calculus of variations*, Adv. Calc. Var., 2 (2009), pp. 311–320.
[22] C. Mariconda and G. Treu, *A Haar-Rado type theorem for minimizers in Sobolev spaces*, ESAIM Control. Optim. Calc. Var., DOI: 10.1051/cocv/2010038.
[23] G. Mingione, *Regularity of minima: An invitation to the dark side of the calculus of variations*, Appl. Math., 51 (2006), pp. 355–426.
[24] M. Miranda, *Un teorema di esistenza e unicità per il problema dell'area minima in n variabili*, Ann. Scuola Norm. Sup. Pisa (3), 19 (1965), pp. 233–249.
[25] G. Stampacchia, *On some regular multiple integral problems in the calculus of variations*, Comm. Pure Appl. Math., 16 (1963), pp. 383–421.