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The Group of Autoprojectivities of the Finite Coxeter Groups

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INTRODUCTION

Let G be a group. We denote by $P(G)$ the group autoprojectives of G, that is, the group of automorphisms of the subgroup lattice $l(G)$ of G. In [1] we studied the group $P(W)$ for a finite irreducible Coxeter group W. The purpose of the present paper is to relax the irreducibility condition, that is, to give a description of the autoprojectivities of W for any finite Coxeter group W.

In [1] we showed that the natural homomorphism $\operatorname{Aut} W \to P(W)$ is injective and, after identifying $\operatorname{Aut} W$ with its image $PA(W)$ in $P(W)$, that the group $P(W)$ is a product of two permutable subgroups, $P(W) = R \operatorname{Aut} W$, with $R \cap \operatorname{Aut} W = 1$.

The present paper is divided into three sections. In Section 1 we give the notion of exceptional prime for a finite Coxeter group W (cf. Definition 1.4) and show that if W has no exceptional primes then $R = \{1\}$ (Proposition 1.7). In Section 2 we obtain our main result: if W is a finite Coxeter group, then every autoprojectivity of W is induced by a (unique) automorphism if and only if W has no exceptional primes (Theorem 2.16). We therefore obtain the complete list of the finite Coxeter groups for which $P(W) = \operatorname{Aut} W$:

(1) W cyclic of order 2,

(2) W dihedral of order $2n$, with $n = 2, 4, 6$, or 12,

(3) W irreducible of rank at least 3,

(4) W reducible with no exceptional primes.

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Finally we give the list of the finite Coxeter groups which are *strongly lattice determined*:

(a) W dihedral of order $2n$, with $n = 2, 4, 6$, or 12,

(b) W irreducible of rank at least 3,

(c) W reducible with no exceptional primes.

In Section 3 we determine the structure of the group R in presence of exceptional primes.

Our notation is standard, relying essentially on [1, 5]. If $X \leq Y \leq G$, $[Y/X]$ denotes the relative subgroup interval. If $X \trianglelefteq Y$, we identify $[Y/X]$ and $l(Y/X)$. p always denotes a prime number. R_δ is the ring $\mathbb{Z}/p^\delta\mathbb{Z}$. $\mathscr{U}(A)$ is the group of units of the ring A. Sym X denotes the group of permutations on a set X; S_k is the symmetric group on k objects.

$PR(X)$ is the group of automorphisms of the partially ordered set of cosets $S(X)$ of the group X.

All groups are assumed to be finite.

1. COXETER GROUPS

Let W be a (finite) Coxeter group with given Coxeter generating set S (for the definitions cf. [1, 5]). The pair (W, S) is called a Coxeter system. To (W, S) there is associated the Coxeter graph, and (W, S) is *irreducible* if the Coxeter graph is connected, *reducible* otherwise. In general let $S_1, \dots, S_t$ be the subsets of S corresponding to the connected components of the Coxeter graph. Then W is the direct product of the parabolic subgroups W_{S_i} and each Coxeter system (W_{S_i}, S_i) is irreducible. The W_{S_i}'s are the *irreducible components* of W.

We fix a Coxeter group W, with Coxeter generating set S, and consider W as a finite reflection group by means of the geometric representation of W. We get the root system Φ, and a fixed simple system $\{\alpha_1, \dots, \alpha_n\}$, so that each $s_i \in S$ is the reflection relative to the vector α_i. By 1.1 in [1], the natural homomorphism $\operatorname{Aut} W \to P(W)$ is injective. This allows us to identify $\operatorname{Aut} W$ with its image $PA(W)$ in $P(W)$ to obtain $\operatorname{Aut} W \leq I(W) \leq P(W)$, where $I(W)$ denotes the group of index-preserving autoprojectivities of W. We introduced the group

$$R_S(W) = \{\varphi \in I(W) \mid \langle s \rangle^\varphi = \langle s \rangle \text{ for every } s \in S\}$$

and proved that

$$I(W) = R_S(W) \operatorname{Aut} W, \qquad R_S(W) \cap \operatorname{Aut} W = \{1\}.$$

The complement $R_S(W)$ of $\operatorname{Aut} W$ in $I(W)$ depends on the chosen Coxeter generating set. However, we shall always keep S fixed and write $R(W)$ for $R_S(W)$.

We recall some results from [1]. Every autoprojectivity of W is index-preserving if and only if W is not dihedral of order $2p$, p an odd prime (Proposition 1.9). In particular we get

$$P(W) = I(W) = R(W) \operatorname{Aut} W \tag{1.1}$$

if W is reducible. We shall call $R(W)$ the group of *exceptional autoprojectivities* of W.

(1.2) Let W be irreducible. Then $R(W) \neq \{1\}$ if and only if W is dihedral of order $2n$, with $n \neq 2, 4, 3, 6,$ or 12 (Theorem 4.6).

From now on we assume W is reducible. Then $S = S_1 \cup \cdots \cup S_t$, $t \geq 2$, $W = W_1 \times \cdots \times W_t$, where the $W_i := W_{S_i}$ are the irreducible components of W. Let γ be in $R(W)$ and let i be in $\{1, \dots, t\}$. Then γ fixes W_i and it induces the autoprojectivity γ_i of W_i which, by definition, lies in $R(W_i)$ (which by our convention is $R_{S_i}(W_i)$). We put

$$\pi \colon R(W) \to R(W_1) \times \cdots \times R(W_t), \qquad \gamma \mapsto (\gamma_1, \dots, \gamma_t).$$

PROPOSITION 1.1. *π is injective.*

Proof. Assume $\gamma \in \ker \pi$. Let Φ_i be the set of roots of W_i. In particular we get $\langle s_\alpha \rangle^{\gamma_i} = \langle s_\alpha \rangle$ for every $\alpha \in \Phi_i$. But then we have $\langle s_\alpha \rangle^{\gamma} = \langle s_\alpha \rangle$ for every $\alpha \in \phi$ since $\Phi = \Phi_1 \cup \cdots \cup \Phi_t$. Hence $\gamma = 1$ by 1.5 in [1]. ∎

By 1.1 and (1.2) we are left to study the case when at least one of the W_i's is dihedral. For this purpose, we recall the structure of the group of index-preserving autoprojectivities of dihedral groups [1, Sect. 3] and give an alternative way to describe this group as a permutation group.

Let D_{2n} be the dihedral group of order $2n$, $D_{2n} = \langle \sigma, \rho \rangle$, with ρ of order n. We consider the Coxeter generating set $\{\sigma, \sigma\rho\}$. Let k be in $\mathbb{Z}$, $k \geq 2$. We put

$$T_k = \left\{ \delta \in S_k \mid L \in S(\mathbb{Z}/k\mathbb{Z}) \quad \Leftrightarrow \quad L^\delta \in S(\mathbb{Z}/k\mathbb{Z}) \right\}.$$

Therefore T_k is isomorphic to the group $PR(\mathbb{Z}/k\mathbb{Z})$. There is a monomorphism $\Delta_n \colon T_n \to I(D_{2n})$ such that for $c \in T_n$, $c\Delta_n$ is the unique autoprojectivity of D_{2n} such that $\langle \sigma\rho^a \rangle^{c\Delta_n} = \langle \sigma\rho^{ac} \rangle$ for every $a \in \mathbb{Z}/n\mathbb{Z}$. Δ_n is an isomorphism if $n \neq 2$ [1, 3.4]. If $n = p_1^{m_1} \cdots p_r^{m_r}$ for distinct primes

$p_1, \ldots, p_r$, then $T_n \cong T_{p_1^{m_1}} \times \cdots \times T_{p_r^{m_r}}$, and each T_{p^m} is a permutational wreath product. We put

$$\Gamma_k = \{ \delta \in T_k \mid 0\delta = 0, 1\delta = 1 \}.$$

Then restricting Δ_k to Γ_k gives rise to an isomorphism from Γ_n to $R(D_{2n})$. This last isomorphism holds also for $n = 2$.

From the above discussion we get

$$(1.3) \qquad \Gamma_n = 1 \quad \Leftrightarrow \quad n = 2, 4, 3, 6, \text{ or } 12.$$

Remark 1.1. For each $i = 1, \ldots, t$, let $\mathscr{D}_i = \{H_{i,0}, \ldots, H_{i, p_i^{m_i}-1}\}$ be the set of dihedral subgroups of W of order $2n/p_i^{m_i}$. We choose the notation so that $H_{i,k} = \langle \rho^{p_i^{m_i}}, \sigma\rho^k \rangle$. If φ is in $I(D_{2n})$, then φ induces a permutation on each set $\mathscr{D}_i$. The homomorphism of $I(D_{2n})$ into $S_{p_1^{m_1}} \times \cdots \times S_{p_r^{m_r}}$ we get is clearly injective. Let $I_i(D_{2n})$ be the subgroup of elements in $I(D_{2n})$ fixing every dihedral subgroup of order $2n/p_j^{m_j}$, $j \neq i$. Then $I_i(D_{2n})$ is isomorphic to the group of permutations of $\mathscr{D}_i$ induced by $I(D_{2n})$, and this group is precisely $T_{p_i^{m_i}}$. In particular $I(D_{2n}) = I_1(D_{2n}) \times \cdots \times I_t(D_{2n})$. If we denote by $R_i(D_{2n})$ the subgroup $\{\gamma \in I_i(D_{2n}) \mid H_{i,0}^{\gamma} = H_{i,0}, H_{i,1}^{\gamma} = H_{i,1}\}$, we get $R(D_{2n}) = R_1(D_{2n}) \times \cdots \times R_r(D_{2n})$, with $R_i(D_{2n}) \cong \Gamma_{p_i^{m_i}}$.

Dually, for each i let $K_i = \langle \rho^{n/p_i^{m_i}}, \sigma \rangle$. Suppose η lies in $I_i(D_{2n})$. Then η fixes K_i, and it induces the autoprojectivity $\tilde{\eta}$ of it. Since for each involution $\sigma\rho^a$ in K_i there exists a unique j such that $H_{i,j} \cap K_i = \langle \sigma\rho^a \rangle$ and, for each k, $H_{i,k} \cap K_i = \langle \sigma\rho^b \rangle$ for some b, the permutation of $\mathscr{D}_i$ induced by η is completely determined by $\tilde{\eta}$, and vice versa. Therefore the map $I_1(D_{2n}) \times \cdots \times I_t(D_{2n}) \to I(K_1) \times \cdots \times I(K_t)$ is an isomorphism. For each i let $R(K_i)$ be the group of exceptional autoprojectivities of K_i with respect to the Coxeter system $\{\sigma, \sigma\rho^b\}$ where $H_{i,1} \cap K_i = \langle \sigma\rho^b \rangle$. Then $R(D_{2n}) \to R(K_1) \times \cdots \times R(K_t)$ is an isomorphism and $R(K_i) \cong \Gamma_{p_i^{m_i}}$.

PROPOSITION 1.2. *Let $n = p^\alpha m$, $(p, m) = 1$ and let γ be in $R(D_{2n})$. If $p^\alpha \leq 4$, then γ induces the identity on $\langle \sigma, \rho^m \rangle$.*

Proof. It is equivalent to prove that $\Gamma_{p^\alpha} = 1$. This comes from (1.3). ∎

To continue the study of the reducible case in presence of dihedral components, we give some definitions.

DEFINITION 1.3. Let X be a finite group, and let p be a prime. We define $v_p(X)$ in the following way. If p is odd, $p^{v_p(X)}$ is the p-exponent of X. If $p = 2$ then $2^{v_2(X)}$ is the 2-exponent of X if X has elements of order 4. Otherwise $v_2(X)$ is 1 if X contains the Klein four group, and is 0 in the remaining case.

DEFINITION 1.4. Let W be a finite Coxeter group. Suppose W has dihedral components and write $W = D_{2n_1} \times \cdots \times D_{2n_r} \times Z$, where Z has no dihedral components. Let p be a prime, and let $p^{\alpha(p)}$ be the maximal power of p dividing at least one of the n_k's. We say that p is *exceptional* if the following conditions hold:

(i) $p^{\alpha(p)} \geq 5$;

(ii) there exists a unique i such that $p^{\alpha(p)}$ divides $i = i(p)$, and Z has no element of order $p^{\alpha(p)}$.

Our aim is to show that $R(W) = 1$ if and only if W has no exceptional primes. We shall use the following result by Schmidt concerning projectivities of products of dihedral groups.

LEMMA 1.5. *Let A, B be isomorphic dihedral groups. Then every projectivity of $A \times B$ onto a group C is induced by a unique isomorphism.*

Proof. See [6, Lemma 3]. ∎

Suppose for the moment $W = D_{2n} \times G$, G any finite Coxeter group.

LEMMA 1.6. *Let $n = p^\alpha m$, $(p, m) = 1$, and let γ be in $R(D_{2n} \times G)$. If G has elements of order p^α, then γ induces the identity on $\langle \sigma, \rho^m \rangle$.*

Proof. Let $A = D_{2n} = \langle \sigma, \rho \rangle$. If $p^\alpha \leq 4$ we are done by 1.2. So assume $p^\alpha \geq 5$. Let g be an element of order p^α in G, and let τ an involution in G such that $\tau g \tau = g^{-1}$. Let $B_0 = \langle \sigma, \rho^{p^\alpha} \rangle$, $B_1 = \langle \sigma\rho, \rho^{p^\alpha} \rangle$, and $A_0 = \langle \sigma, \rho^m \rangle$. By 1.5 applied to $A_0 \times \langle \tau, g \rangle$, there exists an automorphism of A_0 inducing γ on A_0. But A_0 is generated by two involutions fixed by this automorphism, namely the one in $A_0 \cap B_0$ and the one in $A_0 \cap B_1$, and we are done. ∎

PROPOSITION 1.7. *Let W be reducible. If W has no exceptional primes, then every autoprojectivity of W is induced by an automorphism.*

Proof. If W has no dihedral component we know that $R(W) = 1$. So assume $W = D_{2n_1} \times \cdots \times D_{2n_r} \times Z$, where Z has no dihedral components. Let $\eta \in R(W)$. We show that η is the identity on each D_{2n_i}. Let us fix i, and denote by γ the restriction of η to D_{2n_i}. Write $W = D_{2n_i} \times G$. Let q be a prime divisor of n_i, and let q^α be the maximal power of q dividing n_i. It is enough to show that γ fixes every dihedral subgroup of order $2n_i/q^\alpha$ of D_{2n_r}. If $q^\alpha \leq 4$ we are done by 1.2. So assume $q^\alpha \geq 5$. Then $q^{\alpha(q)} \geq 5$, so that there exists an element of order q^α in G, since W has no exceptional prime. We conclude by 1.6. ∎

2. THE MAIN RESULT

In this section we shall show that if W has exceptional primes, then $R(W)$ is not trivial. We introduce some notation.

DEFINITION 2.1. Let p be a prime α, β non-negative integers such that $\alpha \geq \beta$. We denote by $\Theta_{p,\alpha,\beta}$ the group of exceptional autoprojectivities of D_{2p^α} which can be extended to exceptional autoprojectivities of $D_{2p^\alpha} \times D_{2p^\beta}$.

By abuse of notation we consider dihedral also the cyclic group of order 2. We observe that, by 1.1 and Schmidt's result, the restriction map $R(D_{2p^\alpha} \times D_{2p^\beta}) \to \Theta_{p,\alpha,\beta}$ is an isomorphism and $\Theta_{p,\alpha,\alpha} = 1$.

We now assume $W = D_{2n_1} \times \cdots \times D_{2n_r} \times Z$, where Z has no dihedral component, W with exceptional primes $p_1, \ldots, p_h$. For each $i = 1, \ldots, r$, let $D_{2n_i} = \langle \sigma_i, \rho_i \rangle$. Let p be an exceptional prime, and let $i = i(p)$. We put $M_p := \langle \sigma_i, \rho_i^{n_i/p^{\alpha(p)}} \rangle$. We denote by μ_p the restriction map $R(W) \to R(M_p)$, and we put

$$\mu\colon R(W) \to R(M_{p_1}) \times \cdots \times R(M_{p_h}), \qquad \gamma \mapsto (\gamma\mu_{p_1}, \ldots, \gamma\mu_{p_h}).$$

It is clear that μ is injective. We shall determine its image.

DEFINITION 2.2. With the previous notation we put $\beta(p) = v_p(W/D_{2n_{i(p)}})$.

Note that if p is an exceptional prime, then $\beta(p) < \alpha(p)$.

PROPOSITION 2.3. *For every exceptional prime p we have* $\operatorname{Im} \mu_p \leq \Theta_{p,\alpha(p),\beta(p)}$.

Proof. Let $i = i(p)$, $\alpha = \alpha(p)$, $\beta = \beta(p)$, and write $W = D_{2n_i} \times G$. It is enough to show that there exists a dihedral subgroup D of G of order $2p^\beta$ fixed by every $\gamma \in R(W)$. For then, given $\gamma \in R(W)$, γ induces an element of $R(M_p \times D)$, so that $\gamma\mu_p \in \Theta_{p,\alpha,\beta}$.

Suppose p is odd, or $p = 2$ with $\beta \geq 2$. Let X be an irreducible component of G containing an element ρ of order p^β. We can choose an involution $\tau \in X$ such that $\tau\rho\tau = \rho^{-1}$ and such that each $\gamma \in R(W)$ induces the identity on $\langle \tau, \rho \rangle$. We take $D = \langle \tau, \rho \rangle$.

Finally suppose $p = 2$. If $\beta = 0$. Then we take $D = \langle \tau \rangle$, where τ is a Coxeter generator of W not in D_{2n_i}. If $\beta = 1$ there are two commuting involutions τ, τ' in G such that $\langle \tau, \tau' \rangle^\gamma = \langle \tau, \tau' \rangle$ for every $\gamma \in R(W)$. In fact, if G has at least two irreducible components we take τ a simple reflection in one component and τ' a simple reflection in another. If G is irreducible, then it contains by hypothesis the Klein group V. If G is not dihedral then any copy of V in G can be taken for D. If G is dihedral, it has order $4m$ with m odd. If $G = \langle \tau, \rho \rangle$, with ρ of order $2m$ and Coxeter generators τ, $\tau\rho$, we can take $\tau' = \rho^m$. ∎

In particular we get

$$\text{Im}\ \mu \leq \Theta_{p_1,\alpha(p_1),\beta(p_1)} \times \cdots \times \Theta_{p_h,\alpha(p_h),\beta(p_h)}. \tag{2.1}$$

We shall prove that in fact equality holds. We first determine some properties of $\Theta_{p,\alpha,\beta}$.

Let $\gamma \in \Theta_{p,\alpha,\beta}$. For simplicity we write $D_{2p^\alpha} = A = \langle \sigma, \varphi \rangle$, $D_{2p^\beta} = B = \langle \tau, \rho \rangle$. We still denote by γ the unique element of $R(A \times B)$ inducing γ on A, and the element of Γ_{p^α} such that $\langle \sigma\varphi^a \rangle^\gamma = \langle \sigma\varphi^{a\gamma} \rangle$ for every $a \in R_\alpha$.

LEMMA 2.4. *Let X be a dihedral group generated by the involutions x, y. If ψ is an index-preserving projectivity of X onto a group $\overline{X}$, and $\langle x \rangle^\psi = \langle \bar{x} \rangle$, $\langle y \rangle^\psi = \langle \bar{y} \rangle$, then $\langle xy \rangle^\psi = \langle \overline{xy} \rangle$.*

Proof. See [6, 7.7.1]. ∎

PROPOSITION 2.5. *Let γ be in $\Theta_{p,\alpha,\beta}$. Then we have*

$$\langle \sigma\varphi^a\tau\rho^b \rangle^\gamma = \langle \sigma\varphi^{a\gamma}\tau\rho^b \rangle \qquad \textit{and} \qquad \langle \varphi^a\rho^b \rangle^\gamma = \langle \varphi^{a\gamma}\gamma^b \rangle$$

for every $a \in R_\alpha$, $b \in R_\beta$.

Proof. By 1.5 γ induces the identity on B. From $\langle \sigma\varphi^a \rangle^\gamma = \langle \sigma\varphi^{a\gamma} \rangle$, $\langle \tau\rho^b \rangle = \langle \tau\rho^b \rangle$, and 2.4, it follows that $\langle \sigma\varphi^a\tau\rho^b \rangle^\gamma = \langle \sigma\varphi^{a\gamma}\tau\rho^b \rangle$. Moreover $\langle \sigma\tau, \varphi^a\rho^b \rangle^\gamma = \langle \sigma\tau, \sigma\varphi^a\tau\rho^b \rangle^\gamma = \langle \sigma\tau, \sigma\varphi^{a\gamma}\tau\rho^b \rangle = \langle \sigma\tau, \varphi^{a\gamma}\rho^b \rangle$, so that $\langle \varphi^a\rho^b \rangle = \langle \varphi^{a\gamma}\rho^b \rangle$. ∎

In the next proposition we establish a crucial property of the group $\Theta_{p,\alpha,\beta}$.

PROPOSITION 2.6. *Let $\gamma \in \Theta_{p,\alpha,\beta}$, $a, b \in R_\alpha$. If $a \equiv b \quad p^tR_\alpha$ for some $t \leq \alpha - \beta$, then*

$$(b-a)\gamma \equiv b\gamma - a\gamma \quad p^{t+\beta}R_\alpha.$$

Proof. Let $X = \langle \sigma\varphi^a\tau, \varphi^{b-a}\rho \rangle$. By 2.4, $\sigma\varphi^{a\gamma}\tau$, $\varphi^{(b-a)\gamma}\rho$ and $\sigma\varphi^{b\gamma}\tau\rho$ lie in X^γ. Hence $\varphi^{(b-a)\gamma + a\gamma - b\gamma} = \sigma\varphi^{b\gamma}\tau\rho\sigma\varphi^{a\gamma}\tau\varphi^{(b-a)\gamma}$ lies in $X^\gamma \cap \langle \varphi \rangle = (X \cap \langle \varphi \rangle)^\gamma = X \cap \langle \varphi \rangle = \langle (\varphi^{(b-a)}\rho)^{p^\beta} \rangle$. We get $(b-a)\gamma \equiv b\gamma - a\gamma \quad p^{t+\beta}R_\alpha$. ∎

DEFINITION 2.7. Assume $\alpha \geq \beta \geq 0$. We say that an element $\sigma \in S_{p^\alpha}$ satisfies $(*)$ if

$$(b-a)\sigma \equiv b\sigma - a\sigma \quad p^{t+\beta}R_\alpha$$

whenever $b \equiv a \quad p^tR_\alpha$ for some $t \leq \alpha - \beta$.

For every prime p and every pair of non-negative integers (α, β) with $\alpha \geq \beta$ we introduce the group

$$\Gamma_{p,\alpha,\beta} = \{\varepsilon \in \Gamma_{p^\alpha} \mid \varepsilon \text{ satisfies } (*)\}. \tag{2.2}$$

It is clear that

$$\Gamma_{p^\alpha} = \Gamma_{p,\alpha,0} \geq \Gamma_{p,\alpha,1} \geq \cdots \geq \Gamma_{p,\alpha,\alpha} = 1. \tag{2.3}$$

In the special case $p = 2$ we have

PROPOSITION 2.8. $\Gamma_{2,\alpha,1} = \Gamma_{2,\alpha,0}$.

Proof. Let $\gamma \in \Gamma_{2^\alpha}$. Let $a, b \in R_\alpha$ be such that $a \equiv b \quad 2^t R_\alpha$ for some $t \leq \alpha - 1$, but $a \neq b \quad 2^{t+1} R_\alpha$. Since γ is an automorphism of $S(R_\alpha)$ fixing every subgroup, there exist odd integers k, k' such that $(b-a)\gamma = k 2^t$ and $b\gamma = a\gamma + k' 2^t$. Then $(b-a)\gamma \equiv b\gamma - a\gamma \quad 2^{t+1} R_\alpha$. ▮

From 2.4 we get $\Theta_{p,\alpha,\beta} \leq \Gamma_{p,\alpha,\beta}^{\Delta_{p^\alpha}}$. Our aim is to prove that equality holds. This will be a corollary of a more general result. Suppose $D = D_{2p^\alpha m} = \langle \sigma, \varphi \rangle$ with $\alpha \geq 1$, $(p, m) = 1$, and let $A = \langle \sigma, \varphi^m \rangle$. Then for every $\varepsilon \in \Gamma_{p^\alpha}$ there exists a unique element $\gamma_\varepsilon \in R(D)$ fixing the dihedral subgroups of order $2p^\alpha$ of D and inducing ε on A. We call γ_ε the element of $\Gamma_{p^\alpha m}$ induced by ε.

LEMMA 2.9. *Let $\varepsilon \in \Gamma_{p,\alpha,\beta}$, and let γ_ε be the element of $\Gamma_{p^a m}$ induced by ε. Then we have*

$$(b-a)\gamma_\varepsilon \equiv b\gamma_\varepsilon - a\gamma_\varepsilon \quad p^{t+\beta} m\mathbb{Z}/p^\alpha m\mathbb{Z}$$

if $a \equiv b \quad p^t\mathbb{Z}/p^\alpha m\mathbb{Z}$ for some $t \leq \beta - \alpha$.

Proof. Straightforward. ▮

The next proposition is the key step in our construction.

PROPOSITION 2.10. *Let $W = D \times G$, where $D = D_{2p^\alpha m}$, $\alpha \geq 1$, $(p, m) = 1$, and G is a Coxeter group with $v_p(G) = \beta < \alpha$. Let $\varepsilon \in \Gamma_{p,\alpha,\beta}$. Then γ_ε can be uniquely extended to an element of $R(W)$ inducing the identity on G.*

Proof. We write $\gamma = \gamma_\varepsilon$, $D = \langle \sigma, \varphi \rangle$. We define a bijection $\omega: W \to W$ by

$$(\varphi^a g)\omega = \varphi^{a\gamma} g, \qquad (\sigma\varphi^a g)\omega = \sigma\varphi^{a\gamma} g$$

for every $a \in \mathbb{Z}/p^\alpha m\mathbb{Z}$, $g \in G$.

We prove that ω induces an autoprojectivity of W. Let $X \leq W$. We have to show that $X^\omega \leq W$. Now $1 = \varphi^0 = \varphi^{0\gamma}$ is in X^ω. To conclude we have to consider various cases. We first prove a lemma.

LEMMA 2.11. *Let* $d \in p^t\mathbb{Z} \setminus p^{t+1}\mathbb{Z}$ *for some* $t < \alpha - \beta$. *If* $\varphi^d g$ *is in* X *for some* $g \in G$, *then* $\varphi^{mp^{t+\beta}}$ *lies in* X *unless* $p = 2$ *and* $\beta = 0$. *In this case* $\varphi^{m2^{t+1}}$ *lies in* X.

Proof. It is clear that there exists a p-element $x \in G$ such that $\varphi^{mp^t}x$ lies in X. If p is odd, or if $p = 2$ and $\beta \geq 1$, then $v_p(G) = \beta$ means that $\varphi^{mp^{t+\beta}}$ is in X. Finally suppose $p = 2$ and $\beta = 0$. Then $\varphi^{m2^{t+1}} = (\varphi^{m2^t}x)^2$ is in X. ∎

We can now complete the proof of 2.10. Let x, y be in X^ω. We prove that $x^{-1}y \in X^\omega$. We have to consider four cases.

(a_1) $(x, y) = (\varphi^{a\gamma}g, \varphi^{b\gamma}g')$.
We show there exists an element c such that $\varphi^c g^{-1}g' \in X$ and $c\gamma = b\gamma - a\gamma$. Let $\langle \varphi^{b-a} \rangle = \langle \varphi^{m'p^t} \rangle$, with $m' \mid m$. If $t \geq \alpha - \beta$ we get $(b - a)\gamma = b\gamma - a\gamma$ and we can take $c = b - a$. Now assume $t < \alpha - \beta$, and let $b\gamma - a\gamma = (b - a)\gamma + smp^{t+\beta}$. Suppose p is odd, or $p = 2$ and $\beta \geq 1$. Let s' be such that $(b - a + s'mp^{t+\beta})\gamma = (b - a)\gamma + msp^{t+\beta}$, and let $c = b - a + s'mp^{t+\beta}$. Then $c\gamma = b\gamma - a\gamma$, and $\varphi^c g^{-1}g' = \varphi^{b-a}g^{-1}g'\varphi^{s'mp^{t+\beta}} \in X$ by 2.11.

If $p = 2$ and $\beta = 0$, then $\varepsilon \in \Gamma_{2, \alpha, 1}$ by 2.8, so that $b\gamma - a\gamma = (b - a)\gamma + sm2^{t+1}$ for some s. Let s' be such that $(b - a + s'm2^{t+1})\gamma = (b - a)\gamma + ms2^{t+1}$, and let $c = b - a + s'm2^{t+1}$. Then $c\gamma = b\gamma - a\gamma$, and $\varphi^c g^{-1}g' = \varphi^{b-a}g^{-1}g'\varphi^{s'm2^{t+1}} \in X$ by 2.11 and we are done.

(a_2) $(x, y) = (\sigma\varphi^{a\gamma}g, \varphi^{b\gamma}g')$.
We show there exists an element c such that $\sigma\varphi^c g^{-1}g' \in X$ and $c\gamma = a\gamma + b\gamma$. Let $\langle \varphi^b \rangle = \langle \varphi^{m'p^t} \rangle$, with $m' \mid m$. If $t \geq \alpha - \beta$ we can take $c = a + b$. Now assume $t < \alpha - \beta$. Let s be such that $a\gamma + b\gamma = (a + b)\gamma + smp^{t+\beta}$. Then we conclude as in case (a_1).

The remaining two cases are dealt with in a similar way. Note that the same procedure applies to ε^{-1}, so that we have proved that ω induces an autoprojectivity, that we still call ω, of W. It is clear from the definition that ω induces the identity on G and that it lies in $R(W)$. Uniqueness follows from the fact that any exceptional autoprojectivity of W is determined by its action on D and G.

DEFINITION 2.12. With the previous notation, we denote by ι_p the monomorphism $\Gamma_{p, \alpha, \beta} \to R(W)$ sending an element ε of $\Gamma_{p, \alpha, \beta}$ to the unique element of $R(W)$ inducing γ_ε on D and the identity on G.

PROPOSITION 2.13. $\Theta_{p, \alpha, \beta} \cong \Gamma_{p, \alpha, \beta}$.

Proof. We already know that $\Theta_{p, \alpha, \beta} \leq \Gamma^{\Delta_{p^\alpha}}_{p, \alpha, \beta}$. On the other hand, if we take $W = D_{2p^\alpha} \times D_{2p^\beta}$, then given $\varepsilon \in \Gamma_{p, \alpha, \beta}$ we get $\varepsilon\Delta_{p^\alpha} = \varepsilon\iota_p\mu_p \in \Theta_{p, \alpha, \beta}$. Hence $\Theta_{p, \alpha, \beta} = \Gamma^{\Delta_{p^\alpha}}_{p, \alpha, \beta}$. ∎

We can finally prove

THEOREM 2.14. *Let W be a finite reduced Coxeter group with exceptional primes $p_1, \dots, p_h$. Then $R(W)$ is isomorphic to $\Gamma_{p_1, \alpha(p_1), \beta(p_1)} \times \cdots \times \Gamma_{p_h, \alpha(p_h), \beta(p_h)}$.*

Proof. We only have to prove that $\operatorname{Im} \mu = \Gamma_{p_1, \alpha(p_1), \beta(p_1)} \times \cdots \times \Gamma_{p_h, \alpha(p_h), \beta(p_h)}$. It is enough to prove the following: let p be an exceptional prime of W, and let ε be in $\Gamma_{p, \alpha(p), \beta(p)}$. Then there exists $\gamma \in R(W)$ such that $\gamma\mu_p = \varepsilon$ and $\gamma\mu_q = 1$ for every exceptional prime q different from p. We conclude by taking $\gamma = \varepsilon\iota_p$. ∎

Theorem 2.10 gives a complete description of the group of exceptional autoprojectivities of W in terms of the groups $\Gamma_{p, \alpha(p), \beta(p)}$, whose structure we shall determine in the next section. Here we just prove that if p is exceptional, then $\Gamma_{p, \alpha(p), \beta(p)} \neq 1$.

We fix a prime p and integers $\alpha > \beta > 0$. For every $s \in \{1, \dots, \alpha - \beta\}$, $d \in \mathscr{U}(R_{\alpha - s})$ such that $d \equiv 1 \quad p^{\beta - 1}R_{\alpha - s}$, we define $\sigma_{p^s, d}$ in the following way. Let $b \in R_\alpha$, and write $b = b_0 + b_s p^s$, where b_0 is in $\{0, 1, \dots, p^s - 1\}$. We put $b\sigma_{p^s, d} = b_0 + d b_s p^s$.

PROPOSITION 2.15. *$\sigma_{p^s, d}$ lies in $\Gamma_{p, \alpha, \beta}$ and it fixes a for every $0 \leq a < p^s$.*

Proof. We write σ for $\sigma_{p^s, d}$. The fact that σ is bijective and $a\sigma = a$ for every $0 \leq a < p^s$ is clear. We have to prove that σ maps cosets to cosets and it satisfies $(*)$. Let $a, b \in R_\alpha$, $b - a = kp^t$. Write $a = a_0 + a_s p^s$, $b = b_0 + b_s p^s$, $b - a = c_0 + c_s p^s$ with $a_0, b_0, c_0 \in \{0, \dots, p^s - 1\}$. If $t \geq s$, then $a_0 = b_0$ so that $(b - a)\sigma = b\sigma - a\sigma$ and we are done.

Now assume $t < s$. Then $b\sigma - a\sigma - (b - a)\sigma = (b_0 - a_0 - c_0) + d(b_s - a_s - c_s)p^s = (d - 1)(b_s - a_s - c_s)p^s$. But $d - 1 = hp^{\beta - 1}$ for some h, so $b\sigma - a\sigma - (b - a)\sigma = h(b_s - a_s - c_s)p^{s + \beta - 1}$. Hence $b\sigma - a\sigma \equiv (b - a)\sigma p^{t + \beta} R_\alpha$, since $s + \beta - 1 \geq t + \beta$. It is also clear that $b\sigma \in a\sigma + p^t R_\alpha$. ∎

For every $\gamma \geq \beta > 0$ we put

$$C_{p, \gamma, \beta} = \left\{ c \in \mathscr{U}\left(R_\gamma\right) \mid c \equiv 1 \quad p^{\beta - 1} R_\gamma \right\}.$$

If $\beta = 1$, $C_{p, \gamma, \beta} \cong \mathscr{U}(R_\gamma)$. If $\beta \geq 2$, $C_{p, \gamma, \beta}$ has order $p^{\gamma - \beta + 1}$. It is cyclic if p is odd. If $p = 2$ then $C_{2, \gamma, 2} = \mathscr{U}(R_\gamma)$. If $\beta \geq 3$ then $C_{2, \gamma, \beta}$ is cyclic by [7, 5.7.12].

For every $s \in \{1, \dots, \alpha - \beta\}$ we put

$$K_{p^s} = \left\{ \sigma_{p^s, d} \mid d \in C_{p, \alpha - s, \beta} \right\}. \tag{2.4}$$

By 2.15 it follows that

(2.5) $\quad K_{p^s}$ is a subgroup of $\Gamma_{p,\alpha,\beta}$ isomorphic to $C_{p,\alpha-s,\beta}$.

We are in the position to prove

THEOREM 2.16. *Let W be a finite reducible Coxeter group. Then $R(W) = 1$ if and only if W has no exceptional prime.*

Proof. By 1.7, we only have to prove that if W has exceptional primes, then $R(W) \neq 1$. By 2.14 it is enough to show that if p is an exceptional prime of W, then $\Gamma_{p,\alpha(p),\beta(p)} \neq 1$. Let p be an exceptional prime of W, and let $\alpha = \alpha(p)$. We show that $\Gamma_{p,\alpha,\alpha-1} \neq 1$. Suppose first $\alpha \geq 3$. Then $\Gamma_{p,\alpha,\alpha-1} \geq C_{p,\alpha-1,\alpha-1} \cong \mathbb{Z}/p\mathbb{Z}$. If $\alpha = 2$, then $p \geq 3$, since $p^\alpha \geq 5$. Hence $\Gamma_{p,2,1} \geq C_{p,1,1} \cong \mathcal{U}(\mathbb{Z}/p\mathbb{Z}) \neq 1$. If $\alpha = 1$, then $p \geq 5$. Hence $\Gamma_{p,1,0} = \Gamma_p \neq 1$ by (1.3), and we are done. ∎

From the results obtained in [1] in the irreducible case, we get

THEOREM 2.17. *Let W be a finite Coxeter group. Then $P(W) = \mathrm{Aut}\, W$ if and only if W is in the following list*:

(1) *W cyclic of order* 2,

(2) *W dihedral of order $2n$, with $n = 2, 4, 6,$ or* 12,

(3) *W irreducible of rank at least* 3,

(4) *W reducible with no exceptional primes.*

We recall that a group G is said to be *strongly lattice determined* if every projectivity of G onto a group $\overline{G}$ is induced by an isomorphism. Taking into account the results of Uzawa [8] and [1, 4.8], we get

THEOREM 2.18. *Let W be a finite Coxeter group. Then W is strongly lattice determined if and only if W is in the following list*:

(a) *W dihedral of order $2n$, with $n = 2, 4, 6,$ or* 12,

(b) *W irreducible of rank at least* 3,

(c) *W reducible with no exceptional primes.*

3. THE STRUCTURE OF $R(W)$

In this section we take a closer look at the group $R(W)$ in presence of exceptional primes. By 2.14 this amounts to determine the structure of $\Gamma_{p,\alpha,\beta}$. Since for $\beta = 0$, $\Gamma_{p,\alpha,\beta} = \Gamma_{p^\alpha}$ is the stabilizer of 0 and 1 in T_{p^α} which is a permutational wreath product, in our discussion we assume $\beta > 0$.

This kind of problem is somehow similar to a problem studied in [2, 3] in order to determine the group of autoprojectivities of periodic modular groups. In that context we introduced the group (in [3, Sect. 2] called S),

$$S_{p,\alpha,\beta} = \{\gamma \in \operatorname{Sym} R_\alpha \mid i\gamma \equiv i \quad p^\beta R_\alpha \text{ for every } i \in R_\alpha \text{ and}$$
$$b\gamma - a\gamma \equiv b - a \quad p^{t+\beta} R_\alpha \text{ if } b \equiv a \quad p^t R_\alpha \text{ for some } t \leq \alpha - \beta\},$$

where $\alpha > \beta \geq 1$.

We start the investigation of $\Gamma_{p,\alpha,\beta}$. We fix the prime p, and integers α, β such that $\alpha > \beta \geq 1$. We put

$$\text{(3.1)} \qquad A = D_{2p^\alpha} = \langle \sigma, \varphi \rangle, \qquad B = D_{2p^\beta} = \langle \tau, \rho \rangle, \qquad W = A \times B.$$

We know that $R(W) \cong \Theta_{p,\alpha,\beta} \cong \Gamma_{p,\alpha,\beta}$. As usual, we identify these groups.

Suppose $\gamma \in P(W)$. Then γ induces the autoprojectivity $\bar{\gamma}$ of $\overline{W} = W/\langle \varphi^{p^\beta} \rangle$. It is clear that if γ lies in $R(W)$, then $\bar{\gamma}$ lies in $R(W/\langle \varphi^\beta \rangle)$, so that $\bar{\gamma} = 1$. Hence $\langle \sigma\varphi^i \rangle^\gamma \leq \langle \sigma\varphi^i, \epsilon^{p^\beta} \rangle$; that is,

$$\text{(3.2)} \qquad i\gamma \equiv i \quad p^\beta R_\alpha \qquad \text{for every } \gamma \in \Gamma_{p,\alpha,\beta} \text{ and every } i \in R_\alpha.$$

For our discussion it is convenient to introduce the subgroup

$$\text{(3.3)} \qquad K_{p,\alpha,\beta} = \{\gamma \in T_{p^\alpha} \mid \gamma \text{ satisfies } (*), \text{ and } i\sigma \equiv i \quad p^\beta R_\alpha \text{ for every } i \in R_\alpha\}.$$

Hence $\Gamma_{p,\alpha,\beta}$ is the stabilizer of 1 in $K_{p,\alpha,\beta}$. Note that $0\gamma = 0$ for every $\gamma \in K_{p,\alpha,\beta}$, and $K_{p,\beta,\beta} = 1$. In fact $K_{p,\alpha,\beta}$ corresponds to the subgroup

$$K(W) = \{\gamma \in P(W) \mid A^\gamma = A, B^\gamma = B, \bar{\gamma} = 1 \text{ and } \langle \sigma \rangle^\gamma = \langle \sigma \rangle\}.$$

LEMMA 3.1. *Suppose σ in* Sym R_α *satisfies* $(*)$ *and $i\sigma \equiv i \quad p^\beta R_\alpha$ for every $i \in R_\alpha$. Then*

(a) *if $a_1, \ldots, a_r \in p^t R_\alpha$ for some $t \leq \alpha - \beta$, then $(a_1 + \cdots + a_r)\sigma \equiv a_1\sigma + \cdots + a_r\sigma \quad p^{\beta+t} R_\alpha$;*

(b) *$p^s\sigma \equiv p^s \quad p^{\beta+s-1} R_\alpha$ for every $s = 1, \ldots, \alpha - \beta + 1$;*

(c) *if a, b in R_α are such that $a \equiv b \quad p^{\alpha-\beta+1} R_\alpha$, then $b\sigma - a\sigma = b - a$.*

Proof. (a) Follows by induction and the fact that $a \in p^t R_a \Rightarrow (-a)\sigma \equiv -a\sigma \quad p^{\beta+t} R_\alpha$.

(b) True for $s = 1$. Assume the result for $s < \alpha - \beta + 1$. Then $(p^s)\sigma \equiv p^s \quad p^{\beta+s-1}R_\alpha$. But $(p^{s+1})\sigma \equiv p(p^s\sigma) \quad p^{\beta+s}R_\alpha$ by a), so that $(p^{s+1})\sigma \equiv p^{s+1}\sigma \quad p^{\beta+s}R_\alpha$.

(c) By (b) we have $p^{\alpha-\beta+1}\sigma = p^{\alpha-\beta+1}$, and by induction we get $(kp^{\alpha-\beta+1})\sigma = kp^{\alpha-\beta+1}$ for every k. Since $a \equiv b \quad p^{\alpha-\beta}R_\alpha$ we get $b\sigma - a\sigma = (b-a)\sigma$. But $b - a = kp^{\alpha-\beta+1}$, so that $(b-a)\sigma = b - a$. ∎

LEMMA 3.2. *Suppose σ in* Sym R_α *satisfies* $(*)$ *and* $i\sigma \equiv i \quad p^\beta R_\alpha$ *for every* $i \in R_\alpha$. *Then σ lies in* $PR(R_\alpha)$.

Proof. We have to prove that σ maps cosets to cosets. Since σ is invertible, it is enough to show that $(x + H)\sigma \subseteq x\sigma + H$ for every coset $x + H$ of R_α. By 3.1 we have $(kp^t)\sigma \equiv kp^t \quad p^tR_\alpha$ for every $0 \le t \le \alpha$ and every k, since $\beta \ge 1$. Hence $(p^tR_\alpha)\sigma \subseteq p^tR_\alpha$.

Now let $a, b \in R_\alpha$, $b - a = kp^t$. If $t \ge p^{\alpha-\beta+1}$ we get $b\sigma - a\sigma = b - a$ by 3.1(c) so that $b\sigma = a\sigma + kp^t \in a\sigma + p^tR_\alpha$, and $(a + p^tR_\alpha)\sigma \subseteq a\sigma + p^tR_\alpha$. If $t \le \alpha - \beta$, then $b\sigma - a\sigma \equiv (kp^t)\sigma \quad p^{t+\beta}R_\alpha$. But $(kp^t)\sigma \equiv kp^t \quad p^tR_\alpha$, so that $b\sigma - a\sigma \equiv kp^t \quad p^tR_\alpha$. Hence $b\sigma \in a\sigma + p^tR_\alpha$, and we are done. ∎

We begin by considering the case $\alpha = \beta + 1$.

PROPOSITION 3.3. *We have*

$$K_{p,\beta+1,\beta} \cong \begin{cases} PQ \rhd P,\ P \text{ an elementary abelian group of order } p^{p-1}, \\ Q = \langle \alpha \rangle,\ \alpha \text{ a power automorphism of order } p-1 \text{ of } P, \\ \qquad \text{if } \beta = 1 \\ P \text{ an elementary abelian group of order } p^p, \\ \qquad \text{if } \beta \ge 2. \end{cases}$$

Proof. Let $\gamma \in K_{p,\beta+1,\beta}$. Then γ acts trivially on the set of the dihedral subgroups $A_0 = \langle \sigma, \varphi^p \rangle, \dots, A_{p-1} = \langle \sigma\varphi^{p-1}, \varphi^p \rangle$ of order $2p^\beta$ of A, and it induces an automorphism $(\alpha_i, 1)$ on each product $A_i \times B$. Therefore, for each $i \in \{0, \dots, p-1\}$ there exist a unique $d_i \in R_1$ and a unique $c_i \in \mathscr{U}(R_\beta)$ such that $(i + kp)\gamma = i + d_ip^\beta + c_ikp$ for every $k \in \mathbb{Z}$. Since γ fixes 0, we have $d_0 = 0$. Moreover, since φ^p lies in each A_i, we must have $c_i = c_j$ for every i, j. Call c this common value: since $p\gamma \equiv p \quad p^\beta R_\alpha$, we get $c \equiv 1 \quad p^{\beta-1}R_\alpha$.

On the other hand, given $d_0, \dots, d_{p-1} \in R_1$ such that $d_0 = 0$, and $c \in \mathscr{U}(R_{p^\beta})$, $c \equiv 1 \quad p^{\beta-1}R_\alpha$, it is clear that the map given by

$$(i + kp)\gamma = i + d_ip^\beta + ckp$$

for every $i \in \{0, \dots, p-1\}$, $k \in \mathbb{Z}$, is in $K_{p,\beta+1,\beta}$. The structure of $K_{p,\beta+1,\beta}$ follows easily. ∎

COROLLARY 3.4. *We have*

$$\Gamma_{p,s+1,s} \cong \begin{cases} PQ \rhd P,\ P \text{ an elementary abelian group of order } p^{p-2}, \\ Q = \langle \alpha \rangle,\ \alpha \text{ a power automorphism of order } p-1 \text{ of } P, \\ \qquad \text{if } s = 1 \\ P \text{ an elementary abelian group of order } p^{p-1}, \\ \qquad \text{if } s \geq 2 \end{cases}$$

Proof. In the proof of 3.3, if $\gamma \in K_{p,s+1,s}$ corresponds to $(d_0, \dots, d_{p-1}, c)$, then $\gamma \in \Gamma_{p,s+1,s} \Leftrightarrow d_0 = d_1 = 0$. ∎

To deal with the general case, we introduce certain elements of $K_{p,\alpha,\beta}$. In Section 2 we defined $\sigma_{p^s,d}$ for every $s \in \{1, \dots, \alpha - \beta\}$, $d \in C_{p,\alpha-s,\beta}$ and the groups K_{p^s}. Now we consider, with a minor change of notation, the permutations $\sigma_{\xi,z,t}$ introduced in [3]. We recall their definition.

DEFINITION 3.5. For $\xi \in R_\alpha$, t such that $0 \leq t < \alpha - \beta$, $z \in p^t R_{\alpha-\beta}$, set

$$i\sigma_{\xi,z,t} = \begin{cases} i & \text{if } i \notin \xi + p^{t+1}R_\alpha \\ i + p^\beta z & \text{if } i \in \xi + p^{t+1}R_\alpha \end{cases}$$

for every $i \in R_\alpha$.

As already observed in [3], $\sigma_{\xi,z,t} \in PR(R_\alpha)$ and $\sigma_{\xi,z,t}\sigma_{\xi,z',t} = \sigma_{\xi,z+z',t}$, $\sigma_{\xi,z,t}^{-1} = \sigma_{\xi,-z,t}$.

PROPOSITION 3.6. *Assume ξ and t are such that $\xi \notin p^{t+1}R_\alpha$. Then $\sigma_{\xi,z,t}$ lies in $K_{p,\alpha,\beta}$.*

Proof. Clearly $i\sigma_{\xi,z,t} \equiv i \quad p^{\beta+t}R_\alpha$. Let $0 \leq f \leq \alpha - \beta$, and let $i, j \in R_\alpha$ be such that $j \equiv i \quad p^f R_\alpha$.

(a$_1$) $f \leq t$. Here $(j-i)\sigma_{\xi,z,t} \equiv (j-i)$, $j\sigma_{\xi,z,t} \equiv j$, $i\sigma_{\xi,z,t} \equiv i \quad p^{\beta+t}R_\alpha$, so that $(j-i)\sigma_{\xi,z,t} \equiv j\sigma_{\xi,z,t} - i\sigma_{\xi,z,t} \quad p^{\beta+f}R_\alpha$.

(a$_2$) $t+1 \leq f$. Here $j \in \xi + p^{t+1}R_\alpha$ if and only if $i \in \xi + p^{t+1}R_\alpha$; hence $j\sigma_{\xi,z,t} - i\sigma_{\xi,z,t} = j - i$. Moreover, $(j-i)\sigma_{\xi,z,t} = j - i$, since $(\xi + p^{t+1}R_\alpha) \cap p^f R_\alpha = \emptyset$. ∎

We introduce the subsets $I = \{0, 1, \dots, p-1\}$ and $J = \{1, \dots, p^{\alpha-\beta}\}$ of R_α. Moreover, we put $J^* = J \setminus \{p, p^2, \dots, p^{\alpha-\beta}\}$. Given $a \in J$, we put $v(a) = c$ if $p^c \leq j < p^{c+1}$.

DEFINITION 3.7. For $\xi \in J^*$ and $z \in p^{v(\xi)}R_{\alpha-\beta}$ we put $\sigma_{\xi,z} := \sigma_{\xi,z,v(\xi)}$ and $K_\xi = \{\sigma_{\xi,z} \mid z \in p^{v(\xi)}R_\alpha\}$.

Therefore $K_\xi = \Delta_\xi$ as defined in [3, Sect. 2]: it is generated by $\sigma_{\xi, p^{v(\xi)}}$ and has order $p^{\alpha-\beta-v(\xi)}$.

We remark that for i, j in J we have, by [3, (12)] and the definition of K_{p^s},

(3.4) $$iK_j = i \qquad \text{if } i < j.$$

Following [3], we call *elementary transformations* the permutations of the form $\sigma_{\xi, z}$, $\xi \in J^*$, or $\sigma_{p^s, c}$.

In the study of $K_{p, \alpha, \beta}$ we note that

(3.5) if for a $\sigma \in K_{p,\alpha,\beta}$ we have $x\sigma = x$ for every $x \in J$, then $\sigma = 1$.

In fact, by 3.1(a), we have $(kp^{\alpha-\beta})\sigma = kp^{\alpha-\beta}$ for every k. Let $a \in R_\alpha \setminus p^{\alpha-\beta}R_\alpha$. There exists a unique $x \in J \setminus \{p^{\alpha-\beta}\}$ such that $a = x + kp^{\alpha-\beta}$. Then $a\sigma = x\sigma + (kp^{\alpha-\beta})\sigma = x + kp^{\alpha-\beta} = a$.

THEOREM 3.8. *Let $\{\sigma_{i,c_i})_{i\in J}$ and $\{\sigma_{i,c_i'}\}_{i\in J}$ be two families of elementary transformations, and assume $\prod_{i\in J}\sigma_{i,c_i} = \prod_{i\in J}\sigma_{i,c_i'}$, where i describes J in decreasing order. Then $c_i = c_i'$ for every $i \in J$. In particular*

$$\left|\prod_{i\in J} K_i\right| = \begin{cases} (p-1)^{\alpha-1} p^{p+p^2+\cdots+p^{\alpha-1}-(\alpha-1)} & \text{if } \beta = 1 \\ p^{p+p^2+\cdots+p^{\alpha-\beta}} & \text{if } \beta \geq 2. \end{cases}$$

Proof. By (3.4) we have $1 + c_1p^\beta = 1\sigma_{1,c_1} = 1\sigma_{1,c_1'} = 1 + c_1'p^\beta$. Hence $c_1 = c_1'$. Suppose $c_i = c_i'$ for $1 \leq k < i$. Then $\prod_{p^{\alpha-\beta}\geq j\geq i}\sigma_{j,c_j} = \prod_{p^{\alpha-\beta}\geq j\geq i}\sigma_{j,c_j'}$, so that $i\sigma_{i,c_i} = i\sigma_{i,c_i'}$.

(a$_1$) If $i \in J^*$, then $i\sigma_{i,c_i} = i + c_ip^\beta$ and $i\sigma_{i,c_i'} = i + c_i'p^\beta$, so that $c_i = c_i'$.

(a$_2$) If $i = p^s$ for some $1 \leq s \leq \alpha - \beta$, then $p^s\sigma_{p^s, c_p^s} = c_{p^s}p^s$ and $p^s\sigma_{p^s, c_{p^s}'} = c_{p^s}'p^s$, and again $c_{p^s} = c_{p^s}'$.

The result about the order follows, taking into account the orders

$$|K_i| = p^{\alpha-\beta-v(i)} \quad \text{if } i \in J^*, \qquad |K_{p^s}| = \begin{cases} (p-1)p^{\alpha-s-1} & \text{if } \beta = 1 \\ p^{\alpha-s-\beta+1} & \text{if } \beta \geq 2. \end{cases}$$

∎

We now consider the problem of extending autoprojectivities. Suppose $\alpha > \alpha' \geq 1$, and let $\gamma \in P(A)$. Then γ induces the autoprojectivity $\bar{\gamma}$ on $\bar{A} = A/\langle \varphi^{p^{\alpha'}}\rangle$. The Coxeter systems we are considering are the following: $\{\sigma, \sigma\varphi\}$ for A as usual, $\{\bar{\sigma}, \overline{\sigma\varphi}\}$ for $\bar{A}$. We obtain the map $\pi_{\alpha'}^{\alpha}$: $I(D_{2p^\alpha}) \to I(D_{2p^{\alpha'}})$. In terms of permutations, we get the map $r_{\alpha'}^{\alpha}$: $T_{p^\alpha} \to T_{p^{\alpha'}}$, $\gamma \mapsto \bar{\gamma}$

defined in the following way. Let $\gamma \in T_{p^\alpha}$, and $i \in R_{\alpha'}$. Choose $j \in R_\alpha$ such that $j\rho^\alpha_{\alpha'} = i$, where $\rho^\alpha_{\alpha'}\colon R_\alpha \to R_{\alpha'}$ is the canonical epimorphism. Then $i\bar{\gamma} = j\gamma\rho^\alpha_{\alpha'}$. An easy graph theoretical consideration show that $\rho^\alpha_{\alpha'}$ is surjective. If we denote by j_δ the inverse of the isomorphism $\Delta_{p^\delta}\colon T_{p^\delta} \to I(D_{2p^\delta})$ we get the commuting diagram

$$\begin{array}{ccc} I(D_{2p^\alpha}) & \xrightarrow{\pi^\alpha_{\alpha'}} & I(D_{2p^{\alpha'}}) \\ {\scriptstyle j_\alpha}\big\downarrow & & \big\downarrow{\scriptstyle j_{\alpha'}} \\ T_{p^\alpha} & \underset{\rho^\alpha_{\alpha'}}{\longrightarrow} & T_{p^{\alpha'}} \end{array}$$

Moreover, if $\alpha > \alpha' \geq \beta$, then

$$\gamma \in K_{p,\alpha,\beta} \Rightarrow \bar{\gamma} \in K_{p,\alpha',\beta} \qquad \text{and} \qquad \gamma \in \Gamma_{p,\alpha,\beta} \Rightarrow \bar{\gamma} \in \Gamma_{p,\alpha',\beta}.$$

Our aim is to show that also the restrictions $\varphi^\alpha_{\alpha'}\colon K_{p,\alpha,\beta} \to K_{p,\alpha',\beta}$ and $\varphi^\alpha_{\alpha'}\colon \Gamma_{p,\alpha,\beta} \to \Gamma_{p,\alpha',\beta}$ are surjective. Note that

$$\ker \varphi^\alpha_{\alpha'} = \left\{\gamma \in T_{p^\alpha} \mid i\gamma \equiv i \quad p^{\alpha'}R_\alpha \qquad \text{for all } i \in R_\alpha\right\}.$$

An element i of the local ring R_α can be uniquely represented in its p-adic expansion $i = i_0 + i_1 p + \cdots i_{\alpha-1}p^{\alpha-1}$, where $i_k \in I$. Let $\pi\colon R_\alpha \to R_{\alpha-1}$ be the canonical epimorphism. Then, modulo the obvious identifications, we have

$$i\pi = i_0 + i_1 p + \cdots + i_{\alpha-2}p^{\alpha-2}, \tag{3.6}$$

while $\nu\colon\ i_0 + i_1 p + \cdots + i_{\alpha-2}p^{\alpha-2} \mapsto i_0 + i_1 p + \cdots + i_{\alpha-2}p^{\alpha-2}$ defines an injection of $R_{\alpha-1}$ into R_α such that $i\pi\nu = i_0 + i_1 p + \cdots + i_{\alpha-2}p^{\alpha-2}$ and $x_1\nu + \cdots + x_r\nu \equiv 0 \quad p^tR_\alpha$ if $x_1, \ldots, x_r \in R_{\alpha-1}$ are such that $x_1 + \cdots + x_r \equiv 0 \quad p^tR_{\alpha-1}$ for some $0 \leq t \leq \alpha - 1$.

EXTENSION LEMMA 3.9. *Let $\alpha > \alpha' > \beta$ be positive integers. If σ lies in $K_{p,\alpha',\beta}$ then there exists a $\tilde{\sigma}$ in $K_{p,\alpha,\beta}$ such that $\tilde{\sigma}\pi = \pi\sigma$, similarly for $\Gamma_{p,\alpha,\beta}$.*

Proof. It is enough to deal with the case $\alpha - \alpha' = 1$. Let $i \in R_\alpha$, and let $i = i_0 + \cdots + i_{\alpha-1}p^{\alpha-1}$ be its p-adic expansion. Define

$$i\tilde{\sigma} = \left(i_0 + \cdots + i_{\alpha-\beta-1}p^{\alpha-\beta-1}\right)\pi\sigma\nu + i_{\alpha-\beta}p^{\alpha-\beta} + \cdots + i_{\alpha-1}p^{\alpha-1}.$$

Clearly $\tilde{\sigma}$ lies in Sym R_α. For $j := i_0 + \cdots + i_{\alpha-\beta-1}p^{\alpha-\beta-1}$ we have $j \equiv i \quad p^{\alpha-\beta}R_\alpha$, so that

$$i\pi\sigma - j\pi\sigma = (i\pi - j\pi)\sigma = i\pi - j\pi \tag{3.7}$$

by 3.1(c). It follows from (3.7) that $i\tilde{\sigma}\pi = (j\pi\sigma\nu + i - j)\pi = j\pi\sigma + i\pi - j\pi = j\pi\sigma + i\pi\sigma - j\pi\sigma = i\pi\sigma$. Hence

$$\tilde{\sigma}\pi = \pi\sigma. \tag{3.8}$$

In particular for $i \in R_\alpha$ we have $i\tilde{\sigma}\pi = i\pi\sigma \equiv i\pi \quad p^\beta R_{\alpha-1}$ so that $i\tilde{\sigma} \equiv i \quad p^\beta R_\alpha$. Since $i \equiv i\pi\nu \quad p^{\alpha-1}R_\alpha$, we obtain

$$i\tilde{\sigma} \equiv i\tilde{\sigma}\pi\nu = i\pi\sigma\nu \quad p^{\alpha-1}R_\alpha. \tag{3.9}$$

Now suppose $b \equiv a \quad p^f R_\alpha$, $0 \le f \le \alpha - \beta$.

(a_1) $f = \alpha - \beta$. By definition we get $b\tilde{\sigma} - a\tilde{\sigma} = b - a$. On the other hand, $b - a \in p^{\alpha-\beta}R_a \Rightarrow (b-a)\sigma = b - a$, and we are done.

(a_2) $f < \alpha - \beta$. Then $(b\pi - a\pi)\sigma \equiv b\pi\sigma - a\pi\sigma \quad p^{\beta+f}R_{\alpha-1}$ implies

$$(b\pi - a\pi)\sigma\nu \equiv b\pi\sigma\nu - a\pi\sigma\nu \quad p^{\beta+f}R_\alpha.$$

Hence

$$b\tilde{\sigma} - a\tilde{\sigma} \equiv b\pi\sigma\nu - a\pi\sigma\nu \equiv (b\pi - a\pi)\sigma\nu \equiv (b-a)\tilde{\sigma} \quad p^{\beta+f}R_\alpha$$

since $\beta + f \le \alpha - 1$. It is clear that if $\sigma \in \Gamma_{p,\alpha-1,s}$, then $1\tilde{\sigma} = 1\pi\sigma\nu = 1\pi\nu = 1$, and $\tilde{\sigma} \in \Gamma_{p,\alpha,s}$. ∎

In terms of the group W this means that for every $\alpha > \alpha' > \beta$, the natural map $R(W) \to R(W/\langle \varphi^{p^{\alpha'}}\rangle)$ is an epimorphism.

PROPOSITION 3.10. *Suppose $\alpha > \beta$. Then* $\ker \varphi^\alpha_{\alpha-1} \cap K_{p,\alpha,\beta}$ *is an elementary abelian group of order $p^{p^{\alpha-\beta}}$ if $\beta \ge 2$, while* $\ker \varphi^\alpha_{\alpha-1} \cap K_{p,\alpha,\beta} =$ *$PQ \rhd P$, P is an elementary abelian group of order $p^{p^{\alpha-\beta-1}}$, $Q = \langle \alpha \rangle$, and α is a power automorphism of order $p-1$ of P if $\beta = 1$.*

Proof. Argue as in the proof of 3.3, using (3.5). ∎

PROPOSITION 3.11. *We have*

$$|K_{p,\alpha,\beta}| = \begin{cases} (p-1)^{\alpha-1} p^{p+p^2+\cdots+p^{\alpha-1}-(\alpha-1)} & \text{if } \beta = 1 \\ p^{p+p^2+\cdots+p^{\alpha-\beta}} & \text{if } \beta \ge 2. \end{cases}$$

Proof. This follows from 3.9, 3.10, and induction. ∎

We are now in the position to prove that $K_{p,\alpha,\beta}$ is the product of the subgroups K_i.

THEOREM 3.12. *Assume $\alpha > \beta \geq 1$. Then we have*

$$K_{p,\alpha,\beta} = \prod_{i \in J} K_i,$$

i in increasing or decreasing order.

Proof. It is enough to show that $|\prod_{i \in J} K_i| = |K_{p,\alpha,\beta}|$. This follows from 3.8 and 3.11. ∎

Remark 3.1. (a) Given $\sigma \in K_{p,\alpha,\beta}$, there is a recurrent procedure to get the factorization of σ in elementary transformations: c_1 is determined by the relation $1\sigma = 1 + c_1 p^s$ and, knowing $c_1, \dots, c_{i-1}$, c_i is given as follows. Set $\sigma' = \sigma(\sigma_{i-1, c_{i-1}} \cdots \sigma_{1, c_1})^{-1}$, and note that σ' fixes k for $1 \leq k < i$:

(a_1) $i \in J^*$. Then $c_i \in R_{\alpha-\beta}$ is determined by $i\sigma' = i + c_i p^\beta$.

(a_2) $i = p^s$ for some $1 \leq s \leq \alpha - \beta$. Then $c_{p^s} \in C_{\alpha-s,\beta}$ is determined by $p^s\sigma' = c_{p^s} p^s$.

(b) Assume $\beta \geq 2$. Then the p-group $K_{p,\alpha,\beta}$ has a basis (for a definition see [4]).

COROLLARY 3.13. *Let $j \in J$. Then the pointwise stabilizer of the set $\{1, \dots, j\}$ in $K_{p,\alpha,\beta}$ is the product $\prod_{i \in J, i > j} K_i$, where the i's are in decreasing order. In particular $\Gamma_{p,\alpha,\beta} = \prod_{i \in J, i > 1} K_i$ and*

$$|\Gamma_{p,\alpha,\beta}| = \begin{cases} (p-1)^{\alpha-1} p^{p+p^2+\cdots+p^{\alpha-1}-2(\alpha-1)} & \text{if } \beta = 1 \\ p^{p+p^2+\cdots+p^{\alpha-\beta}-(\alpha-\beta)} & \text{if } \beta \geq 2. \end{cases}$$

Proof. Let F denote the pointwise stabilizer. Then for $i \in J$, $i > j$, $K_i \leq F$. On the other hand, if $\sigma \in F$, and $\sigma = \prod_{i \in J} \sigma_{i, c_i}$ is the decomposition of σ in decreasing order, then, starting with $i = 1$, we get $c_i = 0$ if $i \leq j$, $i \in J^*$, $c_i = 1$ if $i \leq j$, $i \notin J^*$. ∎

Let $\gamma \in K(W)$ be a p-element with $i\gamma \equiv i \quad p^t R_\alpha$ for all $i \in R_\alpha$. This is equivalent to $\langle \sigma\varphi^i, \varphi^{p^t} \rangle^\gamma = \langle \sigma\varphi^i, \varphi^{p^t} \rangle$; that is, $\gamma \mid [W/\langle \varphi^{p^t} \rangle] = 1$. Then γ fixes every coset $i + p^t R_\alpha$. Since the orbits of the p-group $\langle \gamma \rangle$ on the set of cosets $i + p^{t+1} R_\alpha$ are of length 1 or p, γ^p fixes every such coset; that is,

(3.10) if $\gamma \in K(W)$ is a p-element then $\gamma \mid \left[W/\langle \varphi^{p^t} \rangle\right] = 1$ implies

$$\gamma^p \mid \left[W/\langle \varphi^{p^{t+1}} \rangle\right] = 1.$$

In particular we get $|\gamma| \le p^{\alpha-\beta}$. Since $|\sigma_{1,1,0}| = p^{\alpha-\beta}$, we have

(3.11) the p-exponent of $K(W)$ is $p^{\alpha-\beta}$.

THEOREM 3.14. *Suppose $\beta \ge 2$, and let $\gamma \in K(W)$. Then, unless $p = 2$, $\beta = 2$ and $\alpha \ge 4$, we have for $\beta \le t < \alpha - 1$*

$$|\gamma| = p^{\alpha-t} \quad \Leftrightarrow \quad \gamma \mid \left[W/\langle \varphi^{p^t} \rangle\right] = 1 \quad \textit{and} \quad \gamma \mid \left[W/\langle \varphi^{p^{t+1}} \rangle\right] \ne 1.$$

Proof. We know that $\gamma \mid [W/\langle \varphi^{p^t} \rangle] = 1 \Rightarrow |\gamma| \le p^{\alpha-t}$. It is enough to show that if $|\gamma| = p$ then $\gamma \mid [W/\langle \varphi^{p^{\alpha-1}} \rangle] = 1$; that is, $i\gamma \equiv i \quad p^{\alpha-1}R_\alpha$ for every $i \in R_\alpha$. We prove this by induction on $r = \alpha - \beta$. If $r = 1$, then the conclusion follows from 3.3. So assume $r > 1$. Set $\chi = \gamma \mid [W/\langle \varphi^{p^{\alpha-1}} \rangle]$ and, for a contradiction, assume $|\chi| = p$. Hence here exists $i \in R_\alpha$ such that $i\gamma \not\equiv i \quad p^{\alpha-1}R_\alpha$. By induction on r, $\chi \mid W/\langle \varphi^{p^{\alpha-2}} \rangle] = 1$, so that $i\gamma \equiv i \quad p^{\alpha-2}R_\alpha$, $i\gamma = i + kp^{\alpha-2}$ say. Let c be such that $p^{\alpha-2}\gamma = cp^{\alpha-2}$. It follows that

(3.12) $$i = i\gamma^p = i + k(1 + c + \cdots + c^{p-1})p^{\alpha-2},$$

being $\varphi^p = 1$. If $\beta \ge 3$, we get $c = 1$ by 3.1, so that $k \in pR_\alpha$, and $i\gamma \equiv i \quad p^{\alpha-1}R_\alpha$, a contradiction. So we are left with $\beta = 2$. Then $c \equiv 1 \quad pR_2$ and $p \ne 2$. Then $1 + c + \cdots + c^{p-1} = p$, so that again $k \in pR_\alpha$, a contradiction. ∎

In 3.14 the case $p = 2$, $\beta = 2$, and $\alpha \ge 4$ cannot be omitted, as the following example shows. Let $p = 2$, $\alpha = 4$, and $\beta = 2$. Then $\sigma = (2,6)(3,7)(4,12)(5,13)(10,14)(11,15)$ lies in $K_{2,4,2}$ has order 2 and $\sigma \mid [W/\langle \varphi^8 \rangle] \ne 1$.

PROPOSITION 3.15. *Assume $\alpha > \alpha' > \beta \ge 2$. Then, unless $p = 2$, $\beta = 2$, and $\alpha > \alpha' + 1$, we have*

$$\ker \varphi^\alpha_{\alpha'} \cap K_{p,\alpha,\beta} = \prod_{i \in J} \Omega_{\alpha-\alpha'}(K_i),$$

$$\ker \varphi^\alpha_{\alpha'} \cap \Gamma_{p,\alpha,\beta} = \prod_{i \in J \setminus \{1\}} \Omega_{\alpha-\alpha'}(K_i),$$

i in decreasing or increasing order.

Proof. The result follows from 3.10 if $\alpha = \alpha' + 1$. So assume $\alpha > \alpha' + 1$. It is clear that for each $i \in J^*$ we have $K_i \cap \ker \varphi^\alpha_{\alpha'} = \Omega_{\alpha-\alpha'}(K_i)$. On the other hand, if $s \in \{1, \dots, \alpha - \beta\}$ and $c \in C_{p,\alpha-s,\beta}$ then $c = 1 + mp^{\beta-1}$, and $\sigma_{p^s,c} \in \ker \varphi^\alpha_{\alpha'}$, if and only if $mp^{\beta-1}p^s \in p^{\alpha'}R_\alpha$. If $\alpha' \le s + \beta - 1$ then $K_{p^s} \le \ker \varphi^\alpha_{\alpha'}$. So assume $s + \beta \le \alpha'$. Then we get $|K_{p^s} \cap \ker \varphi^\alpha_{\alpha'}| = p^{\alpha-\alpha'}$, so that, if we exclude the case $p = 2$ and $\beta = 2$, $K_{p^s} \cap \ker \varphi^\alpha_{\alpha'} = \Omega_{\alpha-\alpha'}(K_{p^s})$.

Now assume $\gamma \in \ker \varphi_{\alpha'}^{\alpha}$, and write $\gamma = \prod_{i \in J} \gamma_i$, $\gamma_i \in K_i$ for every $i \in J$. Applying the procedure of Remark 3.1(a), we can prove that each γ_i lies in $K_i \cap \ker \varphi_{\alpha'}^{\alpha}$, and we are done. ∎

COROLLARY 3.16. *Assume $\beta \geq 2$. Then, unless $p = 2$, $\beta = 2$, and $\alpha \geq 4$, we have for every $t = 1, \ldots, \alpha - \beta$*

$$\Omega_t(K_{p,\alpha,\beta}) = \prod_{i \in J} \Omega_t(K_i),$$

i in increasing or decreasing order.

Proof. By 3.14, we have $K_{p,\alpha,\beta} \cap \ker \varphi_{\alpha-t}^{\alpha} = \Omega_t(K_{p,\alpha,\beta})$. *Then we conclude by* 3.15. ∎

For $s = 1, \ldots, \alpha - \beta$ we consider the quotient $\overline{W}_s = \langle \sigma, \varphi^{p^s} \rangle \times B / \langle \varphi^{p^{s+\beta}} \rangle$. If γ lies in $K_{p,\alpha,\beta}$, then γ induces the autoprojectivity γ_s of $\overline{W}_s$ which, by 1.5, is induced by an automorphism $(\alpha_s, 1)$. Since $\langle \sigma \rangle^\gamma = \langle \sigma \rangle$, α_s is of the form $\bar{\sigma} \mapsto \bar{\sigma}$, $\overline{\varphi^{p^s}} \mapsto d_s \overline{\varphi^{p^s}}$, for a unique $d_s \in \mathscr{U}(R_\beta)$. It is clear that if $p^s\gamma = c_s p^s$, with $c_s \in C_{p,\alpha-s,\beta}$, then d_s is the image of c_s under the projection $R_{\alpha-s} \to R_\beta$. In particular d_s lies in $C_{p,\beta,\beta}$.

We have therefore defined an epimorphism Σ: $K_{p,\alpha,\beta} \to (C_{p,\beta,\beta})^{\alpha-\beta}$. We denote by $F_{p,\alpha,\beta}$ the kernel of Σ. Then

$$F_{p,\alpha,\beta} = \left\{ \gamma \in K_{p,\alpha,\beta} \mid p^s\gamma \equiv p^s \quad p^{s+\beta} \quad \text{for all } s = 1, \ldots, \alpha - \beta \right\}. \tag{3.13}$$

In particular, $F_{p,\alpha,\beta} \leq S_{p,\alpha,\beta}$ and in fact, by the structure of $S_{p,\alpha,\beta}$, $F_{p,\alpha,\beta}$ is the stabilizer of 0 in $S_{p,\alpha,\beta}$.

If $\beta = 1$, by 3.11 it follows that $F_{p,\alpha,\beta}$ is a p-Sylow subgroup of $K_{p,\alpha,\beta}$, and in this case $K_{p,\alpha,\beta}$ splits over $F_{p,\alpha,\beta}$.

PROPOSITION 3.17. *Let q be the integer such that $q\beta < \alpha \leq (q+1)\beta$. Then the derived length of $F_{p,\alpha,\beta}$ ($K_{p,\alpha,\beta}$) is $q(\leq q+1)$.*

Proof. In [3, (8)] we introduced the group $S_1 = \{\sigma \mid 1 + pR_\alpha \mid \sigma \in S_{p,\alpha,\beta}\}$ and showed that $dl(S_{p,\alpha,\beta}) = dl(S_1) = q$ [3, 3.9]. Since $S_1 \hookrightarrow F_{p,\alpha,\beta} \leq S_{p,\alpha,\beta}$ we get $dl(F_{p,\alpha,\beta}) = q$. Since $K_{p,\alpha,\beta}/F_{p,\alpha,\beta}$ is abelian, we conclude. ∎

LEMMA 3.18. *Assume either $p \neq 2$ or $\beta \geq 2$. Then $K_{p,\alpha,\beta}$ is abelian if and only if $\alpha + 1 \leq 2\beta$.*

Proof. Let $E_{p,\alpha,\beta} = \{\sigma \in K_{p,\alpha,\beta} \mid p^{\alpha-\beta}\sigma = p^{\alpha-\beta}\}$. Then $K_{p,\alpha,\beta} = E_{p,\alpha,\beta} \rtimes K_{p^{\alpha-\beta}}$. Assume $\alpha \leq 2\beta$. Let $\sigma, \tau \in E_{p,\alpha,\beta}$ and $i \in R_\alpha$. We get $i\sigma = i + hp^{\alpha-\beta}$, $i\tau = i + kp^{\alpha-\beta}$ for some $h, k \in \mathbb{Z}$. Then $i\sigma\tau = (i + hp^{\alpha-\beta})\tau = i\tau + h(p^{\alpha-\beta}\tau) = i + kp^{\alpha-\beta} + hp^{\alpha-\beta}$, $i\tau\sigma = (i + kp^{\alpha-\beta})\tau = i\tau + k(p^{\alpha-\beta}\tau) = i + hp^{\alpha-\beta} + kp^{\alpha-\beta}$. Therefore $E_{p,\alpha,\beta}$ is abelian.

Now suppose $\alpha + 1 \leq 2\beta$, and let $\sigma \in E_{p,\alpha,\beta}$, $\tau \in K_{p^{\alpha-\beta}}$. To show that $[\sigma, \tau] = 1$ it is enough to show that $i\sigma\tau = i\tau\sigma$ for every $i \in J$. If $i = p^{\alpha-\beta}$, then $p^{\alpha-\beta}\sigma\tau = p^{\alpha-\beta}\tau = cp^{\alpha-\beta} = (cp^{\alpha-\beta})\sigma = p^{\alpha-\beta}\tau\sigma$ for some c and we are done. Now assume $i < p^{\alpha-\beta}$. Then $i\tau = i$, $i\sigma = i + \delta_i p^{\beta}$. But $\beta \geq \alpha - \beta + 1$, so that $(i + \delta_i p^{\beta})\tau = i\tau + \delta_i p^{\beta} = i + \delta_i p^{\beta}$. Hence $i\sigma\tau = (i + \delta_i p^{\beta})\tau = i + \delta_i p^{\beta} = i\sigma = i\tau\sigma$.

On the other hand, if $\alpha \geq 2\beta$, we may choose $\sigma = \sigma_1^{p^{\alpha-2\beta}}$, τ any non-trivial element of $K_{p^{\alpha-\beta}}$. Then $1\sigma\tau = 1 + dp^{\alpha-\beta} \neq 1 + p^{\alpha-\beta} = 1\tau\sigma$. ∎

We finally determine the derived length of $K_{p,\alpha,\beta}$. We note that if $p = 2$ and $\beta = 1$, then $K_{2,\alpha,1} = F_{2,\alpha,1}$, since $C_{2,1,1} = 1$. So, by 3.17, we are left to prove

THEOREM 3.19. *Let q be the integer such that $q\beta \leq \alpha < (q+1)\beta$. Then, unless $p = 2$ and $\beta = 1$, the derived length of $K_{p,\alpha,\beta}$ is q.*

Proof. It is enough to show that $dl(K_{p,(q+1)\beta-1,\beta}) = dl(K_{p,q\beta,\beta}) = q$. We first prove that $dl(K_{p,(q+1)\beta-1,\beta}) = q$. By 3.18 this is true for $q = 1$. Now assume the result for $q - 1 \geq 1$. We consider the kernel M of the surjection π: $K_{p,(q+1)\beta-1,\beta} \to K_{p,q\beta-1,\beta}$. Then $dl(K_{p,(q+1)\beta-1,\beta}/M) = q - 1$. On the other hand $dl(K_{p,(q+1)\beta-1,\beta}/F_{p,(q+1)\beta-1,\beta}) = 1$, so that $dl(K_{p,(q+1)\beta-1,\beta}/M \cap F_{p,(q+1)\beta-1,\beta}) = q - 1$. Since by [3, 3.2] $M \cap F_{p,(q+1)\beta-1,\beta}$ is abelian, we are done. We finally deal with $K_{p,(q+1)\beta,\beta}$. By 3.17 we have $q \leq dl(K_{p,(q+1)\beta,\beta}) \leq q + 1$. To conclude we may use the procedure used in [3, 3.9] to prove that $dl(S_{p,(q+1)\beta,\beta}) = q$. Here we take $\sigma_i = \sigma_{\eta_i, c_i}$, where $\eta_i = 1 + \sum_{k=1,\dots,i} p^{ks}$, $c_i = p^{i\beta}$ for $i = 0, \dots, q-1$. Note that the coset of action (see the definition in [3]) of σ_i is $X_i := \eta_i + p^{is+1}R_{(q+1)\beta}$. Let $\sigma = \sigma_{p,c} \in K_p$. Then we have $[\sigma_i, \sigma] \mid X_i = \sigma_i^{c-1} \mid X_i$. If $\beta \geq 2$ we may take $c = 1 + p^{\beta-1}$, so that $\sigma_i^{c-1} \neq 1$. If $\beta = 1$, then again there exists c such that $\sigma_i^{c-1} \neq 1$ since $p \neq 2$. We have therefore proved that there are elements $f_0, \dots, f_{q-1} \in K'_{p,(q+1)\beta,\beta}$ such that $f_i \mid X_i = \sigma_i^{c-1} \mid X_i \neq 1$. Then we proceed as in the proof of 3.9 in [3] to get $K^{(q)}_{p,(q+1)\beta,\beta} \neq 1$, and we are done. ∎

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