On Dwork cohomology for singular hypersurfaces

Francesco Baldassarri and Pierre Berthelot *

Abstract. Let Z be a projective hypersurface over a finite field. With no smoothness assumption, we relate the p-adic cohomology spaces constructed by Dwork in his study of the zeta function of Z (*cf.* [29], [30], [31]), to the rigid homology spaces of Z. The key result is a general theorem based on the Fourier transform for $\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}$-modules [40], which extends to the rigid context results proved in the algebraic one by Adolphson and Sperber [3], and Dimca, Maaref, Sabbah and Saito [27]. If $\mathcal{V}$, $\mathcal{V}'$ are dual vector bundles over a smooth p-adic formal scheme $\mathcal{X}$, $u : \mathcal{X} \to \mathcal{V}'$ a section, Z the zero locus of its reduction mod p, this theorem gives an identification between the overconvergent local cohomology of $\mathcal{O}_{\mathcal{X},\mathbb{Q}}$ with supports in Z and the relative rigid cohomology of $\mathcal{V}$ with coefficients in the Dwork isocrystal associated to u. Thanks to this result, we also give an interpretation of a canonical filtration on the Dwork complexes in terms of the rigid homology spaces of the intersections of Z with intersections of coordinate hyperplanes.

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Introduction

Following his proof of the rationality of the zeta function of an algebraic variety over a finite field [28], Dwork wrote, between 1962 and 1969, a series of papers ([29], [30], [31], [32]) in which he developed a cohomological theory in order to express the zeta function of a projective hypersurface as an alternating product of characteristic polynomials for a suitable Frobenius action, as predicted by the Weil conjectures. Since Dwork's theory was based on the study of complexes of differential operators, it is natural to ask for the relations between his theory and other cohomological theories based on differential calculus. For non-singular hypersurfaces, this question was answered by Katz's thesis [41], which gave interpretations of Dwork's algebraic and analytic cohomologies in terms of algebraic de Rham cohomology and Monsky-Washnitzer cohomology.

In this article, we revisit this problem and give similar relations without the non-singularity assumption. For algebraic Dwork cohomology, the method we use here was introduced by Adolphson and Sperber in [3], where they generalize Katz's result to the case of smooth complete intersections in a smooth affine variety. It was then generalized to the case of singular subvarieties by Dimca, Maaref, Sabbah and Saito [27] using the techniques of algebraic $\mathcal{D}$-module theory. In particular, they made explicit the role played by the Fourier transform in the Adolphson-Sperber isomorphism. They also introduced and studied a vector bundle $V^{(m)}$ which allowed them to relate algebraic de Rham cohomology spaces with supports in a projective hypersurface Z of degree m with certain algebraic Dwork cohomology groups.

Our main observation is that, thanks to the Fourier transform for $\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}$-modules developed by Huyghe ([36], [38]), [40]), the methods of [27] can be extended to give comparison theorems between rigid cohomology groups with supports in Z and Dwork's analytic cohomology. We also prove that the comparison isomorphisms are compatible with Frobenius actions. This allows us to give a cohomological interpretation of some formulas of Dwork relating Fredholm determinants and zeta functions [29], and, more generally, to complete Dwork's program by proving that the constructions developed in [29] and [30] to treat the smooth case yield the expected cohomology groups in the singular case as well. We note that our methods can also be used to obtain comparison theorems between Dwork's dual theory, used in [31] to deal with the singular case, and de Rham and rigid cohomologies with compact supports. However, in order to keep this article to a reasonable size, we do not include these results here, and we hope to develop them subsequently.

Let us indicate now more precisely the content of the various sections. The first section is devoted to general results underlying the relation between Dwork's algebraic and analytic theories. We explain the construction of the specialization morphisms relating algebraic de Rham cohomology and rigid cohomology, both for ordinary cohomology and cohomology with compact supports. In the case of varieties over number fields, we prove that these specialization morphisms are isomorphisms outside a finite set of primes, as in Dwork's theory in the case of hypersurfaces in a projective space.

In the second section, we recall some general definitions and results about the Fourier transform for coherent $\mathcal{D}^\dagger_{\mathcal{V},\mathbb{Q}}$-modules on a p-adic formal vector bundle $\mathcal{V}$, with dual $\mathcal{V}'$. Our main result here is theorem 2.14, which is the analogue of [27, th. 0.2] in our context. As in the algebraic case, the core of the proof is the identification of the Fourier transform of the structural sheaf (with overconvergence conditions at infinity) of $\mathcal{V}$ with the overconvergent local cohomology sheaf of $\mathcal{O}_{\mathcal{V}'}$ with supports in the zero section. In addition, we prove that these isomorphisms are compatible with Frobenius actions.

The third section gives some consequences of theorem 2.14 in rigid cohomology. The most important is theorem 3.1, which provides a canonical isomorphism, compatible with Frobenius actions, between the rigid homology of the zero locus Z of a section u of a vector bundle V', and the rigid cohomology of the dual vector bundle V with coefficients in the Dwork isocrystal $\mathcal{L}_{\pi,u}$ defined by the section u and the canonical pairing $V' \times V \to \mathbb{A}^1$. We also verify that this isomorphism is compatible under the specialization morphisms with the similar isomorphism defined in [27] for algebraic de Rham cohomology.

The last two sections are devoted to the actual comparison theorems with Dwork cohomology. For simplicity, we only consider here Dwork's original theory for hypersurfaces, although the same methods could clearly be applied to give similar comparison theorems (even in the singular case) for the complexes introduced by Adolphson and Sperber to compute the zeta function of smooth complete intersections [1] (*cf.* also [20], [21]). Let R be the ring of integers in a finite extension K of $\mathbb{Q}_p$, and $f \in R[X_1, \ldots, X_{n+1}]$ an homogeneous polynomial of degree d, defining a projective hypersurface $Z \subset \mathbb{P}^n_R$. Let $Y \subset \mathbb{P}^n_R$ be the complement of the coordinate hyperplanes, and Y_k, Z_k the special fibers of Y, Z. In section 4, we first recall the construction of the Dwork complexes associated to f, as given in [29], and of the operator α which enters in Dwork's computation of the zeta function of $Z_k \cap Y_k$. Dwork's algebraic complex is built from the graded K-algebra $\mathfrak{L}$ generated by monomials $\underline{X}^{\underline{u}} = X_0^{u_0} X_1^{u_1} \cdots X_{n+1}^{u_{n+1}}$ such that $du_0 = u_1 + \cdots + u_{n+1}$. For the analytic complex, we use, as in rigid cohomology, the point of view of Monsky and Washnitzer, and we replace Dwork's spaces $L(b)$, which are Banach spaces of series in the $\underline{X}^{\underline{u}}$ satisfying appropriate growth conditions, by the union $L(0^+)$ of all $L(b)$, $b > 0$. This does not change Dwork's characteristic series.

Let V be the vector bundle associated to the sheaf $\mathcal{O}_{\mathbb{P}^n}(d)$, $D \subset V$ the union of the inverse images of the coordinate hyperplanes in $\mathbb{P}^n$ and of the zero section of V, V_K, D_K the generic fibers, $L_{\pi,f}$ the algebraic module with connection constructed as above using the section of V' defined by f, $\mathcal{L}_{\pi,f}$ the corresponding Dwork isocrystal. We show that the Dwork complexes are isomorphic to the complexes of global algebraic differential forms (resp. analytic with overconvergence at infinity) on V_K, with logarithmic poles along D_K, and coefficients in $L_{\pi,f}$ (resp. $\mathcal{L}_{\pi,f}$). We then use [27, theorem 0.2] and theorem 3.1 to identify the cohomology of the Dwork complexes to the algebraic de Rham cohomology and to the rigid cohomology of the generic and special fibers of Y with supports in Z, in a manner which is compatible with specialization morphisms (theorem 4.6). This provides a cohomological interpretation of Dwork's formula [28, (21)] relating the characteristic series $\det(I - t\alpha)$ with the zeta function of $Z_k \cap Y_k$.

In the last section, we follow Dwork's method to relate $\det(I - t\alpha)$ to the zeta functions of Z_k and of all its intersections with intersections of coordinate hyperplanes. For that purpose, we define an increasing filtration on the Dwork complexes with the following properties. On the one hand, its Fil_0 term computes the primitive algebraic de Rham cohomology and the primitive rigid cohomology of the generic and special fibers of Z (the algebraic statement was proved in [27]). On the other hand, its graded pieces of higher degree decompose as direct sums of Fil_0 terms for the Dwork complexes of the intersections of Z with intersections of coordinate hyperplanes. This also provides a cohomological interpretation of a combinatorial formula of Dwork [29, (4.33)].

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General conventions

Throughout this paper, we will adopt the following conventions:

(i) If E is an abelian group, then $E_{\mathbb{Q}} := E \otimes \mathbb{Q}$.

(ii) All schemes are assumed to be separated and quasi-compact.

(iii) Notation and shift conventions for cohomological operations on $\mathcal{D}$-modules are those of Bernstein and Borel [19].

(iv) In most of this article, a prime number p and a power $q = p^s$ of p will be fixed. For simplicity, we will then call "Frobenius action" an action of the s-th power of the absolute Frobenius endomorphism, and "F-isocrystal" an isocrystal endowed with such an action (*cf.* 1.9 for details).

1. Specialization and cospecialization in rigid cohomology

One of the essential ingredients in Dwork's study of the zeta function for a singular projective hypersurface is the fact that, when the hypersurface is defined over a number field, the analytic cohomology spaces which carry the Frobenius action are isomorphic for almost all prime to their algebraic analogues. We give here a general result from which this comparison theorem follows.

For that purpose, we first construct, for an algebraic variety Z over the ring of integers of a local field of mixed characteristics, a specialization map which relates the algebraic de Rham homology of the generic fiber of Z with the rigid homology of its special fiber. We then prove that, when Z comes from a number field, this map is an isomorphism for almost all primes. We also give a similar result for rigid cohomology with compact supports.

1.1. In this section, we fix a complete discretely valued field K of mixed characteristics $(0,p)$. We denote by R its valuation ring, by $\mathfrak{m}$ its maximal ideal, and by k its residue field. Let $S = \mathrm{Spec}(R)$, and let X be a smooth S-scheme. We first recall how the rigid cohomology of its special fiber X_k can be computed using the analytic space X_K^{an} associated to its generic fiber X_K.

The scheme X defines a p-adic formal scheme $\mathcal{X}$ over R, and we denote by $\mathcal{X}_K$ its generic fiber (in the sense of Raynaud), which is a quasi-compact open rigid analytic subspace of X_K^{an}. For example, if X is a closed S-subscheme of an affine space $\mathbb{A}^r_S$, $\mathcal{X}_K$ is the intersection of X_K^{an} with the closed unit ball in the analytic affine space $\mathbb{A}^{r\,\mathrm{an}}_K$, which is independent of the chosen embedding into an affine space over S. In the general case, the construction can be deduced from the affine case by a glueing argument (*cf.* [12, 0.2] or [14]).

Thanks to results of Nagata ([45], [46]), one can find a proper S-scheme $\overline{X}$ and an open immersion $X \hookrightarrow \overline{X}$. Let $\overline{\mathcal{X}}$ be the formal scheme defined by $\overline{X}$. Note that, since $\overline{X}$ is proper over S, the two analytic spaces $\overline{X}_K^{\mathrm{an}}$ and $\overline{\mathcal{X}}_K$ coincide, and that $\mathcal{X}_K$ is the tube $]X_k[_{\overline{\mathcal{X}}}$ of X_k in $\overline{\mathcal{X}}_K$. We refer to [12, 1.2] for the general notion of a strict neighbourhood of $\mathcal{X}_K$ in $\overline{X}_K^{\mathrm{an}} = \overline{\mathcal{X}}_K$. In particular, X_K^{an} is a strict neighbourhood of $\mathcal{X}_K$ in $\overline{X}_K^{\mathrm{an}}$ [12, (1.2.4) (ii)]. Therefore, the strict neighbourhoods of $\mathcal{X}_K$ contained in X_K^{an} form a fundamental system of strict neighbourhoods of $\mathcal{X}_K$. Moreover, an open subset $V \subset X_K^{\mathrm{an}}$ is a strict neighbourhood of $\mathcal{X}_K$ in $\overline{X}_K^{\mathrm{an}}$ if and only if one of the two following equivalent conditions is satisfied:

(i) The covering $(V, X_K^{\mathrm{an}} \setminus \mathcal{X}_K)$ of X_K^{an} is admissible.

(ii) For any affine open subset $U \subset X$, and any closed embedding $U \subset \mathbb{A}^r_S$, there exists a real number $\rho > 1$ such that $V \cap U_K^{\mathrm{an}}$ contains $U_K^{\mathrm{an}} \cap B(0,\rho)$, where $B(0,\rho)$ is the closed ball of radius ρ in $\mathbb{A}^{r\,\mathrm{an}}_K$.

Since these conditions are intrinsic on X (*i. e.* do not depend upon the compactification $\overline{X}$), it is thus possible to define directly on X_K^{an} the notion of a (fundamental system of) strict neighbourhood(s) of $\mathcal{X}_K$ in X_K^{an}.

If $V' \subset V$ is a pair of strict neighbourhoods of $\mathcal{X}_K$ in $\overline{X}_K^{\mathrm{an}}$, let $j_{V,V'} : V' \hookrightarrow V$ be the inclusion. For any abelian sheaf E on V, we define

$$j^\dagger E \;:=\; \varinjlim_{V' \subset V} \; j_{V,V'*}\, j_{V,V'}^{-1} E,$$

where the limit is taken over all strict neighbourhoods V' of $\mathcal{X}_K$ contained in V. Note that the functor $j^\dagger$ is an exact functor [12, (2.1.3)]. The sheaf $j^\dagger E$ is actually independent of V in the sense that, if $V_1 \subset V$ is a strict neighbourhood of $\mathcal{X}_K$, $j_1^\dagger$ the analogue of $j^\dagger$ on V_1, and $E_1 = j_{V,V_1}^{-1} E$, there is a canonical isomorphism

$$j^\dagger E \xrightarrow{\;\sim\;} \mathbb{R} j_{V,V_1*}\, j_1^\dagger E_1 \qquad (1.1.1)$$

(*cf.* [14, 1.2 (iv)]). Since, for any $j^\dagger\mathcal{O}_V$-module E (resp. $j_1^\dagger\mathcal{O}_{V_1}$-module E_1), the map $E \to j^\dagger E$ (resp. $E_1 \to j_1^\dagger E_1$) is an isomorphism [12, (2.1.3)], it follows that the functors j_{V,V_1}^{-1} and j_{V,V_1*} (resp. $\mathbb{R} j_{V,V_1*}$) give quasi-inverse equivalences between the categories (resp. derived categories) of $j^\dagger\mathcal{O}_V$-modules and $j_1^\dagger\mathcal{O}_{V_1}$-

modules. Moreover, for any $j^\dagger\mathcal{O}_V$-module E, the canonical morphism

$$\mathbb{R}\Gamma(V,E) \longrightarrow \mathbb{R}\Gamma(V_1, j_{V,V_1}^{-1}E)$$

is an isomorphism.

In particular, we can apply this remark to $V = \overline{X}_K^{\mathrm{an}}$ and $V_1 = X_K^{\mathrm{an}}$, and to the de Rham complex of $\overline{X}_K^{\mathrm{an}}$. If $j^\dagger$, $j_X^\dagger$ denote the corresponding functors, we obtain in this way a canonical isomorphism

$$\mathbb{R}\Gamma_{\mathrm{rig}}(X_k/K) := \mathbb{R}\Gamma(\overline{X}_K^{\mathrm{an}}, j^\dagger\Omega^\bullet_{\overline{X}_K^{\mathrm{an}}}) \xrightarrow{\ \sim\ } \mathbb{R}\Gamma(X_K^{\mathrm{an}}, j_X^\dagger\Omega^\bullet_{X_K^{\mathrm{an}}}), \tag{1.1.2}$$

which shows that the rigid cohomology of X_k can be computed directly on X_K^{an} without using a compactification of X_k.

1.2. Let $Z \subset X$ be a closed subscheme, $U = X \setminus Z$, and let $\mathcal{U} = \mathcal{X} \setminus Z_k$ be the formal completion of U. We denote by $j_U^\dagger$ the analogue of $j_X^\dagger$ obtained by taking the limit on strict neighbourhoods of $\mathcal{U}_K$. Thanks to (1.1.2), the rigid cohomology groups of X_k with support in Z_k [14, 2.3] are given by

$$\mathbb{R}\Gamma_{Z_k,\mathrm{rig}}(X_k/K) \simeq \mathbb{R}\Gamma(X_K^{\mathrm{an}}, (j_X^\dagger\Omega^\bullet_{X_K^{\mathrm{an}}} \to j_U^\dagger\Omega^\bullet_{X_K^{\mathrm{an}}})_{\mathrm{t}}), \tag{1.2.1}$$

where the subscript t denotes the total complex associated to a double complex.

On the other hand, we can consider the de Rham cohomology groups of X_K with support in Z_K. Let u denote the inclusion of U in X. If $\mathcal{I}^\bullet$ is an injective resolution of $\Omega^\bullet_{X_K}$ as a complex of sheaves of K-vector spaces over X_K, we obtain by definition

$$\mathbb{R}\Gamma_{Z_K,\mathrm{dR}}(X_K/K) = \Gamma(X_K, (\mathcal{I}^\bullet \to u_{K*}u_K^{-1}\mathcal{I}^\bullet)_{\mathrm{t}}).$$

We now construct a canonical morphism, called the *specialization morphism*:

$$\rho_Z : \mathbb{R}\Gamma_{Z_K,\mathrm{dR}}(X_K/K) \longrightarrow \mathbb{R}\Gamma_{Z_k,\mathrm{rig}}(X_k/K). \tag{1.2.2}$$

Observe that, if $\mathcal{J}$ in a flasque sheaf on X_K^{an}, then, for any $U \subset X$, the sheaf $j_U^\dagger\mathcal{J}$ is acyclic for the functor $\Gamma(X_K^{\mathrm{an}}, -)$. Indeed, the isomorphism (1.1.1) allows to replace X_K^{an} by any strict neighbourhood of $\mathcal{U}_K$. Thus we can replace X_K^{an} by a quasi-compact strict neighbourhood V of $\mathcal{U}_K$ (for example, the complement of an open tube $]\overline{X}_k \setminus U_k[_\lambda$ of radius $\lambda < 1$). This insures that $H^*(V, -)$ commutes with direct limits, and the claim is clear.

Choose an injective resolution $\mathcal{J}^\bullet$ of $\Omega^\bullet_{X_K^{\mathrm{an}}}$, and denote by $\epsilon : X_K^{\mathrm{an}} \to X_K$ the canonical morphism. The functoriality morphism for the de Rham complex can be extended to a morphism

$$\epsilon^{-1}(\mathcal{I}^\bullet \to u_{K*}u_K^{-1}\mathcal{I}^\bullet)_{\mathrm{t}} \longrightarrow (\mathcal{J}^\bullet \to u_{K*}^{\mathrm{an}}u_K^{\mathrm{an}\,-1}\mathcal{J}^\bullet)_{\mathrm{t}},$$

which can then be composed with the canonical morphism

$$(\mathcal{J}^\bullet \to u_{K*}^{\mathrm{an}}u_K^{\mathrm{an}\,-1}\mathcal{J}^\bullet)_{\mathrm{t}} \longrightarrow (j_X^\dagger\mathcal{J}^\bullet \to j_U^\dagger\mathcal{J}^\bullet)_{\mathrm{t}}$$

(we use here the fact that U_K^{an} is a strict neighbourhood of $\mathcal{U}_K$). Taking sections on X_K^{an} and composing with the functoriality map induced by ϵ yields the morphism ρ_Z.

Remark. – By [35], the groups $\mathbb{R}\Gamma_{Z_K,\,\mathrm{dR}}(X_K/K)$ are independent of the embedding of Z_K into the smooth scheme X_K, and define the algebraic de Rham homology of Z. Similarly, the groups $\mathbb{R}\Gamma_{Z_k,\,\mathrm{rig}}(X_k/K)$ depend only upon Z_k, and define the rigid homology of Z_k [47]. It is easy to check that the specialization morphism ρ_Z depends also only upon Z. However, we will not use these facts here.

1.3. Let us change notation, and assume that K is a number field, R its ring of integers, $S = \operatorname{Spec} O_K$, S^0 its set of closed points, and X an S-scheme, with generic fiber X_K. For any $s \in S^0$, the subscript s will denote the special fiber at s. If s corresponds to $\mathfrak{p} \subset R$, let $K(s)$ be the completion of K at $\mathfrak{p}$, $R(s)$ its valuation ring, $X(s) = \operatorname{Spec} R(s) \times_{\operatorname{Spec} R} X$, $X_{K(s)}$ the generic fiber of $X(s)$ over $\operatorname{Spec} R(s)$, $X_{K(s)}^{\mathrm{an}}$ its associated analytic space, $\mathcal{X}(s)$ the formal completion of $X(s)$ with respect to the maximal ideal of $R(s)$, $\mathcal{X}(s)_{K(s)}$ its generic fiber.

Assume that X is smooth over S, and fix a closed subscheme $Z \subset X$. Together with the base change map for algebraic de Rham cohomology, the specialization homomorphism (1.2.2) provides, for each s, a canonical morphism

$$\rho_{Z,s} : \ K(s) \otimes_K \mathbb{R}\Gamma_{Z_K,\,\mathrm{dR}}(X_K/K) \ \longrightarrow \ \mathbb{R}\Gamma_{Z_s,\,\mathrm{rig}}(X_s/K(s)), \tag{1.3.1}$$

which we call the *specialization morphism at s*.

Theorem 1.4. *Under the previous assumptions, there exists a finite subset $\Sigma \subset S^0$ such that the specialization homomorphism* (1.3.1) *is an isomorphism for all $s \notin \Sigma$.*

We begin the proof with the following remarks:

(i) Since the algebraic de Rham cohomology complexes $\mathbb{R}\Gamma_{Z_K,\,\mathrm{dR}}(X_K/K)$ commute with base field extensions, the morphism (1.3.1) is an isomorphism at a point s if and only if, on $K(s)$, the corresponding local morphism (1.2.2) is an isomorphism.

(ii) If there exists a non empty open subset $S' \subset S$ over which X is proper and smooth, it follows from the construction of rigid cohomology and GAGA that the morphism $\rho_{X,s}$ is an isomorphism for all $s \in S'$. In particular, the theorem then holds for the pair (X, X).

(iii) Both algebraic de Rham cohomology and rigid cohomology satisfy the standard excision properties (*cf.* [35, (3.3)] for de Rham cohomology, and [14, 2.5] for rigid cohomology). It follows immediately from the above constructions that the specialization morphisms define a morphism between the corresponding distinguished triangles.

We use an induction argument similar to the one used in [14] to prove the finiteness of rigid cohomology. We will show inductively the following assertions:

(a)$_n$: For any number field K and any smooth O_K-scheme X such that $\dim X_K \leq n$, there exists a finite subset $\Sigma \subset (\operatorname{Spec} O_K)^0$ such that the morphism

$$\rho_{X,s}: \ K(s) \otimes_K \mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K) \ \longrightarrow \ \mathbb{R}\Gamma_{\mathrm{rig}}(X_s/K(s))$$

is an isomorphism for $s \notin \Sigma$.

(b)$_n$: For any number field K, any O_K-scheme Z such that $\dim Z_K \leq n$ and any closed immersion $Z \hookrightarrow X$ into a smooth O_K-scheme X, there exists a finite subset $\Sigma \subset (\operatorname{Spec} O_K)^0$ such that the morphism

$$\rho_{Z,s}: \ K(s) \otimes_K \mathbb{R}\Gamma_{Z_K,\,\mathrm{dR}}(X_K/K) \ \longrightarrow \ \mathbb{R}\Gamma_{Z_s,\,\mathrm{rig}}(X_s/K(s))$$

is an isomorphism for $s \notin \Sigma$.

Let us first check (a)$_0$. The scheme X is then étale over S, and X_K is finite over K. It follows that there exists a non empty open subset in S over which the morphism $X \to S$ is finite. Thus the assertion follows from remark (ii) above.

Let us now prove that (b)$_{n-1}$ implies (a)$_n$. Let X be a smooth S-scheme such that $\dim X_K = n$. Since K is of characteristic zero, we may use resolution of singularities to find an isomorphism between X_K and a dense open subset of a proper and smooth K-scheme Y_K. By general arguments on direct limits, there exists a non empty open subset $S' \subset S$, a proper and smooth S'-scheme Y and an open immersion $X_{|S'} \hookrightarrow Y$ extending over S' the previous immersion $X_K \hookrightarrow Y_K$. By remark (ii), the morphism $\rho_{Y,s}$ is an isomorphism for all $s \in S'$. Let $Z = Y \setminus X$. As X_K is dense in Y_K, we have $\dim Z_K < n$. Therefore, the induction hypothesis implies that the morphism $\rho_{Z,s}$ is an isomorphism for all s outside a finite subset of S^0. Shrinking S' if necessary, the result for X then follows from remark (iii).

We finally prove that (b)$_0$ holds, and that (b)$_{n-1}$ + (a)$_n$ implies (b)$_n$. Let $Z \hookrightarrow X$ be a closed immersion into a smooth S-scheme X, with $\dim Z_K = n$. We may replace Z by Z_{red}, since both source and target of (1.3.1) only depend upon the reduced subscheme. Then, if $T \subset Z$ is the closed subset where $Z \to S$ is not smooth, we have $\dim T_K < n$. Using the excision exact sequences and the induction hypothesis, we are reduced to the case where Z is smooth over S. Let $r = \operatorname{codim}(Z, X)$. We have Gysin isomorphisms for algebraic de Rham cohomology [35, (3.1)] and for rigid cohomology [14, 5.2-5.5]. Moreover, the Gysin isomorphism for rigid cohomology is deduced from the Gysin morphism between algebraic de Rham complexes by taking the analytification and applying suitable $j^\dagger$ functors. Therefore, the specialization morphisms fit in a commutative diagram

$$\begin{array}{ccc} K(s) \otimes_K \mathbb{R}\Gamma_{\mathrm{dR}}(Z_K/K) & \xrightarrow{\ \sim\ } & K(s) \otimes_K \mathbb{R}\Gamma_{Z_K,\,\mathrm{dR}}(X_K/K)[2r] \\ \Big\downarrow{\scriptstyle \rho_{Z,s}} & & \Big\downarrow{\scriptstyle \rho_{Z,s}[2r]} \\ \mathbb{R}\Gamma_{\mathrm{rig}}(Z_s/K(s)) & \xrightarrow{\ \sim\ } & \mathbb{R}\Gamma_{Z_s,\,\mathrm{rig}}(X_s/K(s))[2r]. \end{array}$$

Since Z is smooth, and $\dim Z_K = n$, the induction hypothesis implies that the left vertical arrow is an isomorphism, and the theorem follows.

Remark. – In the step (b)$_{n-1} \Rightarrow$ (a)$_n$, we could use de Jong's theorem on alterations instead of resolution of singularities. We would then argue as in [14, 3.5], using the fact that the specialization morphisms commute with the trace maps associated with a finite étale morphism between two affine schemes.

1.5. We now give an analogue of theorem 1.4 for rigid cohomology with compact supports. Let us first briefly explain the construction of the cospecialization morphism between rigid cohomology with compact supports and algebraic de Rham cohomology with compact supports (the reader can refer to [7, section 6] for more details).

We consider again the situation of 1.1 and 1.2, where K was a complete discretely valued field of mixed characteristics $(0, p)$, and we keep the same notation and hypotheses. Let $\overline{Z}$ be the closure of Z in $\overline{X}$, $T = \overline{Z} \setminus Z$, and let $u :]T_k[_{\overline{\mathcal{X}}} \hookrightarrow]\overline{Z}_k[_{\overline{\mathcal{X}}}$ be the inclusion. By construction [9], the rigid cohomology of Z_k with compact supports is given by

$$\begin{aligned} \mathbb{R}\Gamma_{\mathrm{c,\,rig}}(Z_k/K) &:= \mathbb{R}\Gamma_{]Z_k[}(]\overline{Z}_k[_{\overline{\mathcal{X}}}, \Omega^\bullet_{\overline{X}_K^{\mathrm{an}}}) \\ &\simeq \mathbb{R}\Gamma(]\overline{Z}_k[_{\overline{\mathcal{X}}}, (\Omega^\bullet_{]\overline{Z}_k[} \to u_*\Omega^\bullet_{]T_k[})_{\mathrm{t}}). \end{aligned} \tag{1.5.1}$$

On the other hand, by [7, 1.2], the algebraic de Rham cohomology with compact supports of Z_K is defined as

$$\mathbb{R}\Gamma_{\mathrm{dR,\,c}}(Z_K/K) := \mathbb{R}\Gamma(\overline{X}_K, ((\Omega^\bullet_{\overline{X}_K})_{/\overline{Z}_K} \to (\Omega^\bullet_{\overline{X}_K})_{/T_K})_{\mathrm{t}}), \tag{1.5.2}$$

where $(\Omega^\bullet_{\overline{X}_K})_{/\overline{Z}_K}$ and $(\Omega^\bullet_{\overline{X}_K})_{/T_K}$ are the formal completions along $\overline{Z}_K$ and T_K respectively. Using the functoriality of the de Rham complex and GAGA, we therefore obtain an isomorphism

$$\mathbb{R}\Gamma_{\mathrm{dR,\,c}}(Z_K/K) \simeq \mathbb{R}\Gamma(\overline{X}_K^{\mathrm{an}}, ((\Omega^\bullet_{\overline{X}_K^{\mathrm{an}}})_{/\overline{Z}_K^{\mathrm{an}}} \to (\Omega^\bullet_{\overline{X}_K^{\mathrm{an}}})_{/T_K^{\mathrm{an}}})_{\mathrm{t}}). \tag{1.5.3}$$

Since $\overline{Z}_K^{\mathrm{an}}$ and T_K^{an} are closed analytic subsets of the open subsets $]\overline{Z}_k[_{\overline{\mathcal{X}}}$ and $]T_k[_{\overline{\mathcal{X}}}$ of $\overline{X}_K^{\mathrm{an}}$, we now have a functoriality morphism

$$\mathbb{R}\Gamma(]\overline{Z}_k[_{\overline{\mathcal{X}}}, (\Omega^\bullet_{]\overline{Z}_k[} \to u_*\Omega^\bullet_{]T_k[})_{\mathrm{t}}) \longrightarrow \mathbb{R}\Gamma(\overline{X}_K^{\mathrm{an}}, ((\Omega^\bullet_{\overline{X}_K^{\mathrm{an}}})_{/\overline{Z}_K^{\mathrm{an}}} \to (\Omega^\bullet_{\overline{X}_K^{\mathrm{an}}})_{/T_K^{\mathrm{an}}})_{\mathrm{t}}). \tag{1.5.4}$$

The *cospecialization morphism*

$$\rho_{\mathrm{c},Z} : \mathbb{R}\Gamma_{\mathrm{c,\,rig}}(Z_k/K) \longrightarrow \mathbb{R}\Gamma_{\mathrm{dR,\,c}}(Z_K/K) \tag{1.5.5}$$

is then obtained by composing (1.5.4) with the inverse of (1.5.3).

Remarks. – (i) It is easy to check that $\rho_{\mathrm{c},Z}$ only depends upon Z, and not upon the scheme $\overline{X}$ used to define both cohomologies with compact supports.

(ii) Algebraic de Rham cohomology and rigid cohomology both satisfy Poincaré duality. Indeed, if we assume that X is of constant relative dimension n over S, and if the exponent $^\vee$ denotes the K-linear dual, we have canonical isomorphisms

$$\mathbb{R}\Gamma_{Z_K,\,\mathrm{dR}}(X_K/K) \xrightarrow{\sim} \mathbb{R}\Gamma_{\mathrm{dR,\,c}}(Z_K/K)^\vee[-2n] \tag{1.5.6}$$

(*cf.* [7, 3.4]) and

$$\mathbb{R}\Gamma_{Z_k,\,\mathrm{rig}}(X_k/K) \xrightarrow{\sim} \mathbb{R}\Gamma_{\mathrm{c,\,rig}}(Z_k/K)^\vee[-2n] \tag{1.5.7}$$

(*cf.* [15, 2.4]). One verifies easily that ρ_Z and $\rho_{\mathrm{c},Z}$ are compatible with cup-products on cohomology. On the other hand, the rigid trace map is constructed in [15] starting from Hartshorne's algebraic trace map for projective smooth varieties [34], and this ensures the commutation of the rigid and de Rham trace maps with $\rho_{\mathrm{c},Z}$. It follows that, under the Poincaré duality pairings, ρ_Z and $\rho_{\mathrm{c},Z}$ are dual to each other (a detailed proof can be found in [7, 6.9]).

1.6. We consider now the global situation of 1.3, and we use again the notation and hypotheses of that section. For each $s \in S^0$, the base change map for algebraic de Rham cohomology with compact supports

$$K(s) \otimes_K \mathbb{R}\Gamma_{\mathrm{dR,\,c}}(Z_K/K) \longrightarrow \mathbb{R}\Gamma_{\mathrm{dR,\,c}}(Z_{K(s)}/K(s))$$

is an isomorphism, because algebraic de Rham cohomology commutes with base field extensions [35, 5.2], and this property extends to algebraic de Rham cohomology with compact supports using the standard distinguished triangle defined by a compactification. Composing the inverse of this isomorphism with (1.5.5) gives the *cospecialization morphism at s*

$$\rho_{\mathrm{c},Z,s} \;:\; \mathbb{R}\Gamma_{\mathrm{c,\,rig}}(Z_s/K(s)) \longrightarrow K(s) \otimes_K \mathbb{R}\Gamma_{\mathrm{dR,\,c}}(Z_K/K). \tag{1.6.1}$$

Theorem 1.7. *Under the assumptions of* 1.3*, there exists a finite subset* $\Sigma \subset S^0$ *such that the cospecialization homomorphism* (1.6.1) *is an isomorphism for all* $s \notin \Sigma$.

It follows from remark (ii) of 1.5 that the morphisms $\rho_{Z,s}$ and $\rho_{\mathrm{c},Z,s}$ are dual to each other under Poincaré duality over $K(s)$. Hence $\rho_{\mathrm{c},Z,s}$ is an isomorphism if and only if $\rho_{Z,s}$ is an isomorphism, and theorem 1.4 and theorem 1.7 are equivalent.

One can also give a direct proof of 1.7 as in 1.4. One observes first that, if $Z' \subset Z$ is an open subset, with $T = Z \setminus Z'$, the cospecialization morphisms define a morphism between the corresponding distinguished triangles for cohomologies with compact supports. This allows to proceed by induction on the dimension of Z_K. Indeed, as in 1.4 (ii), the theorem is true if there is an open subset $S' \subset S$ such that Z is proper and smooth over S'. In particular, the theorem is true when $\dim Z_K = 0$. In general, we can remove from Z a closed subscheme T such that $\dim T_K < \dim Z_K$, so as to insure that the generic fiber Z'_K of $Z' := Z \setminus T$ is smooth. It is enough to prove the theorem for Z', and we can use resolution of singularities to find a compactification $\overline{Z}'$ of Z' which is proper and smooth over a non empty open subset of S, and in which Z' is dense. By induction, the result for $\overline{Z}'$ implies the result for Z'.

1.8. In the local case, we will also use the specialization and cospecialization morphisms for some cohomology groups with coefficients. For simplicity, we will only consider here the case where $Z = X$, as this is the only case which will be needed in the present article (the reader interested in the general case will easily generalize our constructions, following the method used in 1.2 and 1.5).

Our notation and hypotheses are again those of 1.1. Let us first observe that, if $V_1 \subset V$ are two strict neighbourhoods of $\mathcal{X}_K$ in $\overline{X}_K^{\mathrm{an}}$, the equivalences j_{V,V_1}^{-1} and j_{V,V_1*} between the categories of $j^\dagger\mathcal{O}_V$-modules and $j_1^\dagger\mathcal{O}_{V_1}$-modules induce quasi-inverse equivalences between the categories of coherent $j^\dagger\mathcal{O}_V$-modules endowed with an integrable and overconvergent connection and of coherent $j_1^\dagger\mathcal{O}_{V_1}$-modules endowed with an integrable and overconvergent connection. The category of overconvergent isocrystals on X_k can thus be realized equivalently on V or on V_1. Therefore, for any overconvergent isocrystal $\mathcal{L}$ on X_k, we get as in 1.1 a canonical isomorphism

$$\mathbb{R}\Gamma_{\mathrm{rig}}(X_k/K,\mathcal{L}) \xrightarrow{\ \sim\ } \mathbb{R}\Gamma(X_K^{\mathrm{an}},\mathcal{L}\otimes\Omega^\bullet_{X_K^{\mathrm{an}}}), \tag{1.8.1}$$

where $\mathcal{L}$ is viewed as a coherent $j_X^\dagger\mathcal{O}_{X_K^{\mathrm{an}}}$-module with an integrable and overconvergent connection.

Let (L,∇) be a locally free finitely generated $\mathcal{O}_{X_K}$-module, endowed with an integrable connection, $(L^{\mathrm{an}},\nabla^{\mathrm{an}})$ its inverse image on X_K^{an}, and $\mathcal{L} = j_X^\dagger L^{\mathrm{an}}$, endowed with the corresponding connection. We assume that this connection on $\mathcal{L}$ is overconvergent, so that $\mathcal{L}$ can be viewed as defining an overconvergent isocrystal on X_k, still denoted by $\mathcal{L}$. The *specialization morphism* for de Rham and rigid cohomologies with coefficients in L is then defined as the composed morphism

$$\begin{aligned}\rho_X^L\ :\ \mathbb{R}\Gamma(X_K,L\otimes\Omega^\bullet_{X_K}) &\longrightarrow \mathbb{R}\Gamma(X_K^{\mathrm{an}},L^{\mathrm{an}}\otimes\Omega^\bullet_{X_K^{\mathrm{an}}}) \\ &\longrightarrow \mathbb{R}\Gamma(X_K^{\mathrm{an}},\mathcal{L}\otimes\Omega^\bullet_{X_K^{\mathrm{an}}}) \ \simeq\ \mathbb{R}\Gamma_{\mathrm{rig}}(X_k/K,\mathcal{L}).\end{aligned} \tag{1.8.2}$$

To define the cospecialization morphism, we choose a compactification $\overline{X}$ of X, and a coherent $\mathcal{O}_{\overline{X}_K}$-module $\overline{L}$ extending L on $\overline{X}_K$. Let I be the ideal of $T := \overline{X}\setminus X$ in $\overline{X}$. In general, the connection ∇ does not extend to $\overline{L}$, but it can be extended as a connection on the pro-$\mathcal{O}_{\overline{X}_K}$-module "$\varprojlim_n$"$I^n\overline{L}$. This allows to define the de Rham pro-complex $I^\bullet\,\overline{L}\otimes\Omega^\bullet_{\overline{X}_K} := (\text{“}\varprojlim_n\text{”} I^n\overline{L})\otimes\Omega^\bullet_{\overline{X}_K}$. The algebraic de Rham cohomology with compact supports and coefficients in L is then defined (*cf.* [4, App. D.2]) as

$$\begin{aligned}\mathbb{R}\Gamma_{\mathrm{dR,\,c}}(X_K/K,L) &:= \mathbb{R}\Gamma(\overline{X}_K,\mathbb{R}\varprojlim I^\bullet\,\overline{L}\otimes\Omega^\bullet_{\overline{X}_K}) \\ &\simeq \mathbb{R}\varprojlim\mathbb{R}\Gamma(\overline{X}_K,I^\bullet\,\overline{L}\otimes\Omega^\bullet_{\overline{X}_K}).\end{aligned} \tag{1.8.3}$$

Note first that GAGA provides a canonical isomorphism

$$\begin{aligned}\mathbb{R}\varprojlim\mathbb{R}\Gamma(\overline{X}_K,I^\bullet\,\overline{L}\otimes\Omega^\bullet_{\overline{X}_K}) &\xrightarrow{\ \sim\ } \mathbb{R}\varprojlim\mathbb{R}\Gamma(\overline{X}_K^{\mathrm{an}},I^\bullet\,\overline{L}^{\mathrm{an}}\otimes\Omega^\bullet_{\overline{X}_K^{\mathrm{an}}}) \\ &\xrightarrow{\ \sim\ } \mathbb{R}\Gamma(\overline{X}_K^{\mathrm{an}},\mathbb{R}\varprojlim I^\bullet\,\overline{L}^{\mathrm{an}}\otimes\Omega^\bullet_{\overline{X}_K^{\mathrm{an}}}).\end{aligned} \tag{1.8.4}$$

Let us now denote by $j : X_K^{\mathrm{an}} \hookrightarrow \overline{X}_K^{\mathrm{an}}$ the given open immersion. We can consider on X_K^{an} and $\overline{X}_K^{\mathrm{an}}$ the functors $\mathbb{R}\underline{\Gamma}_{]X_k[}$ of local sections supported in the tube $]X_k[_{\mathcal{X}}$. As $(X_K^{\mathrm{an}},]T_k[_{\overline{\mathcal{X}}})$ is an admissible covering of $\overline{X}_K^{\mathrm{an}}$, the canonical morphism

$$\mathbb{R}\underline{\Gamma}_{]X_k[}(\mathbb{R}j_*E) \longrightarrow \mathbb{R}j_*(\mathbb{R}\underline{\Gamma}_{]X_k[}E) \tag{1.8.5}$$

is an isomorphism for any complex of abelian sheaves E on X_K^{an}. For the same reason, the canonical morphism $\mathbb{R}\varprojlim(I^\bullet\,\overline{L}^{\mathrm{an}}\otimes\Omega^\bullet_{\overline{X}_K^{\mathrm{an}}})\to\mathbb{R}j_*(L^{\mathrm{an}}\otimes\Omega^\bullet_{X_K^{\mathrm{an}}})$ induces an isomorphism

$$\mathbb{R}\underline{\Gamma}_{]X_k[}(\mathbb{R}\varprojlim(I^\bullet\,\overline{L}^{\mathrm{an}}\otimes\Omega^\bullet_{\overline{X}_K^{\mathrm{an}}}))\xrightarrow{\ \sim\ }\mathbb{R}\underline{\Gamma}_{]X_k[}(\mathbb{R}j_*(L^{\mathrm{an}}\otimes\Omega^\bullet_{X_K^{\mathrm{an}}})). \tag{1.8.6}$$

The *cospecialization morphism* for de Rham and rigid cohomologies with compact supports and coefficients in L is then defined as the composed morphism

$$\begin{aligned}\rho^L_{\mathrm{c},X}\ :\ \mathbb{R}\Gamma_{\mathrm{c,\,rig}}(X_k/K,\mathcal{L}) &:= \mathbb{R}\Gamma_{]X_k[}(X_K^{\mathrm{an}},L^{\mathrm{an}}\otimes\Omega^\bullet_{X_K^{\mathrm{an}}})\\ &\simeq \mathbb{R}\Gamma(\overline{X}_K^{\mathrm{an}},\mathbb{R}\underline{\Gamma}_{]X_k[}(\mathbb{R}j_*(L^{\mathrm{an}}\otimes\Omega^\bullet_{X_K^{\mathrm{an}}})))\\ &\simeq \mathbb{R}\Gamma(\overline{X}_K^{\mathrm{an}},\mathbb{R}\underline{\Gamma}_{]X_k[}(\mathbb{R}\varprojlim(I^\bullet\,\overline{L}^{\mathrm{an}}\otimes\Omega^\bullet_{\overline{X}_K^{\mathrm{an}}})))\\ &\to \mathbb{R}\Gamma(\overline{X}_K^{\mathrm{an}},\mathbb{R}\varprojlim(I^\bullet\,\overline{L}^{\mathrm{an}}\otimes\Omega^\bullet_{\overline{X}_K^{\mathrm{an}}}))\\ &\simeq \mathbb{R}\Gamma_{\mathrm{dR,\,c}}(X_K/K,L)\end{aligned} \tag{1.8.7}$$

deduced from the previous isomorphisms.

Remark. – It is again easy to check that the specialization and cospecialization morphisms for cohomologies with coefficients in L are compatible with pairings on cohomology. Together with the compatibility of the trace maps with cospecialization, this shows that, if the exponent $^\vee$ is used to denote $\mathcal{O}$-linear duals, and if X is of pure relative dimension n, we obtain a commutative diagram

$$\begin{array}{ccc}\mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K,L^\vee) & \longrightarrow & \mathbb{R}\Gamma_{\mathrm{dR,\,c}}(X_K/K,L)^\vee[-2n]\\ {\scriptstyle\rho_X^{L^\vee}}\downarrow & & \downarrow{\scriptstyle(\rho^L_{\mathrm{c},X})^\vee}\\ \mathbb{R}\Gamma_{\mathrm{rig}}(X_k/K,\mathcal{L}^\vee) & \longrightarrow & \mathbb{R}\Gamma_{\mathrm{c,\,rig}}(X_k/K,\mathcal{L})^\vee[-2n],\end{array}$$

and similarly exchanging the roles of cohomology and cohomology with compact supports. In particular, if L is such that $H^*_{\mathrm{rig}}(X_k/K,\mathcal{L}^\vee)$ and $H^*_{\mathrm{c,\,rig}}(X_k/K,\mathcal{L})$ are finite dimensional and satisfy Poincaré duality, then $\rho_X^{L^\vee}$ and $\rho^L_{\mathrm{c},X}$ are dual to each other (note that these properties are not necessarily true without additional assumptions on L).

1.9. Apart from the constant coefficients case, our main interest in this article will be in cohomology groups with coefficients in Dwork's F-isocrystal $\mathcal{L}_\pi$ [10, (1.5)]. We recall briefly here its construction and properties.

We assume now that K is a complete discretely valued field of mixed characteristics $(0,p)$, containing $\mathbb{Q}_p(\zeta_p)$, where ζ_p is a primitive p^{th} root of 1. Let R be the valuation ring of K, $\mathfrak{m}$ its maximal ideal, k its residue field, $S=\operatorname{Spec}R$, $\mathcal{S}=\operatorname{Spf}R$. We recall that, for each root π of the polynomial $t^{p-1}+p$, there exists a unique primitive p^{th} root ζ of 1 such that $\zeta\equiv 1+\pi \mod \pi^2$ (*cf.* [28, p. 636]). Therefore, the choice of an element $\pi\in K$ such that $\pi^{p-1}=-p$ is equivalent to the choice of a non trivial additive K-valued character of $\mathbb{Z}/p\mathbb{Z}$. In the following, we fix such an element π.

To deal with Frobenius actions, we will also assume that there exists an endomorphism $\sigma : R \to R$ lifting a power F^s of the Frobenius endomorphism of k, and such that $\sigma(\pi) = \pi$. In this article, the integer s and the endomorphism σ will be fixed, and we will work systematically with F^s-isocrystals with respect to (K, σ) rather than with F-isocrystals in the usual (absolute) sense. Therefore, we will simplify the terminology, and use the expression "F-isocrystal" to mean "F^s-isocrystal with respect to (K, σ)". Similarily, a Frobenius action will mean a σ-semi-linear action of the s-th power of the absolute Frobenius endomorphism.

The datum of π defines a rank 1 bundle with connection L_π on the affine line $\mathbb{A}^1_S$, by endowing the sheaf $\mathcal{O}_{\mathbb{A}^1_S}$ with the connection ∇_π such that

$$\nabla_\pi(a) = (\frac{da}{dt} + \pi a) \otimes dt, \tag{1.9.1}$$

where t is the canonical coordinate on $\mathbb{A}^1_S$. For any S-morphism $\varphi : X \to \mathbb{A}^1_S$, we will denote by $L_{\pi,\varphi}$ the inverse image of L_π, endowed with the inverse image connection.

Let ${\mathbb{A}^1_K}^{\rm an}$ be the rigid analytic affine line over K, $\widehat{\mathbb{A}}^1_{\mathcal{S}}$ the formal affine line over $\mathcal{S}$, ${\widehat{\mathbb{A}}^1_K}{}^{\rm an}$ its generic fiber (the closed unit disk in ${\mathbb{A}^1_K}^{\rm an}$), and let $j^\dagger_{\mathbb{A}^1}$ be the functor defined as in 1.1 using the strict neighbourhoods of ${\widehat{\mathbb{A}}^1_K}{}^{\rm an}$ in ${\mathbb{A}^1_K}^{\rm an}$. We denote by $L_\pi^{\rm an}$ the analytic vector bundle with connection associated to L_π on ${\mathbb{A}^1_K}^{\rm an}$, and we define $\mathcal{L}_\pi = j^\dagger_{\mathbb{A}^1} L_\pi^{\rm an}$. Then the connection $\nabla_\pi^{\rm an}$ induces an overconvergent connection on $\mathcal{L}_\pi$ [10, (1.5)]. The natural embedding of $\mathbb{A}^1_k$ into $\mathbb{A}^1_S$ allows to realize overconvergent isocrystals on $\mathbb{A}^1_k$ as $j^\dagger_{\mathbb{A}^1}\mathcal{O}_{{\mathbb{A}^1_K}^{\rm an}}$-modules endowed with an integrable and overconvergent connection. Therefore, we can view $\mathcal{L}_\pi$ as an overconvergent isocrystal on $\mathbb{A}^1_k$, defined by the sheaf $j^\dagger_{\mathbb{A}^1}\mathcal{O}_{{\mathbb{A}^1_K}^{\rm an}}$ endowed with the connection (1.9.1). Note that, if ψ is the character of $\mathbb{Z}/p\mathbb{Z}$ corresponding to π as above, then $\mathcal{L}_\pi = \mathcal{L}_{\psi^{-1}}$ in the notation of [10].

In addition, $\mathcal{L}_\pi$ has a canonical structure of F-isocrystal: if one lifts the s-th power of the absolute Frobenius endomorphism of $\mathbb{A}^1_k$ as the σ-linear endomorphism $F_{\mathbb{A}^1} : \mathbb{A}^1_S \to \mathbb{A}^1_S$ such that $F^*_{\mathbb{A}^1}(t) = t^q$, the Frobenius action $\phi : F^*_{\mathbb{A}^1}\mathcal{L}_\pi \to \mathcal{L}_\pi$ is given by

$$\phi(1 \otimes a) = \exp(\pi(t^q - t))a. \tag{1.9.2}$$

Since the category of overconvergent F-isocrystals is functorial with respect to morphisms of k-schemes of finite type, any such morphism $\varphi : X_k \to \mathbb{A}^1_k$ defines by pull-back an overconvergent F-isocrystal on X_k, which will be denoted by $\mathcal{L}_{\pi,\varphi}$. When φ is the reduction mod $\mathfrak{m}$ of a morphism of smooth S-schemes $\tilde{\varphi} : X \to \mathbb{A}^1_S$, then $\mathcal{L}_{\pi,\varphi}$ is obtained as the inverse image of $(j^\dagger_{\mathbb{A}^1}\mathcal{O}_{{\mathbb{A}^1_K}^{\rm an}}, \nabla_\pi^{\rm an})$ by the morphism of ringed spaces

$$\tilde{\varphi}_K^{\rm an} \ : \ (X_K^{\rm an}, j^\dagger_{X_k}\mathcal{O}_{X_K^{\rm an}}) \ \longrightarrow \ ({\mathbb{A}^1_K}^{\rm an}, j^\dagger_{\mathbb{A}^1}\mathcal{O}_{{\mathbb{A}^1_K}^{\rm an}}),$$

i. e. $\mathcal{L}_{\pi,\varphi} = j^\dagger_{X_k} L^{\rm an}_{\pi,\tilde{\varphi}}$ is given by $j^\dagger_{X_k}\mathcal{O}_{X_K^{\rm an}}$ endowed with the inverse image connection $\tilde{\varphi}_K^{\rm an\,*}(\nabla_\pi^{\rm an})$ (*cf.* [12, 2.5.5]). Moreover, if there exists a lifting $F_X : X \to X$ of the s-th power of the absolute Frobenius morphism of X_k as a σ-linear endomorphism of X, the action of Frobenius on $\mathcal{L}_{\pi,\varphi}$ is given by the composite isomorphism

$$F_X^*\tilde{\varphi}^*\mathcal{L}_\pi \xrightarrow{\sim} \tilde{\varphi}^*F_{\mathbb{A}^1}^*\mathcal{L}_\pi \xrightarrow{\sim} \tilde{\varphi}^*\mathcal{L}_\pi,$$

where the first isomorphism is the identification between the two inverse images provided by the Taylor series of the connection ∇_π^{an}, and the second one is the inverse image of ϕ by $\tilde{\varphi}$.

Let us point out that the hypotheses needed in the remark of 1.8 are satisfied by Dwork isocrystals. This is now known to be the case for any F-isocrystal, thanks to Kedlaya's results [42], but it can also be deduced from the case of the constant isocrystal. Indeed, this is a consequence of the relation between Dwork isocrystals and Artin-Schreier coverings, which we recall now in the algebraically liftable case (*cf.* [10, (1.5)], [14, 3.10]). Note that the case of $\mathcal{L}_\pi^\vee$ follows from the case of $\mathcal{L}_\pi$, since $\mathcal{L}_\pi^\vee = \mathcal{L}_{-\pi}$ (and $\mathcal{L}_{-\pi} \simeq \mathcal{L}_\pi$ if $p = 2$). Let $u : C \to \mathbb{A}_S^1$ be the finite covering defined by the equation $y^p - y - t = 0$. Then u is étale outside of the closed subscheme $\mathrm{Spec}(R[y]/(py^{p-1} - 1)) \subset C$, which is quasi-finite over $\mathrm{Spec}\, R$, concentrated in the generic fiber, and whose image in $\mathbb{A}_K^{1\,\mathrm{an}}$ lies outside the open disk of radius $p^{p/(p-1)} > 1$. Let $Y = X \times_{\mathbb{A}_S^1} C$, $v : Y \to X$, and let $\mathcal{Y}$ be the formal completion of Y. Then v_K^{an} is étale in a strict neighbourhood of $]Y_k[_{\mathcal{Y}}$ in Y_K^{an}. The additive group $\mathbb{Z}/p\mathbb{Z}$ acts on the sheaf $j_{X_k}^\dagger v_*^{\mathrm{an}}\mathcal{O}_{Y_K^{\mathrm{an}}}$, and $\mathcal{L}_{\pi,\varphi}$ is the direct factor of $j_{X_k}^\dagger v_*^{\mathrm{an}}\mathcal{O}_{Y_K^{\mathrm{an}}}$ on which $\mathbb{Z}/p\mathbb{Z}$ acts through the character ψ^{-1}. As $v_*^{\mathrm{an}}j_{Y_k}^\dagger\mathcal{O}_{Y_K^{\mathrm{an}}} \simeq j_{X_k}^\dagger v_*^{\mathrm{an}}\mathcal{O}_{Y_K^{\mathrm{an}}}$, it follows that the cohomology spaces $H_{\mathrm{rig}}^*(X_k/K, \mathcal{L}_{\pi,\varphi}^\vee)$ (resp. $H_{\mathrm{c,\,rig}}^*(X_k/K, \mathcal{L}_{\pi,\varphi})$) can be identified with the subspaces of $H_{\mathrm{rig}}^*(Y_k/K)$ (resp. $H_{\mathrm{c,\,rig}}^*(Y_k/K)$ on which $\mathbb{Z}/p\mathbb{Z}$ acts through ψ (resp. ψ^{-1}). Thus the finiteness of the spaces $H_{\mathrm{rig}}^*(X_k/K, \mathcal{L}_{\pi,\varphi}^\vee)$ and $H_{\mathrm{c,\,rig}}^*(X_k/K, \mathcal{L}_{\pi,\varphi})$ follows from the finiteness of rigid cohomology with constant coefficients.

Moreover, the same argument shows that Poincaré duality for $H_{\mathrm{rig}}^*(Y_k/K)$ induces a perfect pairing between the subspaces $H_{\mathrm{rig}}^*(X_k/K, \mathcal{L}_{\pi,\varphi}^\vee)$ and $H_{\mathrm{c,\,rig}}^*(X_k/K, \mathcal{L}_{\pi,\varphi})$. On the other hand, the transitivity of the trace map implies that the Poincaré pairing for $H_{\mathrm{rig}}^*(Y_k/K)$ can be identified with the Poincaré pairing for $H_{\mathrm{rig}}^*(X_k/K, v_*^{\mathrm{an}}j_{Y_k}^\dagger\mathcal{O}_{Y_K^{\mathrm{an}}})$ (defined via the trace map on the finite étale $j_{X_k}^\dagger\mathcal{O}_{X_K^{\mathrm{an}}}$-algebra $v_*^{\mathrm{an}}j_{Y_k}^\dagger\mathcal{O}_{Y_K^{\mathrm{an}}}$). Therefore, the previous pairing is equal to the pairing defined between these cohomology groups by Poincaré duality on X_k.

2. The overconvergent Fourier transform

Unless otherwise specified, we assume for the rest of the paper that the base field K satisfies the hypotheses of 1.9. Our goal in this section is to prove theorem 2.14, which will be the key result to interpret the cohomology of the analytic Dwork complexes for a projective hypersurface in terms of rigid homology groups.

This theorem can be viewed as an analogue for rigid cohomology of [3, th. 1.1] and [27, th. 0.2]. Our main tool here is the theory of $\mathcal{D}_{\mathcal{X},\,\mathbb{Q}}^\dagger$-modules, and our proof follows the method of [27] based on the Fourier transform. Therefore, we begin by briefly recalling some notions about $\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}$-modules and their Fourier transform.

2.1. Let $\mathcal{X}$ be a smooth formal $\mathcal{S}$-scheme of relative dimension n, and $\widehat{\mathcal{D}}_{\mathcal{X}}$ the p-adic completion of the standard sheaf of differential operators on $\mathcal{X}$. The sheaf $\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}$ is the subsheaf of rings of $\widehat{\mathcal{D}}_{\mathcal{X},\mathbb{Q}}$ such that, if $x_1,\dots,x_n$ are local coordinates on an affine open subset $\mathcal{U}\subset\mathcal{X}$, and $\partial_i=\partial/\partial x_i$, $1\le i\le n$, then

$$\Gamma(\mathcal{U},\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}) \;=\; \Big\{P=\sum_{\underline{k}\in\mathbb{N}^n} a_{\underline{k}}\underline{\partial}^{[\underline{k}]}\;\Big|\;\exists\, c,\eta\in\mathbb{R},\eta<1,\text{ such that }\|a_{\underline{k}}\|\le c\eta^{|\underline{k}|}\Big\},$$

where $\underline{\partial}^{[\underline{k}]}=\frac{1}{\underline{k}!}\underline{\partial}^{\underline{k}}$, $a_{\underline{k}}\in\Gamma(\mathcal{U},\mathcal{O}_{\mathcal{X},\mathbb{Q}})$, and $\|-\|$ is a quotient norm on the Tate algebra $\Gamma(\mathcal{U},\mathcal{O}_{\mathcal{X},\mathbb{Q}})$. It can also be written canonically as a union of p-adically complete subsheaves of rings

$$\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}} \;=\; \bigcup_{m\ge 0}\widehat{\mathcal{D}}^{(m)}_{\mathcal{X},\mathbb{Q}},$$

$\widehat{\mathcal{D}}^{(m)}_{\mathcal{X},\mathbb{Q}}$ being defined by

$$\Gamma(\mathcal{U},\widehat{\mathcal{D}}^{(m)}_{\mathcal{X},\mathbb{Q}}) \;=\; \Big\{P=\sum_{\underline{k}}\underline{q}^{(m)}_{\underline{k}}!\,b_{\underline{k}}\underline{\partial}^{[\underline{k}]}\in\Gamma(\mathcal{U},\widehat{\mathcal{D}}_{\mathcal{X},\mathbb{Q}})\;\Big|\;b_{\underline{k}}\to 0\text{ for }|\underline{k}|\to\infty\Big\},$$

with $\underline{k}=p^m\underline{q}^{(m)}_{\underline{k}}+\underline{r}^{(m)}_{\underline{k}}$, $0\le r^{(m)}_{k_i}<p^m$ for all i [13].

One can also introduce overconvergence conditions along a divisor $H\subset X$, where X is the special fiber of $\mathcal{X}$. Let $j:\mathcal{Y}\hookrightarrow\mathcal{X}$ be the inclusion of the complement of H in $\mathcal{X}$. The sheaf $\mathcal{O}_{\mathcal{X},\mathbb{Q}}({}^{\dagger}H)$ of functions with overconvergent singularities along H is the subsheaf of the usual direct image $j_*\mathcal{O}_{\mathcal{Y},\mathbb{Q}}$ such that, if $\mathcal{U}\subset\mathcal{X}$ is an affine open subset, and $h\in\Gamma(\mathcal{U},\mathcal{O}_{\mathcal{X}})$ a lifting of a local equation of H in the special fiber U of $\mathcal{U}$, then

$$\Gamma(\mathcal{U},\mathcal{O}_{\mathcal{X},\mathbb{Q}}({}^{\dagger}H)) \;=\; \Big\{g=\sum_{i\in\mathbb{N}} a_i/h^{i+1}\;\Big|\;\exists\, c,\eta\in\mathbb{R},\eta<1,\text{ such that }\|a_i\|\le c\eta^{i}\Big\},$$

where the a_i's belong to $\Gamma(\mathcal{U},\mathcal{O}_{\mathcal{X},\mathbb{Q}})$ and the norm is again a quotient norm. As for $\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}$, there is a canonical way to write $\mathcal{O}_{\mathcal{X},\mathbb{Q}}({}^{\dagger}H)$ as a union of p-adically complete sub-algebras. Indeed, if we fix $m\ge 0$, there exists a p-adically complete $\mathcal{O}_{\mathcal{X}}$-algebra $\widehat{\mathcal{B}}^{(m)}_{\mathcal{X}}(H)$, depending only on $\mathcal{X}$ and H, such that, on any affine open subset $\mathcal{U}$ as above,

$$\widehat{\mathcal{B}}^{(m)}_{\mathcal{X}}(H)_{|\mathcal{U}} \;\simeq\; \mathcal{O}_{\mathcal{U}}\{T\}/(h^{p^{m+1}}T-p),$$

T being an indeterminate [13]. The algebra $\mathcal{O}_{\mathcal{X},\mathbb{Q}}({}^{\dagger}H)$ is then given by

$$\mathcal{O}_{\mathcal{X},\mathbb{Q}}({}^{\dagger}H) \;=\; \bigcup_{m\ge 0}\widehat{\mathcal{B}}^{(m)}_{\mathcal{X},\mathbb{Q}}(H).$$

It depends only upon the support of H, and not upon the multiplicities of its components.

Moreover, each $\widehat{\mathcal{B}}^{(m)}_{\mathcal{X},\mathbb{Q}}(H)$ is endowed with a natural action of $\widehat{\mathcal{D}}^{(m)}_{\mathcal{X},\mathbb{Q}}$, compatible with its $\mathcal{O}_{\mathcal{X}}$-algebra structure [13]. Therefore, it is possible to endow the completed tensor product $\widehat{\mathcal{B}}^{(m)}_{\mathcal{X},\mathbb{Q}}(H)\widehat{\otimes}_{\mathcal{O}_{\mathcal{X}}}\widehat{\mathcal{D}}^{(m)}_{\mathcal{X},\mathbb{Q}}$ with a ring structure extending those of $\widehat{\mathcal{B}}^{(m)}_{\mathcal{X},\mathbb{Q}}(H)$ and $\widehat{\mathcal{D}}^{(m)}_{\mathcal{X},\mathbb{Q}}$. One can then define the ring of differential operators $\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}({}^{\dagger}H)$ as

$$\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}({}^{\dagger}H) \;:=\; \bigcup_{m\geq 0} \widehat{\mathcal{B}}^{(m)}_{\mathcal{X},\mathbb{Q}}(H)\widehat{\otimes}_{\mathcal{O}_{\mathcal{X}}}\widehat{\mathcal{D}}^{(m)}_{\mathcal{X},\mathbb{Q}}.$$

It follows easily from this definition that, for any affine open subset $\mathcal{U} \subset \mathcal{X}$ on which there exist local coordinates, and a local equation for H in U, the sections of $\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}({}^{\dagger}H)$ on $\mathcal{U}$ can be described as

$$\Gamma(\mathcal{U},\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}({}^{\dagger}H)) \;=\; \Big\{ g = \sum_{i,\underline{k}} \frac{a_{i,\underline{k}}}{h^{i+1}}\,\underline{\partial}^{[\underline{k}]} \;\Big|\; \exists\, c,\eta\in\mathbb{R}, \eta<1, \text{ such that } \|a_{i,\underline{k}}\| \leq c\eta^{i+|\underline{k}|} \Big\},$$

the notation being as above, and $a_{i,\underline{k}} \in \Gamma(\mathcal{U},\mathcal{O}_{\mathcal{X},\mathbb{Q}})$.

When $\mathcal{X}$ is proper, and H is viewed as a divisor at infinity providing a compactification of $Y := X \setminus H$, it is often convenient to replace the notation $\mathcal{O}_{\mathcal{X},\mathbb{Q}}({}^{\dagger}H)$ and $\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}({}^{\dagger}H)$ by $\mathcal{O}_{\mathcal{X},\mathbb{Q}}(\infty)$ and $\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}(\infty)$, if no confusion arises.

Recall that $\mathcal{O}_{\mathcal{X},\mathbb{Q}}({}^{\dagger}H)$ and $\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}({}^{\dagger}H)$ are coherent sheaves of rings, and that coherent modules over these sheaves satisfy the standard A and B theorems [13, 4.3.2 and 4.3.6].

2.2. Let $\mathcal{X}$ be affine, with H, $\mathcal{Y}$ be as before, and let $q : \mathcal{V} = \widehat{\mathbb{A}}^r_{\mathcal{X}} \to \mathcal{X}$ be the formal affine space of relative dimension r over $\mathcal{X}$, $q' : \mathcal{V}' \to \mathcal{X}$ the dual affine space. Assume that $\mathcal{X}$ has local coordinates $x_1,\ldots,x_n$ relative to $\mathcal{S}$, and let $t_1,\ldots,t_r$ (resp. $t'_1,\ldots,t'_r$) denote the standard coordinates on $\mathcal{V}$ (resp. $\mathcal{V}'$) relative to $\mathcal{X}$, ∂_{x_j}, ∂_{t_i} (resp. $\partial_{t'_i}$) the corresponding derivations. Assume also that there exists a section $h \in \Gamma(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ lifting a local equation of H in X.

We will use the results of [36], [37], [38] for the affine space $\mathcal{V}$ over $(\mathcal{X},H)$. While these references are written in the absolute case, *i. e.* $\mathcal{X} = \mathcal{S}$, it is easy to check that the proofs remain valid in our setting, requiring only obvious modifications.

Let us first define the *weakly complete Weyl algebra* $A^{\dagger}_r(\mathcal{X},H)$ associated to the affine space $\mathcal{V}$ over $(\mathcal{X},H)$. Let $\mathcal{W} = q^{-1}(\mathcal{Y}) = \widehat{\mathbb{A}}^r_{\mathcal{Y}}$, $A = \Gamma(\mathcal{X},\mathcal{O}_{\mathcal{X}}) \otimes K$, $\widehat{A}_r(\mathcal{Y}) = \Gamma(\mathcal{W},\widehat{\mathcal{D}}_{\mathcal{W}}) \otimes K$. An element $P \in \widehat{A}_r(\mathcal{Y})$ can be written

$$P \;=\; \sum_{i,\underline{j},\underline{k},\underline{l}} a_{i,\underline{j},\underline{k},\underline{l}}\, h^{-(i+1)}\underline{t}^{\underline{j}}\underline{\partial}_{\underline{t}}^{[\underline{k}]}\underline{\partial}_{\underline{x}}^{[\underline{l}]},$$

with coefficients $a_{i,\underline{j},\underline{k},\underline{l}} \in A$ such that $a_{i,\underline{j},\underline{k},\underline{l}} \to 0$ when $i + |\underline{j}| + |\underline{k}| + |\underline{l}| \to \infty$. Then $P \in A^{\dagger}_r(\mathcal{X},H) \subset \widehat{A}_r(\mathcal{Y})$ iff the $a_{i,\underline{j},\underline{k},\underline{l}}$ can be chosen so that there exists c,

$\eta \in \mathbb{R}$, with $\eta < 1$, such that

$$\|a_{i,\underline{j},\underline{k},\underline{l}}\| \leq c\eta^{i+|\underline{j}|+|\underline{k}|+|\underline{l}|}. \tag{2.2.1}$$

It is easy to check that $A_r^\dagger(\mathcal{X}, H)$ is a sub-K-algebra of $\widehat{A}_r(\mathcal{Y})$.

If $\mathcal{P} = \widehat{\mathbb{P}}^r_{\mathcal{X}}$ is the formal projective space of relative dimension r over $\mathcal{X}$, and $\mathcal{P}'$ the dual projective space, we will keep the notation q and q' for the projections $\mathcal{P} \to \mathcal{X}$ and $\mathcal{P}' \to \mathcal{X}$. Let V, P be the special fibers of $\mathcal{V}$ and $\mathcal{P}$, $H_\infty = P \setminus V$, $H_1 = q^{-1}(H) \cup H_\infty$ (resp. V', P', H'_∞, H'_1). We will use the notation

$$\mathcal{O}_{\mathcal{P},\mathbb{Q}}(\infty) := \mathcal{O}_{\mathcal{P},\mathbb{Q}}({}^\dagger H_1), \qquad \mathcal{D}^\dagger_{\mathcal{P},\mathbb{Q}}(\infty) := \mathcal{D}^\dagger_{\mathcal{P},\mathbb{Q}}({}^\dagger H_1),$$
$$\mathcal{O}_{\mathcal{P}',\mathbb{Q}}(\infty) := \mathcal{O}_{\mathcal{P}',\mathbb{Q}}({}^\dagger H'_1), \qquad \mathcal{D}^\dagger_{\mathcal{P}',\mathbb{Q}}(\infty) := \mathcal{D}^\dagger_{\mathcal{P}',\mathbb{Q}}({}^\dagger H'_1).$$

The following theorem shows that coherent $\mathcal{D}^\dagger_{\mathcal{P},\mathbb{Q}}(\infty)$-modules are determined by their global sections:

Theorem 2.3 (*cf.* [36], [37])**.** (i) *The ring $A_r^\dagger(\mathcal{X}, H)$ is coherent.*

(ii) *There exists a canonical isomorphism of K-algebras*

$$A_r^\dagger(\mathcal{X}, H) \simeq \Gamma(\mathcal{P}, \mathcal{D}^\dagger_{\mathcal{P},\mathbb{Q}}(\infty)). \tag{2.3.1}$$

(iii) *The functor $\Gamma(\mathcal{P}, -)$ induces an equivalence between the category of coherent $\mathcal{D}^\dagger_{\mathcal{P},\mathbb{Q}}(\infty)$-modules and the category of coherent $A_r^\dagger(\mathcal{X}, H)$-modules.*

2.4. Under the previous hypotheses, let us describe the *naive Fourier transform* for coherent $\mathcal{D}^\dagger_{\mathcal{P},\mathbb{Q}}(\infty)$-modules. Let $A'^\dagger_r(\mathcal{X}, H)$ be the weakly complete Weyl algebra associated to the dual affine space $\mathcal{V}'$ over $(\mathcal{X}, H)$. A basic observation is that the p-adic absolute value $|\pi^k/k!|$ satisfies the unequalities

$$1/kp \leq |\pi^k/k!| \leq 1$$

for any $k \in \mathbb{N}$. Comparing to the condition (2.2.1), it follows that the datum of π allows to define a continuous isomorphism

$$\phi: A'^\dagger_r(\mathcal{X}, H) \xrightarrow{\sim} A_r^\dagger(\mathcal{X}, H),$$

characterized by

$$\phi(t'_i) = -\partial_{t_i}/\pi, \qquad \phi(\partial_{t'_i}) = \pi t_i.$$

If $\mathcal{M}$ is a coherent $\mathcal{D}^\dagger_{\mathcal{P},\mathbb{Q}}(\infty)$-module, $\Gamma(\mathcal{P}, \mathcal{M})$ is a coherent $A_r^\dagger(\mathcal{X}, H)$-module. By restriction of scalars via ϕ, it can be viewed as a coherent $A'^\dagger_r(\mathcal{X}, H)$-module. The previous theorem shows that, up to canonical isomorphism, there is a unique coherent $\mathcal{D}^\dagger_{\mathcal{P}',\mathbb{Q}}(\infty)$-module $\mathcal{M}'$ such that $\Gamma(\mathcal{P}', \mathcal{M}') = \phi_*\Gamma(\mathcal{P}, \mathcal{M})$. By definition, the naive Fourier transform $\mathcal{F}_{\text{naive}}(\mathcal{M})$ of $\mathcal{M}$ is the $\mathcal{D}^\dagger_{\mathcal{P}',\mathbb{Q}}(\infty)$-module $\mathcal{M}'$.

2.5. To define the geometric Fourier transform, we will use the standard cohomological operations for $\mathcal{D}^\dagger_{\mathcal{X},\mathbb{Q}}$-modules. We refer to [17] and [18] for their general definitions and basic properties, and we only recall here a few facts needed for our constructions.

a) Let $\mathcal{X}$, $\mathcal{X}'$ be smooth formal schemes of relative dimensions $d_{\mathcal{X}}$, $d_{\mathcal{X}'}$ over $\mathcal{S}$, with special fibers X, X', $f : \mathcal{X} \to \mathcal{X}'$ an $\mathcal{S}$-morphism, and let $H \subset X$, $H' \subset X'$ be divisors such that $f^{-1}(H') \subset H$. We use the notation $\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}(\infty)$, $\mathcal{D}^{\dagger}_{\mathcal{X}',\mathbb{Q}}(\infty)$ for $\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}({}^{\dagger}H)$, $\mathcal{D}^{\dagger}_{\mathcal{X}',\mathbb{Q}}({}^{\dagger}H')$.

In this situation, the morphism f defines transfer bimodules $\mathcal{D}^{\dagger}_{\mathcal{X}\to\mathcal{X}',\mathbb{Q}}(\infty)$ and $\mathcal{D}^{\dagger}_{\mathcal{X}'\leftarrow\mathcal{X},\mathbb{Q}}(\infty)$ (*cf.* [18], or [36, 1.4.1]). The first one is a $(\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}(\infty), f^{-1}\mathcal{D}^{\dagger}_{\mathcal{X}',\mathbb{Q}}(\infty))$-bimodule and can be used to define an inverse image functor, which associates to a left $\mathcal{D}^{\dagger}_{\mathcal{X}',\mathbb{Q}}(\infty)$-module $\mathcal{N}$ the left $\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}(\infty)$-module given by

$$f^*\mathcal{N} \ := \ \mathcal{D}^{\dagger}_{\mathcal{X}\to\mathcal{X}',\mathbb{Q}}(\infty) \otimes_{f^{-1}\mathcal{D}^{\dagger}_{\mathcal{X}',\mathbb{Q}}(\infty)} f^{-1}\mathcal{N}.$$

Note that there is an abuse of notation here, since the definition of $\mathcal{D}^{\dagger}_{\mathcal{X}\to\mathcal{X}',\mathbb{Q}}(\infty)$ involves completions, and therefore this functor cannot be identified in general with the inverse image for $\mathcal{O}_{\mathcal{X}',\mathbb{Q}}$-modules or $\mathcal{O}_{\mathcal{X}',\mathbb{Q}}(\infty)$-modules.

For any complex $\mathcal{N}$ in $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathcal{X}',\mathbb{Q}}(\infty))$, the extraordinary inverse image functor $f^!$ is then defined as usual by

$$f^!(\mathcal{N}) \ := \ \mathbb{L}f^*(\mathcal{N})[d_{\mathcal{X}/\mathcal{X}'}],$$

where $d_{\mathcal{X}/\mathcal{X}'} = d_{\mathcal{X}} - d_{\mathcal{X}'}$.

When f is smooth, $\mathcal{D}^{\dagger}_{\mathcal{X}\to\mathcal{X}',\mathbb{Q}}(\infty)$ is flat over $f^{-1}\mathcal{D}^{\dagger}_{\mathcal{X}',\mathbb{Q}}(\infty)$, and f^* preserves coherence (*cf.* [17], [18]).

b) The bimodule $\mathcal{D}^{\dagger}_{\mathcal{X}'\leftarrow\mathcal{X},\mathbb{Q}}(\infty)$ is a $(f^{-1}\mathcal{D}^{\dagger}_{\mathcal{X}',\mathbb{Q}}(\infty), \mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}(\infty))$-bimodule. It can be used to define a direct image functor f_+ on $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}(\infty))$, which associates to $\mathcal{M} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}(\infty))$ the complex of left $\mathcal{D}^{\dagger}_{\mathcal{X}',\mathbb{Q}}(\infty)$-modules given by

$$f_+(\mathcal{M}) \ := \ \mathbb{R}f_*(\mathcal{D}^{\dagger}_{\mathcal{X}'\leftarrow\mathcal{X},\mathbb{Q}}(\infty) \overset{\mathbb{L}}{\otimes}_{\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}(\infty)} \mathcal{M}).$$

When f is projective, and H is the support of a relatively ample divisor, the acyclicity theorem of Huyghe [39, 5.4.1] shows that, if $n \geq 1$, $R^n f_*$ vanishes for coherent $\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}(\infty)$-modules. On the other hand, f_+ does not preserve coherence in general.

c) Finally, let us recall that overconvergent isocrystals may be viewed as $\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}(\infty)$-modules in the following way. If $\mathcal{X}$ is a smooth formal $\mathcal{S}$-scheme, $H \subset X$ a divisor in its special fiber, $Y = X \setminus H$, there is a specialization morphism $\mathrm{sp} : \mathcal{X}_K \to \mathcal{X}$, which is a continuous map, functorial with respect to $\mathcal{X}$, such that $\mathrm{sp}_* j^{\dagger}_Y \mathcal{O}_{\mathcal{X}_K} \simeq \mathcal{O}_{\mathcal{X},\mathbb{Q}}({}^{\dagger}H)$ (*cf.* [12] or [14, 1.1]). The functor sp_* is exact on the category of coherent $\mathcal{O}_{\mathcal{X}_K}$-modules, and, since H is a divisor, it is also exact on the category of coherent $j^{\dagger}_Y\mathcal{O}_{\mathcal{X}_K}$-modules (*cf.* [14, proof of 4.2]). If $\mathcal{L}$ is an isocrystal on Y which is overconvergent along H, then $\mathrm{sp}_*\mathcal{L}$ is a coherent $\mathcal{O}_{\mathcal{X},\mathbb{Q}}({}^{\dagger}H)$-module, endowed with a canonical structure of $\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}({}^{\dagger}H)$-module [13, 4.4]. By [13, 4.4.5 and 4.4.12], the functor sp_* allows to identify the category of isocrystals on Y which are overconvergent along H with a full subcategory of the category of coherent $\mathcal{D}^\dagger_{\mathcal{X},\mathbb{Q}}({}^\dagger H)$-modules. Moreover, this identification is compatible with inverse images [36, 1.5.4]. Therefore, we will generally misuse notation, and simply keep the letter $\mathcal{L}$ to denote $\mathrm{sp}_*\,\mathcal{L}$.

2.6. We will need the geometric Fourier transform in a more general setup than the situation considered for the definition of the naive Fourier transform. We assume here that $\mathcal{X}$ is a smooth formal scheme of relative dimension n over $\mathcal{S}$, endowed with a divisor $H \subset X$, and that $q : \mathcal{V} \to \mathcal{X}$, $q' : \mathcal{V}' \to \mathcal{X}$ are dual vector bundles of rank r over $\mathcal{X}$. We denote $\mathcal{Y} = \mathcal{X} \setminus H$, $\mathcal{W} = q^{-1}(\mathcal{Y})$, $\mathcal{W}' = {q'}^{-1}(\mathcal{Y})$. Let $q : \mathcal{P} \to \mathcal{X}$ and $q' : \mathcal{P}' \to \mathcal{X}$ be relative projective closures of $\mathcal{V}$ and $\mathcal{V}'$, $\mathcal{P}'' := \mathcal{P}' \times_{\mathcal{X}} \mathcal{P}$, with projections $p : \mathcal{P}'' \to \mathcal{P}$, $p' : \mathcal{P}'' \to \mathcal{P}'$, $q'' : \mathcal{P}'' \to \mathcal{X}$. We write V, V', Y, W, W', P, P', P'' for the special fibers. We define the divisors H_∞, H'_∞, H_1, H'_1 as in 2.2, and we endow P'' with the divisor

$$H_2 \;=\; p^{-1}(H_\infty) \cup p'^{-1}(H'_\infty) \cup q''^{-1}(H),$$

whose support is $p^{-1}(H_1) \cup p'^{-1}(H'_1)$. We will use the notation

$$\mathcal{O}_{\mathcal{P}'',\mathbb{Q}}(\infty) \;:=\; \mathcal{O}_{\mathcal{P}'',\mathbb{Q}}({}^\dagger H_2), \qquad \mathcal{D}^\dagger_{\mathcal{P}'',\mathbb{Q}}(\infty) \;:=\; \mathcal{D}^\dagger_{\mathcal{P}'',\mathbb{Q}}({}^\dagger H_2).$$

To construct the kernel of the geometric Fourier transform, we apply 2.5 c) to $\mathcal{P}''$. Let

$$\mu :\; V' \times_X V \longrightarrow \mathbb{A}^1_X \longrightarrow \mathbb{A}^1_k$$

be the morphism obtained by composing the canonical pairing between V' and V with the projection to $\mathbb{A}^1_k$. As seen in 1.9, μ defines by functoriality a canonical rank 1 overconvergent F-isocrystal $\mathcal{L}_{\pi,\mu}$ over $V' \times_X V$. A fortiori, $j^\dagger_{W'\times W}\mathcal{L}_{\pi,\mu}$ defines an F-isocrystal on $W' \times_Y W$, overconvergent along H_2. We will denote by $\mathcal{L}^W_{\pi,\mu}$ the rank one $\mathcal{O}_{\mathcal{P}'',\mathbb{Q}}(\infty)$-module $\mathrm{sp}_*(j^\dagger_{W'\times W}\mathcal{L}_{\pi,\mu})$, endowed with its natural $\mathcal{D}^\dagger_{\mathcal{P}'',\mathbb{Q}}(\infty)$-module structure, and its Frobenius action. In particular, $\mathcal{L}^W_{\pi,\mu}$ has a canonical basis e, and, above an open subset of $\mathcal{X}$ on which $\mathcal{V}$ has linear coordinates $t_1, \ldots, t_r$ (with dual coordinates $t'_1, \ldots, t'_r$), its underlying connection $\nabla_{\pi,\mu}$ is given by

$$\nabla_{\pi,\mu}(ae) \;=\; e \otimes (da + \pi a(\sum_i t'_i dt_i + t_i dt'_i)). \tag{2.6.1}$$

If $\mathcal{M}''$ is a coherent $\mathcal{D}^\dagger_{\mathcal{P}'',\mathbb{Q}}(\infty)$-module, then $\mathcal{L}^W_{\pi,\mu} \otimes_{\mathcal{O}_{\mathcal{P}'',\mathbb{Q}}(\infty)} \mathcal{M}''$, viewed as a left $\mathcal{D}^\dagger_{\mathcal{P}'',\mathbb{Q}}(\infty)$-module through the standard tensor product structure, is still coherent.

We can now define the *geometric Fourier transform* of a coherent $\mathcal{D}^\dagger_{\mathcal{P},\mathbb{Q}}(\infty)$-module $\mathcal{M}$ by

$$\mathcal{F}_{\mathrm{geom}}(\mathcal{M}) \;:=\; p'_+(\mathcal{L}^W_{\pi,\mu} \otimes_{\mathcal{O}_{\mathcal{P}'',\mathbb{Q}}(\infty)} p^*(\mathcal{M})). \tag{2.6.2}$$

For simplicity, we do not use the standard shifts here, so that, when $\mathcal{M}$ consists in a single coherent $\mathcal{D}^\dagger_{\mathcal{P},\mathbb{Q}}(\infty)$-module placed in degree 0, $\mathcal{L}^W_{\pi,\mu} \otimes p^*(\mathcal{M})$ is a coherent $\mathcal{D}^\dagger_{\mathcal{P}'',\mathbb{Q}}(\infty)$-module placed in degree 0.

A priori, $\mathcal{F}_{\mathrm{geom}}(\mathcal{M})$ is only known to be a complex in $D^{\mathrm{b}}(\mathcal{D}^\dagger_{\mathcal{P}',\mathbb{Q}}(\infty))$. The following theorem, due to Huyghe, shows that $\mathcal{F}_{\mathrm{geom}}$ transforms a coherent $\mathcal{D}^\dagger_{\mathcal{P},\mathbb{Q}}(\infty)$-module into a coherent $\mathcal{D}^\dagger_{\mathcal{P}',\mathbb{Q}}(\infty)$-module, and provides the comparison between the naive and geometric Fourier transforms:

Theorem 2.7 ([36], [40])**.** *Let $\mathcal{M}$ be a coherent $\mathcal{D}^\dagger_{\mathcal{P},\mathbb{Q}}(\infty)$-module.*

(i) *The complex $\mathcal{F}_{\mathrm{geom}}(\mathcal{M})$ is acyclic in degrees $\neq 0$. In degree 0, its cohomology sheaf is a coherent $\mathcal{D}^\dagger_{\mathcal{P}',\mathbb{Q}}(\infty)$-module.*

(ii) *Under the assumptions of 2.4, there is a natural isomorphism of $\mathcal{D}^\dagger_{\mathcal{P}',\mathbb{Q}}(\infty)$-modules*

$$\mathcal{F}_{\mathrm{geom}}(\mathcal{M}) \;\simeq\; \mathcal{F}_{\mathrm{naive}}(\mathcal{M}). \tag{2.7.1}$$

One of the main steps in the proof of (ii) is the computation of the geometric Fourier transform of $\mathcal{D}^\dagger_{\mathcal{P},\mathbb{Q}}(\infty)$. We will actually use this result under the more general assumptions of 2.6. By construction, the bimodule $\mathcal{D}^\dagger_{\mathcal{P}'\leftarrow\mathcal{P}'',\mathbb{Q}}(\infty)$ is isomorphic to $\mathcal{D}^{\dagger,\,\mathrm{d}}_{\mathcal{P}''\to\mathcal{P}',\mathbb{Q}}(\infty) \otimes_{\mathcal{O}_{\mathcal{P}''}} \omega_{\mathcal{P}''/\mathcal{P}'}$, where $\mathcal{D}^{\dagger,\,\mathrm{d}}_{\mathcal{P}''\to\mathcal{P}',\mathbb{Q}}(\infty)$ is the analogue of $\mathcal{D}^\dagger_{\mathcal{P}''\to\mathcal{P}',\mathbb{Q}}(\infty)$ obtained using the right $\mathcal{O}_{\mathcal{P}',\mathbb{Q}}(\infty)$-module structure of $\mathcal{D}^\dagger_{\mathcal{P}',\mathbb{Q}}(\infty)$ rather than the left one, and $\omega_{\mathcal{P}''/\mathcal{P}'} = \wedge^r \Omega^1_{\mathcal{P}''/\mathcal{P}'}$. If we write $\mathcal{V} = \mathrm{Spf}(\widehat{\mathbb{S}(\mathcal{E})})$, where $\mathcal{E}$ is locally free of rank r over $\mathcal{O}_{\mathcal{X}}$, then there is a canonical isomorphism

$$\mathcal{O}_{\mathcal{P}'',\mathbb{Q}}(\infty) \otimes_{\mathcal{O}_{\mathcal{P}''}} \Omega^1_{\mathcal{P}''/\mathcal{P}'} \;\simeq\; \mathcal{O}_{\mathcal{P}'',\mathbb{Q}}(\infty) \otimes_{\mathcal{O}_{\mathcal{P}''}} q''^*\mathcal{E}.$$

Since $\mathcal{L}^W_{\pi,\mu}$ has a canonical section, one can use this remark to define a canonical map

$$\mathcal{D}^\dagger_{\mathcal{P}',\mathbb{Q}}(\infty) \otimes q'^*(\wedge^r \mathcal{E}) \;\to\; p'_*(\mathcal{D}^\dagger_{\mathcal{P}'\leftarrow\mathcal{P}'',\mathbb{Q}}(\infty) \otimes \mathcal{L}^W_{\pi,\mu}).$$

On the other hand, $\mathcal{D}^\dagger_{\mathcal{P}'\leftarrow\mathcal{P}'',\mathbb{Q}}(\infty) \overset{\mathbb{L}}{\otimes} (\mathcal{L}^W_{\pi,\mu} \otimes \mathcal{D}^\dagger_{\mathcal{P}''\to\mathcal{P},\mathbb{Q}}(\infty))$ can be computed using the Spencer resolution of $\mathcal{D}^\dagger_{\mathcal{P}''\to\mathcal{P},\mathbb{Q}}(\infty)$

$$\cdots \;\to\; \mathcal{D}^\dagger_{\mathcal{P}'',\mathbb{Q}}(\infty) \otimes \mathcal{T}_{\mathcal{P}''/\mathcal{P}} \;\to\; \mathcal{D}^\dagger_{\mathcal{P}'',\mathbb{Q}}(\infty) \;\to\; \mathcal{D}^\dagger_{\mathcal{P}''\to\mathcal{P},\mathbb{Q}}(\infty) \;\to\; 0,$$

which gives a canonical map

$$p'_*(\mathcal{D}^\dagger_{\mathcal{P}'\leftarrow\mathcal{P}'',\mathbb{Q}}(\infty) \otimes \mathcal{L}^W_{\pi,\mu}) \;\to\; \mathcal{H}^0(p'_+(\mathcal{L}^W_{\pi,\mu} \otimes \mathcal{D}^\dagger_{\mathcal{P}''\to\mathcal{P},\mathbb{Q}}(\infty))).$$

Using appropriate division theorems, one can then prove that the composite map is an isomorphism

$$\mathcal{D}^\dagger_{\mathcal{P}',\mathbb{Q}}(\infty) \otimes q'^*(\wedge^r \mathcal{E}) \;\xrightarrow{\;\sim\;}\; \mathcal{F}_{\mathrm{geom}}(\mathcal{D}^\dagger_{\mathcal{P},\mathbb{Q}}(\infty)) \tag{2.7.2}$$

(*cf.* [40], and [22] for the complex analytic case). When $\mathcal{E}$ is a free $\mathcal{O}_{\mathcal{X}}$-module, the choice of a basis of $\mathcal{E}$ provides a trivialisation of $\wedge^r\mathcal{E}$ and the isomorphism (2.7.2) reduces to the inverse of (2.7.1) for $\mathcal{D}^\dagger_{\mathcal{P},\mathbb{Q}}(\infty)$.

Remark. – One can also give a Gauss-Manin style description of $\mathcal{F}_{\mathrm{geom}}(\mathcal{M})$, using the de Rham resolution of the bimodule $\mathcal{D}^\dagger_{\mathcal{P}'\leftarrow\mathcal{P}'',\mathbb{Q}}(\infty)$ to compute p'_+ :

$$\cdots \;\to\; \Omega^{r-1}_{\mathcal{P}''/\mathcal{P}'}\otimes\mathcal{D}^\dagger_{\mathcal{P}'',\mathbb{Q}}(\infty) \;\to\; \Omega^{r}_{\mathcal{P}''/\mathcal{P}'}\otimes\mathcal{D}^\dagger_{\mathcal{P}'',\mathbb{Q}}(\infty) \;\to\; \mathcal{D}^\dagger_{\mathcal{P}'\leftarrow\mathcal{P}'',\mathbb{Q}}(\infty) \;\to\; 0.$$

In general, this is only a resolution in the category of $(p'^{-1}\mathcal{O}_{\mathcal{P}',\mathbb{Q}}(\infty),\mathcal{D}^\dagger_{\mathcal{P}'',\mathbb{Q}}(\infty))$-bimodules. However, when $\mathcal{V}$ is the trivial bundle $\widehat{\mathbb{A}}^r_{\mathcal{X}}$, it can be viewed as a resolution in the category of $(p'^{-1}\mathcal{D}^\dagger_{\mathcal{P}',\mathbb{Q}}(\infty),\mathcal{D}^\dagger_{\mathcal{P}'',\mathbb{Q}}(\infty))$-bimodules. Indeed, the product structure $\mathcal{P}''=\mathcal{P}'\times_{\mathcal{S}}\widehat{\mathbb{P}}^r_{\mathcal{S}}$ and the fact that $H_2=p^{-1}(H_1)\cup p'^{-1}(H'_1)$ allow to define a ring homomorphism $\mathcal{D}^\dagger_{\mathcal{P}',\mathbb{Q}}(\infty)\to p'_*\mathcal{D}^\dagger_{\mathcal{P}'',\mathbb{Q}}(\infty)$ (this is a consequence of [17, 2.3.1]). The ring $\mathcal{D}^\dagger_{\mathcal{P}'',\mathbb{Q}}(\infty)$ is thus endowed with a natural structure of left $(p'^{-1}\mathcal{D}^\dagger_{\mathcal{P}',\mathbb{Q}}(\infty),p^{-1}\mathcal{O}_{\mathcal{P},\mathbb{Q}}(\infty))$-bimodule from which the claim follows easily.

Therefore, one obtains for $\mathcal{F}_{\mathrm{geom}}(\mathcal{M})$ the $\mathcal{O}_{\mathcal{P}',\mathbb{Q}}(\infty)$-linear presentation

$$\mathcal{F}_{\mathrm{geom}}(\mathcal{M}) \;\simeq\; \mathrm{Coker}\left(p'_*(\Omega^{r-1}_{\mathcal{P}''/\mathcal{P}'}(\infty)\otimes(\mathcal{L}^W_{\pi,\mu}\otimes p^*\mathcal{M}))\to p'_*(\Omega^{r}_{\mathcal{P}''/\mathcal{P}'}(\infty)\otimes(\mathcal{L}^W_{\pi,\mu}\otimes p^*\mathcal{M}))\right), \tag{2.7.3}$$

where $\Omega^\bullet_{\mathcal{P}''/\mathcal{P}'}(\infty)=\Omega^\bullet_{\mathcal{P}''/\mathcal{P}'}\otimes\mathcal{O}_{\mathcal{P}'',\mathbb{Q}}(\infty)$, all tensor products are taken over $\mathcal{O}_{\mathcal{P}'',\mathbb{Q}}(\infty)$, and the arrows are defined by the tensor product connection on $\mathcal{L}^W_{\pi,\mu}\otimes p^*\mathcal{M}$. Over an open subset on which $\mathcal{V}$ is trivial, the choice of a trivialisation turns this presentation into a $\mathcal{D}^\dagger_{\mathcal{P}',\mathbb{Q}}(\infty)$-linear presentation, which induces on the cokernel the canonical $\mathcal{D}^\dagger_{\mathcal{P}',\mathbb{Q}}(\infty)$-module structure of $\mathcal{F}_{\mathrm{geom}}(\mathcal{M})$. In particular, this induced structure is independent of the trivialisation, and can be glued on variable open subsets of $\mathcal{X}$.

2.8. In view of our applications to Dwork cohomology, we want now to describe the geometric Fourier transform of the constant $\mathcal{D}^\dagger_{\mathcal{P},\mathbb{Q}}(\infty)$-module $\mathcal{O}_{\mathcal{P},\mathbb{Q}}(\infty)$ as a local cohomology sheaf.

So let us first recall (in the smooth and liftable case, and for an overconvergent isocrystal) the definition of the *overconvergent local cohomology sheaves* with supports in a closed subvariety. As before, we denote by $\mathcal{X}$ a smooth formal scheme, $H\subset X$ a divisor in its special fiber, $\mathcal{Y}=\mathcal{X}\setminus H$. Let $Z\subset X$ be a closed subscheme, $\mathcal{U}=\mathcal{Y}\setminus Z=\mathcal{X}\setminus(H\cup Z)$. If $\mathcal{L}$ is an isocrystal on Y overconvergent along H, the overconvergent local cohomology of $\mathcal{L}$ with support in Z is the complex of $\mathcal{O}_{\mathcal{X},\mathbb{Q}}({}^\dagger H)$-modules given by

$$\mathbb{R}\underline{\Gamma}^\dagger_Z(\mathcal{L}):=\mathbb{R}\,\mathrm{sp}_*(\mathcal{L}\to j^\dagger_U(\mathcal{L})).$$

This complex can be endowed with a natural structure of complex of $\mathcal{D}^\dagger_{\mathcal{X},\mathbb{Q}}({}^\dagger H)$-module: the case where $H=\emptyset$ is treated in [11, (4.1.5)], and one proceeds in the same way in the general case, using [13, 4.4.3]. Its cohomology sheaves will be denoted by $\mathcal{H}^{\dagger\, i}_Z(\mathcal{L})$. When $\mathcal{L}$ is the constant isocrystal, we will use the notation $\mathbb{R}\underline{\Gamma}^{\dagger}_Z(\mathcal{O}_{\mathcal{X},\mathbb{Q}}({}^\dagger H))$, $\mathcal{H}^{\dagger\, i}_Z(\mathcal{O}_{\mathcal{X},\mathbb{Q}}({}^\dagger H))$.

Remark. – Using the method of [17, 4.4.4], the definition of overconvergent local cohomology can be extended to coherent $\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}({}^\dagger H)$-modules (we refer to [18] for the comparison between the two methods for overconvergent isocrystals). If $i : \mathcal{Z} \hookrightarrow \mathcal{X}$ is a closed immersion of smooth formal $\mathcal{S}$-schemes, and $\mathcal{M} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}({}^\dagger H))$, we obtain with this definition a canonical isomorphism [17, (4.4.5.2)]

$$i_+ i^! \mathcal{M} \xrightarrow{\ \sim\ } \mathbb{R}\underline{\Gamma}^{\dagger}_Z(\mathcal{M}). \tag{2.8.1}$$

For smooth subvarieties, the local structure of overconvergent local cohomology is similar to the local structure of algebraic local cohomology:

Proposition 2.9. *With the previous notation, assume that Z is smooth of codimension r in X. Then:*

(i) *For all $i \neq r$, $\mathcal{H}^{\dagger\, i}_Z(\mathcal{O}_{\mathcal{X},\mathbb{Q}}({}^\dagger H)) = 0$.*

(ii) *Let $t_1, \dots, t_n$ be local coordinates on $\mathcal{X}$ such that $Z = V(\bar{t}_1, \dots, \bar{t}_r)$, where $\bar{t}_i$ is the reduction of t_i mod $\mathfrak{m}$. Then the map sending 1 to $1/t_1 \cdots t_r$ provides a $\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}({}^\dagger H)$-linear isomorphism*

$$\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}({}^\dagger H)\Big/\Big(\sum_{i=1}^{r}\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}({}^\dagger H)t_i + \sum_{i=r+1}^{n}\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}({}^\dagger H)\partial_{t_i}\Big) \xrightarrow{\ \sim\ } \mathcal{H}^{\dagger\, r}_Z(\mathcal{O}_{\mathcal{X},\mathbb{Q}}({}^\dagger H)). \tag{2.9.1}$$

We may assume that $\mathcal{X}$ is affine and has local coordinates $t_1, \dots, t_n$ as in (ii). For $1 \le i_1 < \dots < i_k \le r$, let $H_{i_1\dots i_k} = H \cup V(t_{i_1}) \cup \dots \cup V(t_{i_k})$, and $\mathcal{U}_{i_1\dots i_k} = \mathcal{X} \setminus H_{i_1\dots i_k}$. Using the open covering of $\mathcal{U}$ given by $\mathcal{U}_1, \dots, \mathcal{U}_r$, we get a Čech exact sequence [14, (1.2.2)]

$$0 \to j^\dagger_U \mathcal{O}_{\mathcal{X}_K} \to \bigoplus_{i=1}^{r} j^\dagger_{U_i}\mathcal{O}_{\mathcal{X}_K} \to \dots \to j^\dagger_{U_{1\dots r}}\mathcal{O}_{\mathcal{X}_K} \to 0 .$$

Since $H_{i_1\dots i_k}$ is the support of a divisor, the complex $\mathbb{R}\,\mathrm{sp}_*\, j^\dagger_{U_{i_1\dots i_k}}\mathcal{O}_{\mathcal{X}_K}$ is reduced to its cohomology sheaf in degree 0, which is $\mathcal{O}_{\mathcal{X},\mathbb{Q}}({}^\dagger H_{i_1\dots i_k})$. Thus the complex $\mathbb{R}\underline{\Gamma}^{\dagger}_Z(\mathcal{O}_{\mathcal{X},\mathbb{Q}}({}^\dagger H))$ is isomorphic to

$$\mathcal{O}_{\mathcal{X},\mathbb{Q}}({}^\dagger H) \to \bigoplus_{i=1}^{r}\mathcal{O}_{\mathcal{X},\mathbb{Q}}({}^\dagger H_i) \to \dots \to \mathcal{O}_{\mathcal{X},\mathbb{Q}}({}^\dagger H_{1\dots r}) \to 0 \to \dots .$$

On the other hand, the sequence $t_1, \dots, t_r$ is regular on $\mathcal{O}_{\mathcal{X},\mathbb{Q}}({}^\dagger H)$, hence the complex

$$0 \to \mathcal{O}_{\mathcal{X},\mathbb{Q}}({}^\dagger H) \to \bigoplus_{i=1}^{r}\mathcal{O}_{\mathcal{X},\mathbb{Q}}({}^\dagger H)[1/t_i] \to \dots \to \mathcal{O}_{\mathcal{X},\mathbb{Q}}({}^\dagger H)[1/t_1 \dots t_r] \to 0$$

is acyclic in degrees $\neq r$. Note that this is a complex of $\mathcal{O}_{\mathcal{X},\mathbb{Q}}({}^\dagger H)\otimes_{\mathcal{O}_{\mathcal{X},\mathbb{Q}}}\mathcal{D}_{\mathcal{X},\mathbb{Q}}$-modules; let $\mathcal{D}_{\mathcal{X},\mathbb{Q}}({}^\dagger H)=\mathcal{O}_{\mathcal{X},\mathbb{Q}}({}^\dagger H)\otimes_{\mathcal{O}_{\mathcal{X},\mathbb{Q}}}\mathcal{D}_{\mathcal{X},\mathbb{Q}}$. Since $\widehat{\mathcal{B}}^{(m)}_{\mathcal{X}}(H)\widehat{\otimes}_{\mathcal{O}_{\mathcal{X}}}\widehat{\mathcal{D}}^{(m)}_{\mathcal{X}}$ is flat over $\widehat{\mathcal{B}}^{(m)}_{\mathcal{X}}(H)\otimes_{\mathcal{O}_{\mathcal{X}}}\mathcal{D}^{(m)}_{\mathcal{X}}$ for all m [13, (3.3.4)], $\mathcal{D}^\dagger_{\mathcal{X},\mathbb{Q}}({}^\dagger H)$ is flat over $\mathcal{D}_{\mathcal{X},\mathbb{Q}}({}^\dagger H)$. Hence, assertion (i) will follow if we prove that, for any sequence $i_1<\ldots<i_k$, the canonical map

$$\mathcal{D}^\dagger_{\mathcal{X},\mathbb{Q}}({}^\dagger H)\otimes_{\mathcal{D}_{\mathcal{X},\mathbb{Q}}({}^\dagger H)}\mathcal{O}_{\mathcal{X},\mathbb{Q}}({}^\dagger H)[1/t_{i_1}\ldots t_{i_k}]\ \to\ \mathcal{O}_{\mathcal{X},\mathbb{Q}}({}^\dagger H_{i_1\ldots i_k}) \tag{2.9.2}$$

is an isomorphism. A standard computation shows that the map sending 1 to $1/t_{i_1}\ldots t_{i_k}$ yields an isomorphism

$$\mathcal{D}_{\mathcal{X},\mathbb{Q}}({}^\dagger H)\Big/\Big(\sum_{j=1}^{k}\mathcal{D}_{\mathcal{X},\mathbb{Q}}({}^\dagger H)\partial_{t_{i_j}}t_{i_j}+\sum_{i\neq i_1,\ldots,i_k}\mathcal{D}_{\mathcal{X},\mathbb{Q}}({}^\dagger H)\partial_{t_i}\Big)\\ \xrightarrow{\ \sim\ }\ \mathcal{O}_{\mathcal{X},\mathbb{Q}}({}^\dagger H)[1/t_{i_1}\ldots t_{i_k}].$$

Similarly, the map sending 1 to $1/t_{i_1}\cdots t_{i_k}$ gives an isomorphism

$$\mathcal{D}^\dagger_{\mathcal{X},\mathbb{Q}}({}^\dagger H)\Big/\Big(\sum_{j=1}^{k}\mathcal{D}^\dagger_{\mathcal{X},\mathbb{Q}}({}^\dagger H)\partial_{t_{i_j}}t_{i_j}+\sum_{i\neq i_1,\ldots,i_k}\mathcal{D}^\dagger_{\mathcal{X},\mathbb{Q}}({}^\dagger H)\partial_{t_i}\Big)\xrightarrow{\ \sim\ }\mathcal{O}_{\mathcal{X},\mathbb{Q}}({}^\dagger H_{i_1\ldots i_k}). \tag{2.9.3}$$

Indeed, this is proposition (4.3.2) of [11] if $H=\emptyset$; as observed in the remark of [14, 4.7], this remains true in the general case, using the method of the proof of [14, 4.6]. These presentations imply that the map (2.9.2) is an isomorphism.

Finally, the presentation (2.9.3), combined with the exact sequence

$$\bigoplus_{i=1}^{r}\mathcal{O}_{\mathcal{X},\mathbb{Q}}({}^\dagger H_{1\ldots\widehat{i}\ldots r})\ \to\ \mathcal{O}_{\mathcal{X},\mathbb{Q}}({}^\dagger H_{1\ldots r})\ \to\ \mathcal{H}^{\dagger r}_{Z}(\mathcal{O}_{\mathcal{X},\mathbb{Q}}({}^\dagger H))\ \to\ 0,$$

implies assertion (ii).

Returning to the setting of 2.6, our next result gives the Fourier transform of $\mathcal{O}_{\mathcal{P},\mathbb{Q}}(\infty)$:

Proposition 2.10. *Under the assumptions of* 2.6*, let $i:\mathcal{X}\to\mathcal{V}'$ be the zero section, and let us identify $\mathcal{X}$ with its image in $\mathcal{V}'\subset\mathcal{P}'$. Then there exists a canonical isomorphism of $\mathcal{D}^\dagger_{\mathcal{P}',\mathbb{Q}}(\infty)$-modules*

$$\mathcal{F}_{\mathrm{geom}}(\mathcal{O}_{\mathcal{P},\mathbb{Q}}(\infty))\ \simeq\ \mathcal{H}^{\dagger r}_{X}(\mathcal{O}_{\mathcal{P}',\mathbb{Q}}(\infty)). \tag{2.10.1}$$

Let us first assume that $\mathcal{X}$ is affine, with local coordinates $x_1,\ldots,x_n$ defining derivations $\partial_{x_1},\ldots,\partial_{x_n}$ and that $\mathcal{V}=\mathbb{A}^r_{\mathcal{X}}$, with standard linear coordinates $t_1,\ldots,t_r$ defining derivations $\partial_{t_1},\ldots,\partial_{t_r}$. The Spencer resolution of $\mathcal{O}_{\mathcal{P},\mathbb{Q}}(\infty)$ over $\mathcal{D}^\dagger_{\mathcal{P},\mathbb{Q}}(\infty)$ yields an isomorphism

$$\Gamma(\mathcal{P},\mathcal{O}_{\mathcal{P},\mathbb{Q}}(\infty))\ \simeq\ A^\dagger_r(\mathcal{X},H)\Big/\Big(\sum_{i=1}^{r}A^\dagger_r(\mathcal{X},H)\partial_{t_i}+\sum_{j}A^\dagger_r(\mathcal{X},H)\partial_{x_j}\Big).$$

Therefore, the naive Fourier transform of $\mathcal{O}_{\mathcal{P},\mathbb{Q}}(\infty)$ is defined by

$$\Gamma(\mathcal{P}, \mathcal{F}_{\text{naive}}(\mathcal{O}_{\mathcal{P},\mathbb{Q}}(\infty))) \;\simeq\; A_r^\dagger(\mathcal{X}, H)\Big/\Big(\sum_{i=1}^r A_r^\dagger(\mathcal{X}, H)t_i + \sum_j A_r^\dagger(\mathcal{X}, H)\partial_{x_j}\Big).$$

On the other hand, proposition 2.9 (ii) shows that $\Gamma(\mathcal{P}, \mathcal{H}_X^{\dagger\, r}(\mathcal{O}_{\mathcal{P}',\mathbb{Q}}(\infty)))$ has precisely the same presentation. Using 2.7, we obtain a $\mathcal{D}^\dagger_{\mathcal{P}',\mathbb{Q}}(\infty)$-linear isomorphism

$$\mathcal{F}_{\text{geom}}(\mathcal{O}_{\mathcal{P},\mathbb{Q}}(\infty)) \;\simeq\; \mathcal{F}_{\text{naive}}(\mathcal{O}_{\mathcal{P},\mathbb{Q}}(\infty)) \;\simeq\; \mathcal{H}_X^{\dagger\, r}(\mathcal{O}_{\mathcal{P}',\mathbb{Q}}(\infty)) \tag{2.10.2}$$

as in (2.10.1).

To complete the construction of (2.10.1) in the general case, we need to glue the previous isomorphisms (2.10.2) on variable open subsets where $\mathcal{V}$ is trivial, and therefore to prove that they are independent of the choice of coordinates.

Let $\mathcal{E} = \mathcal{O}^r_{\mathcal{X}}$, with basis $t_1, \ldots, t_r$, defining the dual basis $t'_1, \ldots, t'_r$. Thus $t_1, \ldots, t_r$ are coordinates on $\mathbb{A}^r_{\mathcal{X}} = \mathrm{Spf}(\widehat{\mathbb{S}(\mathcal{E})})$, and $t'_1, \ldots, t'_r$ are a regular sequence of generators for the ideal of the zero section in $\mathcal{V}'$. The $\mathcal{D}^\dagger_{\mathcal{P},\mathbb{Q}}(\infty)$-linear surjective map $\mathcal{D}^\dagger_{\mathcal{P},\mathbb{Q}}(\infty) \to \mathcal{O}_{\mathcal{P},\mathbb{Q}}(\infty)$ provides the following diagram

$$\begin{array}{ccccc}
\mathcal{F}_{\text{geom}}(\mathcal{O}_{\mathcal{P},\mathbb{Q}}(\infty)) & \xrightarrow{\sim} & \mathcal{F}_{\text{naive}}(\mathcal{O}_{\mathcal{P},\mathbb{Q}}(\infty)) & \xrightarrow{\sim} & \mathcal{H}_X^{\dagger\, r}(\mathcal{O}_{\mathcal{P}',\mathbb{Q}}(\infty)) \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{F}_{\text{geom}}(\mathcal{D}^\dagger_{\mathcal{P},\mathbb{Q}}(\infty)) & \xrightarrow{\sim} & \mathcal{F}_{\text{naive}}(\mathcal{D}^\dagger_{\mathcal{P},\mathbb{Q}}(\infty)) & = & \mathcal{D}^\dagger_{\mathcal{P}',\mathbb{Q}}(\infty) \\
\sim\uparrow & & & \nearrow\sim & \\
\mathcal{D}^\dagger_{\mathcal{P}',\mathbb{Q}}(\infty) \otimes q'^*(\wedge^r \mathcal{E}) & & & &
\end{array} .$$

In this diagram, the upper composite arrow is the isomorphism (2.10.2), and the right vertical arrow is the map corresponding to (2.9.1). The left square is commutative by functoriality, and the right one because of the definition of the isomorphism (2.10.2). The oblique arrow is the trivialisation given by the basis $t_1 \wedge \ldots \wedge t_r$ of $\wedge^r \mathcal{E}$, and yields a commutative triangle as explained in 2.7. Since the left composite arrow is canonical, it suffices to check that the composite map $\mathcal{D}^\dagger_{\mathcal{P}',\mathbb{Q}}(\infty) \otimes q'^*(\wedge^r \mathcal{E}) \to \mathcal{H}_X^{\dagger\, r}(\mathcal{O}_{\mathcal{P}',\mathbb{Q}}(\infty))$ is independent of the choice of coordinates. Equivalently, it suffices to check that the image of 1 under the corresponding map $\mathcal{D}^\dagger_{\mathcal{P}',\mathbb{Q}}(\infty) \to \mathcal{H}_X^{\dagger\, r}(\mathcal{O}_{\mathcal{P}',\mathbb{Q}}(\infty)) \otimes q'^*(\wedge^r(\mathcal{E}^\vee))$ is independent of the choice of coordinates. Since it is the section $\frac{1}{t'_1 \cdots t'_r} \otimes t'_1 \wedge \ldots \wedge t'_r$, this is clear.

Remarks. – (i) It follows from this local calculation that, if one computes the Fourier transform of $\mathcal{O}_{\mathcal{P},\mathbb{Q}}(\infty)$ using the isomorphism

$$\mathcal{F}_{\text{geom}}(\mathcal{O}_{\mathcal{P},\mathbb{Q}}(\infty)) \;\simeq\; \mathcal{H}^0(p'_*(\Omega^\bullet_{\mathcal{P}''/\mathcal{P}'}(\infty) \otimes \mathcal{L}^W_{\pi,\mu}[r])) \tag{2.10.3}$$

as in (2.7.3), the isomorphism

$$\varepsilon \;:\; \mathcal{H}^r(p'_*(\Omega^\bullet_{\mathcal{P}''/\mathcal{P}'}(\infty) \otimes \mathcal{L}^W_{\pi,\mu})) \;\xrightarrow{\;\sim\;}\; \mathcal{H}_X^{\dagger\, r}(\mathcal{O}_{\mathcal{P}',\mathbb{Q}}(\infty)) \tag{2.10.4}$$

defined by (2.10.1) is the unique $\mathcal{D}^{\dagger}_{\mathcal{P}',\mathbb{Q}}(\infty)$-linear isomorphism such that, for any basis $t_1, \ldots, t_r$ of $\mathcal{E}$,

$$\varepsilon((dt_1 \wedge \ldots \wedge dt_r) \otimes e) = \frac{1}{t'_1 \cdots t'_r},$$

where e is the canonical section of $\mathcal{L}^W_{\pi,\mu}$.

(ii) Following the method of [8], one could also give a more conceptual proof of proposition 2.10. However, our local computation will be useful to check the compatibility of (2.10.1) with Frobenius actions.

2.11. To define Frobenius actions, we will use the fact that the inverse image functor for $\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}(\infty)$-modules can be defined with respect to non necessarily liftable morphisms of schemes between the special fibers, as explained in [16, 2.1.6]. In particular, the inverse image $F^*\mathcal{M}$ of a left $\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}(\infty)$-module $\mathcal{M}$ by the s-th power of the absolute Frobenius endomorphism of X can be defined without assuming that it can be lifted to $\mathcal{X}$. When such a lifting F exists, $F^*\mathcal{M}$ is the usual inverse image by F, and, up to canonical isomorphism, it is independent of the choice of F. Applying this remark to $\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}(\infty)$, one can associate to the s-th power of the absolute Frobenius endomorphism of X a transfer bimodule $\mathcal{D}^{\dagger}_{\mathcal{X}\to\mathcal{X},\mathbb{Q}}(\infty)$ which can be locally identified to $F^*\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}(\infty)$, for any local lifting F. This allows to extend globally the definition of the functor F^* to the derived category $D^{\mathrm{b}}(\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}(\infty))$ by the usual formula $F^*\mathcal{M} = \mathcal{D}^{\dagger}_{\mathcal{X}\to\mathcal{X},\mathbb{Q}}(\infty) \overset{\mathbb{L}}{\otimes}_{\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}(\infty)} \mathcal{M}$. A Frobenius action on a complex $\mathcal{M} \in D^{\mathrm{b}}(\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}(\infty))$ can then be defined as an isomorphism $\Phi : F^*\mathcal{M} \xrightarrow{\sim} \mathcal{M}$ in $D^{\mathrm{b}}(\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}(\infty))$.

The existence of Frobenius actions will generally follow from the functoriality properties of rigid cohomology. Thus, the isomorphism

$$\mathcal{F}_{\mathrm{geom}}(\mathcal{O}_{\mathcal{P},\mathbb{Q}}(\infty)) \;=\; p'_+(\mathcal{L}^W_{\pi,\mu}) \;\simeq\; p'_*(\Omega^{\bullet}_{\mathcal{P}''/\mathcal{P}'}(\infty) \otimes \mathcal{L}^W_{\pi,\mu})[r]$$

provides a Frobenius action on $\mathcal{F}_{\mathrm{geom}}(\mathcal{O}_{\mathcal{P},\mathbb{Q}}(\infty))$ coming from the F-isocrystal structure of $\mathcal{L}^W_{\pi,\mu}$ and the functoriality properties of rigid cohomomology for V'' relatively to V'. Similarily, the canonical F-isocrystal structure of $\mathcal{O}_{\mathcal{P}',\mathbb{Q}}(\infty)$ and the fact that F^* commutes with the $j^{\dagger}$ functors provides a Frobenius action on $\mathcal{H}^{\dagger\, r}_X(\mathcal{O}_{\mathcal{P}',\mathbb{Q}}(\infty))$.

Proposition 2.12. *The canonical isomorphism* (2.10.1) *commutes with the Frobenius actions defined above on* $\mathcal{F}_{\mathrm{geom}}(\mathcal{O}_{\mathcal{P},\mathbb{Q}}(\infty)) = \mathcal{H}^r(p'_*(\Omega^{\bullet}_{\mathcal{P}''/\mathcal{P}'}(\infty) \otimes \mathcal{L}^W_{\pi,\mu}))$ *and* $\mathcal{H}^{\dagger\, r}_X(\mathcal{O}_{\mathcal{P}',\mathbb{Q}}(\infty))$.

This is a local property, hence we may assume that $\mathcal{X}$ is affine and that $\mathcal{V} = \mathbb{A}^r_{\mathcal{X}}$; let $\mathcal{X} = \mathrm{Spf}\, A$. We may also assume that there exists $h \in A$ lifting a local equation of H in X. We fix a lifting $F : \mathcal{X} \to \mathcal{X}$ of the s-th power of the absolute Frobenius endomorphism of X, and we extend it to $\mathcal{V}$ and $\mathcal{V}'$ by setting $F^*(t_i) = t_i^q$, $F^*(t'_i) = t'^{\,q}_i$.

The Frobenius action on $\mathcal{H}^r(p'_*(\Omega^{\bullet}_{\mathcal{P}''/\mathcal{P}'}(\infty) \otimes \mathcal{L}^W_{\pi,\mu}))$ is deduced by functoriality from the chosen liftings of Frobenius, and the given action on $\mathcal{L}_{\pi,\mu}$. The latter is the composite isomorphism

$$F^*\mathcal{L}_{\pi,\mu} \;=\; F^*\mu^*\mathcal{L}_\pi \;\xrightarrow{\sim}\; \mu^*F^*\mathcal{L}_\pi \;\xrightarrow{\sim}\; \mu^*\mathcal{L}_\pi \;=\; \mathcal{L}_{\pi,\mu},$$

where the first isomorphism is the Taylor isomorphism comparing the two inverse images (since $F \circ \mu \neq \mu \circ F$), and the second is the pull-back of the Frobenius action on $\mathcal{L}_\pi$ given by (1.9.2). It follows that the Frobenius action on $\mathcal{L}_{\pi,\mu}$ is given by multiplication by

$$\exp\pi((\sum_i t_i t'_i)^q - \sum_i t_i t'_i)\exp\pi(\sum_i t_i^q {t'_i}^q - (\sum_i t_i t'_i)^q) \;=\; \prod_i \exp\pi(t_i^q {t'_i}^q - t_i t'_i). \tag{2.12.1}$$

Thus we want to compare the action induced by (2.12.1) on $\mathcal{H}^r(p'_*(\Omega^\bullet_{\mathcal{P}''/\mathcal{P}'}(\infty) \otimes \mathcal{L}^W_{\pi,\mu}))$ with the canonical action of Frobenius on $\mathcal{H}^{\dagger r}_X(\mathcal{O}_{\mathcal{P}',\mathbb{Q}}(\infty))$.

In order to follow the action of Frobenius, we will use the following description of (2.10.1). Let B be the weak completion of $A[h^{-1}, t'_1, \ldots, t'_r, t_1, \ldots, t_r]$. Then the complex $\Gamma(\mathcal{P}', p'_*(\Omega^\bullet_{\mathcal{P}''/\mathcal{P}'}(\infty) \otimes \mathcal{L}^W_{\pi,\mu}))$ can be identified with the total complex $K^\bullet$ associated to the r-uple complex such that $K^{j_1,\ldots,j_r} = B_\mathbb{Q}$ if $(j_1, \ldots, j_r) \in \{0,1\}^r$ and 0 otherwise, the differentials being defined by

$$\nabla_i \;=\; \partial_{t_i} + \pi t'_i \;:\; K^{j_1,\ldots,j_{i-1},0,j_{i+1},\ldots,j_r} \;\longrightarrow\; K^{j_1,\ldots,j_{i-1},1,j_{i+1},\ldots,j_r}.$$

On the other hand, the covering of $V' \times V \setminus X \times V$ by the open subsets $D(t'_i)$ provides (thanks to [14, (1.2.3)]) a Čech exact sequence

$$0 \;\to\; B_\mathbb{Q} \;\to\; \bigoplus_i B[{t'_i}^{-1}]^\dagger_\mathbb{Q} \;\to\; \ldots \;\to\; B[{t'_1}^{-1}, \ldots, {t'_r}^{-1}]^\dagger_\mathbb{Q} \;\to$$
$$B[{t'_1}^{-1}, \ldots, {t'_r}^{-1}]^\dagger_\mathbb{Q} / \sum_i B[{t'_1}^{-1}, \ldots, \widehat{{t'_i}^{-1}}, \ldots, {t'_r}^{-1}]^\dagger_\mathbb{Q} \;\to\; 0,$$

in which the last term is equal to $\Gamma(\mathcal{P}', p'_*(\mathcal{H}^{\dagger r}_P(\mathcal{O}_{\mathcal{P}'',\mathbb{Q}}(\infty))))$; here $\mathcal{P}$ is embedded into $\mathcal{P}''$ thanks to the closed immersion $i' = i \times \mathrm{Id}_\mathcal{P}$. Since the arrows commute with the ∇_i's, we can build out of this sequence a similar exact sequence of r-uple complexes. Thus, the total complexes associated to these r-uple complexes sit in a similar exact sequence of complexes

$$0 \;\to\; K^\bullet \;\to\; \bigoplus_i K^\bullet_i \;\to\; \ldots \;\to\; K^\bullet_{1,\ldots,r} \;\to\; K'^\bullet \;\to\; 0, \tag{2.12.2}$$

in which the complexes $K^\bullet$ and $K'^\bullet$ are respectively

$$\Gamma(\mathcal{P}', p'_*(\Omega^\bullet_{\mathcal{P}''/\mathcal{P}'}(\infty) \otimes \mathcal{L}^W_{\pi,\mu}))$$

and

$$\Gamma(\mathcal{P}', p'_*(\Omega^\bullet_{\mathcal{P}''/\mathcal{P}'}(\infty) \otimes \mathcal{H}^{\dagger r}_P(\mathcal{L}^W_{\pi,\mu}))).$$

It is clear from its construction that this is an exact sequence of complexes of $\Gamma(\mathcal{P}', \mathcal{D}^\dagger_{\mathcal{P}',\mathbb{Q}}(\infty))$-modules (if we view here $\Gamma(\mathcal{P}', \mathcal{D}^\dagger_{\mathcal{P}',\mathbb{Q}}(\infty))$ as a subring of

$\Gamma(\mathcal{P}', p'_*\mathcal{D}^\dagger_{\mathcal{P}'',\mathbb{Q}}(\infty))$ thanks to the choice of coordinates as in the remark of 2.7), and that it is compatible with the action of Frobenius.

In the next lemma, we will show that ∇_i is an isomorphism on any term of the form $B[t'^{-1}_{i_1}, \ldots, t'^{-1}_{i_k}]^\dagger_{\mathbb{Q}}$ when i is one of the i_j's. Therefore, all complexes $K^\bullet_{i_1,\ldots,i_k}$ are acyclic, and the exact sequence (2.12.2) provides in $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathcal{P}',\mathbb{Q}}(\infty))$ an isomorphism $K^\bullet[r] \xrightarrow{\sim} K'^\bullet$. Both complexes are actually reduced to a single cohomology sheaf in degree 0, and we obtain a $\mathcal{D}^\dagger_{\mathcal{P}',\mathbb{Q}}(\infty)$-linear isomorphism

$$\mathcal{H}^r(p'_*(\Omega^\bullet_{\mathcal{P}''/\mathcal{P}'}(\infty) \otimes \mathcal{L}^W_{\pi,\mu})) \xrightarrow{\sim} \mathcal{H}^0(p'_*(\Omega^\bullet_{\mathcal{P}''/\mathcal{P}'}(\infty) \otimes \mathcal{H}^{\dagger r}_P(\mathcal{L}^W_{\pi,\mu}))), \quad (2.12.3)$$

compatible with the natural Frobenius actions. Using 2.9 (i) and (2.8.1), we obtain

$$\mathcal{H}^{\dagger r}_P(\mathcal{L}^W_{\pi,\mu}) \simeq \mathbb{R}\underline{\Gamma}^\dagger_P(\mathcal{L}^W_{\pi,\mu})[r] \simeq i'_+ i'^!(\mathcal{L}^W_{\pi,\mu})[r] \simeq i'_+ i'^*(\mathcal{L}^W_{\pi,\mu}).$$

But $i'^*(\mathcal{L}^W_{\pi,\mu})$ is the trivial F-isocrystal $\mathcal{O}_{\mathcal{P},\mathbb{Q}}(\infty)$, as follows from (2.6.1) and (2.12.1) (or from the fact that the restriction of μ to $V \hookrightarrow V' \times_X V$ factors through the zero section of $\mathbb{A}^1_k$). Thus (2.12.3) can be written as a Frobenius compatible isomorphism

$$\mathcal{H}^r(p'_*(\Omega^\bullet_{\mathcal{P}''/\mathcal{P}'}(\infty) \otimes \mathcal{L}^W_{\pi,\mu})) \xrightarrow{\sim} \mathcal{H}^0(p'_*(\Omega^\bullet_{\mathcal{P}''/\mathcal{P}'}(\infty) \otimes \mathcal{H}^{\dagger r}_P(\mathcal{O}_{\mathcal{P}'',\mathbb{Q}}(\infty)))).$$

The target can be computed using the canonical isomorphisms

$$\begin{aligned}\mathcal{H}^0(p'_*(\Omega^\bullet_{\mathcal{P}''/\mathcal{P}'}(\infty) \otimes \mathcal{H}^{\dagger r}_P(\mathcal{O}_{\mathcal{P}'',\mathbb{Q}}(\infty)))) &\xrightarrow{\sim} \mathcal{H}^0(p'_+ i'_+(\mathcal{O}_{\mathcal{P},\mathbb{Q}}(\infty))[-r]) \\ &\xrightarrow{\sim} \mathcal{H}^0(i_+ q_+(\mathcal{O}_{\mathcal{P},\mathbb{Q}}(\infty))[-r])\end{aligned}$$

(*cf.* [17, 4.3.6, 4.3.7]). The complex $q_+(\mathcal{O}_{\mathcal{P},\mathbb{Q}}(\infty))[-r]$ is given by the relative de Rham cohomology of an overconvergent power series algebra over $\mathcal{O}_{\mathcal{X},\mathbb{Q}}(\infty)$. Therefore, it is isomorphic to $\mathcal{O}_{\mathcal{X},\mathbb{Q}}(\infty)$, and we obtain

$$\mathcal{H}^0(p'_*(\Omega^\bullet_{\mathcal{P}''/\mathcal{P}'}(\infty) \otimes \mathcal{H}^{\dagger r}_P(\mathcal{O}_{\mathcal{P}'',\mathbb{Q}}(\infty)))) \xrightarrow{\sim} \mathcal{H}^{\dagger r}_X(\mathcal{O}_{\mathcal{P}',\mathbb{Q}}(\infty)).$$

It is easy to check that this isomorphism is compatible with the functoriality actions of Frobenius. By composition, we finally obtain an isomorphism

$$\mathcal{H}^r(p'_*(\Omega^\bullet_{\mathcal{P}''/\mathcal{P}'}(\infty) \otimes \mathcal{L}^W_{\pi,\mu})) \xrightarrow{\sim} \mathcal{H}^{\dagger r}_X(\mathcal{O}_{\mathcal{P}',\mathbb{Q}}(\infty)) \quad (2.12.4)$$

which is compatible with the Frobenius actions.

To end the proof, we only have to check that this isomorphism is equal to (2.10.4). Remark 2.10 (i) shows that it suffices to check that (2.12.4) maps $(dt_1 \wedge \ldots \wedge dt_r) \otimes e$ to $1/t'_1 \cdots t'_r$. If $r = 1$ (which will be the case in our application), the sequence (2.12.2) is a short exact sequence of length 1 complexes, and the claim follows from an easy computation based on the snake lemma. In the general case, one can first observe that it is enough to prove the analogous claim in the algebraic situation, where each $B[t'^{-1}_{i_1}, \ldots, t'^{-1}_{i_k}]^\dagger$ is replaced by $A[h^{-1}, t'_1, \ldots, t'_r, t_1, \ldots, t_r, t'^{-1}_{i_1}, \ldots, t'^{-1}_{i_k}]$, because it provides a complex similar to (2.12.2), mapping to (2.12.2). Thus one can define for algebraic de Rham cohomology a morphism similar to (2.12.4) and mapping to it. It is then enough to observe that, in the algebraic situation, the rank r case can be reduced to the rank 1 case by a multiplicativity argument.

We now check the acyclicity lemma used in the above proof.

Lemma 2.13. *For any sequence* $1 \leq i_1 < \ldots < i_s \leq r$, *and any* $i \in \{i_1, \ldots, i_s\}$, *the map*

$$\nabla_i \ = \ \partial_{t_i} + \pi t'_i \ : \ B[{t'_{i_1}}^{-1}, \ldots, {t'_{i_s}}^{-1}]^\dagger_{\mathbb{Q}} \ \to \ B[{t'_{i_1}}^{-1}, \ldots, {t'_{i_s}}^{-1}]^\dagger_{\mathbb{Q}}$$

is an isomorphism.

We may assume that $i = i_1 = 1$, and write t, t', ∂, ∇ for $t_1, t'_1, \partial_{t_1}, \nabla_1$. Let $C = A[h^{-1}, t', \ldots, t'_r, t_2, \ldots, t_r, t'^{-1}, \ldots, {t'_{i_s}}^{-1}]^\dagger$. We endow the Tate algebra $\widehat{A[h^{-1}]}_{\mathbb{Q}}$ with any Banach norm, extend it by setting $\|t_i\| = \|t'_i\| = \|{t'_i}^{-1}\| = 1$ for all i, and take the induced norm on $C_{\mathbb{Q}}$. Then any element $\varphi \in B[t'^{-1}, \ldots, {t'_{i_s}}^{-1}]^\dagger_{\mathbb{Q}}$ can be written uniquely as a series $\varphi = \sum_{k \geq 0} \alpha_k t^k$, where the coefficients $\alpha_k \in C_{\mathbb{Q}}$ are such that $\|\alpha_k\| \leq c\eta^k$ for some constants $c, \eta \in \mathbb{R}, \eta < 1$. If $\nabla(\varphi) = 0$, then

$$(k+1)\alpha_{k+1} + \pi t' \alpha_k = 0$$

for all $k \geq 0$. Then the coefficient α_k is given by

$$\alpha_k = (-1)^k \alpha_0 \frac{\pi^k}{k!} {t'}^k,$$

and $\|\alpha_k\| = \|\alpha_0\| |\pi^k/k!|$. As $\overline{\lim} |\pi^k/k!|^{1/k} = 1$, the α_k cannot be the coefficients of an element of $B[t'^{-1}, \ldots, {t'_{i_s}}^{-1}]^\dagger_{\mathbb{Q}}$ if $\alpha_0 \neq 0$. Therefore, $\varphi = 0$.

To check the surjectivity of ∇, let $\psi = \sum_{k \geq 0} \beta_k t^k$ be a given element in $B[t'^{-1}, \ldots, {t'_{i_s}}^{-1}]^\dagger_{\mathbb{Q}}$. We must find a sequence of elements $\alpha_k \in C_{\mathbb{Q}}$ such that

$$(k+1)\alpha_{k+1} + \pi t' \alpha_k = \beta_k$$

for all $k \geq 0$. Because there exists $c, \eta \in \mathbb{R}$ such that $\|\beta_k\| \leq c\eta^k$, with $\eta < 1$, we can define α_k as the sum of the series

$$\alpha_k \ := \ \frac{1}{\pi t'} \frac{(-\pi t')^k}{k!} \sum_{j \geq k} \frac{j!}{(-\pi t')^j} \beta_j,$$

which converges in $C_{\mathbb{Q}}$. The coefficients α_k satisfy the previous relation, and it is easy to check that, for any η' such that $\eta < \eta' < 1$, there exists $c' \in \mathbb{R}$ such that $\|\alpha_k\| \leq c' {\eta'}^k$. Thus they define a series $\varphi \in B[t'^{-1}, \ldots, {t'_{i_s}}^{-1}]^\dagger_{\mathbb{Q}}$ such that $\nabla(\varphi) = \psi$.

Remark. – Similar computations show that the algebraic analogue of lemma 2.13, where $B[{t'_{i_1}}^{-1}, \ldots, {t'_{i_s}}^{-1}]^\dagger$ is replaced by $A[h^{-1}, t'_1, \ldots, t'_r, t_1, \ldots, t_r, {t'_{i_1}}^{-1}, \ldots, {t'_{i_k}}^{-1}]$, is also true.

We can now deduce from 2.10 the main result of this section. Our proof follows the method of [27].

Theorem 2.14. *Under the assumptions of* 2.6*, let* $u : \mathcal{X} \to \mathcal{V}'$ *be a section,* $\mathcal{L}^W_{\pi,u} = (u \times \mathrm{Id})^* \mathcal{L}^W_{\pi,\mu}$ *the overconvergent* F*-isocrystal on* W *obtained by functoriality,* $\mathcal{Z} \subset \mathcal{X}$ *the zero locus of* u*,* Z *the special fiber of* $\mathcal{Z}$*. Assume that* Z *is locally a complete intersection of codimension* r *in* X*. Then there exists in* $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathcal{X},\mathbb{Q}}(\infty))$ *a canonical isomorphism*

$$q_+(\mathcal{L}^W_{\pi,u}) \ \simeq \ \mathbb{R}\underline{\Gamma}^\dagger_Z(\mathcal{O}_{\mathcal{X},\mathbb{Q}}(\infty))[r], \tag{2.14.1}$$

compatible with the Frobenius actions on both sides.

As in 2.11, the Frobenius actions are defined by functoriality using comparison with rigid cohomology.

Let $u' = u \times \mathrm{Id} : \mathcal{P} \hookrightarrow \mathcal{P}''$. We consider the cartesian square

$$\begin{array}{ccc} \mathcal{P} & \xhookrightarrow{u'} & \mathcal{P}'' \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle p'} \\ \mathcal{X} & \xhookrightarrow{u} & \mathcal{P}', \end{array}$$

and we apply the functor $u^!$ to the isomorphism (2.10.1). In view of 2.9, we obtain an isomorphism

$$u^!(p'_+(\mathcal{L}^W_{\pi,\mu})) \simeq u^!(\mathcal{H}^{\dagger\, r}_X(\mathcal{O}_{\mathcal{P}',\mathbb{Q}}(\infty))) \simeq u^!(\mathbb{R}\underline{\Gamma}^\dagger_X(\mathcal{O}_{\mathcal{P}',\mathbb{Q}}(\infty))[r]) \tag{2.14.2}$$

in $D^{\mathrm{b}}(\mathcal{D}^\dagger_{\mathcal{X},\mathbb{Q}}(\infty))$. Thus it is enough to check that there exists canonical isomorphisms

$$\begin{aligned} u^!(p'_+(\mathcal{L}^W_{\pi,\mu})) \ &\simeq \ q_+(\mathcal{L}^W_{\pi,u})[-r], & (2.14.3) \\ u^!(\mathbb{R}\underline{\Gamma}^\dagger_X(\mathcal{O}_{\mathcal{P}',\mathbb{Q}}(\infty))) \ &\simeq \ \mathbb{R}\underline{\Gamma}^\dagger_Z(\mathcal{O}_{\mathcal{X},\mathbb{Q}}(\infty))[-r]. & (2.14.4) \end{aligned}$$

We only give a rough sketch here, refering to [18] for more details. Using the techniques of [17] to handle direct and inverse limits, one can reduce to proving the analogs of (2.14.3) and (2.14.4) in $D^{\mathrm{b}}(\mathcal{D}^{(m)}_{X_i})$, where the subscript i denotes the reduction mod p^i, and m is any positive integer. The first isomorphism is a base change result, which follows from the following two facts:

a) If $\widetilde{\mathcal{D}}^{(m)}_{P'_i} = \mathcal{D}^{(m)}_{P'_i} \otimes \mathcal{B}^{(m)}_{P'_i}(H'_1)$, $\widetilde{\mathcal{D}}^{(m)}_{P''_i} = \mathcal{D}^{(m)}_{P''_i} \otimes \mathcal{B}^{(m)}_{P''_i}(H_2)$ and $\widetilde{\mathcal{D}}^{(m)}_{P'_i \leftarrow P''_i} = \mathcal{D}^{(m)}_{P'_i \leftarrow P''_i} \otimes \mathcal{B}^{(m)}_{P''_i}(H_2)$, then $\widetilde{\mathcal{D}}^{(m)}_{P'_i \leftarrow P''_i}$ is a flat $\widetilde{\mathcal{D}}^{(m)}_{P'_i}$-module, whose formation commutes with base changes;

b) If $\widetilde{\mathcal{D}}^{(m)}_{P_i} = \mathcal{D}^{(m)}_{P_i} \otimes \mathcal{B}^{(m)}_{P_i}(H_1)$, $\widetilde{\mathcal{D}}^{(m)}_{X_i \leftarrow P_i} = \mathcal{D}^{(m)}_{X_i \leftarrow P_i} \otimes \mathcal{B}^{(m)}_{P_i}(H_1)$ and $\mathcal{M}$ is a flat quasi-coherent $\widetilde{\mathcal{D}}^{(m)}_{P''_i}$-module, then the canonical base change morphism

$$\mathbb{L}u^*(\mathbb{R}p'_*(\widetilde{\mathcal{D}}^{(m)}_{P'_i \leftarrow P''_i} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{P''_i}} \mathcal{M})) \ \to \ \mathbb{R}q_*(\widetilde{\mathcal{D}}^{(m)}_{X_i \leftarrow P_i} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{P_i}} u'^*(\mathcal{M}))$$

is an isomorphism.

The proof of the second isomorphism is more delicate, and uses the description of overconvergent local cohomology with support in a closed subscheme defined by an ideal $\mathcal{I}$ in terms of the $\mathbb{R}\mathcal{H}om$ of the divided power envelopes $\mathcal{P}^n_{(m)}(\mathcal{I})$ of $\mathcal{I}$ (*cf.* [17, 4.4.4]). This allows to reduce the assertion to the following claim:

c) If $\mathcal{J}$ and $\mathcal{I}$ are the ideals of X_i and Z_i in P'_i and X_i respectively, the canonical morphism

$$\mathbb{L}u^*(\mathbb{R}\mathcal{H}om_{\mathcal{O}_{P'_i}}(\mathcal{P}^n_{(m)}(\mathcal{J}), \mathcal{O}_{P'_i})) \ \to \ \mathbb{R}\mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{P}^n_{(m)}(\mathcal{I}), \mathcal{O}_{X_i})$$

is an isomorphism.

The key point here is that, thanks to our complete intersection hypothesis for Z in X, the two copies of $\mathcal{O}_{X_i}$ viewed as $\mathcal{O}_{P'_i}$-modules via the section u and the zero section, are Tor-independent over $\mathcal{O}_{P'_i}$. Using known results on the structure of divided power envelopes in the case of complete intersections [13, 1.5.3], it follows that the canonical map

$$\mathbb{L}u^*(\mathcal{P}^n_{(m)}(\mathcal{J})) \ \to \ \mathcal{P}^n_{(m)}(\mathcal{I})$$

is an isomorphism, which implies our claim.

This provides the construction of (2.14.1) in $D^{\mathrm{b}}(\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}(\infty))$. However, the right hand side of (2.14.1) is known to have coherent cohomology (thanks to a straightforward generalization of [17, 4.4.9] adding overconvergent poles along some divisor). Thus, (2.14.1) is an isomorphism in $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}(\infty))$.

Since (2.10.1) is compatible with Frobenius actions, the isomorphism (2.14.2) defined by applying $u^!$ to (2.10.1) is compatible with the Frobenius actions obtained by inverse image (thanks to [17, 4.3.4]). Using the construction of functoriality maps in rigid cohomology [14, 1.5] to define Frobenius actions, it is easy to check that the isomorphism (2.14.3) identifies the inverse image of the Frobenius action on $\mathbb{R}p'_*(\Omega^{\bullet}_{\mathcal{P}''/\mathcal{P}'}(\infty) \otimes \mathcal{L}^W_{\pi,\mu})$ with the Frobenius action on $\mathbb{R}q_*(\Omega^{\bullet}_{\mathcal{P}/\mathcal{X}}(\infty) \otimes \mathcal{L}^W_{\pi,u})$. On the other hand, using the rigid analytic construction of overconvergent local cohomology, it is also immediate to check that the isomorphism (2.14.4) identifies the inverse image of the Frobenius action on $\mathbb{R}\underline{\Gamma}^{\dagger}_X(\mathcal{O}_{\mathcal{P}',\mathbb{Q}}(\infty))$ with the Frobenius action on $\mathbb{R}\underline{\Gamma}^{\dagger}_Z(\mathcal{O}_{\mathcal{X},\mathbb{Q}}(\infty))$. It follows that the isomorphism (2.14.1) commutes with Frobenius actions.

Remark. – The complete intersection hypothesis on Z has only been used to give a simple proof of the isomorphism (2.14.4). While (2.14.4) has not yet been checked in the general case, there is no doubt that it should be true in full generality, and therefore that 2.14 should remain valid without the complete intersection hypothesis.

For example, it is worth noting that the theorem is true when the section u reduces to the zero section $X \hookrightarrow V'$, hence $Z = X$. Indeed, the functors u_+ and $u^!$ only depend upon the reduction of u over $\mathrm{Spec}(k)$ [16], so we may assume that u itself is the zero section. Then, thanks to [17, (4.4.5.2)], there is a canonical isomorphism

$$\mathbb{R}\underline{\Gamma}^{\dagger}_X(\mathcal{O}_{\mathcal{P}',\mathbb{Q}}(\infty)) \ \xrightarrow{\ \sim\ } \ u_+(u^!(\mathcal{O}_{\mathcal{P}',\mathbb{Q}}(\infty))).$$

Moreover, the functors u_+ and $u^!$ are quasi-inverse equivalences between coherent $\mathcal{D}^{\dagger}_{\mathcal{X},\mathbb{Q}}(\infty)$-modules and coherent $\mathcal{D}^{\dagger}_{\mathcal{P}',\mathbb{Q}}(\infty)$-modules with support in X [17, 5.3.3].

Therefore, the previous isomorphism gives an isomorphism

$$u^!(\mathbb{R}\underline{\Gamma}^\dagger_X(\mathcal{O}_{\mathcal{P}',\mathbb{Q}}(\infty))) \xrightarrow{\ \sim\ } u^!(\mathcal{O}_{\mathcal{P}',\mathbb{Q}}(\infty)) = \mathcal{O}_{\mathcal{X},\mathbb{Q}}(\infty)[-r],$$

and (2.14.4) is an isomorphism. As $\mathcal{L}^W_{\pi,u} = \mathcal{O}_{\mathcal{P},\mathbb{Q}}(\infty)$ in this case, the isomorphism (2.14.1) is simply the isomorphism

$$q_+(\mathcal{O}_{\mathcal{P},\mathbb{Q}}(\infty)) \xrightarrow{\ \sim\ } \mathcal{O}_{\mathcal{X},\mathbb{Q}}(\infty)[r]$$

resulting from the triviality of the relative de Rham cohomology of a vector bundle.

3. Applications to rigid cohomology

We now derive consequences of 2.14 for rigid cohomology, including rigid cohomology with compact supports. We will also check the compatibility between our isomorphism and its algebraic analog, constructed in [27].

Theorem 3.1. *With the notation and hypotheses of* 2.14*, assume in addition that* $\mathcal{X}$ *is proper over* $\operatorname{Spf} R$*. Then there exists a canonical isomorphism*

$$\mathbb{R}\Gamma_{\mathrm{rig}}(W/K, \mathcal{L}_{\pi,u}) \simeq \mathbb{R}\Gamma_{Z\cap Y,\,\mathrm{rig}}(Y/K) \tag{3.1.1}$$

which commutes with the natural Frobenius actions F^* *on both cohomology spaces.*

Let $f : \mathcal{X} \to \mathcal{S}$ be the structural morphism, and n the relative dimension of $\mathcal{X}$ over $\mathcal{S}$. Since (2.14.1) is an isomorphism in $D^{\mathrm{b}}(\mathcal{D}^\dagger_{\mathcal{X},\mathbb{Q}}(\infty))$, it defines an isomorphism

$$f_+(q_+(\mathcal{L}^W_{\pi,u})[-r])[-n] \simeq f_+(\mathbb{R}\underline{\Gamma}^\dagger_Z(\mathcal{O}_{\mathcal{X},\mathbb{Q}}(\infty)))[-n].$$

As $q_+(\mathcal{L}^W_{\pi,u})$ belongs to $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathcal{X},\mathbb{Q}}(\infty))$, we obtain

$$\begin{aligned} f_+(q_+(\mathcal{L}^W_{\pi,u})[-r])[-n] &\simeq (f\circ q)_+(\mathcal{L}^W_{\pi,u})[-r-n] \\ &\simeq \mathbb{R}\Gamma(\mathcal{P}, \Omega^\bullet_{\mathcal{P}} \otimes \mathcal{L}^W_{\pi,u}) \\ &\simeq \mathbb{R}\Gamma(\mathcal{P}, \mathbb{R}\,\mathrm{sp}_*(\Omega^\bullet_{\mathcal{P}_K} \otimes j^\dagger_W L^{\mathrm{an}}_{\pi,\mu\circ(u\times \mathrm{Id})})) \\ &\simeq \mathbb{R}\Gamma(\mathcal{P}_K, \Omega^\bullet_{\mathcal{P}_K} \otimes j^\dagger_W L^{\mathrm{an}}_{\pi,\mu\circ(u\times \mathrm{Id})}) \\ &= \mathbb{R}\Gamma_{\mathrm{rig}}(W/K, \mathcal{L}_{\pi,u}), \end{aligned}$$

the latter isomorphism being due to the fact that $(\mathcal{P}, H_2)$ is a smooth compactification of $\mathcal{W}$. On the other hand, if $\mathcal{U} = \mathcal{X} \setminus Z$, we obtain

$$\begin{aligned} f_+(\mathbb{R}\underline{\Gamma}^\dagger_Z(\mathcal{O}_{\mathcal{X},\mathbb{Q}}(\infty)))[-n] &\simeq \mathbb{R}\Gamma(\mathcal{X}, \Omega^\bullet_{\mathcal{X}} \otimes \mathbb{R}\underline{\Gamma}^\dagger_Z(\mathcal{O}_{\mathcal{X},\mathbb{Q}}(\infty))) \\ &\simeq \mathbb{R}\Gamma(\mathcal{X}, \mathbb{R}\,\mathrm{sp}_*((\Omega^\bullet_{\mathcal{X}_K} \otimes (j^\dagger_Y \mathcal{O}_{\mathcal{X}_K} \to j^\dagger_{U\cap Y}\mathcal{O}_{\mathcal{X}_K}))_{\mathrm{t}})) \\ &= \mathbb{R}\Gamma_{Z\cap Y,\,\mathrm{rig}}(Y/K). \end{aligned}$$

Therefore, we obtain the isomorphism (3.1.1). As (2.14.1) is compatible with Frobenius actions, the same holds for (3.1.1).

Remark. – As for theorem 2.14, theorem 3.1 remains valid when the reduction of u over $\mathrm{Spec}(k)$ is the zero section.

Corollary 3.2. *Under the assumptions of* 3.1*, there exists a canonical isomorphism*

$$\mathbb{R}\Gamma_{\mathrm{c,\,rig}}(W/K, \mathcal{L}_{\pi,u}) \;\simeq\; \mathbb{R}\Gamma_{\mathrm{c,\,rig}}(Z \cap Y/K)[-2r], \tag{3.2.1}$$

which commutes with the Frobenius actions F^* on $\mathbb{R}\Gamma_{\mathrm{c,\,rig}}(W/K, \mathcal{L}_{\pi,u})$ and $q^r F^*$ on $\mathbb{R}\Gamma_{\mathrm{c,\,rig}}(Z \cap Y/K)$.

Replacing $\mathcal{L}_{\pi}$ by $\mathcal{L}_{-\pi}$ in (3.1.1) and taking K-linear duals yields an isomorphism

$$\mathbb{R}\Gamma_{\mathrm{rig}}(W/K, \mathcal{L}_{-\pi,u})^{\vee} \;\simeq\; \mathbb{R}\Gamma_{Z\cap Y,\,\mathrm{rig}}(Y/K)^{\vee}$$

which commutes with the dual actions of Frobenius $F_* = F^{*\,\vee}$ on both sides. Poincaré duality is compatible with F^*, and provides isomorphisms

$$\begin{aligned} \mathbb{R}\Gamma_{\mathrm{rig}}(W/K, \mathcal{L}_{-\pi,u})^{\vee} &\;\simeq\; \mathbb{R}\Gamma_{\mathrm{c,\,rig}}(W/K, \mathcal{L}_{\pi,u})[2n+2r], \\ \mathbb{R}\Gamma_{Z\cap Y,\,\mathrm{rig}}(Y/K)^{\vee} &\;\simeq\; \mathbb{R}\Gamma_{\mathrm{c,\,rig}}(Z \cap Y/K)[2n]. \end{aligned}$$

Since $F^* = q^{n+r}\sigma$ on $H^{2(n+r)}_{\mathrm{c,\,rig}}(W/K)$ (resp. $q^n\sigma$ on $H^{2n}_{\mathrm{c,\,rig}}(Y/K)$), these isomorphisms identify F_* on $\mathbb{R}\Gamma_{\mathrm{rig}}(W/K, \mathcal{L}_{-\pi,u})^{\vee}$ to $q^{n+r}(F^*)^{-1}$ on $\mathbb{R}\Gamma_{\mathrm{c,\,rig}}(W/K, \mathcal{L}_{\pi,u})$, and F_* on $\mathbb{R}\Gamma_{Z\cap Y,\,\mathrm{rig}}(Y/K)^{\vee}$ to $q^n(F^*)^{-1}$ on $\mathbb{R}\Gamma_{\mathrm{c,\,rig}}(Z \cap Y/K)$. The corollary follows.

3.3. We now want to check that, when the previous situation is algebraizable, the isomorphism (3.1.1) is compatible with specialization. As we are returning to a situation similar to 1.1, we change notation. For any S-scheme X, we denote by X_K and X_k the generic and special fibers of X, $\alpha_X : X_K \hookrightarrow X$ the inclusion of the generic fiber, and $\mathcal{X}$ the (p-adic) formal completion of X.

Let $f : X \to S$ be a proper and smooth morphism of relative dimension n, $q : V \to X$ a vector bundle of rank r over X, $q' : V' \to X$ the dual vector bundle, P and P' their projective closures over X, $u : X \hookrightarrow V'$ a section, $Z \hookrightarrow X$ its zero locus. We also assume that $H \subset X$ is a relative divisor over S, and we set $Y = X \setminus H$, $W = q^{-1}(Y)$, $W' = q'^{-1}(Y)$.

Let $L_{\pi,\mu}$ be the rank 1 module with integrable connection on $V'_K \times V_K$ defined as the inverse image of L_π by the canonical pairing $V'_K \times_{X_K} V_K \to \mathbb{A}^1_K$, and $L_{\pi,u}$ its inverse image by the section $u_K \times \mathrm{Id} : V_K \hookrightarrow V'_K \times_{X_K} V_K$. Note that, on $\mathbb{A}^1_K$, L_π is the inverse image of the usual exponential module under the automorphism defined by multiplication by π. Therefore, we can deduce from [27, th. 0.2] canonical isomorphisms

$$q_{K,+}(L_{\pi,u})[-r] \;\simeq\; \mathbb{R}\underline{\Gamma}_{Z_K}(\mathcal{O}_{X_K}), \tag{3.3.1}$$

$$\mathbb{R}\Gamma_{\mathrm{dR}}(W_K/K, L_{\pi,u}) \;\simeq\; \mathbb{R}\Gamma_{Z_K\cap Y_K,\,\mathrm{dR}}(Y_K/K). \tag{3.3.2}$$

Proposition 3.4. *Under the previous assumptions, the square*

$$\begin{array}{ccc} \mathbb{R}\Gamma_{\mathrm{dR}}(W_K/K, L_{\pi,u}) & \xrightarrow[\sim]{(3.3.2)} & \mathbb{R}\Gamma_{Z_K\cap Y_K,\,\mathrm{dR}}(Y_K/K) \\ \rho_W^{L_{\pi,u}}\big\downarrow & & \big\downarrow\rho_{Z\cap Y} \\ \mathbb{R}\Gamma_{\mathrm{rig}}(W_k/K, \mathcal{L}_{\pi,u}) & \xrightarrow[\sim]{(3.1.1)} & \mathbb{R}\Gamma_{Z_k\cap Y_k,\,\mathrm{rig}}(Y_k/K), \end{array} \qquad (3.4.1)$$

where the vertical arrows are the specialization morphisms defined in 1.8 *and* 1.2*, is commutative.*

To check this compatibility, we will first give an interpretation of the specialization morphisms in terms of $\mathcal{D}$-modules.

3.5. Let X be a smooth S-scheme, $H \subset X$ a relative divisor, $j : Y \hookrightarrow X$ the inclusion of $Y := X \setminus H$ in X, $i : \mathcal{X} \to X$ the canonical morphism, and $i^{\mathrm{an}} : \mathcal{X}_K \hookrightarrow X_K^{\mathrm{an}}$ the inclusion. We consistently regard $\mathcal{O}_{\mathcal{X}}$-modules as $\mathcal{O}_X$-modules via i_*. If $\mathcal{D}_X^{(m)}$ (resp. $\mathcal{D}_{X_K}$) is the sheaf of differential operators of level m on X (resp. the sheaf of differential operators on X_K), we will use the notation $\mathcal{D}_X^{(m)}(\infty) = j_*(\mathcal{D}_Y^{(m)})$, $\mathcal{D}_{X_K}(\infty) = j_{K*}(\mathcal{D}_{Y_K})$, $\mathcal{D}_{X,\mathbb{Q}}(\infty) = j_*(\mathcal{D}_Y) \otimes \mathbb{Q} \simeq \alpha_{X*}(\mathcal{D}_{X_K}(\infty))$, as well as $\mathcal{O}_{X_K}(\infty) = j_{K*}(\mathcal{O}_{Y_K})$, $\mathcal{O}_{X,\mathbb{Q}}(\infty) = j_*(\mathcal{O}_Y) \otimes \mathbb{Q} \simeq \alpha_{X*}(\mathcal{O}_{X_K}(\infty))$. For any m, there is a canonical ring isomorphism

$$\mathcal{D}_{X,\mathbb{Q}}(\infty) \ \simeq \ \mathcal{D}_X^{(m)}(\infty) \otimes \mathbb{Q}.$$

On the other hand, the construction of $\mathcal{D}^\dagger_{\mathcal{X},\mathbb{Q}}(\infty)$ provides a ring homomorphism

$$\mathcal{D}_X^{(0)}(\infty) \ \to \ \widehat{\mathcal{B}}_{\mathcal{X}}^{(0)}(H_k)\widehat{\otimes}\widehat{\mathcal{D}}_{\mathcal{X}}^{(0)} \otimes \mathbb{Q} \ \to \ \mathcal{D}^\dagger_{\mathcal{X},\mathbb{Q}}(\infty).$$

Thus we obtain a canonical ring homomorphism

$$\mathcal{D}_{X,\mathbb{Q}}(\infty) \ \to \ \mathcal{D}^\dagger_{\mathcal{X},\mathbb{Q}}(\infty). \qquad (3.5.1)$$

If $M \in D^{\mathrm{b}}(\mathcal{D}_{X_K}(\infty))$, and $\mathcal{M} \in D^{\mathrm{b}}(\mathcal{D}^\dagger_{\mathcal{X},\mathbb{Q}}(\infty))$, a *specialization morphism from M to $\mathcal{M}$* is by definition a morphism

$$\mathbb{R}\alpha_{X*}(M) \ \to \ \mathcal{M}, \qquad (3.5.2)$$

in $D^{\mathrm{b}}(\mathcal{D}_{X,\mathbb{Q}}(\infty))$; note that $\mathbb{R}\alpha_{X*}(M) = \alpha_{X*}(M)$ if M is a quasi-coherent $\mathcal{D}_{X_K}(\infty)$-module. For example, the morphism (3.5.1) itself, as well as the canonical morphism

$$\mathcal{O}_{X,\mathbb{Q}}(\infty) \ \to \ \mathcal{O}_{\mathcal{X},\mathbb{Q}}(\infty)$$

defined similarily, are specialization morphisms.

More generally, let M be a $\mathcal{D}_{X_K}(\infty)$-module, M^{an} the associated analytic sheaf, which is a $(\mathcal{D}_{X_K}(\infty))^{\mathrm{an}}$-module. Note that, for any open subset $U \subset X$, with formal completion $\mathcal{U}$, we have $\mathcal{U}_K \subset \mathcal{X}_K \cap U_K^{\mathrm{an}}$. It follows that there is a natural $\mathcal{D}_{X,\mathbb{Q}}(\infty)$-linear morphism

$$\alpha_{X\,*}(M) \ \to\ \mathrm{sp}_*(i^{\,\mathrm{an}\,*}(M^{\,\mathrm{an}})).$$

Therefore, if $\mathcal{M}^{\mathrm{an}}$ is a $(\mathcal{D}_{X_K}(\infty))^{\mathrm{an}}$-module such that $\mathcal{M} := \mathrm{sp}_*(i^{\,\mathrm{an}\,*}(\mathcal{M}^{\,\mathrm{an}}))$ is endowed with a structure of $\mathcal{D}^\dagger_{\mathcal{X},\mathbb{Q}}(\infty)$-module inducing its natural $\mathcal{D}_{X,\mathbb{Q}}(\infty)$-module structure, the datum of a $(\mathcal{D}_{X_K}(\infty))^{\mathrm{an}}$-linear morphism $M^{\mathrm{an}} \to \mathcal{M}^{\mathrm{an}}$ defines a specialization morphism from M to $\mathcal{M}$. In particular, we will use this remark in the following situations:

a) Let L be a coherent $\mathcal{O}_{Y_K}$-module endowed with an integrable connection, such that the induced connection on $\mathcal{L} = j^\dagger_{Y_k}(L^{\mathrm{an}})$ is overconvergent along H_k. If $M = j_{K\,*}(L)$ and $\mathcal{M} = \mathrm{sp}_*(i^{\,\mathrm{an}\,*}(\mathcal{L}))$, then there is a canonical specialization morphism from M to $\mathcal{M}$.

b) Let $Z \subset X$ be a closed subscheme, $U = Y \setminus Z = X \setminus H \cup Z$, $M = \mathbb{R}\underline{\Gamma}_{Z_K}(\mathcal{O}_{X_K}(\infty))$, $\mathcal{M} = \mathbb{R}\underline{\Gamma}^\dagger_{Z_k}(\mathcal{O}_{\mathcal{X},\mathbb{Q}}(\infty))$. If $J^\bullet$ is an injective resolution of $\mathcal{O}_{X_K}(\infty)$ over $\mathcal{D}_{X_K}(\infty)$, and $\mathcal{J}^\bullet$ an injective resolution of $(\mathcal{O}_{X_K}(\infty))^{\mathrm{an}}$ over $(\mathcal{D}_{X_K}(\infty))^{\mathrm{an}}$, one can choose a $(\mathcal{D}_{X_K}(\infty))^{\mathrm{an}}$-morphism $\varphi : J^{\bullet\,\mathrm{an}} \to \mathcal{J}^\bullet$ inducing the identity on $(\mathcal{O}_{X_K}(\infty))^{\mathrm{an}}$. As $X_K^{\mathrm{an}} \setminus Z_K^{\mathrm{an}}$ is a strict neighbourhood of $]U_k[_{\mathcal{X}}$, φ induces a morphism $(\underline{\Gamma}_{Z_K}(J^\bullet))^{\mathrm{an}} \to (\mathcal{J}^\bullet \to j^\dagger_{U_k}(\mathcal{J}^\bullet))_{\mathrm{t}}$. One obtains in this way a canonical specialization morphism from M to $\mathcal{M}$.

Specialization morphisms are functorial in X in the following sense. Let $f : X \to X'$ be an S-morphism, $H \subset X$, $H' \subset X'$ relative divisors such that $f^{-1}(H') \subset H$, $Y = X \setminus H$, $Y' = X' \setminus H'$. As for (3.5.1), there are natural specialization morphisms on X

$$\begin{aligned} \alpha_{X\,*}(\mathcal{D}_{X_K\to X'_K}(\infty)) &\ \to\ \mathcal{D}^\dagger_{\mathcal{X}\to\mathcal{X}',\mathbb{Q}}(\infty)\\ \alpha_{X\,*}(\mathcal{D}_{X'_K\leftarrow X_K}(\infty)) &\ \to\ \mathcal{D}^\dagger_{\mathcal{X}'\leftarrow\mathcal{X},\mathbb{Q}}(\infty), \end{aligned}$$

where $\mathcal{D}_{X_K\to X'_K}(\infty) = j_{K\,*}(\mathcal{D}_{Y_K\to Y'_K})$, $\mathcal{D}_{X'_K\leftarrow X_K}(\infty) = j_{K\,*}(\mathcal{D}_{Y'_K\leftarrow Y_K})$. Moreover, these morphisms are semi-linear with respect to the ring homomorphism $f^{-1}(\mathcal{D}_{X',\mathbb{Q}}(\infty)) \to f^{-1}(\mathcal{D}^\dagger_{\mathcal{X}',\mathbb{Q}}(\infty))$ deduced from (3.5.1) on X'. It follows that, for any complexes $M' \in D^{\mathrm{b}}(\mathcal{D}_{X'_K}(\infty))$, $\mathcal{M}' \in D^{\mathrm{b}}(\mathcal{D}^\dagger_{\mathcal{X}',\mathbb{Q}}(\infty))$ (resp. $M \in D^{\mathrm{b}}(\mathcal{D}_{X_K}(\infty))$, $\mathcal{M} \in D^{\mathrm{b}}(\mathcal{D}^\dagger_{\mathcal{X},\mathbb{Q}}(\infty))$), a specialization morphism from M' to $\mathcal{M}'$ (resp. from M to $\mathcal{M}$) defines canonically specialization morphisms

$$\begin{aligned} \alpha_{X\,*}(f_K^*(M')) &\ \to\ f^*(\mathcal{M}'),\\ \alpha_{X'\,*}(f_{K\,+}(M)) &\ \to\ f_+(\mathcal{M}). \end{aligned}$$

Finally, a specialization morphism defines a morphism between de Rham cohomologies. This is the particular case of the previous situation where $X' = S$, and it can be described in the following way. A specialization morphism from $M \in D^{\mathrm{b}}(\mathcal{D}_{X_K}(\infty))$ to $\mathcal{M} \in D^{\mathrm{b}}(\mathcal{D}^\dagger_{\mathcal{X},\mathbb{Q}}(\infty))$ defines a morphism

$$\omega_X \otimes^{\mathbb{L}}_{\mathcal{D}^{(0)}_X} \mathbb{R}\alpha_{X\,*}(M) \ \to\ \omega_X \otimes^{\mathbb{L}}_{\mathcal{D}^{(0)}_X} \mathcal{M}.$$

Using the de Rham resolution of ω_X over $\mathcal{D}_X^{(0)}$ and taking global sections, we obtain a morphism

$$\mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K, M) \ \to\ \mathbb{R}\Gamma_{\mathrm{dR}}(\mathcal{X}/\mathcal{S}, \mathcal{M}). \tag{3.5.3}$$

When X is proper, and M, $\mathcal{M}$ come from an $\mathcal{O}_{Y_K}$-module with connection L as in a) above, the morphism (3.5.3) can be written as

$$\mathbb{R}\Gamma_{\mathrm{dR}}(Y_K/K, L) \ \to\ \mathbb{R}\Gamma_{\mathrm{rig}}(Y_k/K, \mathcal{L}).$$

The computation based on de Rham resolutions shows that this morphism is the specialization morphism ρ_Y^L defined in (1.8.2). Similarily, if X is proper, $M = \mathbb{R}\underline{\Gamma}_{Z_K}(\mathcal{O}_{X_K}(\infty))$ and $\mathcal{M} = \mathbb{R}\underline{\Gamma}^\dagger_{Z_k}(\mathcal{O}_{\mathcal{X},\mathbb{Q}}(\infty))$ as in b) above, the morphism (3.5.3) can be written as

$$\mathbb{R}\Gamma_{Z_K\cap Y_K,\,\mathrm{dR}}(Y_K/K) \ \to\ \mathbb{R}\Gamma_{Z_k\cap Y_k,\,\mathrm{rig}}(Y_k/K),$$

and this morphism is the morphism $\rho_{Z\cap Y}$ defined in (1.2.2).

3.6. We now return to the proof of 3.4. We endow P, P' and $P'' = P' \times_X P$ with the divisors defined by H and the hyperplanes at infinity as in 2.6. Using the natural specialisation morphism for $L_{\pi,\mu}$, and applying the previous remarks, we obtain a specialization morphism

$$\alpha_{P'\,*}(p'_+(L^W_{\pi,\mu})) \ \to\ p'_+(\mathcal{L}^W_{\pi,\mu}),$$

where $\mathcal{L}^W_{\pi,\mu}$ is defined as in 2.6, $L^W_{\pi,\mu}$ denotes the direct image of $(L_{\pi,\mu})_{|W'_K\times W_K}$ by the inclusion $W'_K \times W_K \hookrightarrow P'_K \times P_K$, and we keep the notation p' for the projections $P''_K \to P'_K$ and $\mathcal{P}'' \to \mathcal{P}'$. On the other hand, we also obtain a specialization morphism

$$\alpha_{P'\,*}(\mathcal{H}^r_{X_K}(\mathcal{O}_{P'_K}(\infty))) \ \to\ \mathcal{H}^{\dagger\,r}_{X_k}(\mathcal{O}_{\mathcal{P}',\mathbb{Q}}(\infty)).$$

These morphisms fit in a commutative square

$$\begin{array}{ccc} \alpha_{P'\,*}(p'_+(L^W_{\pi,\mu})) & \xrightarrow[\sim]{} & \alpha_{P'\,*}(\mathcal{H}^r_{X_K}(\mathcal{O}_{P'_K}(\infty))) \\ \downarrow & & \downarrow \\ p'_+(\mathcal{L}^W_{\pi,\mu}) & \xrightarrow[\sim]{(2.10.1)} & \mathcal{H}^{\dagger\,r}_{X_k}(\mathcal{O}_{\mathcal{P}',\mathbb{Q}}(\infty)), \end{array} \tag{3.6.1}$$

where the upper isomorphism is the algebraic analogue of (2.10.1) (*cf.* [27, 2.3]). Indeed, this commutativity is a local property on X, hence one may assume that $V = \mathbb{A}^r_X$, with coordinates $t_1, \ldots, t_r$, and then it follows from the fact that both isomorphisms send the section $(dt_1 \wedge \ldots \wedge dt_r) \otimes e$, where e is the basis of $L^W_{\pi,\mu}$ (resp. $\mathcal{L}^W_{\pi,\mu}$), to the section $1/t'_1 \ldots t'_r$ of the corresponding local cohomology sheaf.

Using the isomorphisms (2.14.3) and (2.14.4), and their algebraic analogues, it follows by functoriality that the specialization morphisms defined in 3.5 fit in a commutative square

$$\begin{array}{ccc} \mathbb{R}\alpha_{X\,*}(q_+(L^W_{\pi,u})) & \xrightarrow[\sim]{} & \mathbb{R}\alpha_{X\,*}(\mathbb{R}\underline{\Gamma}_{Z_K}(\mathcal{O}_{X_K}(\infty)))[r] \\ \downarrow & & \downarrow \\ q_+(\mathcal{L}^W_{\pi,u}) & \xrightarrow[\sim]{(2.14.1)} & \mathbb{R}\underline{\Gamma}^\dagger_{Z_k}(\mathcal{O}_{\mathcal{X},\mathbb{Q}}(\infty))[r], \end{array} \tag{3.6.2}$$

where the upper isomorphism is the image by $\mathbb{R}\alpha_{X\,*}$ of the isomorphism defined in [27, 0.2]. Taking de Rham cohomology, the proposition follows as explained in 3.5.

3.7. Let us now assume that we are in the situation considered in 1.3, where K is a number field, with ring of integers R, and $S = \operatorname{Spec} R$. Consider a proper and smooth S-scheme X, endowed with a divisor H, $Y = X \setminus H$, a vector bundle V of rank r over X, and a section $u : X \hookrightarrow V'$ of the dual vector bundle, such that the zero locus Z of u is flat over S, and locally a complete intersection of codimension r in X. For each closed point $s \in S^0$, let $K(s)$ be the completion of K at s, $k(s)$ its residue field, p_s the characteristic of $k(s)$. We choose for each s a root π_s of the polynomial $X^{p_s-1} + p_s$ in a finite extension $K'(s)$ of $K(s)$, with residue field $k'(s)$. If R' is an R-algebra, we denote by the subscript R' objects deduced from S-objects by base change from $\operatorname{Spec}(R)$ to $\operatorname{Spec}(R')$. Then, combining 1.4 with the previous proposition, and using Poincaré duality, we obtain the following corollary:

Corollary 3.8. *Under the previous assumptions, there exists a finite subset $\Sigma \subset S^0$ such that the morphisms*

$$\rho_W^{L_{\pi_s,u}} \;:\; \mathbb{R}\Gamma_{\mathrm{dR}}(W_{K'(s)}/K'(s), L_{\pi_s,u}) \;\to\; \mathbb{R}\Gamma_{\mathrm{rig}}(W_{k'(s)}/K'(s), \mathcal{L}_{\pi_s,u}), \tag{3.8.1}$$

$$\rho_{\mathrm{c},W}^{L_{\pi_s,u}} \;:\; \mathbb{R}\Gamma_{\mathrm{c,\,rig}}(W_{k'(s)}/K'(s), \mathcal{L}_{\pi_s,u}) \;\to\; \mathbb{R}\Gamma_{\mathrm{dR,\,c}}(W_{K'(s)}/K'(s), L_{\pi_s,u}) \tag{3.8.2}$$

are isomorphisms for all $s \notin \Sigma$.

4. The algebraic and analytic Dwork complexes

We will now use the results of the previous sections to explain the geometric interpretation of the algebraic and analytic complexes constructed by Dwork to obtain a rationaliy formula for the zeta function of a projective hypersurface over a finite field.

In this section, K will be a finite extension of $\mathbb{Q}_p$, R its ring of integers, k its residue field, of cardinality $q = p^s$. We assume that K contains the primitive p-th roots of 1, and we fix an element $\pi \in K$ such that $\pi^{p-1} = -p$. Let $X = \mathbb{P}^n_S$ be the projective space of relative dimension n over $S = \operatorname{Spec}(R)$, $X_1, \ldots, X_{n+1}$ the standard projective coordinates on X, $H_1, \dots, H_{n+1}$ the corresponding coordinate hyperplanes, $U_i = X \setminus H_i$, $H = H_1 \cup \dots \cup H_{n+1}$, $Y = X \setminus H$.

We fix an homogeneous polynomial $f \in R[x_1, \dots, x_{n+1}]$ of degree $d \geq 1$, and we denote by $Z \subset X$ the projective hypersurface defined by f. As before, the subscripts K and k will denote the generic and special fibers. In [28, (21)], Dwork introduces a characteristic series $\chi_F(t)$ defined by a Frobenius operator, such that the zeta function of the affine hypersurface $Z_k \cap Y_k$ can be expressed by the formula

$$\zeta(Z_k \cap Y_k, qt) = (1-t)^{-(-\delta)^n} \chi_F(t)^{-(-\delta)^{n+1}}, \tag{4.0.1}$$

where the operator δ on the multiplicative group $K[[t]]^\times$ is defined by $A(t)^\delta = A(t)/A(qt)$. Although the proof given in [28] is non-cohomological, Dwork gave in subsequent articles a cohomological interpretation of this formula when Z_k is non singular ([29], [30]). We will show here that, using Dwork's computations and our previous results, this formula has an interpretation in terms of rigid cohomology which holds also in the singular case.

4.1. We first recall the construction of the algebraic and analytic Dwork complexes associated to f (*cf.* [29, §3]). Let $\mathfrak{T}$ be the set of multi-indexes $\underline{u} = (u_0, u_1, \dots, u_{n+1}) \in \mathbb{N}^{d+2}$ such that

$$du_0 = u_1 + \dots + u_{n+1}.$$

a) We denote by $\mathfrak{L}$ the graded sub-algebra of $K[X_0, X_1, \dots, X_{n+1}]$ whose elements are polynomials of the form

$$P(X_0, X_1, \dots, X_{n+1}) = \sum_{\underline{u} \in \mathfrak{T}} a_{\underline{u}} \underline{X}^{\underline{u}}.$$

For any $b \in \mathbb{R}$, $b > 0$, we denote by $L(b)$ the sub-algebra of the power series algebra $K[[X_0, X_1, \dots, X_{n+1}]]$ defined by

$$L(b) = \big\{\xi = \sum_{\underline{u} \in \mathfrak{T}} a_{\underline{u}} \underline{X}^{\underline{u}} \bigm| \exists\, c \in \mathbb{R} \text{ such that } \operatorname{ord}(a_{\underline{u}}) \geq bu_0 + c\big\}.$$

The algebra $L(b)$ can be endowed with the norm $\|\xi\| = \sup_{\underline{u}} |a_{\underline{u}}| p^{bu_0}$, for which it is a p-adic Banach algebra. If $b < b'$, then $L(b') \subset L(b)$, and the inclusion is a completely continuous map [48]. We define

$$L(0^+) = \bigcup_{b>0} L(b).$$

b) For any $i \geq 1$, the differential operator

$$D_i = X_i \frac{\partial}{\partial X_i} + \pi X_0 X_i \frac{\partial f}{\partial X_i} \tag{4.1.1}$$

acts on $\mathfrak{L}$ and $L(b)$, for all b, hence also on $L(0^+)$. Moreover, we have $D_i \circ D_j = D_j \circ D_i$ for all i, j. Therefore, we can form "Koszul complexes"

$$K^\bullet(\mathfrak{L}; \underline{D}) \subset K^\bullet(L(b); \underline{D}) \subset K^\bullet(L(0^+); \underline{D}).$$

using the sequence $\underline{D} = (D_1, \ldots, D_{n+1})$. For example, $K^\bullet(\mathfrak{L}; \underline{D})$ is defined as

$$\mathfrak{L} \to \bigoplus_{i=1}^{n+1} \mathfrak{L} \cdot e_i \to \bigoplus_{i<j} \mathfrak{L} \cdot e_i \wedge e_j \to \cdots \to \mathfrak{L} \cdot e_1 \wedge \cdots \wedge e_{n+1},$$

the differential being defined by

$$d(Pe_{i_1} \wedge \ldots \wedge e_{i_k}) = \sum_{i=1}^{n+1} (D_i P) e_i \wedge e_{i_1} \wedge \ldots \wedge e_{i_k}.$$

The definition of $K^\bullet(L(b); \underline{D})$ and $K^\bullet(L(0^+); \underline{D})$ is similar. We will consider these complexes as being concentrated in degrees in $[0, n+1]$.

c) Let $F(X_0, \ldots, X_{n+1}) \in K[[X_0, \ldots X_{n+1}]]$ be the formal power series

$$F(X_0, \ldots, X_{n+1}) = \exp(\pi(X_0 f(X_1, \ldots, X_{n+1}) - X_0^q f(X_1^q, \ldots, X_{n+1}^q))).$$

If b_0 is small enough, then $F \in L(b_0)$, and multiplication by F in $K[[\underline{X}]]$ induces a continuous endomorphism of $L(b')$ for any $b' \leq b_0$. On the other hand, one can define an endomorphism ψ of $K[[\underline{X}]]$ by $\psi(\sum_{\underline{u}} a_{\underline{u}} \underline{X}^{\underline{u}}) = \sum_{\underline{u}} a_{q\underline{u}} \underline{X}^{\underline{u}}$, and ψ induces a continuous homomorphism $\psi : L(b/q) \to L(b)$ for any $b \leq b_0$. Finally, one can define an endomorphism $\alpha : K[[\underline{X}]] \to K[[\underline{X}]]$ by $\alpha(\xi) = \psi(F\xi)$. If $b \leq qb_0$, then α induces a completely continuous endomorphism of $L(b)$, which is the composite

$$\alpha : L(b) \hookrightarrow L(b/q) \xrightarrow{F} L(b/q) \xrightarrow{\psi} L(b).$$

Dwork defines the characteristic series $\chi_F(t)$ of α as the limit (for the topology of pointwise convergence of coefficients in K) of the characteristic polynomials $\det(I - t\alpha_N)$, where α_N is the endomorphism of the vector space of polynomials of degree $\leq N$ defined by replacing F by its truncation in degrees $\leq (q-1)N$ [29, §2]. In particular, χ_F does not depend upon b, and will not change if K is replaced by an extension which is complete under a valuation inducing the p-adic valuation on K. On the other hand, we can view α as a completely continuous endomorphism of $L(b)$ for small b, which allows to define its Fredholm determinant $\det(I - t\alpha)$. For any such b, we have $\det(I - t\alpha) = \chi_F(t)$, thanks to [48, §9].

Since α is a completely continuous endomorphism of $L(b)$, it endows $L(b)$ with the structure of a nuclear space in Monsky's sense [44, 1.3], and $\det(I - t\alpha)$ is equal to Monsky's characteristic series for nuclear operators. Moreover, since $\det(I - t\alpha)$ is independent of b, the space $L(0^+) = \cup_b L(b)$ endowed with α is still nuclear, and the characteristic series of α on $L(0^+)$ is still equal to $\det(I - t\alpha) = \chi_F(t)$ [44, 1.6].

For any i, we have $\alpha \circ D_i = qD_i \circ \alpha$. Thus, one can define an endomorphism (again be denoted by α) of the complex $K^\bullet(L(0^+); \underline{D})$, by setting

$$\alpha(Pe_{i_1} \wedge \ldots \wedge e_{i_k}) = q^{n+1-k} \alpha(P) e_{i_1} \wedge \ldots \wedge e_{i_k} \qquad (4.1.2)$$

in degree k. The characteristic series of α on $K^\bullet(L(0^+);\underline{D})$ is then defined by

$$\begin{aligned}\det(I - t\alpha|K^\bullet(L(0^+);\underline{D})) &:= \prod_{i=0}^{n+1}\det(I - t\alpha|K^i(L(0^+);\underline{D}))^{(-1)^i} \\ &= \chi_F(t)^{(-\delta)^{n+1}}. \qquad (4.1.3)\end{aligned}$$

On the other hand, α acts on the cohomology spaces $H^i(K^\bullet(L(0^+);\underline{D}))$. By [44, 1.4], they are still nuclear spaces, and we have

$$\det(I - t\alpha|K^\bullet(L(0^+);\underline{D})) = \prod_{i=0}^{n+1}\det(I - t\alpha|H^i(K^\bullet(L(0^+);\underline{D})))^{(-1)^i}.$$

hence

$$\prod_{i=0}^{n+1}\det(I - t\alpha|H^i(K^\bullet(L(0^+);\underline{D})))^{(-1)^i} = \chi_F(t)^{(-\delta)^{n+1}}. \qquad (4.1.4)$$

4.2. To give a geometric interpretation of Dwork's complexes, we follow [27], and we introduce the vector bundle $V = \mathrm{Spec}(\mathbb{S}(\mathcal{O}_X(d)))$ over X. Let $q : V \to X$ be the projection, $V_i = q^{-1}(U_i)$, $H'_i = q^{-1}(H_i)$, $H' = \cup_i H'_i = q^{-1}(H)$. We view X as a closed subscheme of V via the zero section, and we define $D = H' \cup X$, $W = V \setminus H' = q^{-1}(Y)$, $W^* = V \setminus D = W \setminus Y$. We observe that D is a relative normal crossings divisor in V above $S = \mathrm{Spec}(R)$. For all $r \geq 0$, we will denote by $\Omega^r_V(\log D)$ the sheaf of differential forms of degree r over V with logarithmic poles along D.

Note also that these definitions, as well as the definition of $\mathfrak{L}$ given above, make sense over any base ring R.

Lemma 4.3. *With the above notation, we have:*

(i) *Over any base ring R, there is a natural isomorphism of R-algebras*

$$\mathfrak{L} \xrightarrow{\sim} \Gamma(V, \mathcal{O}_V), \qquad (4.3.1)$$

and $H^i(V, \mathcal{O}_V) = 0$ for all $i \geq 1$.

(ii) *If d is invertible in R, the sheaf $\Omega^1_V(\log D)$ is a free $\mathcal{O}_V$-module.*

Since $V = \mathrm{Spec}(\mathbb{S}(\mathcal{O}_X(d)))$, we have

$$\mathbb{R}q_*(\mathcal{O}_V) = q_*(\mathcal{O}_V) = \bigoplus_{m\geq 0}\mathcal{O}_X(md),$$

hence

$$\mathbb{R}\Gamma(V, \mathcal{O}_V) \simeq \mathbb{R}\Gamma(X, \bigoplus_{m\geq 0}\mathcal{O}_X(md)) \simeq \bigoplus_{m\geq 0}\Gamma(X, \mathcal{O}_X(md)) \simeq \Gamma(V, \mathcal{O}_V).$$

If we denote by $k[X_1, \dots, X_{n+1}]_i$ the R-submodule of homogeneous polynomials of degre i, we can compose the standard isomorphism

$$\bigoplus_{m\geq 0} k[X_1, \dots, X_{n+1}]_{md} \xrightarrow{\sim} \bigoplus_{m\geq 0} \Gamma(X, \mathcal{O}_X(md))$$

with the obvious isomorphism

$$\mathfrak{L} \xrightarrow{\sim} \bigoplus_{m\geq 0} k[X_1, \dots, X_{n+1}]_{md}$$

defined by the substitution $X_0 \mapsto 1$, and the first assertion follows.

To construct a basis for $\Omega^1_V(\log D)$, one can use the following local coordinates on the open subsets V_j, $1 \leq j \leq n+1$. For $i \neq j$, $1 \leq i \leq n+1$, let $x_{i,j} = X_i/X_j$, so that the $x_{i,j}$ are local coordinates on U_j. On the other hand, the section $t_j = X_j^d \in \Gamma(X, \mathcal{O}_X(d)) \subset \Gamma(V, \mathcal{O}_V)$ defines a relative coordinate on V above U_j. Keeping the notation $x_{i,j}$ for the inverse images of the $x_{i,j}$ on V_j, we obtain a system of local coordinates $(x_{i,j}, t_j)_{i\neq j}$ on V_j. It will sometimes be convenient to use also the notation $x_{j,j} = X_j/X_j = 1$. Thus, the image in $\Gamma(V_j, \mathcal{O}_V)$ of a monomial $\underline{X}^{\underline{u}} \in \mathfrak{L}$ is the section

$$t_j^{u_0} x_{1,j}^{u_1} \cdots x_{n+1,j}^{u_{n+1}}. \tag{4.3.2}$$

If d is invertible in R, we can now define a family $(\omega_i)_{1\leq i\leq n+1}$ of global differential forms on V by setting

$$\begin{aligned} \omega_i|_{V_j} &= \frac{1}{d}\frac{dt_j}{t_j} + \frac{dx_{i,j}}{x_{i,j}} \quad \text{if } j \neq i, \\ \omega_i|_{V_i} &= \frac{1}{d}\frac{dt_i}{t_i} \quad \left(= \frac{1}{d}\frac{dt_i}{t_i} + \frac{dx_{i,i}}{x_{i,i}} \right). \end{aligned} \tag{4.3.3}$$

When j varies, the $\omega_i|_{V_j}$ glue to define global sections $\omega_i = \frac{1}{d}\frac{dt_i}{t_i} \in \Gamma(V, \Omega^1_V(\log D))$, and it is clear that, on any V_j, they form a basis of $\Omega^1_V(\log D)$.

4.4. Returning to our initial context where R is the ring of integers of K, we denote by $\mathcal{X}$, $\mathcal{V}$, $\mathcal{U}_j$, $\mathcal{V}_j$ the formal p-adic completions of X, V, U_j, V_j. Over $\mathcal{U}_{j,K}$, the generic fiber $\mathcal{V}_K$ of $\mathcal{V}$ is the compact open subset of V_K^{an} defined by the condition $|t_j(y)| \leq 1$, so that the map $\mathcal{V}_K \to \mathcal{X}_K$ is a locally trivial fibration whose fibers are closed unit discs. Let P be the projective closure of V and $\mathcal{P}$ be the formal completion of P along its special fiber. We recall that the tube $]V_k[_{\mathcal{P}}$ of V_k in $\mathcal{P}_K$ is equal to $\mathcal{V}_K$. Let $j_V^\dagger$ be the functor of overconvergent sections around $\mathcal{V}_K$, as defined in 1.1. The first assertion of the previous lemma has the following analytic counterpart:

Lemma 4.5. *With the previous notation, there is a natural isomorphism of K-algebras*

$$L(0^+) \xrightarrow{\sim} \Gamma(V_K^{\mathrm{an}}, j_V^\dagger(\mathcal{O}_{V_K^{\mathrm{an}}})) \tag{4.5.1}$$

extending the isomorphism (4.3.1)*. Furthermore,* $H^i(V_K^{\rm an}, j_V^\dagger(\mathcal{O}_{V_K^{\rm an}})) = 0$ *for all* $i \geq 1$.

For any $\rho \in \mathbb{R}$, $\rho \geq 1$, and any $j \in \{1, \ldots, n+1\}$, let $\mathcal{V}_{j,\rho} \subset (q^{\rm an})^{-1}(\mathcal{U}_{j,K})$ be the affinoid open subset defined as $\mathcal{V}_{j,\rho} = \{y \mid |t_j(y)| \leq \rho\}$. If $j, j' \in \{1, \ldots, n+1\}$, then $t_{j'} = x_{j',j}^d t_j$. As $|x_{j',j}(x)| = 1$ for all $x \in \mathcal{U}_{j',K} \cap \mathcal{U}_{j,K}$, it follows that $\mathcal{V}_{j,\rho} \cap (q^{\rm an})^{-1}(\mathcal{U}_{j,K} \cap \mathcal{U}_{j',K}) = \mathcal{V}_{j',\rho} \cap (q^{\rm an})^{-1}(\mathcal{U}_{j,K} \cap \mathcal{U}_{j',K})$. Hence the $\mathcal{V}_{j,\rho}$ for variable j glue together to define an open subset $\mathcal{V}_\rho \subset V_K^{\rm an}$. Note that $\mathcal{V}_1 = \mathcal{V}_K$. It is clear that, for $\rho \overset{>}{\to} 1$, the $\mathcal{V}_\rho$ are a fundamental system of strict neighbourhoods of $\mathcal{V}_K$. Thus, if $j_\rho : \mathcal{V}_\rho \hookrightarrow V_K^{\rm an}$ denotes the inclusion morphism, there is a natural isomorphism $j_V^\dagger(\mathcal{O}_{V_K^{\rm an}}) \xrightarrow{\sim} \varinjlim_{\rho \overset{>}{\to} 1} j_{\rho *}(\mathcal{O}_{\mathcal{V}_\rho})$.

For $b > 0$, let $\xi = \sum_{\underline{u} \in \mathfrak{T}} a_{\underline{u}} \underline{X}^{\underline{u}} \in L(b)$, and let ρ be such that $1 < \rho < p^b$. Since $\|x_{i,j}\| \leq 1$ in $\Gamma(\mathcal{U}_{j,K}, \mathcal{O}_{\mathcal{X}_K^{\rm an}})$, the series

$$\sum_{\underline{u} \in \mathfrak{T}} a_{\underline{u}} (t_j^{u_0} \prod_i x_{i,j}^{u_i}) \ = \ \sum_{u_0} (\sum_{u_1 + \ldots + u_{n+1} = du_0} a_{\underline{u}} \prod_i x_{i,j}^{u_i}) t_j^{u_0}$$

converges towards an element $\xi_{j,\rho} \in \mathcal{V}_{j,\rho}$. Moreover, since $t_{j'} = x_{j',j}^d t_j$ and $x_{i,j'} = x_{i,j} x_{j,j'}$ above $U_j \cap U_{j'}$, these series glue for variable j to define an element $\xi_\rho \in \Gamma(\mathcal{V}_\rho, \mathcal{O}_{V_K^{\rm an}})$. Then the homomorphism (4.5.1) is obtained by sending $\xi \in L(b) \subset L(0^+)$ to the image of ξ_ρ in $\Gamma(V_K^{\rm an}, j_V^\dagger(\mathcal{O}_{V_K^{\rm an}}))$, for any ρ such that $1 < \rho < p^b$.

If $\xi \neq 0$, then $\xi_{j,\rho} \neq 0$, hence (4.5.1) is injective. To prove it is surjective, we define, for $b \geq 0$,

$$L'(b) \ = \ \big\{ \xi = \sum_{\underline{u} \in \mathfrak{T}} a_{\underline{u}} \underline{X}^{\underline{u}} \ \big| \ {\rm ord}(a_{\underline{u}}) - b u_0 \to +\infty \text{ if } u_0 \to +\infty \big\}.$$

Thus $L'(b) \subset L(b)$ for all $b > 0$, and $L(0^+) = \bigcup_{b>0} L'(b)$. The previous construction provides a natural homomorphism

$$L'(\log_p(\rho)) \ \longrightarrow \ \Gamma(\mathcal{V}_\rho, \mathcal{O}_{\mathcal{V}_\rho}) \tag{4.5.2}$$

for any $\rho \geq 1$. Then it suffices to construct a decreasing sequence of real numbers ρ_m, with limit 1, such that the following holds when ρ is one of the ρ_m's:

a) The homomorphism (4.5.2) is an isomorphism.

On the other hand, it follows from 1.1 that, for any fixed $\rho_0 > 1$, we have

$$\begin{aligned} H^i(V_K^{\rm an}, j_V^\dagger(\mathcal{O}_{V_K^{\rm an}})) & \simeq \ H^i(\mathcal{V}_{\rho_0}, j_V^\dagger(\mathcal{O}_{V_K^{\rm an}})) \\ & \simeq \ \varinjlim_{\rho \overset{>}{\to} 1} H^i(\mathcal{V}_{\rho_0}, j_{\rho *}(\mathcal{O}_{\mathcal{V}\rho})) \\ & \simeq \ \varinjlim_{\rho \overset{>}{\to} 1} H^i(\mathcal{V}_\rho, \mathcal{O}_{\mathcal{V}\rho}), \end{aligned}$$

where the second isomorphism is due to the fact that $\mathcal{V}_{\rho_0}$ is quasi-compact and separated, and the third one to the fact that, for any affinoid A in $\mathcal{V}_{\rho_0}$, $A \cap \mathcal{V}_\rho$ is affinoid. As above, the vanishing of $H^i(V_K^{\rm an}, j_V^\dagger(\mathcal{O}_{V_K^{\rm an}}))$ will follow if we construct a decreasing sequence of real numbers ρ_m, with limit 1, such that the following holds when ρ is one of the ρ_m's:

b) For any $i > 0$, $H^i(\mathcal{V}_\rho, \mathcal{O}_{\mathcal{V}\rho}) = 0$.

Let us prove that assertions a) and b) hold when ρ belongs to the sequence $\rho_m = p^{1/m}$. As it suffices to prove these properties after a finite extension of the base field K, we may assume that there exists an element $\lambda \in K$ such that $|\lambda| = \rho$. Then multiplication by λ in the vector bundle V_K^{an} induces an isomorphism of rigid spaces $h_\lambda : \mathcal{V}_1 \xrightarrow{\sim} \mathcal{V}_\rho$. Moreover, the substitution $X_0 \mapsto \lambda X_0$ defines an isomorphism $h'_\lambda : L'(\log_p(\rho)) \xrightarrow{\sim} L'(0)$, and the homomorphisms (4.5.2) are compatible with h'_λ and h^*_λ. Therefore, it suffices to prove a) and b) when $\rho = 1$.

In this case, we have $\mathcal{V}_1 = \mathcal{V}_K$, and $H^i(\mathcal{V}_1, \mathcal{O}_{\mathcal{V}_1}) = H^i(\mathcal{V}, \mathcal{O}_{\mathcal{V}}) \otimes K$ for all $i \geq 0$. On the other hand, $L'(0) = \widehat{\mathfrak{L}_R} \otimes K$, where $\widehat{\mathfrak{L}_R}$ is the p-adic completion of the algebra $\mathfrak{L}$ constructed over the base ring R. Denoting by R_j, V_j the reductions modulo p^j of R, $\mathcal{V}$, and applying lemma 4.3 over R_j, we obtain that

$$\mathfrak{L}_{R_j} \xrightarrow{\sim} \Gamma(V_j, \mathcal{O}_{V_j}), \qquad H^i(V_j, \mathcal{O}_{V_j}) = 0 \quad \text{if } i \geq 1.$$

In particular, the cohomology groups $H^i(V_j, \mathcal{O}_{V_j})$ satisfy the Mittag-Leffler condition for all $i \geq 0$, and therefore this gives an isomorphism

$$H^i(\mathcal{V}, \mathcal{O}_{\mathcal{V}}) \xrightarrow{\sim} \varprojlim_j H^i(V_j, \mathcal{O}_{V_j})$$

for all i. Since $\widehat{\mathfrak{L}_R} = \varprojlim_j \mathfrak{L}_{R_j}$, assertions a) and b) follow.

Theorem 4.6. *Under the assumptions of* 4.2 *and* 4.4*, let $q' : V' \to X$ be the dual vector bundle of V, $u : X \hookrightarrow V'$ the section defined by the homogeneous polynomial $f \in \Gamma(X, \mathcal{O}_X(d))$, $L_{\pi,f}$ the rank one module with connection on V obtained as the inverse image of L_π by the morphism $V \hookrightarrow V' \times V \to \mathbb{A}^1_S$ defined by u, $\mathcal{L}_{\pi,f} = j_V^\dagger(L_{\pi,f}^{\mathrm{an}})$ the corresponding overconvergent F-isocrystal on V_k. We denote again by the subscripts K and k the generic fiber and the special fiber of an S-scheme.*

(i) *There exists an isomorphism of complexes*

$$\theta : \; K^\bullet(\mathfrak{L}; \underline{D}) \xrightarrow{\sim} \Gamma(V_K, \Omega^\bullet_{V_K}(\log D_K) \otimes L_{\pi,f}), \tag{4.6.1}$$

which can be identified in degree 0 *to the isomorphism* (4.3.1) (*using the canonical basis of $L_{\pi,f}$*)*. In the derived category of K-vector spaces, θ defines an isomorphism*

$$K^\bullet(\mathfrak{L}; \underline{D}) \xrightarrow{\sim} \mathbb{R}\Gamma_{\mathrm{dR}}(W_K^*/K, L_{\pi,f}). \tag{4.6.2}$$

(ii) *There exists an isomorphism of complexes*

$$\theta^\dagger : \; K^\bullet(L(0^+); \underline{D}) \xrightarrow{\sim} \Gamma(V_K^{\mathrm{an}}, j_V^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}}(\log D_K^{\mathrm{an}}) \otimes L_{\pi,f}^{\mathrm{an}})) \tag{4.6.3}$$

which can be identified in degree 0 *to the isomorphism* (4.5.1)*. In the derived category of K-vector spaces, $\theta^\dagger$ defines an isomorphism*

$$K^\bullet(L(0^+); \underline{D}) \xrightarrow{\sim} \mathbb{R}\Gamma_{\mathrm{rig}}(W_k^*/K, \mathcal{L}_{\pi,f}), \tag{4.6.4}$$

in which the endomorphism α *of* $K^\bullet(L(0^+);\underline{D})$ *corresponds to the endomorphism* $F_* = q^{n+1}(F^*)^{-1}$ *on rigid cohomology.*

Let us recall that, by construction, $L_{\pi,f}$ is a rank 1 $\mathcal{O}_V$-module endowed with a natural basis e. If we use this basis to identify $L_{\pi,f}$ to $\mathcal{O}_V$, then the isomorphisms θ and $\theta^\dagger$ have been defined in degree 0 by the previous lemmas. Thus, if θ_j is the composed homomorphism

$$\theta_j \ : \ \mathfrak{L} \ \xrightarrow{\sim} \ \Gamma(V_K, \mathcal{O}_{V_K}) \ \hookrightarrow \Gamma(V_{j,K}, \mathcal{O}_{V_K}),$$

and $\underline{u} \in \mathfrak{T}$, we obtain in $\Gamma(V_{j,K}, \mathcal{L}_{\pi,f})$

$$\theta(\underline{X}^{\underline{u}}) \ = \ \theta_j(\underline{X}^{\underline{u}}) \otimes e \ = \ t_j^{u_0} x_{1,j}^{u_1} \cdots x_{n+1,j}^{u_{n+1}} \otimes e, \tag{4.6.5}$$

thanks to (4.3.2). In higher degrees, we define θ (resp. $\theta^\dagger$) as the unique isomorphism which is semi-linear with respect to (4.3.1) (resp. (4.5.1)), and sends any product $e_{i_1} \wedge \ldots \wedge e_{i_k}$ to $\omega_{i_1} \wedge \ldots \wedge \omega_{i_k} \otimes e$, where $(\omega_i)_i$ is the basis defined in (4.3.3). We obtain in this way isomorphisms of graded modules θ and $\theta^\dagger$.

For each j, let t'_j be the dual coordinate associated to t_j on $V'_j = q'^{-1}(U_j)$. Under the composed morphism $\varphi_j : V_j \hookrightarrow V' \times V \to \mathbb{A}^1_S$, the inverse image of the coordinate $t \in \Gamma(\mathbb{A}^1_S, \mathcal{O}_{\mathbb{A}^1_S})$ is

$$\varphi_j^*(t) \ = \ u^*(t'_j)t_j \ = \ t_j f(x_{1,j}, \ldots, x_{n+1,j}) \ = \ \theta_j(f).$$

It follows that, viewing f as an element of $\Gamma(V, \mathcal{O}_V)$ through (4.3.1), the connection $\nabla_{\pi,f}$ of $L_{\pi,f}$ is given by

$$\nabla_{\pi,f}(ge) \ = \ (d(g) + \pi g\, d(f)) \otimes e \tag{4.6.6}$$

for any section g of $\mathcal{O}_V$.

Since the ω_i are a basis of $\Omega^1_{V_K}(\log D_K)$ over $\mathcal{O}_{V_K}$, we can define derivations ∂_i of $\mathcal{O}_{V_K}$ by setting

$$d(g) = \sum_{i=1}^{n+1} \partial_i(g)\, \omega_i,$$

so that $\nabla_{\pi,f}$ is given by

$$\nabla_{\pi,f}(ge) \ = \ \sum_{i=1}^{n+1} (\partial_i(g) + \pi g \partial_i(f))\, \omega_i \otimes e.$$

To prove the commutation of θ and $\theta^\dagger$ with the differentials, it is then enough to prove that, when $g \in K[X_1, \ldots, X_{n+1}]$ is homogeneous of degree dk, the isomorphism (4.3.1) maps $X_0^k X_i\, \partial g/\partial X_i$ to $\partial_i(g)$ for all i. We can compute in $\Gamma(V_{n+1,K}, \mathcal{O}_{V_K})$, and use the coordinates $t_{n+1}, x_{1,n+1}, \ldots, x_{n,n+1}$ to write $g =$

$t_{n+1}^k g(\underline{x}_{n+1}, 1)$, where $\underline{x}_{n+1}$ stands for $x_{1,n+1}, \ldots, x_{n,n+1}$. Then we obtain

$$\begin{aligned} d(g) &= kt_{n+1}^k g(\underline{x}_{n+1}, 1)\frac{dt_{n+1}}{t_{n+1}} + \sum_{i=1}^n t_{n+1}^k x_{i,n+1}\frac{\partial g}{\partial X_i}(\underline{x}_{n+1}, 1)\frac{dx_{i,n+1}}{x_{i,n+1}} \\ &= dkt_{n+1}^k g(\underline{x}_{n+1}, 1)\,\omega_{n+1} + \sum_{i=1}^n t_{n+1}^k x_{i,n+1}\frac{\partial g}{\partial X_i}(\underline{x}_{n+1}, 1)(\omega_i - \omega_{n+1}) \\ &= t_{n+1}^k(dkg(\underline{x}_{n+1}, 1) - \sum_{i=1}^n x_{i,n+1}\frac{\partial g}{\partial X_i}(\underline{x}_{n+1}, 1))\,\omega_{n+1} \\ &\qquad\qquad + \sum_{i=1}^n t_{n+1}^k x_{i,n+1}\frac{\partial g}{\partial X_i}(\underline{x}_{n+1}, 1)\,\omega_i, \end{aligned}$$

from which the claim follows.

The acyclicity property of lemma 4.3 implies that

$$\Gamma(V_K, \Omega^\bullet_{V_K}(\log D_K) \otimes L_{\pi,f}) \;=\; \mathbb{R}\Gamma(V_K, \Omega^\bullet_{V_K}(\log D_K) \otimes L_{\pi,f})$$

in the derived category. As $L_{\pi,f}$ has no singularities along D, Deligne's theorem [25, II 3.14] shows that the canonical morphism

$$\mathbb{R}\Gamma(V_K, \Omega^\bullet_{V_K}(\log D_K) \otimes L_{\pi,f}) \;\longrightarrow\; \mathbb{R}\Gamma(W_K^*, \Omega^\bullet_{W_K^*} \otimes L_{\pi,f})$$

is an isomorphism. Combined with (4.6.1), it provides the isomorphism (4.6.2).

Similarily, lemma 4.5 implies that

$$\Gamma(V_K^{\mathrm{an}}, j_V^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}}(\log D_K^{\mathrm{an}}) \otimes L_{\pi,f}^{\mathrm{an}})) \;=\; \mathbb{R}\Gamma(V_K^{\mathrm{an}}, j_V^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}}(\log D_K^{\mathrm{an}}) \otimes L_{\pi,f}^{\mathrm{an}})).$$

in the derived category. On the other hand, corollary A.4 of the Appendix provides an isomorphism

$$\mathbb{R}\Gamma(V_K^{\mathrm{an}}, j_V^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}}(\log D_K^{\mathrm{an}}) \otimes L_{\pi,f}^{\mathrm{an}})) \;\xrightarrow{\ \sim\ }\; \mathbb{R}\Gamma_{\mathrm{rig}}(W_k^*/K, \mathcal{L}_{\pi,f}),$$

so that this gives (4.6.4) by composition as in the algebraic case. Indeed, let T be the infinity divisor in the special fiber of P, and $\overline{D}$ the closure of D in P. As $L_{\pi,f}$ has no singularities along D, $j_V^\dagger(L_{\pi,f}^{\mathrm{an}})$ satisfies the hypothesis of A.1, and we can apply corollary A.4 to $\mathcal{P}$ endowed with the divisors T, $\overline{D}$, and to $j_V^\dagger(L_{\pi,f}^{\mathrm{an}})$ on the strict neighbourhood V_K^{an} of $\mathcal{V}_K$. Then (A.4.1) gives the above isomorphism.

To compare the Frobenius actions on $K^\bullet(L(0^+); \underline{D})$ and $\mathbb{R}\Gamma_{\mathrm{rig}}(W_k^*/K, \mathcal{L}_{\pi,f})$, we must describe explicitly the F-isocrystal structure of $\mathcal{L}_{\pi,f}$. We first observe that V can be endowed with a global lifting F_V of the Frobenius morphism of V_k by setting $F_V^*(X_i) = X_i^q$ for all i, hence $F_V^*(x_{i,j}) = x_{i,j}^q$, $F_V^*(t_j) = t_j^q$ for all i, j. Let $u' = u \times \mathrm{Id}_V : V \hookrightarrow V' \times V$. Then the Frobenius action $\phi_{\pi,f} : F_V^*(\mathcal{L}_{\pi,f}) \xrightarrow{\ \sim\ } \mathcal{L}_{\pi,f}$ is given by the composed isomorphism

$$F_V^*(\mathcal{L}_{\pi,f}) = F_V^*\, u'^*\mu^*(\mathcal{L}_\pi) \;\xrightarrow{\ \sim\ }\; u'^*\mu^* F_{\mathbb{A}^1}^*(\mathcal{L}_\pi) \;\xrightarrow{\ \sim\ }\; u'^*\mu^*(\mathcal{L}_\pi) = \mathcal{L}_{\pi,f},$$

where the first isomorphism is the Taylor isomorphism relating the two inverse images of L_π under the morphisms $\mu \circ u' \circ F_V$ and $F_{\mathbb{A}^1} \circ \mu \circ u'$, and the second one is the pull-back of $\phi : F_{\mathbb{A}^1}^*(\mathcal{L}_\pi) \xrightarrow{\sim} \mathcal{L}_\pi$. Over V_j, the inverse images of the coordinate t on $\mathbb{A}^1$ under the morphisms $\mu \circ u' \circ F_V$ and $F_{\mathbb{A}^1} \circ \mu \circ u'$ are equal respectively to $t_j^q f(x_{i,j}^q)$ and $t_j^q f(x_{i,j})^q$. Thus, the restriction of the first isomorphism over $(q^{\mathrm{an}})^{-1}(\mathcal{U}_{j,K})$ is given in the canonical basis by multiplication by $\exp(\pi t_j^q(f(x_{i,j}^q) - f(x_{i,j})^q))$. From (1.9.2) we deduce that the restriction of the second one is given by multiplication by $\exp(\pi(t_j^q f(x_{i,j})^q - t_j f(x_{i,j})))$. Therefore, $\phi_{\pi,f}$ is given by

$$\phi_{\pi,f}(1 \otimes e) \;=\; \exp(\pi(t_j^q f(x_{i,j}^q) - t_j f(x_{i,j})))\, e. \tag{4.6.7}$$

The inverse image morphism F^* on $\mathbb{R}\Gamma_{\mathrm{rig}}(W_k^*, \mathcal{L}_{\pi,f})$ is obtained by applying the functor $\mathbb{R}\Gamma(V_K^{\mathrm{an}}, -)$ to the morphism of complexes $\Phi_{\pi,f}$:

$$\begin{array}{ccc} j_{W^*}^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}}) \otimes_{j_V^\dagger(\mathcal{O}_{V_K^{\mathrm{an}}})} \mathcal{L}_{\pi,f} & \xrightarrow{F^*\otimes \mathrm{Id}} & F_*(j_{W^*}^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}}) \otimes_{j_V^\dagger(\mathcal{O}_{V_K^{\mathrm{an}}})} F^*(\mathcal{L}_{\pi,f})) \\ & & \wr\big\downarrow \mathrm{Id}\otimes\phi_{\pi,f} \\ & & F_*(j_{W^*}^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}}) \otimes_{j_V^\dagger(\mathcal{O}_{V_K^{\mathrm{an}}})} \mathcal{L}_{\pi,f}). \end{array}$$

The direct image morphism F_* on $\mathbb{R}\Gamma_{\mathrm{rig}}(W_k^*, \mathcal{L}_{\pi,f})$ is defined as the Poincaré dual of the morphism F^* on $\mathbb{R}\Gamma_{\mathrm{c,\,rig}}(W_k^*, \mathcal{L}_{\pi,f}^\vee)$, and it is easy to check that $F_* = q^{n+1}(F^*)^{-1}$ (since $F^* = q^{n+1}$ on $H^{2n+2}_{\mathrm{c,\,rig}}(W_k^*/K)$ [47, 6.5], and F^* is an isomorphism compatible with pairings). On the other hand, F_V is finite étale of rank q^{n+1} over W_K^*, and the corresponding trace morphism extends to a morphism of complexes

$$\mathrm{Tr}_F : F_*(j_{W^*}^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}})) \to j_{W^*}^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}})$$

such that the composed morphism

$$j_{W^*}^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}}) \xrightarrow{F^*} F_*(j_{W^*}^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}})) \xrightarrow{\mathrm{Tr}_F} j_{W^*}^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}})$$

is multiplication by q^{n+1}. Let $\Psi_{\pi,f}$ be the composed morphism

$$\begin{array}{ccc} F_*(j_{W^*}^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}}) \otimes_{j_V^\dagger(\mathcal{O}_{V_K^{\mathrm{an}}})} F^*(\mathcal{L}_{\pi,f})) & \xleftarrow[\sim]{\mathrm{Id}\otimes\phi_{\pi,f}^{-1}} & F_*(j_{W^*}^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}}) \otimes_{j_V^\dagger(\mathcal{O}_{V_K^{\mathrm{an}}})} \mathcal{L}_{\pi,f}) \\ \wr\big\downarrow & & \\ F_*(j_{W^*}^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}})) \otimes_{j_V^\dagger(\mathcal{O}_{V_K^{\mathrm{an}}})} \mathcal{L}_{\pi,f} & \xrightarrow{\mathrm{Tr}_F\otimes\mathrm{Id}} & j_{W^*}^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}}) \otimes_{j_V^\dagger(\mathcal{O}_{V_K^{\mathrm{an}}})} \mathcal{L}_{\pi,f}. \end{array}$$

It is clear that $\Psi_{\pi,f} \circ \Phi_{\pi,f} = q^{n+1}$, hence $\Psi_{\pi,f}$ induces F_* on $\mathbb{R}\Gamma_{\mathrm{rig}}(W_k^*/K, \mathcal{L}_{\pi,f})$.

As $F^*(dt_j/t_j) = q\, dt_j/t_j$ and $F^*(dx_{i,j}/x_{i,j}) = q\, dx_{i,j}/x_{i,j}$, the morphism Tr_F can also be defined on $j_V^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}}(\log D_K^{\mathrm{an}}))$, and the canonical morphism

$$j_V^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}}(\log D_K^{\mathrm{an}})) \longrightarrow j_{W^*}^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}})$$

commutes with the morphisms Tr_F on both complexes. Repeating the definition of $\Psi_{\pi,f}$, we obtain an endomorphism F_* of the complex $\Gamma(V_K^{\mathrm{an}}, j_V^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}}(\log D_K^{\mathrm{an}})) \otimes$

$\mathcal{L}_{\pi,f}$) such that the canonical isomorphism

$$\Gamma(V_K^{\mathrm{an}}, j_V^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}}(\log D_K^{\mathrm{an}})) \otimes \mathcal{L}_{\pi,f}) \xrightarrow{\sim} \mathbb{R}\Gamma_{\mathrm{rig}}(W_k^*/K, \mathcal{L}_{\pi,f})$$

commutes with F_*. Therefore, it suffices to show that the isomorphism $\theta^\dagger$ identifies α to F_*. But on the one hand $\theta^\dagger$ identifies $q^{n+1}\psi$ on $L(0^+)$ with the trace of F on the algebra $\Gamma(V_K^{\mathrm{an}}, \mathcal{O}_{V_K^{\mathrm{an}}})$. On the other hand we have on $(q^{\mathrm{an}})^{-1}(\mathcal{U}_{j,K})$

$$\theta^\dagger(F(X_0, \dots, X_{n+1})) \;=\; \exp(\pi(t_j f(x_{i,j}) - t_j^q f(x_{i,j}^q))),$$

which by (4.6.7) is the series defining $\phi_{\pi,f}^{-1}$. The claim then follows easily from (4.1.2).

As a consequence, general results known for rigid cohomology also apply to Dwork cohomology:

Corollary 4.7. *Without assumption on f, the Dwork cohomology spaces $H^i(K^\bullet(L(0^+);\underline{D}))$ are finite dimensional K-vector spaces, and α induces an automorphism on these spaces.*

Thanks to (4.6.4), this follows from 1.9, or from [42].

Corollary 4.8. *Let K be a number field, R its ring of integers, $S = \mathrm{Spec}(R)$, S^0 the set of closed points in S. For each $s \in S^0$, let $K(s)$ be the completion of K at s, p_s its residue characteristic, $K'(s)$ a finite extension of $K(s)$ containing a root π_s of the polynomial $X^{p_s-1}+p_s$. Assume that $f \in R[X_1, \dots, X_{n+1}]$ is a homogeneous polynomial of degree $d \geq 1$, and denote by $K^\bullet(\mathfrak{L}_s;\underline{D})$, $K^\bullet(L_s(0^+);\underline{D})$ the Dwork complexes built with f on $K'(s)$. Then there exists a finite subset $\Sigma \subset S^0$ such that, for all $s \in S^0 \setminus \Sigma$, the inclusion*

$$K^\bullet(\mathfrak{L}_s;\underline{D}) \;\subset\; K^\bullet(L_s(0^+);\underline{D})$$

induces an isomorphism on the cohomology spaces.

Using again (4.6.4), this is a consequence of 3.7.

Our next theorem relates Dwork's cohomology with the rigid homology of the affine hypersurface $Z_k \cap Y_k$.

Theorem 4.9. *Under the assumptions of* 4.2 *and* 4.4*, there exists distinguished triangles*

$$\mathbb{R}\Gamma_{Z_K\cap Y_K,\,\mathrm{dR}}(Y_K/K) \;\longrightarrow\; K^\bullet(\mathfrak{L};\underline{D}) \;\longrightarrow\; \mathbb{R}\Gamma_{\mathrm{dR}}(Y_K/K)[-1] \xrightarrow{+1} \quad (4.9.1)$$

$$\mathbb{R}\Gamma_{Z_k\cap Y_k,\,\mathrm{rig}}(Y_k/K) \;\longrightarrow\; K^\bullet(L(0^+);\underline{D}) \;\longrightarrow\; \mathbb{R}\Gamma_{\mathrm{rig}}(Y_k/K)[-1] \xrightarrow{+1} \quad (4.9.2)$$

and a canonical morphism of distinguished triangles

$$\begin{array}{ccccccc}
\mathbb{R}\Gamma_{Z_K\cap Y_K,\,\mathrm{dR}}(Y_K/K) & \longrightarrow & K^\bullet(\mathfrak{L};\underline{D}) & \longrightarrow & \mathbb{R}\Gamma_{\mathrm{dR}}(Y_K/K)[-1] & \xrightarrow{+1} & \\
\downarrow{\scriptstyle \rho_{Z\cap Y}} & & \downarrow & & \downarrow{\scriptstyle \rho_Y[-1]} & & \\
\mathbb{R}\Gamma_{Z_k\cap Y_k,\,\mathrm{rig}}(Y_k/K) & \longrightarrow & K^\bullet(L(0^+);\underline{D}) & \longrightarrow & \mathbb{R}\Gamma_{\mathrm{rig}}(Y_k/K)[-1] & \xrightarrow{+1} & ,
\end{array} \tag{4.9.3}$$

where the left and right vertical maps are the specialisation morphisms (1.2.2) *and the middle one is the canonical inclusion.*

Moreover, the arrows in the triangle (4.9.2) *commute with the following endomorphisms:*

a) $q^{n+1}F^{*\,-1}$ *on* $\mathbb{R}\Gamma_{Z_k\cap Y_k,\,\mathrm{rig}}(Y_k/K)$,
b) α *on* $K^\bullet(L(0^+);\underline{D})$,
c) $q^nF^{*\,-1}$ *on* $\mathbb{R}\Gamma_{\mathrm{rig}}(Y_k/K)[-1]$.

With the notation of 4.2, the decomposition $D = H' \cup X$ gives rise to an exact sequence of logarithmic de Rham complexes

$$0 \longrightarrow \Omega^\bullet_V(\log H') \longrightarrow \Omega^\bullet_V(\log D) \xrightarrow{\mathrm{Res}} \Omega^\bullet_X(\log H)(-1)[-1] \longrightarrow 0, \tag{4.9.4}$$

where Res is the residue map [25, II 3.7]. This sequence is compatible with the action of F_V by functoriality, provided that the functoriality map on $\Omega^\bullet_X(\log H)$ is multiplied by q, as indicated by the -1 twist: this is due to the fact that, on each V_j, we have $F_V^*(dt_j/t_j) = q\,dt_j/t_j$.

Tensoring with $L_{\pi,f}$, which is trivial on the zero section $X \subset V$, and taking cohomology on V_K, we obtain a distinguished triangle

$$\begin{array}{ccc}
 & \mathbb{R}\Gamma(X_K,\Omega^\bullet_{X_K}(\log H_K))[-1] & \\
{\scriptstyle +1}\swarrow & & \nwarrow \\
\mathbb{R}\Gamma(V_K,\Omega^\bullet_{V_K}(\log H'_K)\otimes L_{\pi,f}) & \longrightarrow & \mathbb{R}\Gamma(V_K,\Omega^\bullet_{V_K}(\log D_K)\otimes L_{\pi,f}).
\end{array} \tag{4.9.5}$$

Using the Grothendieck-Deligne theorem, we obtain the following isomorphisms:

$$\begin{aligned}
\mathbb{R}\Gamma(V_K,\Omega^\bullet_{V_K}(\log H'_K)\otimes L_{\pi,f}) &\xrightarrow{\sim} \mathbb{R}\Gamma(W_K,\Omega^\bullet_{W_K}\otimes L_{\pi,f}),\\
\mathbb{R}\Gamma(V_K,\Omega^\bullet_{V_K}(\log D_K)\otimes L_{\pi,f}) &\xrightarrow{\sim} \mathbb{R}\Gamma(W_K^*,\Omega^\bullet_{W_K^*}\otimes L_{\pi,f}),\\
\mathbb{R}\Gamma(X_K,\Omega^\bullet_{X_K}(\log H_K)) &\xrightarrow{\sim} \mathbb{R}\Gamma(Y_K,\Omega^\bullet_{Y_K}).
\end{aligned}$$

On the other hand, (3.3.2) gives a canonical isomorphism

$$\mathbb{R}\Gamma(W_K,\Omega^\bullet_{W_K}\otimes L_{\pi,f}) \xrightarrow{\sim} \mathbb{R}\Gamma_{Z_K\cap Y_K,\,\mathrm{dR}}(Y_K/K).$$

Thus the triangle (4.9.5) can be written as

$$\mathbb{R}\Gamma_{Z_K\cap Y_K,\,\mathrm{dR}}(Y_K/K) \longrightarrow \mathbb{R}\Gamma_{\mathrm{dR}}(W_K^*/K,L_{\pi,f}) \longrightarrow \mathbb{R}\Gamma_{\mathrm{dR}}(Y_K/K)[-1] \xrightarrow{+1}. \tag{4.9.6}$$

The exact sequence (4.9.4) defines a similar sequence on V_K^{an}. Applying $j_V^\dagger$, tensoring with $\mathcal{L}_{\pi,f}$ and taking cohomology on V_K^{an} provides a distinguished triangle

$$\begin{array}{ccc} & \mathbb{R}\Gamma(X_K^{\mathrm{an}}, \Omega^\bullet_{X_K^{\mathrm{an}}}(\log H_K^{\mathrm{an}}))(-1)[-1] & \\ {\scriptstyle +1}\swarrow & & \nwarrow \\ \mathbb{R}\Gamma(V_K^{\mathrm{an}}, j_V^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}}(\log H_K'^{\mathrm{an}})) \otimes \mathcal{L}_{\pi,f}) & \longrightarrow & \mathbb{R}\Gamma(V_K^{\mathrm{an}}, j_V^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}}(\log D_K^{\mathrm{an}})) \otimes \mathcal{L}_{\pi,f}), \end{array} \tag{4.9.7}$$

in which the arrows are compatible with the action of F_V by functoriality. Using A.4, we obtain as in the algebraic case isomorphisms

$$\begin{aligned} \mathbb{R}\Gamma(V_K^{\mathrm{an}}, j_V^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}}(\log H_K'^{\mathrm{an}})) \otimes \mathcal{L}_{\pi,f}) &\xrightarrow{\ \sim\ } \mathbb{R}\Gamma(V_K^{\mathrm{an}}, j_W^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}}) \otimes \mathcal{L}_{\pi,f}), \\ \mathbb{R}\Gamma(V_K^{\mathrm{an}}, j_V^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}}(\log D_K^{\mathrm{an}})) \otimes \mathcal{L}_{\pi,f}) &\xrightarrow{\ \sim\ } \Gamma(V_K^{\mathrm{an}}, j_{W^*}^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}}) \otimes \mathcal{L}_{\pi,f}) \\ \mathbb{R}\Gamma(X_K^{\mathrm{an}}, \Omega^\bullet_{X_K^{\mathrm{an}}}(\log H_K^{\mathrm{an}})) &\xrightarrow{\ \sim\ } \mathbb{R}\Gamma(X_K^{\mathrm{an}}, j_Y^\dagger(\Omega^\bullet_{X_K^{\mathrm{an}}})), \end{aligned}$$

and the targets are respectively equal by construction to $\mathbb{R}\Gamma_{\mathrm{rig}}(W_k/K, \mathcal{L}_{\pi,f})$, $\mathbb{R}\Gamma_{\mathrm{rig}}(W_k^*/K, \mathcal{L}_{\pi,f})$, $\mathbb{R}\Gamma_{\mathrm{rig}}(Y_k/K)$. Note that these isomorphisms identify F_V^* to F^*. Thanks to (3.1.1), we also have an isomorphism

$$\mathbb{R}\Gamma_{\mathrm{rig}}(W_k/K, \mathcal{L}_{\pi,f}) \ \simeq\ \mathbb{R}\Gamma_{Z_k\cap Y_k,\,\mathrm{rig}}(Y_k/K)$$

which is compatible to F^*. Taking the Frobenius actions into account, the triangle (4.9.7) can therefore be rewritten as

$$\mathbb{R}\Gamma_{Z_k\cap Y_k,\,\mathrm{rig}}(Y_k/K) \ \longrightarrow\ \mathbb{R}\Gamma_{\mathrm{rig}}(W_k^*/K, \mathcal{L}_{\pi,f}) \ \longrightarrow\ \mathbb{R}\Gamma_{\mathrm{rig}}(Y_k/K)(-1)[-1] \ \xrightarrow{+1}. \tag{4.9.8}$$

Because of the functoriality of the constructions used to build the triangles (4.9.5) and (4.9.7) from the exact sequence (4.9.4), it is clear from the definition of the specialization morphisms given in 1.2 and 1.8 that they fit in a morphism of triangles

$$\begin{array}{ccccccc} \mathbb{R}\Gamma_{\mathrm{dR}}(W_K/K, L_{\pi,f}) & \longrightarrow & \mathbb{R}\Gamma_{\mathrm{dR}}(W_K^*/K, L_{\pi,f}) & \longrightarrow & \mathbb{R}\Gamma_{\mathrm{dR}}(Y_K/K)[-1] & \xrightarrow{+1} & \\ \downarrow{\scriptstyle \rho_W^{L_{\pi,f}}} & & \downarrow{\scriptstyle \rho_{W^*}^{L_{\pi,f}}} & & \downarrow{\scriptstyle \rho_Y[-1]} & & \\ \mathbb{R}\Gamma_{\mathrm{rig}}(W_k/K, \mathcal{L}_{\pi,f}) & \longrightarrow & \mathbb{R}\Gamma_{\mathrm{rig}}(W_k^*/K, \mathcal{L}_{\pi,f}) & \longrightarrow & \mathbb{R}\Gamma_{\mathrm{rig}}(Y_k/K)[-1] & \xrightarrow{+1} & . \end{array}$$

On the other hand, proposition 3.4 shows that the isomorphisms (3.3.2) and (3.1.1) identify $\rho_{Z\cap Y}$ with $\rho_W^{L_{\pi,f}}$. Therefore, the specialization morphisms define a morphism of triangles from (4.9.6) to (4.9.8).

Finally, we can use the isomorphisms (4.6.2) and (4.6.4) to rewrite triangle (4.9.6) as (4.9.1), and triangle (4.9.8) as (4.9.2). Under these isomorphisms, the inclusion $K^\bullet(\mathfrak{L}; \underline{D}) \subset K^\bullet(L(0^+); \underline{D})$ corresponds to the inclusion

$$\Gamma(V_K, \Omega^\bullet_{V_K}(\log D_K) \otimes L_{\pi,f}) \subset \Gamma(V_K^{\mathrm{an}}, j_V^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}}(\log D_K^{\mathrm{an}}) \otimes \mathcal{L}_{\pi,f})),$$

hence to the specialization morphism $\rho^{L_{\pi,f}}_{W^*_K}$. Therefore, we obtain the morphism of triangles (4.9.3). As the endomorphism α of $K^\bullet(L(0^+);\underline{D})$ corresponds to $q^{n+1}(F^*_V)^{-1}$ in the derived category, the morphisms of the triangle (4.9.2) satisfy the announced compatibilities with Frobenius actions.

Using the triangle (4.9.2), one can give an interpretation of Dwork's formula (4.0.1) in terms of rigid cohomology:

Corollary 4.10. *The characteristic series $\chi_F(t)$ of α satisfies the relation*

$$\begin{aligned}\chi_F(t)^{(-\delta)^{n+1}} &= (1-t)^{-(-\delta)^n}\prod_{i=n-1}^{2n-2}\det(I-t\,qF^*|H^i_{\mathrm{c,\,rig}}(Z_k\cap Y_k/K))^{(-1)^i}\\ &= (1-t)^{-(-\delta)^n}\zeta(Z_k\cap Y_k,qt)^{-1}. \qquad (4.10.1)\end{aligned}$$

Since all complexes in the triangle (4.9.2) have finite dimensional cohomology groups, we obtain the relation

$$\begin{aligned}\det(I-t\,\alpha|K^\bullet(L(0^+);\underline{D})) &= \det(I-t\,q^{n+1}F^{*\,-1}|\mathbb{R}\Gamma_{Z_k\cap Y_k,\,\mathrm{rig}}(Y_k/K))\\ &\quad\times\det(I-t\,q^nF^{*\,-1}|\mathbb{R}\Gamma_{\mathrm{rig}}(Y_k/K)[-1]),\end{aligned}$$

and, by (4.1.4), we have

$$\det(I-t\,\alpha|K^\bullet(L(0^+);\underline{D})) = \chi_F(t)^{(-\delta)^{n+1}}.$$

The affine variety Y_k is an n-dimensional torus, hence we have for any i

$$H^i_{\mathrm{rig}}(Y_k/K) = K^{\binom{n}{i}},$$

and F^* is multiplication by q^i on $H^i_{\mathrm{rig}}(Y_k/K)$. It follows that

$$\det(I-t\,q^nF^{*\,-1}|\mathbb{R}\Gamma_{\mathrm{rig}}(Y_k/K)[-1]) = (1-t)^{-(-\delta)^n}.$$

Finally, Poincaré duality provides a perfect pairing

$$\mathbb{R}\Gamma_{Z_k\cap Y_k,\,\mathrm{rig}}(Y_k/K)\otimes_K\mathbb{R}\Gamma_{\mathrm{c,\,rig}}(Z_k\cap Y_k/K)\longrightarrow K(-n)[-2n],$$

which is compatible to F^*. Therefore the automorphism $q^{n+1}F^{*\,-1}$ is dual to qF^* on $\mathbb{R}\Gamma_{\mathrm{c,\,rig}}(Z_k/K)$, and we obtain

$$\det(I-t\,q^{n+1}F^{*\,-1}|\mathbb{R}\Gamma_{Z_k\cap Y_k,\,\mathrm{rig}}(Y_k/K)) = \prod_i\det(I-t\,qF^*|H^i_{\mathrm{c,\,rig}}(Z_k\cap Y_k/K))^{(-1)^i}.$$

Thanks to [33, 6.3 I], the second relation of the corollary follows. To complete the proof of the first one, we observe first that $H^i_{\mathrm{c,\,rig}}(Z_k\cap Y_k/K)=0$ for $i>2n-2$ because $\dim(Z_k\cap Y_k)=n-1$. On the other hand, $H^i_{\mathrm{c,\,rig}}(Y_k\setminus Z_k/K)=H^i_{\mathrm{c,\,rig}}(Y_k/K)=0$ for $i<n$, because $Y_k\setminus Z_k$ and Y_k are affine and smooth of dimension n, so that their rigid cohomology with compact supports is Poincaré dual to their Monsky-Washnitzer cohomology, which is concentrated in degrees $\leq n$. Hence $H^i_{\mathrm{c,\,rig}}(Z_k\cap Y_k/K)=0$ for $i<n-1$.

Corollary 4.11. *For any i, the eigenvalues of α acting on the Dwork cohomology space $H^i(K^\bullet(L(0^+);\underline{D}))$ are Weil numbers, whose weights belong to the interval $[2n-2i+2, 2n-i+2]$.*

The first assertion results from the fact that the eigenvalues of F^* on $\mathbb{R}\Gamma_{Z_k\cap Y_k,\,\mathrm{rig}}(Y_k/K)$ are Weil numbers, a result proved by Chiarellotto using comparison with ℓ-adic cohomology [24, I 2.3], and more recently by direct p-adic methods by Kedlaya [43]. To get estimates on the weights, we use the exact sequences

$$H^i_{Z_k\cap Y_k,\,\mathrm{rig}}(Y_k/K) \longrightarrow H^i(K^\bullet(L(0^+);\underline{D})) \longrightarrow H^{i-1}_{\mathrm{rig}}(Y_k/K).$$

From the above discussion, it follows that $q^nF^{*\,-1}$ is pure of weight $2n-2(i-1)$ on $H^{i-1}_{\mathrm{rig}}(Y_k/K)$. From [24, I 2.3], we get that $q^{n+1}F^{*\,-1}$ is mixed of weights in $[2n-2i+4, 2n-i+2]$ on $H^i_{Z_k\cap Y_k,\,\mathrm{rig}}(Y_k/K)$. The above estimate follows.

5. The coordinate filtration

While theorems 4.6 and 4.9 relate the cohomology of Dwork complexes to the de Rham and rigid cohomologies of the open subsets $Z_K\cap Y_K$ and $Z_k\cap Y_k$, the cohomologies of the projective hypersurfaces Z_K and Z_k themselves can also be computed using the Dwork complexes, and the same holds for the cohomologies of all their intersections with any intersection of coordinate hyperplanes. The method developed by Dwork relies on the construction of certain subcomplexes of $\mathfrak{L}$ and $L(0^+)$, and we want now to give a cohomological interpretation of these subcomplexes, and of the following formula (5.0.1) of Dwork, which is closely related.

We denote by N the subset $\{1,\dots,n+1\}\subset\mathbb{N}$. For each non empty subset $A\subset N$, we define

$$H_A = \bigcap_{i\notin A} H_i, \qquad Z_A \;=\; Z\cap H_A.$$

In particular, $Z_N = Z$. Let $m(A) = \#(A)-1$, so that H_A is a linear projective subspace of dimension $m(A)$ of $\mathbb{P}^n_S$, and let $P_A(t)$ be the rational function defined by

$$\begin{aligned}\zeta(Z_{A,k},t) &= P_A(t)^{(-1)^{m(A)}}(1-q^{m(A)}t)\Big/\prod_{i=0}^{m(A)}(1-q^it)\\ &= P_A(t)^{(-1)^{m(A)}}(1-q^{m(A)}t)\zeta(\mathbb{P}^{m(A)}_k,t).\end{aligned}$$

Then a combinatorial argument based on (4.0.1) shows that

$$\chi_F(t)^{\delta^{n+1}} \;=\; (1-t)\prod_{\substack{A\subset N\\ A\neq\emptyset}} P_A(qt) \tag{5.0.1}$$

(*cf.* [29, (4.33)]).

5.1. For all $A \subset N$, let $\Pi_A : \mathfrak{L} \to \mathfrak{L}$, $\Pi_A^\dagger : L(0^+) \to L(0^+)$ be the homomorphisms defined by sending X_i to 0 if $i \notin A$, and X_i to X_i otherwise. We define

$$\mathfrak{L}_A \;=\; \mathrm{Im}(\Pi_A : \mathfrak{L} \to \mathfrak{L}), \qquad L_A(0^+) \;=\; \mathrm{Im}(\Pi_A^\dagger : L(0^+) \to L(0^+)).$$

Thus we obtain $\mathfrak{L}_\emptyset = K$, $\mathfrak{L}_N = \mathfrak{L}$.

We denote respectively by f_A and $D_{A,i}$, for $i \in A$, the polynomial and the differential operator deduced from f and D_i by substituting 0 to X_j if $j \notin A$. Thus f_A is the equation defining Z_A as an hypersurface in H_A, and $D_{A,i}$ is the differential operator defined by f_A as in (4.1.1). We can introduce the algebras $\mathfrak{L}_A$ and $L_A(0^+)$ built on the variables X_0 and X_i for $i \in A$, and define the Koszul complexes $K^\bullet(\mathfrak{L}_A; \underline{D}_A)$, $K^\bullet(L_A(0^+); \underline{D}_A)$ associated to the differential operators $D_{A,i}$, $i \in A$. We denote by α_A the endomorphism of $K^\bullet(L_A(0^+); \underline{D}_A)$ defined by the polynomial f_A.

For all $B \subset A \subset N$, let $M_B = \prod_{i \in B} X_i$, let $(M_B) \subset K[[X_0, \ldots, X_{n+1}]]$ be the ideal generated by M_B, and let

$$\mathfrak{L}_A^B \;=\; \mathfrak{L}_A \cap (M_B), \qquad L_A^B(0^+) \;=\; L_A(0^+) \cap (M_B).$$

In particular, $\mathfrak{L}_A^\emptyset = \mathfrak{L}_A$.

We want to define an increasing filtration on the complexes $K^\bullet(\mathfrak{L}; \underline{D})$ and $K^\bullet(L(0^+); \underline{D})$. We first define for each subset $A \subset N$ an increasing filtration $\mathrm{Fil}_r^A \mathfrak{L}$ (resp. $\mathrm{Fil}_r^A L(0^+)$) on $\mathfrak{L}$ (resp. $L(0^+)$) by

$$\mathrm{Fil}_r^A \mathfrak{L} \;=\; \begin{cases} 0 & \text{if } r < 0, \\ \sum_{\substack{B \subset A \\ \#(A \setminus B) = r}} \mathfrak{L}_N^B & \text{if } 0 \le r \le \#A, \\ \mathfrak{L} & \text{if } r > \#A \end{cases}$$

(resp. $L(0^+)$, $L_N^B(0^+)$). Note that $\mathrm{Fil}_0^A \mathfrak{L} = \mathfrak{L}_N^A$, and $\mathrm{Fil}_a^A \mathfrak{L} = \mathfrak{L}$ for $a = \#A$.

We can now define $\mathrm{Fil}_r K^\bullet(\mathfrak{L}; \underline{D})$ (resp. $L(0^+)$) by

$$\mathrm{Fil}_r K^j(\mathfrak{L}; \underline{D}) \;=\; \bigoplus_{\#A = j} \mathrm{Fil}_r^A \mathfrak{L} \cdot e_A \tag{5.1.1}$$

(resp. $L(0^+)$), where $e_A = e_{i_1} \wedge \ldots \wedge e_{i_j}$ if $A = \{i_1, \ldots, i_j\}$ with $i_1 < \ldots < i_j$. To check that this is indeed a sub-complex, we observe that

$$d(P \cdot e_A) \;=\; \sum_{i \notin A} D_i(P)\, e_i \wedge e_A.$$

Assume $i \notin A$, and write $A_i = A \cup \{i\}$. If P is divisible by M_B for some $B \subset A$ with $\#(A \setminus B) = r$, then $i \notin B$, and $D_i(P)$ is divisible by X_i, hence by $X_i M_B = M_{B \cup \{i\}}$. Therefore $D_i(P) \in \mathrm{Fil}_r^{A_i}$, hence Fil_r is a subcomplex.

For all $A \subset N$, $L_N^A(0^+)$ is stable under the endomorphism α of $L(0^+)$. Indeed, this is clearly the case for both multiplication by $F(X_0, \ldots, X_{n+1})$ and ψ.

It follows that, for all r, the subcomplex $\mathrm{Fil}_r K^\bullet(L(0^+);\underline{D})$ is stable under the endomorphism α of $K^\bullet(L(0^+);\underline{D})$.

Finally, we observe that, since $\mathfrak{L}_A$ and $L_A(0^+)$ are built from f_A as $\mathfrak{L}$ and $L(0^+)$ from f, the complexes $K^\bullet(\mathfrak{L}_A;\underline{D}_A)$ and $K^\bullet(L_A(0^+);\underline{D}_A)$ can be endowed with an analogous filtration.

To describe the associated graded complex, we first prove a lemma.

Lemma 5.2. *Let $A \subset N$ be a subset, and $A' = N \setminus A$. For all r, there exists a canonical isomorphism*

$$\bigoplus_{\substack{B\subset A\\ \#(A\setminus B)=r}} \mathfrak{L}^B_{A'\cup B} \xrightarrow{\ \sim\ } \mathrm{gr}^A_r \mathfrak{L} \tag{5.2.1}$$

(*resp.* $L(0^+)$).

The proofs for $\mathfrak{L}$ and $L(0^+)$ are completely parallel, so we limit ourselves here to the case of $\mathfrak{L}$. For $B \subset A$, with $\#(A \setminus B) = r$, we have $\mathfrak{L}^B_{A'\cup B} \subset \mathfrak{L}^B_N \subset \mathrm{Fil}^A_r \mathfrak{L}$, which defines the map.

Let $a = \#A$. We first observe that $\mathrm{Fil}^A_r \mathfrak{L}$ has a K-basis formed by monomials $\underline{X}^{\underline{u}}$, $\underline{u} \in \mathfrak{T}$, such that $u_i \geq 1$ for at least $a - r$ indices in A. Therefore $\mathrm{gr}^A_r \mathfrak{L}$ has a basis formed by the classes of monomials $\underline{X}^{\underline{u}}$ such that $u_i \geq 1$ for exactly $(a - r)$ indices in A. This set of monomials is the disjoint union, for all $B \subset A$ with $\#(A \setminus B) = r$, of the subsets of monomials divisible by M_B and not by any X_i for $i \in A \setminus B$. It follows immediately that the map (5.2.1) is an isomorphism.

Proposition 5.3. *With the notation of* 5.1*, there exists for all r a canonical isomorphism of complexes*

$$\bigoplus_{\#A=n+1-r} \mathrm{Fil}_0 K^\bullet(\mathfrak{L}_A;\underline{D}_A)[-r] \xrightarrow{\ \sim\ } \mathrm{gr}_r K^\bullet(\mathfrak{L};\underline{D}) \tag{5.3.1}$$

(*resp.* $L_A(0^+)$, $L(0^+)$). *In the case of $L(0^+)$, this isomorphism is compatible with the actions of the endomorphisms α.*

We first describe $\mathrm{Fil}_0 K^\bullet(\mathfrak{L}_A;\underline{D}_A)$, with $A \subset N$. For all $B \subset A$, we have $\mathrm{Fil}^B_0 \mathfrak{L}_A = \mathfrak{L}^B_A$, hence $\mathrm{Fil}_0 K^\bullet(\mathfrak{L}_A;\underline{D}_A)$ is the complex

$$\mathfrak{L}_A \longrightarrow \bigoplus_{\substack{B\subset A\\ \#B=1}} \mathfrak{L}^B_A \cdot e_B \longrightarrow \dots \longrightarrow \bigoplus_{\substack{B\subset A\\ \#B=\#A-1}} \mathfrak{L}^B_A \cdot e_B \longrightarrow \mathfrak{L}^A_A \cdot e_A \longrightarrow 0.$$

Using lemma 5.2, we obtain for each r and each j

$$\begin{aligned}\mathrm{gr}_r K^j(\mathfrak{L};\underline{D}) &= \bigoplus_{\#A=j} \mathrm{gr}^A_r \mathfrak{L} \cdot e_A \\ &\simeq \bigoplus_{\#A=j} \bigoplus_{\substack{B\subset A\\ \#B=j-r}} \mathfrak{L}^B_{A'\cup B} \cdot e_A,\end{aligned}$$

with $A' = N \setminus A$. Let $A'' = A' \cup B$. Then $\#A'' = n + 1 - r$, and the datum of (A, B) is equivalent to the datum of (A'', B). Let $C = A \setminus B$. If we map e_B to

$(-1)^{rj}\varepsilon(B,C)e_A$, where $\varepsilon(B,C)$ is the signature of the permutation $\{B,C\}$ of A, we obtain an isomorphism

$$\begin{aligned}\mathrm{gr}_r\, K^j(\mathfrak{L};\underline{D}) &\simeq \bigoplus_{\#A''=n+1-r}\ \bigoplus_{\substack{B\subset A''\\ \#B=j-r}} \mathfrak{L}^B_{A''}\cdot e_B\\ &\simeq \bigoplus_{\#A''=n+1-r} \mathrm{Fil}_0\, K^{j-r}(\mathfrak{L}_{A''};\underline{D}_{A''}).\end{aligned}$$

It is then easy to check that, for variable j, these isomorphisms define an isomorphism of complexes as claimed in (5.3.1).

The same proof applies to $\mathrm{gr}_r\, K^\bullet(L(0^+);\underline{D})$, and the compatibility with the endomorphisms α is straightforward.

Corollary 5.4. *The characteristic series $\chi_F(t)$ satisfies the relation*

$$\chi_F(t)^{\delta^{n+1}} \;=\; (1-t)\prod_{A\neq\emptyset}\det(I-t\alpha|\,\mathrm{Fil}_0\, K^\bullet(L_A(0^+);\underline{D}_A))^{(-1)^{\#A}}. \qquad (5.4.1)$$

By definition (*cf.* (4.1.3)), $\chi_F(t)^{(-\delta)^{n+1}} = \det(I-t\alpha|K^\bullet(L(0^+);\underline{D}))$. Since the filtration $\mathrm{Fil}_\bullet\, K^\bullet(L(0^+);\underline{D})$ is stable under α, the corollary follows from the proposition by observing that

$$\mathrm{gr}_{n+1}\, K^\bullet(L(0^+);\underline{D}) \;\simeq\; \mathrm{Fil}_0\, K^\bullet(L_\emptyset(0^+);0)[-n-1],$$

and that $\mathrm{Fil}_0\, K^\bullet(L_\emptyset(0^+);0) = K^\bullet(L_\emptyset(0^+);0) = K$, with $\alpha_\emptyset = \mathrm{Id}$.

We now want to give a geometric interpretation of the complexes $\mathrm{Fil}_0\, K^\bullet(\mathfrak{L}_A;\underline{D}_A)$ and $\mathrm{Fil}_0\, K^\bullet(L(0^+)_A;\underline{D}_A)$, providing a computation of $\det(I-t\alpha)$ in terms of zeta functions. It is clearly sufficient to treat the case where $A=N$.

5.5. We first introduce a notation used in our next theorem. Let $\mathcal{C}$ be an abelian category, and $C(\mathcal{C})$ the category of complexes of objects of $\mathcal{C}$. For any $n\in\mathbb{Z}$, the truncation functor $\tau_{\geq n}$ associates to a complex $E^\bullet\in C(\mathcal{C})$ the complex

$$\tau_{\geq n}E^\bullet \;:\; \ldots \longrightarrow 0 \longrightarrow E^n/d(E^{n-1}) \longrightarrow E^{n+1} \longrightarrow \ldots\,.$$

The category $C(\mathcal{C})$ is itself an abelian category. If $E^{\bullet,\bullet}$ is a double complex of objects of $\mathcal{C}$, we can view it as a complex of objects of $C(\mathcal{C})$ indexed by the second index. We will then denote by $\tau^{\mathrm{II}}_{\geq n}E^{\bullet,\bullet}$ the double complex obtained by applying the truncation functor in $C(C(\mathcal{C}))$. If $g: E^{\bullet,\bullet}\to E'^{\bullet,\bullet}$ is a morphism of double complexes, then g induces a morphism $\tau^{\mathrm{II}}_{\geq n}g : \tau^{\mathrm{II}}_{\geq n}E^{\bullet,\bullet}\to\tau^{\mathrm{II}}_{\geq n}E'^{\bullet,\bullet}$. It is easy to check that, if $g,g' : E^{\bullet,\bullet}\to E'^{\bullet,\bullet}$ are homotopic in the sense of [23, IV 4], then the morphisms $\tau^{\mathrm{II}}_{\geq n}g$ and $\tau^{\mathrm{II}}_{\geq n}g'$ are also homotopic. In particular, the morphisms $\tau^{\mathrm{II}}_{\geq n}g_{\mathrm{t}}$ and $\tau^{\mathrm{II}}_{\geq n}g'_{\mathrm{t}}$ induced between the associated simple complexes are homotopic (in the usual sense).

Let $\mathcal{C}'$ be another abelian category, and $F : \mathcal{C} \to \mathcal{C}'$ a left exact functor. Assume that $\mathcal{C}$ has enough injectives. Then any complex $E^{\bullet} \in C^+(\mathcal{C})$ has an injective Cartan-Eilenberg resolution $I^{\bullet,\bullet}$ for which there exists an index i_0 such that $I^{i,j} = 0$ for $i < i_0$. We can apply the functor $\tau^{\mathrm{II}}_{\geq n}$ to the double complex $F(I^{\bullet,\bullet})$ defined by such a resolution, and take the associated total complex $\tau^{\mathrm{II}}_{\geq n}F(I^{\bullet,\bullet})_{\mathrm{t}}$. The above remarks show that, viewed as an object in the derived category $D(\mathcal{C}')$, this complex is independent of the choice of the resolution $I^{\bullet,\bullet}$, and depends functorially upon $E^{\bullet} \in C^+(\mathcal{C})$. We will denote

$$\tau^{\mathrm{II}}_{\geq n}\mathbb{R}F_{\mathrm{t}} \ : \ C^+(\mathcal{C}) \ \longrightarrow \ D^+(\mathcal{C}').$$

the functor defined in this way. For $n = 1$ and any $E^{\bullet} \in C^+(\mathcal{C})$, this construction gives rise to a distinguished triangle

$$F(E^{\bullet}) \ \longrightarrow \ \mathbb{R}F(E^{\bullet}) \ \longrightarrow \ \tau^{\mathrm{II}}_{\geq 1}\mathbb{R}F_{\mathrm{t}}(E^{\bullet}) \ \xrightarrow{+1} \tag{5.5.1}$$

in $D^+(\mathcal{C}')$. This triangle is functorial with respect to $E^{\bullet}$ when $E^{\bullet}$ varies in $C^+(\mathcal{C})$ (but of course not in $D^+(\mathcal{C})$).

When $E^{\bullet} \in C^{\geq 0}(\mathcal{C})$, one can find a Cartan-Eilenberg resolution $I^{\bullet,\bullet}$ of $E^{\bullet}$ such that $I^{i,j} = 0$ for $i < 0$. Therefore, $\tau^{\mathrm{II}}_{\geq n}\mathbb{R}F_{\mathrm{t}}(E^{\bullet})$ is concentrated in degree $\geq n$. Applying the usual truncation functor $\tau_{\geq n}$ to the second morphism in (5.5.1), one obtains a canonical morphism

$$\tau_{\geq n}\mathbb{R}F(E^{\bullet}) \ \longrightarrow \ \tau^{\mathrm{II}}_{\geq n}\mathbb{R}F_{\mathrm{t}}(E^{\bullet}) \tag{5.5.2}$$

in $D^{\geq n}(\mathcal{C})$.

Theorem 5.6. (i) *There exists natural isomorphisms of complexes*

$$\mathrm{Fil}_0\, K^{\bullet}(\mathfrak{L}; \underline{D}) \ \xrightarrow{\sim} \ \Gamma(V_K, \Omega^{\bullet}_{V_K} \otimes L_{\pi,f}), \tag{5.6.1}$$

$$\mathrm{Fil}_0\, K^{\bullet}(L(0^+); \underline{D}) \ \xrightarrow{\sim} \ \Gamma(V_K^{\mathrm{an}}, j_V^{\dagger}(\Omega^{\bullet}_{V_K^{\mathrm{an}}}) \otimes \mathcal{L}_{\pi,f}), \tag{5.6.2}$$

the latter being compatible with the Frobenius actions α on $\mathrm{Fil}_0\, K^{\bullet}(L(0^+); \underline{D})$ and $F_* = \Psi_{\pi,f}$ on $\Gamma(V_K^{\mathrm{an}}, j_V^{\dagger}(\Omega^{\bullet}_{V_K^{\mathrm{an}}}) \otimes \mathcal{L}_{\pi,f})$ (cf. 4.6 for the definition of $\Psi_{\pi,f}$).

(ii) *Let $s : X \hookrightarrow V$ be the zero section. With the notation of 5.5, the functoriality morphisms*

$$q_*(\Omega^{\bullet}_V \otimes L_{\pi,f}) \ \longrightarrow \ q_*s_*(\Omega^{\bullet}_X) \ \simeq \ \Omega^{\bullet}_{\mathbb{P}^n_S},$$

$$q_*^{\mathrm{an}}(j_V^{\dagger}(\Omega^{\bullet}_{V_K^{\mathrm{an}}}) \otimes \mathcal{L}_{\pi,f}) \ \longrightarrow \ q_*^{\mathrm{an}}s_*^{\mathrm{an}}(\Omega^{\bullet}_{X_K^{\mathrm{an}}}) \ \simeq \ \Omega^{\bullet}_{\mathbb{P}^{n\,\mathrm{an}}_K},$$

induced by s give rise to a commutative diagram of isomorphisms

$$\begin{array}{ccc} \tau^{\mathrm{II}}_{\geq 1}\mathbb{R}\Gamma_{\mathrm{t}}(V_K, \Omega^{\bullet}_{V_K} \otimes L_{\pi,f}) & \xrightarrow{\sim} & \tau_{\geq 1}\mathbb{R}\Gamma(\mathbb{P}^n_K, \Omega^{\bullet}_{\mathbb{P}^n_K}) \\ \downarrow \wr & & \downarrow \wr \\ \tau^{\mathrm{II}}_{\geq 1}\mathbb{R}\Gamma_{\mathrm{t}}(V_K^{\mathrm{an}}, j_V^{\dagger}(\Omega^{\bullet}_{V_K^{\mathrm{an}}}) \otimes \mathcal{L}_{\pi,f}) & \xrightarrow{\sim} & \tau_{\geq 1}\mathbb{R}\Gamma(\mathbb{P}^{n\,\mathrm{an}}_K, \Omega^{\bullet}_{\mathbb{P}^{n\,\mathrm{an}}_K}), \end{array} \tag{5.6.3}$$

in which the lower horizontal isomorphism is compatible with the Frobenius actions F^*.

The algebraic part of this theorem is essentially [27, 3.3]. We give here a proof which extends to the rigid context.

To prove assertion (i), we consider the isomorphisms of complexes

$$\begin{array}{ccc} K^\bullet(\mathfrak{L};\underline{D}) & \xrightarrow{\sim} & \Gamma(V_K, \Omega^\bullet_{V_K}(\log D_K)\otimes L_{\pi,f}) \\ \downarrow & & \downarrow \\ K^\bullet(L(0^+);\underline{D}) & \xrightarrow{\sim} & \Gamma(V_K^{\mathrm{an}}, j_V^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}}(\log D_K^{\mathrm{an}}))\otimes \mathcal{L}_{\pi,f}) \end{array}$$

provided by (4.6.1) and (4.6.3), and we look at the images of Fil_0 under these isomorphisms. For all $A\subset N$, the element e_A is mapped to

$$\omega_A \;=\; \bigwedge_{i\in A}\omega_i \;=\; \bigwedge_{i\in A}(\frac{1}{d}\frac{dt_j}{t_j}+\frac{dx_{i,j}}{x_{i,j}}),$$

where the last equality takes place in $\Gamma(V_{j,K},\Omega^\bullet_{V_K}(\log D_K))$ for any j. Expanding this form, one obtains a sum of terms whose denominator is of degree ≤ 1 with respect to any of the coordinates $t_j, x_{i,j}, i\in A$. As any monomial $\underline{X}^{\underline{u}}\in \mathrm{Fil}_0^A\,\mathfrak{L} = \mathfrak{L}_N^A$ is divisible by the X_i's for $i\in A$, its image in $\Gamma(V_{j,K},\mathcal{O}_{V_K})$ is divisible by the $x_{i,j}$, $i\in A$. If $A\neq\emptyset$, $\sum_{i\in A}u_i>0$, hence $u_0>0$, and the image of $\underline{X}^{\underline{u}}$ is also divisible by t_j. It follows that the image of $\underline{X}^{\underline{u}}e_A$ is a differential form which has no poles along D_K. Therefore the isomorphisms (4.6.1) and (4.6.3) map respectively $\mathrm{Fil}_0\,K^\bullet(\mathfrak{L};\underline{D})$ and $\mathrm{Fil}_0\,K^\bullet(L(0^+);\underline{D})$ to $\Gamma(V_K,\Omega^\bullet_{V_K}\otimes L_{\pi,f})$ and $\Gamma(V_K^{\mathrm{an}}, j_V^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}})\otimes\mathcal{L}_{\pi,f})$.

Conversely, let $\underline{X}^{\underline{u}}\in\mathfrak{L}$ be a monomial such that the image of $\underline{X}^{\underline{u}}e_A$ is in $\Gamma(V_K,\Omega^\bullet_{V_K}\otimes L_{\pi,f})$. Then, for each j, the form $t_j^{u_0}x_{1,j}^{u_1}\dots x_{n+1,j}^{u_{n+1}}\omega_A$ belongs to $\Gamma(V_{j,K},\Omega^\bullet_{V_K})$. Using (4.3.3), it follows that, for each j, $u_i\ge 1$ for all $i\in A, i\neq j$, hence that $u_i\ge1$ for all $i\in A$. Therefore $\underline{X}^{\underline{u}}e_A$ belongs to $\mathrm{Fil}_0\,K^\bullet(\mathfrak{L};\underline{D})$, and (5.6.1) is an isomorphism. Applying the same argument with a series in $L(0^+)$, we see that (5.6.2) is also an isomorphism. The compatibility with Frobenius actions results from the proof of 4.6.

Using a Cartan-Eilenberg resolution of $\Omega^\bullet_{V_K}\otimes L_{\pi,f}$, we obtain a canonical isomorphism

$$\tau^{\mathrm{II}}_{\ge1}\mathbb{R}\Gamma_{\mathrm{t}}(V_K,\Omega^\bullet_{V_K}\otimes L_{\pi,f}) \;\xrightarrow{\sim}\; \tau^{\mathrm{II}}_{\ge1}\mathbb{R}\Gamma_{\mathrm{t}}(\mathbb{P}^n_K, q_*(\Omega^\bullet_{V_K}\otimes L_{\pi,f}))$$

which defines by functoriality the upper horizontal morphism in (5.6.3). Using the first cohomology spectral sequences, the proof that it is an isomorphism is reduced to proving that the maps $H^i(\mathbb{P}^n_K, q_*(\Omega^r_{V_K}))\to H^i(\mathbb{P}^n_K,\Omega^r_{\mathbb{P}^n_K})$ are isomorphisms for all $i\ge1$ and all r. As $\Omega^1_{V/\mathbb{P}^n_S}\simeq q^*(\mathcal{O}_{\mathbb{P}^n_S}(d))$, there are exact sequences

$$0 \longrightarrow q^*(\Omega^r_{\mathbb{P}^n_S}) \longrightarrow \Omega^r_V \longrightarrow q^*(\Omega^{r-1}_{\mathbb{P}^n_S}(d)) \longrightarrow 0 \qquad (5.6.4)$$

for all r. They give rise to exact sequences

$$0 \longrightarrow \bigoplus_{k\ge0}\Omega^r_{\mathbb{P}^n_S}(kd) \longrightarrow q_*(\Omega^r_V) \longrightarrow \bigoplus_{k>0}\Omega^{r-1}_{\mathbb{P}^n_S}(kd) \longrightarrow 0,$$

in which the map $\Omega^r_{\mathbb{P}^n_S} \to q_*(\Omega^r_V)$ is a section of the map $q_*(\Omega^r_V) \to \Omega^r_{\mathbb{P}^n_S}$ defined by s. Since $H^i(\mathbb{P}^n_S, \Omega^r_{\mathbb{P}^n_S}(m)) = 0$ for all r and $i, m > 0$ (and any basis S) [26, 1.1], the claim follows. Moreover, since $\Omega^\bullet_{\mathbb{P}^n_K}$ is concentrated in positive degrees, (5.5.2) gives a canonical morphism

$$\tau_{\geq 1}\mathbb{R}\Gamma(\mathbb{P}^n_K, \Omega^\bullet_{\mathbb{P}^n_K}) \longrightarrow \tau^{\mathrm{II}}_{\geq 1}\mathbb{R}\Gamma_{\mathrm{t}}(\mathbb{P}^n_K, \Omega^\bullet_{\mathbb{P}^n_K}).$$

It is an isomorphism since $\Gamma(\mathbb{P}^n_K, \Omega^r_{\mathbb{P}^n_K}) = 0$ for $r \geq 1$, and this allows to replace $\tau^{\mathrm{II}}_{\geq 1}\mathbb{R}\Gamma_{\mathrm{t}}$ by $\tau_{\geq 1}\mathbb{R}\Gamma$ in the right column of (5.6.3).

The lower horizontal morphism in (5.6.3) is defined similarily. To prove that it is an isomorphism, one takes the exact sequence of analytic sheaves corresponding to (5.6.4) and one applies the exact functor $j^\dagger_V$. Taking direct images, one obtains an exact sequence

$$0 \to q^{\mathrm{an}}_*(j^\dagger_V(q^{\mathrm{an}\,*}(\Omega^r_{\mathbb{P}^{n\,\mathrm{an}}_K}))) \to q^{\mathrm{an}}_*(j^\dagger_V(\Omega^r_{V^{\mathrm{an}}_K})) \to q^{\mathrm{an}}_*(j^\dagger_V(q^{\mathrm{an}\,*}(\Omega^{r-1}_{\mathbb{P}^{n\,\mathrm{an}}_K}(d)))) \to 0.$$

The surjectivity here results from the fact that $R^i q^{\mathrm{an}}_*(j^\dagger_V(E)) = 0$ for $i \geq 1$ and any coherent $\mathcal{O}_{V^{\mathrm{an}}_K}$-module E. Indeed, if we denote by $\mathcal{V}_\rho$ the strict neighbourhoods of $\mathcal{V}$ introduced in the proof of lemma 4.5, and if $U \subset \mathbb{P}^{n\,\mathrm{an}}_K$ is any affinoid open subset, then $q^{\mathrm{an}\,-1}(U) \cap \mathcal{V}_\rho$ is affinoid, which implies that $H^i(q^{\mathrm{an}\,-1}(U), j^\dagger_V(E)) = 0$ for $i \geq 1$.

In this exact sequence, we have isomorphisms

$$\begin{aligned} q^{\mathrm{an}}_*(j^\dagger_V(\mathcal{O}_{V^{\mathrm{an}}_K})) \otimes \Omega^r_{\mathbb{P}^{n\,\mathrm{an}}_K} &\xrightarrow{\ \sim\ } q^{\mathrm{an}}_*(j^\dagger_V(q^{\mathrm{an}\,*}(\Omega^r_{\mathbb{P}^{n\,\mathrm{an}}_K}))), \\ q^{\mathrm{an}}_*(j^\dagger_V(\mathcal{O}_{V^{\mathrm{an}}_K})) \otimes \Omega^{r-1}_{\mathbb{P}^{n\,\mathrm{an}}_K}(d) &\xrightarrow{\ \sim\ } q^{\mathrm{an}}_*(j^\dagger_V(q^{\mathrm{an}\,*}(\Omega^{r-1}_{\mathbb{P}^{n\,\mathrm{an}}_K}(d)))). \end{aligned}$$

Furthermore, the canonical morphism $\mathcal{O}_{\mathbb{P}^{n\,\mathrm{an}}_K}(d) \to q^{\mathrm{an}}_*(j^\dagger_V(\mathcal{O}_{V^{\mathrm{an}}_K}))$ induces an isomorphism

$$q^{\mathrm{an}}_*(j^\dagger_V(\mathcal{O}_{V^{\mathrm{an}}_K}))(d) \xrightarrow{\ \sim\ } q^{\mathrm{an}}_*(j^\dagger_V(\mathcal{O}_{V^{\mathrm{an}}_K}))/\mathcal{O}_{\mathbb{P}^{n\,\mathrm{an}}_K}.$$

Therefore, proving that the lower horizontal line of (5.6.3) is an isomorphism is reduced as above to proving that

$$H^i(\mathbb{P}^{n\,\mathrm{an}}_K, q^{\mathrm{an}}_*(j^\dagger_V(\mathcal{O}_{V^{\mathrm{an}}_K})) \otimes \Omega^r_{\mathbb{P}^{n\,\mathrm{an}}_K}(d)) = 0$$

for all $i \geq 1$ and all r. One can then proceed as in the proof of lemma 4.5 to reduce this claim to a similar vanishing statement for the (algebraic) projective space over a $\mathbb{Z}/p^n\mathbb{Z}$-scheme, which results again from the vanishing of the spaces $H^i(\mathbb{P}^n_S, \Omega^r_{\mathbb{P}^n_S}(m))$ for all $i, m > 0$, all r, and any basis S.

Finally, one obtains the commutative square (5.6.3) by functoriality. As the right vertical arrow is an isomorphism, the same holds for the left one. It is clear that the lower isomorphism commutes with the actions of F by functoriality.

Corollary 5.7. *There exists distinguished triangles, related by a morphism of triangles defined by the specialization morphisms,*

$$\begin{array}{ccccccc}
\mathrm{Fil}_0\, K^\bullet(\mathfrak{L};\underline{D}) & \longrightarrow & \mathbb{R}\Gamma_{Z_K,\,\mathrm{dR}}(\mathbb{P}^n_K/K) & \longrightarrow & \tau_{\geq 1}\mathbb{R}\Gamma_{\mathrm{dR}}(\mathbb{P}^n_K/K) & \xrightarrow{+1} & \\
\downarrow & & \downarrow{\scriptstyle \rho_Z} & & \downarrow{\scriptstyle \wr} & & \\
\mathrm{Fil}_0\, K^\bullet(L(0^+);\underline{D}) & \longrightarrow & \mathbb{R}\Gamma_{Z_k,\,\mathrm{rig}}(\mathbb{P}^n_k/K) & \longrightarrow & \tau_{\geq 1}\mathbb{R}\Gamma_{\mathrm{rig}}(\mathbb{P}^n_k/K) & \xrightarrow{+1} & .
\end{array} \tag{5.7.1}$$

Moreover, the morphisms in the lower triangle commute with the following Frobenius actions:

a) α *on* $\mathrm{Fil}_0\, K^\bullet(L(0^+);\underline{D})$,
b) $q^{n+1}(F^*)^{-1}$ *on* $\mathbb{R}\Gamma_{Z,\,\mathrm{rig}}(\mathbb{P}^n_k/K)$,
c) $q^{n+1}(F^*)^{-1}$ *on* $\tau_{\geq 1}\mathbb{R}\Gamma_{\mathrm{rig}}(\mathbb{P}^n_k/K)$.

We only explain the construction of the analytic triangle, since the argument is similar in the algebraic case.

Applying (5.5.1) to the functor $\Gamma(\mathbb{P}^{n\,\mathrm{an}}_K,-)$ and to the complex $q_*^{\mathrm{an}}(j_V^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}})\otimes \mathcal{L}_{\pi,f})$, we obtain a distinguished triangle

$$\begin{array}{c}
\tau^{\mathrm{II}}_{\geq 1}\mathbb{R}\Gamma_{\mathrm{t}}(\mathbb{P}^{n\,\mathrm{an}}_K, q_*^{\mathrm{an}}(j_V^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}})\otimes \mathcal{L}_{\pi,f})) \\
{\scriptstyle +1}\swarrow \qquad\qquad \nwarrow \\
\Gamma(\mathbb{P}^{n\,\mathrm{an}}_K, q_*^{\mathrm{an}}(j_V^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}})\otimes \mathcal{L}_{\pi,f})) \longrightarrow \mathbb{R}\Gamma(\mathbb{P}^{n\,\mathrm{an}}_K, q_*^{\mathrm{an}}(j_V^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}})\otimes \mathcal{L}_{\pi,f}))
\end{array} \tag{5.7.2}$$

in which all arrows are compatible with F^*. Thanks to (5.6.2), we have an isomorphism

$$\mathrm{Fil}_0\, K^\bullet(L(0^+);\underline{D}) \xrightarrow{\ \sim\ } \Gamma(\mathbb{P}^{n\,\mathrm{an}}_K, q_*^{\mathrm{an}}(j_V^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}})\otimes \mathcal{L}_{\pi,f}))$$

which identifies α on $\mathrm{Fil}_0\, K^\bullet(L(0^+);\underline{D})$ to the endomorphism $\Psi_{\pi,f}$ defined by Tr_F. Therefore the composed morphism

$$\begin{aligned}
\mathrm{Fil}_0\, K^\bullet(L(0^+);\underline{D}) &\longrightarrow \mathbb{R}\Gamma(\mathbb{P}^{n\,\mathrm{an}}_K, q_*^{\mathrm{an}}(j_V^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}})\otimes \mathcal{L}_{\pi,f})) \\
&\simeq \mathbb{R}\Gamma(V_K^{\mathrm{an}}, j_V^\dagger(\Omega^\bullet_{V_K^{\mathrm{an}}})\otimes \mathcal{L}_{\pi,f}) \\
&= \mathbb{R}\Gamma_{\mathrm{rig}}(V_k/K, \mathcal{L}_{\pi,f})
\end{aligned}$$

commutes with α and $q^{n+1}(F^*)^{-1}$. On the other hand, theorem 3.1 (applied for $H=\emptyset$) provides an isomorphism

$$\mathbb{R}\Gamma_{\mathrm{rig}}(V_k/K, \mathcal{L}_{\pi,f}) \xrightarrow{\ \sim\ } \mathbb{R}\Gamma_{Z_k,\,\mathrm{rig}}(\mathbb{P}^n_k/K)$$

which commutes with F^*. This gives the definition of the first arrow in (5.7.1) and shows its compatibility with Frobenius actions.

To complete the construction of (5.7.1), we use the bottom line of (5.6.3), which gives an isomorphism

$$\begin{aligned}\tau^{\mathrm{II}}_{\geq 1}\mathbb{R}\Gamma_{\mathrm{t}}(\mathbb{P}^{n\,\mathrm{an}}_K, q^{\mathrm{an}}_*(j^\dagger_V(\Omega^\bullet_{V^{\mathrm{an}}_K})\otimes\mathcal{L}_{\pi,f})) &\xrightarrow{\ \sim\ } \tau_{\geq 1}\mathbb{R}\Gamma(\mathbb{P}^{n\,\mathrm{an}}_K, \Omega^\bullet_{\mathbb{P}^{n\,\mathrm{an}}_K})\\ &= \tau_{\geq 1}\mathbb{R}\Gamma_{\mathrm{rig}}(\mathbb{P}^n_k/K)\end{aligned}$$

compatible with F^*. The corollary follows.

5.8. To conclude, we explain how corollary 5.7 implies Dwork's formula (5.0.1).

Thanks to (5.4.1), it suffices to prove that, for any $A\subset N$, $A\neq\emptyset$,

$$\det(I-t\alpha|\operatorname{Fil}_0 K^\bullet(L_A(0^+);\underline{D}_A)) \;=\; P_A(qt)^{(-1)^{\#A}}.$$

Recall that $m(A)=\#A-1$. The triangle (5.7.1) relative to f_A gives

$$\begin{aligned}\det(I-t\alpha|\operatorname{Fil}_0 K^\bullet(L_A(0^+);\underline{D}_A)) \;=&\\ &\det(I-qt(q^{m(A)}(F^*)^{-1})|\mathbb{R}\Gamma_{Z_{A,k},\,\mathrm{rig}}(\mathbb{P}^{m(A)}_k/K))\\ &\cdot\det(I-qt(q^{m(A)}(F^*)^{-1})|\tau_{\geq 1}\mathbb{R}\Gamma_{\mathrm{rig}}(\mathbb{P}^{m(A)}_k/K))^{-1}.\end{aligned}$$

Since $q^{m(A)}(F^*)^{-1}=F_*$, the first factor is equal to $\zeta(Z_{A,k},qt)^{-1}$, while the second one is equal to $\zeta(\mathbb{P}^{m(A)}_k,qt)(1-q^{m(A)+1}t)$. The claim then follows from the definition of $P_A(t)$.

Remark. – When $A=N$, the previous equation can be written as

$$\det(I-t\alpha|\operatorname{Fil}_0 K^\bullet(L(0^+);\underline{D}))^{-1} \;=\; \zeta(Z_k,qt)(1-qt)\cdots(1-q^nt), \qquad (5.8.1)$$

a formula which does not seem to appear explicitly in Dwork's papers, but was pointed out by Adolphson and Sperber (*cf.* [1, p. 289]).

Appendix: Logarithmic versus rigid cohomology of an overconvergent isocrystal with logarithmic singularities

In the body of this paper, we made use of the following strengthened version of the comparison theorems proved in [5] and [14, 4.2].

We place ourselves in the local situation of 1.1 and consider a smooth formal scheme $\mathcal{X}$ over $\mathcal{S}=\operatorname{Spf} O_K$. We fix a divisor T of the special fiber X of $\mathcal{X}$ and let $V=X\setminus T$ denote the complementary open subscheme of X. We also choose a divisor $\mathcal{Z}$ in $\mathcal{X}$ with relative normal crossings over $\mathcal{S}$, union of smooth irreducible components $\mathcal{Z}_1,\dots,\mathcal{Z}_n$, and let $U=X\setminus Z$. We have the functor $j^\dagger_U$ (resp. $j^\dagger_V$) defined for abelian sheaves on $\mathcal{X}_K$, and more generally for abelian sheaves on any strict neighborhood of $]U[_{\mathcal{X}}$ (resp. of $]V[_{\mathcal{X}}$) in $\mathcal{X}_K$, by using the full system of strict neighborhoods of $]U[_{\mathcal{X}}$ (resp. of $]V[_{\mathcal{X}}$) in $\mathcal{X}_K$. We let $\mathrm{sp}:\mathcal{X}_K\longrightarrow\mathcal{X}$ be the specialization morphism.

Theorem A.1. *Let $\mathcal{L}$ be a coherent $j^\dagger_V\mathcal{O}_{\mathcal{X}_K}$-module, such that $\mathrm{sp}_*\mathcal{L}$ is a locally free $\mathcal{O}_{\mathcal{X},\mathbb{Q}}({}^\dagger T)$-module (of finite type). Assume $\mathcal{L}$ is endowed with an integrable* *connexion*

$$\nabla : \mathcal{L} \longrightarrow j_V^\dagger(\Omega^1_{\mathcal{X}_K}(\log \mathcal{Z}_K)) \otimes_{j_V^\dagger \mathcal{O}_{\mathcal{X}_K}} \mathcal{L} \tag{A.1.1}$$

with logarithmic singularities along $\mathcal{Z}_K$ *and that* $j_U^\dagger(\mathcal{L},\nabla) = j_{U\cap V}^\dagger(\mathcal{L},\nabla)$ *is overconvergent along* $Z \cup T$. *Consider the following assumption:*

> $(\mathbf{NL})_{\mathbf{G}}$ *The additive subgroup* Λ *of* K^{alg} *generated by the exponents of monodromy of* $(\mathcal{L},\nabla)$ *around the components of* $\mathcal{Z}_K$, *consists of p-adically non-Liouville numbers.*

Then, if the assumption $(\mathbf{NL})_{\mathbf{G}}$ *is satisfied, and none of the exponents of monodromy of* $(\mathcal{L},\nabla)$ *around the components of* $\mathcal{Z}_K$ *is a negative integer, the canonical inclusion of complexes of* K*-vector spaces on* $\mathcal{X}$

$$\iota_{\mathcal{X}} : \mathrm{sp}_*(j_V^\dagger(\Omega^\bullet_{\mathcal{X}_K}(\log \mathcal{Z}_K)) \otimes_{j_V^\dagger \mathcal{O}_{\mathcal{X}_K}} \mathcal{L}) \longrightarrow \mathrm{sp}_*(j_{U\cap V}^\dagger(\Omega^\bullet_{\mathcal{X}_K}) \otimes_{j_V^\dagger \mathcal{O}_{\mathcal{X}_K}} \mathcal{L}) \tag{A.1.2}$$

is a quasi-isomorphism.

We have the following general result [14, p. 355].

Proposition A.2. *Let* E *be a coherent* $j_V^\dagger \mathcal{O}_{\mathcal{X}_K}$*-module. For any open affine* $\mathcal{Y} \subset \mathcal{X}$, *and any* $q \geq 1$,

$$H^q(\mathcal{Y}_K, E) \;=\; 0. \tag{A.2.1}$$

Therefore:

(i) $R^q\mathrm{sp}_*E = 0$, $\forall q \geq 1$;

(ii) $H^q(\mathcal{Y}, \mathrm{sp}_*E) = 0$, *for any open affine* $\mathcal{Y} \subset \mathcal{X}$, *and* $q \geq 1$.

Let $h \in \mathcal{O}(\mathcal{Y})$ be a lifting of an equation of the divisor $T \cap Y$ of Y, and let $\Gamma_0 \subset \mathbb{R}^\times$ be the multiplicative group of absolute values of non zero elements of K, $\Gamma = \Gamma_0 \otimes \mathbb{Q} \subset \mathbb{R}^\times$. For $\lambda \in \Gamma$, let $j_\lambda : U_\lambda \hookrightarrow \mathcal{Y}_K$ be the inclusion of the open affinoid $U_\lambda = \{x \in \mathcal{Y}_K \mid |h(x)| \geq \lambda\}$. Then there exists a $\lambda_0 \in \Gamma$ and a coherent $\mathcal{O}_{U_{\lambda_0}}$-module $\mathcal{E}$ such that

$$E_{|\mathcal{Y}_K} \;=\; \varinjlim_{\substack{\lambda_0 \leq \lambda < 1 \\ \lambda \in \Gamma}} j_{\lambda *}\, j_{\lambda,\lambda_0}^{-1} \mathcal{E}, \tag{A.2.2}$$

where $j_{\lambda,\lambda_0} : U_\lambda \hookrightarrow U_{\lambda_0}$ denotes the immersion. So, $\mathcal{Y}_K$ being quasi-compact and separated, and by Kiehl's acyclicity theorem applied to the affinoid U_λ and the coherent sheaf $j_{\lambda,\lambda_0}^{-1}\mathcal{E}$,

$$H^q(\mathcal{Y}_K, E) \;=\; \varinjlim_{\substack{\lambda_0 \leq \lambda < 1 \\ \lambda \in \Gamma}} H^q(\mathcal{Y}_K, j_{\lambda *}\, j_{\lambda,\lambda_0}^{-1}\mathcal{E}) \;=\; \varinjlim_{\substack{\lambda_0 \leq \lambda < 1 \\ \lambda \in \Gamma}} H^q(U_\lambda, j_{\lambda,\lambda_0}^{-1}\mathcal{E}) = 0.$$

This proves (A.2.1). Then (i) follows because $R^q\mathrm{sp}_*E$ is the sheaf associated to the presheaf $\mathcal{U} \mapsto H^q(\mathcal{U}_K, E)$ on $\mathcal{X}$. Finally,

$$H^q(\mathcal{Y}, \mathrm{sp}_*E) \;=\; \mathcal{H}^q(\mathbb{R}\Gamma(\mathcal{Y}, -) \circ \mathbb{R}\mathrm{sp}_*(E)) \;=\; \mathcal{H}^q(\mathbb{R}\Gamma(\mathcal{Y}_K, E)) \;=\; H^q(\mathcal{Y}_K, E).$$

A.3. We now prove theorem A.1. Let $u : \mathcal{Y} \hookrightarrow \mathcal{X}$ be an open formal subscheme of $\mathcal{X}$. The inverse image on $\mathcal{Y}_K$ of the morphism $\iota_{\mathcal{X}}$ of (A.1.2) via u_K is the morphism $\iota_{\mathcal{Y}}$ for the data $(u_K)^{-1}(\mathcal{L})$, $T \cap \mathcal{Y}$, $\mathcal{Z} \cap \mathcal{Y}$ (*cf.* [12, 2.1.4.2]). So, it will suffice to prove that $\Gamma(\mathcal{Y}, \iota_{\mathcal{Y}})$ is a quasi isomorphism for all $\mathcal{Y}$ in a basis of open affine formal subschemes of $\mathcal{X}$. When $\mathcal{X}$ is affine, the previous proposition shows that $\Gamma(\mathcal{X}, \iota_{\mathcal{X}})$ identifies with the natural morphism of hypercohomology

$$\mathbb{R}\Gamma(\mathcal{X}_K, j_V^\dagger(\Omega^\bullet_{\mathcal{X}_K}(\log \mathcal{Z}_K)) \otimes_{j_V^\dagger \mathcal{O}_{\mathcal{X}_K}} \mathcal{L}) \longrightarrow \mathbb{R}\Gamma(\mathcal{X}_K, j_{U\cap V}^\dagger(\Omega^\bullet_{\mathcal{X}_K}) \otimes_{j_V^\dagger \mathcal{O}_{\mathcal{X}_K}} \mathcal{L}). \tag{A.3.1}$$

We will therefore prove that, for any $q \geq 0$, the natural morphism of hypercohomology groups

$$H^q(\mathcal{X}_K, j_V^\dagger(\Omega^\bullet_{\mathcal{X}_K}(\log \mathcal{Z}_K)) \otimes_{j_V^\dagger \mathcal{O}_{\mathcal{X}_K}} \mathcal{L}) \longrightarrow H^q(\mathcal{X}_K, j_{U\cap V}^\dagger(\Omega^\bullet_{\mathcal{X}_K}) \otimes_{j_V^\dagger \mathcal{O}_{\mathcal{X}_K}} \mathcal{L}) \tag{A.3.2}$$

is an isomorphism of K-vector spaces, when $\mathcal{X}$ is replaced by any element of a basis of open affine formal subschemes of the original formal scheme $\mathcal{X}$. We may then assume that $\mathcal{L}$ is a free $j_V^\dagger \mathcal{O}_{\mathcal{X}_K}$-module of finite type and that there exists a system of coordinates $t_1, \ldots, t_d$ on $\mathcal{X}$ such that $\mathcal{Z}$ is defined by $t_1 \cdots t_r = 0$. We choose a lifting $h \in \mathcal{O}(\mathcal{X})$ of an equation of T.

We now follow the arguments of [5], with minor modifications. For $\lambda, \sigma \in (0,1) \cap \Gamma$ we define strict neighborhoods $\{\mathbf{W}_{\lambda,\sigma}\}$ (resp. $\{\mathbf{V}_\sigma\}$) of $]U \cap V[_{\mathcal{X}}$ (resp. $]V[_{\mathcal{X}}$) in $\mathcal{X}_K$, by

$$\mathbf{W}_{\lambda,\sigma} \;=\; \{x \in \mathcal{X}_K \mid |h(x)| \geq \sigma, |t_1(x)| > \lambda, \ldots, |t_r(x)| > \lambda\},$$

while

$$\mathbf{V}_\sigma \;=\; \{x \in \mathcal{X}_K \mid |h(x)| \geq \sigma\},$$

and we denote by $j_{\lambda,\sigma} : \mathbf{W}_{\lambda,\sigma} \hookrightarrow \mathbf{V}_\sigma$, the open embedding. For σ sufficiently close to 1, say $\sigma \geq \sigma_0$, there is a free $\mathcal{O}_{\mathbf{V}_\sigma}$-module of finite type L_σ, equipped with a logarithmic connection

$$\nabla_\sigma : L_\sigma \longrightarrow \Omega^1_{\mathbf{V}_\sigma}(\log \mathcal{Z}_K \cap \mathbf{V}_\sigma) \otimes_{\mathcal{O}_{\mathbf{V}_\sigma}} L_\sigma, \tag{A.3.3}$$

such that $j_V^\dagger(L_\sigma, \nabla_\sigma) = (\mathcal{L}, \nabla)$. The condition of overconvergence of $(\mathcal{L}, \nabla)$ along $T \cup Z$ means that:

> (**SC**) There exist functions $\sigma, \lambda : (0,1) \cap \Gamma \longrightarrow (0,1) \cap \Gamma$ such that for any $\eta \in (0,1) \cap \Gamma$, $\sigma(\eta) \geq \sigma_0$, and for any open affinoid $C \subset \mathbf{W}_{\lambda(\eta),\sigma(\eta)}$ and any section $e \in \Gamma(C, L_{\sigma(\eta)})$
>
> $$\lim_{|\alpha| \to \infty} ||(\alpha!)^{-1} \nabla(\partial/\partial t)^\alpha(e)||_C \, \eta^{|\alpha|} \;=\; 0, \tag{A.3.4}$$
>
> where $\nabla(\partial/\partial t)^\alpha = \prod_{i=1}^d \nabla(\partial/\partial t_i)^{\alpha_i}$ and $||\ \ ||_C$ denotes any Banach norm on $\Gamma(C, L_{\sigma(\eta)})$.

For any $\alpha \in \mathbb{N}^d$ and $e \in \Gamma(\mathbf{V}_\sigma, L_\sigma)$, $t^\alpha \nabla(\partial/\partial t)^\alpha(e)$ has no poles along $\mathcal{Z}_K \cap \mathbf{V}_\sigma$. On the other hand, if C is a strict affinoid neighborhood of $]V[_{\mathcal{X}}$ of the form $C = \{x \in \mathcal{X}_K \mid |h(x)| \geq \epsilon\}$, for any $\epsilon \in \Gamma$, it is easy to check that, for any holomorphic function $f \in \Gamma(C, \mathcal{O}_C)$, $||f||_C = ||f||_{C\cap]U[_\mathcal{X}}$, where the Banach norms $||\ \ ||_C$ and $||\ \ ||_{C\cap]U[_\mathcal{X}}$ are related by the presentation $\Gamma(C\cap]U[_\mathcal{X}, \mathcal{O}_C) \simeq \Gamma(C, \mathcal{O}_C)\{T\}/(Tt_1\cdots t_r - 1)$. So, the condition of overconvergence of $(\mathcal{L}, \nabla)$ along $T \cup Z$ may be reformulated as follows:

$(\mathbf{SC})'$ There exists a function $\sigma : (0,1)\cap\Gamma \longrightarrow (0,1)\cap\Gamma$ such that for any $\eta \in (0,1)\cap\Gamma$, any open affinoid $C \subset \mathbf{V}_{\sigma(\eta)}$ and any section $e \in \Gamma(C, L_{\sigma(\eta)})$

$$\lim_{|\alpha|\to\infty} ||(\alpha!)^{-1}t^\alpha\nabla(\partial/\partial t)^\alpha(e)||_C\ \eta^{|\alpha|} \ = \ 0 \qquad \text{(A.3.5)}$$

where $||\ \ ||_C$ denotes any Banach norm on $\Gamma(C, L_{\sigma(\eta)})$.

Let $\eta_0, \eta_1 : (0,1)\cap\Gamma \longrightarrow (0,1)\cap\Gamma$ be non-decreasing functions such that $\lambda < \eta_0(\lambda) < \eta_1(\lambda)$, for all λ. We parametrize a fundamental system of strict neighborhoods $\{W_\lambda\}$ (resp. $\{V_\lambda\}$) of $]U\cap V[_\mathcal{X}$ (resp. $]V[_\mathcal{X}$) in $\mathcal{X}_K$, by $\lambda \in (0,1)\cap\Gamma$, as follows:

$$W_\lambda \ = \ \mathbf{W}_{\lambda,\sigma(\eta_1(\lambda))} \ = \ \{x \in \mathcal{X}_K \mid |h(x)| \geq \sigma(\eta_1(\lambda)), |t_1(x)| > \lambda, \ldots, |t_r(x)| > \lambda\},$$

while

$$V_\lambda \ = \ \mathbf{V}_{\sigma(\eta_1(\lambda))} \ = \ \{x \in \mathcal{X}_K \mid |h(x)| \geq \sigma(\eta_1(\lambda))\},$$

where σ is a function satisfying property $(\mathbf{SC})'$ such that $\sigma(\eta) > \eta$ for any $\eta \in (0,1)\cap\Gamma$, and we denote by $j_\lambda : W_\lambda \hookrightarrow V_\lambda$ the open embedding. We abusively write $(L_\lambda, \nabla_\lambda)$ for $(L_{\sigma(\eta_1(\lambda))}, \nabla_{\sigma(\eta_1(\lambda))})$.

It will be enough to compare, for any $\lambda \in (0,1)\cap\Gamma$, the effect in hypercohomology of the morphism of complexes of abelian sheaves on V_λ

$$\Omega^\bullet_{V_\lambda}(\log(\mathcal{Z}_K \cap V_\lambda)) \otimes L_\lambda \ \longrightarrow \ j_{\lambda *}(\Omega^\bullet_{W_\lambda} \otimes L_{\lambda|W_\lambda}) \qquad \text{(A.3.6)}$$

We slightly modify the construction of [5, §4], and, for any fixed $\lambda \in (0,1)\cap\Gamma$, we construct an admissible covering $\{\mathbf{U}_{\mathcal{S},\lambda}\}$ of $\mathcal{X}_K$ parametrized by the subsets $\mathcal{S}$ of $\mathcal{T} = \{1, \ldots, r\}$, where

$$\mathbf{U}_{\mathcal{S},\lambda} = \{x \in \mathcal{X}_K \mid |t_i(x)| < \eta_1(\lambda), \text{for } i \in \mathcal{S}, |t_j(x)| \geq \eta_0(\lambda), \text{for } j \in \mathcal{T} \setminus \mathcal{S}\}. \qquad \text{(A.3.7)}$$

As shown in [5, 4.2], $\mathbf{U}_{\mathcal{S},\lambda}$ is a trivial bundle in open polydisks of radius $\eta_1(\lambda)$ of relative dimension $s = \operatorname{card}\mathcal{S}$ over the affinoid space

$$\mathbf{V}_{\mathcal{S},\lambda} = \{x \in \mathcal{X}_K \mid |t_i(x)| = 0, \text{for } i \in \mathcal{S}, |t_j(x)| \geq \eta_0(\lambda), \text{for } j \in \mathcal{T} \setminus \mathcal{S}\}. \qquad \text{(A.3.8)}$$

Let h_0 be the restriction of the function h to the closed analytic subspace $\mathbf{V}_{\mathcal{S},\lambda}$. Under the previous description of $\mathbf{V}_{\mathcal{S},\lambda}$, $\mathbf{U}_{\mathcal{S},\lambda}\cap V_\lambda$ identifies with $\mathbf{W}_{\mathcal{S},\lambda}\times D^s(0, \eta_1(\lambda)^-)$, the trivial bundle in open polydisks of radius $\eta_1(\lambda)$ of relative dimension s over the affinoid space

$$\mathbf{W}_{\mathcal{S},\lambda} = \{x \in \mathbf{V}_{\mathcal{S},\lambda} \mid |h_0(x)| \geq \sigma(\eta_1(\lambda))\}. \qquad \text{(A.3.9)}$$

Under the previous identification, $\mathbf{U}_{\mathcal{S},\lambda} \cap W_\lambda$ identifies with

$$\mathbf{W}_{\mathcal{S},\lambda} \times C^s(0,(\lambda,\eta_1(\lambda))),$$

the trivial bundle in open polyannuli of interior radius λ and exterior radius $\eta_1(\lambda)$ of relative dimension s over $\mathbf{W}_{\mathcal{S},\lambda}$. Let now $A := \mathcal{O}(\mathbf{W}_{\mathcal{S},\lambda})$ be the affinoid algebra corresponding to $\mathbf{W}_{\mathcal{S},\lambda}$ and $|| \quad ||_A$ be any Banach norm on A. We set $A_{\eta_1(\lambda)}\{\{x\}\} = A_{\eta_1(\lambda)}\{\{x_1,\ldots,x_s\}\} := \mathcal{O}(\mathbf{W}_{\mathcal{S},\lambda} \times D^s(0,\eta_1(\lambda)^-))$, where $x = (x_1,\ldots,x_s)$ denotes an ordered arrangement of $\{t_i, i \in \mathcal{S}\}$, a topological K-algebra with the locally convex topology defined by the family of seminorms

$$|| \sum_{\alpha\in\mathbb{N}^s} a_\alpha x^\alpha ||_\epsilon = \sup_\alpha ||a_\alpha||_A \epsilon^{|\alpha|},$$

indexed by $\epsilon \in (0,\eta_1(\lambda))$. Let $\mathcal{L}_A$ be the K-Lie-algebra of continuous derivations of A/K, trivially extended to $A_{\eta_1(\lambda)}\{\{x\}\}$, and $\mathcal{L}_{A,x}$ be the K-Lie-subalgebra of the K-Lie-algebra of continuous derivations of $A_{\eta_1(\lambda)}\{\{x\}\}/K$ generated by $\mathcal{L}_A$ and the $x_i\partial/\partial x_i$, for $i = 1,\ldots,s$. So, the free $A_{\eta_1(\lambda)}\{\{x\}\}$-module of finite type $M := L_\lambda(\mathbf{W}_{\mathcal{S},\lambda} \times C^s(0,(\lambda,\eta_1(\lambda))))$, carries an action of $\mathcal{L}_{A,x}$. By a trivial dilatation, we may assume that $\eta_1(\lambda) = 1$, so that $A_{\eta_1(\lambda)}\{\{x\}\} = A\{\{x\}\}$ in the notation of [6, 4.1.1]. It readily follows from condition $(\mathbf{SC})'$ that M satisfies the local overconvergence condition [6, 5.1]

($\mathbf{SC_L}$) For all $m \in M$ and ϵ, $\eta \in (0,1)$

$$\lim_{|\alpha|\to\infty} ||(\alpha!)^{-1} x^\alpha \nabla(\partial/\partial x)^\alpha(m)||_\epsilon \, \eta^{|\alpha|} \;=\; 0, \qquad \text{(A.3.10)}$$

So, in the notation of [6, §5], M is an object of $SCMod^f_{A,\Sigma}(A\{\{x\}\},\mathcal{L}_{A,x})$, where $\Sigma := \Lambda + \mathbb{Z}$ is an additive subgroup of K^{alg} which, by the assumption $(\mathbf{NL})_\mathbf{G}$, does not contain any p-adically Liouville number. So, the classification result [6, 6.5.2] applies and we may assume that M is a λ-simple object of $SCMod^f_{A,\Sigma}(A\{\{x\}\},\mathcal{L}_{A,x})$ [6, §4.2], for $\lambda = (\lambda_1,\ldots,\lambda_s) \in \Sigma^s$. As in the proof of [5, 6.4], we can assume, by localisation on an admissible affinoid covering of $\mathbf{W}_{\mathcal{S},\lambda}$, that $M = A\{\{x\}\}$ with trivial action of $\mathcal{L}_A$, while $x_i\partial/\partial x_i$, for $i = 1,\ldots,s$, acts via $\nabla(x_i\partial/\partial x_i) = x_i\partial/\partial x_i + \lambda_i$. Our result then follows from [5, 6.6, (i)].

Corollary A.4. *Under the hypothesis of* A.1*, let us assume in addition that $\mathcal{X}$ is proper over $\mathcal{S}$. Then, for any strict neighborhoood W of $]V[_\mathcal{X}$, there exists a canonical isomorphism*

$$\mathbb{R}\Gamma(W, j_V^\dagger(\Omega^\bullet_{\mathcal{X}_K}(\log \mathcal{Z}_K)) \otimes_{j_V^\dagger\mathcal{O}_{\mathcal{X}_K}} \mathcal{L}) \xrightarrow{\;\sim\;} \mathbb{R}\Gamma_{\mathrm{rig}}(U\cap V,\mathcal{L}), \qquad \text{(A.4.1)}$$

where we still denote by $\mathcal{L}$ the overconvergent isocrystal on $U \cap V$ defined by $j^\dagger_{U\cap V}(\mathcal{L})$.

From (1.1.1), we deduce that the canonical morphism

$$\mathbb{R}\Gamma(\mathcal{X}_K, j_V^\dagger(\Omega^\bullet_{\mathcal{X}_K}(\log \mathcal{Z}_K)) \otimes_{j_V^\dagger\mathcal{O}_{\mathcal{X}_K}} \mathcal{L}) \longrightarrow \mathbb{R}\Gamma(W, j_V^\dagger(\Omega^\bullet_{\mathcal{X}_K}(\log \mathcal{Z}_K)) \otimes_{j_V^\dagger\mathcal{O}_{\mathcal{X}_K}} \mathcal{L})$$

is an isomorphism. On the other hand, applying the functor $\mathbb{R}\Gamma(\mathcal{X}, -)$ to the isomorphism (A.1.2) and using A.2 (i), we get an isomorphism

$$\mathbb{R}\Gamma(\mathcal{X}_K, j_V^\dagger(\Omega^\bullet_{\mathcal{X}_K}(\log \mathcal{Z}_K)) \otimes_{j_V^\dagger \mathcal{O}_{\mathcal{X}_K}} \mathcal{L}) \xrightarrow{\sim} \mathbb{R}\Gamma(\mathcal{X}_K, j_{U\cap V}^\dagger(\Omega^\bullet_{\mathcal{X}_K}) \otimes_{j_V^\dagger \mathcal{O}_{\mathcal{X}_K}} \mathcal{L}).$$

Since $\mathcal{X}$ is proper, the target space is equal to $\mathbb{R}\Gamma_{\mathrm{rig}}(U \cap V, \mathcal{L})$, and the corollary follows.

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